Maxwell's Equations

Ampere's Law

$$\nabla \times \underline{H} = \underline{J} \qquad \oint_C \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{ds} + \underbrace{\frac{d}{dt}\int \underline{D} \cdot \underline{ds}}_{\text{displacement current}}$$

$$\nabla \cdot \underline{B} = 0 \qquad \oint_S \underline{B} \cdot \underline{ds} = 0$$

Faraday's Law:

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \oint_C \underline{E} \cdot \underline{dl} = -\frac{d}{dt}\int_S \underline{B} \cdot \underline{ds}$$

$$\nabla \cdot \underline{D} = \rho \qquad \oint_S \underline{D} \cdot \underline{ds} = \int_V \rho \, dv$$

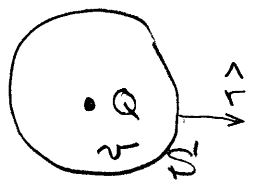
$$\underline{D} = \epsilon \underline{E}$$

$$\underline{B} = \mu \underline{H}$$

$$\underline{J} = \sigma \underline{E}$$

Electric fields from charge distributions

Electric field from a point charge



1) Field has spherical symmetry, so $\underline{E} = E_r \hat{r}$
2) Use Gauss' Law

$$\oint_S \underline{D} \cdot \underline{ds} = \int_V \rho \, dv$$

$$\oint_S \underline{D} \cdot \underline{ds} = \oint D_r \hat{r} \cdot \hat{r} \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r r^2 \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{4\pi} = 4\pi D_r r^2$$

$$\int_v \rho \, dv = Q$$

$$\therefore \quad D_r 4\pi r^2 = Q \quad \text{or} \quad D_r = \frac{Q}{4\pi r^2}$$

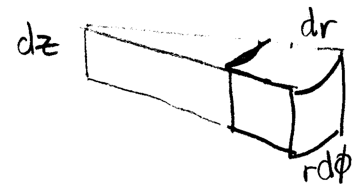
$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

NOTE

cylindrical coordinates

$ds = r \, dr \, d\phi$

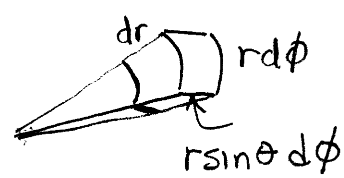
$dv = r \, dr \, d\phi \, dz$



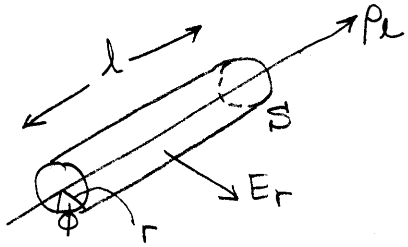
spherical coordinates

$ds = r^2 \sin\theta \, dr \, d\theta$

$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$



Example 4.12 Line charge



Whenever there is symmetry use Gauss Law

$$\oint_S \underline{D}\cdot \underline{ds} = \oint \epsilon_0 \underline{E}\cdot \underline{ds}$$

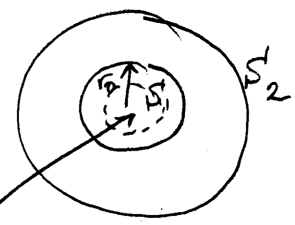
$$= \epsilon_0 \int_0^{2\pi} E_r \; l r d\phi = Q_{enc}$$

$$Q_{enc} = \rho_l \, l$$

$$\therefore \quad \epsilon_0 E_r \, l r \, 2\pi = \rho_l \, l$$

$$E_r = \frac{\rho_l}{2\pi\epsilon_0 r}$$

Example 4.13 Spherical cloud of charge



(1) Field has spherical symmetry whether $r > r_0$ or $r < r_0$

(2) Use Gauss' Law

charge density ρ_v for $r < r_0$ $\qquad Q = \frac{4}{3}\pi a^3 \rho_v$

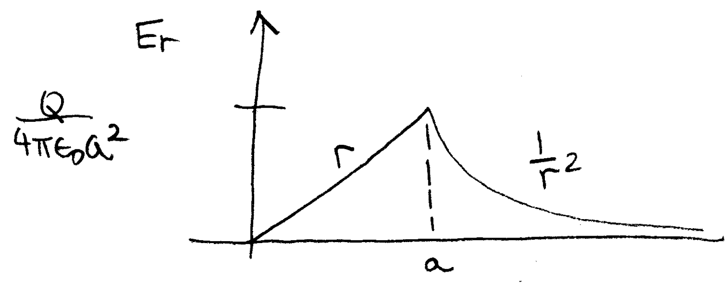
$$\underbrace{\oint_{S_1 \text{ or } S_2} \underline{D} \cdot d\underline{S} = \oint_{S_1 \text{ or } S_2} \epsilon_0 \underline{E} \cdot r^2 \sin\theta \, d\theta \, d\phi}_{4\pi\epsilon_0 E_r r^2} = \begin{cases} \int_0^r \int_0^{2\pi} \int_0^{\pi} \rho_v r^2 \sin\theta \, d\theta \, d\phi \, dr & r < a \\ \rho_v \cdot \frac{4}{3}\pi a^3 & r > a \end{cases}$$

for $r > a$ $\quad 4\pi\epsilon_0 E_r r^2 = \rho_v \frac{4}{3}\pi a^3$ or $E_r = \dfrac{\rho_v \frac{4}{3}\pi a^3}{4\pi\epsilon_0 r^2} = \dfrac{Q}{4\pi\epsilon_0 r^2}$

for $r < a$ $\quad Q_{inc} = \frac{4}{3}\pi r^3 \rho_v$

$$\cancel{4\pi}\epsilon_0 E_r r^2 = \cancel{\tfrac{4}{3}}\pi r^3 \rho_v$$

$$\text{or } E_r = \frac{r\rho_v}{3\epsilon_0} = \frac{r\,Q}{\frac{4}{\cancel{3}}\pi a^3 \cdot \cancel{3}\epsilon_0} = \frac{Q\,r}{4\pi\epsilon_0 a^3}$$



When we don't have symmetry we use electric potential

electric potential at any point P is given by

$$\Phi(P) \equiv \left[\frac{W}{q}\right] = -\int_{\infty}^{P} \underline{E}\cdot d\underline{l}$$

work per unit charge in putting unit charge at P

$$\Phi(P) = \Phi(x,y,z) = -\int_{\infty}^{r}\left(\hat{r}\,\frac{Q}{4\pi\epsilon_0|\underline{r}|^2}\right)\cdot\left(\hat{r}\,dr\right) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

the work done in moving a charge from $a \to b$ is the electrostatic potential difference between the two points

$$\left[\frac{W}{q}\right]_{a\to b} = \Phi_{ab} = \int_{a}^{b} \underline{E}\cdot d\underline{l} = \Phi(b) - \Phi(a)$$

Electrostatic potential is very useful as it is a scalar.

It can be readily shown that

$$(dW)_x = q\,\Phi(x+\Delta x, y, z) - q\,\Phi(x,y,z) = q\,\frac{\partial\Phi}{\partial x}\Delta x$$

In three dimensions

$$dW = q\left(\frac{\partial\Phi}{\partial x}\Delta x + \frac{\partial\Phi}{\partial y}\Delta y + \frac{\partial\Phi}{\partial z}\Delta z\right) = -q\underline{E}\cdot\underline{\Delta l}$$

where $\underline{\Delta l} = (\hat{x}\,\Delta x + \hat{y}\,\Delta y + \hat{z}\,\Delta z)$

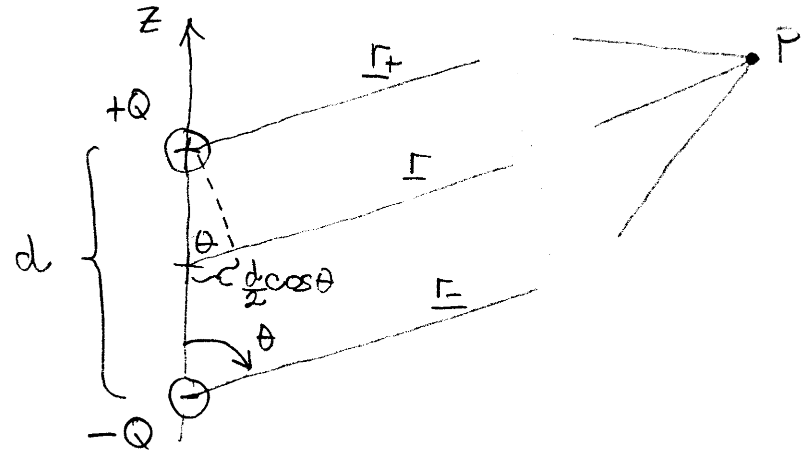
$$\Rightarrow\ \underline{E} = -\left(\hat{x}\frac{\partial\Phi}{\partial x} + \hat{y}\frac{\partial\Phi}{\partial y} + \hat{z}\frac{\partial\Phi}{\partial z}\right)$$

or $\underline{E} = -\nabla\Phi$

4.4.3. Electrostatic Potential resulting from multiple point charges,

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\underline{r} - \underline{r}_k'|}$$

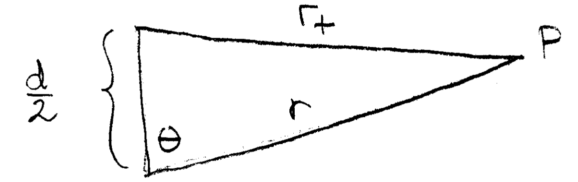
The electric dipole



Summing the potentials

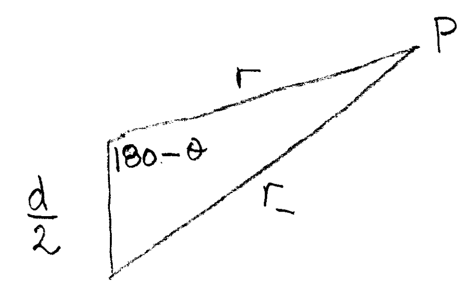
$$\Phi = \frac{+Q}{4\pi\epsilon_0 r_+} + \frac{-Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

Now use law of cosines



$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2(r)\left(\frac{d}{2}\right)\cos\theta$$

$$r_+ = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}$$



$$r_-^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2(r)\left(\frac{d}{2}\right)\cos(\pi - \theta)$$

$$r_- = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta}$$

In almost every case $r \gg d$, i.e. P is far away

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+\left(\frac{d}{2}\right)^2 - rd\cos\theta}} - \frac{1}{\sqrt{r^2+\left(\frac{d}{2}\right)^2 + rd\cos\theta}}\right)$$

Rewrite denominators and expand as a Taylor series

$$\frac{1}{r\sqrt{1+\left(\frac{d}{2r}\right)^2 - \left(\frac{d}{r}\cos\theta\right)}} \cong \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{d}{2r}\right)^2 + \frac{d}{2r}\cos\theta + \dots\right]$$

neglect this term

$$(1+u)^{-\frac{1}{2}} = 1 - \frac{1}{2}u + \dots$$

Then

$$\Phi \cong \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\left[1+\frac{d}{2r}\cos\theta\right] - \frac{1}{r}\left[1-\frac{d}{2r}\cos\theta\right]\right)$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left(\frac{d}{r}\cos\theta\right)$$

$$\Phi = \frac{Qd\cos\theta}{4\pi\epsilon_0 r}$$

$$\underline{E} = -\underline{\nabla}\Phi$$

$$= -\left[\hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]$$ in spherical coordinates no ϕ dependence

$$= +\hat{r}\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3} + \hat{\theta}\frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}$$

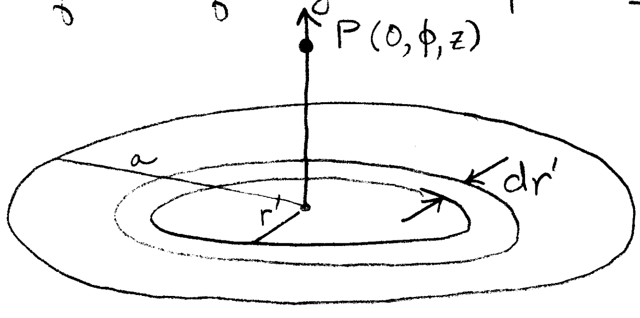
$$\underline{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right].$$

In general we can extend discrete charge distributions to continuous ones.

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum \frac{Q}{|\underline{r}-\underline{r}_k|}$$

$$+ \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\underline{r}')\,dv'}{|\underline{r}-\underline{r}'|} \quad \text{etc.}$$

4-10 Potential of a disk of charge at a point $\underline{P}$ on axis



$$\Phi = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s(r')\,ds'}{|\underline{r}-\underline{r}'|}$$

$$\Phi(r=0) = \frac{1}{4\pi\epsilon_0} \int_0^a \int_0^{2\pi} \frac{\rho_s \quad r'dr'\, d\phi}{\sqrt{z^2+(r')^2}}$$

$$= \frac{\rho_s\, 2\pi}{4\pi\epsilon_0} \int_0^a \frac{r'dr'}{\sqrt{z^2+(r')^2}} + C$$

constant of integration

$$= \frac{\rho_s}{2\epsilon_0} (r'^2 + z^2)^{\frac{1}{2}} \Big|_0^a + C$$

$$= \frac{\rho_s}{2\epsilon_0} \left[(a^2+z^2)^{\frac{1}{2}} - |z| \right] + C$$

we require $\Phi \to 0$ as $z \to \infty$ $\Rightarrow C = 0$

$$\Phi(r=0) = \frac{\rho_s}{2\epsilon_0} \left[\sqrt{a^2+z^2} - |z| \right].$$

$$E_z(P) = -\frac{\partial \Phi}{\partial z} = \begin{cases} \frac{\rho}{2\epsilon_0} \left[1 - z(a^2+z^2)^{-\frac{1}{2}} \right] & z > 0 \\ \frac{\rho}{2\epsilon_0} \left[1 + z(a^2+z^2)^{-\frac{1}{2}} \right] & r < 0 \end{cases}$$

Poisson's & Laplace's Equations

Gauss' Law $\quad \underline{\nabla} \cdot \underline{D} = \rho \quad$ or $\quad \underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0}$

but $\underline{E} = -\underline{\nabla}\Phi$ so $\underline{\nabla} \cdot (-\underline{\nabla}\Phi) = \frac{\rho}{\epsilon_0}$ or $\nabla \cdot \nabla\Phi = -\frac{\rho}{\epsilon_0}$

in rectangular coordinates

$$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\Phi = 0$$

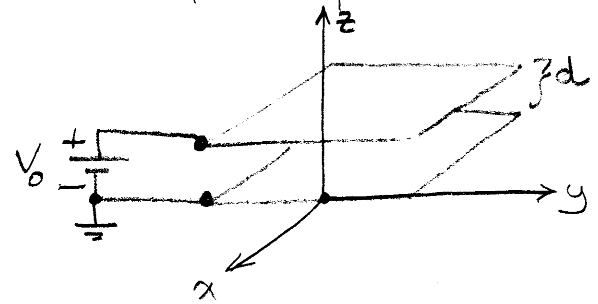
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = -\rho/\epsilon_0$$

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} \qquad \text{Poisson's equation}$$

In charge-free regions this reduces to

$$\nabla^2\Phi = 0 \qquad \text{Laplace's Equation}$$

Example 4-23: Two parallel plates



Find $\Phi(x,y,z)$ and $\underline{E}(x,y,z)$ between the plates.

In rectangular coordinates Laplace's equation is

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \quad \text{since charge free}$$

From symmetry no dependence on x or y

$$\therefore \quad \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial z^2} = 0$$

$$\Rightarrow \Phi(z) = C_1 z + C_2 \quad \text{where } C_1, C_2 \text{ come from the B.C.'s}$$

$$\Phi(0) = 0 \qquad \Phi(+d) = +V_o$$

$$\Phi(0) = 0 = C_2$$

$$\Phi(+d) = +V_o = C_1 d \quad \therefore \; C_1 = \frac{V_o}{d}$$

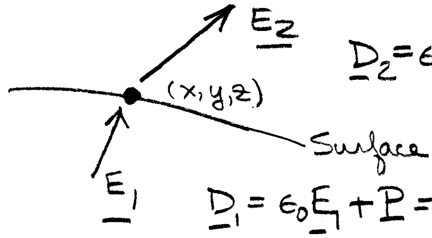
$$\Phi(z) = \frac{V_o}{d} z$$

$$\underline{E} = -\underline{\nabla} \Phi(z) = -\hat{z} \frac{\partial \Phi}{\partial z} = -\hat{z} \frac{V_o}{d}$$

To find the charge densities on the plates we use the boundary conditions

p. 82-84 (my notes), Section 4.11 Inan & Inan #1

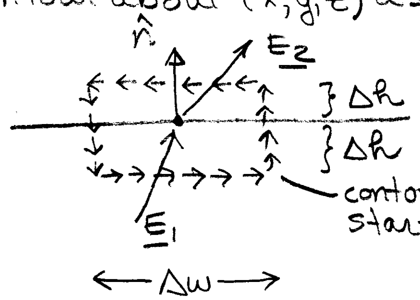
Consider the case of E fields on different sides of a surface



$\underline{D}_2 = \epsilon_0 \underline{E}_2$

$\underline{D}_1 = \epsilon_0 \underline{E}_1 + \underline{P} = \epsilon \underline{E}_1$

Choose a contour about (x,y,z) as



$\hat{n}$ from 1 to 2

contour C
start here and go clockwise

Since $\underline{E}$ is conservative $\oint \underline{E} \cdot d\underline{l} = 0$ (no time dependent B field)

Doing contour integral

$$\underbrace{E_{1n}\Delta h + E_{2n}\Delta h}_{} - \underbrace{E_{2t}\Delta w - E_{2n}\Delta h}_{\text{cancel}} - \underbrace{E_{1n}\Delta h}_{\text{cancel}} + E_{1t}\Delta w = 0$$

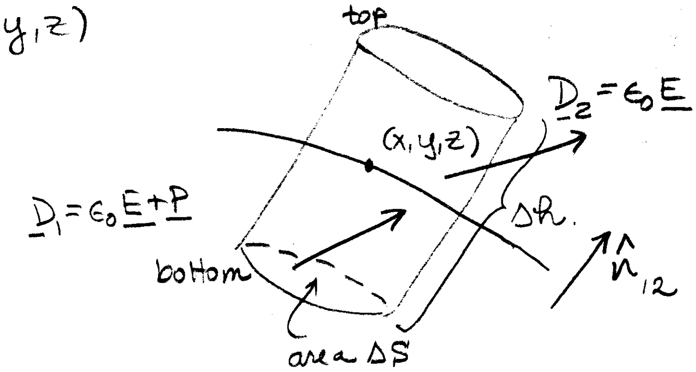
Assume Δh and Δw are small enough that $\underline{E}_1$ & $\underline{E}_2$ are constant on C

This leaves $E_{1t}\Delta w - E_{2t}\Delta w = 0$

or $E_{1t} = E_{2t}$

mathematically $\hat{n}_{12} \times (\underline{E}_2 - \underline{E}_1) = 0$

For the normal component of E choose a small volume centered on (x, y, z)



Using Gauss' Law

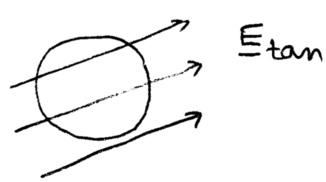
$$\oint_S \underline{D} \cdot \underline{ds} = \rho_f \, \Delta h \, \Delta S$$

↑ any charge density in volume

$$\int_{top} \underline{D} \cdot \underline{ds} + \int_{sides} \underline{D} \cdot \underline{ds} + \int_{bottom} \underline{D} \cdot \underline{ds} = \rho_f \, \Delta h \, \Delta S$$

Consider the integral over the sides as $\Delta S \to 0$

top view



consider the integral $\underline{D} \cdot \underline{ds}$

$$\underline{D} \cdot \underline{ds} = \epsilon \underline{E}_{tan} \cdot \hat{r}$$

since E_{tan} approx. constant
and $\hat{r}$ rotates as it goes around the surface
then $\oint \underline{D} \cdot \underline{ds} \to 0$

This leaves

$$\int_{top} \underline{D} \cdot \underline{ds} + \int_{bottom} \underline{D} \cdot \underline{ds} = \rho_f \, \Delta h \, \Delta S$$

Since $\Delta h \, \Delta S \to 0$ we can assume $\underline{D}$ is constant on each surface

$$D_{2n} \cancel{\Delta S} - D_{1n} \cancel{\Delta S} = \rho_f \, \Delta h \, \cancel{\Delta S}$$

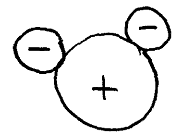
minus sign due to surface

$$D_{2n} - D_{1n} = \rho_f \, \Delta h \to \rho_s$$

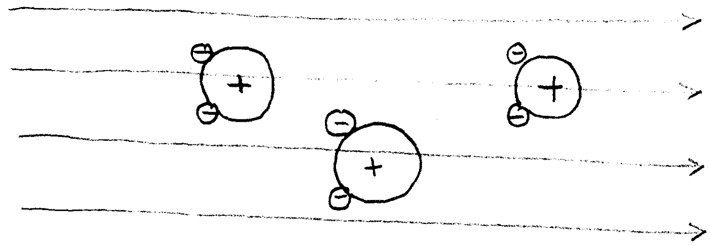
$$\hat{n}_{12} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_s$$

Polarization

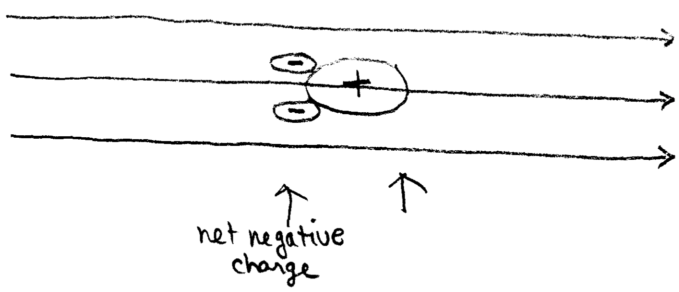
dielectric - insulating material in which charge separation occurs at the microscopic level due to an applied $\underline{E}$ field



consider a water molecule



alignment with applied fields



notice also that field can stretch the atoms

(this creates electric dipoles)

define a macroscopic average

$$\underline{P} \triangleq \lim_{\Delta v \to 0} \frac{\sum_i \underline{p}_i}{\Delta v}$$

dipole moment

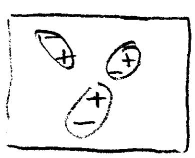
$$= N \underline{p}$$

number density of dipoles

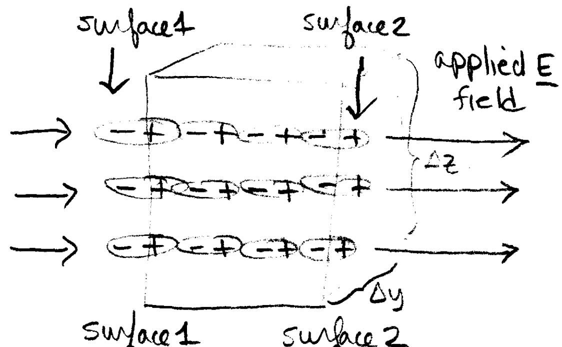
Note: $\underline{p} = Qd\,\hat{z}$

in bulk materials

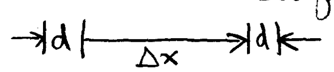
randomly distributed



align with applied field



This gives rise to surface charge



net bound charge is zero!

The charge on each surface is given by

$$dq_1 = \underbrace{-\,(Nqd)}_{\text{dipole density}}\Big|_{x_0} \underbrace{\Delta y\, \Delta z}_{\text{volume of surface layer.}}$$

surface #1

Relate this charge to polarization

We have to relate this to bound charge

$$dq_1 = +\,(Nqd)\Big|_{x_0} \Delta y\, \Delta z$$

multiply $\underline{P}$ by surface area to get Q at surface 1.

$$P_x(x_0) = Nqd\Big|_{x_0}$$

$$P_x(x_0+\Delta x) = Nqd\Big|_{x_0+\Delta x}$$

polarization is a linear quantity, i.e. dipole moment

The net bound charge is then (this is for volume).

$$dq_{total} = \underbrace{P_x(x_0)\,\Delta y\, \Delta z}_{\substack{\text{note sign is +}\\ \text{since this is}\\ \text{bound charge}}} - \underbrace{P_x(x_0+\Delta x)\,\Delta y\, \Delta z}_{\substack{\text{this sign is −}\\ \text{since it is bound}}}$$

$$= -\,\frac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x}\,\underbrace{\Delta x\, \Delta y\, \Delta z}_{\Delta v}$$

The bound volume charge density is then given by

$$\rho_b = \lim_{\Delta v \to 0} \frac{dq_{total}}{\Delta v} = \lim_{\Delta v \to 0} -\,\frac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x}$$

$$= -\,\frac{dP_x}{dx}\Big|_{x_0}$$

Extending to three dimensions

$$\rho_b = -\,\underline{\nabla} \cdot \underline{P}$$

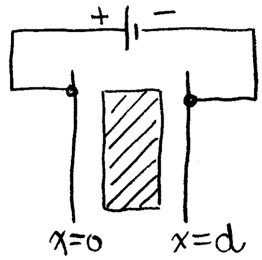
This can be written at the macroscopic level

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$$= \epsilon_0 \underline{E} + \chi_e \epsilon_0 \underline{E}$$

$$= \epsilon \underline{E}$$

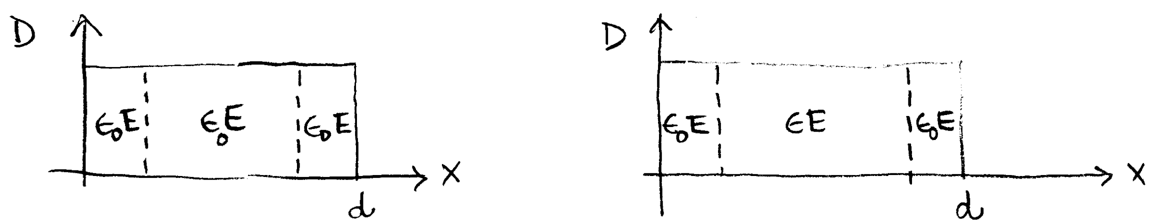
Now lets go back to the parallel plate capacitor with a dielectric, polarizable block between the plates.



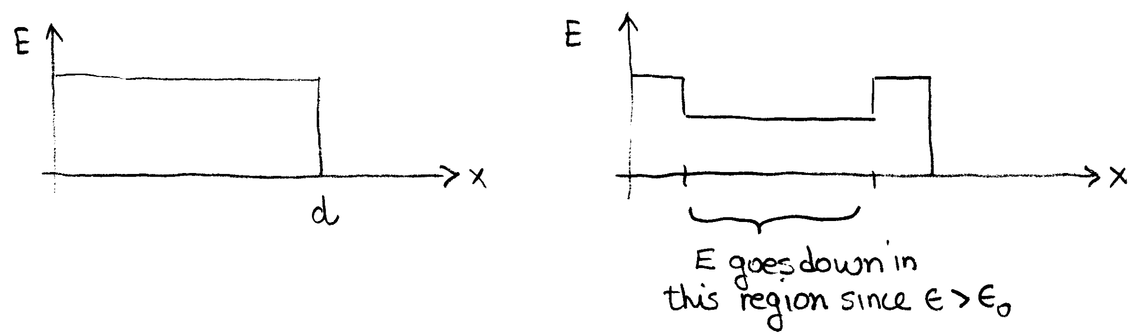
capacitor with no block

with dielectric block

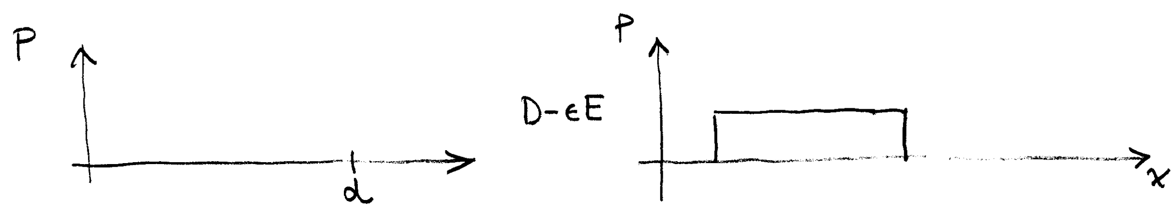
since normal D is continuous



but E is not continuous

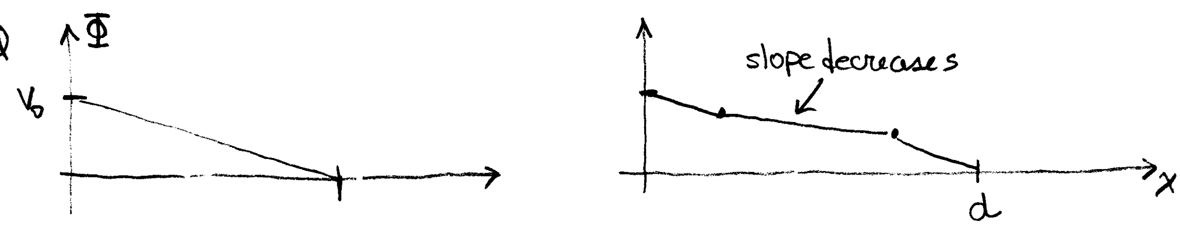


The difference between E and D is P

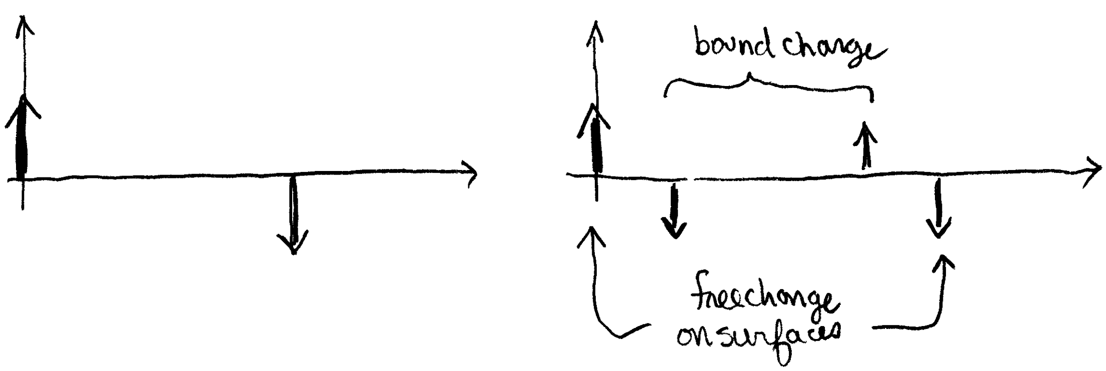


Φ is the integral of E

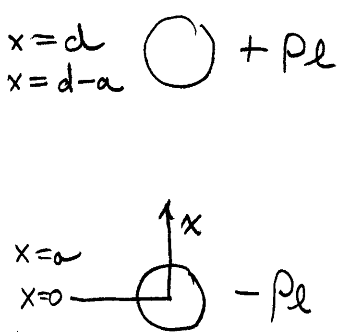
$E = -\frac{d\Phi}{dx}$



derivative of E gives surface charge density



Example 4-29. Capacitance of a Two-Wire Line



Very useful problem for telephony, radio transmission lines, and power transmission.

General solution is complicated because "proximity effect" causes charge densities to be larger on facing sides

We will solve for $d \gg a$ to avoid this effect

This is a very difficult problem to solve for the potential directly.

See Inan & Inan, Engineering Electromagnetics, Example 4-11

The potential from a finite length of charge $-l$ to $+l$ is given by

$$\Phi(\underline{P}) = -\frac{\rho_l}{4\pi\epsilon_0} \ln\left[\frac{z-a+[r^2+(z-a)^2]^{\frac{1}{2}}}{z+a+[r^2+(z+a)^2]^{\frac{1}{2}}}\right]$$

You could compute $\underline{E} = -\underline{\nabla}\Phi$ from this function but it is VERY complex as shown in this example.

Furthermore, we are interested in infinite length lines where $l \to \infty$

See Paris & Hurd, Basic Electromagnetic Theory, Example 3-3

As $l \to \infty$ the arguement of the $\ln[\ ]$ function increases without limit and Φ is undefined. The problem is that our previous expressions for Φ, i.e. $\Phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho}{r}\,dv$, work only for bounded charge distributions.

The best method to find the potential associated with an infinite line charge is the integral form of Gauss' Law.

If we use this expression for the E field between the two conductors we can write an expression for the E field between the two conductors as

$$E_x(x,0,0) \doteq \frac{-\rho_\ell}{2\pi\epsilon_0 x} \underset{\text{direction}}{\mp} \frac{\rho_\ell}{2\pi\epsilon_0 (d-x)}$$

ignores a

We will need to use the most general definition of capacitance

$$C \triangleq \frac{Q}{\Phi_{12}} = \frac{\oint \underline{D}\cdot \underline{ds}}{-\int_L \underline{E}\cdot \underline{dl}}$$

Gauss Law

defn of potential

Since these are infinite lines Q is readily given as ρ_ℓ, the charge per unit length.

Φ_{12} can be integrated as follows (remember we can use <u>any path</u>)

$$\Phi_{12} = -\frac{1}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{-\rho_\ell}{x} - \frac{\rho_\ell}{d-x}\right]dx$$

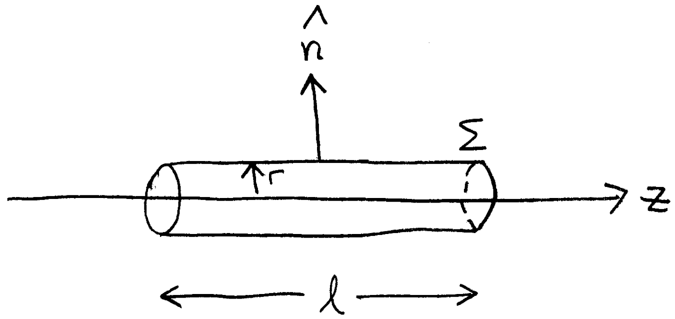
$$= \frac{\rho_\ell}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{1}{x} - \frac{1}{d-x}\right]dx = \frac{\rho_\ell}{2\pi\epsilon_0}\Big[\ln x - \ln(d-x)\Big]_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left(\frac{x}{d-x}\right)\Bigg|_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\left[\ln\left(\frac{d-a}{+a}\right) - \ln\left(\frac{a}{d-a}\right)\right] = \frac{\rho_\ell}{2\pi\epsilon_0}\, 2\ln\left(\frac{d-a}{a}\right)$$

and, for $d \gg a$,

$$\Phi_{12} \cong \frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)$$



$$\oint_S \underline{D}\cdot\hat{n}\,ds = \int \rho\, d\tau = Q_{enclosed}$$

By symmetry $\quad D_r \cdot 2\pi r \cdot l = \rho_l\, l \quad$ where ρ_l is the line charge density

$$D_r = \frac{\rho_l}{2\pi r}$$

$$\underline{D} = \frac{\rho_l}{2\pi r}\hat{r}$$

$$\underline{E} = \frac{\rho_l}{2\pi\epsilon_0 r}\hat{r}$$

Since $\underline{E} = -\underline{\nabla}\Phi$ and the field is circularly symmetric, i.e.

$\frac{\partial\Phi}{\partial\phi} \to 0$ and $\frac{\partial\Phi}{\partial z} = 0$ since it does not matter where we put our origin.

$$\frac{\rho_l}{2\pi\epsilon_0 r}\hat{r} = \underline{E} = -\underline{\nabla}\Phi = -\frac{\partial\Phi}{\partial r}\hat{r}$$

or, $\quad \Phi(r) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r + c$

If we pick a reference potential $\Phi(r=r_0)=0$

$$0 = \Phi(r=r_0) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r_0 + c$$

and $\Phi(r) = \frac{\rho_l}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right)$

We can compute the capacitance <u>per unit length</u> as

$$C \cong \frac{\rho_\ell}{\frac{\rho_\ell}{\pi \epsilon_0} \ln\left(\frac{d}{a}\right)} = \frac{\pi \epsilon_0}{\ln\left(\frac{d}{a}\right)}$$

For example, a 115kV transmission line uses two 1.407cm aluminum conductors separated by 3meters

$$\text{or } C \cong \frac{\pi \left(8.854 \times 10^{-12}\ F/m\right)}{\ln\left(\frac{3}{.01407}\right)} = 5.19\ nF/km.$$

<u>Hint for future problems</u>

You can compute C for conductor configurations for which you have derived $\underline{E}$ or Φ. For example, a single cylindrical conductor above a ground plane is that of this twin line with a infinitely large conducting sheet between the two conductors. You can also use the "method of images."

Laplace's Equation solved using separation of variables

$$\nabla^2 \Phi = 0$$

Assume $\Phi = f(u)\, g(v)$

Then $$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = g \frac{\partial^2 f}{\partial u^2} + f \frac{\partial^2 g}{\partial v^2} = 0$$

Divide by fg to get

$$\frac{1}{f} \frac{\partial^2 f}{\partial u^2} + \frac{1}{g} \frac{\partial^2 g}{\partial v^2} = 0$$

For this to be true the first term must equal the second term independent of variables, i.e.

$$\frac{1}{g} \frac{\partial^2 g}{\partial v^2} = -\frac{1}{f} \frac{\partial^2 f}{\partial u^2} \quad \text{independent of } u \text{ and } v$$

This can be written as

$$\frac{1}{f} \frac{\partial^2 f}{\partial u^2} = k_u^2 \quad \text{where } k_u \text{ is a constant}$$

and $$\frac{1}{g} \frac{\partial^2 g}{\partial v^2} = k_v^2 \quad \text{where } k_u^2 = -k_v^2$$

These are differential equations which need to have the boundary conditions specified. You can specify either Φ or $\frac{\partial \Phi}{\partial n}$

Φ Dirichlet boundary condition

$\frac{\partial \Phi}{\partial n}$ Neumann boundary condition

Since $D_{normal} = \epsilon E_{normal} = -\epsilon \frac{\partial \Phi}{\partial n}$

equivalent to specifying charge density.

mixed combination of Dirichlet & Neumann

The solution is dependent upon the actual value of k.

If $k_u^2 > 0$

$$\frac{d^2f}{du^2} - k_u^2 f = 0$$

$$f(u) = c_1 e^{+k_u u} + c_2 e^{-k_u u}$$

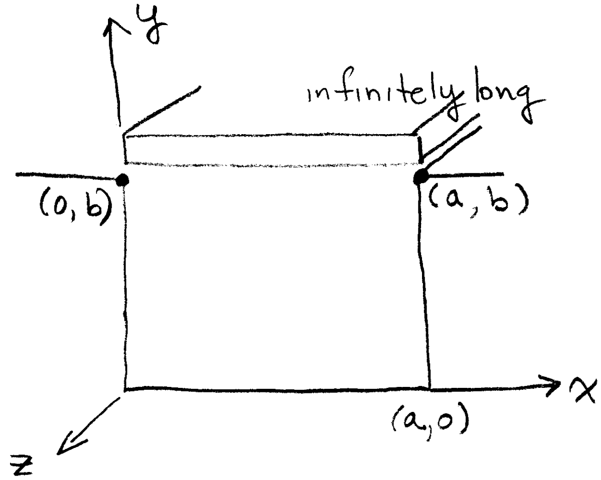
If $k_u^2 < 0$

$$\frac{d^2f}{du^2} + k_u^2 f = 0$$

$$f(u) = c_1 \sin k_u u + c_2 \cos k_u u$$

If $k_u^2 = 0$

$$f(u) = c_1 + c_2 u$$



Use these boundary conditions

$$\Phi(0,y) = 0$$
$$\Phi(x,0) = 0$$
$$\Phi(a,y) = 0$$
$$\Phi(x,b) = V \sin \frac{\pi x}{a}$$

Let $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$ and further let $\Phi = f(x)\, g(y)$

As before $\quad g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0$

and dividing $\quad \frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} = 0$

Separate to get

$$\frac{1}{f} \frac{d^2 f}{dx^2} = k_x^2 \quad \text{and} \quad \frac{1}{g} \frac{d^2 g}{dy^2} = 0 \quad \text{where } k_x^2 + k_y^2 = 0$$

Which one is positive depends upon the boundary conditions. Since $\Phi(x,b) = V \sin \frac{\pi x}{a}$ we want the x solution to be sinusoidal, or $k_x^2 < 0$.

$$\frac{d^2 f}{dx^2} - k_x^2 f = 0 \Rightarrow f = c_1 \sin k_x x + c_2 \cos k_x x$$

This requires $k_y^2 > 0$ and

$$\frac{d^2 g}{dy^2} + k_y^2 g = 0 \Rightarrow g = c_3 e^{+k_y y} + c_4 e^{-k_y y}$$

$$\Phi(x,y) = (c_1 \sin k_x x + c_2 \cos k_x x)(c_3 e^{+k_y y} + c_4 e^{-k_y y})$$

Since we want $\Phi(x,b) = V \sin \frac{\pi x}{a}$
this allows us to set $c_2 = 0$ and pick $k_x = \frac{\pi}{a}$

Then

$$\Phi(x,y) = (\sin k_x x)(c_3 e^{+k_y y} + c_4 e^{-k_y y})$$

At this point the problem becomes interesting

You cannot use C_3 and C_4 to make $\Phi(0,y) = 0$ and $\Phi(a,y) = 0$ using exponentials.
Solution use the sine function

For $x = 0$ $\sin k_x x = 0$ always

For $x = a$ $\sin k_x a = 0$ when $k_x a = n\pi$ $n = 1, 2, 3$

Using these results we can write $\Phi(x,y)$ as

$$\Phi(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)\left[c_3 e^{-\frac{n\pi}{a} y} + c_4 e^{+\frac{n\pi}{a} y}\right]$$

since $k_y^2 = \left(\frac{n\pi}{a}\right)^2$ Note that my sign is correct since if $k_x^2 < 0$ for sinusoidal solutions, then $k_y^2 > 0$.

At $y = 0$ $\Phi(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)(c_3 + c_4)$

Since this is independent of x $c_3 = -c_4$.

$$\therefore \Phi(x,y) = \sum_{n=1}^{\infty} c \sin\left(\frac{n\pi}{a} x\right)\left[e^{-\frac{n\pi}{a} y} - e^{+\frac{n\pi}{a} y}\right]$$

Now look at $y=b$ where $\Phi(x,b) = V \sin \frac{\pi x}{a}$

$$\Phi(x,b) = \sum_{n=1}^{\infty} c \sin\left(\frac{n\pi x}{a}\right)\left[e^{-\frac{n\pi}{a}b} - e^{+\frac{n\pi}{a}b}\right] = V\sin\left(\frac{\pi x}{a}\right)$$

Now this is only true if $n=1$ where

$$c \, \cancel{\sin\left(\frac{\pi x}{a}\right)}\left[e^{-\frac{\pi b}{a}} - e^{+\frac{\pi}{a}b}\right] = V\cancel{\sin\left(\frac{\pi x}{a}\right)}$$

$$\therefore \; c = \frac{V}{e^{-\frac{\pi b}{a}} - e^{+\frac{\pi b}{a}}} = \frac{V}{2\sinh\left(\frac{\pi b}{a}\right)}$$

Final solution,

$$\Phi(x,y) = c \sin\left(\frac{\pi}{a}x\right)\left[e^{-\frac{\pi}{a}y} - e^{+\frac{\pi}{a}y}\right]$$