in problems which include the axis $r = 0$ in the range over which the solution is to apply. The solution to the z equation (7) when $T^2 = -\tau^2$ is

$$Z = C_3' \sin \tau z + C_4' \cos \tau z \tag{18}$$

Summarizing, either of the following forms satisfies Laplace's equation in the two cylindrical coordinates r and z:

$$\Phi(r, z) = [C_1 J_0(Tr) + C_2 N_0(Tr)][C_3 \sinh Tz + C_4 \cosh Tz] \tag{19}$$

$$\Phi(r, z) = [C_1' I_0(\tau r) + C_2' K_0(\tau r)][C_3' \sin \tau z + C_4' \cos \tau z] \tag{20}$$

As was the case with the rectangular harmonics, the two forms are not really different since (19) includes (20) if T is allowed to become imaginary, but the two separate ways of writing the solution are useful, as will be demonstrated in later examples. The case with no assumed symmetries is discussed in the following section.

7.14 BESSEL FUNCTIONS

In Sec. 7.13 an example of a Bessel function was shown as a solution of the differential equation 7.13(8) which describes the radial variations in Laplace's equation for axially symmetric fields where a product solution is assumed. This is just one of a whole family of functions which are solutions of the general Bessel differential equation.

Bessel Functions with Real Arguments For certain problems, as, for example, the solution for field between the two halves of a longitudinally split cylinder, it may be necessary to retain the ϕ variations in the equation. The solution may be assumed in product form again, $RF_\phi Z$, where R is a function of r alone, F_ϕ of ϕ alone, and Z of z alone, Z has solutions in hyperbolic functions as before, and F_ϕ may also be satisfied by sinusoids:

$$Z = C \cosh Tz + D \sinh Tz \tag{1}$$

$$F_\phi = E \cos \nu\phi + F \sin \nu\phi \tag{2}$$

The differential equation for R is then slightly different from the zero-order Bessel equation obtained previously:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(T^2 - \frac{\nu^2}{r^2}\right)R = 0 \tag{3}$$

It is apparent at once that Eq. 7.13(8) is a special case of this more general equation, that is, $\nu = 0$. A series solution to the general equation carried through as in Sec. 7.13 shows that the function defined by the series

$$J_\nu(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{\nu+2m}}{m!\Gamma(\nu + m + 1)} \tag{4}$$

is a solution to the equation.

$\Gamma(\nu + m + 1)$ is the gamma function of $(\nu + m + 1)$ and, for ν integral, is equivalent to the factorial of $(\nu + m)$. Also for ν nonintegral, values of this gamma function are

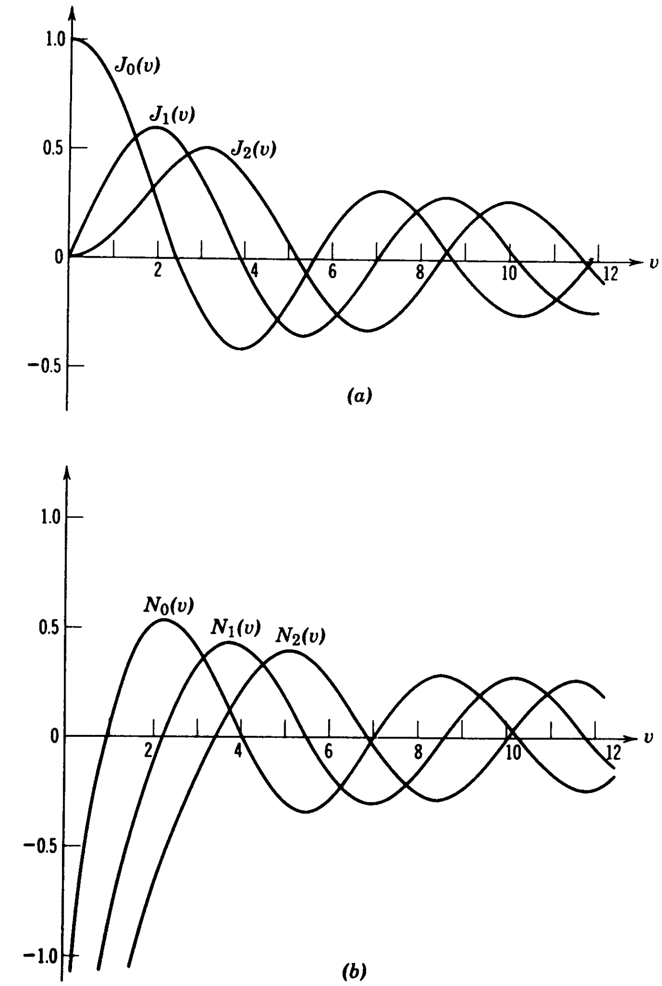

FIG. 7.14 (*a*) Bessel functions of the first kind. (*b*) Bessel functions of the second kind.

tabulated. If ν is an integer n,

$$J_n(Tr) = \sum_{m=0}^{\infty} \frac{(-1)^m (Tr/2)^{n+2m}}{m!(n+m)!} \tag{5}$$

It can be shown that $J_{-n} = (-1)^n J_n$. A few of these functions are plotted in Fig. 7.14*a*. Similarly, a second independent solution[13] to the equation is

$$N_\nu(Tr) = \frac{\cos \nu\pi J_\nu(Tr) - J_{-\nu}(Tr)}{\sin \nu\pi} \tag{6}$$

[13] *If ν is nonintegral, $J_{-\nu}$ is not linearly related to J_ν, and it is then proper to use either $J_{-\nu}$ or N_ν as the second solution; for ν integral, N_ν must be used. Equation (6) is indeterminate for ν integral but is subject to evaluation by usual methods.*

and $N_{-n} = (-1)^n N_n$. As may be noted in Fig. 7.14*b* these are infinite at the origin. A complete solution to (3) may be written

$$R = AJ_\nu(Tr) + BN_\nu(Tr) \tag{7}$$

The constant ν is known as the order of the equation. J_ν is then called a Bessel function of first kind, order ν; N_ν is a Bessel function of second kind, order ν. Of most interest for this chapter are cases in which $\nu = n$, an integer.

It is useful to keep in mind that, in the physical problem considered here, ν is the number of radians of the sinusoidal variation of the potential per radian of angle about the axis.

The functions $J_\nu(v)$ and $N_\nu(v)$ are tabulated in the references.[14,15] Some care should be observed in using these references, for there is a wide variation in notation for the second solution, and not all the functions used are equivalent, since they differ in the values of arbitrary constants selected for the series. The $N_\nu(v)$ is chosen here because it is the form most common in current mathematical physics and also the form most commonly tabulated. Of course, it is quite proper to use any one of the second solutions throughout a given problem, since all the differences will be absorbed in the arbitrary constants of the problem, and the same final numerical result will be obtained; but it is necessary to be consistent in the use of only one of these throughout any given analysis.

It is of interest to observe the similarity between (3) and the simple harmonic equation, the solutions of which are sinusoids. The difference between these two differential equations lies in the term $(1/r)(dR/dr)$ which produces its major effect as $r \to 0$. Note that for regions far removed from the axis as, for example, near the outer edge of Fig. 1.19*a*, the region bounded by surfaces of a cylindrical coordinate system approximates a cube. For these reasons, it may be expected that, away from the origin, the Bessel functions are similar to sinusoids. That this is true may be seen in Figs. 7.14*a* and *b*. For large values of the arguments, the Bessel functions approach sinusoids with magnitude decreasing as the square root of radius, as will be seen in the asymptotic forms, Eqs. 7.15(1) and 7.15(2).

Hankel Functions It is sometimes convenient to take solutions to the simple harmonic equation in the form of complex exponentials rather than sinusoids. That is, the solution of

$$\frac{d^2Z}{dz^2} + K^2 Z = 0 \tag{8}$$

can be written as

$$Z = Ae^{+jKz} + Be^{-jKz} \tag{9}$$

[14] *E. Jahnke, F. Emde, and F. Lösch,* Tables of Higher Functions, *6th ed. revised by F. Lösch, McGraw-Hill, New York, 1960.*

[15] *M. Abramowitz and I. A. Stegun (Eds.),* Handbook of Mathematical Functions, *Dover, New York, 1964.*

where

$$e^{\pm jKz} = \cos Kz \pm j \sin Kz \tag{10}$$

Since the complex exponentials are linear combinations of cosine and sine functions, we may also write the general solution of (8) as

$$Z = A'e^{jKz} + B' \sin Kz$$

or other combinations.

Similarly, it is convenient to define new Bessel functions which are linear combinations of the $J_\nu(Tr)$ and $N_\nu(Tr)$ functions. By direct analogy with the definition (10) of the complex exponential, we write

$$H_\nu^{(1)}(Tr) = J_\nu(Tr) + jN_\nu(Tr) \tag{11}$$

$$H_\nu^{(2)}(Tr) = J_\nu(Tr) - jN_\nu(Tr) \tag{12}$$

These are called Hankel functions of the first and second kinds, respectively. Since they both contain the function $N_\nu(Tr)$, they are both singular at $r = 0$. Negative and positive orders are related by

$$H_{-\nu}^{(1)}(Tr) = e^{j\pi\nu}H_\nu^{(1)}(Tr)$$

$$H_{-\nu}^{(2)}(Tr) = e^{-j\pi\nu}H_\nu^{(2)}(Tr)$$

For large values of the argument, these can be approximated by complex exponentials, with magnitude decreasing as square root of radius. For example,

$$\underset{Tr\to\infty}{H_\nu^{(1)}(Tr)} = \sqrt{\frac{2}{\pi Tr}}\, e^{j(Tr - \pi/4 - \nu\pi/2)}$$

This asymptotic form suggests that Hankel functions may be useful in wave propagation problems as the complex exponential is in plane-wave propagation. It is also sometimes convenient to use Hankel functions as alternate independent solutions in static problems. Complete solutions of (3) may be written in a variety of ways using combinations of Bessel and Hankel functions.

Bessel and Hankel Functions of Imaginary Arguments If T is imaginary, $T = j\tau$, and (3) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\tau^2 + \frac{\nu^2}{r^2}\right)R = 0 \tag{13}$$

The solution to (3) is valid here if T is replaced by $j\tau$ in the definitions of $J_\nu(Tr)$ and $N_\nu(Tr)$. In this case $N_\nu(j\tau r)$ is complex and so requires two numbers for each value of the argument, whereas $j^{-\nu}J_\nu(j\tau r)$ is always a purely real number. It is convenient to replace $N_\nu(j\tau r)$ by a Hankel function. The quantity $j^{\nu-1}H_\nu^{(1)}(j\tau r)$ is also purely real and so requires tabulation of only one value for each value of the argument. If ν is not an integer, $j^\nu J_{-\nu}(j\tau r)$ is independent of $j^{-\nu}J_\nu(j\tau r)$ and may be used as a second solution.

Thus, for nonintegral ν two possible complete solutions are

$$R = A_2 J_\nu(j\tau r) + B_2 J_{-\nu}(j\tau r) \tag{14}$$

and

$$R = A_3 J_\nu(j\tau r) + B_3 H_\nu^{(1)}(j\tau r) \tag{15}$$

where powers of j are included in the constants. For $\nu = n$, an integer, the two solutions in (14) are not independent but (15) is still a valid solution.

It is common practice to denote these solutions as

$$I_{\pm\nu}(v) = j^{\mp\nu} J_{\pm\nu}(jv) \tag{16}$$

$$K_\nu(v) = \frac{\pi}{2} j^{\nu+1} H_\nu^{(1)}(jv) \tag{17}$$

where $v = \tau r$.

As is noted in Sec. 7.15 some of the formulas relating Bessel functions and Hankel functions must be changed for these modified Bessel functions. Special cases of these functions were seen as $I_0(\tau r)$ and $K_0(\tau r)$ in Sec. 7.13 for the axially symmetric field. The forms of $I_\nu(\tau r)$ and $K_\nu(\tau r)$ for $\nu = 0, 1$ are shown in Fig. 7.14c. As is suggested by these curves, the asymptotic forms of the modified Bessel functions are related to growing and decaying real exponentials, as will be seen in Eqs. 7.15(5) and 7.15(6). It is also clear from the figure that $K_\nu(\tau r)$ is singular at the origin.

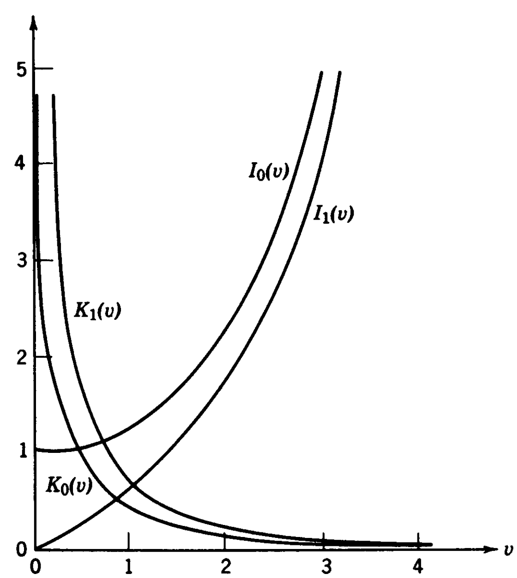

FIG. 7.14c Modified Bessel functions.

7.15 BESSEL FUNCTION ZEROS AND FORMULAS[16]

The first several zeros of the low-order Bessel functions and of the derivatives of Bessel functions are given in Tables 7.15a and 7.15b, respectively.

Table 7.15a
Zeros of Bessel Functions

J_0	J_1	J_2	N_0	N_1	N_2
2.405	3.832	5.136	0.894	2.197	3.384
5.520	7.016	8.417	3.958	5.430	6.794
8.654	10.173	11.620	7.086	8.596	10.023

Table 7.15b
Zeros of Derivatives of Bessel Functions

J_0'	J_1'	J_2'	N_0'	N_1'	N_2'
0.000	1.841	3.054	2.197	3.683	5.003
3.832	5.331	6.706	5.430	6.942	8.351
10.173	8.536	9.969	8.596	10.123	11.574

Asymptotic Forms

$$J_\nu(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}} \cos\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{1}$$

$$N_\nu(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}} \sin\left(v - \frac{\pi}{4} - \frac{\nu\pi}{2}\right) \tag{2}$$

$$H_\nu^{(1)}(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}}\, e^{j[v-(\pi/4)-(\nu\pi/2)]} \tag{3}$$

$$H_\nu^{(2)}(v) \underset{v\to\infty}{\to} \sqrt{\frac{2}{\pi v}}\, e^{-j[v-(\pi/4)-(\nu\pi/2)]} \tag{4}$$

$$j^{-\nu} \underset{v\to\infty}{J_\nu(jv)} = \underset{v\to\infty}{I_\nu(v)} \to \sqrt{\frac{1}{2\pi v}}\, e^{v} \tag{5}$$

$$j^{\nu+1} \underset{v\to\infty}{H_\nu^{(1)}(jv)} = \frac{2}{\pi} \underset{v\to\infty}{K_\nu(v)} \to \sqrt{\frac{2}{\pi v}}\, e^{-v} \tag{6}$$

[16] *More extensive tabulations are found in the sources given in footnotes 14 and 15.*

Derivatives The following formulas which may be found by differentiating the appropriate series, term by term, are valid for any of the functions $J_\nu(v)$, $N_\nu(v)$, $H_\nu^{(1)}(v)$, and $H_\nu^{(2)}(v)$. Let $R_\nu(v)$ denote any one of these, and R_ν' denote $(d/dv)[R_\nu(v)]$.

$$R_0' = -R_1(v) \tag{7}$$

$$R_1'(v) = R_0(v) - \frac{1}{v} R_1(v) \tag{8}$$

$$vR_\nu'(v) = \nu R_\nu(v) - vR_{\nu+1}(v) \tag{9}$$

$$vR_\nu'(v) = -\nu R_\nu(v) + vR_{\nu-1}(v) \tag{10}$$

$$\frac{d}{dv}[v^{-\nu}R_\nu(v)] = -v^{-\nu}R_{\nu+1}(v) \tag{11}$$

$$\frac{d}{dv}[v^{\nu}R_\nu(v)] = v^{\nu}R_{\nu-1}(v) \tag{12}$$

Note that

$$R_\nu'(Tr) = \frac{d}{d(Tr)}[R_\nu(Tr)] = \frac{1}{T}\frac{d}{dr}[R_\nu(Tr)] \tag{13}$$

For the I and K functions, different forms for the foregoing derivatives must be used. They may be obtained from these formulas by substituting Eqs. 7.14(16) and 7.14(17) in the preceding expressions. Some of these are

$$\begin{aligned} vI_\nu'(v) &= \nu I_\nu(v) + vI_{\nu+1}(v) \\ vI_\nu'(v) &= -\nu I_\nu(v) + vI_{\nu-1}(v) \end{aligned} \tag{14}$$

$$\begin{aligned} vK_\nu'(v) &= \nu K_\nu(v) - vK_{\nu+1}(v) \\ vK_\nu'(v) &= -\nu K_\nu(v) - vK_{\nu-1}(v) \end{aligned} \tag{15}$$

Recurrence Formulas By recurrence formulas, it is possible to obtain the values for Bessel functions of any order, when the values of functions for any two other orders, differing from the first by integers, are known. For example, subtract (10) from (9). The result may be written

$$\frac{2\nu}{v} R_\nu(v) = R_{\nu+1}(v) + R_{\nu-1}(v) \tag{16}$$

As before, R_ν may denote J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$, but not I_ν or K_ν. For these, the recurrence formulas are

$$\frac{2\nu}{v} I_\nu(v) = I_{\nu-1}(v) - I_{\nu+1}(v) \tag{17}$$

$$\frac{2\nu}{v} K_\nu(v) = K_{\nu+1}(v) - K_{\nu-1}(v) \tag{18}$$

Integrals Integrals that will be useful in solving later problems are given below. R_ν denotes J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$:

$$\int v^{-\nu} R_{\nu+1}(v)\, dv = -v^{-\nu} R_\nu(v) \tag{19}$$

$$\int v^{\nu} R_{\nu-1}(v)\, dv = v^{\nu} R_\nu(v) \tag{20}$$

$$\begin{aligned}\int vR_\nu(\alpha v)R_\nu(\beta v)\, dv &= \frac{v}{\alpha^2 - \beta^2} \\ &\times [\beta R_\nu(\alpha v)R_{\nu-1}(\beta v) - \alpha R_{\nu-1}(\alpha v)R_\nu(\beta v)],\ \alpha \neq \beta\end{aligned} \tag{21}$$

$$\begin{aligned}\int vR_\nu^2(\alpha v)\, dv &= \frac{v^2}{2}[R_\nu^2(\alpha v) - R_{\nu-1}(\alpha v)R_{\nu+1}(\alpha v)] \\ &= \frac{v^2}{2}\left[R_\nu'^2(\alpha v) + \left(1 - \frac{\nu^2}{\alpha^2 v^2}\right)R_\nu^2(\alpha v)\right]\end{aligned} \tag{22}$$

7.16 Expansion of a Function as a Series of Bessel Functions

In Sec. 7.11 a study was made of the method of Fourier series by which a function may be expressed over a given region as a series of sines or cosines. It is possible to evaluate the coefficients in such a case because of the orthogonality property of sinusoids. A study of the integrals, Eqs. 7.15(21) and 7.15(22), shows that there are similar orthogonality expressions for Bessel functions. For example, these integrals may be written for zero-order Bessel functions, and if α and β are taken as p_m/a and p_q/a, where p_m and p_q are the mth and qth roots of $J_0(v) = 0$, that is, $J_0(p_m) = 0$ and $J_0(p_q) = 0$, $p_m \neq p_q$, then Eq. 7.15(21) gives

$$\int_0^a rJ_0\left(\frac{p_m r}{a}\right)J_0\left(\frac{p_q r}{a}\right) dr = 0 \tag{1}$$

So, a function $f(r)$ may be expressed as an infinite sum of zero-order Bessel functions

$$f(r) = b_1 J_0\left(p_1 \frac{r}{a}\right) + b_2 J_0\left(p_2 \frac{r}{a}\right) + b_3 J_0\left(p_3 \frac{r}{a}\right) + \cdots$$

or

$$f(r) = \sum_{m=1}^{\infty} b_m J_0\left(\frac{p_m r}{a}\right) \tag{2}$$

The coefficients b_m may be evaluated in a manner similar to that used for Fourier coefficients by multiplying each term of (2) by $rJ_0(p_m r/a)$ and integrating from 0 to

a. Then by (1) all terms on the right disappear except the mth term:

$$\int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr = \int_0^a b_m r\left[J_0\left(\frac{p_m r}{a}\right)\right]^2 dr$$

From Eq. 7.15(22),

$$\int_0^a b_m r J_0^2\left(\frac{p_m r}{a}\right) dr = \frac{a^2}{2} b_m J_1^2(p_m) \tag{3}$$

or

$$b_m = \frac{2}{a^2 J_1^2(p_m)} \int_0^a rf(r)J_0\left(\frac{p_m r}{a}\right) dr \tag{4}$$

In the above, as in the Fourier series, the orthogonality relations enabled us to obtain coefficients of the series under the assumption that the series is a proper representation of the function to be expanded, but two additional points are required to show that the representation is valid. The series must of course converge, and the set of orthogonal functions must be *complete*, that is, sufficient to represent an arbitrary function over the interval of concern. These points have been shown for the Bessel series of (2) and for other orthogonal sets of functions to be used in this text.[17]

Expansions similar to (2) can be made with Bessel functions of other orders and types (Prob. 7.16a).

Example 7.16
BESSEL FUNCTION EXPANSION FOR CONSTANT IN RANGE 0 < r < A

If the function $f(r)$ in (4) is a constant V_0 in the range $0 < r < a$, we have

$$b_m = \frac{2V_0}{a^2 J_1^2(p_m)} \int_0^a rJ_0\left(\frac{p_m r}{a}\right) dr \tag{5}$$

Using Eq. 7.15(20) with $R = J$, $\nu = 1$, and $v = p_m r/a$, the integral in (5) becomes

$$\left(\frac{a}{p_m}\right)^2 \int_0^a \left(\frac{p_m r}{a}\right) J_0\left(\frac{p_m r}{a}\right) d\left(\frac{p_m r}{a}\right) = \left[\left(\frac{a}{p_m}\right)^2 \left(\frac{p_m r}{a}\right) J_1\left(\frac{p_m r}{a}\right)\right]_0^a$$
$$= \frac{a^2}{p_m} J_1(p_m) \tag{6}$$

[17] *See, for example, E. T. Whittaker and G. N. Watson,* A Course in Modern Analysis, *4th ed., pp. 374–378, University Press, Cambridge, 1927.*

and the series expansion (2) for the constant V_0 is

$$f(r) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m)} J_0\left(\frac{p_m r}{a}\right) \tag{7}$$

or, using the values of the zeros of J_0 in Table 7.15a,

$$\begin{aligned} f(r) = {} & \frac{0.832V_0}{J_1(2.405)} J_0\left(\frac{2.405r}{a}\right) + \frac{0.362V_0}{J_1(5.520)} J_0\left(\frac{5.520r}{a}\right) \\ & + \frac{0.231V_0}{J_1(8.654)} J_0\left(\frac{8.654r}{a}\right) + \cdots \end{aligned} \tag{8}$$

Further evaluation of (8) requires reference to tables in the sources given in footnotes 14 and 15 or numerical evaluation of Eq. 7.13(11).

7.17 Fields Described by Cylindrical Harmonics

We will consider here the two basic types of boundary value problems which exist in axially symmetric cylindrical systems. These can be understood by reference to Fig. 7.17*a*. In one type both Φ_1 and Φ_2, the potentials on the ends, are zero and a nonzero potential Φ_3 is applied to the cylindrical surface. In the second type $\Phi_3 = 0$ and either (or both) Φ_1 or Φ_2 are nonzero. The gaps between ends and side are considered negligibly small. For simplicity, the nonzero potentials will be taken to be independent of the coordinate along the surface. In the first type, a Fourier series of sinusoids is used to expand the boundary potentials as was done in the rectangular problems. In the second situation, a series of Bessel functions is used to expand the boundary potential along the radial coordinate.

Nonzero Potential on Cylindrical Surface Since the boundary potentials are axially symmetric, zero-order Bessel functions should be used. The repeated zeros along the z coordinate dictate the use of sinusoidal functions of z. The potential in Eq. 7.13(20) is the appropriate form. Certain of the constants can be evaluated immediately. Since $K_0(\tau r)$ is singular on the axis, C_2' must be identically zero to give a finite potential there. The cos τz equals unity at $z = 0$ but the potential must be zero there so $C_4' = 0$. As in the problem discussed in Sec. 7.10 the repeated zeros at $z = l$ require that $\tau = m\pi/l$. Therefore the general harmonic which fits all boundary conditions except $\Phi = V_0$ at $r = a$ is

$$\Phi_m = A_m I_0\left(\frac{m\pi r}{l}\right) \sin\left(\frac{m\pi z}{l}\right) \tag{1}$$

Figure 7.17*b* shows a sketch of this harmonic for $m = 1$ and with the nonzero boundary potential on the cylinder. It is clear that we have here the problem of expanding the

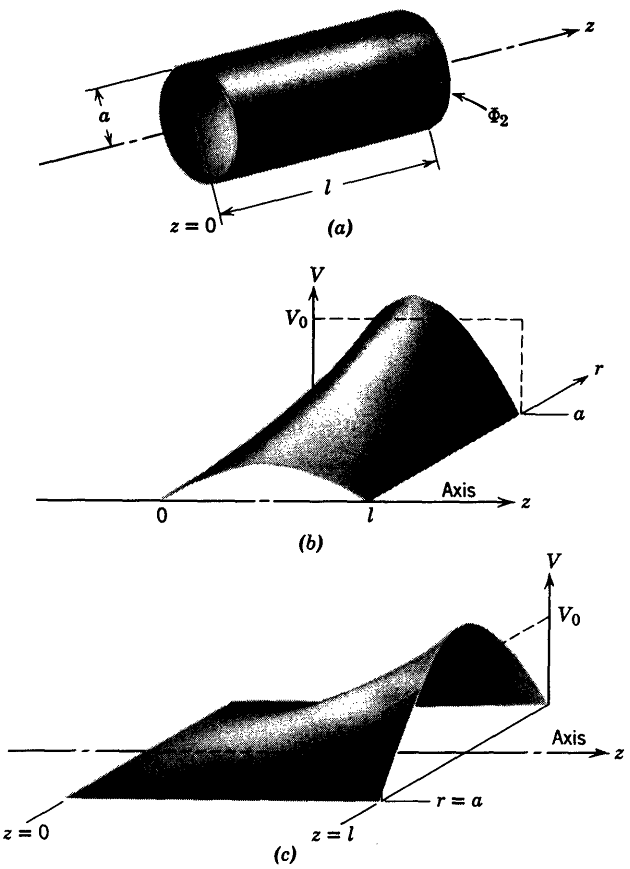

Fig. 7.17 (*a*) Cylinder with conducting boundaries. (*b*) One harmonic component for matching boundary conditions when nonzero potential is applied to cylindrical surface in (*a*). (*c*) One harmonic component for matching boundary conditions when nonzero potential is applied to end surface in (*a*).

boundary potential in sinusoids just as in the rectangular problem of Sec. 7.12. Following the procedure used there we obtain

$$\Phi(r, z) = \sum_{m \text{ odd}} \frac{4V_0}{m\pi} \frac{I_0(m\pi r/l)}{I_0(m\pi a/l)} \sin \frac{m\pi z}{l} \tag{2}$$

Nonzero Potential on End In this situation if we refer to Fig. 7.17*a*, we see that $\Phi_1 = \Phi_3 = 0$ and $\Phi_0 = V_0$. In selecting the proper form for the solution from Sec. 7.13, the boundary condition that $\Phi = 0$ at $r = a$ for all values of z indicates that the R function must become zero at $r = a$. Thus, we select the J_0 functions since the I_0's do not ever become zero. (The corresponding second solution, N_0, does not appear since potential must remain finite on the axis.) The value of T in Eq. 7.13(19) is determined from the condition that $\Phi = 0$ at $r = a$ for all values of z. Thus, if p_m is the mth root of $J_0(v) = 0$, T must be p_m/a. The corresponding solution for Z is in hyperbolic functions. The coefficient of the hyperbolic cosine term must be zero since Φ is zero at $z = 0$ for all values of r. Thus, a sum of all cylindrical harmonics with arbitrary amplitudes which satisfy the symmetry of the problem and the boundary conditions so far imposed may be written

$$\Phi(r, z) = \sum_{m=1}^{\infty} B_m J_0\left(\frac{p_m r}{a}\right) \sinh\left(\frac{p_m z}{a}\right) \tag{3}$$

One of the harmonics and the required boundary potentials are shown in Fig. 7.17*c*.

The remaining condition is that, at $z = l$, $\Phi = 0$ at $r = a$ and $\Phi = V_0$ at $r < a$. Here we can use the general technique of expanding the boundary potential in a series of the same form as that used for the potentials inside the region, as regards the dependence on the coordinate along the boundary. In Ex. 7.16 we expanded a constant over the range $0 < r < a$ in J_0 functions so that result can be used here to evaluate the constants in (3). Evaluating (3) at the boundary $z = l$, we have

$$\Phi(r, l) = \sum_{m=1}^{\infty} B_m \sinh\left(\frac{p_m l}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{4}$$

Equations (4) and 7.16(7) must be equivalent for all values of r. Consequently, coefficients of corresponding terms of $J_0(p_m r/a)$ must be equal. The constant B_m is now completely determined, and the potential at any point inside the region is

$$\Phi(r, z) = \sum_{m=1}^{\infty} \frac{2V_0}{p_m J_1(p_m) \sinh(p_m l/a)} \sinh\left(\frac{p_m z}{a}\right) J_0\left(\frac{p_m r}{a}\right) \tag{5}$$

7.18 Spherical Harmonics

Consider next Laplace's equation in spherical coordinates for regions with symmetry about the axis so that variations with azimuthal angle ϕ may be neglected. Laplace's equation in the two remaining spherical coordinates r and θ then becomes (obtainable from form of inside front cover)

$$\frac{\partial^2(r\Phi)}{\partial r^2} + \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) = 0 \tag{1}$$

or

$$r \frac{\partial^2\Phi}{\partial r^2} + 2 \frac{\partial\Phi}{\partial r} + \frac{1}{r} \frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r \tan\theta} \frac{\partial\Phi}{\partial\theta} = 0 \tag{2}$$

Assume a product solution

$$\Phi = R\Theta$$

where R is a function of r alone, and Θ of θ alone:

$$rR''\Theta + 2R'\Theta + \frac{1}{r}R\Theta'' + \frac{1}{r\tan\theta}R\Theta' = 0$$

and

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = -\frac{\Theta''}{\Theta} - \frac{\Theta'}{\Theta\tan\theta} \tag{3}$$

From the previous logic, if the two sides of the equations are to be equal to each other for all values of r and θ, both sides can be equal only to a constant. Since the constant may be expressed in any nonrestrictive way, let it be $m(m + 1)$. The two resulting ordinary differential equations are then

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - m(m + 1)R = 0 \tag{4}$$

$$\frac{d^2\Theta}{d\theta^2} + \frac{1}{\tan\theta}\frac{d\Theta}{d\theta} + m(m + 1)\Theta = 0 \tag{5}$$

Equation (4) has a solution which is easily verified to be

$$R = C_1r^m + C_2r^{-(m+1)} \tag{6}$$

A solution to (5) in terms of simple functions is not obvious, so, as with the Bessel equation, a series solution may be assumed. The coefficients of this series must be determined so that the differential equation (5) is satisfied and the resulting series made to define a new function. There is one departure here from an exact analog with the Bessel functions, for it turns out that a proper selection of the arbitrary constants will make the series for the new function terminate in a finite number of terms if m is an integer. Thus, for any integer m, the polynomial defined by

$$P_m(\cos\theta) = \frac{1}{2^m m!}\left[\frac{d}{d(\cos\theta)}\right]^m(\cos^2\theta - 1)^m \tag{7}$$

is a solution to the differential equation (5). The equation is known as Legendre's equation; the solutions are called Legendre polynomials of order m. Their forms for the first few values of m are tabulated below and are shown in Fig. 7.18a. Since they are polynomials and not infinite series, their values can be calculated easily if desired, but values of the polynomials are also tabulated in many references.

$$\begin{aligned}
P_0(\cos\theta) &= 1\\
P_1(\cos\theta) &= \cos\theta\\
P_2(\cos\theta) &= \tfrac{1}{2}(3\cos^2\theta - 1)\\
P_3(\cos\theta) &= \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)\\
P_4(\cos\theta) &= \tfrac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)\\
P_5(\cos\theta) &= \tfrac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)
\end{aligned} \tag{8}$$

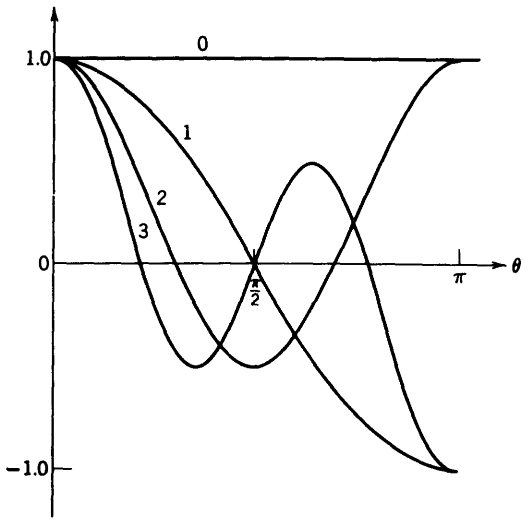

FIG. 7.18a Legendre polynomials.

It is recognized that $\Theta = C_1 P_m(\cos\theta)$ is only one solution to the second-order differential equation (5). There must be a second independent solution, which may be obtained from the first in the same manner as for Bessel functions, but it turns out that this solution becomes infinite for $\theta = 0$. Consequently it is not needed when the axis of spherical coordinates is included in the region over which the solution applies. When the axis is excluded, the second solution must be included. It is typically denoted $Q_n(\cos\theta)$ and tabulated in the references.[18]

An orthogonality relation for Legendre polynomials is quite similar to those for sinusoids and Bessel functions which led to the Fourier series and expansion in Bessel functions, respectively.

$$\int_0^\pi P_m(\cos\theta)P_n(\cos\theta)\sin\theta\,d\theta = 0, \qquad m \neq n \tag{9}$$

$$\int_0^\pi [P_m(\cos\theta)]^2 \sin\theta\,d\theta = \frac{2}{2m+1} \tag{10}$$

It follows that, if a function $f(\theta)$ defined between the limits of 0 to π is written as a series of Legendre polynomials,

$$f(\theta) = \sum_{m=0}^{\infty} \alpha_m P_m(\cos\theta), \qquad 0 < \theta < \pi \tag{11}$$

[18] *W. R. Smythe,* Static and Dynamic Electricity, *3rd ed., Hemisphere Publishing Co., Washington, DC, 1989.*

the coefficients must be given by the formula

$$\alpha_m = \frac{2m + 1}{2} \int_0^\pi f(\theta) P_m(\cos\theta) \sin\theta \, d\theta \qquad (12)$$

Example 7.18a
HIGH-PERMEABILITY SPHERE IN UNIFORM FIELD

We will examine the field distribution in and around a sphere of permeability $\mu \neq \mu_0$ when it is placed in an otherwise uniform magnetic field in free space. The uniform field is disturbed by the sphere as indicated in Fig. 7.18*b*. The reason for choosing this example is threefold. It shows, first, an application of spherical harmonics. Second, it is an example of a situation in which the constants in series solutions for two regions are evaluated by matching across a boundary. Finally, it is an example of a magnetic boundary-value problem.

Since there are no currents in the region to be studied, we may use the scalar magnetic potential introduced in Sec. 2.13. The magnetic intensity is given by

$$\mathbf{H} = -\nabla \Phi_m \qquad (13)$$

As the problem is axially symmetric and the axis is included in the region of interest, the solutions $P_m(\cos\theta)$ are applicable. The series solutions with these restrictions are

$$\Phi_m(r, \theta) = \sum_m P_m(\cos\theta)[C_{1m} r^m + C_{2m} r^{-(m+1)}] \qquad (14)$$

The procedure is to write general forms for the potential inside and outside the sphere and match these across the boundary. Since the potential must remain finite at $r = 0$,

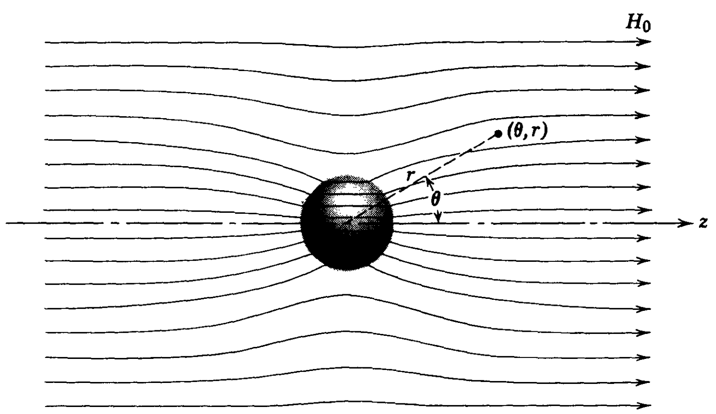

FIG. 7.18*b* Sphere of magnetic material in an otherwise uniform magnetic field.