There are three methods of calculating magnetic fields

① Amperes Law $\oint_C \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{ds}$

② Biot-Savart Law

$$\underline{B} = \frac{\mu_0}{4\pi} \int \frac{\underline{I}\, dl_1 \times \hat{r}}{r^2}$$

where the B field at a point P is calculated by summing (vectorially) the differential current vectors crossed with the vector pointing from the current element to P

③ Vector potential

$\underline{\nabla} \times \underline{B} \neq 0$ for a magnetic field $\left(= \underline{J} + \frac{\partial \underline{D}}{\partial t} \right)$

We cannot use a scalar potential like $\underline{B} = -\nabla \Phi$

since $\nabla \times (-\nabla \Phi) \equiv 0$

Use a potential function $\underline{B} = \underline{\nabla} \times \underline{A}$

Let's see if this gives a $\underline{B}$ field

Look at divergence $\nabla \cdot B = \nabla \cdot \nabla \times A \equiv 0$ which is good since $\nabla \cdot B = 0$

Now do curl $\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A})$

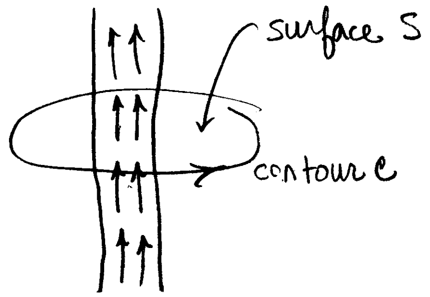
but $\underline{\nabla} \times \underline{B} = \underline{J}$ (in static problems)

$$\underline{\nabla} \times \underline{\nabla} \times \underline{A} = \nabla \nabla \cdot \underline{A} - \nabla^2 \underline{A}$$

Magnetostatics

Ampere's Law $\quad \oint_C \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{ds} \quad$ (static case)

Magnetic Field from infinite uniform current density in wire of radius r_0



total current I

cylindrical coordinates & symmetry
so use $\underline{H} = H_\theta \hat{\theta}$

$$\underline{J} = \begin{cases} \frac{I}{\pi r^2} \hat{z} & r < r_0 \\ 0 & r > r_0 \end{cases}$$

For $r > r_0$

$$\oint_C \underline{H} \cdot \underline{dl} = H_\theta \cdot 2\pi r$$

$$\int_S \underline{J} \cdot \hat{n}\, da = \underbrace{\int_0^{r_0}\int_0^{2\pi} \frac{I}{\pi r_0^2} \hat{z} \cdot \hat{z}\, r dr d\theta}_{\text{This is just the current } I} + \int_{r_0}^{r}\int_0^{2\pi} 0 \cdot \hat{z}\, r dr d\theta$$

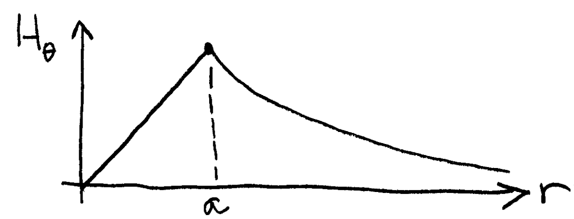
$\therefore$ for $r > r_0 \quad H_\theta \cdot 2\pi r = I \quad$ or $\quad \underline{H} = \frac{I}{2\pi r} \hat{\theta}$

For $r < r_0$ the first integral must be evaluated

$$\int_0^{r_0}\int_0^{2\pi} \frac{I}{\pi r_0^2} r dr d\theta = \int_0^{r} \frac{I}{\pi r_0^2} r dr\, 2\pi = \frac{I}{r_0^2} r^2 \Big|_0^r = I\frac{r^2}{r_0^2}$$

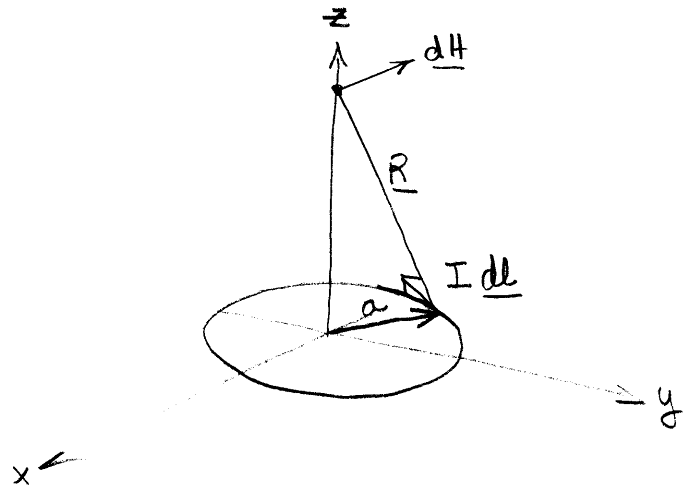
$\therefore$ for $r < r_0 \quad H_\theta \cdot 2\pi r = I \frac{r^2}{r_0^2} \quad$ or $\quad \underline{H} = \frac{I}{2\pi r_0^2} r \hat{\theta}$

Plotting these results



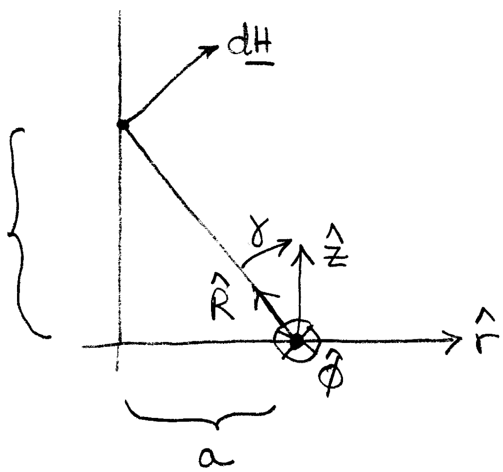
Biot Savart Law (and vector potential) usually involve a lot of geometry

Example: field from a current loop of radius a along z-axis



$$\underline{dH} = \frac{I\underline{dl} \times \hat{R}}{4\pi R^2}$$

Let's do a better drawing from side.



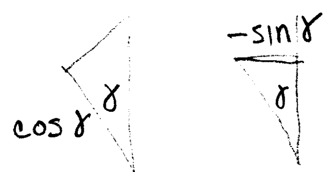
$\underline{dH}$ is in the direction shown if $\underline{dl}$ is in the $+\hat{\phi}$ direction

Need unit vectors for $\underline{dl}$ & $\underline{R}$

By inspection

$$\underline{dl} = a\,d\phi\,\hat{\phi}$$

However $\hat{R} = c_1\hat{r} + c_2\hat{z}$



to find these components formally note that

$$\hat{R}\cdot\hat{z} = c_1 \cancel{\hat{r}\cdot\hat{z}}^{\,0} + c_2\hat{z}\cdot\hat{z} \quad \text{or} \quad c_2 = \hat{R}\cdot\hat{z} = \cos\gamma$$

$$\hat{R}\cdot\hat{r} = c_1\hat{R}\cdot\hat{r} + c_2\cancel{\hat{z}\cdot\hat{r}}^{\,0} \qquad c_1 = \hat{R}\cdot\hat{r} = -\sin\gamma$$

$$\therefore\ \hat{R} = \cos\gamma\,\hat{z} - \sin\gamma\,\hat{r}$$

Then
$$d\underline{H} = \frac{I \; a\, d\phi\; \hat{\phi} \times (\cos\gamma\, \hat{z} - \sin\gamma\, \hat{r})}{4\pi(z^2+a^2)}$$

$$= \frac{I a\, d\phi}{4\pi(z^2+a^2)} \left[\cos\gamma\, \hat{r} + \sin\gamma\, \hat{z} \right]$$

$$d\underline{H} = \frac{I a\, d\phi}{4\pi} \left[\frac{\cos\gamma}{a^2+z^2}\, \hat{r} + \frac{\sin\gamma}{a^2+z^2}\, \hat{z} \right]$$

but note that $\cos\gamma = \dfrac{z}{R} = \dfrac{z}{(a^2+z^2)^{1/2}}$

$$\sin\gamma = \frac{a}{R} = \frac{a}{(a^2+z^2)^{1/2}}$$

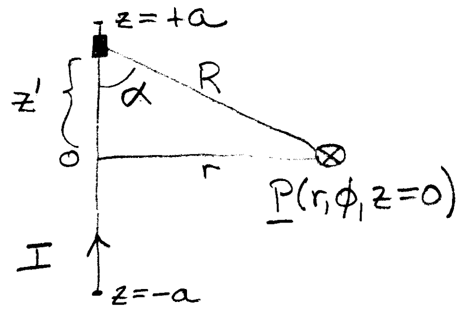
Integrating

$$H_r(\text{total}) = \frac{Ia}{4\pi} \int_0^{2\pi} \frac{z}{(a^2+z^2)^{3/2}}\, d\phi = 0$$

$$H_z(\text{total}) = \frac{Ia}{4\pi} \int_0^{2\pi} \frac{a}{(a^2+z^2)^{3/2}}\, d\phi = \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

Biot - Savart Law (cont.)

Finite length straight wire



From symmetry we expect no dependence on ϕ.

Compute $\underline{H}$ at point $\underline{P}$

$$d\underline{H}(P) = \frac{1}{4\pi} \frac{I\,\hat{z}\,dz' \times \hat{R}}{R^2}$$

where $R^2 = r^2 + (z')^2$, $\hat{z} \times \hat{R} = \hat{\phi}\ |\hat{z}||\hat{R}| \sin\alpha = \hat{\phi} \sin\alpha$

$$d\underline{H}(P) = \frac{1}{4\pi} I \frac{dz'\ \hat{\phi} \sin\alpha}{r^2 + (z')^2}$$

and from the geometry $\sin\alpha = \dfrac{r}{(r^2 + (z')^2)^{1/2}}$

$$d\underline{H}(\underline{P}) = \frac{I}{4\pi} \frac{r\,dz'}{(r^2 + (z')^2)^{3/2}}\ \hat{\phi}$$

The total field at P due to the wire from $-a$ to $+a$ can be gotten by integrating

$$\underline{H}_P = \hat{\phi}\ \frac{Ir}{4\pi} \int_{z'=-a}^{z'=+a} \frac{dz'}{(r^2 + (z')^2)^{3/2}}$$

$$= \hat{\phi}\ \frac{Ir}{4\pi} \left[\frac{z'}{r^2 \sqrt{r^2 + (z')^2}} \right]_{z'=-a}^{z'=+a} = \hat{\phi}\ \frac{I}{4\pi r} \frac{2a}{\sqrt{r^2 + a^2}}$$

$$\underline{H}_P = \hat{\phi}\ \frac{Ia}{2\pi r \sqrt{r^2 + a^2}}$$

If the wire is infinitely long, or for very close to the wire, i.e. $r \ll a$ we get

$$\underline{H}_P \cong \hat{\phi}\ \frac{I}{2\pi r}$$

Vector potential

Others have proven that Maxwell's Equations are satisified if

$$\underline{\nabla} \cdot \underline{A} = -\mu\epsilon \frac{\partial \Phi}{\partial t}$$

For static (time-independent) fields $\frac{\partial}{\partial t} \to 0$ and $\underline{\nabla} \cdot \underline{A} = 0$

Then, the vector potential is defined by

$$\underline{\nabla} \times \underline{B} = -\nabla^2 \underline{A} = \mu \underline{J}$$

We get three component equations

$$\nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

which is similar to Poisson's Equation.

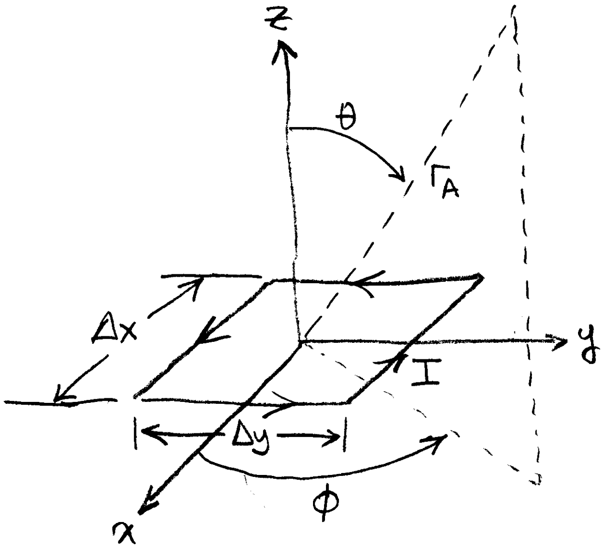
The general solution is $A_x = \int \frac{\mu J_x \, dv}{4\pi r}$

etc.

In general $\underline{A} = \int \frac{\mu \underline{J} \, dv}{4\pi r}$

This is known as the Green's function solution for $\underline{A}$

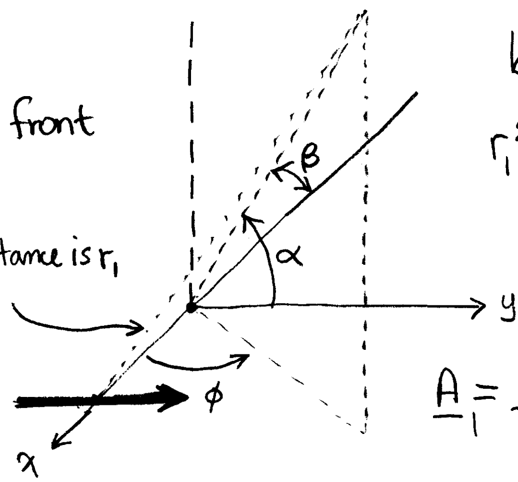
Example: square magnetic dipole



Use vector potential to sum up each current element separately.

(a) current element in front

this distance is r_1



by law of cosines

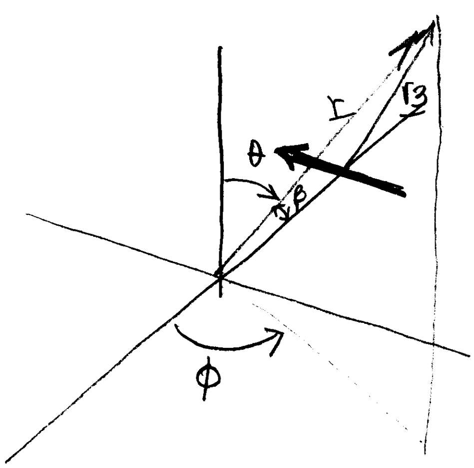
$$r_1^2 = r^2 + \left(\frac{\Delta x}{2}\right)^2 - 2r\left(\frac{\Delta x}{2}\right)\cos(\pi - \beta)$$

$$= r^2 + \frac{\Delta x^2}{4} + r\Delta x \cos\beta$$

$$\underline{A}_1 = \frac{\mu_0 (I\Delta y)}{4\pi r_1}\hat{y}$$

This assumes that the current element is concentrated at a point, i.e. $\Delta y \to 0$

(b) Current element in back.



Again using law of cosines

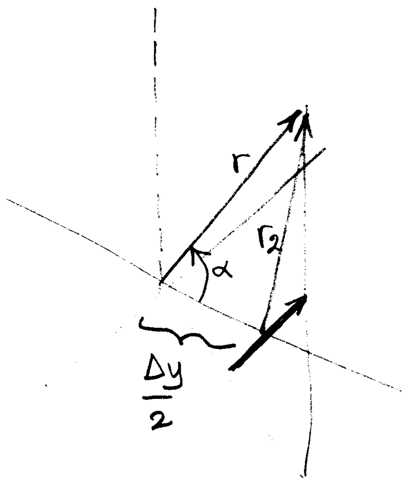
$$r_3^2 = r^2 + \left(\frac{\Delta x}{2}\right)^2 - 2r\left(\frac{\Delta x}{2}\right)\cos\beta$$

$$r_3^2 = r^2 + \left(\frac{\Delta x}{2}\right)^2 - r\Delta x \cos\beta$$

$$\underline{A}_3 = -\frac{\mu_0 I \Delta y}{4\pi r_3}\hat{y}$$

Note − sign since I is in $-\hat{y}$ direction

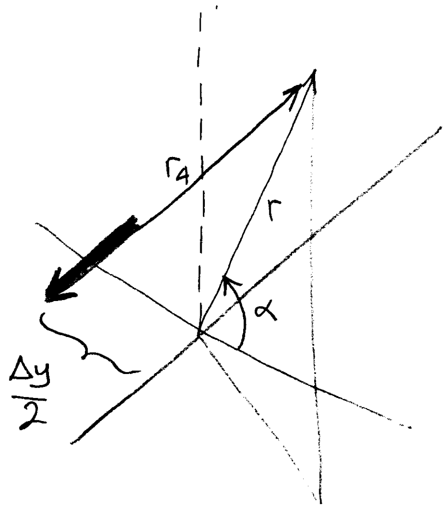
Now we do the other two sides



$$r_2^{\,2} = r^2 + \left(\frac{\Delta y}{2}\right)^2 - 2r\left(\frac{\Delta y}{2}\right)\cos\alpha$$

$$r_2^{\,2} = r^2 + \frac{(\Delta y)^2}{4} - r\Delta y\cos\alpha$$

$$\underline{A}_2 = -\frac{\mu_0 I\Delta x}{4\pi r_2}\hat{x}$$



$$r_4^{\,2} = r^2 + \left(\frac{\Delta y}{2}\right)^2 - 2r\left(\frac{\Delta y}{2}\right)\cos(\pi-\alpha)$$

$$r_4^{\,2} = r^2 + \left(\frac{\Delta y}{2}\right)^2 + r\Delta y\cos\alpha$$

$$\underline{A}_4 = \frac{\mu_0 I\Delta x}{4\pi r_4}\hat{x}$$

Just like the electric dipole we want to add the potentials in the limit as $\frac{\Delta x}{r}, \frac{\Delta y}{r} \to 0$

Let's just examine $\underline{A}_1$

$$\underline{A}_1 = \frac{\mu_0 (I\Delta y)\hat{y}}{4\pi\sqrt{r^2 + \frac{\Delta x^2}{4} + r\Delta x\cos\beta}} = \frac{\mu_0 (I\Delta y)\hat{y}}{4\pi r\sqrt{1 + \frac{1}{4}\left(\frac{\Delta x}{r}\right)^2 + \left(\frac{\Delta x}{r}\right)\cos\beta}}$$

$$\cong \frac{\mu_0 I\Delta y\,\hat{y}}{4\pi r\sqrt{1+\left(\frac{\Delta x}{r}\right)\cos\beta}} = \frac{\mu_0 I\Delta y\,\hat{y}}{4\pi r}\left(1+\left(\frac{\Delta x}{r}\right)\cos\beta\right)^{-\frac{1}{2}}$$

$$\approx \frac{\mu_0}{4\pi} I\left(\frac{\Delta y}{r}\right)\hat{y}\left[1 - \frac{1}{2}\frac{\Delta x}{r}\cos\beta\right].$$

We can re-arrange this to make it more convenient

$$\underline{A}_1 \cong \frac{\mu_0 I}{4\pi}\,\hat{y}\left[\left(\frac{\Delta y}{r}\right) - \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right]$$

Similarly

$$\underline{A}_2 \cong -\frac{\mu_0 I}{4\pi}\,\hat{x}\left[\left(\frac{\Delta x}{r}\right) + \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right].$$

$$\underline{A}_3 \cong -\frac{\mu_0 I}{4\pi}\,\hat{y}\left[\left(\frac{\Delta y}{r}\right) + \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right]$$

$$\underline{A}_4 \cong \frac{\mu_0 I}{4\pi}\,\hat{x}\left[\left(\frac{\Delta x}{r}\right) - \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right]$$

Which can be summed up

$$\underline{A} = \underline{A}_1 + \underline{A}_2 + \underline{A}_3 + \underline{A}_4$$

$$= \frac{\mu_0 I}{4\pi}\left[\hat{y}\left\{\left(\frac{\Delta y}{r}\right) - \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right\} - \hat{x}\left\{\left(\frac{\Delta x}{r}\right) + \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right\}\right.$$
$$\left. - \hat{y}\left\{\frac{\Delta y}{r} + \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right\} + \hat{x}\left\{\left(\frac{\Delta x}{r}\right) - \frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right\}\right]$$

$$= \frac{\mu_0 I}{4\pi}\left[-\hat{x}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha - \hat{y}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right]$$

$$\underline{A} = -\frac{\mu_0 I}{4\pi}\,\Delta x\Delta y\left[\hat{x}\cos\alpha + \hat{y}\cos\beta\right]$$

This can be converted to spherical coordinates (not easy)
See EEAP 210, Spring '84 notes, p. 137–139

$$\underline{A} = \frac{\mu_0 I}{4\pi r^2}\,\Delta S\,\sin\theta\,\hat{\phi}$$ where $\Delta S = \Delta x \Delta y$ the area of the loop.

Can now compute $\underline{B}$ in spherical coordinates

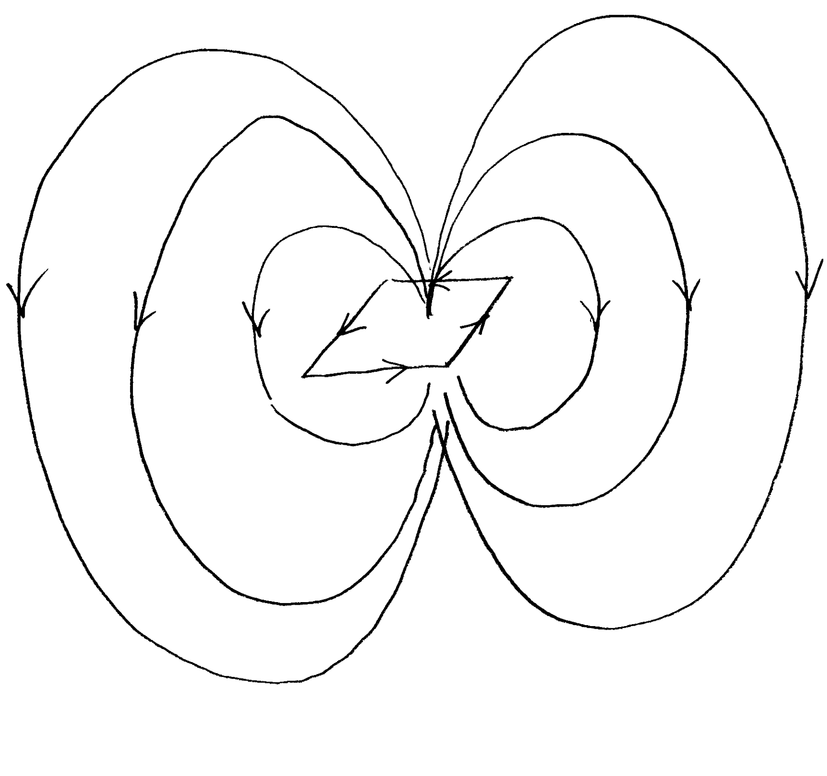
$$\underline{B} = \underline{\nabla} \times \underline{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0 I}{4\pi r^2}\Delta S \sin^2\theta\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I}{4\pi}\Delta S \sin\theta \frac{1}{r}\right)\hat{\theta}$$

$$= \frac{1}{r\sin\theta}\frac{\mu_0 I}{4\pi r^2}\Delta S\, 2\sin\theta\cos\theta\,\hat{r} - \frac{1}{r}\frac{\mu_0 I}{4\pi}\Delta S \sin\theta\left(-\frac{1}{r^2}\right)\hat{\theta}$$

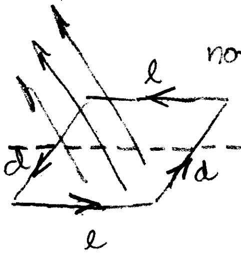
$$= \frac{\mu_0 I}{4\pi r^3}\Delta S\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

This is exactly the field we got for the electric dipole moment.

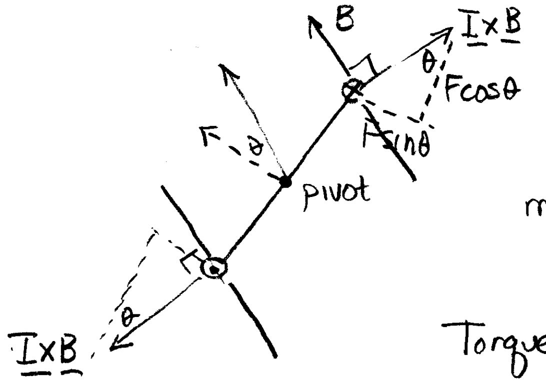


Torque on a magnetic dipole

$\underline{F} = q\underline{v} \times \underline{B} = I\underline{l} \times \underline{B}$



consider a simple loop.
from the side



magnetic dipole $\underline{m} = I\underline{A}$

Torque on magnetic dipole T

$T = 2 \times F \times$ moment arm

2: two sides

F: force ⊥ to loop

moment arm: $d/2$ about pivot

$= 2 \quad F\sin\theta \quad \frac{d}{2}$

$= 2 \, (IlB) \sin\theta \, \frac{d}{2}$

$T = IA \, B\sin\theta$

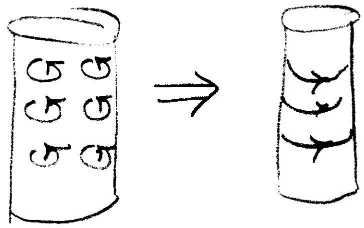
or in vector form $T = I\,\underline{A} \times \underline{B} = \underline{m} \times \underline{B}$

define the macroscopic polarization

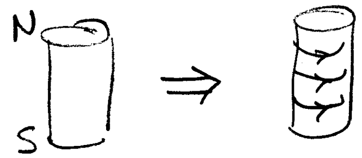
$$\underline{m} = \lim_{\Delta v \to 0} \frac{\sum_i \underline{m_i}}{\Delta v}$$

complicated because of

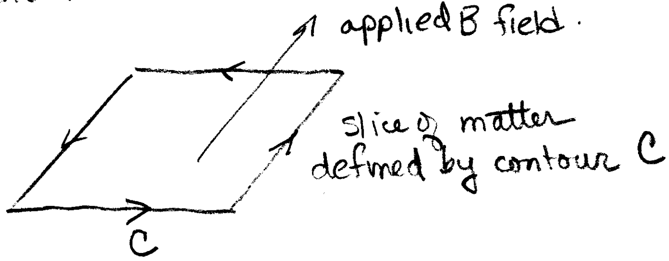
(i) currents combining



(ii) permanent magnets look like currents



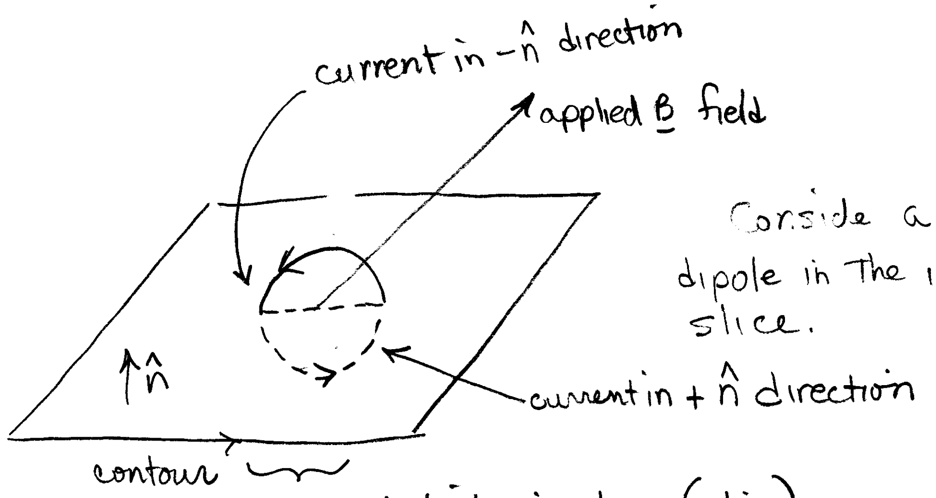
For polarization we considered a cube of electric dipoles. For magnetization we must consider a slice of magnetic dipoles because Ampère's Law is different than Gauss' Law.



$$\underline{M} = N\underline{m} = N\,\underbrace{\underline{I\,ds}}$$

microscopic dipole moment

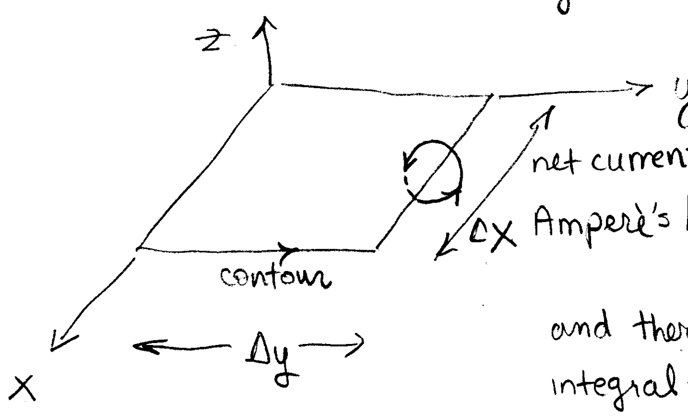
density of magnetic dipoles/unit volume



microscopic dipole in plane (slice)

Conside a magnetic dipole in The interior of the slice.

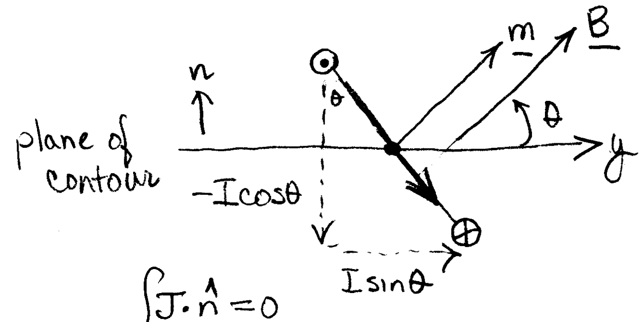
The only place a microscopic current loop will give a non-zero net current is if it is at The edges.



net current is in $-\hat{n}$ direction since Ampere's Law requires $\int \underline{J} \cdot \hat{n}\, ds$

and there is only a contribution to the integral in the $-\hat{n}$ direction.

We need to know an expression for this net current

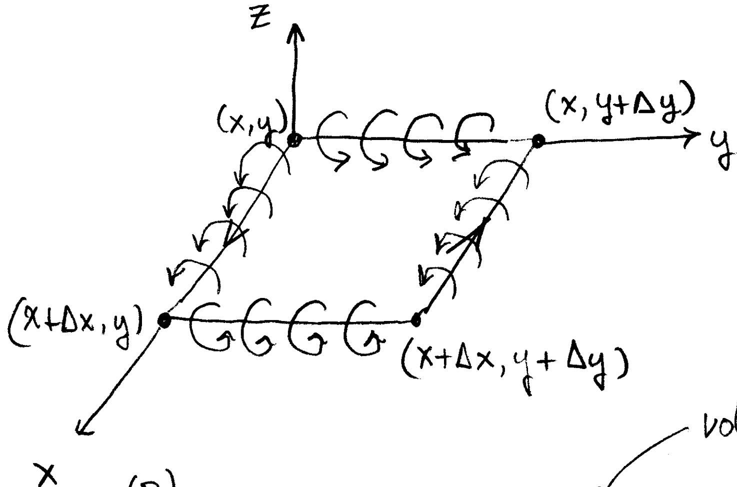


$\int \underline{J} \cdot \hat{n} = 0$

magnetic dipole at angle θ to plane

Note that there is no dependence on the x-y dependence of $\underline{B}$ only the angle.

EXCEPT AT EDGES! For the back side and B in the xz plane the current contribution would be $-I\cos\theta$.



volume density of dipoles

$$I_z^{(B)} = \int \underline{J} \cdot \hat{n}\, ds \quad = N \left(-I\cos\theta\right)\Big|_x \underbrace{dS\,\Delta y}.$$

along backedge at $x = 0$ — volume is cross section times Δy.

re-write in terms of definitions $\quad \underline{M} = \underbrace{N I}\, d\underline{S}$

$NI\cos\theta\, dS$ is y-component of $\underline{M}$ at x

as $\quad I_z^{(B)} = -\, M_y\Big|_x \,\Delta y$

Do exactly same thing on front edge at $x+\Delta x$

$$I_z^{(F)} = +\, M_y\Big|_{x+\Delta x} \Delta y.$$

If we look at dipoles on side edges and do same thing we get.

$$I_z^{(L)} = +\, M_x\Big|_y \,\Delta x$$

$$I_z^{(R)} = -\, M_x\Big|_{y+\Delta y} \Delta x$$

$$I_z\,(\text{total}) = \left(M_y\Big|_{x+\Delta x} - M_y\Big|_x\right)\Delta y - \left(M_x\Big|_{y+\Delta y} - M_x\Big|_y\right)\Delta x$$

$$J_z(\text{total}) = \frac{I_z\,(\text{total})}{\Delta x\,\Delta y} = \frac{M_y\big|_{x+\Delta x} - M_y\big|_x}{\Delta x} - \frac{M_x\big|_{y+\Delta y} - M_x\big|_y}{\Delta y} \rightarrow \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}.$$

to recognize what this actually is mathematically consider

$$(\underline{\nabla} \times \underline{M})_z = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M_x & M_y & M_z \end{vmatrix} \bullet \hat{z} = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}$$

$\Rightarrow$ surface current density in the z-direction is given by the z-component of the curl of the magnetization.

$$J_z = (\nabla \times M)_z$$

Can do in other directions as well

$$J_x = (\nabla \times M)_x$$

$$J_y = (\nabla \times M)_y$$

so vectorially

$$\underline{J} = \underline{\nabla} \times \underline{M}$$

Relationship between $\underline{B}$ and $\underline{H}$

for free space $\quad \underline{\nabla} \times \underline{B} = \mu_0 \underline{J}$

Magnetic material can have currents $\underline{J}_m = \underline{\nabla} \times \underline{M}$ of magnetic origin as well as free currents.

In general,

$$\underline{J} = \underline{J}_f + \underline{J}_m$$

$\underline{J}_f$: free currents sources for magnetic materials

$\underline{J}_m$: induced currents from magnetic dipoles or magnetic materials

$$\underline{\nabla} \times \underline{H} = \underline{J}_f + \underline{J}_m$$

$$\underline{\nabla} \times \frac{\underline{B}}{\mu_0} = \underline{J}_f + \underline{J}_m = \underline{J}_f + \underline{\nabla} \times \underline{m}$$

$$\nabla \times \left(\frac{\underline{B}}{\mu_0} - \underline{M} \right) = \underline{J}_f$$

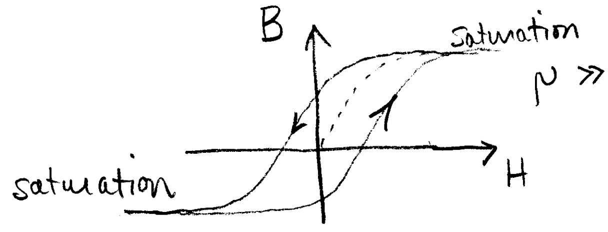
$$\therefore \quad \frac{\underline{B}}{\mu_0} - \underline{M} = \underline{H} \quad \text{or} \quad \underline{B} = \mu_0 (\underline{H} + \underline{M}) = \mu \underline{H}$$

this is deceptively written as $\mu = \mu(H)$

$\mu \lesssim \mu_0$ diamagnetic, orbital motion of electrons

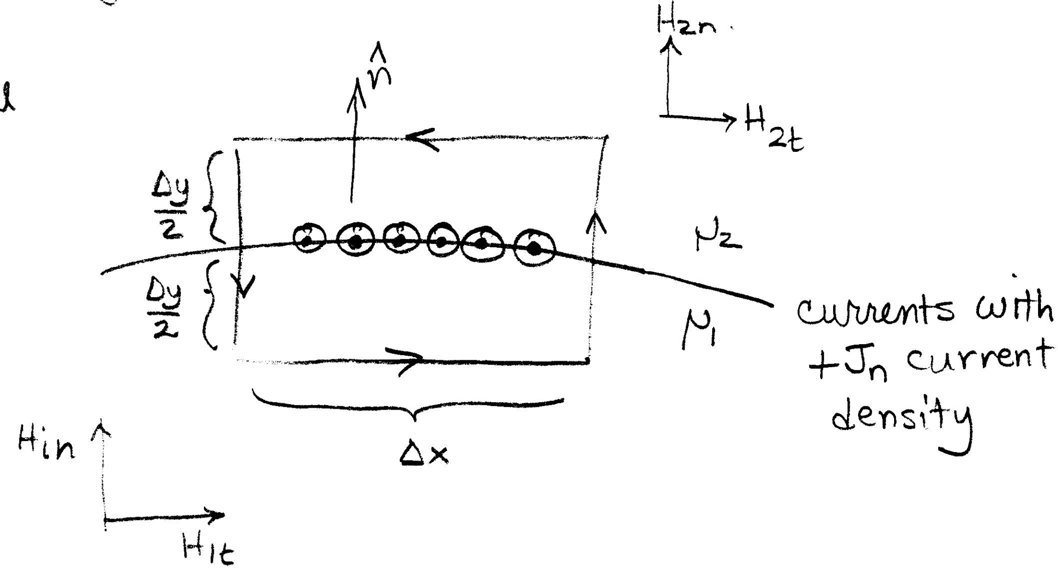
$\mu \gtrsim \mu_0$ paramagnetic, due to electron spin

$\mu \gg \mu_0$ ferromagnetic, ferrimagnetic due to electron spin



Magnetic Boundary Conditions

tangential



Ampere's Law $\oint_C \underline{H} \cdot d\underline{l} = \int_S \underline{J} \cdot \hat{n}\, ds$

$$\oint \underline{H} \cdot d\underline{l} = H_{1t}\Delta x + \cancel{H_{1n}\frac{\Delta y}{2}} + \cancel{H_{2n}\frac{\Delta y}{2}} - H_{2t}\Delta x - \cancel{H_{2n}\frac{\Delta y}{2}} - \cancel{H_{1n}\frac{\Delta y}{2}}$$

$$= (H_{1t} - H_{2t})\Delta x$$

$\oint \underline{J} \cdot \hat{n}\, ds = +J_n \Delta x \Delta y$ where J_n is the normal current density

Equating $(H_{1t} - H_{2t})\cancel{\Delta x} = J_n \cancel{\Delta x} \Delta y$

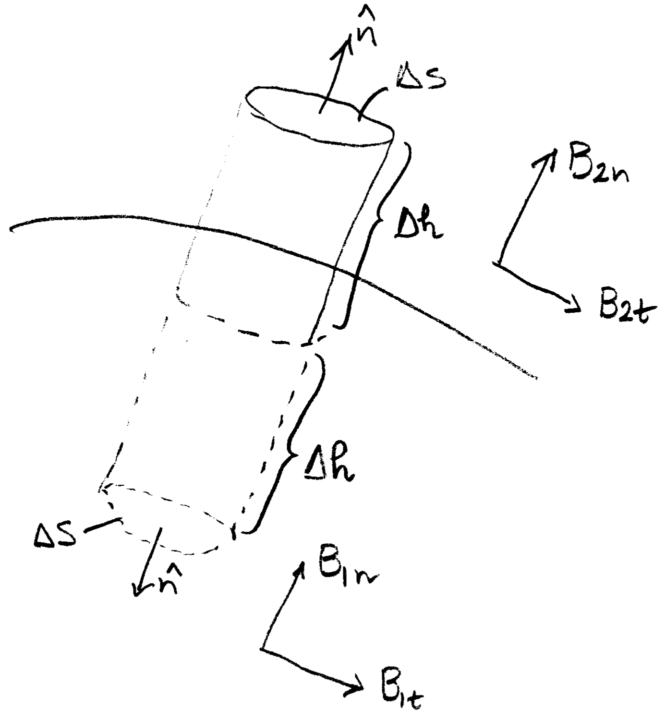
$$H_{1t} - H_{2t} = \underbrace{J_n \Delta y}_{\text{leaves surface current density}} \rightarrow K_s \quad \text{as } \Delta y \rightarrow 0$$

Vectorially $\hat{n} \times (\underline{H_2} - \underline{H_1}) = \underline{K}$

For permanent magnets

$\hat{n} \times (\underline{M_2} - \underline{M}) = \underline{K_m}$ (equivalent surface current).

normal



$$\oint \underline{B} \cdot \hat{n}\, ds = 0$$

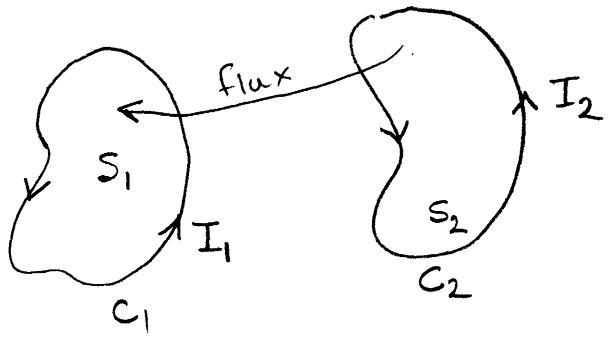
For this surface there will be NO contribution from sides as long as cylinder is small enough that $\underline{B}$ is uniform

From the ends only

$$\oint \underline{B} \cdot \underline{ds} = -B_{1n}\,\Delta S + B_{2n}\,\Delta S = 0$$

$$\therefore \quad B_{1n} = B_{2n}$$

Inductance



Self Inductance $\quad L_{11} \triangleq \dfrac{\varphi_{11}}{I_1}$ flux linking C_1 due to current in C_1

Mutual Inductance $\quad L_{ij} \triangleq \dfrac{\varphi_{ij}}{I_i}$ $\dfrac{\text{flux linking } C_j \text{ due to current in } C_i}{\text{current in } C_i}$

Example of self-inductance.

single loop $\quad H_z(\text{total})\Big|_{z=0} = \dfrac{Ia^2}{2(a^2)^{3/2}} = \dfrac{Ia^2}{2a^3} = \dfrac{Ia}{2}$

$$B_z\,(\text{total})\Big|_{z=0} = \mu_0 \frac{Ia}{2}$$

$$\varphi_{11} = \frac{\mu_0 Ia}{2} \cdot \pi a^2$$

$$L_{11} = \frac{\frac{\mu_0 Ia\, \pi a^2}{2}}{I} = \frac{\mu_0 \pi a^3}{2}$$