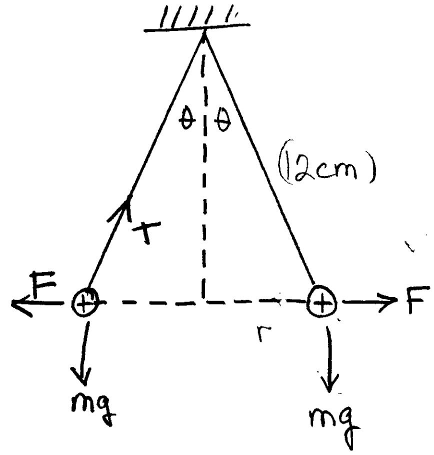
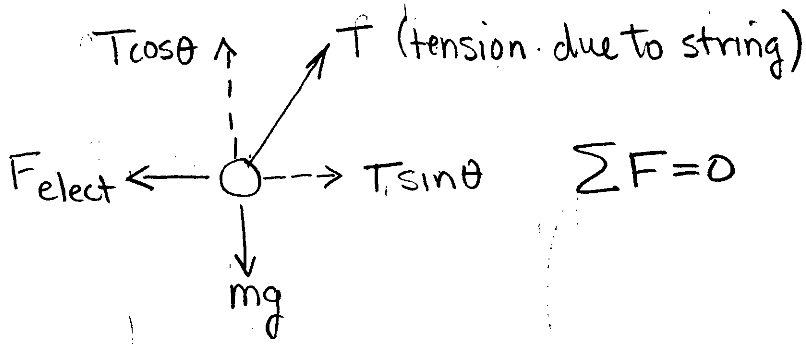
4.3 Two small identical electrically charged conducting spheres of mass 2.5g and charge +150nC are suspended by weightless strings of length 12cm each, as shown below. Calculate the deflection angle θ.



At rest the horizontal forces on the spheres will be zero. but mg is vertical.



$\sum F = 0$

$T\cos\theta = mg \qquad F_{elect} = T\sin\theta$

Solving $\dfrac{\cancel{T}\cos\theta}{\cancel{T}\sin\theta} = \dfrac{mg}{F_{elect}}$ or $F_{elect} = mg\dfrac{\sin\theta}{\cos\theta}$

Computing Felect

$$F_{elect} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} = \frac{(q)^2}{4\pi\epsilon_0 r^2} \quad \text{where } \sin\theta = \frac{(r/2)}{l} \leftarrow 12\text{cm.}$$

$$= \frac{q^2}{4\pi\epsilon_0 (2l\sin\theta)^2} = \frac{q^2}{16\pi\epsilon_0 l^2 \sin^2\theta}$$

$$mg\frac{\sin\theta}{\cos\theta} = \frac{q^2}{16\pi\epsilon_0 l \sin^2\theta}$$

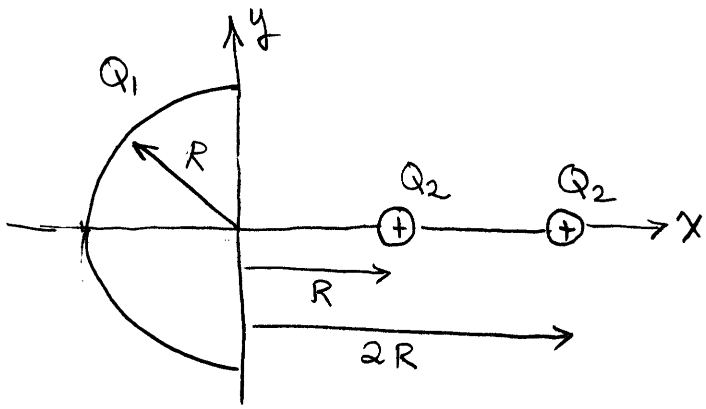
Note: mass in kg!

$$\sin^3\theta = \frac{q^2}{mg\,16\pi\epsilon_0 l^2}\cos\theta = \frac{(150\times10^{-9})^2}{(2.5\times10^{-3})(9.8\frac{m}{sec^2})\,16\pi(8.854\times10^{-12})(.12)^2}\cos\theta$$

$$\sin^3\theta = 0.1433\cos\theta$$

Solve iteratively to get $\theta \cong 29^\circ$.

4.16 A total charge of Q_1 is distributed uniformly along a half-circular ring as shown below. Two point charges, each of magnitude Q_2, are situated as shown. The surrounding medium is free space.



(a) Find Q_2 in terms of Q_1 so that the potential Φ at the center of the ring is zero.

We know that the potential due to a point charge is $\Phi = \dfrac{Q}{4\pi\epsilon_0 R}$

See p. 5 of notes. We will need to calculate the potential of the half-circular ring.

$$\Phi_{\text{half-ring}}(\text{origin}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_s(r')\,ds'}{|r-r'|} = \frac{1}{4\pi\epsilon_0}\int_{\pi/2}^{3\pi/2} \frac{\rho_\ell\, R\,d\phi}{R}$$

$$= \frac{1}{4\pi\epsilon_0}\rho_\ell \cdot \pi \qquad \text{where } \rho_\ell = \frac{Q_1}{\pi R} \begin{array}{l}\leftarrow \text{total charge} \\ \leftarrow \text{length of wire}\end{array}$$

$$\therefore \underset{\text{at origin}}{\text{Total }\Phi} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R} + \frac{1}{4\pi\epsilon_0}\frac{Q_2}{R} + \frac{1}{4\pi\epsilon_0}\frac{Q_2}{2R}$$

for $\Phi_{\text{total}} = 0$ $\qquad \dfrac{Q_1}{R} + \dfrac{Q_2}{R} + \dfrac{Q_2}{2R} = 0$ or $\dfrac{Q_1}{R} = -\dfrac{3Q_2}{2R}$

$$\therefore\ Q_2 = -\frac{2}{3}Q_1$$

(b) Find Q_2 in terms of Q_1 so that the electric field $\underline{E}$ at the center of the ring is zero.

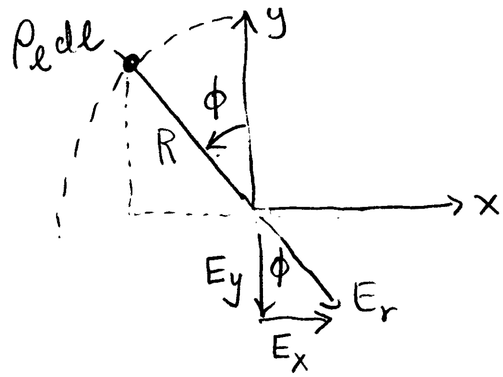
From symmetry there is no y component of $\underline{E}$ at the origin

The field from the point charges is

$$\underline{E}_1 = -\frac{Q_2}{4\pi\epsilon_0 R^2}\,\hat{x} \quad x \text{ field component due to charge at } R$$

$$\underline{E}_2 = -\frac{Q_2}{4\pi\epsilon_0 (2R)^2}\,\hat{x} \quad x \text{ field component due to charge at } 2R$$

Need to integrate to find E_x due to half-circular ring.



From p. 2 of notes $E_r = \dfrac{Q}{4\pi\epsilon_0 r^2}$

From figure $r = R$

$$Q = \rho_e dl = \frac{Q_1}{\pi R}\cdot R\,d\phi = \frac{Q_1\,d\phi}{\pi}$$

$$E_x = E_r \sin\phi$$

$$\therefore (E_{ring})_x = \frac{1}{4\pi\epsilon_0}\int_0^{\pi} \frac{Q_1\,d\phi}{\pi R^2}\sin\phi = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{\pi R^2}\int_0^{\pi}\sin\phi\,d\phi$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q_1}{\pi R^2}\left.-\cos\phi\right|_0^{\pi} = \frac{1}{4\pi\epsilon_0}\frac{Q_1}{\pi R^2}\cdot 2$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{2Q_1}{\pi R^2}\right)$$

Summing the field components $\underline{E}_{1,x} + \underline{E}_{2,x} + \underline{E}_{ring,x} = 0$

$$\frac{1}{4\pi\epsilon_0}\left(-\frac{Q_2}{R^2}\right) + \frac{1}{4\pi\epsilon_0}\left(-\frac{Q_2}{4R^2}\right) + \frac{1}{4\pi\epsilon_0}\left(\frac{2Q_1}{\pi R^2}\right) = 0$$

$$-\frac{Q_2}{R^2} - \frac{Q_2}{4R^2} + \frac{2Q_1}{\pi R^2} = 0 \quad \text{or} \quad Q_2 = \frac{8Q_1}{5\pi}$$

4.21 Assume that the electron charge distribution in a hydrogen atom is given by

$$\rho(r) = \frac{q_e}{\pi a_0^3} e^{-\frac{2r}{a_0}}$$

where q_e is the charge of an electron, a is the Bohr radius ($a_0 \cong .529 \times 10^{-10} m$).

(a) Find the electric potential and the electric field due to the electron cloud only.

This problem is best solved using Gauss' Law.

$$\oint_S \underline{D} \cdot \underline{ds} = \int_V \rho dv$$

The left hand side is simply $\int_S \epsilon_0 \underline{E} \cdot \underline{ds} = \epsilon_0 E_r 4\pi r^2$

The calculation for $\int \rho dv$ is complicated.

$$\int \rho dv = \frac{q_e}{\pi a_0^3} \int_0^{2\pi}\int_0^{\pi}\int_0^{r} e^{-\frac{2r'}{a_0}} \underbrace{r'^2 \sin\theta' dr' d\theta' d\phi'}_{dv \text{ in spherical coordinates}}$$

$$= \frac{q_e}{\pi a_0^3} 2\pi \int_0^{\pi}\int_0^{r} e^{-\frac{2r'}{a_0}} r'^2 \underline{\sin\theta' d\theta'}\, dr'$$

$$= \frac{q_e}{\pi a_0^3} 2\pi \int_0^{r} e^{-\frac{2r'}{a_0}} r'^2 \left(-\cos\theta'\right)\Big|_0^{\pi} dr'$$

$$= \frac{q_e}{\pi a_0^3} (2\pi)(2) \int_0^{r} r'^2 e^{-\frac{2r'}{a_0}} dr'$$

We have to look up this integral. It is actually #521 in the CRC Handbook.

521. $\int x^2 e^{ax}\,dx = \frac{x^2 e^{ax}}{a} - \frac{2}{a}\int x e^{ax}\,dx$

The integral of $\int x e^{ax} dx$ is given by 520. as.

520. $\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$

The overall integral is then

$$\int x^2 e^{ax} dx = \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left[\frac{e^{ax}}{a^2}(ax-1)\right]$$

$$= e^{ax}\left[\frac{x^2}{a} - \frac{2}{\cancel{a}}\cdot\frac{1}{a^2}\cancel{a}x + \frac{2}{a}\frac{1}{a^2}\right]$$

$$\int x^2 e^{ax} dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right]$$

$$\therefore \int \rho\, dv = \frac{4q_e}{a_0^3}\int_0^r r'^2 e^{-\frac{2r'}{a_0}} dr'. \qquad \text{Identify } a = -\frac{2}{a_0}$$

$$= \frac{4q_e}{a_0^3}\left[e^{-\frac{2}{a_0}r}\left[\frac{r^2}{\left(-\frac{2}{a_0}\right)} - 2\frac{r}{\left(-\frac{2}{a_0}\right)^2} + \frac{2}{\left(-\frac{2}{a_0}\right)^3}\right]\right]_0^r$$

$$= \frac{4q_e}{a_0^3}\left[e^{-\frac{2r}{a_0}}\left[-\frac{a_0 r^2}{2} - \frac{a_0^2 r}{2} - \frac{a_0^3}{4}\right] - \left[-\frac{a_0^3}{4}\right]\right]$$

Finishing the calculations

$$\epsilon_0 E_r \cancel{4}\pi r^2 = \frac{\cancel{4}q_e}{a_0^3}\left[\frac{a_0^3}{4} - e^{-\frac{2r}{a_0}}\left[\frac{a_0 r^2}{2} + \frac{a_0^2 r}{2} + \frac{a_0^3}{4}\right]\right]$$

$$E_r = \frac{q_e}{4\pi\epsilon_0 r^2}\left[1 - e^{-\frac{2r}{a_0}}\left[2\frac{r^2}{a_0^2} + 2\frac{r}{a_0} + 1\right]\right]$$

Integrate this to determine the electric potential Φ

$$\Phi(r) = -\int_{\infty}^{r} E_r(r)\,dr$$

This is where Mathematica makes the problem easy. By hand, unfortunately,

$$\Phi^{(1)} = \frac{-q_e}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{q_e}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r} = -\frac{q_e}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)$$

$$\Phi^{(2)} = \frac{-q_e}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{-1}{r^2}\frac{2r^2}{a_0^2}e^{-\frac{2r}{a_0}}dr = \frac{-q_e}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{-2}{a_0^2}e^{-\frac{2r}{a_0}}dr$$

$$= -\frac{q_e}{4\pi\epsilon_0}\frac{-2}{a_0^2}\left[\frac{e^{-\frac{2r}{a_0}}}{-\frac{2}{a_0}}\right]_{\infty}^{r} = -\frac{q_e}{4\pi\epsilon_0}\left[+\frac{e^{-\frac{2r}{a_0}}}{a_0}\right]_{\infty}^{r}$$

$$= -\frac{q_e}{4\pi\epsilon_0}\left[+\frac{e^{-\frac{2r}{a_0}}}{a_0}\right]$$

$$\Phi^{(3)} = -\frac{q_e}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{1}{r^2}\left(-\frac{2r}{a_0}\right)e^{-\frac{2r}{a_0}} = \frac{-q_e}{4\pi\epsilon_0}\int_{\infty}^{r}-\frac{2}{a_0}\frac{e^{-\frac{2r}{a_0}}}{r}dr$$

$$\Phi^{(4)} = -\frac{q_e}{4\pi\epsilon_0}\int_{\infty}^{r}\frac{1}{r^2}(-1)e^{-\frac{2r}{a_0}}dr$$

$$= -\frac{q_e}{4\pi\epsilon_0}\left[+\frac{e^{-\frac{2}{a_0}r}}{r} + \frac{2}{a_0}\int_{\infty}^{r}\frac{e^{-\frac{2r}{a_0}}}{r}dr\right].$$

$$\Phi = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(3)} + \Phi^{(4)}$$

$$= -\frac{q_e}{4\pi\epsilon_0}\left[-\frac{1}{r} + \frac{e^{-\frac{2}{a_0}r}}{a_0} - \frac{2}{a_0}\int_\infty^r \frac{e^{-\frac{2r}{a_0}}}{r}dr + \frac{e^{-\frac{2}{a_0}r}}{r} + \frac{2}{a_0}\int_\infty^r \frac{e^{-\frac{2r}{a_0}}}{r}\right]$$

$$= -\frac{q_e}{4\pi\epsilon_0}\left[-\frac{1}{r} + e^{-\frac{2}{a_0}r}\left[\frac{1}{a_0} + \frac{1}{r}\right]\right]$$

$$= \frac{q_e}{4\pi\epsilon_0 r} - \frac{q_e}{4\pi\epsilon_0} e^{-\frac{2}{a_0}r}\left[\frac{1}{a_0} + \frac{1}{r}\right]$$

(b) Find the total electric potential and the electric field in the atom, assuming that the nucleus (proton) is localized at the origin.

We use superposition

$$E_{r,atom} = E_{r,nucleus} + E_{r,electroncloud}$$

$$= \frac{-q_e}{4\pi\epsilon_0 r^2} + \frac{q_e}{4\pi\epsilon_0 r^2}\left[1 - e^{-\frac{2r}{a_0}}\left[\frac{2r^2}{a_0^2} + 2\frac{r}{a_0} + 1\right]\right]$$

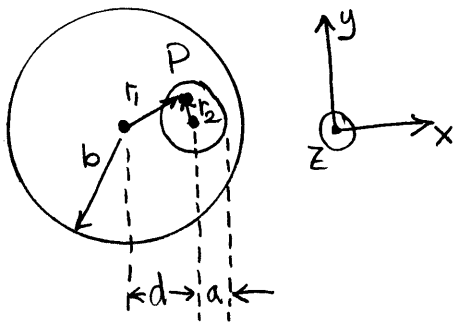
one of these has to have a negative sign and cancel.

$$E_{r,atom} = -\frac{q_e}{4\pi\epsilon_0 r^2} e^{-\frac{2r}{a_0}}\left[\frac{2r^2}{a_0^2} + 2\frac{r}{a_0} + 1\right].$$

We integrate this to get Φ_{atom}. All that happens is that the $\frac{1}{r}$ term disappears

$$\Phi_{atom} = -\frac{q_e}{4\pi\epsilon_0} e^{-\frac{2r}{a_0}}\left[\frac{1}{a_0} + \frac{1}{r}\right].$$

4.24 A spherical region of radius b in free space is uniformly charged with a charge density of $\rho = K$, where K is a constant. The sphere contains an uncharged spherical cavity of radius a. The centers of the two spheres are separated by a distance d such that $d + a < b$. Find the electric field inside the cavity.



This is a classic problem in superposition. Assume that there exists a second charge density $\rho_2 = -K$ inside the spherical cavity such that $\rho_1 + \rho_2 = K - K = 0$ there.

Let P be a point in the uncharged cavity $\underline{r}_1$ from the center of the sphere of radius b, and $\underline{r}_2$ from the center of the uncharged cavity of radius a.

We solved for the field of a spherical cloud of charge on p. 4 of the class notes.

For the uncharged cavity the field from charge density ρ_1 is given by

$$\underline{E}_1 = \frac{\rho_1}{3\epsilon_0}\underline{r}_1 = \frac{K}{3\epsilon_0}\underline{r}_1$$

Similarly, the field from charge density ρ_2 is

$$\underline{E}_2 = \frac{\rho_2}{3\epsilon_0}\underline{r}_2 = -\frac{K}{3\epsilon_0}\underline{r}_2$$

The electric field is then

$$\underline{E}_{tot} = \underline{E}_1 + \underline{E}_2 = \frac{K}{3\epsilon_0}(\underline{r}_1 - \underline{r}_2)$$

If we choose $\underline{r}_1$ & $\underline{r}_2$ along x axis $\underline{r}_1 - \underline{r}_2 = d\hat{x}$ so $E = \frac{Kd}{3\epsilon_0}\hat{x}$

4.27 A basic MOS transistor consists of a gate conductor and a semiconductor (which is the other conductor), separated by a gate dielectric. Consider a MOS transistor using silicon dioxide (SiO_2) ($\epsilon_r = 3.9$) as the gate oxide. The gate oxide capacitance can be approximated as a parallel plate capacitor. The gate-oxide capacitance per unit area is given by

$$C_{ox} = \frac{\epsilon_{ox}}{t_{ox}}$$

where ϵ_{ox} and t_{ox} are the permittivity and the thickness of the gate dielectric.

(a) If the thickness of the SiO_2 layer is 2×10^{-6} cm, find the gate oxide capacitance per unit area.

$$C_{ox} = \frac{\epsilon_r \epsilon_0}{t_{ox}} = \frac{(3.9)(8.85\times10^{-12})}{2\times10^{-8}\,\text{m}} = 1.73\times10^{-3}\ \frac{\text{F}}{\text{m}^2}$$

units

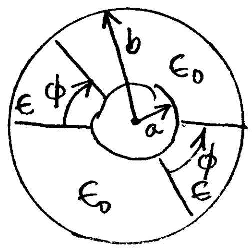
(b) If the length and the width of the gate region are $L = 5\times10^{-4}$ cm and $W = 2\times10^{-3}$ cm respectively, find the total gate capacitance.

$$C_g = C_{ox}A = C_{ox}LW = \left(1.73\times10^{-3}\ \frac{\text{F}}{\text{m}^2}\right)\left(5\times10^{-6}\,\text{m}\right)\left(2\times10^{-5}\,\text{m}\right)$$

$$= 1.73\times10^{-13} = 0.173\ \text{pf}.$$

4.34. The cross-sectional view of an air-filled coaxial capacitor with spacers made out of material of permittivity ϵ is shown below.

(a) Find the capacitance of this coaxial line in terms of ϵ, a, b, ϕ.



The best way to solve any capacitance is usually to start from the potential.
Assume the inner conductor is at $\Phi = +V$ and the outer conductor is at $\Phi = 0$.

Use Laplace's equation in cylindrical coordinates

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2}}_{\text{symmetric in } \phi} + \underbrace{\frac{\partial^2 \Phi}{\partial z^2}}_{\text{no } z\text{-dependence}} = 0 .$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) = 0$$

Integrating $\quad r\dfrac{d\Phi}{dr} = c_1$

Integrating again

$$d\Phi = c_1 \frac{dr}{r}$$

$$\Phi = c_1 \ln r + c_2$$

Matching the boundary conditions

$$\Phi(a) = c_1 \ln a + c_2 = V$$

$$\Phi(b) = c_1 \ln b + c_2 = 0$$

$$c_1(\ln a - \ln b) = V$$

$$c_1 = \frac{V}{\ln a - \ln b} = \frac{V}{\ln(a/b)}$$

$$c_2 = -c_1 \ln b = -\frac{V \ln b}{\ln(a/b)}$$

$$\Phi = \frac{V \ln r}{\ln(a/b)} - \frac{V \ln b}{\ln(a/b)} = V\frac{\ln(r/b)}{\ln(a/b)}$$

The electric field is computed as

$$\underline{E} = -\nabla \Phi = -\hat{r}\frac{\partial \Phi}{\partial r} = -\hat{r}\frac{V}{\ln\left(\frac{a}{b}\right)}\frac{1}{\left(\frac{r}{b}\right)}\cdot\frac{1}{b}$$

$$\underline{D} = \epsilon\,\underline{E} = -\frac{\epsilon V}{\ln\left(\frac{a}{b}\right)}\frac{1}{r}\hat{r} = \frac{\epsilon V}{r\ln\frac{b}{a}}\hat{r}$$

The surface charge density ρ_s is then given as

$$\rho_s = \hat{n}\cdot(\underline{D}_1 - \underline{D}_2) = \hat{r}\cdot\left(\frac{\epsilon V}{b\ln\frac{b}{a}}\hat{r} - 0\right) = \frac{\epsilon V}{b\ln\left(\frac{b}{a}\right)}$$

To compute the capacitance/unit length we need to compute $\dfrac{C}{\Delta z} = \dfrac{Q}{V}\dfrac{1}{\Delta z}$

so we need to compute the charge density/unit length.

$$Q = \underbrace{\rho_{s1}\cdot 2\phi\cdot b\cdot\Delta z}_{\text{surface charge over spacers}} + \underbrace{\rho_{s2}\,(2\pi - 2\phi)\,b\,\Delta z}_{\text{surface charge elsewhere.}}$$

$$= \frac{\epsilon V}{b\ln\left(\frac{b}{a}\right)}\cdot b\cdot 2\phi\,\Delta z + \frac{\epsilon_0 V}{b\ln\left(\frac{b}{a}\right)}\cdot b\cdot(2\pi - 2\phi)\,\Delta z$$

$$Q = \frac{2V\Delta z}{\ln\left(\frac{b}{a}\right)}\left[\epsilon\phi + \epsilon_0(\pi - \phi)\right].$$

$$\frac{C}{\Delta z} = \frac{Q}{V\Delta z} = \frac{2\cancel{V}\cancel{\Delta z}\left[\epsilon\phi + \epsilon_0(\pi - \phi)\right]}{\ln\left(\frac{b}{a}\right)\cancel{V}\cancel{\Delta z}}$$

$$\frac{C}{\Delta z} = \frac{2\left[\epsilon\phi + \epsilon_0(\pi - \phi)\right]}{\ln\left(\frac{b}{a}\right)}$$

(b) If the spacers are to be made out of mica ($\epsilon = 6\epsilon_0$), determine the angle ϕ such that only 10% of the total energy stored by the capacitor is stored in the spacers.

Many ways to do this. I will do from basic principles

Energy stored in electric field

$$W_e = \int_V \frac{1}{2}\epsilon |\underline{E}|^2 dv$$

For angular wedge of capacitor, i.e. 2ϕ

$$W_e = \int_0^{\Delta z}\int_0^{2\phi}\int_a^b \frac{1}{2}\epsilon \left| -\hat{r}\frac{V}{r\ln\left(\frac{a}{b}\right)}\right|^2 r\,dr\,d\phi\,dz$$

$$= \int_0^{\Delta z}\int_0^{2\phi}\int_a^b \frac{1}{2}\epsilon \frac{V^2}{r^2\ln^2\left(\frac{a}{b}\right)} \cdot r\,dr\,d\phi\,dz$$

$$= \Delta z\, 2\phi \frac{\epsilon V^2}{2\ln^2\left(\frac{a}{b}\right)}\int_a^b \frac{dr}{r} = \frac{\Delta z\,\phi\,\epsilon\, V^2}{\ln^2\left(\frac{a}{b}\right)}\ln\left(\frac{b}{a}\right)$$

$$= \frac{\Delta z\,\phi\,\epsilon\, V^2}{\ln\left(\frac{b}{a}\right)}$$

For the capacitor the total energy is given by

$$W_e = \underbrace{\frac{\Delta z\,\phi\,\epsilon\, V^2}{\ln\left(\frac{b}{a}\right)}}_{\text{energy in spacers}} + \underbrace{\frac{\Delta z\,(\pi-\phi)\epsilon_0\, V^2}{\ln\left(\frac{b}{a}\right)}}_{\text{energy in free-space}}$$

Then we want

$$\frac{\dfrac{\Delta z\,\phi\,\epsilon\, V^2}{\ln\left(\frac{b}{a}\right)}}{\dfrac{\Delta z\,\phi\,\epsilon\, V^2}{\ln\left(\frac{b}{a}\right)} + \dfrac{\Delta z\,(\pi-\phi)\epsilon_0\, V^2}{\ln\left(\frac{b}{a}\right)}} = 0.1$$

This reduces to

$$\frac{\phi\epsilon}{\phi\epsilon + (\pi-\phi)\epsilon_0} = 0.1$$

$$\phi(6\epsilon_0) = 0.1\,\phi(6\epsilon_0) + 0.1\pi\epsilon_0 - 0.1\phi\epsilon_0$$

$$6\cancel{\epsilon_0}\,\phi = 0.6\cancel{\epsilon_0}\,\phi + \frac{\pi}{10}\cancel{\epsilon_0} - 0.1\cancel{\epsilon_0}\,\phi$$

$$(6-0.6+0.1)\phi = \frac{\pi}{10}$$

$$\phi = \frac{\pi}{5.5(10)} = \frac{\pi}{55} \cong 3.3^\circ$$

(c) Consider the capacitor without the spacers (i.e., $\phi=0$). For a given potential difference V_0 between the inner and outer conductors and for a given fixed value of b, determine the inner radius a for which the largest value of the electric field is a minimum.

We already know that $\underline{E} = \hat{r}\dfrac{V}{r\ln\left(\frac{b}{a}\right)}$

Maximum electric field occurs when r is smallest, i.e. $r=a$

$$E_{max} = \frac{V}{a\ln\left(\frac{b}{a}\right)}$$

To minimize E_{max} as a function of a compute $\dfrac{dE_{max}}{da} = 0$.

$$\frac{d}{da}\left[\frac{V}{a\ln\left(\frac{b}{a}\right)}\right] = \frac{d}{da}\left[Va^{-1}\ln^{-1}\left(\frac{b}{a}\right)\right]$$

$$= V(-1)a^{-2}\ln^{-1}\left(\frac{b}{a}\right) + Va^{-1}(-1)\ln^{-2}\left(\frac{b}{a}\right)\frac{1}{\frac{\cancel{b}}{a}}\cdot\cancel{b}(-1)a^{-2}$$

$$= -\frac{\cancel{V}}{\cancel{a^2}\cancel{\ln\left(\frac{b}{a}\right)}} + \frac{\cancel{V}}{a}\,\frac{1}{\ln^{\cancel{2}}\left(\frac{b}{a}\right)}\,\frac{a}{\cancel{a^2}} = 0$$

$$\therefore\quad -1 + \frac{1}{\ln(b/a)} = 0 \quad \text{or} \quad \ln\left(\frac{b}{a}\right) = 1$$

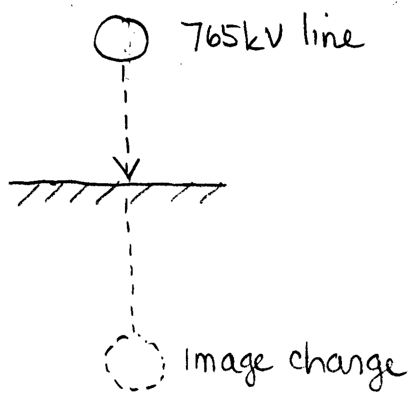
$$\therefore\quad \frac{b}{a} = e \quad \text{or} \quad a = \frac{b}{e}$$

Incidentially this maximizes the breakdown voltage.

4.39 Many 60 Hz high-voltage transmission lines operate at a rms alternating voltage of 765 kV.

(a) What is the peak electric field at ground level under such a line if the wire is 12 m above the ground?

We solved for the electric field between two conductors on p. 20-21 of notes. We can use the method of images, i.e. a + conductor 12 m above the ground and a second conductor 12 m below ground to model this.



From p. 21

$$E_x(x,0,0) = \frac{-\rho_\ell}{2\pi\epsilon_0 x} - \frac{\rho_\ell}{2\pi\epsilon_0 (d-x)}$$

if $d = 2h$, $x = h$ this becomes

$$E_x(h,0,0) = -\frac{\rho_\ell}{\pi\epsilon_0 h}$$

We can estimate the charge density from the voltage relationship (p. 21).

$$\Phi_{12} = 2\Phi_{1gnd} \cong \frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)$$

The factor of 2 is because the potential is wrt ground and not the other conductor

Need to assume wire radius. Let $a = 0.25$ meter

$$\rho_\ell = \frac{2\pi\epsilon_0 \Phi_{1gnd}}{\ln\left(\frac{d}{a}\right)} = \frac{2\pi(8.854\times10^{-12}\,F/m)(765\times10^{3})}{\ln\left(\frac{2\cdot 12}{0.25}\right)} =$$

$$= \frac{2\pi(8.854\times10^{-12})(765\times10^{3})}{\ln(96)} = 9.32\,\mu C$$

$$E_x = -\frac{\rho_\ell}{\pi\epsilon_0 h} = -\frac{(9.32\times10^{-6})}{\pi(8.854\times10^{-12})(12)} = 27922\,\frac{\text{volts}}{m}$$

(b) What is the peak potential difference between the head and feet of a 6 ft-tall person?

Assume the E field remains constant

$$\Phi \cong E_x h_{person} = \left(27992 \frac{\text{volts}}{m}\right)(2m) \cong 55.8kV$$

(c) Is the field sufficient to ignite a standard (110V) fluorescent lamp of 2ft. length?

The bolb requires $\dfrac{110 \text{ volts}}{2ft \times .3048 \frac{m}{ft}} = 180 \frac{V}{m}.$

The light will certainly light.

Repeat for $a = 1.5cm = .015m$.

$$\rho_\ell = \frac{2\pi\,(8.854\times10^{-12})(765\times10^{3})}{\ln\left(\frac{2.12}{.015}\right)} = 5.768\,\mu C$$

$$E_x = -\frac{\rho_\ell}{\pi\epsilon_0 h} = -\frac{5.768\times10^{-6}}{\pi(8.854\times10^{-12})(12)} = 17281 \text{ volts}/m.$$

Still more than enough to light lamp.

4.42 Consider a pair of small conducting spheres with radii a, b small compared to the separation distance between their centers (i.e., $a, b \ll d$).

(a) Determine the electrostatic energy stored by this configuration, assuming that the spheres with radii a and b carry changes of Q and $-Q$, respectively. Your answer should depend on d. State all assumptions.

I woued do this by computing the discrete energy. Specifically, $W_e = \frac{1}{2}\sum_{i=1}^{2} q_i \Phi_i = \frac{1}{2}(q_1\Phi_1 + q_2\Phi_2)$

The only trick to the problem is writing Φ_1 and Φ_2.

$$\Phi_1 = \underbrace{\frac{-Q}{4\pi\epsilon_0 b}}_{\text{potential of sphere}} + \underbrace{\frac{Q}{4\pi\epsilon_0(d-b)}}_{\text{potential due to other sphere.}}$$

of radii b has change $-Q$

$$\Phi_2 = \underbrace{\frac{Q}{4\pi\epsilon_0 a}}_{\text{potential of sphere}} + \underbrace{\frac{-Q}{4\pi\epsilon_0(d-a)}}_{\text{potential due to other sphere.}}$$

$$W_e = \frac{1}{2}(-Q)\Phi_1 + \frac{1}{2}(+Q)\Phi_2 = \frac{Q^2}{8\pi\epsilon_0}\left[+\frac{1}{b} - \frac{1}{d-b} + \frac{1}{a} - \frac{1}{d-a}\right]$$

(b) Repeat part (a) assuming the spheres with radii a and b carry charges of $+Q$ and $+2Q$ respectively.

$$\Phi_1 = \frac{2Q}{4\pi\epsilon_0 b} + \frac{Q}{4\pi\epsilon_0(d-b)}$$

of radii b has charge 2Q

$$\Phi_2 = \frac{Q}{4\pi\epsilon_0 a} + \frac{2Q}{4\pi\epsilon_0(d-a)}$$

of radii a has charge Q

$$W_e = \frac{1}{2}(2Q)\Phi_1 + \frac{1}{2}(Q)\Phi_2 = \frac{Q^2}{8\pi\epsilon_0}\left[\frac{4}{b} + \frac{2}{d-b} + \frac{1}{a} + \frac{2}{d-a}\right].$$