Standard Mathematical Tables

Eighteenth Edition

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*Derivatives

In the following formulas u, v, w represent functions of x, while a, c, n represent fixed real numbers. All arguments in the trigonometric functions are measured in radians, and all inverse trigonometric and hyperbolic functions represent principal values.

1. $\frac{d}{dx}(a) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(au) = a\frac{du}{dx}$

4. $\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$

5. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

6. $\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$

7. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{1}{v}\frac{du}{dx} - \frac{u}{v^2}\frac{dv}{dx}$

8. $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

9. $\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

10. $\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$

11. $\frac{d}{dx}\left(\frac{1}{u^n}\right) = -\frac{n}{u^{n+1}}\frac{du}{dx}$

12. $\frac{d}{dx}\left(\frac{u^n}{v^m}\right) = \frac{u^{n-1}}{v^{m+1}}\left(nv\frac{du}{dx} - mu\frac{dv}{dx}\right)$

13. $\frac{d}{dx}(u^n v^m) = u^{n-1}v^{m-1}\left(nv\frac{du}{dx} + mu\frac{dv}{dx}\right)$

14. $\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\cdot\frac{du}{dx}$

* Let $y = f(x)$ and $\frac{dy}{dx} = \frac{d[f(x)]}{dx} = f'(x)$ define respectively a function and its derivative for any value x in their common domain. The differential for the function at such a value x is accordingly defined as

$$dy = d[f(x)] = \frac{dy}{dx}dx = \frac{d[f(x)]}{dx}dx = f'(x)\,dx$$

Each derivative formula has an associated differential formula. For example, formula 6 above has the differential formula

$$d(uvw) = uv\,dw + vw\,du + uw\,dv$$

DERIVATIVES (Continued)

15. $\frac{d^2}{dx^2}[f(u)] = \frac{df(u)}{du} \cdot \frac{d^2u}{dx^2} + \frac{d^2f(u)}{du^2} \cdot \left(\frac{du}{dx}\right)^2$

16. $\frac{d^n}{dx^n}[uv] = \binom{n}{0} v \frac{d^n u}{dx^n} + \binom{n}{1} \frac{dv}{dx} \frac{d^{n-1}u}{dx^{n-1}} + \binom{n}{2} \frac{d^2v}{dx^2} \frac{d^{n-2}u}{dx^{n-2}}$

$$+ \cdots + \binom{n}{k} \frac{d^kv}{dx^k} \frac{d^{n-k}u}{dx^{n-k}} + \cdots + \binom{n}{n} u \frac{d^nv}{dx^n}$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ the binomial coefficient, n non-negative integer and $\binom{n}{0} = 1$.

17. $\frac{du}{dx} = \frac{1}{\frac{dx}{du}} \quad \text{if} \frac{dx}{du} \neq 0$

18. $\frac{d}{dx}(\log_a u) = (\log_a e)\frac{1}{u}\frac{du}{dx}$

19. $\frac{d}{dx}(\log_e u) = \frac{1}{u}\frac{du}{dx}$

20. $\frac{d}{dx}(a^u) = a^u(\log_e a)\frac{du}{dx}$

21. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

22. $\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + (\log_e u)u^v\frac{dv}{dx}$

23. $\frac{d}{dx}(\sin u) = \frac{du}{dx}(\cos u)$

24. $\frac{d}{dx}(\cos u) = -\frac{du}{dx}(\sin u)$

25. $\frac{d}{dx}(\tan u) = \frac{du}{dx}(\sec^2 u)$

26. $\frac{d}{dx}(\cot u) = -\frac{du}{dx}(\csc^2 u)$

27. $\frac{d}{dx}(\sec u) = \frac{du}{dx}\sec u \cdot \tan u$

28. $\frac{d}{dx}(\csc u) = -\frac{du}{dx}\csc u \cdot \cot u$

29. $\frac{d}{dx}(\text{vers } u) = \frac{du}{dx}\sin u$

30. $\frac{d}{dx}(\arcsin u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad \left(-\frac{\pi}{2} \leq \arcsin u \leq \frac{\pi}{2}\right)$

DERIVATIVES (Continued)

31. $\frac{d}{dx}(\text{arc cos } u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad (0 \le \text{arc cos } u \le \pi)$

32. $\frac{d}{dx}(\text{arc tan } u) = \frac{1}{1+u^2}\frac{du}{dx}, \quad \left(-\frac{\pi}{2} < \text{arc tan } u < \frac{\pi}{2}\right)$

33. $\frac{d}{dx}(\text{arc cot } u) = -\frac{1}{1+u^2}\frac{du}{dx}, \quad (0 \le \text{arc cot } u \le \pi)$

34. $\frac{d}{dx}(\text{arc sec } u) = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \quad \left(0 \le \text{arc sec } u < \frac{\pi}{2}, -\pi \le \text{arc sec } u < -\frac{\pi}{2}\right)$

35. $\frac{d}{dx}(\text{arc csc } u) = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}, \quad \left(0 < \text{arc csc } u \le \frac{\pi}{2}, -\pi < \text{arc csc } u \le -\frac{\pi}{2}\right)$

36. $\frac{d}{dx}(\text{arc vers } u) = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx}, \quad (0 \le \text{arc vers } u \le \pi)$

37. $\frac{d}{dx}(\sinh u) = \frac{du}{dx}(\cosh u)$

38. $\frac{d}{dx}(\cosh u) = \frac{du}{dx}(\sinh u)$

39. $\frac{d}{dx}(\tanh u) = \frac{du}{dx}(\text{sech}^2 u)$

40. $\frac{d}{dx}(\coth u) = -\frac{du}{dx}(\text{csch}^2 u)$

41. $\frac{d}{dx}(\text{sech } u) = -\frac{du}{dx}(\text{sech } u \cdot \tanh u)$

42. $\frac{d}{dx}(\text{csch } u) = -\frac{du}{dx}(\text{csch } u \cdot \coth u)$

43. $\frac{d}{dx}(\sinh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2+1})] = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

44. $\frac{d}{dx}(\cosh^{-1} u) = \frac{d}{dx}[\log(u + \sqrt{u^2-1})] = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad (u > 1, \cosh^{-1} u > 0)$

45. $\frac{d}{dx}(\tanh^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{1+u}{1-u}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 < 1)$

46. $\frac{d}{dx}(\coth^{-1} u) = \frac{d}{dx}\left[\frac{1}{2}\log\frac{u+1}{u-1}\right] = \frac{1}{1-u^2}\frac{du}{dx}, \quad (u^2 > 1)$

47. $\frac{d}{dx}(\text{sech}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1-u^2}}{u}\right] = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad (0 < u < 1, \text{sech}^{-1} u > 0)$

48. $\frac{d}{dx}(\text{csch}^{-1} u) = \frac{d}{dx}\left[\log\frac{1+\sqrt{1+u^2}}{u}\right] = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$

49. $\dfrac{d}{dq}\displaystyle\int_p^q f(x)\,dx = f(q),$ [p constant]

50. $\dfrac{d}{dp}\displaystyle\int_p^q f(x)\,dx = -f(p),$ [q constant]

51. $\dfrac{d}{da}\displaystyle\int_p^q f(x,a)\,dx = \int_p^q \frac{\partial}{\partial a}[f(x,a)]\,dx + f(q,a)\frac{dq}{da} - f(p,a)\frac{dp}{da}$

INTEGRALS

ELEMENTARY FORMS

1. $\int a\,dx = ax$

2. $\int a \cdot f(x)\,dx = a\int f(x)\,dx$

3. $\int \phi(y)\,dx = \int \frac{\phi(y)}{y'}\,dy, \qquad \text{where } y' = \frac{dy}{dx}$

4. $\int (u + v)\,dx = \int u\,dx + \int v\,dx, \qquad$ where u and v are any functions of x

5. $\int u\,dv = u\int dv - \int v\,du = uv - \int v\,du$

6. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

7. $\int x^n\,dx = \frac{x^{n+1}}{n+1}, \qquad \text{except } n = -1$

8. $\int \frac{f'(x)\,dx}{f(x)} = \log f(x), \qquad (df(x) = f'(x)\,dx)$

9. $\int \frac{dx}{x} = \log x$

10. $\int \frac{f'(x)\,dx}{2\sqrt{f(x)}} = \sqrt{f(x)}, \qquad (df(x) = f'(x)\,dx)$

11. $\int e^x\,dx = e^x$

12. $\int e^{ax}\,dx = e^{ax}/a$

13. $\int b^{ax}\,dx = \frac{b^{ax}}{a\log b}, \qquad (b > 0)$

14. $\int \log x\,dx = x\log x - x$

15. $\int a^x \log a\,dx = a^x, \qquad (a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$

INTEGRALS (Continued)

17. $$\int \frac{dx}{a^2 - x^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{a+x}{a-x}, \quad (a^2 > x^2) \end{cases}$$

18. $$\int \frac{dx}{x^2 - a^2} = \begin{cases} -\frac{1}{a}\coth^{-1}\frac{x}{a} \\ \text{or} \\ \frac{1}{2a}\log\frac{x-a}{x+a}, \quad (x^2 > a^2) \end{cases}$$

19. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1}\frac{x}{|a|} \\ \text{or} \\ -\cos^{-1}\frac{x}{|a|}, \quad (a^2 > x^2) \end{cases}$$

20. $$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$$

21. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

22. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

FORMS CONTAINING $(a + bx)$

For forms containing $a + bx$, but not listed in the table, the substitution $u = \frac{a + bx}{x}$ may prove helpful.

23. $$\int (a + bx)^n\, dx = \frac{(a + bx)^{n+1}}{(n+1)b}, \quad (n \neq -1)$$

24. $$\int x(a + bx)^n\, dx$$

$$= \frac{1}{b^2(n+2)}(a + bx)^{n+2} - \frac{a}{b^2(n+1)}(a + bx)^{n+1}, \quad (n \neq -1, -2)$$

25. $$\int x^2(a + bx)^n\, dx = \frac{1}{b^3}\left[\frac{(a + bx)^{n+3}}{n+3} - 2a\frac{(a + bx)^{n+2}}{n+2} + a^2\frac{(a + bx)^{n+1}}{n+1}\right]$$

INTEGRALS (Continued)

26. $$\int x^m(a+bx)^n\,dx = \begin{cases} \dfrac{x^{m+1}(a+bx)^n}{m+n+1} + \dfrac{an}{m+n+1}\displaystyle\int x^m(a+bx)^{n-1}\,dx \\ \text{or} \\ \dfrac{1}{a(n+1)}\left[-x^{m+1}(a+bx)^{n+1} + (m+n+2)\displaystyle\int x^m(a+bx)^{n+1}\,dx\right] \\ \text{or} \\ \dfrac{1}{b(m+n+1)}\left[x^m(a+bx)^{n+1} - ma\displaystyle\int x^{m-1}(a+bx)^n\,dx\right] \end{cases}$$

27. $$\int \frac{dx}{a+bx} = \frac{1}{b}\log(a+bx)$$

28. $$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

29. $$\int \frac{dx}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}$$

30. $$\int \frac{x\,dx}{a+bx} = \begin{cases} \dfrac{1}{b^2}[a+bx-a\log(a+bx)] \\ \text{or} \\ \dfrac{x}{b} - \dfrac{a}{b^2}\log(a+bx) \end{cases}$$

31. $$\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left[\log(a+bx) + \frac{a}{a+bx}\right]$$

32. $$\int \frac{x\,dx}{(a+bx)^n} = \frac{1}{b^2}\left[\frac{-1}{(n-2)(a+bx)^{n-2}} + \frac{a}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2$$

33. $$\int \frac{x^2\,dx}{a+bx} = \frac{1}{b^3}\left[\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\log(a+bx)\right]$$

34. $$\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left[a+bx-2a\log(a+bx) - \frac{a^2}{a+bx}\right]$$

35. $$\int \frac{x^2\,dx}{(a+bx)^3} = \frac{1}{b^3}\left[\log(a+bx) + \frac{2a}{a+bx} - \frac{a^2}{2(a+bx)^2}\right]$$

36. $$\int \frac{x^2\,dx}{(a+bx)^n} = \frac{1}{b^3}\left[\frac{-1}{(n-3)(a+bx)^{n-3}} + \frac{2a}{(n-2)(a+bx)^{n-2}} - \frac{a^2}{(n-1)(a+bx)^{n-1}}\right], \qquad n \neq 1, 2, 3$$

INTEGRALS (Continued)

37. $\int \frac{dx}{x(a+bx)} = -\frac{1}{a}\log\frac{a+bx}{x}$

38. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\log\frac{a+bx}{x}$

39. $\int \frac{dx}{x(a+bx)^3} = \frac{1}{a^3}\left[\frac{1}{2}\left(\frac{2a+bx}{a+bx}\right)^2 + \log\frac{x}{a+bx}\right]$

40. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\log\frac{a+bx}{x}$

41. $\int \frac{dx}{x^3(a+bx)} = \frac{2bx-a}{2a^2x^2} + \frac{b^2}{a^3}\log\frac{x}{a+bx}$

42. $\int \frac{dx}{x^2(a+bx)^2} = -\frac{a+2bx}{a^2x(a+bx)} + \frac{2b}{a^3}\log\frac{a+bx}{x}$

FORMS CONTAINING $c^2 \pm x^2$, $x^2 - c^2$

43. $\int \frac{dx}{c^2+x^2} = \frac{1}{c}\tan^{-1}\frac{x}{c}$

44. $\int \frac{dx}{c^2-x^2} = \frac{1}{2c}\log\frac{c+x}{c-x}, \qquad (c^2 > x^2)$

45. $\int \frac{dx}{x^2-c^2} = \frac{1}{2c}\log\frac{x-c}{x+c}, \qquad (x^2 > c^2)$

46. $\int \frac{x\,dx}{c^2 \pm x^2} = \pm\frac{1}{2}\log(c^2 \pm x^2)$

47. $\int \frac{x\,dx}{(c^2 \pm x^2)^{n+1}} = \mp\frac{1}{2n(c^2 \pm x^2)^n}$

48. $\int \frac{dx}{(c^2 \pm x^2)^n} = \frac{1}{2c^2(n-1)}\left[\frac{x}{(c^2 \pm x^2)^{n-1}} + (2n-3)\int\frac{dx}{(c^2 \pm x^2)^{n-1}}\right]$

49. $\int \frac{dx}{(x^2 - c^2)^n} = \frac{1}{2c^2(n-1)}\left[-\frac{x}{(x^2 - c^2)^{n-1}} - (2n-3)\int\frac{dx}{(x^2 - c^2)^{n-1}}\right]$

50. $\int \frac{x\,dx}{x^2-c^2} = \frac{1}{2}\log(x^2-c^2)$

51. $\int \frac{x\,dx}{(x^2 - c^2)^{n+1}} = -\frac{1}{2n(x^2 - c^2)^n}$

INTEGRALS (Continued)

FORMS CONTAINING $a + bx$ and $c + dx$

$$u = a + bx, \qquad v = c + dx, \qquad k = ad - bc$$

If $k = 0$, then $v = \frac{c}{a}u$

52. $$\int \frac{dx}{u \cdot v} = \frac{1}{k} \cdot \log\left(\frac{v}{u}\right)$$

53. $$\int \frac{x\,dx}{u \cdot v} = \frac{1}{k}\left[\frac{a}{b}\log(u) - \frac{c}{d}\log(v)\right]$$

54. $$\int \frac{dx}{u^2 \cdot v} = \frac{1}{k}\left(\frac{1}{u} + \frac{d}{k}\log\frac{v}{u}\right)$$

55. $$\int \frac{x\,dx}{u^2 \cdot v} = \frac{-a}{bku} - \frac{c}{k^2}\log\frac{v}{u}$$

56. $$\int \frac{x^2\,dx}{u^2 \cdot v} = \frac{a^2}{b^2ku} + \frac{1}{k^2}\left[\frac{c^2}{d}\log(v) + \frac{a(k - bc)}{b^2}\log(u)\right]$$

57. $$\int \frac{dx}{u^n \cdot v^m} = \frac{1}{k(m-1)}\left[\frac{-1}{u^{n-1} \cdot v^{m-1}} - (m + n - 2)b\int \frac{dx}{u^n \cdot v^{m-1}}\right]$$

58. $$\int \frac{u}{v}\,dx = \frac{bx}{d} + \frac{k}{d^2}\log(v)$$

59. $$\int \frac{u^m\,dx}{v^n} = \begin{cases} \dfrac{-1}{k(n-1)}\left[\dfrac{u^{m+1}}{v^{n-1}} + b(n - m - 2)\displaystyle\int \dfrac{u^m}{v^{n-1}}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n-m-1)}\left[\dfrac{u^m}{v^{n-1}} + mk\displaystyle\int \dfrac{u^{m-1}}{v^n}\,dx\right] \\ \text{or} \\ \dfrac{-1}{d(n-1)}\left[\dfrac{u^m}{v^{n-1}} - mb\displaystyle\int \dfrac{u^{m-1}}{v^{n-1}}\,dx\right] \end{cases}$$

FORMS CONTAINING $(a + bx^n)$

60. $$\int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a}, \qquad (ab > 0)$$

61. $$\int \frac{dx}{a + bx^2} = \begin{cases} \dfrac{1}{2\sqrt{-ab}}\log\dfrac{a + x\sqrt{-ab}}{a - x\sqrt{-ab}}, & (ab < 0) \\ \text{or} & \\ \dfrac{1}{\sqrt{-ab}}\tanh^{-1}\dfrac{x\sqrt{-ab}}{a}, & (ab < 0) \end{cases}$$

INTEGRALS (Continued)

62. $$\int \frac{dx}{a^2 + b^2x^2} = \frac{1}{ab}\tan^{-1}\frac{bx}{a}$$

63. $$\int \frac{x\,dx}{a + bx^2} = \frac{1}{2b}\log(a + bx^2)$$

64. $$\int \frac{x^2\,dx}{a + bx^2} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a + bx^2}$$

65. $$\int \frac{dx}{(a + bx^2)^2} = \frac{x}{2a(a + bx^2)} + \frac{1}{2a}\int \frac{dx}{a + bx^2}$$

66. $$\int \frac{dx}{a^2 - b^2x^2} = \frac{1}{2ab}\log\frac{a + bx}{a - bx}$$

67. $$\int \frac{dx}{(a + bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2ma}\dfrac{x}{(a + bx^2)^m} + \dfrac{2m - 1}{2ma}\displaystyle\int \frac{dx}{(a + bx^2)^m} \\ \text{or} \\ \dfrac{(2m)!}{(m!)^2}\left[\dfrac{x}{2a}\displaystyle\sum_{r=1}^{m}\frac{r!(r - 1)!}{(4a)^{m-r}(2r)!(a + bx^2)^r} + \frac{1}{(4a)^m}\int \frac{dx}{a + bx^2}\right] \end{cases}$$

68. $$\int \frac{x\,dx}{(a + bx^2)^{m+1}} = -\frac{1}{2bm(a + bx^2)^m}$$

69. $$\int \frac{x^2\,dx}{(a + bx^2)^{m+1}} = \frac{-x}{2mb(a + bx^2)^m} + \frac{1}{2mb}\int \frac{dx}{(a + bx^2)^m}$$

70. $$\int \frac{dx}{x(a + bx^2)} = \frac{1}{2a}\log\frac{x^2}{a + bx^2}$$

71. $$\int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a + bx^2}$$

72. $$\int \frac{dx}{x(a + bx^2)^{m+1}} = \begin{cases} \dfrac{1}{2am(a + bx^2)^m} + \dfrac{1}{a}\displaystyle\int \frac{dx}{x(a + bx^2)^m} \\ \text{or} \\ \dfrac{1}{2a^{m+1}}\left[\displaystyle\sum_{r=1}^{m}\frac{a^r}{r(a + bx^2)^r} + \log\frac{x^2}{a + bx^2}\right] \end{cases}$$

73. $$\int \frac{dx}{x^2(a + bx^2)^{m+1}} = \frac{1}{a}\int \frac{dx}{x^2(a + bx^2)^m} - \frac{b}{a}\int \frac{dx}{(a + bx^2)^{m+1}}$$

74. $$\int \frac{dx}{a + bx^3} = \frac{k}{3a}\left[\frac{1}{2}\log\frac{(k + x)^3}{a + bx^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

75. $$\int \frac{x\,dx}{a + bx^3} = \frac{1}{3bk}\left[\frac{1}{2}\log\frac{a + bx^3}{(k + x)^3} + \sqrt{3}\tan^{-1}\frac{2x - k}{k\sqrt{3}}\right], \qquad \left(k = \sqrt[3]{\frac{a}{b}}\right)$$

INTEGRALS (Continued)

76. $\int \frac{x^2\,dx}{a + bx^3} = \frac{1}{3b} \log (a + bx^3)$

77. $\int \frac{dx}{a + bx^4} = \frac{k}{2a}\left[\frac{1}{2} \log \frac{x^2 + 2kx + 2k^2}{x^2 - 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2}\right],$

$\left(ab > 0, k = \sqrt[4]{\frac{a}{4b}}\right)$

78. $\int \frac{dx}{a + bx^4} = \frac{k}{2a}\left[\frac{1}{2} \log \frac{x + k}{x - k} + \tan^{-1} \frac{x}{k}\right], \quad \left(ab < 0, k = \sqrt[4]{-\frac{a}{b}}\right)$

79. $\int \frac{x\,dx}{a + bx^4} = \frac{1}{2bk} \tan^{-1} \frac{x^2}{k}, \quad \left(ab > 0, k = \sqrt{\frac{a}{b}}\right)$

80. $\int \frac{x\,dx}{a + bx^4} = \frac{1}{4bk} \log \frac{x^2 - k}{x^2 + k}, \quad \left(ab < 0, k = \sqrt{-\frac{a}{b}}\right)$

81. $\int \frac{x^2\,dx}{a + bx^4} = \frac{1}{4bk}\left[\frac{1}{2} \log \frac{x^2 - 2kx + 2k^2}{x^2 + 2kx + 2k^2} + \tan^{-1} \frac{2kx}{2k^2 - x^2}\right],$

$\left(ab > 0, k = \sqrt[4]{\frac{a}{4b}}\right)$

82. $\int \frac{x^2\,dx}{a + bx^4} = \frac{1}{4bk}\left[\log \frac{x - k}{x + k} + 2 \tan^{-1} \frac{x}{k}\right], \quad \left(ab < 0, k = \sqrt[4]{-\frac{a}{b}}\right)$

83. $\int \frac{x^3\,dx}{a + bx^4} = \frac{1}{4b} \log (a + bx^4)$

84. $\int \frac{dx}{x(a + bx^n)} = \frac{1}{an} \log \frac{x^n}{a + bx^n}$

85. $\int \frac{dx}{(a + bx^n)^{m+1}} = \frac{1}{a} \int \frac{dx}{(a + bx^n)^m} - \frac{b}{a} \int \frac{x^n\,dx}{(a + bx^n)^{m+1}}$

86. $\int \frac{x^m\,dx}{(a + bx^n)^{p+1}} = \frac{1}{b} \int \frac{x^{m-n}\,dx}{(a + bx^n)^p} - \frac{a}{b} \int \frac{x^{m-n}\,dx}{(a + bx^n)^{p+1}}$

87. $\int \frac{dx}{x^m(a + bx^n)^{p+1}} = \frac{1}{a} \int \frac{dx}{x^m(a + bx^n)^p} - \frac{b}{a} \int \frac{dx}{x^{m-n}(a + bx^n)^{p+1}}$

INTEGRALS (Continued)

88. $$\int x^m(a+bx^n)^p\,dx = \begin{cases} \dfrac{1}{b(np+m+1)}\Big[x^{m-n+1}(a+bx^n)^{p+1} \\ \qquad - a(m-n+1)\displaystyle\int x^{m-n}(a+bx^n)^p\,dx\Big] \\ \text{or} \\ \dfrac{1}{np+m+1}\Big[x^{m+1}(a+bx^n)^p \\ \qquad + anp\displaystyle\int x^m(a+bx^n)^{p-1}\,dx\Big] \\ \text{or} \\ \dfrac{1}{a(m+1)}\Big[x^{m+1}(a+bx^n)^{p+1} \\ \qquad - (m+1+np+n)b\displaystyle\int x^{m+n}(a+bx^n)^p\,dx\Big] \\ \text{or} \\ \dfrac{1}{an(p+1)}\Big[-x^{m+1}(a+bx^n)^{p+1} \\ \qquad + (m+1+np+n)\displaystyle\int x^m(a+bx^n)^{p+1}\,dx\Big] \end{cases}$$

FORMS CONTAINING $c^3 \pm x^3$

89. $$\int \frac{dx}{c^3 \pm x^3} = \pm\frac{1}{6c^2}\log\frac{(c \pm x)^3}{c^3 \pm x^3} + \frac{1}{c^2\sqrt{3}}\tan^{-1}\frac{2x \mp c}{c\sqrt{3}}$$

90. $$\int \frac{dx}{(c^3 \pm x^3)^2} = \frac{x}{3c^3(c^3 \pm x^3)} + \frac{2}{3c^3}\int\frac{dx}{c^3 \pm x^3}$$

91. $$\int \frac{dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3}\left[\frac{x}{(c^3 \pm x^3)^n} + (3n-1)\int\frac{dx}{(c^3 \pm x^3)^n}\right]$$

92. $$\int \frac{x\,dx}{c^3 \pm x^3} = \frac{1}{6c}\log\frac{c^3 \pm x^3}{(c \pm x)^3} \pm \frac{1}{c\sqrt{3}}\tan^{-1}\frac{2x \mp c}{c\sqrt{3}}$$

93. $$\int \frac{x\,dx}{(c^3 \pm x^3)^2} = \frac{x^2}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^3}\int\frac{x\,dx}{c^3 \pm x^3}$$

94. $$\int \frac{x\,dx}{(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3}\left[\frac{x^2}{(c^3 \pm x^3)^n} + (3n-2)\int\frac{x\,dx}{(c^3 \pm x^3)^n}\right]$$

95. $$\int \frac{x^2\,dx}{c^3 \pm x^3} = \pm\frac{1}{3}\log(c^3 \pm x^3)$$

INTEGRALS (Continued)

96. $$\int \frac{x^2\,dx}{(c^3 \pm x^3)^{n+1}} = \mp \frac{1}{3n(c^3 \pm x^3)^n}$$

97. $$\int \frac{dx}{x(c^3 \pm x^3)} = \frac{1}{3c^3} \log \frac{x^3}{c^3 \pm x^3}$$

98. $$\int \frac{dx}{x(c^3 \pm x^3)^2} = \frac{1}{3c^3(c^3 \pm x^3)} + \frac{1}{3c^6} \log \frac{x^3}{c^3 \pm x^3}$$

99. $$\int \frac{dx}{x(c^3 \pm x^3)^{n+1}} = \frac{1}{3nc^3(c^3 \pm x^3)^n} + \frac{1}{c^3} \int \frac{dx}{x(c^3 \pm x^3)^n}$$

100. $$\int \frac{dx}{x^2(c^3 \pm x^3)} = -\frac{1}{c^3 x} \mp \frac{1}{c^3} \int \frac{x\,dx}{c^3 \pm x^3}$$

101. $$\int \frac{dx}{x^2(c^3 \pm x^3)^{n+1}} = \frac{1}{c^3} \int \frac{dx}{x^2(c^3 \pm x^3)^n} \mp \frac{1}{c^3} \int \frac{x\,dx}{(c^3 \pm x^3)^{n+1}}$$

FORMS CONTAINING $c^4 \pm x^4$

102. $$\int \frac{dx}{c^4 + x^4} = \frac{1}{2c^3\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 + cx\sqrt{2} + c^2}{x^2 - cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$$

103. $$\int \frac{dx}{c^4 - x^4} = \frac{1}{2c^3} \left[\frac{1}{2} \log \frac{c + x}{c - x} + \tan^{-1} \frac{x}{c} \right]$$

104. $$\int \frac{x\,dx}{c^4 + x^4} = \frac{1}{2c^2} \tan^{-1} \frac{x^2}{c^2}$$

105. $$\int \frac{x\,dx}{c^4 - x^4} = \frac{1}{4c^2} \log \frac{c^2 + x^2}{c^2 - x^2}$$

106. $$\int \frac{x^2\,dx}{c^4 + x^4} = \frac{1}{2c\sqrt{2}} \left[\frac{1}{2} \log \frac{x^2 - cx\sqrt{2} + c^2}{x^2 + cx\sqrt{2} + c^2} + \tan^{-1} \frac{cx\sqrt{2}}{c^2 - x^2} \right]$$

107. $$\int \frac{x^2\,dx}{c^4 - x^4} = \frac{1}{2c} \left[\frac{1}{2} \log \frac{c + x}{c - x} - \tan^{-1} \frac{x}{c} \right]$$

108. $$\int \frac{x^3\,dx}{c^4 \pm x^4} = \pm \frac{1}{4} \log (c^4 \pm x^4)$$

FORMS CONTAINING $(a + bx + cx^2)$

$X = a + bx + cx^2$ and $q = 4ac - b^2$

If $q = 0$, then $X = c\left(x + \frac{b}{2c}\right)^2$, and formulas starting with 23 should be used in place of these.

109. $$\int \frac{dx}{X} = \frac{2}{\sqrt{q}} \tan^{-1} \frac{2cx + b}{\sqrt{q}}, \qquad (q > 0)$$

INTEGRALS (Continued)

110. $$\int \frac{dx}{X} = \begin{cases} \dfrac{-2}{\sqrt{-q}} \tanh^{-1} \dfrac{2cx + b}{\sqrt{-q}} \\ \text{or} \\ \dfrac{1}{\sqrt{-q}} \log \dfrac{2cx + b - \sqrt{-q}}{2cx + b + \sqrt{-q}}, \qquad (q < 0) \end{cases}$$

111. $$\int \frac{dx}{X^2} = \frac{2cx + b}{qX} + \frac{2c}{q} \int \frac{dx}{X}$$

112. $$\int \frac{dx}{X^3} = \frac{2cx + b}{q} \left(\frac{1}{2X^2} + \frac{3c}{qX} \right) + \frac{6c^2}{q^2} \int \frac{dx}{X}$$

113. $$\int \frac{dx}{X^{n+1}} = \begin{cases} \dfrac{2cx + b}{nqX^n} + \dfrac{2(2n - 1)c}{qn} \displaystyle\int \frac{dx}{X^n} \\ \text{or} \\ \dfrac{(2n)!}{(n!)^2} \left(\dfrac{c}{q} \right)^n \left[\dfrac{2cx + b}{q} \displaystyle\sum_{r=1}^{n} \left(\frac{q}{cX} \right)^r \left(\frac{(r - 1)!r!}{(2r)!} \right) + \int \frac{dx}{X} \right] \end{cases}$$

114. $$\int \frac{x\,dx}{X} = \frac{1}{2c} \log X - \frac{b}{2c} \int \frac{dx}{X}$$

115. $$\int \frac{x\,dx}{X^2} = -\frac{bx + 2a}{qX} - \frac{b}{q} \int \frac{dx}{X}$$

116. $$\int \frac{x\,dx}{X^{n+1}} = -\frac{2a + bx}{nqX^n} - \frac{b(2n - 1)}{nq} \int \frac{dx}{X^n}$$

117. $$\int \frac{x^2}{X}\,dx = \frac{x}{c} - \frac{b}{2c^2} \log X + \frac{b^2 - 2ac}{2c^2} \int \frac{dx}{X}$$

118. $$\int \frac{x^2}{X^2}\,dx = \frac{(b^2 - 2ac)x + ab}{cqX} + \frac{2a}{q} \int \frac{dx}{X}$$

119. $$\int \frac{x^m\,dx}{X^{n+1}} = -\frac{x^{m-1}}{(2n - m + 1)cX^n} - \frac{n - m + 1}{2n - m + 1} \cdot \frac{b}{c} \int \frac{x^{m-1}\,dx}{X^{n+1}} + \frac{m - 1}{2n - m + 1} \cdot \frac{a}{c} \int \frac{x^{m-2}\,dx}{X^{n+1}}$$

120. $$\int \frac{dx}{xX} = \frac{1}{2a} \log \frac{x^2}{X} - \frac{b}{2a} \int \frac{dx}{X}$$

121. $$\int \frac{dx}{x^2X} = \frac{b}{2a^2} \log \frac{X}{x^2} - \frac{1}{ax} + \left(\frac{b^2}{2a^2} - \frac{c}{a} \right) \int \frac{dx}{X}$$

122. $$\int \frac{dx}{xX^n} = \frac{1}{2a(n - 1)X^{n-1}} - \frac{b}{2a} \int \frac{dx}{X^n} + \frac{1}{a} \int \frac{dx}{xX^{n-1}}$$

INTEGRALS (Continued)

123. $$\int \frac{dx}{x^m X^{n+1}} = -\frac{1}{(m-1)ax^{m-1}X^n} - \frac{n+m-1}{m-1}\cdot\frac{b}{a}\int\frac{dx}{x^{m-1}X^{n+1}} - \frac{2n+m-1}{m-1}\cdot\frac{c}{a}\int\frac{dx}{x^{m-2}X^{n+1}}$$

FORMS CONTAINING $\sqrt{a+bx}$

124. $$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}$$

125. $$\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}$$

126. $$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}$$

127. $$\int x^m\sqrt{a+bx}\,dx = \begin{cases} \dfrac{2}{b(2m+3)}\left[x^m\sqrt{(a+bx)^3} - ma\displaystyle\int x^{m-1}\sqrt{a+bx}\,dx\right] \\ \text{or} \\ \dfrac{2}{b^{m+1}}\sqrt{a+bx}\displaystyle\sum_{r=0}^{m}\dfrac{m!(-a)^{m-r}}{r!(m-r)!(2r+3)}(a+bx)^{r+1} \end{cases}$$

128. $$\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}$$

129. $$\int \frac{\sqrt{a+bx}}{x^2}dx = -\frac{\sqrt{a+bx}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{a+bx}}$$

130. $$\int \frac{\sqrt{a+bx}}{x^m}dx = -\frac{1}{(m-1)a}\left[\frac{\sqrt{(a+bx)^3}}{x^{m-1}} + \frac{(2m-5)b}{2}\int\frac{\sqrt{a+bx}}{x^{m-1}}dx\right]$$

131. $$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}$$

132. $$\int \frac{x\,dx}{\sqrt{a+bx}} = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx}$$

133. $$\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2(8a^2-4abx+3b^2x^2)}{15b^3}\sqrt{a+bx}$$

INTEGRALS (Continued)

134. $$\int \frac{x^m\,dx}{\sqrt{a+bx}} = \begin{cases} \dfrac{2}{(2m+1)b}\left[x^m\sqrt{a+bx} - ma\displaystyle\int \frac{x^{m-1}\,dx}{\sqrt{a+bx}}\right] \\ \text{or} \\ \dfrac{2(-a)^m\sqrt{a+bx}}{b^{m+1}} \displaystyle\sum_{r=0}^{m} \frac{(-1)^r m!(a+bx)^r}{(2r+1)r!(m-r)!a^r} \end{cases}$$

135. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right), \qquad (a > 0)$$

136. $$\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{a}}, \qquad (a < 0)$$

137. $$\int \frac{dx}{x^2\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{ax} - \frac{b}{2a}\int \frac{dx}{x\sqrt{a+bx}}$$

138. $$\int \frac{dx}{x^n\sqrt{a+bx}} = \begin{cases} -\dfrac{\sqrt{a+bx}}{(n-1)ax^{n-1}} - \dfrac{(2n-3)b}{(2n-2)a}\displaystyle\int \frac{dx}{x^{n-1}\sqrt{a+bx}} \\ \text{or} \\ \dfrac{(2n-2)!}{[(n-1)!]^2}\left[-\dfrac{\sqrt{a+bx}}{a}\displaystyle\sum_{r=1}^{n-1}\frac{r!(r-1)!}{x^r(2r)!}\left(-\frac{b}{4a}\right)^{n-r-1} + \left(-\frac{b}{4a}\right)^{n-1}\int\frac{dx}{x\sqrt{a+bx}}\right] \end{cases}$$

139. $$\int (a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2(a+bx)^{\frac{2\pm n}{2}}}{b(2\pm n)}$$

140. $$\int x(a+bx)^{\pm\frac{n}{2}}\,dx = \frac{2}{b^2}\left[\frac{(a+bx)^{\frac{4\pm n}{2}}}{4\pm n} - \frac{a(a+bx)^{\frac{2\pm n}{2}}}{2\pm n}\right]$$

141. $$\int \frac{dx}{x(a+bx)^{\frac{m}{2}}} = \frac{1}{a}\int \frac{dx}{x(a+bx)^{\frac{m-2}{2}}} - \frac{b}{a}\int \frac{dx}{(a+bx)^{\frac{m}{2}}}$$

142. $$\int \frac{(a+bx)^{\frac{n}{2}}\,dx}{x} = b\int (a+bx)^{\frac{n-2}{2}}\,dx + a\int \frac{(a+bx)^{\frac{n-2}{2}}}{x}\,dx$$

143. $$\int f(x, \sqrt{a+bx})\,dx = \frac{2}{b}\int f\left(\frac{z^2-a}{b}, z\right)z\,dz, \qquad (z = \sqrt{a+bx})$$

INTEGRALS (Continued)

FORMS CONTAINING $\sqrt{a+bx}$ and $\sqrt{c+dx}$

$$u = a + bx \qquad v = c + dx \qquad k = ad - bc$$

If $k = 0$, then $v = \frac{c}{a}u$, and formulas starting with 124 should be used in place of these.

144. $$\int \frac{dx}{\sqrt{uv}} = \begin{cases} \dfrac{2}{\sqrt{bd}} \tanh^{-1} \dfrac{\sqrt{bduv}}{bv} \\ \text{or} \\ \dfrac{1}{\sqrt{bd}} \log \dfrac{bv + \sqrt{bduv}}{bv - \sqrt{bduv}} \\ \text{or} \\ \dfrac{1}{\sqrt{bd}} \log \dfrac{(bv + \sqrt{bduv})^2}{v}, \qquad (bd > 0) \end{cases}$$

145. $$\int \frac{dx}{\sqrt{uv}} = \begin{cases} \dfrac{2}{\sqrt{-bd}} \tan^{-1} \dfrac{\sqrt{-bduv}}{bv} \\ \text{or} \\ -\dfrac{1}{\sqrt{-bd}} \sin^{-1} \left(\dfrac{2bdx + ad + bc}{|k|} \right), \qquad (bd < 0) \end{cases}$$

146. $$\int \sqrt{uv}\, dx = \frac{k + 2bv}{4bd} \sqrt{uv} - \frac{k^2}{8bd} \int \frac{dx}{\sqrt{uv}}$$

147. $$\int \frac{dx}{v\sqrt{u}} = \begin{cases} \dfrac{1}{\sqrt{kd}} \log \dfrac{d\sqrt{u} - \sqrt{kd}}{d\sqrt{u} + \sqrt{kd}} \\ \text{or} \\ \dfrac{1}{\sqrt{kd}} \log \dfrac{(d\sqrt{u} - \sqrt{kd})^2}{v}, \qquad (kd > 0) \end{cases}$$

148. $$\int \frac{dx}{v\sqrt{u}} = \frac{2}{\sqrt{-kd}} \tan^{-1} \frac{d\sqrt{u}}{\sqrt{-kd}}, \qquad (kd < 0)$$

149. $$\int \frac{x\, dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{bd} - \frac{ad + bc}{2bd} \int \frac{dx}{\sqrt{uv}}$$

150. $$\int \frac{dx}{v\sqrt{uv}} = \frac{-2\sqrt{uv}}{kv}$$

INTEGRALS (Continued)

151. $\int \frac{v\,dx}{\sqrt{uv}} = \frac{\sqrt{uv}}{b} - \frac{k}{2b}\int \frac{dx}{\sqrt{uv}}$

152. $\int \sqrt{\frac{v}{u}}\,dx = \frac{v}{|v|}\int \frac{v\,dx}{\sqrt{uv}}$

153. $\int v^m\sqrt{u}\,dx = \frac{1}{(2m+3)d}\left(2v^{m+1}\sqrt{u} + k\int \frac{v^m\,dx}{\sqrt{u}}\right)$

154. $\int \frac{dx}{v^m\sqrt{u}} = -\frac{1}{(m-1)k}\left(\frac{\sqrt{u}}{v^{m-1}} + \left(m-\frac{3}{2}\right)b\int \frac{dx}{v^{m-1}\sqrt{u}}\right)$

155. $$\int \frac{v^m\,dx}{\sqrt{u}} = \begin{cases} \frac{2}{b(2m+1)}\left[v^m\sqrt{u} - mk\int \frac{v^{m-1}}{\sqrt{u}}\,dx\right] \\ \text{or} \\ \frac{2(m!)^2\sqrt{u}}{b(2m+1)!}\sum_{r=0}^{m}\left(-\frac{4k}{b}\right)^{m-r}\frac{(2r)!}{(r!)^2}v^r \end{cases}$$

FORMS CONTAINING $\sqrt{x^2 \pm a^2}$

156. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2 \log(x + \sqrt{x^2 \pm a^2})]$

157. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log(x + \sqrt{x^2 \pm a^2})$

158. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{|a|}\sec^{-1}\frac{x}{a}$

159. $\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$

160. $\int \frac{\sqrt{x^2 + a^2}}{x}\,dx = \sqrt{x^2 + a^2} - a\log\left(\frac{a + \sqrt{x^2 + a^2}}{x}\right)$

161. $\int \frac{\sqrt{x^2 - a^2}}{x}\,dx = \sqrt{x^2 - a^2} - |a|\sec^{-1}\frac{x}{a}$

162. $\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$

163. $\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3}$

INTEGRALS (Continued)

164. $$\int \sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(x^2 \pm a^2)^3} \pm \frac{3a^2x}{2}\sqrt{x^2 \pm a^2} + \frac{3a^4}{2}\log(x + \sqrt{x^2 \pm a^2})\right]$$

165. $$\int \frac{dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$$

166. $$\int \frac{x\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-1}{\sqrt{x^2 \pm a^2}}$$

167. $$\int x\sqrt{(x^2 \pm a^2)^3}\,dx = \tfrac{1}{5}\sqrt{(x^2 \pm a^2)^5}$$

168. $$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\log(x + \sqrt{x^2 \pm a^2})$$

169. $$\int x^3\sqrt{x^2 + a^2}\,dx = (\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2)\sqrt{(a^2 + x^2)^3}$$

170. $$\int x^3\sqrt{x^2 - a^2}\,dx = \frac{1}{5}\sqrt{(x^2 - a^2)^5} + \frac{a^2}{3}\sqrt{(x^2 - a^2)^3}$$

171. $$\int \frac{x^2\,dx}{\sqrt{x^2 \pm a^2}} = \frac{x}{2}\sqrt{x^2 \pm a^2} \mp \frac{a^2}{2}\log(x + \sqrt{x^2 \pm a^2})$$

172. $$\int \frac{x^3\,dx}{\sqrt{x^2 \pm a^2}} = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} \mp a^2\sqrt{x^2 \pm a^2}$$

173. $$\int \frac{dx}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2x}$$

174. $$\int \frac{dx}{x^3\sqrt{x^2 + a^2}} = -\frac{\sqrt{x^2 + a^2}}{2a^2x^2} + \frac{1}{2a^3}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

175. $$\int \frac{dx}{x^3\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2x^2} + \frac{1}{2|a^3|}\sec^{-1}\frac{x}{a}$$

176. $$\int x^2\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{x}{6}\sqrt{(x^2 \pm a^2)^5} \mp \frac{a^2x}{24}\sqrt{(x^2 \pm a^2)^3} - \frac{a^4x}{16}\sqrt{x^2 \pm a^2} \mp \frac{a^6}{16}\log(x + \sqrt{x^2 \pm a^2})$$

177. $$\int x^3\sqrt{(x^2 \pm a^2)^3}\,dx = \frac{1}{7}\sqrt{(x^2 \pm a^2)^7} \mp \frac{a^2}{5}\sqrt{(x^2 \pm a^2)^5}$$

INTEGRALS (Continued)

178. $$\int \frac{\sqrt{x^2 \pm a^2}\,dx}{x^2} = -\frac{\sqrt{x^2 \pm a^2}}{x} + \log(x + \sqrt{x^2 \pm a^2})$$

179. $$\int \frac{\sqrt{x^2 + a^2}}{x^3}\,dx = -\frac{\sqrt{x^2 + a^2}}{2x^2} - \frac{1}{2a}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

180. $$\int \frac{\sqrt{x^2 - a^2}}{x^3}\,dx = -\frac{\sqrt{x^2 - a^2}}{2x^2} + \frac{1}{2|a|}\sec^{-1}\frac{x}{a}$$

181. $$\int \frac{\sqrt{x^2 \pm a^2}}{x^4}\,dx = \mp\frac{\sqrt{(x^2 \pm a^2)^3}}{3a^2x^3}$$

182. $$\int \frac{x^2\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \frac{-x}{\sqrt{x^2 \pm a^2}} + \log(x + \sqrt{x^2 \pm a^2})$$

183. $$\int \frac{x^3\,dx}{\sqrt{(x^2 \pm a^2)^3}} = \sqrt{x^2 \pm a^2} \pm \frac{a^2}{\sqrt{x^2 \pm a^2}}$$

184. $$\int \frac{dx}{x\sqrt{(x^2 + a^2)^3}} = \frac{1}{a^2\sqrt{x^2 + a^2}} - \frac{1}{a^3}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

185. $$\int \frac{dx}{x\sqrt{(x^2 - a^2)^3}} = -\frac{1}{a^2\sqrt{x^2 - a^2}} - \frac{1}{|a^3|}\sec^{-1}\frac{x}{a}$$

186. $$\int \frac{dx}{x^2\sqrt{(x^2 \pm a^2)^3}} = -\frac{1}{a^4}\left[\frac{\sqrt{x^2 \pm a^2}}{x} + \frac{x}{\sqrt{x^2 \pm a^2}}\right]$$

187. $$\int \frac{dx}{x^3\sqrt{(x^2 + a^2)^3}} = -\frac{1}{2a^2x^2\sqrt{x^2 + a^2}} - \frac{3}{2a^4\sqrt{x^2 + a^2}} + \frac{3}{2a^5}\log\frac{a + \sqrt{x^2 + a^2}}{x}$$

188. $$\int \frac{dx}{x^3\sqrt{(x^2 - a^2)^3}} = \frac{1}{2a^2x^2\sqrt{x^2 - a^2}} - \frac{3}{2a^4\sqrt{x^2 - a^2}} - \frac{3}{2|a^5|}\sec^{-1}\frac{x}{a}$$

189. $$\int \frac{x^m}{\sqrt{x^2 \pm a^2}}\,dx = \frac{1}{m}x^{m-1}\sqrt{x^2 \pm a^2} \mp \frac{m-1}{m}a^2\int \frac{x^{m-2}}{\sqrt{x^2 \pm a^2}}\,dx$$

190. $$\int \frac{x^{2m}}{\sqrt{x^2 \pm a^2}}\,dx = \frac{(2m)!}{2^{2m}(m!)^2}\left[\sqrt{x^2 \pm a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{(2r)!}(\mp a^2)^{m-r}(2x)^{2r-1} + (\mp a^2)^m \log(x + \sqrt{x^2 \pm a^2})\right]$$

191. $$\int \frac{x^{2m+1}}{\sqrt{x^2 \pm a^2}}\,dx = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(\mp 4a^2)^{m-r}x^{2r}$$

INTEGRALS (Continued)

192. $$\int \frac{dx}{x^m\sqrt{x^2 \pm a^2}} = \mp \frac{\sqrt{x^2 \pm a^2}}{(m-1)a^2x^{m-1}} \mp \frac{(m-2)}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{x^2 \pm a^2}}$$

193. $$\int \frac{dx}{x^{2m}\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}\sum_{r=0}^{m-1} \frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!(\mp a^2)^{m-r}x^{2r+1}}$$

194. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 + a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 + a^2}}{a^2}\sum_{r=1}^{m}(-1)^{m-r+1}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{(-1)^{m+1}}{2^{2m}a^{2m+1}}\log\frac{\sqrt{x^2+a^2}+a}{x}\right]$$

195. $$\int \frac{dx}{x^{2m+1}\sqrt{x^2 - a^2}} = \frac{(2m)!}{(m!)^2}\left[\frac{\sqrt{x^2 - a^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}|a|^{2m+1}}\sec^{-1}\frac{x}{a}\right]$$

196. $$\int \frac{dx}{(x-a)\sqrt{x^2 - a^2}} = -\frac{\sqrt{x^2 - a^2}}{a(x-a)}$$

197. $$\int \frac{dx}{(x+a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x+a)}$$

198. $$\int f(x, \sqrt{x^2 + a^2})\,dx = a\int f(a\tan u, a\sec u)\sec^2 u\,du, \qquad \left(u = \tan^{-1}\frac{x}{a}, a > 0\right)$$

199. $$\int f(x, \sqrt{x^2 - a^2})\,dx = a\int f(a\sec u, a\tan u)\sec u\tan u\,du, \qquad \left(u = \sec^{-1}\frac{x}{a}, a > 0\right)$$

FORMS CONTAINING $\sqrt{a^2 - x^2}$

200. $$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2\sin^{-1}\frac{x}{|a|}\right]$$

201. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \sin^{-1}\dfrac{x}{|a|} \\ \text{or} \\ -\cos^{-1}\dfrac{x}{|a|} \end{cases}$$

202. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$$

INTEGRALS (Continued)

203. $\int \frac{\sqrt{a^2 - x^2}}{x}\,dx = \sqrt{a^2 - x^2} - a \log\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right)$

204. $\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$

205. $\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}$

206. $\int \sqrt{(a^2 - x^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(a^2 - x^2)^3} + \frac{3a^2x}{2}\sqrt{a^2 - x^2} + \frac{3a^4}{2}\sin^{-1}\frac{x}{|a|}\right]$

207. $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}}$

208. $\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}}$

209. $\int x\sqrt{(a^2 - x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2 - x^2)^5}$

210. $\int x^2\sqrt{a^2 - x^2}\,dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{|a|}\right)$

211. $\int x^3\sqrt{a^2 - x^2}\,dx = (-\frac{1}{5}x^2 - \frac{2}{15}a^2)\sqrt{(a^2 - x^2)^3}$

212. $\int x^2\sqrt{(a^2 - x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2 - x^2)^5} + \frac{a^2x}{24}\sqrt{(a^2 - x^2)^3} + \frac{a^4x}{16}\sqrt{a^2 - x^2} + \frac{a^6}{16}\sin^{-1}\frac{x}{|a|}$

213. $\int x^3\sqrt{(a^2 - x^2)^3}\,dx = \frac{1}{7}\sqrt{(a^2 - x^2)^7} - \frac{a^2}{5}\sqrt{(a^2 - x^2)^5}$

214. $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{|a|}$

215. $\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\frac{\sqrt{a^2 - x^2}}{a^2x}$

216. $\int \frac{\sqrt{a^2 - x^2}}{x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{x} - \sin^{-1}\frac{x}{|a|}$

217. $\int \frac{\sqrt{a^2 - x^2}}{x^3}\,dx = -\frac{\sqrt{a^2 - x^2}}{2x^2} + \frac{1}{2a}\log\frac{a + \sqrt{a^2 - x^2}}{x}$

218. $\int \frac{\sqrt{a^2 - x^2}}{x^4}\,dx = -\frac{\sqrt{(a^2 - x^2)^3}}{3a^2x^3}$

INTEGRALS (Continued)

219. $$\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{|a|}$$

220. $$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = -\frac{2}{3}(a^2-x^2)^{\frac{3}{2}} - x^2(a^2-x^2)^{\frac{1}{2}} - \frac{1}{3}\sqrt{a^2-x^2}(x^2+2a^2)$$

221. $$\int \frac{x^3\,dx}{\sqrt{(a^2-x^2)^3}} = 2(a^2-x^2)^{\frac{1}{2}} + \frac{x^2}{(a^2-x^2)^{\frac{1}{2}}} = \frac{a^2}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}$$

222. $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

223. $$\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

224. $$\int \frac{dx}{x^2\sqrt{(a^2-x^2)^3}} = \frac{1}{a^4}\left[-\frac{\sqrt{a^2-x^2}}{x} + \frac{x}{\sqrt{a^2-x^2}}\right]$$

225. $$\int \frac{dx}{x^3\sqrt{(a^2-x^2)^3}} = -\frac{1}{2a^2x^2\sqrt{a^2-x^2}} + \frac{3}{2a^4\sqrt{a^2-x^2}} - \frac{3}{2a^5}\log\frac{a+\sqrt{a^2-x^2}}{x}$$

226. $$\int \frac{x^m}{\sqrt{a^2-x^2}}\,dx = -\frac{x^{m-1}\sqrt{a^2-x^2}}{m} + \frac{(m-1)a^2}{m}\int \frac{x^{m-2}}{\sqrt{a^2-x^2}}\,dx$$

227. $$\int \frac{x^{2m}}{\sqrt{a^2-x^2}}\,dx = \frac{(2m)!}{(m!)^2}\left[-\sqrt{a^2-x^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2^{2m-2r+1}(2r)!}a^{2m-2r}x^{2r-1} + \frac{a^{2m}}{2^{2m}}\sin^{-1}\frac{x}{|a|}\right]$$

228. $$\int \frac{x^{2m+1}}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2}\sum_{r=0}^{m}\frac{(2r)!(m!)^2}{(2m+1)!(r!)^2}(4a^2)^{m-r}x^{2r}$$

229. $$\int \frac{dx}{x^m\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{(m-1)a^2x^{m-1}} + \frac{m-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}\sqrt{a^2-x^2}}$$

230. $$\int \frac{dx}{x^{2m}\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}\sum_{r=0}^{m-1}\frac{(m-1)!m!(2r)!2^{2m-2r-1}}{(r!)^2(2m)!a^{2m-2r}x^{2r+1}}$$

231. $$\int \frac{dx}{x^{2m+1}\sqrt{a^2-x^2}} = \frac{(2m)!}{(m!)^2}\left[-\frac{\sqrt{a^2-x^2}}{a^2}\sum_{r=1}^{m}\frac{r!(r-1)!}{2(2r)!(4a^2)^{m-r}x^{2r}} + \frac{1}{2^{2m}a^{2m+1}}\log\frac{a-\sqrt{a^2-x^2}}{x}\right]$$

INTEGRALS (Continued)

232. $$\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{2b\sqrt{a^2 - b^2}} \log \frac{(b\sqrt{a^2 - x^2} + x\sqrt{a^2 - b^2})^2}{b^2 - x^2}, \quad (a^2 > b^2)$$

233. $$\int \frac{dx}{(b^2 - x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{b^2 - a^2}} \tan^{-1} \frac{x\sqrt{b^2 - a^2}}{b\sqrt{a^2 - x^2}}, \quad (b^2 > a^2)$$

234. $$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 - x^2}} = \frac{1}{b\sqrt{a^2 + b^2}} \tan^{-1} \frac{x\sqrt{a^2 + b^2}}{b\sqrt{a^2 - x^2}}$$

235. $$\int \frac{\sqrt{a^2 - x^2}}{b^2 + x^2} dx = \frac{\sqrt{a^2 + b^2}}{|b|} \sin^{-1} \frac{x\sqrt{a^2 + b^2}}{|a|\sqrt{x^2 + b^2}} - \sin^{-1} \frac{x}{|a|}$$

236. $$\int f(x, \sqrt{a^2 - x^2})\, dx = a \int f(a \sin u, a \cos u) \cos u\, du, \quad \left(u = \sin^{-1} \frac{x}{a}, a > 0\right)$$

FORMS CONTAINING $\sqrt{a + bx + cx^2}$

$$X = a + bx + cx^2, q = 4ac - b^2, \text{and } k = \frac{4c}{q}$$

If $q = 0$, then $\sqrt{X} = \sqrt{c}\left|x + \frac{b}{2c}\right|$

237. $$\int \frac{dx}{\sqrt{X}} = \begin{cases} \frac{1}{\sqrt{c}} \log(2\sqrt{cX} + 2cx + b) \\ \text{or} \\ \frac{1}{\sqrt{c}} \sinh^{-1} \frac{2cx + b}{\sqrt{q}}, \quad (c > 0) \end{cases}$$

238. $$\int \frac{dx}{\sqrt{X}} = -\frac{1}{\sqrt{-c}} \sin^{-1} \frac{2cx + b}{\sqrt{-q}}, \quad (c < 0)$$

239. $$\int \frac{dx}{X\sqrt{X}} = \frac{2(2cx + b)}{q\sqrt{X}}$$

240. $$\int \frac{dx}{X^2\sqrt{X}} = \frac{2(2cx + b)}{3q\sqrt{X}}\left(\frac{1}{X} + 2k\right)$$

241. $$\int \frac{dx}{X^n\sqrt{X}} = \begin{cases} \frac{2(2cx + b)\sqrt{X}}{(2n - 1)qX^n} + \frac{2k(n - 1)}{2n - 1} \int \frac{dx}{X^{n-1}\sqrt{X}} \\ \text{or} \\ \frac{(2cx + b)(n!)(n - 1)!4^n k^{n-1}}{q[(2n)!]\sqrt{X}} \sum_{r=0}^{n-1} \frac{(2r)!}{(4kX)^r (r!)^2} \end{cases}$$

INTEGRALS (Continued)

242. $$\int \sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{4c} + \frac{1}{2k}\int \frac{dx}{\sqrt{X}}$$

243. $$\int X\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{8c}\left(X + \frac{3}{2k}\right) + \frac{3}{8k^2}\int \frac{dx}{\sqrt{X}}$$

244. $$\int X^2\sqrt{X}\,dx = \frac{(2cx+b)\sqrt{X}}{12c}\left(X^2 + \frac{5X}{4k} + \frac{15}{8k^2}\right) + \frac{5}{16k^3}\int \frac{dx}{\sqrt{X}}$$

245. $$\int X^n\sqrt{X}\,dx = \begin{cases} \dfrac{(2cx+b)X^n\sqrt{X}}{4(n+1)c} + \dfrac{2n+1}{2(n+1)k}\displaystyle\int X^{n-1}\sqrt{X}\,dx \\ \text{or} \\ \dfrac{(2n+2)!}{[(n+1)!]^2(4k)^{n+1}}\left[\dfrac{k(2cx+b)\sqrt{X}}{c}\displaystyle\sum_{r=0}^{n}\dfrac{r!(r+1)!(4kX)^r}{(2r+2)!} + \displaystyle\int \dfrac{dx}{\sqrt{X}}\right] \end{cases}$$

246. $$\int \frac{x\,dx}{\sqrt{X}} = \frac{\sqrt{X}}{c} - \frac{b}{2c}\int \frac{dx}{\sqrt{X}}$$

247. $$\int \frac{x\,dx}{X\sqrt{X}} = -\frac{2(bx+2a)}{q\sqrt{X}}$$

248. $$\int \frac{x\,dx}{X^n\sqrt{X}} = -\frac{\sqrt{X}}{(2n-1)cX^n} - \frac{b}{2c}\int \frac{dx}{X^n\sqrt{X}}$$

249. $$\int \frac{x^2\,dx}{\sqrt{X}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right)\sqrt{X} + \frac{3b^2-4ac}{8c^2}\int \frac{dx}{\sqrt{X}}$$

250. $$\int \frac{x^2\,dx}{X\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{cq\sqrt{X}} + \frac{1}{c}\int \frac{dx}{\sqrt{X}}$$

251. $$\int \frac{x^2\,dx}{X^n\sqrt{X}} = \frac{(2b^2-4ac)x+2ab}{(2n-1)cqX^{n-1}\sqrt{X}} + \frac{4ac+(2n-3)b^2}{(2n-1)cq}\int \frac{dx}{X^{n-1}\sqrt{X}}$$

252. $$\int \frac{x^3\,dx}{\sqrt{X}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3} - \frac{2a}{3c^2}\right)\sqrt{X} + \left(\frac{3ab}{4c^2} - \frac{5b^3}{16c^3}\right)\int \frac{dx}{\sqrt{X}}$$

253. $$\int \frac{x^n\,dx}{\sqrt{X}} = \frac{1}{nc}x^{n-1}\sqrt{X} - \frac{(2n-1)b}{2nc}\int \frac{x^{n-1}\,dx}{\sqrt{X}} - \frac{(n-1)a}{nc}\int \frac{x^{n-2}\,dx}{\sqrt{X}}$$

INTEGRALS (Continued)

254. $$\int x\sqrt{X}\,dx = \frac{X\sqrt{X}}{3c} - \frac{b(2cx+b)}{8c^2}\sqrt{X} - \frac{b}{4ck}\int\frac{dx}{\sqrt{X}}$$

255. $$\int xX\sqrt{X}\,dx = \frac{X^2\sqrt{X}}{5c} - \frac{b}{2c}\int X\sqrt{X}\,dx$$

256. $$\int xX^n\sqrt{X}\,dx = \frac{X^{n+1}\sqrt{X}}{(2n+3)c} - \frac{b}{2c}\int X^n\sqrt{X}\,dx$$

257. $$\int x^2\sqrt{X}\,dx = \left(x - \frac{5b}{6c}\right)\frac{X\sqrt{X}}{4c} + \frac{5b^2-4ac}{16c^2}\int\sqrt{X}\,dx$$

258. $$\int\frac{dx}{x\sqrt{X}} = -\frac{1}{\sqrt{a}}\log\frac{2\sqrt{aX}+bx+2a}{x}, \qquad (a > 0)$$

259. $$\int\frac{dx}{x\sqrt{X}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bx+2a}{|x|\sqrt{-q}}\right), \qquad (a < 0)$$

260. $$\int\frac{dx}{x\sqrt{X}} = -\frac{2\sqrt{X}}{bx}, \qquad (a = 0)$$

261. $$\int\frac{dx}{x^2\sqrt{X}} = -\frac{\sqrt{X}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{X}}$$

262. $$\int\frac{\sqrt{X}\,dx}{x} = \sqrt{X} + \frac{b}{2}\int\frac{dx}{\sqrt{X}} + a\int\frac{dx}{x\sqrt{X}}$$

263. $$\int\frac{\sqrt{X}\,dx}{x^2} = -\frac{\sqrt{X}}{x} + \frac{b}{2}\int\frac{dx}{x\sqrt{X}} + c\int\frac{dx}{\sqrt{X}}$$

FORMS INVOLVING $\sqrt{2ax - x^2}$

264. $$\int\sqrt{2ax-x^2}\,dx = \frac{1}{2}\left[(x-a)\sqrt{2ax-x^2} + a^2\sin^{-1}\frac{x-a}{|a|}\right]$$

265. $$\int\frac{dx}{\sqrt{2ax-x^2}} = \begin{cases}\cos^{-1}\dfrac{a-x}{|a|}\\ \text{or}\\ \sin^{-1}\dfrac{x-a}{|a|}\end{cases}$$

INTEGRALS (Continued)

266. $$\int x^n\sqrt{2ax-x^2}\,dx = \begin{cases} -\dfrac{x^{n-1}(2ax-x^2)^{\frac{3}{2}}}{n+2} + \dfrac{(2n+1)a}{n+2}\displaystyle\int x^{n-1}\sqrt{2ax-x^2}\,dx \\ \text{or} \\ \sqrt{2ax-x^2}\left[\dfrac{x^{n+1}}{n+2} - \displaystyle\sum_{r=0}^{n}\dfrac{(2n+1)!(r!)^2a^{n-r+1}}{2^{n-r}(2r+1)!(n+2)!n!}x^r\right] \\ \qquad + \dfrac{(2n+1)!a^{n+2}}{2^n n!(n+2)!}\sin^{-1}\dfrac{x-a}{|a|} \end{cases}$$

267. $$\int\frac{\sqrt{2ax-x^2}}{x^n}dx = \frac{(2ax-x^2)^{\frac{3}{2}}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int\frac{\sqrt{2ax-x^2}}{x^{n-1}}dx$$

268. $$\int\frac{x^n\,dx}{\sqrt{2ax-x^2}} = \begin{cases} \dfrac{-x^{n-1}\sqrt{2ax-x^2}}{n} + \dfrac{a(2n-1)}{n}\displaystyle\int\frac{x^{n-1}}{\sqrt{2ax-x^2}}dx \\ \text{or} \\ -\sqrt{2ax-x^2}\displaystyle\sum_{r=1}^{n}\dfrac{(2n)!r!(r-1)!a^{n-r}}{2^{n-r}(2r)!(n!)^2}x^{r-1} \\ \qquad + \dfrac{(2n)!a^n}{2^n(n!)^2}\sin^{-1}\dfrac{x-a}{|a|} \end{cases}$$

269. $$\int\frac{dx}{x^n\sqrt{2ax-x^2}} = \begin{cases} \dfrac{\sqrt{2ax-x^2}}{a(1-2n)x^n} + \dfrac{n-1}{(2n-1)a}\displaystyle\int\frac{dx}{x^{n-1}\sqrt{2ax-x^2}} \\ \text{or} \\ -\sqrt{2ax-x^2}\displaystyle\sum_{r=0}^{n-1}\dfrac{2^{n-r}(n-1)!n!(2r)!}{(2n)!(r!)^2a^{n-r}x^{r+1}} \end{cases}$$

270. $$\int\frac{dx}{(2ax-x^2)^{\frac{3}{2}}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}$$

271. $$\int\frac{x\,dx}{(2ax-x^2)^{\frac{3}{2}}} = \frac{x}{a\sqrt{2ax-x^2}}$$

MISCELLANEOUS ALGEBRAIC FORMS

272. $$\int\frac{dx}{\sqrt{2ax+x^2}} = \log(x+a+\sqrt{2ax+x^2})$$

273. $$\int\sqrt{ax^2+c}\,dx = \frac{x}{2}\sqrt{ax^2+c} + \frac{c}{2\sqrt{a}}\log(x\sqrt{a}+\sqrt{ax^2+c}), \qquad (a>0)$$

274. $$\int\sqrt{ax^2+c}\,dx = \frac{x}{2}\sqrt{ax^2+c} + \frac{c}{2\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \qquad (a<0)$$

INTEGRALS (Continued)

275. $$\int\sqrt{\frac{1+x}{1-x}}\,dx = \sin^{-1}x - \sqrt{1-x^2}$$

276. $$\int\frac{dx}{x\sqrt{ax^n+c}} = \begin{cases} \dfrac{1}{n\sqrt{c}}\log\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{ax^n+c}+\sqrt{c}} \\ \text{or} \\ \dfrac{2}{n\sqrt{c}}\log\dfrac{\sqrt{ax^n+c}-\sqrt{c}}{\sqrt{x^n}}, \qquad (c>0) \end{cases}$$

277. $$\int\frac{dx}{x\sqrt{ax^n+c}} = \frac{2}{n\sqrt{-c}}\sec^{-1}\sqrt{-\frac{ax^n}{c}}, \qquad (c<0)$$

278. $$\int\frac{dx}{\sqrt{ax^2+c}} = \frac{1}{\sqrt{a}}\log\left(x\sqrt{a}+\sqrt{ax^2+c}\right), \qquad (a>0)$$

279. $$\int\frac{dx}{\sqrt{ax^2+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(x\sqrt{-\frac{a}{c}}\right), \qquad (a<0)$$

280. $$\int(ax^2+c)^{m+\frac{1}{2}}\,dx = \begin{cases} \dfrac{x(ax^2+c)^{m+\frac{1}{2}}}{2(m+1)} + \dfrac{(2m+1)c}{2(m+1)}\displaystyle\int(ax^2+c)^{m-\frac{1}{2}}\,dx \\ \text{or} \\ x\sqrt{ax^2+c}\displaystyle\sum_{r=0}^{m}\frac{(2m+1)!(r!)^2c^{m-r}}{2^{2m-2r+1}m!(m+1)!(2r+1)!}(ax^2+c)^r \\ \qquad + \dfrac{(2m+1)!c^{m+1}}{2^{2m+1}m!(m+1)!}\displaystyle\int\frac{dx}{\sqrt{ax^2+c}} \end{cases}$$

281. $$\int x(ax^2+c)^{m+\frac{1}{2}}\,dx = \frac{(ax^2+c)^{m+\frac{3}{2}}}{(2m+3)a}$$

282. $$\int\frac{(ax^2+c)^{m+\frac{1}{2}}}{x}\,dx = \begin{cases} \dfrac{(ax^2+c)^{m+\frac{1}{2}}}{2m+1} + c\displaystyle\int\frac{(ax^2+c)^{m-\frac{1}{2}}}{x}\,dx \\ \text{or} \\ \sqrt{ax^2+c}\displaystyle\sum_{r=0}^{m}\frac{c^{m-r}(ax^2+c)^r}{2r+1} + c^{m+1}\int\frac{dx}{x\sqrt{ax^2+c}} \end{cases}$$

283. $$\int\frac{dx}{(ax^2+c)^{m+\frac{1}{2}}} = \begin{cases} \dfrac{x}{(2m-1)c(ax^2+c)^{m-\frac{1}{2}}} + \dfrac{2m-2}{(2m-1)c}\displaystyle\int\frac{dx}{(ax^2+c)^{m-\frac{1}{2}}} \\ \text{or} \\ \dfrac{x}{\sqrt{ax^2+c}}\displaystyle\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2c^{m-r}(ax^2+c)^r} \end{cases}$$

INTEGRALS (Continued)

284. $\int \frac{dx}{x^m\sqrt{ax^2+c}} = -\frac{\sqrt{ax^2+c}}{(m-1)cx^{m-1}} - \frac{(m-2)a}{(m-1)c}\int \frac{dx}{x^{m-2}\sqrt{ax^2+c}}$

285. $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\log \frac{x\sqrt{2}+\sqrt{1+x^4}}{1-x^2}$

286. $\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1+x^4}}$

287. $\int \frac{dx}{x\sqrt{x^n+a^2}} = -\frac{2}{na}\log \frac{a+\sqrt{x^n+a^2}}{\sqrt{x^n}}$

288. $\int \frac{dx}{x\sqrt{x^n-a^2}} = -\frac{2}{na}\sin^{-1}\frac{a}{\sqrt{x^n}}$

289. $\int \sqrt{\frac{x}{a^3-x^3}}\,dx = \frac{2}{3}\sin^{-1}\left(\frac{x}{a}\right)^{\frac{3}{2}}$

FORMS INVOLVING TRIGONOMETRIC FUNCTIONS

290. $\int (\sin ax)\,dx = -\frac{1}{a}\cos ax$

291. $\int (\cos ax)\,dx = \frac{1}{a}\sin ax$

292. $\int (\tan ax)\,dx = -\frac{1}{a}\log\cos ax = \frac{1}{a}\log\sec ax$

293. $\int (\cot ax)\,dx = \frac{1}{a}\log\sin ax = -\frac{1}{a}\log\csc ax$

294. $\int (\sec ax)\,dx = \frac{1}{a}\log(\sec ax + \tan ax) = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$

295. $\int (\csc ax)\,dx = \frac{1}{a}\log(\csc ax - \cot ax) = \frac{1}{a}\log\tan\frac{ax}{2}$

296. $\int (\sin^2 ax)\,dx = -\frac{1}{2a}\cos ax\sin ax + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$

297. $\int (\sin^3 ax)\,dx = -\frac{1}{3a}(\cos ax)(\sin^2 ax + 2)$

298. $\int (\sin^4 ax)\,dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

299. $\int (\sin^n ax)\,dx = -\frac{\sin^{n-1} ax\cos ax}{na} + \frac{n-1}{n}\int (\sin^{n-2} ax)\,dx$

INTEGRALS (Continued)

300. $\int (\sin^{2m} ax)\,dx = -\frac{\cos ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\sin^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

301. $\int (\sin^{2m+1} ax)\,dx = -\frac{\cos ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\sin^{2r} ax$

302. $\int (\cos^2 ax)\,dx = \frac{1}{2a}\sin ax\cos ax + \frac{1}{2}x = \frac{1}{2}x + \frac{1}{4a}\sin 2ax$

303. $\int (\cos^3 ax)\,dx = \frac{1}{3a}(\sin ax)(\cos^2 ax + 2)$

304. $\int (\cos^4 ax)\,dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$

305. $\int (\cos^n ax)\,dx = \frac{1}{na}\cos^{n-1} ax\sin ax + \frac{n-1}{n}\int (\cos^{n-2} ax)\,dx$

306. $\int (\cos^{2m} ax)\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\cos^{2r+1} ax + \frac{(2m)!}{2^{2m}(m!)^2}x$

307. $\int (\cos^{2m+1} ax)\,dx = \frac{\sin ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\cos^{2r} ax$

308. $\int \frac{dx}{\sin^2 ax} = \int (\csc^2 ax)\,dx = -\frac{1}{a}\cot ax$

309. $\int \frac{dx}{\sin^m ax} = \int (\csc^m ax)\,dx = -\frac{1}{(m-1)a}\cdot\frac{\cos ax}{\sin^{m-1} ax} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} ax}$

310. $\int \frac{dx}{\sin^{2m} ax} = \int (\csc^{2m} ax)\,dx = -\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\sin^{2r+1} ax}$

311. $\int \frac{dx}{\sin^{2m+1} ax} = \int (\csc^{2m+1} ax)\,dx =$

$$-\frac{1}{a}\cos ax\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)!\sin^{2r+2} ax} + \frac{1}{a}\cdot\frac{(2m)!}{2^{2m}(m!)^2}\log\tan\frac{ax}{2}$$

312. $\int \frac{dx}{\cos^2 ax} = \int (\sec^2 ax)\,dx = \frac{1}{a}\tan ax$

313. $\int \frac{dx}{\cos^n ax} = \int (\sec^n ax)\,dx = \frac{1}{(n-1)a}\cdot\frac{\sin ax}{\cos^{n-1} ax} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} ax}$

314. $\int \frac{dx}{\cos^{2m} ax} = \int (\sec^{2m} ax)\,dx = \frac{1}{a}\sin ax\sum_{r=0}^{m-1}\frac{2^{2m-2r-1}(m-1)!m!(2r)!}{(2m)!(r!)^2\cos^{2r+1} ax}$

INTEGRALS (Continued)

315. $$\int \frac{dx}{\cos^{2m+1} ax} = \int (\sec^{2m+1} ax)\, dx = \frac{1}{a} \sin ax \sum_{r=0}^{m-1} \frac{(2m)!(r!)^2}{2^{2m-2r}(m!)^2(2r+1)! \cos^{2r+2} ax} + \frac{1}{a} \cdot \frac{(2m)!}{2^{2m}(m!)^2} \log(\sec ax + \tan ax)$$

316. $$\int (\sin mx)(\sin nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

317. $$\int (\cos mx)(\cos nx)\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

318. $$\int (\sin ax)(\cos ax)\, dx = \frac{1}{2a} \sin^2 ax$$

319. $$\int (\sin mx)(\cos nx)\, dx = -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)}, \qquad (m^2 \neq n^2)$$

320. $$\int (\sin^2 ax)(\cos^2 ax)\, dx = -\frac{1}{32a} \sin 4ax + \frac{x}{8}$$

321. $$\int (\sin ax)(\cos^m ax)\, dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

322. $$\int (\sin^m ax)(\cos ax)\, dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

323. $$\int (\cos^m ax)(\sin^n ax)\, dx = \begin{cases} \dfrac{\cos^{m-1} ax \sin^{n+1} ax}{(m+n)a} + \dfrac{m-1}{m+n} \displaystyle\int (\cos^{m-2} ax)(\sin^n ax)\, dx \\ \text{or} \\ -\dfrac{\sin^{n-1} ax \cos^{m+1} ax}{(m+n)a} + \dfrac{n-1}{m+n} \displaystyle\int (\cos^m ax)(\sin^{n-2} ax)\, dx \end{cases}$$

324. $$\int \frac{\cos^m ax}{\sin^n ax}\, dx = \begin{cases} -\dfrac{\cos^{m+1} ax}{(n-1)a \sin^{n-1} ax} - \dfrac{m-n+2}{n-1} \displaystyle\int \frac{\cos^m ax}{\sin^{n-2} ax}\, dx \\ \text{or} \\ \dfrac{\cos^{m-1} ax}{a(m-n) \sin^{n-1} ax} + \dfrac{m-1}{m-n} \displaystyle\int \frac{\cos^{m-2} ax}{\sin^n ax}\, dx \end{cases}$$

INTEGRALS (Continued)

325. $$\int \frac{\sin^m ax}{\cos^n ax} dx = \begin{cases} \dfrac{\sin^{m+1} ax}{a(n-1)\cos^{n-1} ax} + \dfrac{m-n+2}{n-1} \displaystyle\int \frac{\sin^m ax}{\cos^{n-2} ax} dx \\ \text{or} \\ -\dfrac{\sin^{m-1} ax}{a(m-n)\cos^{n-1} ax} - \dfrac{m-1}{m-n} \displaystyle\int \frac{\sin^{m-2} ax}{\cos^n ax} dx \end{cases}$$

326. $$\int \frac{\sin ax}{\cos^2 ax} dx = \frac{1}{a \cos ax} = \frac{\sec ax}{a}$$

327. $$\int \frac{\sin^2 ax}{\cos ax} dx = -\frac{1}{a} \sin ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

328. $$\int \frac{\cos ax}{\sin^2 ax} dx = -\frac{1}{a \sin ax} = -\frac{\csc ax}{a}$$

329. $$\int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a} \log \tan ax$$

330. $$\int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a}\left(\sec ax + \log \tan \frac{ax}{2}\right)$$

331. $$\int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}$$

332. $$\int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a} \csc ax + \frac{1}{a} \log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

333. $$\int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a} \cot 2ax$$

334. $$\int \frac{dx}{\sin^m ax \cos^n ax} = \begin{cases} -\dfrac{1}{a(m-1)(\sin^{m-1} ax)(\cos^{n-1} ax)} \\ \qquad + \dfrac{m+n-2}{m-1} \displaystyle\int \frac{dx}{(\sin^{m-2} ax)(\cos^n ax)} \\ \text{or} \\ \dfrac{1}{a(n-1)\sin^{m-1} ax \cos^{n-1} ax} \\ \qquad - \dfrac{m+n-2}{n-1} \displaystyle\int \frac{dx}{\sin^m ax \cos^{n-2} ax} \end{cases}$$

335. $$\int \sin(a+bx)\,dx = -\frac{1}{b} \cos(a+bx)$$

336. $$\int \cos(a+bx)\,dx = \frac{1}{b} \sin(a+bx)$$

337. $$\int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

INTEGRALS (Continued)

338. $$\int \frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$$

339. $$\int \frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$$

*340. $$\int \frac{dx}{a+b\sin x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\dfrac{a\tan\frac{x}{2}+b}{\sqrt{a^2-b^2}} \\ \text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}}\log\dfrac{a\tan\frac{x}{2}+b-\sqrt{b^2-a^2}}{a\tan\frac{x}{2}+b+\sqrt{b^2-a^2}} \end{cases}$$

*341. $$\int \frac{dx}{a+b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2-b^2}}\tan^{-1}\dfrac{\sqrt{a^2-b^2}\tan\frac{x}{2}}{a+b} \\ \text{or} \\ \dfrac{1}{\sqrt{b^2-a^2}}\log\left(\dfrac{\sqrt{b^2-a^2}\tan\frac{x}{2}+a+b}{\sqrt{b^2-a^2}\tan\frac{x}{2}-a-b}\right) \end{cases}$$

*342. $$\int \frac{dx}{a+b\sin x+c\cos x}$$
$$= \begin{cases} \dfrac{1}{\sqrt{b^2+c^2-a^2}}\log\dfrac{b-\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}{b+\sqrt{b^2+c^2-a^2}+(a-c)\tan\frac{x}{2}}, & \text{if } a^2<b^2+c^2, a\neq c \\ \text{or} \\ \dfrac{2}{\sqrt{a^2-b^2-c^2}}\tan^{-1}\dfrac{b+(a-c)\tan\frac{x}{2}}{\sqrt{a^2-b^2-c^2}}, & \text{if } a^2>b^2+c^2 \\ \text{or} \\ \dfrac{1}{a}\left[\dfrac{a-(b+c)\cos x-(b-c)\sin x}{a-(b-c)\cos x+(b+c)\sin x}\right], & \text{if } a^2=b^2+c^2, a\neq c. \end{cases}$$

*343. $$\int \frac{\sin^2 x\,dx}{a+b\cos^2 x} = \frac{1}{b}\sqrt{\frac{a+b}{a}}\tan^{-1}\left(\sqrt{\frac{a}{a+b}}\tan x\right)-\frac{x}{b}, \qquad (ab>0, \text{ or } |a|>|b|)$$

* See note p. 403

INTEGRALS (Continued)

*344. $$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{b\tan x}{a}\right)$$

*345. $$\int \frac{\cos^2 cx}{a^2 + b^2\sin^2 cx}dx = \frac{\sqrt{a^2+b^2}}{ab^2c}\tan^{-1}\frac{\sqrt{a^2+b^2}\tan cx}{a} - \frac{x}{b^2}$$

346. $$\int \frac{\sin cx \cos cx}{a\cos^2 cx + b\sin^2 cx}dx = \frac{1}{2c(b-a)}\log(a\cos^2 cx + b\sin^2 cx)$$

347. $$\int \frac{\cos cx}{a\cos cx + b\sin cx}dx = \int \frac{dx}{a + b\tan cx} = \frac{1}{c(a^2+b^2)}[acx + b\log(a\cos cx + b\sin cx)]$$

348. $$\int \frac{\sin cx}{a\sin cx + b\cos cx}dx = \int \frac{dx}{a + b\cot cx} = \frac{1}{c(a^2+b^2)}[acx - b\log(a\sin cx + b\cos cx)]$$

*349. $$\int \frac{dx}{a\cos^2 x + 2b\cos x\sin x + c\sin^2 x} = \begin{cases} \dfrac{1}{2\sqrt{b^2-ac}}\log\dfrac{c\tan x + b - \sqrt{b^2-ac}}{c\tan x + b + \sqrt{b^2-ac}}, & (b^2 > ac) \\ \text{or} & \\ \dfrac{1}{\sqrt{ac-b^2}}\tan^{-1}\dfrac{c\tan x + b}{\sqrt{ac-b^2}}, & (b^2 < ac) \\ \text{or} & \\ -\dfrac{1}{c\tan x + b}, & (b^2 = ac) \end{cases}$$

350. $$\int \frac{\sin ax}{1 \pm \sin ax}dx = \pm x + \frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

351. $$\int \frac{dx}{(\sin ax)(1 \pm \sin ax)} = \frac{1}{a}\tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + \frac{1}{a}\log\tan\frac{ax}{2}$$

352. $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) - \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

353. $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a}\cot\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

354. $$\int \frac{\sin ax}{(1+\sin ax)^2}dx = -\frac{1}{2a}\tan\left(\frac{\pi}{4} - \frac{ax}{2}\right) + \frac{1}{6a}\tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

* See note p. 403.

INTEGRALS (Continued)

355. $$\int \frac{\sin ax}{(1-\sin ax)^2}\,dx = -\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

356. $$\int \frac{\sin x\,dx}{a+b\sin x} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\sin x}$$

357. $$\int \frac{dx}{(\sin x)(a+b\sin x)} = \frac{1}{a}\log\tan\frac{x}{2} - \frac{b}{a}\int \frac{dx}{a+b\sin x}$$

358. $$\int \frac{dx}{(a+b\sin x)^2} = \frac{b\cos x}{(a^2-b^2)(a+b\sin x)} + \frac{a}{a^2-b^2}\int \frac{dx}{a+b\sin x}$$

359. $$\int \frac{\sin x\,dx}{(a+b\sin x)^2} = \frac{a\cos x}{(b^2-a^2)(a+b\sin x)} + \frac{b}{b^2-a^2}\int \frac{dx}{a+b\sin x}$$

*360. $$\int \frac{dx}{a^2+b^2\sin^2 cx} = \frac{1}{ac\sqrt{a^2+b^2}}\tan^{-1}\frac{\sqrt{a^2+b^2}\tan cx}{a}$$

*361. $$\int \frac{dx}{a^2-b^2\sin^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2-b^2}}\tan^{-1}\dfrac{\sqrt{a^2-b^2}\tan cx}{a}, & (a^2>b^2) \\ \text{or} & \\ \dfrac{1}{2ac\sqrt{b^2-a^2}}\log\dfrac{\sqrt{b^2-a^2}\tan cx+a}{\sqrt{b^2-a^2}\tan cx-a}, & (a^2<b^2) \end{cases}$$

362. $$\int \frac{\cos ax}{1+\cos ax}\,dx = x - \frac{1}{a}\tan\frac{ax}{2}$$

363. $$\int \frac{\cos ax}{1-\cos ax}\,dx = -x - \frac{1}{a}\cot\frac{ax}{2}$$

364. $$\int \frac{dx}{(\cos ax)(1+\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\tan\frac{ax}{2}$$

365. $$\int \frac{dx}{(\cos ax)(1-\cos ax)} = \frac{1}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right) - \frac{1}{a}\cot\frac{ax}{2}$$

366. $$\int \frac{dx}{(1+\cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

367. $$\int \frac{dx}{(1-\cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

368. $$\int \frac{\cos ax}{(1+\cos ax)^2}\,dx = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}$$

369. $$\int \frac{\cos ax}{(1-\cos ax)^2}\,dx = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

* See note p. 403.

INTEGRALS (Continued)

370. $$\int \frac{\cos x\,dx}{a+b\cos x} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\cos x}$$

371. $$\int \frac{dx}{(\cos x)(a+b\cos x)} = \frac{1}{a}\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - \frac{b}{a}\int \frac{dx}{a+b\cos x}$$

372. $$\int \frac{dx}{(a+b\cos x)^2} = \frac{b\sin x}{(b^2-a^2)(a+b\cos x)} - \frac{a}{b^2-a^2}\int \frac{dx}{a+b\cos x}$$

373. $$\int \frac{\cos x}{(a+b\cos x)^2}dx = \frac{a\sin x}{(a^2-b^2)(a+b\cos x)} - \frac{b}{a^2-b^2}\int \frac{dx}{a+b\cos x}$$

*374. $$\int \frac{dx}{a^2+b^2-2ab\cos cx} = \frac{2}{c(a^2-b^2)}\tan^{-1}\left(\frac{a+b}{a-b}\tan\frac{cx}{2}\right)$$

*375. $$\int \frac{dx}{a^2+b^2\cos^2 cx} = \frac{1}{ac\sqrt{a^2+b^2}}\tan^{-1}\frac{a\tan cx}{\sqrt{a^2+b^2}}$$

*376. $$\int \frac{dx}{a^2-b^2\cos^2 cx} = \begin{cases} \dfrac{1}{ac\sqrt{a^2-b^2}}\tan^{-1}\dfrac{a\tan cx}{\sqrt{a^2-b^2}}, & (a^2>b^2) \\ \text{or} & \\ \dfrac{1}{2ac\sqrt{b^2-a^2}}\log\dfrac{a\tan cx-\sqrt{b^2-a^2}}{a\tan cx+\sqrt{b^2-a^2}}, & (b^2>a^2) \end{cases}$$

377. $$\int \frac{\sin ax}{1\pm\cos ax}dx = \mp\frac{1}{a}\log(1\pm\cos ax)$$

378. $$\int \frac{\cos ax}{1\pm\sin ax}dx = \pm\frac{1}{a}\log(1\pm\sin ax)$$

379. $$\int \frac{dx}{(\sin ax)(1\pm\cos ax)} = \pm\frac{1}{2a(1\pm\cos ax)} + \frac{1}{2a}\log\tan\frac{ax}{2}$$

380. $$\int \frac{dx}{(\cos ax)(1\pm\sin ax)} = \mp\frac{1}{2a(1\pm\sin ax)} + \frac{1}{2a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

381. $$\int \frac{\sin ax}{(\cos ax)(1\pm\cos ax)}dx = \frac{1}{a}\log(\sec ax\pm 1)$$

382. $$\int \frac{\cos ax}{(\sin ax)(1\pm\sin ax)}dx = -\frac{1}{a}\log(\csc ax\pm 1)$$

383. $$\int \frac{\sin ax}{(\cos ax)(1\pm\sin ax)}dx = \frac{1}{2a(1\pm\sin ax)} \pm \frac{1}{2a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

384. $$\int \frac{\cos ax}{(\sin ax)(1\pm\cos ax)}dx = -\frac{1}{2a(1\pm\cos ax)} \pm \frac{1}{2a}\log\tan\frac{ax}{2}$$

* See note p. 403.

INTEGRALS (Continued)

385. $\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}} \log \tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$

386. $\int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a} \tan\left(ax \mp \frac{\pi}{4}\right)$

387. $\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm \frac{1}{a} \log\left(1 \pm \tan \frac{ax}{2}\right)$

388. $\int \frac{dx}{a^2 \cos^2 cx - b^2 \sin^2 cx} = \frac{1}{2abc} \log \frac{b \tan cx + a}{b \tan cx - a}$

389. $\int x(\sin ax)\, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax$

390. $\int x^2(\sin ax)\, dx = \frac{2x}{a^2} \sin ax - \frac{a^2x^2 - 2}{a^3} \cos ax$

391. $\int x^3(\sin ax)\, dx = \frac{3a^2x^2 - 6}{a^4} \sin ax - \frac{a^2x^3 - 6x}{a^3} \cos ax$

392. $$\int x^m \sin ax\, dx = \begin{cases} -\frac{1}{a} x^m \cos ax + \frac{m}{a} \int x^{m-1} \cos ax\, dx \\ \text{or} \\ \cos ax \sum_{r=0}^{\left[\frac{m}{2}\right]} (-1)^{r+1} \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ \quad + \sin ax \sum_{r=0}^{\left[\frac{m-1}{2}\right]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$; $[3\frac{1}{2}] = 3$, $[\frac{1}{2}] = 0$, etc.

393. $\int x(\cos ax)\, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax$

394. $\int x^2(\cos ax)\, dx = \frac{2x \cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3} \sin ax$

395. $\int x^3(\cos ax)\, dx = \frac{3a^2x^2 - 6}{a^4} \cos ax + \frac{a^2x^3 - 6x}{a^3} \sin ax$

396. $$\int x^m(\cos ax)\, dx = \begin{cases} \frac{x^m \sin ax}{a} - \frac{m}{a} \int x^{m-1} \sin ax\, dx \\ \text{or} \\ \sin ax \sum_{r=0}^{\left[\frac{m}{2}\right]} (-1)^r \frac{m!}{(m-2r)!} \cdot \frac{x^{m-2r}}{a^{2r+1}} \\ \quad + \cos ax \sum_{r=0}^{\left[\frac{m-1}{2}\right]} (-1)^r \frac{m!}{(m-2r-1)!} \cdot \frac{x^{m-2r-1}}{a^{2r+2}} \end{cases}$$

See note integral 392.

INTEGRALS (Continued)

397. $\int \frac{\sin ax}{x}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}$

398. $\int \frac{\cos ax}{x}\,dx = \log x + \sum_{n=1}^{\infty} (-1)^n \frac{(ax)^{2n}}{2n(2n)!}$

399. $\int x(\sin^2 ax)\,dx = \frac{x^2}{4} - \frac{x \sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$

400. $\int x^2(\sin^2 ax)\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax - \frac{x \cos 2ax}{4a^2}$

401. $\int x(\sin^3 ax)\,dx = \frac{x \cos 3ax}{12a} - \frac{\sin 3ax}{36a^2} - \frac{3x \cos ax}{4a} + \frac{3 \sin ax}{4a^2}$

402. $\int x(\cos^2 ax)\,dx = \frac{x^2}{4} + \frac{x \sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$

403. $\int x^2(\cos^2 ax)\,dx = \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right) \sin 2ax + \frac{x \cos 2ax}{4a^2}$

404. $\int x(\cos^3 ax)\,dx = \frac{x \sin 3ax}{12a} + \frac{\cos 3ax}{36a^2} + \frac{3x \sin ax}{4a} + \frac{3 \cos ax}{4a^2}$

405. $\int \frac{\sin ax}{x^m}\,dx = -\frac{\sin ax}{(m-1)x^{m-1}} + \frac{a}{m-1}\int \frac{\cos ax}{x^{m-1}}\,dx$

406. $\int \frac{\cos ax}{x^m}\,dx = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1}\int \frac{\sin ax}{x^{m-1}}\,dx$

407. $\int \frac{x}{1 \pm \sin ax}\,dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2} \log (1 \pm \sin ax)$

408. $\int \frac{x}{1 + \cos ax}\,dx = \frac{x}{a} \tan \frac{ax}{2} + \frac{2}{a^2} \log \cos \frac{ax}{2}$

409. $\int \frac{x}{1 - \cos ax}\,dx = -\frac{x}{a} \cot \frac{ax}{2} + \frac{2}{a^2} \log \sin \frac{ax}{2}$

410. $\int \frac{x + \sin x}{1 + \cos x}\,dx = x \tan \frac{x}{2}$

411. $\int \frac{x - \sin x}{1 - \cos x}\,dx = -x \cot \frac{x}{2}$

412. $\int \sqrt{1 - \cos ax}\,dx = -\frac{2 \sin ax}{a\sqrt{1 - \cos ax}}$

413. $\int \sqrt{1 + \cos ax}\,dx = \frac{2 \sin ax}{a\sqrt{1 + \cos ax}}$

INTEGRALS (Continued)

414. $\int \sqrt{1+\sin x}\,dx = \pm 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right),$

[use + if $(8k-1)\frac{\pi}{2} < x \leq (8k+3)\frac{\pi}{2}$, otherwise − ; k an integer]

415. $\int \sqrt{1-\sin x}\,dx = \pm 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right),$

[use + if $(8k-3)\frac{\pi}{2} < x \leq (8k+1)\frac{\pi}{2}$, otherwise − ; k an integer]

416. $\int \frac{dx}{\sqrt{1-\cos x}} = \pm\sqrt{2}\log\tan\frac{x}{4},$

[use + if $4k\pi < x < (4k+2)\pi$, otherwise − ; k an integer]

417. $\int \frac{dx}{\sqrt{1+\cos x}} = \pm\sqrt{2}\log\tan\left(\frac{x+\pi}{4}\right),$

[use + if $(4k-1)\pi < x < (4k+1)\pi$, otherwise − ; k an integer]

418. $\int \frac{dx}{\sqrt{1-\sin x}} = \pm\sqrt{2}\log\tan\left(\frac{x}{4} - \frac{\pi}{8}\right),$

[use + if $(8k+1)\frac{\pi}{2} < x < (8k+5)\frac{\pi}{2}$, otherwise − ; k an integer]

419. $\int \frac{dx}{\sqrt{1+\sin x}} = \pm\sqrt{2}\log\tan\left(\frac{x}{4} + \frac{\pi}{8}\right),$

[use + if $(8k-1)\frac{\pi}{2} < x < (8k+3)\frac{\pi}{2}$, otherwise − ; k an integer]

420. $\int (\tan^2 ax)\,dx = \frac{1}{a}\tan ax - x$

421. $\int (\tan^3 ax)\,dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log\cos ax$

422. $\int (\tan^4 ax)\,dx = \frac{\tan^3 ax}{3a} - \frac{1}{a}\tan x + x$

423. $\int (\tan^n ax)\,dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int (\tan^{n-2} ax)\,dx$

424. $\int (\cot^2 ax)\,dx = -\frac{1}{a}\cot ax - x$

425. $\int (\cot^3 ax)\,dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\log\sin ax$

426. $\int (\cot^4 ax)\,dx = -\frac{1}{3a}\cot^3 ax + \frac{1}{a}\cot ax + x$

INTEGRALS (Continued)

427. $$\int (\cot^n ax)\,dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int (\cot^{n-2} ax)\,dx$$

428. $$\int \frac{x}{\sin^2 ax}\,dx = \int x(\csc^2 ax)\,dx = -\frac{x \cot ax}{a} + \frac{1}{a^2}\log \sin ax$$

429. $$\int \frac{x}{\sin^n ax}\,dx = \int x(\csc^n ax)\,dx = -\frac{x \cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{(n-2)}{(n-1)}\int \frac{x}{\sin^{n-2} ax}\,dx$$

430. $$\int \frac{x}{\cos^2 ax}\,dx = \int x(\sec^2 ax)\,dx = \frac{1}{a}x \tan ax + \frac{1}{a^2}\log \cos ax$$

431. $$\int \frac{x}{\cos^n ax}\,dx = \int x(\sec^n ax)\,dx = \frac{x \sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x}{\cos^{n-2} ax}\,dx$$

432. $$\int \frac{\sin ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}$$

433. $$\int \frac{\sin ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = -\frac{1}{ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right)$$

434. $$\int (\sin ax)\sqrt{1+b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1+b^2\sin^2 ax} - \frac{1+b^2}{2ab}\sin^{-1}\frac{b\cos ax}{\sqrt{1+b^2}}$$

435. $$\int (\sin ax)\sqrt{1-b^2\sin^2 ax}\,dx = -\frac{\cos ax}{2a}\sqrt{1-b^2\sin^2 ax} - \frac{1-b^2}{2ab}\log\left(b\cos ax + \sqrt{1-b^2\sin^2 ax}\right)$$

436. $$\int \frac{\cos ax}{\sqrt{1+b^2\sin^2 ax}}\,dx = \frac{1}{ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

437. $$\int \frac{\cos ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = \frac{1}{ab}\sin^{-1}(b\sin ax)$$

438. $$\int (\cos ax)\sqrt{1+b^2\sin^2 ax}\,dx = \frac{\sin ax}{2a}\sqrt{1+b^2\sin^2 ax} + \frac{1}{2ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

INTEGRALS (Continued)

439. $\int (\cos ax)\sqrt{1 - b^2 \sin^2 ax}\, dx = \frac{\sin ax}{2a}\sqrt{1 - b^2 \sin^2 ax} + \frac{1}{2ab}\sin^{-1}(b \sin ax)$

440. $\int \frac{dx}{\sqrt{a + b \tan^2 cx}} = \frac{\pm 1}{c\sqrt{a - b}} \sin^{-1}\left(\sqrt{\frac{a - b}{a}} \sin cx\right), \quad (a > |b|)$

[use + if $(2k - 1)\frac{\pi}{2} < x \leq (2k + 1)\frac{\pi}{2}$, otherwise − ; k an integer]

FORMS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

441. $\int (\sin^{-1} ax)\, dx = x \sin^{-1} ax + \frac{\sqrt{1 - a^2x^2}}{a}$

442. $\int (\cos^{-1} ax)\, dx = x \cos^{-1} ax - \frac{\sqrt{1 - a^2x^2}}{a}$

443. $\int (\tan^{-1} ax)\, dx = x \tan^{-1} ax - \frac{1}{2a}\log(1 + a^2x^2)$

444. $\int (\cot^{-1} ax)\, dx = x \cot^{-1} ax + \frac{1}{2a}\log(1 + a^2x^2)$

445. $\int (\sec^{-1} ax)\, dx = x \sec^{-1} ax - \frac{1}{a}\log(ax + \sqrt{a^2x^2 - 1})$

446. $\int (\csc^{-1} ax)\, dx = x \csc^{-1} ax + \frac{1}{a}\log(ax + \sqrt{a^2x^2 - 1})$

447. $\int \left(\sin^{-1}\frac{x}{a}\right) dx = x \sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2}, \quad (a > 0)$

448. $\int \left(\cos^{-1}\frac{x}{a}\right) dx = x \cos^{-1}\frac{x}{a} - \sqrt{a^2 - x^2}, \quad (a > 0)$

449. $\int \left(\tan^{-1}\frac{x}{a}\right) dx = x \tan^{-1}\frac{x}{a} - \frac{a}{2}\log(a^2 + x^2)$

450. $\int \left(\cot^{-1}\frac{x}{a}\right) dx = x \cot^{-1}\frac{x}{a} + \frac{a}{2}\log(a^2 + x^2)$

451. $\int x[\sin^{-1}(ax)]\, dx = \frac{1}{4a^2}[(2a^2x^2 - 1)\sin^{-1}(ax) + ax\sqrt{1 - a^2x^2}]$

452. $\int x[\cos^{-1}(ax)]\, dx = \frac{1}{4a^2}[(2a^2x^2 - 1)\cos^{-1}(ax) - ax\sqrt{1 - a^2x^2}]$

INTEGRALS (Continued)

453. $$\int x^n[\sin^{-1}(ax)]\,dx = \frac{x^{n+1}}{n+1}\sin^{-1}(ax) - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad (n \neq -1)$$

454. $$\int x^n[\cos^{-1}(ax)]\,dx = \frac{x^{n+1}}{n+1}\cos^{-1}(ax) + \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad (n \neq -1)$$

455. $$\int x(\tan^{-1} ax)\,dx = \frac{1+a^2x^2}{2a^2}\tan^{-1} ax - \frac{x}{2a}$$

456. $$\int x^n(\tan^{-1} ax)\,dx = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx$$

457. $$\int x(\cot^{-1} ax)\,dx = \frac{1+a^2x^2}{2a^2}\cot^{-1} ax + \frac{x}{2a}$$

458. $$\int x^n(\cot^{-1} ax)\,dx = \frac{x^{n+1}}{n+1}\cot^{-1} ax + \frac{a}{n+1}\int\frac{x^{n+1}}{1+a^2x^2}\,dx$$

459. $$\int\frac{\sin^{-1}(ax)}{x^2}\,dx = a\log\left(\frac{1-\sqrt{1-a^2x^2}}{x}\right) - \frac{\sin^{-1}(ax)}{x}$$

460. $$\int\frac{\cos^{-1}(ax)\,dx}{x^2} = -\frac{1}{x}\cos^{-1}(ax) + a\log\frac{1+\sqrt{1-a^2x^2}}{x}$$

461. $$\int\frac{\tan^{-1}(ax)\,dx}{x^2} = -\frac{1}{x}\tan^{-1}(ax) - \frac{a}{2}\log\frac{1+a^2x^2}{x^2}$$

462. $$\int\frac{\cot^{-1} ax}{x^2}\,dx = -\frac{1}{x}\cot^{-1} ax - \frac{a}{2}\log\frac{x^2}{a^2x^2+1}$$

463. $$\int(\sin^{-1} ax)^2\,dx = x(\sin^{-1} ax)^2 - 2x + \frac{2\sqrt{1-a^2x^2}}{a}\sin^{-1} ax$$

464. $$\int(\cos^{-1} ax)^2\,dx = x(\cos^{-1} ax)^2 - 2x - \frac{2\sqrt{1-a^2x^2}}{a}\cos^{-1} ax$$

465. $$\int(\sin^{-1} ax)^n\,dx = \begin{cases} x(\sin^{-1} ax)^n + \dfrac{n\sqrt{1-a^2x^2}}{a}(\sin^{-1} ax)^{n-1} \\ \qquad\qquad -n(n-1)\displaystyle\int(\sin^{-1} ax)^{n-2}\,dx \\ \text{or} \\ \displaystyle\sum_{r=0}^{[\frac{n}{2}]}(-1)^r\frac{n!}{(n-2r)!}\,x(\sin^{-1} ax)^{n-2r} \\ \qquad + \displaystyle\sum_{r=0}^{[\frac{n-1}{2}]}(-1)^r\frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!\,a}(\sin^{-1} ax)^{n-2r-1} \end{cases}$$

Note: $[s]$ means greatest integer $\leq s$. Thus $[3.5]$ means 3; $[5] = 5$, $[\frac{1}{2}] = 0$.

INTEGRALS (Continued)

466. $$\int (\cos^{-1} ax)^n\,dx = \begin{cases} x(\cos^{-1} ax)^n - \dfrac{n\sqrt{1-a^2x^2}}{a}(\cos^{-1} ax)^{n-1} \\ \qquad\qquad -n(n-1)\displaystyle\int (\cos^{-1} ax)^{n-2}\,dx \\ \text{or} \\ \displaystyle\sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r \frac{n!}{(n-2r)!} x(\cos^{-1} ax)^{n-2r} \\ \qquad - \displaystyle\sum_{r=0}^{\left[\frac{n-1}{2}\right]} (-1)^r \frac{n!\sqrt{1-a^2x^2}}{(n-2r-1)!a}(\cos^{-1} ax)^{n-2r-1} \end{cases}$$

467. $$\int \frac{1}{\sqrt{1-a^2x^2}}(\sin^{-1} ax)\,dx = \frac{1}{2a}(\sin^{-1} ax)^2$$

468. $$\int \frac{x^n}{\sqrt{1-a^2x^2}}(\sin^{-1} ax)\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\sin^{-1} ax + \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}}{\sqrt{1-a^2x^2}}\sin^{-1} ax\,dx$$

469. $$\int \frac{1}{\sqrt{1-a^2x^2}}(\cos^{-1} ax)\,dx = -\frac{1}{2a}(\cos^{-1} ax)^2$$

470. $$\int \frac{x^n}{\sqrt{1-a^2x^2}}(\cos^{-1} ax)\,dx = -\frac{x^{n-1}}{na^2}\sqrt{1-a^2x^2}\cos^{-1} ax - \frac{x^n}{n^2a} + \frac{n-1}{na^2}\int \frac{x^{n-2}}{\sqrt{1-a^2x^2}}\cos^{-1} ax\,dx$$

471. $$\int \frac{\tan^{-1} ax}{a^2x^2+1}\,dx = \frac{1}{2a}(\tan^{-1} ax)^2$$

472. $$\int \frac{\cot^{-1} ax}{a^2x^2+1}\,dx = -\frac{1}{2a}(\cot^{-1} ax)^2$$

473. $$\int x\sec^{-1} ax\,dx = \frac{x^2}{2}\sec^{-1} ax - \frac{1}{2a^2}\sqrt{a^2x^2-1}$$

474. $$\int x^n\sec^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sec^{-1} ax - \frac{1}{n+1}\int \frac{x^n\,dx}{\sqrt{a^2x^2-1}}$$

475. $$\int \frac{\sec^{-1} ax}{x^2}\,dx = -\frac{\sec^{-1} ax}{x} + \frac{\sqrt{a^2x^2-1}}{x}$$

476. $$\int x\csc^{-1} ax\,dx = \frac{x^2}{2}\csc^{-1} ax + \frac{1}{2a^2}\sqrt{a^2x^2-1}$$

477. $$\int x^n\csc^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\csc^{-1} ax + \frac{1}{n+1}\int \frac{x^n\,dx}{\sqrt{a^2x^2-1}}$$

INTEGRALS (Continued)

478. $\displaystyle\int \frac{\csc^{-1} ax}{x^2}\,dx = -\frac{\csc^{-1} ax}{x} - \frac{\sqrt{a^2x^2 - 1}}{x}$

FORMS INVOLVING TRIGONOMETRIC SUBSTITUTIONS

479. $\displaystyle\int f(\sin x)\,dx = 2\int f\left(\frac{2z}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(z = \tan\frac{x}{2}\right)$

480. $\displaystyle\int f(\cos x)\,dx = 2\int f\left(\frac{1-z^2}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(z = \tan\frac{x}{2}\right)$

***481.** $\displaystyle\int f(\sin x)\,dx = \int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad (u = \sin x)$

***482.** $\displaystyle\int f(\cos x)\,dx = -\int f(u)\frac{du}{\sqrt{1-u^2}}, \qquad (u = \cos x)$

***483.** $\displaystyle\int f(\sin x, \cos x)\,dx = \int f(u, \sqrt{1-u^2})\frac{du}{\sqrt{1-u^2}}, \qquad (u = \sin x)$

484. $\displaystyle\int f(\sin x, \cos x)\,dx = 2\int f\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right)\frac{dz}{1+z^2}, \qquad \left(\quad = \tan\frac{x}{2}\right)$

LOGARITHMIC FORMS

485. $\displaystyle\int (\log x)\,dx = x\log x - x$

486. $\displaystyle\int x(\log x)\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$

487. $\displaystyle\int x^2(\log x)\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9}$

488. $\displaystyle\int x^n(\log ax)\,dx = \frac{x^{n+1}}{n+1}\log ax - \frac{x^{n+1}}{(n+1)^2}$

489. $\displaystyle\int (\log x)^2\,dx = x(\log x)^2 - 2x\log x + 2x$

490. $$\int (\log x)^n\,dx = \begin{cases} x(\log x)^n - n\displaystyle\int (\log x)^{n-1}\,dx, & (n \neq -1) \\ \quad \text{or} & \\ (-1)^n n!\,x \displaystyle\sum_{r=0}^{n} \frac{(-\log x)^r}{r!} & \end{cases}$$

* The square roots appearing in these formulas may be plus or minus, depending on the quadrant of x. Care must be used to give them the proper sign.

INTEGRALS (Continued)

491. $$\int \frac{(\log x)^n}{x}\,dx = \frac{1}{n+1}(\log x)^{n+1}$$

492. $$\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2\cdot 2!} + \frac{(\log x)^3}{3\cdot 3!} + \cdots$$

493. $$\int \frac{dx}{x\log x} = \log(\log x)$$

494. $$\int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}$$

495. $$\int \frac{x^m\,dx}{(\log x)^n} = -\frac{x^{m+1}}{(n-1)(\log x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m\,dx}{(\log x)^{n-1}}$$

496. $$\int x^m(\log x)^n\,dx = \begin{cases} \dfrac{x^{m+1}(\log x)^n}{m+1} - \dfrac{n}{m+1}\displaystyle\int x^m(\log x)^{n-1}\,dx \\ \quad\text{or} \\ (-1)^n\dfrac{n!}{m+1}x^{m+1}\displaystyle\sum_{r=0}^{n}\dfrac{(-\log x)^r}{r!(m+1)^{n-r}} \end{cases}$$

497. $$\int \sin(\log x)\,dx = \tfrac{1}{2}x\sin(\log x) - \tfrac{1}{2}x\cos(\log x)$$

498. $$\int \cos(\log x)\,dx = \tfrac{1}{2}x\sin(\log x) + \tfrac{1}{2}x\cos(\log x)$$

499. $$\int [\log(ax+b)]\,dx = \frac{ax+b}{a}\log(ax+b) - x$$

500. $$\int \frac{\log(ax+b)}{x^2}\,dx = \frac{a}{b}\log x - \frac{ax+b}{bx}\log(ax+b)$$

501. $$\int x^m[\log(ax+b)]\,dx = \frac{1}{m+1}\left[x^{m+1} - \left(-\frac{b}{a}\right)^{m+1}\right]\log(ax+b) - \frac{1}{m+1}\left(-\frac{b}{a}\right)^{m+1}\sum_{r=1}^{m+1}\frac{1}{r}\left(-\frac{ax}{b}\right)^r$$

502. $$\int \frac{\log(ax+b)}{x^m}\,dx = -\frac{1}{m-1}\,\frac{\log(ax+b)}{x^{m-1}} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\log\frac{ax+b}{x} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\sum_{r=1}^{m-2}\frac{1}{r}\left(-\frac{b}{ax}\right)^r,\ (m>2)$$

503. $$\int \left[\log\frac{x+a}{x-a}\right]dx = (x+a)\log(x+a) - (x-a)\log(x-a)$$

504. $$\int x^m\left[\log\frac{x+a}{x-a}\right]dx = \frac{x^{m+1}-(-a)^{m+1}}{m+1}\log(x+a) - \frac{x^{m+1}-a^{m+1}}{m+1}\log(x-a) + \frac{2a^{m+1}}{m+1}\sum_{r=1}^{\left[\frac{m+1}{2}\right]}\frac{1}{m-2r+2}\left(\frac{x}{a}\right)^{m-2r+2}$$

INTEGRALS (Continued)

505. $$\int \frac{1}{x^2}\left[\log\frac{x+a}{x-a}\right]dx = \frac{1}{x}\log\frac{x-a}{x+a} - \frac{1}{a}\log\frac{x^2-a^2}{x^2}$$

506. $$\int (\log X)\,dx = \begin{cases} \left(x+\dfrac{b}{2c}\right)\log X - 2x + \dfrac{\sqrt{4ac-b^2}}{c}\tan^{-1}\dfrac{2cx+b}{\sqrt{4ac-b^2}}, \\ \qquad (b^2-4ac<0) \\ \text{or} \\ \left(x+\dfrac{b}{2c}\right)\log X - 2x + \dfrac{\sqrt{b^2-4ac}}{c}\tanh^{-1}\dfrac{2cx+b}{\sqrt{b^2-4ac}}, \\ \qquad (b^2-4ac>0) \\ \text{where} \\ X = a+bx+cx^2 \end{cases}$$

507. $$\int x^n(\log X)\,dx = \frac{x^{n+1}}{n+1}\log X - \frac{2c}{n+1}\int\frac{x^{n+2}}{X}dx - \frac{b}{n+1}\int\frac{x^{n+1}}{X}dx$$

where $X = a + bx + cx^2$

508. $$\int [\log(x^2+a^2)]\,dx = x\log(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$$

509. $$\int [\log(x^2-a^2)]\,dx = x\log(x^2-a^2) - 2x + a\log\frac{x+a}{x-a}$$

510. $$\int x[\log(x^2\pm a^2)]\,dx = \tfrac{1}{2}(x^2\pm a^2)\log(x^2\pm a^2) - \tfrac{1}{2}x^2$$

511. $$\int [\log(x+\sqrt{x^2\pm a^2})]\,dx = x\log(x+\sqrt{x^2\pm a^2}) - \sqrt{x^2\pm a^2}$$

512. $$\int x[\log(x+\sqrt{x^2\pm a^2})]\,dx = \left(\frac{x^2}{2}\pm\frac{a^2}{4}\right)\log(x+\sqrt{x^2\pm a^2}) - \frac{x\sqrt{x^2\pm a^2}}{4}$$

513. $$\int x^m[\log(x+\sqrt{x^2\pm a^2})]\,dx = \frac{x^{m+1}}{m+1}\log(x+\sqrt{x^2\pm a^2}) - \frac{1}{m+1}\int\frac{x^{m+1}}{\sqrt{x^2\pm a^2}}dx$$

514. $$\int \frac{\log(x+\sqrt{x^2+a^2})}{x^2}dx = -\frac{\log(x+\sqrt{x^2+a^2})}{x} - \frac{1}{a}\log\frac{a+\sqrt{x^2+a^2}}{x}$$

515. $$\int \frac{\log(x+\sqrt{x^2-a^2})}{x^2}dx = -\frac{\log(x+\sqrt{x^2-a^2})}{x} + \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

INTEGRALS (Continued)

516. $$\int x^n \log(x^2 - a^2)\,dx = \frac{1}{n+1}\left[x^{n+1}\log(x^2-a^2) - a^{n+1}\log(x-a) - (-a)^{n+1}\log(x+a) - 2\sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{a^{2r}x^{n-2r+1}}{n-2r+1}\right]$$

EXPONENTIAL FORMS

517. $$\int e^x\,dx = e^x$$

518. $$\int e^{-x}\,dx = -e^{-x}$$

519. $$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

520. $$\int x\,e^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

521. $$\int x^m e^{ax}\,dx = \begin{cases} \dfrac{x^m e^{ax}}{a} - \dfrac{m}{a}\displaystyle\int x^{m-1} e^{ax}\,dx \\ \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}(-1)^r \frac{m!\,x^{m-r}}{(m-r)!\,a^{r+1}} \end{cases}$$

522. $$\int \frac{e^{ax}\,dx}{x} = \log x + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots$$

523. $$\int \frac{e^{ax}}{x^m}\,dx = -\frac{1}{m-1}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{e^{ax}}{x^{m-1}}\,dx$$

524. $$\int e^{ax}\log x\,dx = \frac{e^{ax}\log x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

525. $$\int \frac{dx}{1+e^x} = x - \log(1+e^x) = \log\frac{e^x}{1+e^x}$$

526. $$\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\log(a+be^{px})$$

527. $$\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}}\tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right), \qquad (a>0, b>0)$$

528. $$\int \frac{dx}{ae^{mx}-be^{-mx}} = \begin{cases} \dfrac{1}{2m\sqrt{ab}}\log\dfrac{\sqrt{a}\,e^{mx}-\sqrt{b}}{\sqrt{a}\,e^{mx}+\sqrt{b}} \\ \text{or} \\ \dfrac{-1}{m\sqrt{ab}}\tanh^{-1}\left(\sqrt{\dfrac{a}{b}}\,e^{mx}\right), \qquad (a>0, b>0) \end{cases}$$

INTEGRALS (Continued)

529. $\int (a^x - a^{-x})\,dx = \dfrac{a^x + a^{-x}}{\log a}$

530. $\int \dfrac{e^{ax}}{b + ce^{ax}}\,dx = \dfrac{1}{ac}\log(b + ce^{ax})$

531. $\int \dfrac{x\,e^{ax}}{(1 + ax)^2}\,dx = \dfrac{e^{ax}}{a^2(1 + ax)}$

532. $\int x\,e^{-x^2}\,dx = -\tfrac{1}{2}e^{-x^2}$

533. $\int e^{ax}[\sin(bx)]\,dx = \dfrac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2 + b^2}$

534. $\int e^{ax}[\sin(bx)][\sin(cx)]\,dx = \dfrac{e^{ax}[(b - c)\sin(b - c)x + a\cos(b - c)x]}{2[a^2 + (b - c)^2]} - \dfrac{e^{ax}[(b + c)\sin(b + c)x + a\cos(b + c)x]}{2[a^2 + (b + c)^2]}$

535. $$\int e^{ax}[\sin(bx)][\cos(cx)]\,dx = \begin{cases} \dfrac{e^{ax}[a\sin(b - c)x - (b - c)\cos(b - c)x]}{2[a^2 + (b - c)^2]} + \dfrac{e^{ax}[a\sin(b + c)x - (b + c)\cos(b + c)x]}{2[a^2 + (b + c)^2]} \\ \text{or} \\ \dfrac{e^{ax}}{\rho}[(a\sin bx - b\cos bx)[\cos(cx - \alpha)] - c(\sin bx)\sin(cx - \alpha)] \\ \text{where} \\ \rho = \sqrt{(a^2 + b^2 - c^2)^2 + 4a^2c^2}, \\ \rho\cos\alpha = a^2 + b^2 - c^2, \qquad \rho\sin\alpha = 2ac \end{cases}$$

536. $\int e^{ax}[\sin(bx)][\sin(bx + c)]\,dx$

$$= \frac{e^{ax}\cos c}{2a} - \frac{e^{ax}[a\cos(2bx + c) + 2b\sin(2bx + c)]}{2(a^2 + 4b^2)}$$

537. $\int e^{ax}[\sin(bx)][\cos(bx + c)]\,dx$

$$= \frac{-e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx + c) - 2b\cos(2bx + c)]}{2(a^2 + 4b^2)}$$

538. $\int e^{ax}[\cos(bx)]\,dx = \dfrac{e^{ax}}{a^2 + b^2}[a\cos(bx) + b\sin(bx)]$

INTEGRALS (Continued)

539. $$\int e^{ax}[\cos(bx)][\cos(cx)]\,dx = \frac{e^{ax}[(b-c)\sin(b-c)x + a\cos(b-c)x]}{2[a^2+(b-c)^2]} + \frac{e^{ax}[(b+c)\sin(b+c)x + a\cos(b+c)x]}{2[a^2+(b+c)^2]}$$

540. $$\int e^{ax}[\cos(bx)][\cos(bx+c)]\,dx = \frac{e^{ax}\cos c}{2a} + \frac{e^{ax}[a\cos(2bx+c) + 2b\sin(2bx+c)]}{2(a^2+4b^2)}$$

541. $$\int e^{ax}[\cos(bx)][\sin(bx+c)]\,dx = \frac{e^{ax}\sin c}{2a} + \frac{e^{ax}[a\sin(2bx+c) - 2b\cos(2bx+c)]}{2(a^2+4b^2)}$$

542. $$\int e^{ax}[\sin^n bx]\,dx = \frac{1}{a^2+n^2b^2}\left[(a\sin bx - nb\cos bx)\,e^{ax}\sin^{n-1} bx + n(n-1)b^2\int e^{ax}[\sin^{n-2} bx]\,dx\right]$$

543. $$\int e^{ax}[\cos^n bx]\,dx = \frac{1}{a^2+n^2b^2}\left[(a\cos bx + nb\sin bx)\,e^{ax}\cos^{n-1} bx + n(n-1)b^2\int e^{ax}[\cos^{n-2} bx]\,dx\right]$$

544. $$\int x^m e^x \sin x\,dx = \frac{1}{2}x^m e^x(\sin x - \cos x) - \frac{m}{2}\int x^{m-1} e^x \sin x\,dx + \frac{m}{2}\int x^{m-1} e^x \cos x\,dx$$

545. $$\int x^m e^{ax}[\sin bx]\,dx = \begin{cases} x^m e^{ax}\dfrac{a\sin bx - b\cos bx}{a^2+b^2} - \dfrac{m}{a^2+b^2}\displaystyle\int x^{m-1} e^{ax}(a\sin bx - b\cos bx)\,dx \\ \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}\frac{(-1)^r m!\,x^{m-r}}{\rho^{r+1}(m-r)!}\sin[bx-(r+1)\alpha] \\ \text{where} \\ \rho = \sqrt{a^2+b^2}, \qquad \rho\cos\alpha = a, \qquad \rho\sin\alpha = b \end{cases}$$

546. $$\int x^m e^x \cos x\,dx = \frac{1}{2}x^m e^x(\sin x + \cos x) - \frac{m}{2}\int x^{m-1} e^x \sin x\,dx - \frac{m}{2}\int x^{m-1} e^x \cos x\,dx$$

INTEGRALS (Continued)

547. $$\int x^m e^{ax}\cos bx\,dx = \begin{cases} x^m e^{ax}\dfrac{a\cos bx + b\sin bx}{a^2+b^2} \\ \qquad -\dfrac{m}{a^2+b^2}\displaystyle\int x^{m-1}e^{ax}(a\cos bx + b\sin bx)\,dx \\ \qquad \text{or} \\ e^{ax}\displaystyle\sum_{r=0}^{m}\dfrac{(-1)^r m!\,x^{m-r}}{\rho^{r+1}(m-r)!}\cos[bx-(r+1)\alpha] \\ \qquad \text{where} \\ \rho=\sqrt{a^2+b^2},\qquad \rho\cos\alpha = a,\qquad \rho\sin\alpha = b \end{cases}$$

548. $$\int e^{ax}(\cos^m x)(\sin^n x)\,dx = \begin{cases} \dfrac{e^{ax}\cos^{m-1}x\sin^n x[a\cos x + (m+n)\sin x]}{(m+n)^2+a^2} \\ \qquad -\dfrac{na}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)\,dx \\ \qquad +\dfrac{(m-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^n x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}\cos^m x\sin^{n-1}x[a\sin x - (m+n)\cos x]}{(m+n)^2+a^2} \\ \qquad +\dfrac{ma}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)\,dx \\ \qquad +\dfrac{(n-1)(m+n)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)(a\sin x\cos x + m\sin^2 x - n\cos^2 x)}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^n x)\,dx \\ \qquad +\dfrac{n(n-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \\ \qquad \text{or} \\ \dfrac{e^{ax}(\cos^{m-1}x)(\sin^{n-1}x)(a\cos x\sin x + m\sin^2 x - n\cos^2 x)}{(m+n)^2+a^2} \\ \qquad +\dfrac{m(m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^{m-2}x)(\sin^{n-2}x)\,dx \\ \qquad +\dfrac{(n-m)(n+m-1)}{(m+n)^2+a^2}\displaystyle\int e^{ax}(\cos^m x)(\sin^{n-2}x)\,dx \end{cases}$$

INTEGRALS (Continued)

549. $$\int x\, e^{ax}(\sin bx)\, dx = \frac{x\, e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2)\sin bx - 2ab \cos bx]$$

550. $$\int x\, e^{ax}(\cos bx)\, dx = \frac{x\, e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) - \frac{e^{ax}}{(a^2 + b^2)^2}[(a^2 - b^2)\cos bx + 2ab \sin bx]$$

551. $$\int \frac{e^{ax}}{\sin^n x} dx = -\frac{e^{ax}[a \sin x + (n-2)\cos x]}{(n-1)(n-2)\sin^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\sin^{n-2} x} dx$$

552. $$\int \frac{e^{ax}}{\cos^n x} dx = -\frac{e^{ax}[a \cos x - (n-2)\sin x]}{(n-1)(n-2)\cos^{n-1} x} + \frac{a^2 + (n-2)^2}{(n-1)(n-2)} \int \frac{e^{ax}}{\cos^{n-2} x} dx$$

553. $$\int e^{ax} \tan^n x\, dx = e^{ax} \frac{\tan^{n-1} x}{n-1} - \frac{a}{n-1} \int e^{ax} \tan^{n-1} x\, dx - \int e^{ax} \tan^{n-2} x\, dx$$

HYPERBOLIC FORMS

554. $$\int (\sinh x)\, dx = \cosh x$$

555. $$\int (\cosh x)\, dx = \sinh x$$

556. $$\int (\tanh x)\, dx = \log \cosh x$$

557. $$\int (\coth x)\, dx = \log \sinh x$$

558. $$\int (\operatorname{sech} x)\, dx = \tan^{-1} (\sinh x)$$

559. $$\int \operatorname{csch} x\, dx = \log \tanh \left(\frac{x}{2}\right)$$

560. $$\int x(\sinh x)\, dx = x \cosh x - \sinh x$$

561. $$\int x^n(\sinh x)\, dx = x^n \cosh x - n \int x^{n-1}(\cosh x)\, dx$$

562. $$\int x(\cosh x)\, dx = x \sinh x - \cosh x$$

563. $$\int x^n(\cosh x)\, dx = x^n \sinh x - n \int x^{n-1}(\sinh x)\, dx$$

INTEGRALS (Continued)

564. $\int (\operatorname{sech} x)(\tanh x)\,dx = -\operatorname{sech} x$

565. $\int (\operatorname{csch} x)(\coth x)\,dx = -\operatorname{csch} x$

566. $\int (\sinh^2 x)\,dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

567. $$\int (\sinh^m x)(\cosh^n x)\,dx = \begin{cases} \frac{1}{m+n}(\sinh^{m+1} x)(\cosh^{n-1} x) \\ \qquad + \frac{n-1}{m+n}\int (\sinh^m x)(\cosh^{n-2} x)\,dx \\ \quad \text{or} \\ \frac{1}{m+n}\sinh^{m-1} x \cosh^{n+1} x \\ \quad - \frac{m-1}{m+n}\int (\sinh^{m-2} x)(\cosh^n x)\,dx, \qquad (m+n \neq 0) \end{cases}$$

568. $$\int \frac{dx}{(\sinh^m x)(\cosh^n x)} = \begin{cases} -\frac{1}{(m-1)(\sinh^{m-1} x)(\cosh^{n-1} x)} \\ \qquad -\frac{m+n-2}{m-1}\int \frac{dx}{(\sinh^{m-2} x)(\cosh^n x)}, \qquad (m \neq 1) \\ \text{or} \\ \frac{1}{(n-1)\sinh^{m-1} x \cosh^{n-1} x} \\ \qquad + \frac{m+n-2}{n-1}\int \frac{dx}{(\sinh^m x)(\cosh^{n-2} x)}, \qquad (n \neq 1) \end{cases}$$

569. $\int (\tanh^2 x)\,dx = x - \tanh x$

570. $\int (\tanh^n x)\,dx = -\frac{\tanh^{n-1} x}{n-1} + \int (\tanh^{n-2} x)\,dx, \qquad (n \neq 1)$

571. $\int (\operatorname{sech}^2 x)\,dx = \tanh x$

572. $\int (\cosh^2 x)\,dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

573. $\int (\coth^2 x)\,dx = x - \coth x$

574. $\int (\coth^n x)\,dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x\,dx, \qquad (n \neq 1)$

INTEGRALS (Continued)

575. $\int (\operatorname{csch}^2 x)\,dx = -\operatorname{ctnh} x$

576. $\int (\sinh mx)(\sinh nx)\,dx = \frac{\sinh (m+n)x}{2(m+n)} - \frac{\sinh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

577. $\int (\cosh mx)(\cosh nx)\,dx = \frac{\sinh (m+n)x}{2(m+n)} + \frac{\sinh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

578. $\int (\sinh mx)(\cosh nx)\,dx = \frac{\cosh (m+n)x}{2(m+n)} + \frac{\cosh (m-n)x}{2(m-n)}, \qquad (m^2 \neq n^2)$

579. $\int \left(\sinh^{-1}\frac{x}{a}\right) dx = x\sinh^{-1}\frac{x}{a} - \sqrt{x^2+a^2}, \qquad (a > 0)$

580. $\int x\left(\sinh^{-1}\frac{x}{a}\right) dx = \left(\frac{x^2}{2} + \frac{a^2}{4}\right)\sinh^{-1}\frac{x}{a} - \frac{x}{4}\sqrt{x^2+a^2}, \qquad (a > 0)$

581. $\int x^n(\sinh^{-1} x)\,dx = \frac{x^{n+1}}{n+1}\sinh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{(1+x^2)^{\frac{1}{2}}}\,dx, \qquad (n \neq -1)$

582. $\int \left(\cosh^{-1}\frac{x}{a}\right) dx = \begin{cases} x\cosh^{-1}\frac{x}{a} - \sqrt{x^2-a^2}, & \left(\cosh^{-1}\frac{x}{a} > 0\right) \\ \text{or} & \\ x\cosh^{-1}\frac{x}{a} + \sqrt{x^2-a^2}, & \left(\cosh^{-1}\frac{x}{a} < 0\right), \qquad (a > 0) \end{cases}$

583. $\int x\left(\cosh^{-1}\frac{x}{a}\right) dx = \frac{2x^2-a^2}{4}\cosh^{-1}\frac{x}{a} - \frac{x}{4}(x^2-a^2)^{\frac{1}{2}}$

584. $\int x^n(\cosh^{-1} x)\,dx = \frac{x^{n+1}}{n+1}\cosh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{(x^2-1)^{\frac{1}{2}}}\,dx, \qquad (n \neq -1)$

585. $\int \left(\tanh^{-1}\frac{x}{a}\right) dx = x\tanh^{-1}\frac{x}{a} + \frac{a}{2}\log(a^2-x^2), \qquad \left(\left|\frac{x}{a}\right| < 1\right)$

586. $\int \left(\coth^{-1}\frac{x}{a}\right) dx = x\coth^{-1}\frac{x}{a} + \frac{a}{2}\log(x^2-a^2), \qquad \left(\left|\frac{x}{a}\right| > 1\right)$

587. $\int x\left(\tanh^{-1}\frac{x}{a}\right) dx = \frac{x^2-a^2}{2}\tanh^{-1}\frac{x}{a} + \frac{ax}{2}, \qquad \left(\left|\frac{x}{a}\right| < 1\right)$

588. $\int x^n\left(\tanh^{-1} x\right) dx = \frac{x^{n+1}}{n+1}\tanh^{-1} x - \frac{1}{n+1}\int \frac{x^{n+1}}{1-x^2}\,dx, \qquad (n \neq -1)$

589. $\int x\left(\coth^{-1}\frac{x}{a}\right) dx = \frac{x^2-a^2}{2}\coth^{-1}\frac{x}{a} + \frac{ax}{2}, \qquad \left(\left|\frac{x}{a}\right| > 1\right)$

590. $\int x^n(\coth^{-1} x)\,dx = \frac{x^{n+1}}{n+1}\coth^{-1} x + \frac{1}{n+1}\int \frac{x^{n+1}}{x^2-1}\,dx, \qquad (n \neq -1)$

DEFINITE INTEGRALS

591. $\int (\operatorname{sech}^{-1} x)\, dx = x \operatorname{sech}^{-1} x + \sin^{-1} x$

592. $\int x \operatorname{sech}^{-1} x\, dx = \frac{x^2}{2} \operatorname{sech}^{-1} x - \frac{1}{2}(1 - x^2)$

593. $\int x^n \operatorname{sech}^{-1} x\, dx = \frac{x^{n+1}}{n+1} \operatorname{sech}^{-1} x + \frac{1}{n+1} \int \frac{x^n}{(1 - x^2)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

594. $\int \operatorname{csch}^{-1} x\, dx = x \operatorname{csch}^{-1} x + \frac{x}{|x|} \sinh^{-1} x$

595. $\int x \operatorname{csch}^{-1} x\, dx = \frac{x^2}{2} \operatorname{csch}^{-1} x + \frac{1}{2} \frac{x}{|x|} \sqrt{1 + x^2}$

596. $\int x^n \operatorname{csch}^{-1} x\, dx = \frac{x^{n+1}}{n+1} \operatorname{csch}^{-1} x + \frac{1}{n+1} \frac{x}{|x|} \int \frac{x^n}{(x^2 + 1)^{\frac{1}{2}}}\, dx, \qquad (n \neq -1)$

DEFINITE INTEGRALS

597. $$\int_0^\infty x^{n-1} e^{-x}\, dx = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \frac{1}{n} \prod_{m=1}^{\infty} \frac{\left(1 + \frac{1}{m}\right)^n}{1 + \frac{n}{m}}$$

$$= \Gamma(n), n \neq 0, -1, -2, -3, \ldots \qquad \text{(Gamma Function)}$$

598. $\int_0^\infty t^n p^{-t}\, dt = \frac{n!}{(\log p)^{n+1}}, \qquad (n = 0, 1, 2, 3, \ldots \text{ and } p > 0)$

599. $\int_0^\infty t^{n-1} e^{-(a+1)t}\, dt = \frac{\Gamma(n)}{(a+1)^n}, \qquad (n > 0, a > -1)$

600. $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \qquad (m > -1, n > -1)$

601. $\Gamma(n)$ is finite if $n > 0$, $\Gamma(n+1) = n\Gamma(n)$

602. $\Gamma(n) \cdot \Gamma(1 - n) = \frac{\pi}{\sin n\pi}$

603. $\Gamma(n) = (n - 1)!$ if $n =$ integer > 0

604. $\Gamma(\frac{1}{2}) = 2 \int_0^\infty e^{-t^2}\, dt = \sqrt{\pi} = 1.7724538509 \cdots = (-\frac{1}{2})!$

605. $\Gamma(n + \frac{1}{2}) = \frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{2^n} \sqrt{\pi} \qquad n = 1, 2, 3, \ldots$

606. $\Gamma(-n + \frac{1}{2}) = \frac{(-1)^n 2^n \sqrt{\pi}}{1 \cdot 3 \cdot 5 \ldots (2n - 1)} \qquad n = 1, 2, 3, \ldots$

DEFINITE INTEGRALS (Continued)

607. $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = B(m,n)$

(Beta function)

608. $B(m,n) = B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, where m and n are any positive real numbers.

609. $\int_a^b (x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}\frac{\Gamma(m+1)\cdot\Gamma(n+1)}{\Gamma(m+n+2)}$,

$(m > -1, n > -1, b > a)$

610. $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \qquad [m > 1]$

611. $\int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \qquad [p < 1]$

612. $\int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \qquad [p < 1]$

613. $\int_0^\infty \frac{x^{p-1}\,dx}{1+x} = \frac{\pi}{\sin p\pi}$

$= B(p, 1-p) = \Gamma(p)\Gamma(1-p), \qquad [0 < p < 1]$

614. $\int_0^\infty \frac{x^{m-1}\,dx}{1+x^n} = \frac{\pi}{n \sin \frac{m\pi}{n}}, \qquad [0 < m < n]$

615. $\int_0^\infty \frac{x^a\,dx}{(m+x^b)^c} = \frac{m}{b}^{\left[\frac{a+1-bc}{b}\right]} \cdot \frac{\Gamma\left(\frac{a+1}{b}\right)\Gamma\left(c - \frac{a+1}{b}\right)}{\Gamma(c)}$,

$\left(a > -1, b > 0, m > 0, c > \frac{a+1}{b}\right)$

616. $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$

617. $\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}$, if $a > 0$; 0, if $a = 0$; $-\frac{\pi}{2}$, if $a < 0$

618. $\int_0^a (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{1}{2}\int_{-a}^a (a^2-x^2)^{\frac{n}{2}}\,dx = \frac{1\cdot 3\cdot 5\ldots n}{2\cdot 4\cdot 6\ldots(n+1)}\cdot\frac{\pi}{2}\cdot a^{n+1}$ (n odd)

619. $\int_0^a x^m(a^2-x^2)^{\frac{n}{2}}\,dx = \begin{cases} \frac{1}{2}a^{m+n+1}B\left(\frac{m+1}{2}, \frac{n+2}{2}\right) \\ \text{or} \\ \frac{1}{2}a^{m+n+1}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{m+n+3}{2}\right)} \end{cases}$

DEFINITE INTEGRALS (Continued)

620. $$\int_0^{\pi/2} (\sin^n x)\,dx = \begin{cases} \displaystyle\int_0^{\pi/2} (\cos^n x)\,dx \\ \text{or} \\ \dfrac{1\cdot 3\cdot 5\cdot 7\ldots(n-1)}{2\cdot 4\cdot 6\cdot 8\ldots(n)}\,\dfrac{\pi}{2}, & (n \text{ an even integer}, n \neq 0) \\ \text{or} \\ \dfrac{2\cdot 4\cdot 6\cdot 8\ldots(n-1)}{1\cdot 3\cdot 5\cdot 7\ldots(n)}, & (n \text{ an odd integer}, n \neq 1) \\ \text{or} \\ \dfrac{\sqrt{\pi}}{2}\,\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}, & (n > -1) \end{cases}$$

621. $$\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}, \text{ if } m > 0;\ 0, \text{ if } m = 0;\ -\frac{\pi}{2}, \text{ if } m < 0$$

622. $$\int_0^\infty \frac{\cos x\,dx}{x} = \infty$$

623. $$\int_0^\infty \frac{\tan x\,dx}{x} = \frac{\pi}{2}$$

624. $$\int_0^\pi \sin ax \cdot \sin bx\,dx = \int_0^\pi \cos ax \cdot \cos bx\,dx = 0, \qquad (a \neq b;\ a, b \text{ integers})$$

625. $$\int_0^{\pi/a} [\sin(ax)][\cos(ax)]\,dx = \int_0^\pi [\sin(ax)][\cos(ax)]\,dx = 0$$

626. $$\int_0^\pi [\sin(ax)][\cos(bx)]\,dx = \frac{2a}{a^2 - b^2}, \text{ if } a - b \text{ is odd, or 0 if } a - b \text{ is even}$$

627. $$\int_0^\infty \frac{\sin x \cos mx\,dx}{x}$$

$$= 0, \text{ if } m < -1 \text{ or } m > 1;\ \frac{\pi}{4}, \text{ if } m = \pm 1;\ \frac{\pi}{2}, \text{ if } m^2 < 1$$

628. $$\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx = \frac{\pi a}{2}, \qquad (a \leq b)$$

629. $$\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}$$

630. $$\int_0^\infty \frac{\sin^2(px)}{x^2}\,dx = \frac{\pi p}{2}$$

DEFINITE INTEGRALS (Continued)

631. $\int_0^\infty \frac{\sin x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \qquad 0 < p < 1$

632. $\int_0^\infty \frac{\cos x}{x^p}\,dx = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)}, \qquad 0 < p < 1$

633. $\int_0^\infty \frac{1-\cos px}{x^2}\,dx = \frac{\pi p}{2}$

634. $\int_0^\infty \frac{\sin px \cos qx}{x}\,dx = \left\{0, \quad q > p > 0; \quad \frac{\pi}{2}, \quad p > q > 0; \quad \frac{\pi}{4}, \quad p = q > 0\right\}$

635. $\int_0^\infty \frac{\cos(mx)}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-|m|a}, \qquad (a > 0)$

636. $\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

637. $\int_0^\infty \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \qquad n > 1$

638. $\int_0^\infty \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \qquad n > 1$

639. $\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$

640. $\int_0^\infty \frac{\sin^3 x}{x^2}\,dx = \frac{3}{4}\log 3$

641. $\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$

642. $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$

643. $\int_0^{\pi/2} \frac{dx}{1+a\cos x} = \frac{\cos^{-1} a}{\sqrt{1-a^2}}, \qquad (a < 1)$

644. $\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}, \qquad (a > b \geq 0)$

645. $\int_0^{2\pi} \frac{dx}{1+a\cos x} = \frac{2\pi}{\sqrt{1-a^2}}, \qquad (a^2 < 1)$

646. $\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}$

647. $\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}$

DEFINITE INTEGRALS (Continued)

648. $\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi(a^2 + b^2)}{4a^3b^3}, \qquad (a, b > 0)$

649. $\int_0^{\pi/2} \sin^{n-1} x \cos^{m-1} x \, dx = \frac{1}{2} \mathrm{B}\left(\frac{n}{2}, \frac{m}{2}\right),$ m and n positive integers

650. $\int_0^{\pi/2} (\sin^{2n+1} \theta) \, d\theta = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{1 \cdot 3 \cdot 5 \ldots (2n+1)}, \qquad (n = 1, 2, 3 \ldots)$

651. $\int_0^{\pi/2} (\sin^{2n} \theta) \, d\theta = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \ldots (2n)} \left(\frac{\pi}{2}\right), \qquad (n = 1, 2, 3 \ldots)$

652. $\int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2\left\{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots\right\}$

653. $\int_0^{\pi/2} \frac{dx}{1 + \tan^m x} = \frac{\pi}{4}$

654. $\int_0^{\pi/2} \sqrt{\cos \theta} \, d\theta = \frac{(2\pi)^{\frac{3}{2}}}{[\Gamma(\frac{1}{4})]^2}$

655. $\int_0^{\pi/2} (\tan^h \theta) \, d\theta = \frac{\pi}{2 \cos\left(\frac{h\pi}{2}\right)}, \qquad (0 < h < 1)$

656. $\int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} \, dx = \frac{\pi}{2} \log \frac{a}{b}, \qquad (a, b > 0)$

657. The area enclosed by a curve defined through the equation $x^{\frac{b}{c}} + y^{\frac{b}{c}} = a^{\frac{b}{c}}$ where $a > 0$, c a positive odd integer and b a positive even integer is given by

$$\frac{\left[\Gamma\left(\frac{c}{b}\right)\right]^2}{\Gamma\left(\frac{2c}{b}\right)} \left(\frac{2ca^2}{b}\right)$$

658. $I = \iiint_R x^{h-1} y^{m-1} z^{n-1} \, dv$, where R denotes the region of space bounded by the co-ordinate planes and that portion of the surface $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^k = 1$, which lies in the first octant and where h, m, n, p, q, k, a, b, c, denote positive real numbers is given by

$$\int_0^a x^{h-1} \, dx \int_0^{b\left[1-\left(\frac{x}{a}\right)^p\right]^{\frac{1}{q}}} y^m \, dy \int_0^{c\left[1-\left(\frac{x}{a}\right)^p-\left(\frac{y}{b}\right)^q\right]^{\frac{1}{k}}} z^{n-1} \, dz$$

$$= \frac{a^h b^m c^n}{pqk} \frac{\Gamma\left(\frac{h}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{k}\right)}{\Gamma\left(\frac{h}{p} + \frac{m}{q} + \frac{n}{k} + 1\right)}$$

DEFINITE INTEGRALS (Continued)

659. $\int_0^\infty e^{-ax}\,dx = \frac{1}{a}, \qquad (a > 0)$

660. $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\frac{b}{a}, \qquad (a, b > 0)$

661. $\int_0^\infty x^n e^{-ax}\,dx = \begin{cases} \frac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ \text{or} & \\ \frac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$

662. $\int_0^\infty x^n \exp(-ax^p)\,dx = \frac{\Gamma(k)}{pa^k}, \qquad \left(n > -1, p > 0, a > 0, k = \frac{n+1}{p}\right)$

663. $\int_0^\infty e^{-a^2x^2}\,dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right), \qquad (a > 0)$

664. $\int_0^\infty x\,e^{-x^2}\,dx = \frac{1}{2}$

665. $\int_0^\infty x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$

666. $\int_0^\infty x^{2n} e^{-ax^2}\,dx = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$

667. $\int_0^\infty x^{2n+1} e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}, \qquad (a > 0)$

668. $\int_0^1 x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}\left[1 - e^{-a}\sum_{r=0}^{m}\frac{a^r}{r!}\right]$

669. $\int_0^\infty e^{\left(-x^2 - \frac{a^2}{x^2}\right)}\,dx = \frac{e^{-2a}\sqrt{\pi}}{2}, \qquad (a \geq 0)$

670. $\int_0^\infty e^{-nx}\sqrt{x}\,dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$

671. $\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}$

672. $\int_0^\infty e^{-ax}(\cos mx)\,dx = \frac{a}{a^2 + m^2}, \qquad (a > 0)$

673. $\int_0^\infty e^{-ax}(\sin mx)\,dx = \frac{m}{a^2 + m^2}, \qquad (a > 0)$

DEFINITE INTEGRALS (Continued)

674. $\int_0^\infty x\,e^{-ax}[\sin(bx)]\,dx = \frac{2ab}{(a^2+b^2)^2}, \qquad (a>0)$

675. $\int_0^\infty x\,e^{-ax}[\cos(bx)]\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}, \qquad (a>0)$

676. $\int_0^\infty x^n\,e^{-ax}[\sin(bx)]\,dx = \frac{n![(a-ib)^{n+1}-(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}, \qquad (i^2=-1, a>0)$

677. $\int_0^\infty x^n\,e^{-ax}[\cos(bx)]\,dx = \frac{n![(a-ib)^{n+1}+(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}, \qquad (i^2=-1, a>0)$

678. $\int_0^\infty \frac{e^{-ax}\sin x}{x}\,dx = \cot^{-1} a, \qquad (a>0)$

679. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right), \qquad (ab \neq 0)$

680. $\int_0^\infty e^{-t\cos\phi}\,t^{b-1}\sin(t\sin\phi)\,dt = [\Gamma(b)]\sin(b\phi), \qquad \left(b>0, -\frac{\pi}{2}<\phi<\frac{\pi}{2}\right)$

681. $\int_0^\infty e^{-t\cos\phi}\,t^{b-1}[\cos(t\sin\phi)]\,dt = [\Gamma(b)]\cos(b\phi), \qquad \left(b>0, -\frac{\pi}{2}<\phi<\frac{\pi}{2}\right)$

682. $\int_0^\infty t^{b-1}\cos t\,dt = [\Gamma(b)]\cos\left(\frac{b\pi}{2}\right), \qquad (0<b<1)$

683. $\int_0^\infty t^{b-1}(\sin t)\,dt = [\Gamma(b)]\sin\left(\frac{b\pi}{2}\right), \qquad (0<b<1)$

684. $\int_0^1 (\log x)^n\,dx = (-1)^n \cdot n!$

685. $\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$

686. $\int_0^1 \left(\log\frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$

687. $\int_0^1 \left(\log\frac{1}{x}\right)^{n} dx = n!$

688. $\int_0^1 x\log(1-x)\,dx = -\frac{3}{4}$

689. $\int_0^1 x\log(1+x)\,dx = \frac{1}{4}$

690. $\int_0^1 x^m(\log x)^n\,dx = \frac{(-1)^n n!}{(m+1)^{n+1}}, \qquad m>-1, n=0,1,2,\ldots$

If $n \neq 0, 1, 2, \ldots$ replace $n!$ by $\Gamma(n+1)$.

DEFINITE INTEGRALS (Continued)

691. $\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$

692. $\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$

693. $\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{12}$

694. $\int_0^1 \frac{\log(1-x)}{x}\,dx = -\frac{\pi^2}{6}$

695. $\int_0^1 (\log x)[\log(1+x)]\,dx = 2 - 2\log 2 - \frac{\pi^2}{12}$

696. $\int_0^1 (\log x)[\log(1-x)]\,dx = 2 - \frac{\pi^2}{6}$

697. $\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}$

698. $\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}$

699. $\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2$

700. $\int_0^1 x^m\left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \qquad \text{if } m+1>0, n+1>0$

701. $\int_0^1 \frac{(x^p - x^q)\,dx}{\log x} = \log\left(\frac{p+1}{q+1}\right), \qquad (p+1>0, q+1>0)$

702. $\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}$

703. $\int_0^\infty \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$

704. $\int_0^{\pi/2} (\log\sin x)\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2$

705. $\int_0^{\pi/2} (\log\sec x)\,dx = \int_0^{\pi/2} \log\csc x\,dx = \frac{\pi}{2}\log 2$

706. $\int_0^\pi x(\log\sin x)\,dx = -\frac{\pi^2}{2}\log 2$

707. $\int_0^{\pi/2} (\sin x)(\log\sin x)\,dx = \log 2 - 1$

DEFINITE INTEGRALS (Continued)

708. $\int_0^{\pi/2} (\log \tan x)\, dx = 0$

709. $\int_0^{\pi} \log (a \pm b \cos x)\, dx = \pi \log \left(\frac{a + \sqrt{a^2 - b^2}}{2} \right), \qquad (a \geq b)$

710. $\int_0^{\pi} \log (a^2 - 2ab \cos x + b^2)\, dx = \begin{cases} 2\pi \log a, & a \geq b > 0 \\ 2\pi \log b, & b \geq a > 0 \end{cases}$

711. $\int_0^{\infty} \frac{\sin ax}{\sinh bx} dx = \frac{\pi}{2b} \tanh \frac{a\pi}{2b}$

712. $\int_0^{\infty} \frac{\cos ax}{\cosh bx} dx = \frac{\pi}{2b} \operatorname{sech} \frac{a\pi}{2b}$

713. $\int_0^{\infty} \frac{dx}{\cosh ax} = \frac{\pi}{2a}$

714. $\int_0^{\infty} \frac{x\, dx}{\sinh ax} = \frac{\pi^2}{4a^2}$

715. $\int_0^{\infty} e^{-ax} (\cosh bx)\, dx = \frac{a}{a^2 - b^2}, \qquad (0 \leq |b| < a)$

716. $\int_0^{\infty} e^{-ax} (\sinh bx)\, dx = \frac{b}{a^2 - b^2}, \qquad (0 \leq |b| < a)$

717. $\int_0^{\infty} \frac{\sinh ax}{e^{bx} + 1} dx = \frac{\pi}{2b} \csc \frac{a\pi}{b} - \frac{1}{2a}$

718. $\int_0^{\infty} \frac{\sinh ax}{e^{bx} - 1} dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$

719. $\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} = \frac{\pi}{2} \left[1 + \left(\frac{1}{2} \right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \cdots \right], \text{ if } k^2 < 1$

720. $\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\, dx = \frac{\pi}{2} \left[1 - \left(\frac{1}{2} \right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4} \right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 \frac{k^6}{5} - \cdots \right], \text{ if } k^2 < 1$

721. $\int_0^{\infty} e^{-x} \log x\, dx = -\gamma = -0.5772157\ldots$

722. $\int_0^{\infty} e^{-x^2} \log x\, dx = -\frac{\sqrt{\pi}}{4} (\gamma + 2 \log 2)$

DEFINITE INTEGRALS (Continued)

723. $$\int_0^{\infty} \log\left(\frac{e^x+1}{e^x-1}\right) dx = \frac{\pi^2}{4}$$

724. $$\int_0^{\infty} \left(\frac{1}{1-e^{-x}} - \frac{1}{x}\right) e^{-x}\, dx = \gamma = 0.5772157\ldots$$ [Euler's Constant]

725. $$\int_0^{\infty} \frac{1}{x}\left(\frac{1}{1+x} - e^{-x}\right) dx = \gamma = 0.5772157\ldots$$