Maxwell's Equations (general differential)

$\nabla \times \bar{\mathscr{E}} = -\dfrac{\partial \bar{\mathscr{B}}}{\partial t}$ [1.11a]

$\nabla \cdot \bar{\mathscr{D}} = \tilde{\rho}$ [1.11b]

$\nabla \times \bar{\mathscr{H}} = \bar{\mathscr{J}} + \dfrac{\partial \bar{\mathscr{D}}}{\partial t}$ [1.11c]

$\nabla \cdot \bar{\mathscr{B}} = 0$ [1.11d]

Maxwell's Equations (time harmonic)

$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$ [1.11a]

$\nabla \cdot \mathbf{D} = \rho$ [1.11b]

$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$ [1.11c]

$\nabla \cdot \mathbf{B} = 0$ [1.11d]

Maxwell's Equations (integral)

$\oint_C \bar{\mathscr{E}} \cdot d\mathbf{l} = -\int_S \dfrac{\partial \bar{\mathscr{B}}}{\partial t} \cdot d\mathbf{s}$ [1.1]

$\oint_S \bar{\mathscr{D}} \cdot d\mathbf{s} = \int_V \rho dv$ [1.2]

$\oint_C \bar{\mathscr{H}} \cdot d\mathbf{l} = \int_S \bar{\mathscr{J}} \cdot d\mathbf{s} + \int_S \dfrac{\partial \bar{\mathscr{D}}}{\partial t} \cdot d\mathbf{s}$ [1.3]

$\oint_S \bar{\mathscr{B}} \cdot d\mathbf{s} = 0$ [1.4]

Electromagnetic Boundary Conditions

$\hat{\mathbf{n}} \times \left[\bar{\mathscr{E}}_1 - \bar{\mathscr{E}}_2\right] = 0$ [1.12]

$\hat{\mathbf{n}} \times \left[\bar{\mathscr{H}}_1 - \bar{\mathscr{H}}_2\right] = 0$ [1.13]

$\hat{\mathbf{n}} \times \bar{\mathscr{H}}_1 = \bar{\mathscr{J}}$ [1.14]

$\hat{\mathbf{n}} \cdot \left[\bar{\mathscr{D}}_1 - \bar{\mathscr{D}}_2\right] = \tilde{\rho}_s$ [1.15]

$\hat{\mathbf{n}} \cdot \left[\bar{\mathscr{B}}_1 - \bar{\mathscr{B}}_2\right] = 0$ [1.16]

$\hat{\mathbf{n}} \cdot \left[\bar{\mathscr{J}}_1 - \bar{\mathscr{J}}_2\right] = -\dfrac{\partial \tilde{\rho}_s}{\partial t}$ [1.17]

Reflection & transmission (simple dielectric)

$\Gamma = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ [3.2]

$T = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2}{\eta_2 + \eta_1}$ [3.3]

Basic waves

$n \equiv \dfrac{c}{v_p} = \beta \dfrac{c}{\omega} = \sqrt{\mu_r \varepsilon_r}$ p.135

$\beta = \dfrac{2\pi}{\lambda} = \dfrac{2\pi f}{v_p} = \omega\sqrt{\mu\varepsilon}$

$f\lambda = v_p = \dfrac{1}{\sqrt{\mu\varepsilon}}$

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Reflection & transmission (multiple dielectrics)

$\Gamma_{eff} = \dfrac{(\eta_2 - \eta_1)(\eta_3 + \eta_2) + (\eta_2 + \eta_1)(\eta_3 - \eta_2)e^{-2j\beta_2 d}}{(\eta_2 + \eta_1)(\eta_3 + \eta_2) + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-2j\beta_2 d}}$ [3.7]

$T_{eff} = \dfrac{4\eta_2\eta_3 e^{-j\beta_2 d}}{(\eta_2 + \eta_1)(\eta_3 + \eta_2) + (\eta_2 - \eta_1)(\eta_3 - \eta_2)e^{-2j\beta_2 d}}$ [3.8]

$Z_2(-d) = \eta_2 \dfrac{\eta_3 + j\eta_2 \tan(\beta_2 d)}{\eta_2 + j\eta_3 \tan(\beta_2 d)} = \eta_{23}$ [3.9]

$\Gamma_{eff} = \rho_{eff} e^{j\phi_r} = \dfrac{Z_2(-d) - \eta_1}{Z_2(-d) + \eta_1} = \dfrac{\eta_{23} - \eta_1}{\eta_{23} + \eta_1}$ [3.10]

Plane waves in lossy materials

$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$ [2.19]

$\alpha = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]^{1/2}$ [2.20]

$\beta = \omega\sqrt{\dfrac{\mu\varepsilon}{2}}\left[\sqrt{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]^{1/2}$ [2.21]

Lossy materials

Polarization currents:

$\varepsilon_c = \varepsilon' - j\varepsilon''$ [p.46]

$\tan\delta_c = \dfrac{\sigma_{eff}}{\omega\varepsilon'} = \dfrac{\varepsilon''}{\varepsilon'}$ [p.48]

Conduction Currents:

$\sigma_{eff} = \sigma + \omega\varepsilon''$ [p.48]

$\tan\delta_c = \dfrac{\sigma + \omega\varepsilon''}{\omega\varepsilon'}$ [p.48]

Use $\tan\delta_c$ instead of $\dfrac{\sigma}{\omega\varepsilon}$ in expression for complex imepdance

$\eta_c = \sqrt{\dfrac{\mu}{\varepsilon_{eff}}} = \sqrt{\dfrac{\mu}{\varepsilon - j\dfrac{\sigma}{\omega}}} = \dfrac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\left(1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right)^{1/4}} e^{j(1/2)Tan^{-1}(\sigma/\omega\varepsilon)}$ [2.22]

Good conductor approximations

$\alpha = \beta \approx \sqrt{\dfrac{\omega\mu\sigma}{2}}$ [p.54]

$\delta \equiv \dfrac{1}{\alpha} \approx \sqrt{\dfrac{2}{\omega\mu\sigma}} = \dfrac{1}{\sqrt{\pi f \mu\sigma}}$ [2.26]

$\eta_c = \sqrt{\dfrac{\mu}{\varepsilon}} \dfrac{1}{\left(1 - j\dfrac{\sigma}{\omega\varepsilon}\right)^{1/2}} = \sqrt{\dfrac{j\omega\mu}{\sigma}} = \sqrt{\dfrac{\omega\mu}{\sigma}} e^{j45^\circ}$ [p.54]

Poynting Theorem

$$\int_V \bar{\mathscr{E}} \cdot \bar{\mathscr{J}} dv = -\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\mu\left|\bar{\mathscr{H}}\right|^2 + \frac{1}{2}\varepsilon\left|\bar{\mathscr{E}}\right|^2\right)dv - \oint_S \left(\bar{\mathscr{E}} \times \bar{\mathscr{H}}\right)\cdot d\mathbf{s} \qquad [2.31]$$

$$\mathbf{S}_{av} = \hat{z}\frac{E_0^2}{2\eta} \qquad [2.32]$$

$$\mathbf{S}_{av} = \frac{1}{2}\mathscr{R}e\{\mathbf{E}\times\mathbf{H}^*\} \qquad [2.43]$$

Arbitrarily directed uniform plane waves

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\beta\hat{\mathbf{k}}\cdot\mathbf{r}} \qquad [2.58]$$

$$\hat{\mathbf{k}} = \frac{\hat{\mathbf{x}}\beta_x + \hat{\mathbf{y}}\beta_y + \hat{\mathbf{z}}\beta_z}{\omega\sqrt{\mu\varepsilon}} \qquad [p.97]$$

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\eta}\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r}) \qquad [2.61]$$

Reflection and refraction of oblique waves at planar dielectric interfaces

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_{p1}}{v_{p2}} = \sqrt{\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}} \qquad [3.19]$$

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\cos\theta_i - \sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}} \qquad [3.24]$$

$$T_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}} \qquad [3.25]$$

$$\Gamma_\parallel = \frac{E_{r0}}{E_{i0}} = \frac{-\eta_1\cos\theta_i + \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{-\cos\theta_i + \varepsilon_1/\varepsilon_2\sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}}{\cos\theta_i + \varepsilon_1/\varepsilon_2\sqrt{\varepsilon_2/\varepsilon_1 - \sin^2\theta_i}} \qquad [3.26]$$

$$T_\parallel = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t} = \frac{2\sqrt{\varepsilon_1/\varepsilon_2}\cos\theta_i}{\cos\theta_i + \sqrt{\varepsilon_1/\varepsilon_2}\sqrt{1 - \varepsilon_1/\varepsilon_2\sin^2\theta_i}} \qquad [3.27]$$

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Total internal reflection

$$\Gamma_\perp = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}} = 1e^{j\phi_\perp} \qquad [3.32]$$

$$\tan\frac{\phi_\perp}{2} = \frac{\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\cos\theta_i} \qquad [p.192]$$

$$\Gamma_\parallel = -\frac{\varepsilon_2/\varepsilon_1\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\varepsilon_2/\varepsilon_1\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}} = 1e^{j\phi_\parallel} \qquad [3.33]$$

$$\tan\frac{\phi_\parallel}{2} = \frac{\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\varepsilon_2/\varepsilon_1\cos\theta_i} \qquad [p.192]$$

Normal incidence on a lossy medium

$$\gamma_2 = j\omega\sqrt{\mu\varepsilon_{eff}} \approx j\omega\left(\frac{\mu_2\sigma_2}{j\omega}\right)^{1/2} = \left(j\omega\mu_2\sigma_2\right)^{1/2} = \frac{1+j}{\delta} \qquad [3.39]$$

$$\eta_c = Z_s = R_s + jX_s = \sqrt{\frac{\mu_2}{\varepsilon_{eff}}} \approx \left(\frac{j\omega\mu_2}{\sigma_2}\right)^{1/2} = \frac{\gamma_2}{\sigma_2} = \frac{1+j}{\sigma_2\delta} \qquad [3.40]$$

$$\Gamma_\parallel = -\frac{\varepsilon_2/\varepsilon_1\cos\theta_i + j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\varepsilon_2/\varepsilon_1\cos\theta_i - j\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}} = 1e^{j\phi_\parallel} \qquad [3.33]$$

$$\tan\frac{\phi_\parallel}{2} = \frac{\sqrt{\sin^2\theta_i - \varepsilon_2/\varepsilon_1}}{\varepsilon_2/\varepsilon_1\cos\theta_i} \qquad [p.192]$$

Parallel plate slab waveguide

Parallel-plate TE_m modes: m=0,±1,±2,…

$$E_y = C_1 \sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.12a]$$

$$H_z = -\frac{1}{j\omega\mu}\frac{\partial E_y}{\partial x} = -\frac{m\pi}{j\omega\mu a}C_1\cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.12b]$$

$$H_x = -\frac{\bar{\gamma}}{j\omega\mu}E_y = -\frac{\bar{\gamma}}{j\omega\mu}C_1\sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.12c]$$

Parallel-plate TM_m modes: m=0,±1,±2,…

$$H_y = C_4\cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.13a]$$

$$E_x = \frac{\bar{\gamma}}{j\omega\varepsilon}H_y = \frac{\bar{\gamma}}{j\omega\varepsilon}\cos\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.13b]$$

$$E_z = \frac{jm\pi}{\omega\varepsilon a}C_4\sin\left(\frac{m\pi}{a}x\right)e^{-\bar{\gamma}z} \qquad [4.13c]$$

Parallel-plate TEM mode

$$H_y = C_4 e^{-\bar{\gamma}z} \qquad [4.14a]$$

$$E_x = \frac{\bar{\gamma}}{j\omega\varepsilon}C_4 e^{-\bar{\gamma}z} \qquad [4.14b]$$

$$E_z = 0 \qquad [4.14c]$$

Propagation constants

$$f_{c_m} = \frac{m\upsilon_p}{2a} = \frac{m}{2a\sqrt{\mu\varepsilon}} \qquad [4.15]$$

$$\bar{\gamma} = j\bar{\beta}_m = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} = j\beta\sqrt{1-\left(\frac{f_{c_m}}{f}\right)^2},\ f > f_{c_m} \qquad [4.16]$$

$$\bar{\gamma} = \bar{\alpha}_m = j\sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon} = \beta\sqrt{\left(\frac{f_{c_m}}{f}\right)^2 - 1},\ f < f_{c_m} \qquad [4.14c]$$

$$\bar{\lambda}_m = \frac{2\pi}{\bar{\beta}_m} = \frac{\lambda}{\sqrt{1-\left(f_{c_m}/f\right)^2}} \qquad [4.18]$$

$$\bar{\upsilon}_{p_m} = \frac{\omega}{\bar{\beta}_m} = \frac{\bar{\upsilon}_p}{\sqrt{1-\left(f_{c_m}/f\right)^2}} \qquad [4.18]$$

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Conduction losses

$$\alpha_{c_{TEM}} = \frac{R_s}{\eta a} = \frac{1}{\eta a}\sqrt{\frac{\omega\mu_o}{2\sigma}} \qquad [4.22]$$

$$\alpha_{c_{TE_m}} = \frac{2R_s\left(\frac{f_{c_m}}{f}\right)^2}{\eta a\sqrt{1-\left(\frac{f_{c_m}}{f}\right)^2}} \qquad [4.23]$$

$$\alpha_{c_{TM_m}} = \frac{2R_s}{\eta a\sqrt{1-\left(\frac{f_{c_m}}{f}\right)^2}} \qquad [4.24]$$

Dielectric losses

$$\alpha_d = \frac{\omega\sqrt{\mu\varepsilon'}\,\varepsilon''/\varepsilon'}{2\sqrt{1-\left(\frac{\omega_{c_m}}{\omega}\right)^2}} \qquad [4.27]$$

For parallel plate TE modes the total power through the guide is

$$P_{av} = \int_0^b\int_0^a \frac{\bar{\beta}C_1^2}{2\omega\mu}\sin^2\left(\frac{m\pi}{a}x\right)dxdy = \frac{\bar{\beta}C_1^2 ab}{4\omega\mu} \qquad [p.277]$$

For the parallel plate TEM (TM_0) mode the total power through the guide is

$$P_{av} = \frac{1}{2}\eta C_4^2 ba \qquad [p.277]$$

Dielectric slab waveguide

TM Modes The non-zero field components are E_z, E_x, and H_y

For x≤-d/2

$$E_z^0(x) = C_o \sin(\beta_x x) + C_e \cos(\beta_x x) \quad [4.34]$$

where the transverse propagation constant is given by

$$\beta_x^2 = \omega^2 \mu_d \varepsilon_d - \bar{\beta}^2 = h_d^2 \quad [4.35]$$

For x≥-d/2

$$E_z^0(x) = \begin{cases} C_a e^{-\alpha_x (x - d/2)}, x \geq d/2 \\ C_a e^{\alpha_x (x + d/2)}, x \leq -d/2 \end{cases} \quad [4.36]$$

where the transverse attenuation constant is given by

$$\alpha_x^2 = \bar{\beta}^2 - \omega^2 \mu_0 \varepsilon_0 = -h_0^2 \text{ or } \alpha_x = \sqrt{\bar{\beta}_x^2 - \omega^2 \mu_0 \varepsilon_0} \quad [4.37]$$

The cutoff frequencies are given by

$$f_{C_{TMm}} = \frac{(m-1)}{2d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \begin{matrix} m = 1,3,5,\ldots odd \\ m = 2,4,6\ldots even \end{matrix} \quad [4.45]$$

TE Modes The non-zero field components are H_z, H_x, and E_y

For x≤-d/2

$$H_z^0(x) = C_o \sin(\beta_x x) + C_e \cos(\beta_x x) \quad [4.46]$$

where the transverse propagation constant

$$\beta_x^2 = \omega^2 \mu_d \varepsilon_d - \bar{\beta}^2 = h_d^2$$

For x≥-d/2

$$H_z^0(x) = \begin{cases} C_a e^{-\alpha_x (x - d/2)}, x \geq d/2 \\ C_a e^{\alpha_x (x + d/2)}, x \leq -d/2 \end{cases}$$

where the transverse propagation constant

$$\alpha_x^2 = \bar{\beta}^2 - \omega^2 \mu_0 \varepsilon_0 = -h_0^2 \text{ or } \alpha_x = \sqrt{\bar{\beta}_x^2 - \omega^2 \mu_0 \varepsilon_0} \quad [4.37]$$

For Odd TM Modes:

$$\frac{\alpha_x}{\beta_x} = \frac{\varepsilon_0}{\varepsilon_d} \tan\left(\frac{\beta_x d}{2}\right) \quad [4.40]$$

For Even TM Modes:

$$\frac{\alpha_x}{\beta_x} = -\frac{\varepsilon_0}{\varepsilon_d} \cot\left(\frac{\beta_x d}{2}\right) \quad [4.44]$$

For ALL TM Modes:

$$\alpha_x = \sqrt{\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - \beta_x^2} \quad [4.42]$$

The cutoff frequencies are given by

$$f_{C_{TEm}} = \frac{(m-1)}{2d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \begin{matrix} m = 1,3,5,\ldots odd \\ m = 2,4,6\ldots even \end{matrix} \quad [4.49]$$

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Dielectric covered ground plane

$$f_{C_{TEm}} = \frac{(m-1)}{2d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}} \begin{matrix} m = 1,3,5,\ldots odd\_TM_m \\ m = 2,4,6\ldots even\_TE_m \end{matrix} \quad [4.50]$$

Dielectric slab waveguide ray theory

$$\tan\theta_i = \frac{\bar{\beta}}{\beta_x} \quad [\text{p.306}]$$

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Detailed example of odd TM Modes for slab dielectric waveguide: free space above the guide

For x≥d/2

$$E_z^0(x) = C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x - d/2\right)} \quad [4.39a]$$

$$E_x^0(x) = -\frac{j\bar{\beta}}{\alpha_x} C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x - d/2\right)} \quad [4.39b]$$

$$H_y^0(x) = -\frac{j\omega\varepsilon_0}{\alpha_x} C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x - d/2\right)} \quad [4.39c]$$

For |x|≤d/2

$$E_z^0(x) = C_0 \sin(\beta_x x) \quad [4.39d]$$

$$E_x^0(x) = -\frac{j\bar{\beta}}{\beta_x} C_0 \cos(\beta_x x) \quad [4.39e]$$

$$H_y^0(x) = \frac{j\omega\varepsilon_d}{\beta_x} C_0 \cos(\beta_x x) \quad [4.39f]$$

For x≤-d/2

$$E_z^0(x) = C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x + d/2\right)} \quad [4.39g]$$

$$E_x^0(x) = -\frac{j\bar{\beta}}{\alpha_x} C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x + d/2\right)} \quad [4.39h]$$

$$H_y^0(x) = -\frac{j\omega\varepsilon_0}{\alpha_x} C_0 \sin\left(\frac{\beta_x d}{2}\right) e^{-\alpha_x\left(x + d/2\right)} \quad [4.39i]$$