Power Flow on a transmission line

You usually want to maximize the time-average power delivered to the load.

You can calculate time average power as

$$P_{av}(z) = \frac{1}{T}\int_0^{T_p} \mathcal{V}(z,t)\,\mathcal{I}(z,t)\,dt$$

where $T_p = \frac{2\pi}{\omega}$ is the period of the signal.

In terms of phasors $\quad P_{AV}(z) = \frac{1}{2}\operatorname{Re}\{V(z)I^*(z)\}$

We can re-write this in terms of transmission line parameters as

$$V(z) = V^+e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z}$$

$$I(z) = \underbrace{\frac{V^+}{Z_0}e^{-j\beta z}}_{\substack{+z \text{ traveling wave} \\ P^+}} - \underbrace{\Gamma_L \frac{V^+}{Z_0}e^{+j\beta z}}_{\substack{-z \text{ traveling wave} \\ P^-}}$$

$$P^+ = \frac{1}{2}\operatorname{Re}\left\{\frac{V^+e^{-j\beta z}(V^+e^{-j\beta z})^*}{Z_0}\right\} = \frac{V^+(V^+)^*}{2Z_0} = \frac{|V^+|^2}{2Z_0}$$

$$P^- = \frac{1}{2}\operatorname{Re}\left\{\frac{\Gamma_L V^+e^{+j\beta z}(-\Gamma_L V^+e^{+j\beta z})^*}{Z_0}\right\} = -\frac{\Gamma_L\Gamma_L^* V^+(V^+)^*}{2Z_0} = -\rho^2\frac{|V^+|^2}{2Z_0}$$

$$\Gamma_L\Gamma_L^* = \rho^2$$

power is going in opposite direction to $V(z)$ and $I(z)$

Net power in forward direction

$$P_{AV} = P^+ + P^- = \frac{|V^+|^2}{2Z_0} - \rho^2\frac{|V^+|^2}{2Z_0} = (1-\rho^2)\frac{|V^+|^2}{2Z_0}$$

$\Rightarrow$ you maximize power flow by minimizing ρ.

i.e. $Z_L = Z_0$

$\Gamma_L = 0$

$S = 1$

Consider what happens if the load is NOT matched to the load.

$$\frac{P_L}{P^+} \qquad \frac{\text{power dissipated in load}}{\text{power delivered to matched load}}$$

$$P_L = \frac{1}{2}\,\mathrm{Re}\left\{V_L I_L^*\right\} = \frac{1}{2}\,\mathrm{Re}\left\{V^+(1+\Gamma_L)\cdot\left[\frac{V^+(1-\Gamma_L)}{Z_0}\right]^*\right\}$$

$$= \frac{1}{2}\,\mathrm{Re}\left\{\frac{V^+(V^+)^*(1+\Gamma_L)(1-\Gamma_L^*)}{Z_0}\right\}$$

gives twice imaginary part

$$= \frac{1}{2}\,\mathrm{Re}\left\{|V^+|^2\frac{\left[1+(\Gamma_L-\Gamma_L^*)-\Gamma_L\Gamma_L^*\right]}{Z_0}\right\}$$

$$= \frac{1}{2}\frac{|V^+|^2}{Z_0}(1-\rho^2)$$

$$\therefore \frac{P_L}{P^+} = \frac{\frac{1}{2}\frac{|V^+|^2}{Z_0}(1-\rho^2)}{\frac{1}{2}\frac{|V^+|^2}{Z_0}} = 1-\rho^2$$

$$\frac{P_L}{P^+} = 1-\rho^2 = 1-\left(\frac{S-1}{S+1}\right)^2 = \frac{(S+1)^2-(S-1)^2}{(S+1)^2} = \frac{(S^2+2S+1)-(S^2-2S+1)}{(S+1)^2}$$

$$\frac{P_L}{P^+} = \frac{4S}{(S+1)^2}$$ this is a maximum when $S=1$, i.e. matched.

Example. 3-12

A VHF transmitter operating at 125 MHz and developing $V_0 = 100\,e^{j0^\circ}$ volts with a source resistance of $R_s = 50\Omega$ feeds an antenna with a feed-point impedance of $Z_L = 100 - j60$ through a 50Ω, polyethylene-filled coaxial line that is 17m long.

(a) Find the voltage $V(z)$ on the line.

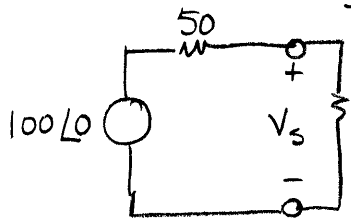
$$\lambda = \frac{v_p}{f} = \frac{20\ \text{cm/ns}}{125\times10^6} = \frac{20\times\frac{\text{m}}{100\ \text{cm}}\times\frac{\text{ns}}{10^9\,\text{s}}}{125\times10^6} = \frac{2\times10^8}{125\times10^6} = 1.6\text{m}$$

$$l = \frac{17\text{m}}{1.6\,\text{m}/\lambda} = 10.625\lambda \qquad \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{1.6} = \frac{5\pi}{4}$$

$$\tan\beta l = \tan\left(\frac{2\pi}{\lambda}\cdot 10.625\lambda\right) = 1$$

$$Z_{in} = Z_0\,\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l} = 50\,\frac{(100-j60)+j50(1)}{50+j(100-j60)(1)}$$

$$Z_{in} = 50\,\frac{100-j10}{110+j100} = 22.62 - j25.1\ \Omega.$$



We can first solve for V_s using a voltage divider

$$V_s = \frac{22.62-j25.1}{22.62-j25.1+50}\,100e^{j0}$$

$$V_s = 38.51 - j21.27 \text{ Volts}$$

$$V_s = 43.99\,e^{-j0.504}$$

But V_s also can be written using the wave expressions for $V(z)$

$$V_s = V(z) \qquad = V^+e^{-j\beta z}\left(1+\Gamma_L e^{+j2\beta z}\right)$$ remember $z=-l$, so this becomes $V^+e^{+j\beta l}\left(1+\Gamma_L e^{-j2\beta l}\right)$

$$\beta l = \frac{2\pi}{\lambda}\cdot 10.625\lambda = 21.25\pi = \frac{5}{4}\pi = -\frac{3}{4}\pi$$ ← note sign change

$$2\beta l = 2\cdot\frac{2\pi}{\lambda}\cdot 10.625\lambda = 42.5\pi = \frac{5}{2}\pi = \frac{1}{2}\pi$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100-j60)-50}{(100-j60)+50} = \frac{50-j60}{150-j60} = 0.42 - j0.23 = 0.483e^{-j28.4^\circ}$$

$$V_s = V(z=-17\text{m}) = V^+e^{-j(+\frac{3}{4}\pi)}\left(1+(0.42-j0.23)e^{-j\frac{1}{2}\pi}\right)$$

$$V_s = V^+e^{-j\frac{3\pi}{4}}\left(1+(-.23-j.42)\right) = V^+e^{-j\frac{3\pi}{4}}(0.77-j0.42)$$

These expressions for V_s can be equated and solved for V^+

$$V^+ e^{-j\frac{3\pi}{4}}(0.77 - j0.42) = 43.99\, e^{-j0.504}$$

$$V^+ = \frac{43.99\, e^{-j0.504}}{(0.77 - j0.42)}\, e^{j\frac{3\pi}{4}} = (50.15 - 0.23)\, e^{j\frac{3\pi}{4}}$$

$$V^+ = 50.15\, e^{-j.004}\, e^{j\frac{3\pi}{4}} \approx 50.1\, e^{j\frac{3\pi}{4}}$$

$$\Rightarrow V(z) = V^+ e^{-j\beta z}(1 + \Gamma(z))$$

$$V(z) = 50.1\, e^{j\frac{3\pi}{4}}\, e^{-j\frac{5}{4}\pi z}\left(1 + .483\, e^{-j28.4^\circ}\, e^{-j\frac{5}{2}\pi z}\right)$$

(b) Find the load voltage V_L.

$$V_L = V(z=0) = 50.1\, e^{j\frac{3\pi}{4}}\left(1 + .483\, e^{-j28.4^\circ}\right)$$

$$V_L = 50.1\, e^{j\frac{3\pi}{4}}(1.426 - j0.23) = 50.1\, e^{j\frac{3\pi}{4}}\, 1.444\, e^{-j9.17^\circ} =$$

$$V_L = 72.3\, e^{+j125.8} \text{ volts.}$$

(c) Find the time average power absorbed by the VHF antenna.

$$P_L = \frac{1}{2}|I_L|^2 R_L = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L \longleftarrow$$ since it is absorbed use only real part.

$$Z_L = 100 - j60 = 116.6\, e^{-j31.0^\circ}$$

$$P_L = \frac{1}{2}\frac{(72.3)^2}{(116.6)^2}\cdot(100) = 19.2 \text{ watts.}$$

(d) Find the power absorbed by the source impedance R_s

To find this we need to know I_s

$$I_s = \frac{V_0}{R_s + Z_{in}} = \frac{100\, e^{j0^\circ}}{50 + 22.62 - j25.1} = 1.301\, e^{j19.1^\circ}$$

$$P_{RS} = \frac{1}{2}|I_s|^2 R_s = \frac{1}{2}(1.301)^2(50) = 42.3 \text{ watts.}$$

3.5 Impedance matching

Why do impedance matching

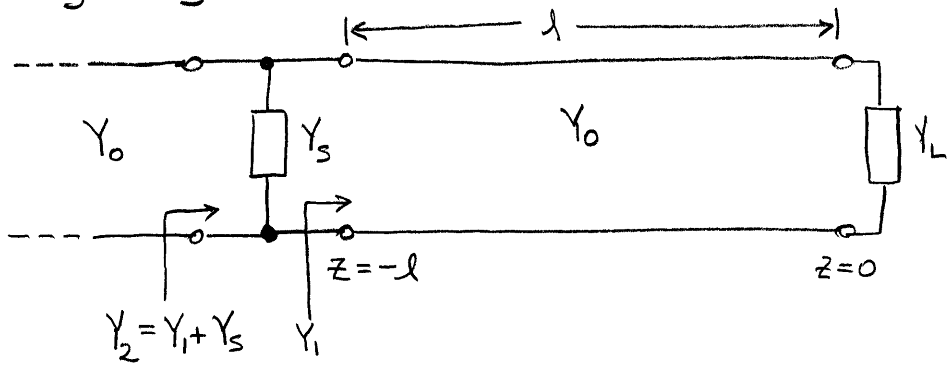
(1) reduce reflections & standing waves that jeopardize the power-handing capabilities of the line

(b) maximize power delivered to the load

(c) improve signal-to-noise ratio of system

(d) reduce amplitude and phase errors of system.

3.5.1. Matching using Lumped Reactive Elements



To impedance match connect Y_s @ $z=-l$ such that

$$Y_2(z=-l) = Y_1(z=-l) + Y_s = Y_o$$

Do in terms of normalized admittance

$$\overline{Y}(z) = \frac{Y(z)}{Y_o} = \frac{1-\Gamma_L e^{j2\beta z}}{1+\Gamma_L e^{j2\beta z}}$$

$\Rightarrow$ find $z=-l$ such that $\overline{Y}_1 = \overline{Y}(z=-l) = 1 - j\overline{B}$

want real part to be 1

$\overline{B}$ is whatever Y_L and Y_o and l cause it to be.

<u>Choose</u> $\overline{Y}_s = +j\overline{B}$ so that $\overline{Y}_2 = \overline{Y}_1 + \overline{Y}_s = 1 - j\overline{B} + j\overline{B} = 1$

We can calculate the distance ℓ from the load we wish to place Y_s.

$$\overline{Y}_1 = \overline{Y}(z=-\ell) = \frac{1-\Gamma_L e^{-j2\beta\ell}}{1+\Gamma_L e^{-j2\beta\ell}} = 1-j\overline{B} \quad \text{from the problem}$$

Since $\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \rho e^{j\psi}$ we can substitute it into $\overline{Y}_1$

$$\overline{Y}_1 = \frac{1-\rho e^{j\psi} e^{-j2\beta\ell}}{1+\rho e^{j\psi} e^{-j2\beta\ell}} = \frac{1-\rho e^{j\theta}}{1+\rho e^{j\theta}}$$

where $\theta = \psi - 2\beta\ell$

we can rationalize this and separate it into real and imaginary parts

$$\overline{Y}_1 = \frac{1-\rho e^{j\theta}}{1+\rho e^{j\theta}} \cdot \frac{1+\rho e^{-j\theta}}{1+\rho e^{-j\theta}} = \frac{1+\rho e^{-j\theta} - \rho e^{j\theta} - \rho^2}{1+\rho e^{j\theta} + \rho e^{-j\theta} + \rho^2}$$

$$\overline{Y}_1 = \frac{1+\rho(-2j\sin\theta) - \rho^2}{1+2\rho\cos\theta + \rho^2} = \frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} - j\frac{2\rho\sin\theta}{1+2\rho\cos\theta+\rho^2}$$

For matching we want to pick ℓ such that the first term is 1.

$$\frac{1-\rho^2}{1+2\rho\cos\theta+\rho^2} = 1$$

$$\cancel{1} - \rho^2 = \cancel{1} + 2\rho\cos\theta + \rho^2$$

$$2\rho\cos\theta + 2\rho^2 = 0$$

$$\cos\theta + \rho = 0$$

$$\cos\theta = -\rho$$

$$\theta = \cos^{-1}(-\rho)$$

$$\theta = \psi - 2\beta\ell = \cos^{-1}(-\rho)$$

Solving for ℓ gives

this will actually have two solutions

$$\ell = \frac{\psi - \theta}{2\beta} = \frac{\psi - \cos^{-1}(-\rho)}{2\beta} = \frac{\lambda}{4\pi}\left[\psi - \cos^{-1}(-\rho)\right]$$

This relationship has two solutions $\theta = \cos(-\rho)$ since cos is an even function

$$\left.\begin{array}{l} \frac{\pi}{2} \le \theta_1 \le \pi \\ \text{and} \quad -\pi \le \theta_2 \le -\frac{\pi}{2} \end{array}\right\}$$ since $-\rho < 0$ these solutions must lie in the 2nd and 3rd quadrants as shown

This specifies l by

$$l = \frac{\lambda}{4\pi}\left[\psi - \cos^{-1}(-\rho)\right]$$

Now we assumed that $z = -l$. If l comes out negative add $\frac{\lambda}{2}$ since this simply adds $2\beta\left(\frac{\lambda}{2}\right) = 2\left(\frac{2\pi}{\lambda}\right)\left(\frac{\lambda}{2}\right) = 2\pi$ to the arguement.

Once the l at which $\mathrm{Re}\{\bar{Y}_2\} = 1$ is determined we can calculate the corresponding susceptance $\bar{B}$

$$\bar{B} = -\mathrm{Im}\left\{\bar{Y}_1\right\}\Big|_{\cos\theta=-\rho} = \frac{2\rho\sin\theta}{1+2\rho\cos\theta+\rho}\Big|_{\cos\theta=-\rho}$$

since $\cos\theta = -\rho$

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-\rho^2}$$

$$\bar{B} = \frac{\pm 2\rho\sqrt{1-\rho^2}}{1-2\rho^2+\rho^2} = \frac{\pm 2\rho\sqrt{1-\rho^2}}{1-\rho^2} = \pm\frac{2\rho}{\sqrt{1-\rho^2}}$$

The sign of $\bar{B}$ comes from the angle θ

if θ in the 2nd quadrant $\sin\theta > 0$ and we use $+$ (capacitor)

if θ in the 3rd quadrant $\sin\theta < 0$ and we use $-$ (inductor)

We did this for parallel (shunt) matching.

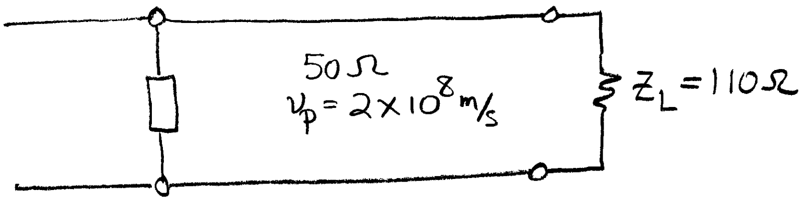
You can also do this with series elements, except you would calculate $\bar{Z}_1 = \bar{Z}(z=-l) = 1 - j\bar{X}$ and match with a series element.

$$l = \frac{\psi - \cos^{-1}(\rho)}{2\beta} = \frac{\lambda}{4\pi}\left[\psi - \cos^{-1}(\rho)\right]$$

$$\bar{X} = \pm\frac{2\rho}{\sqrt{1-\rho^2}}$$

Example 3-15

An antenna having a feed-point impedance of $110\,\Omega$ is to be matched to a $50\,\Omega$ coaxial line with $v_p = 2\times10^8$ m/s using a single shunt lumped reactive element as shown below. Find the position (nearest the load) and the appropriate value of the reactive element for operation at 30 MHz using (a) a capacitor, and (b) an inductor.



(a) $\Gamma_L = \rho e^{j\psi} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{110-50}{110+50} = 0.375$ (real)

the location of the capacitor will be

$$l_1 = \frac{\psi - \theta}{2\beta} = \frac{\lambda}{4\pi}\left[\cancel{\psi}^{\,0} - \cos^{-1}(-\rho)\right] = -\frac{\lambda}{4\pi}\cos^{-1}(.375) = \frac{-\lambda(1.955)}{4\pi}$$

for a capacitor the sign of B is positive so this is a 2nd quadrant angle. $\frac{\pi}{2} < \theta < \pi$

(since $B_c = j\omega C$)

since $\lambda = \dfrac{v_p}{f} = \dfrac{2\times10^8 \text{ m/s}}{30\times10^6/\text{s}} = 6.67$ m.

Since l_1 is negative add $+\frac{\lambda}{2}$

$$l_1 = -0.156\lambda + 0.5\lambda = +0.344\lambda = 0.344(6.67\text{m}) = 2.29\text{m}$$

the capacitance is given by

$$(j\bar{B}_1) = j\omega C\,(Z_0)$$

$$\bar{B}_1 = +\frac{2\rho}{\sqrt{1-\rho^2}} = \frac{2(0.375)}{\sqrt{1-(.375)^2}} = 0.809$$

where I chose the + sign to correspond to a shunt capacitance.

$$(0.809) = 2\pi(30\times10^6)\,C\,(50)$$

$$C = 85.8\text{ pf.}$$

(b) For an inductor we much choose a 3rd quadrant angle.

i.e. $\cos^{-1}(-.375) = -1.955$

$\uparrow$

different sign

The corresponding l is given by

$$l_2 = \frac{\cancelto{0}{\phi} - \theta_2}{2\beta} = \frac{\lambda}{4\pi}\left[\cancelto{0}{\phi} - \cos^{-1}(-\rho)\right] = -\frac{\lambda}{4\pi}(-1.955) = +0.1555\lambda$$

$$l_2 = +0.1555(6.67) = 1.04\text{m}$$

The value of the susceptance is $\bar{B}_2 = -\frac{2\rho}{\sqrt{1-\rho^2}} = -0.809$

The corresponding value is

$$\frac{-j}{\omega L_s} Z_0 = j\bar{B}_2 = -j\,0.809$$

$$L_s = \frac{-jZ_0}{\omega(-j0.809)} = \frac{50}{2\pi(30\times10^6)(0.809)} = 0.328\,\mu\text{H}$$

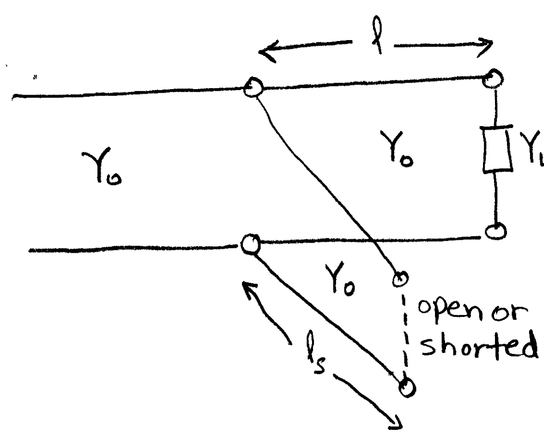
3.5.2 Matching using Series or Shunt Stubs

At microwave frequencies we commonly use open- or short-circuited stubs (short lengths of transmission line) connected in series or parallel to match.

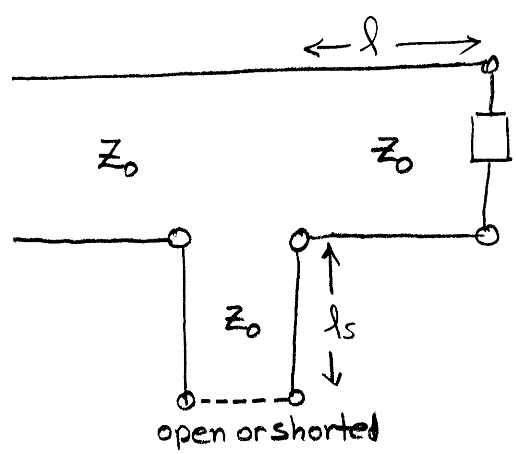
short-circuited stub – used for coax and waveguides because a short is less sensitive to pick-up and radiates less than an open

open-circuit stub – used for micro strips and striplines because it is easier to fabricate

A shunt (parallel) stub is usually better than a series stub since breaking the line to add the stub may create discontinuities and lead to reflections.



parallel stub



series stub

From the previous expressions for l and $\overline{B}$ we only need to determine l_s of the stub to give $\overline{Y}_s = +j\overline{B}$ at the junction.

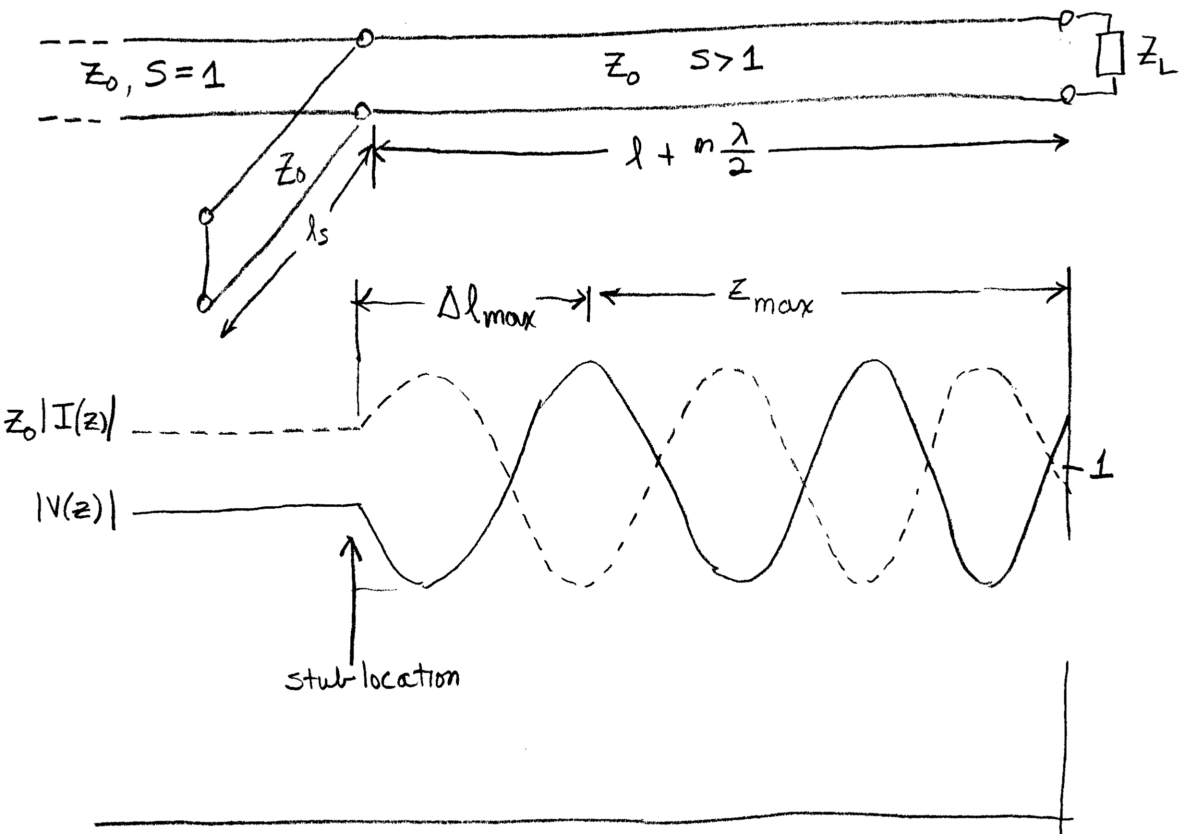
For a short circuited line we know

$$Z_{in} = j Z_o \tan(\beta l)$$

Inverting

$$\overline{Y}_s = j\overline{B} = \frac{Z_o}{j Z_o \tan(\beta l_s)} = \frac{1}{j\tan(\beta l_s)}$$

so you can readily calculate $\tan(\beta l_s) = -\frac{1}{\overline{B}}$ for a short-circuited stub



You can measure the stub location l relative to the position z_{max} of the nearest voltage maximum toward the load since

$$l + m\frac{\lambda}{2}$$

You can specify

sign of l

$$\theta = \psi - 2\beta l = \psi + 2\beta z_{max} - 2\beta\Delta l_{max}$$

$$\theta = -m2\pi - 2\beta\Delta l_{max}$$

$$\Delta l_{max} = -\frac{\theta + m2\pi}{2\beta} = -\frac{1}{2\beta}\left[\cos^{-1}(-\rho) + m2\pi\right]$$

can be measured.

Example 3-16

Design a single stub system to match a load consisting of a resistance $R_L = 200\Omega$ in parallel with an inductance $L_L = 200/\pi$ nH to a transmission line with characteristic impedance $Z_0 = 100\Omega$ and operating at 500 MHz. Connect the stub in parallel with the line.

Express the load as an admittance

$$Y_L = \frac{1}{R_L} - j\frac{1}{\omega L_L} = \frac{1}{200} - j\frac{1}{2\pi(500\times10^6)(\frac{200}{\pi}\times10^{-9})} = .005 - j.005$$

The reflection coefficient at the load is

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = \frac{.01-(.005-j.005)}{.01+(.005-j.005)} = \frac{.005+j.005}{.015-j.005} = \frac{1+j}{3-j} = 0.2+j0.4$$

$$\Gamma_L = 0.447\,e^{j63.43^\circ} = 0.447\,e^{j1.107}$$

The location of the stub is given as

$$l = \frac{\lambda}{4\pi}\left[\psi - \cos^{-1}(-\rho)\right] = \frac{\lambda}{4\pi}\left[1.107 \mp 2.034\right] = \begin{cases} -0.074\lambda + .5\lambda = .426\lambda \\ +0.250\lambda \end{cases}$$

can be 2nd or 3nd quadrant angle i.e. C or L

$$\bar{B} = \pm\frac{2\rho}{\sqrt{1-\rho^2}} = \pm\frac{2(.447)}{\sqrt{1-(.447)^2}} \cong \pm 1$$

The length of the first stub is $\tan(\beta l_{s1}) = -\frac{1}{\bar{B}} = -1$ for a short-circuited line

$$\beta l_s = \frac{2\pi}{\lambda} l_{s1} = -0.7853$$

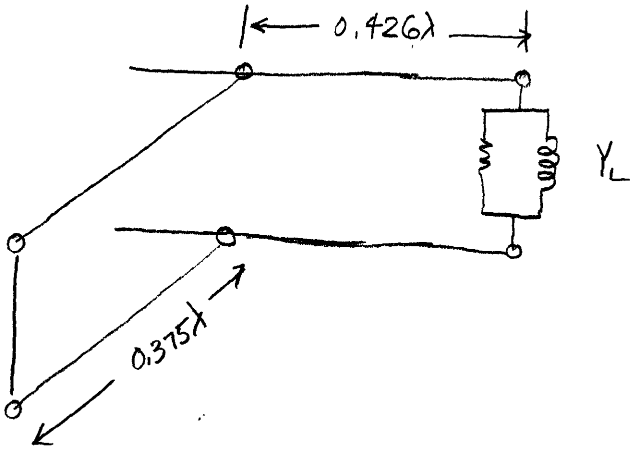
$$l_{s1} = -0.125\lambda + .5\lambda = .375\lambda$$

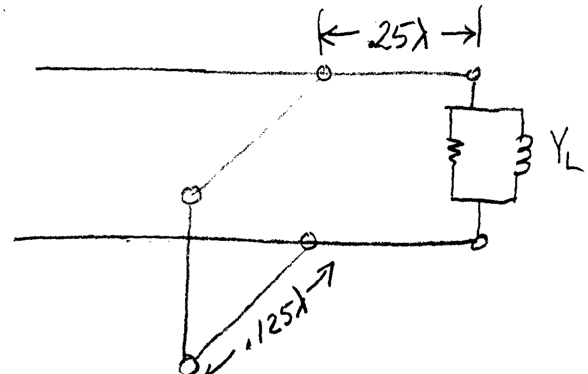
add $\frac{1}{2}\lambda$ since $l<0$

The length of the second stub is $\tan(\beta l_{s2}) = -\frac{1}{\bar{B}} = -\frac{1}{-1} = +1$

$$\beta l_{s2} = \frac{2\pi}{\lambda} l_{s2} = +0.7854$$

$$l_{s2} = +0.125\lambda$$





Preferred solution.

One thing we have NOT discussed is the frequency dependence of a solution.

Consider these two designs

$$Y_L(f) = \frac{1}{200} - j\frac{1}{2\pi f\left(\frac{200}{\pi}\times 10^{-9}\right)} = \frac{1}{200} - j\frac{10^9}{400f}$$

On the line $\beta l = \frac{2\pi}{\lambda} \cdot l = 2\pi \frac{f}{c} l$

Then just to the right of the stub

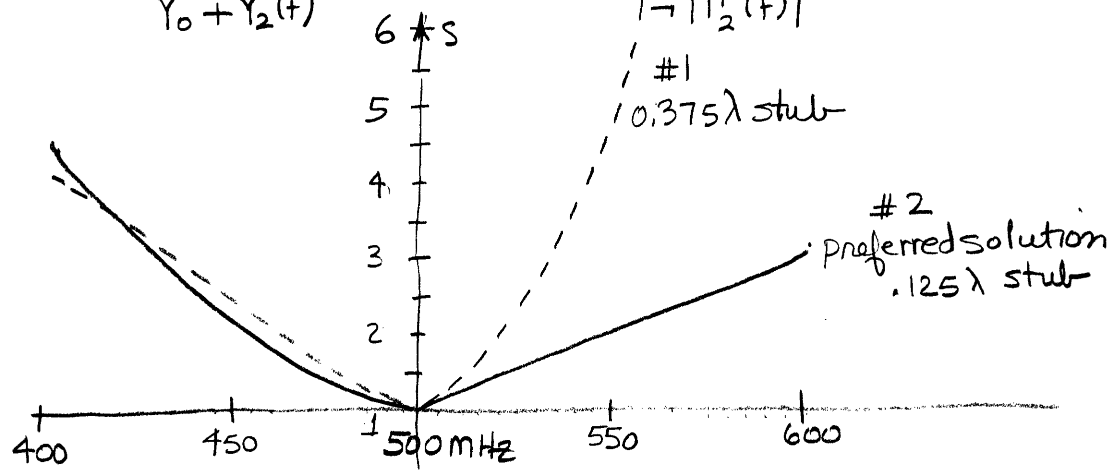
$$Y_1(f) = Y_0 \frac{Y_L + jY_0 \tan(2\pi f l/c)}{Y_0 + jY_L \tan(2\pi f l/c)}$$

The total line admittance see just to the left of the stub is then

$$Y_2(f) = Y_S + Y_1 = \frac{-jY_0}{\tan(2\pi f l_s/c)} + Y_0 \frac{Y_L + jY_0 \tan(2\pi f l/c)}{Y_0 + jY_L \tan(2\pi f l/c)}$$

We can now compute

$$\Gamma_2(f) = \frac{Y_0 - Y_2(f)}{Y_0 + Y_2(f)} \quad \text{and} \quad S_2(f) = \frac{1 + |\Gamma_2(f)|}{1 - |\Gamma_2(f)|}$$



Most engineers would consider solution #2 to be better since it works over a broader range of frequencies.

3.5.3. Quarter Wave Transformer Matching

A common and powerful technique for matching a load to a transmission line is to use a quarter-wave length long transmission line.

For $l = \frac{\lambda}{4}$ the input impedance is

$$Z_{in}\Big|_{l=\frac{\lambda}{4}} = Z_0 \left.\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right|_{l=\frac{\lambda}{4}} \rightarrow Z_0 \frac{jZ_0 \cancel{\tan\beta l}}{jZ_L \cancel{\tan\beta l}} = \frac{Z_0^2}{Z_L}$$

$$\beta l = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

In terms of normalized impedances

$$Z_{in} = \frac{Z_0^2}{Z_L}$$

or

$$\frac{Z_{in}}{Z_0} = \frac{Z_0}{Z_L}$$

$$\bar{Z}_{in} = \frac{1}{\bar{Z}_L}$$

we have two cases, $Z_L = R_L$ and $Z_L = R_L + jX_L$

Assume that the impedance of the matching section is Z_Q

(a) For $Z_L = R_L$ $\quad Z_{in} = \dfrac{Z_Q^2}{R_L}$

If we want Z_{in} to match R_1 (a cable impedance)

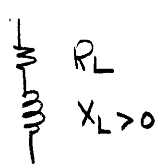
$$Z_Q = \sqrt{R_1 R_L}$$

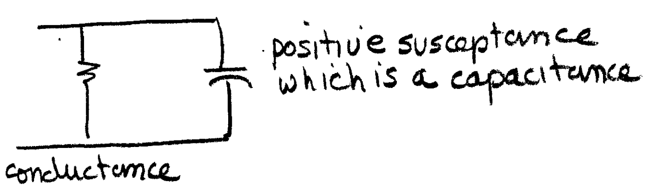
(b) for $Z_L = R_L + jX_L$

in this case $\quad Z_{in} = \dfrac{Z_Q^2}{R_L + jX_L}$

This is better written in terms of admittances.

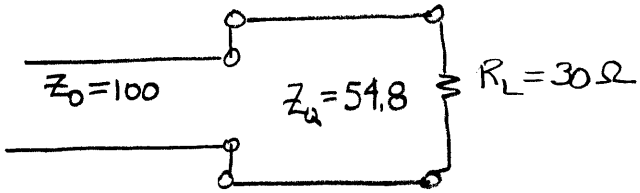
$$Y_{in} = \frac{R_L + jX_L}{Z_Q^2} = \frac{R_L}{Z_Q^2} + j\frac{X_L}{Z_Q^2}$$

if Z_L is 

then Y_{in} is 

Example 3-17

Design a quarter-wavelength section to match a thin monopole antenna of length 0.24λ having a purely resistive feed-point impedance of $R_L = 30\Omega$ to a transmission line having a characteristic impedance of $Z_0 = 100\Omega$.



$S=1$ $\quad\quad$ $S=1.82$

The matching section is $\frac{\lambda}{4}$ so $Z_{in} = \dfrac{Z_Q^2}{R_L} = 100$ for matching

$$Z_Q = \sqrt{(100)(R_L)} = \sqrt{(100)(30)} = 54.77\Omega$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 - 54.77}{30 + 54.77} = \frac{-24.77}{84.77} = -0.292 = 0.29e^{j\pi}$$

$$S = \frac{1+\rho}{1-\rho} = \frac{1+.29}{1-.29} = \frac{1.29}{0.71} = 1.82$$

Example 3-18

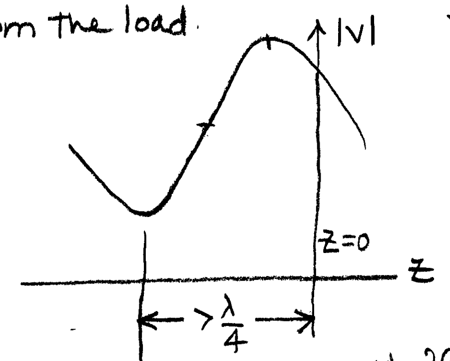
A thin wire have wave dipole antenna has an input impedance of $Z_L = 73 + j\,42.5\,\Omega$. Design a quarter-wave transformer to match this antenna to a transmission line with characteristic impedance $Z_0 = 100\Omega$.

At $z=0$ $\quad \Gamma_L = \rho e^{j\psi} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{73 + j42.5 - 100}{73 + j\,42.5 + 100} = -.09 + j.26 = .283\, e^{j108.6^\circ}$

$\psi = 108.6^\circ = 1.896$ radians

The load is inductive since $j42.5 > 0$

For an inductive load the voltage minimum is more than $\frac{\lambda}{4}$ away from the load.



goes at $2\beta z$

The maximum occurs when

$$\psi + 2\beta z_{max} = 0 \qquad (1+\Gamma \text{ aligns})$$

$$z_{max} = -\frac{\psi}{2\beta} = -\frac{1.896}{2\left(\frac{2\pi}{\lambda}\right)} = -0.151\lambda$$

We picked a maximum since $\mathcal{Im}(Z_{in}) = 0$ at a voltage maximum.

We need to calculate $Z(z = -.151\lambda)$ and switch to normalized impedances

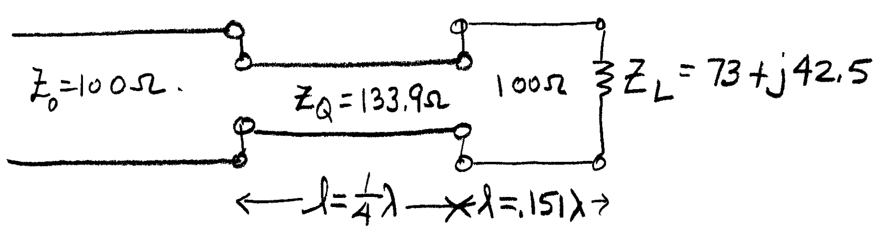
$$\bar{Z}(z=-.151\lambda) = \left.\frac{\bar{Z}_L - j\tan\beta z}{1 - j\bar{Z}_L \tan\beta z}\right|_{z=-.151\lambda} = \frac{(.73 + j.425) + j\,1.395}{1 - j(.73 + j.425)(-1.395)}$$

$$\tan(\beta z) = \tan\left[\frac{2\pi}{\lambda}(-.151\lambda)\right] = -\tan(.302\pi) = -1.395$$

$$\bar{Z} = (z = -.151\lambda) = \frac{.73 + j\,1.82}{.407 + j\,1.018} = 1.7936$$

We insert a $\frac{\lambda}{4}$ section at $z = -0.151\lambda$. The impedance of this section will be

$$\bar{Z}_Q = \sqrt{(1.7936)(1)} = 1.339 \qquad Z_Q = 133.9\Omega$$



3,6 The Smith Chart

$$\bar{Z}(z) = \frac{1+\Gamma(z)}{1-\Gamma(z)} \quad \text{where } \Gamma(z) = \rho e^{j(\psi + 2\beta z)}$$

Let $\bar{Z} = \bar{R} + j\bar{X}$ and $\Gamma = u + jv$

The relationship between these variables can be calculated by equating these expressions

$$\bar{z} = \bar{R} + j\bar{X} = \frac{1+(u+jv)}{1-(u+jv)} = \frac{(1+u)+jv}{(1-u)-jv}\;\frac{(1-u)+jv}{(1-u)+jv} = \frac{(1-u^2)+jv2-v^2}{(1-u)^2+v^2}$$

$$\bar{R} = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$\bar{R}(1-u^2) + \bar{R}v^2 = 1-u^2-v^2$$

$$-1+\bar{R}(1-u^2)+u^2+\bar{R}v^2+v^2 = 0$$

$$\bar{R}(1-u^2)-(1-u^2)+(\bar{R}+1)v^2 = 0$$

$$(1-u^2)(\bar{R}-1)+(\bar{R}+1)v^2 = 0$$

$$(1-u^2)(\bar{R}-1)(\bar{R}+1)+(\bar{R}+1)^2v^2 = 0$$

$$1+(1-u^2)(\bar{R}-1)(\bar{R}+1)+(\bar{R}+1)^2v^2 = 1$$

$$\frac{1+(1-u^2)(\bar{R}^2-1)}{(\bar{R}+1)^2} + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$\frac{1+(\bar{R}^2-u^2\bar{R}^2-1+u^2)}{(\bar{R}+1)^2} + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$\frac{-u^2\bar{R}^2+u^2+\bar{R}^2}{(\bar{R}+1)^2} + v^2 = \frac{1}{(\bar{R}+1)^2}$$

$$\left(u-\frac{\bar{R}}{\bar{R}+1}\right)^2 + v^2 = \frac{1}{(1+\bar{R})^2}$$

$$\bar{X} = \frac{2v}{(1-u)^2+v^2}$$

$$(1-u)^2+v^2 = \frac{2v}{\bar{X}}$$

$$(1-u)^2+v^2-\frac{2v}{\bar{X}}+\frac{1}{\bar{X}^2} = \frac{1}{\bar{X}^2}$$

$$(u-1)^2+\left(v-\frac{1}{\bar{X}}\right)^2 = \frac{1}{\bar{X}^2}$$

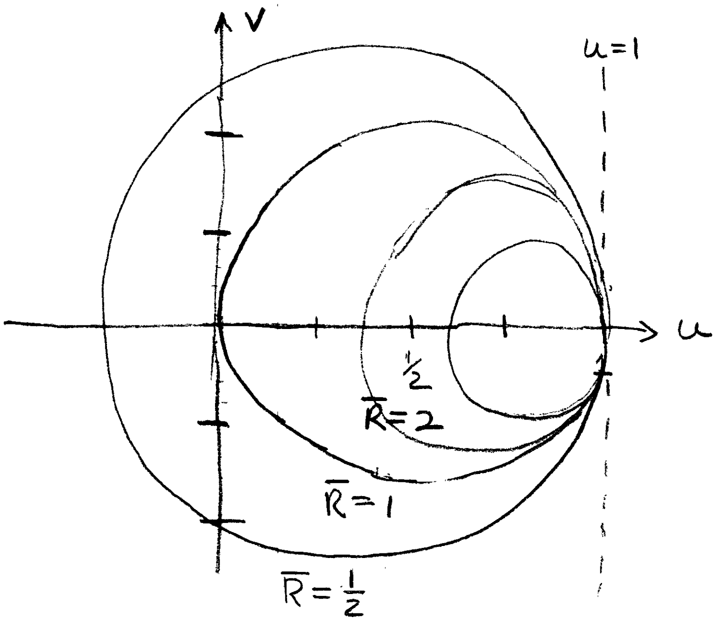
These are equations of circles in uv plane

centered at $u = \frac{\bar{R}}{\bar{R}+1}$, $v = 0$

radius $\frac{1}{1+\bar{R}}$

centered at $u = 1$, $v = \frac{1}{\bar{X}}$

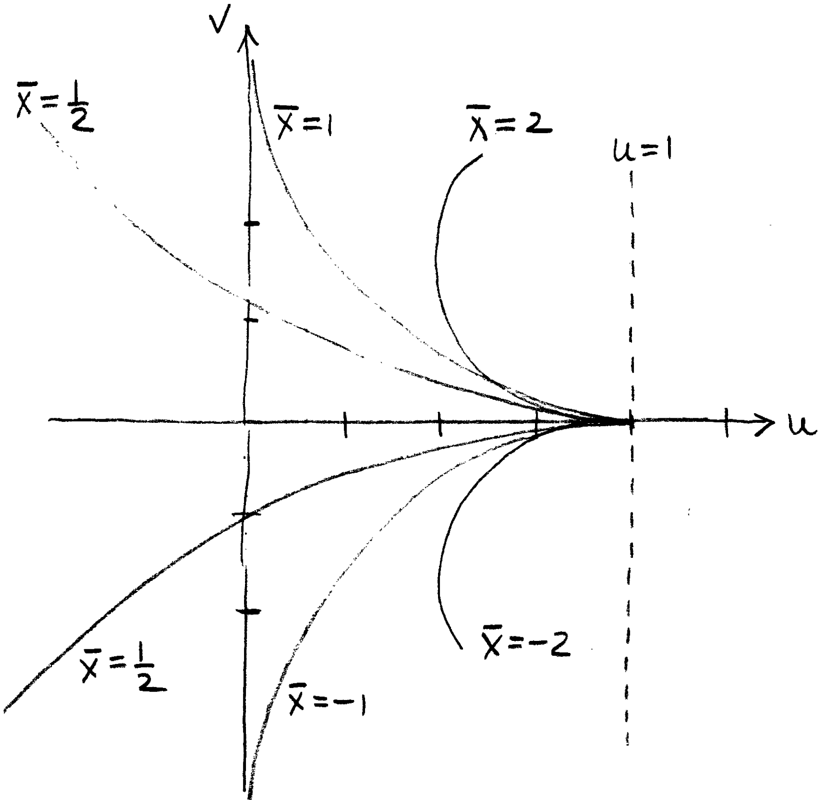
radius $\frac{1}{\bar{X}}$



These are the circles of u, the real part of Γ

$$\left(u-\frac{\bar{R}}{1+\bar{R}}\right)^2+v^2=\left(\frac{1}{1+\bar{R}}\right)^2$$

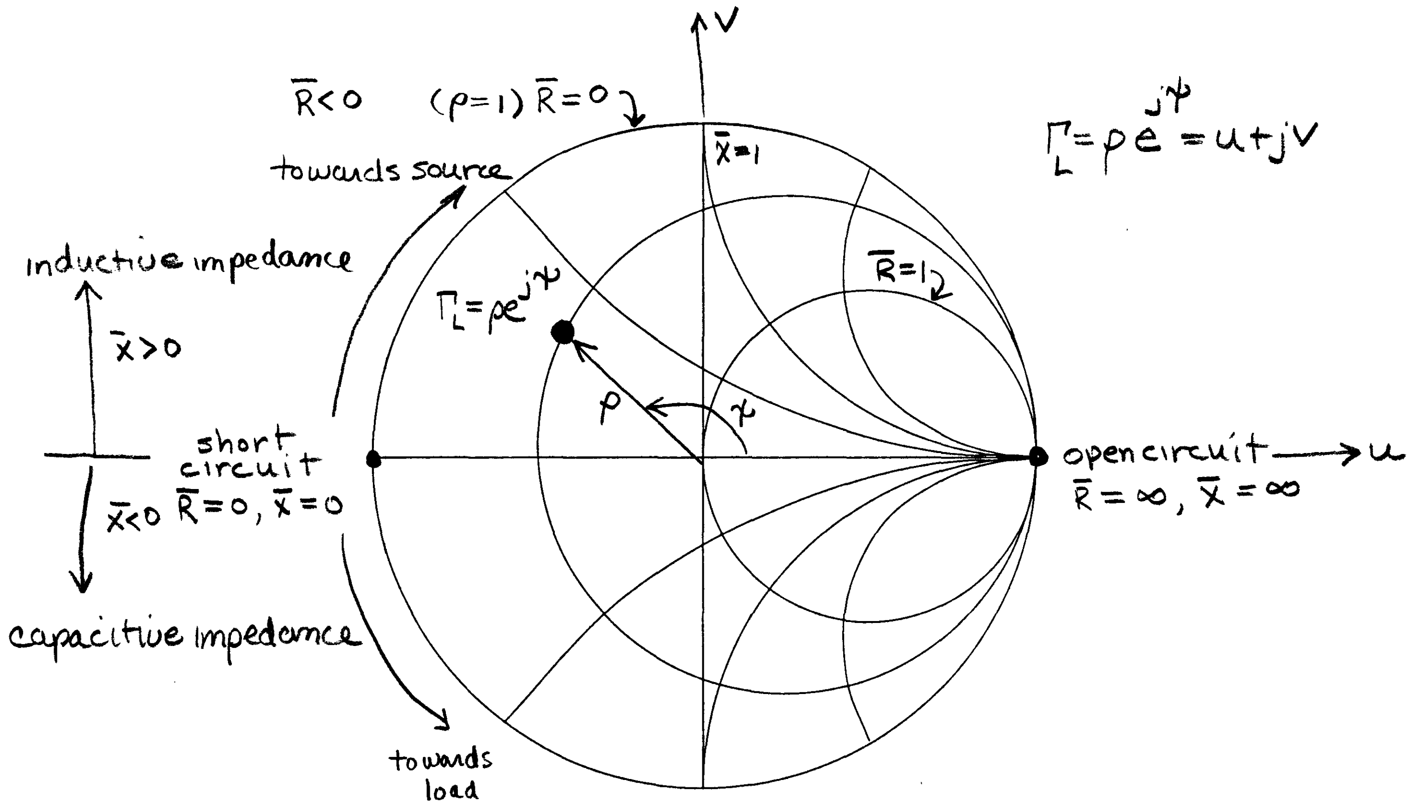
The circles of Γ given $\bar{R}$, (constant $\bar{R}$ circles)
The circles are tangent to the $u=1$ line

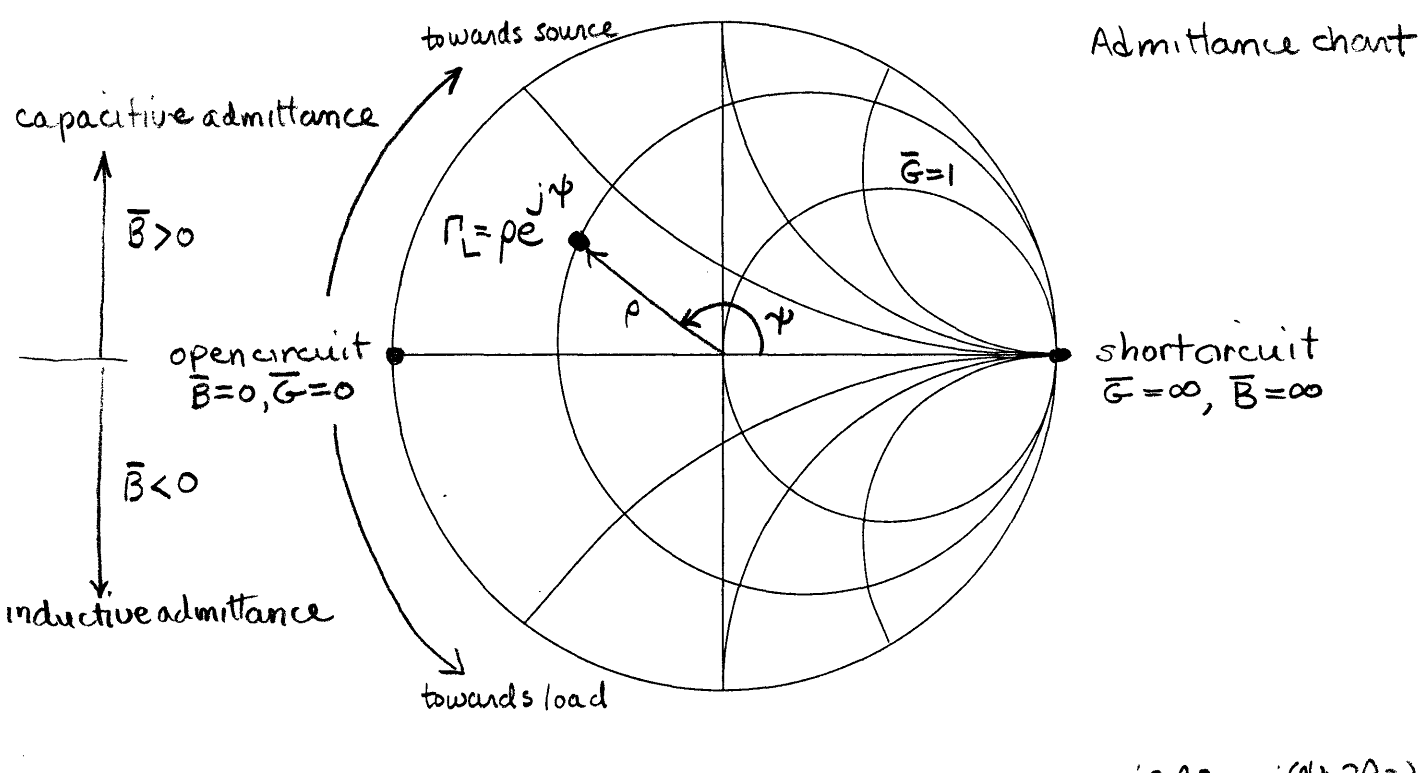


$$(u-1)^2+\left(v-\frac{1}{\bar{X}}\right)^2=\underbrace{\frac{1}{\bar{X}^2}}_{}$$

this allows $\pm v$ symmetry

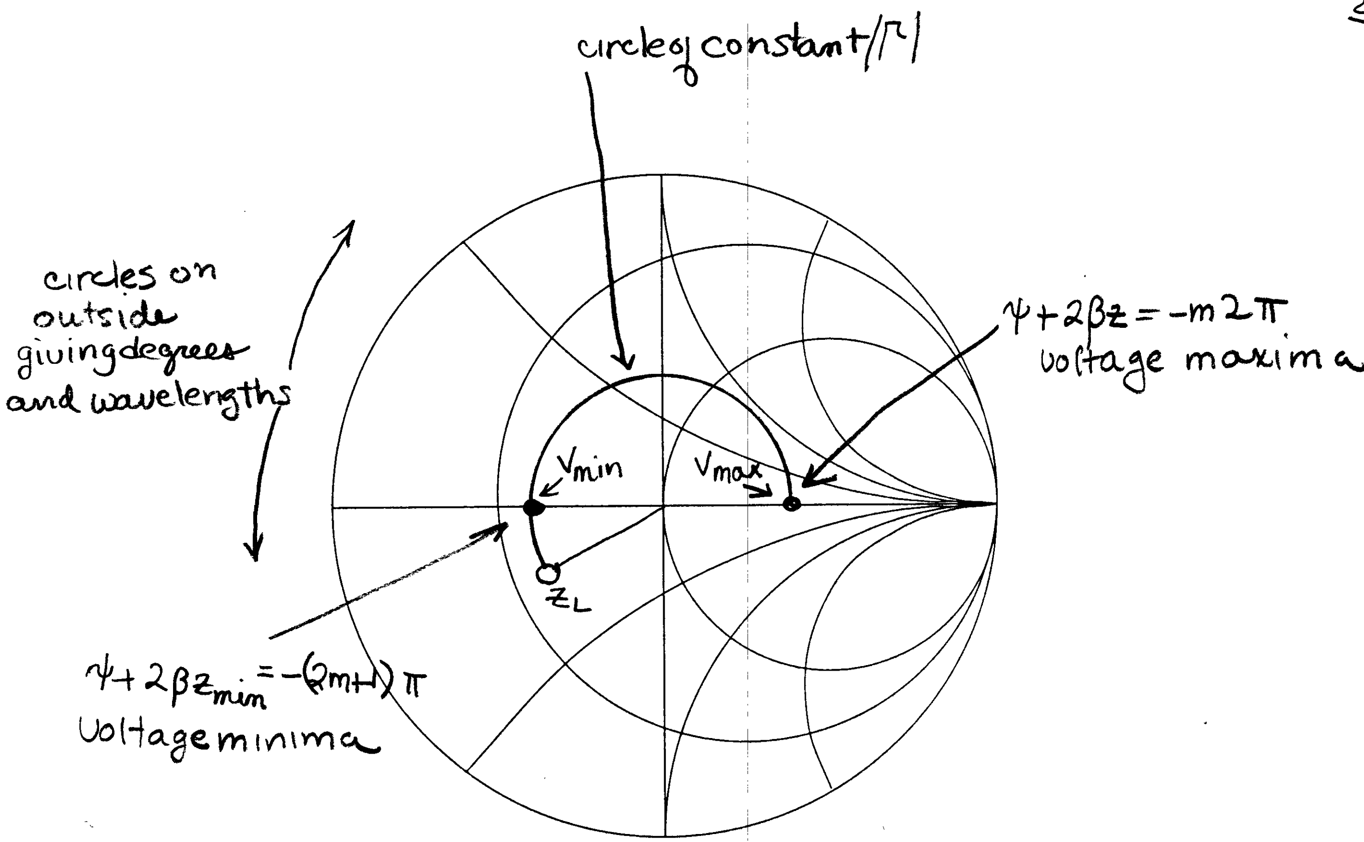
The circles of Γ given $\bar{X}$, (constant $\bar{X}$ circles)
There are two circles





rotation is understood by remembering that $\Gamma = \Gamma_L e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)}$

as you move away from the load z is increasing negative reducing the angle ψ



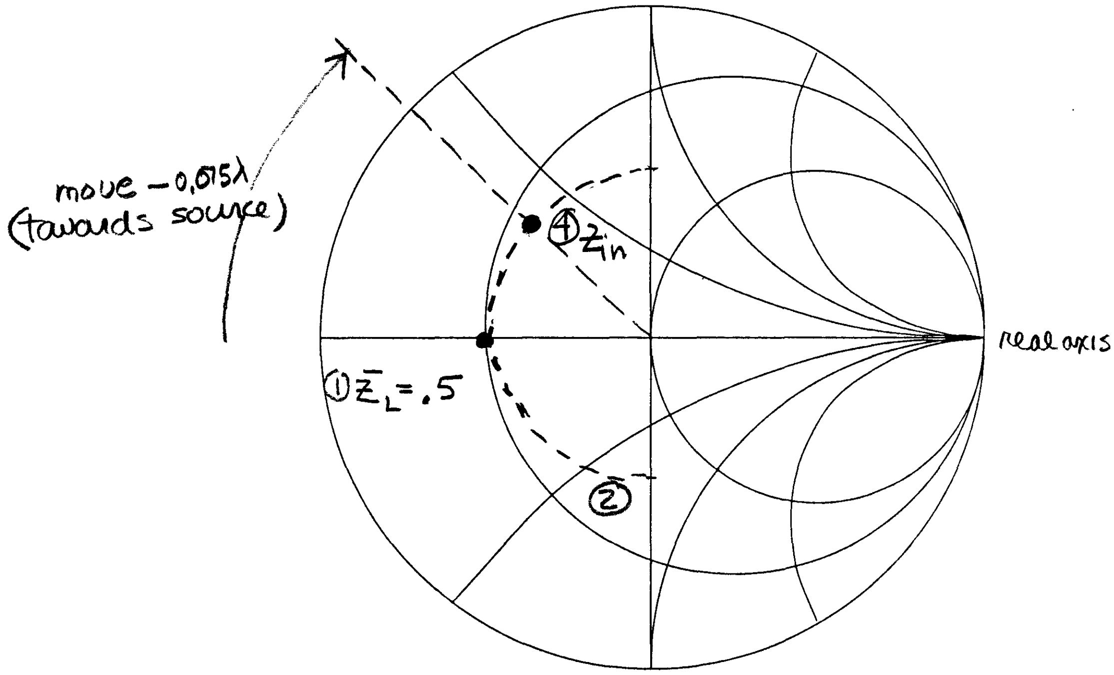
What is very easy with a Smith chart is locating voltage maxima and minima

Recall that $S = \dfrac{1+\rho}{1-\rho}$

At a voltage maxima $\bar{Z} = \bar{R}_{max} = S$

so the S circles are the same as the ρ circles.

Example 3-20



Find the input impedance of a lossless transmission line with the parameters $z_0 = 100\,\Omega$, $z_L = 50 + j0\,\Omega$, $l = 86.25$ cm, and $\lambda = 1.5$ m.

Electrical length of line is $\frac{0.8625}{1.5} = 0.575\lambda$

① $\bar{z}_L = \frac{z_L}{z_0} = \frac{50}{100} = 0.5$ and enter on chart

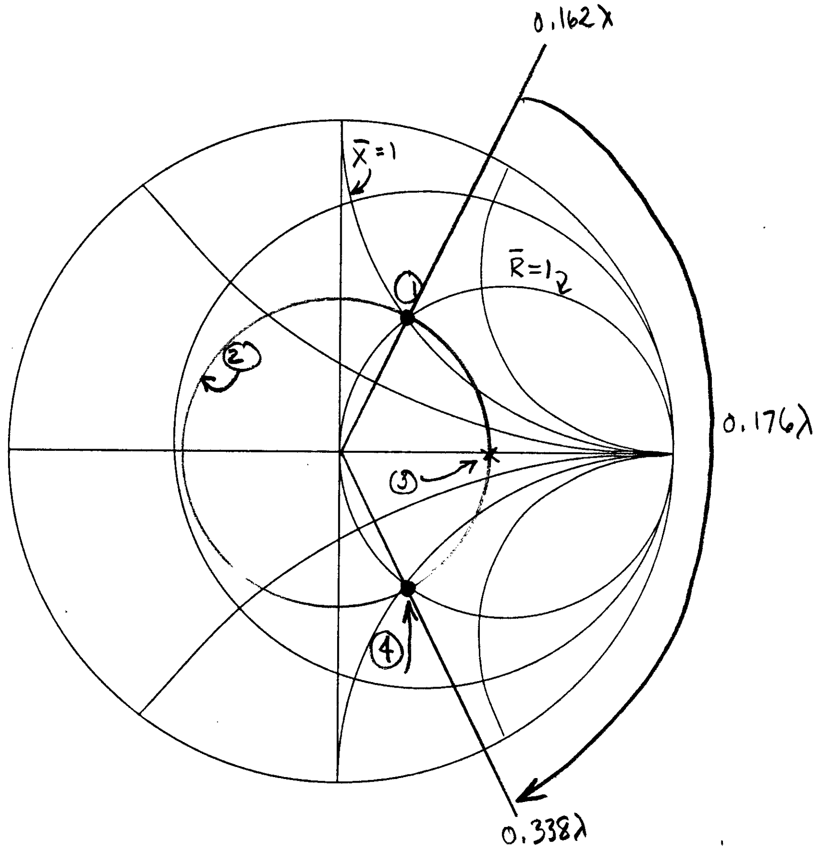
② draw constant ρ circle

③ impedance goes as $2\beta l$ so a full cycle every $\frac{\lambda}{2}$
so examine line length $0.575\lambda - 0.5\lambda = 0.075\lambda$
move 0.075λ away from load (towards source)

④ read off $\bar{z}_{in} \cong 0.59 + j.36$

$z_{in} = 59 + j36\,\Omega$

Example 3-22



Find the input impedance of a lossless transmission line given the following parameters: $Z_0 = 100\Omega$, $Z_L = 100 + j100$, line length $l = 0.676\lambda$ (i.e., $0.5\lambda + 0.176\lambda$).

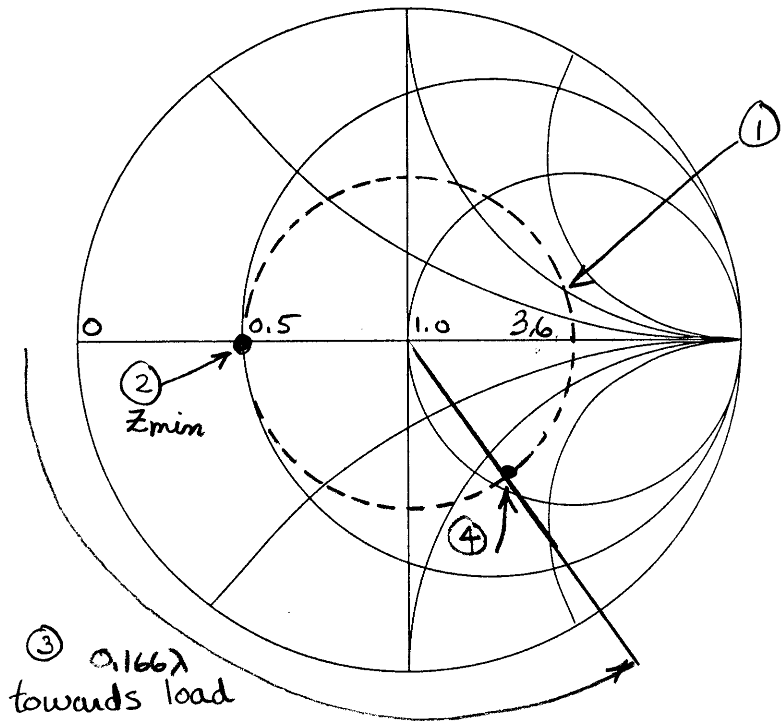
① The normalized load impedance is $\bar{Z}_L = \dfrac{100 + j100}{100} = 1 + j1$
enter on chart.

② Draw circle of constant ρ (centered at origin) through this point

③ Note that the intersection of this circle with the real axis gives $\bar{R} \cong 2.62$. This is also the value of S.
S = 2.62 circle

④ To find Z_{in} which goes as $2\beta l$ we move along this circle (clockwise) towards source. Remember that it repeats every $\frac{\lambda}{2}$ so we go 0.176λ. We start at 0.162λ from chart and add 0.176λ to get 0.338λ.
This corresponds to $\bar{Z}_{in} = 1 - j1 = 100 - j100$.

Example 3-23 Find the normalized load impedance on a transmission line with the following measured parameters; standing wave ratio $S = 3.6$ and first voltage minimum $z_{min} = -0.166\lambda$.



① Draw constant ρ circle corresponding to $S = 3.6$

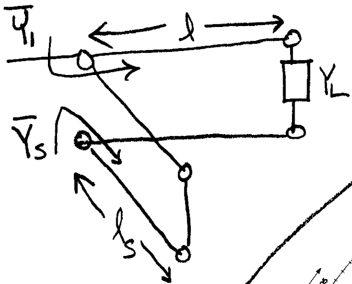
② z_{min} is when this circle intersects negative u axis.

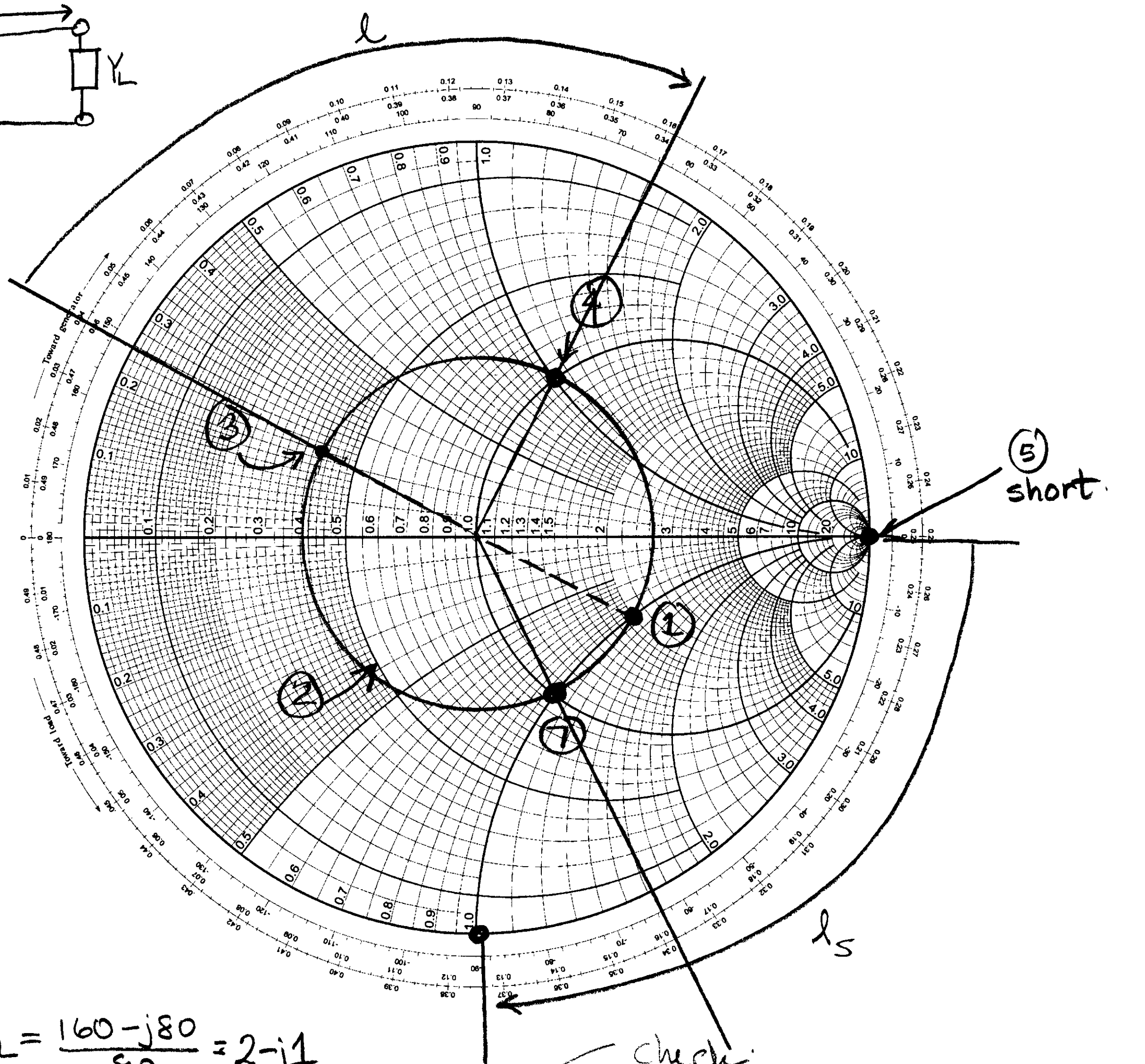
③ Start with z_{min} and move towards load (counterclockwise) a distance $+0.166\lambda$.

④ This location gives unknown (now known) load impedance to be $\bar{Z}_L = 0.89 - j1.13$

Example 3-24

Given a characteristic impedance $Z_0 = 80\Omega$ and a load admittance $Z_L = 160 - j80$, match the line to the load by using a short circuited stub.





① compute $\overline{Z}_L = \dfrac{160 - j80}{80} = 2 - j1$
and plot on chart.

② Draw circle of constant ρ through Z_L.

③ Convert $\overline{Z}_L$ to $\overline{Y}_L$. $\overline{Y}_L = \dfrac{1}{\overline{Z}_L} = 0.4 + j.2$ and plot
Note: this corresponds to reflection through the origin.

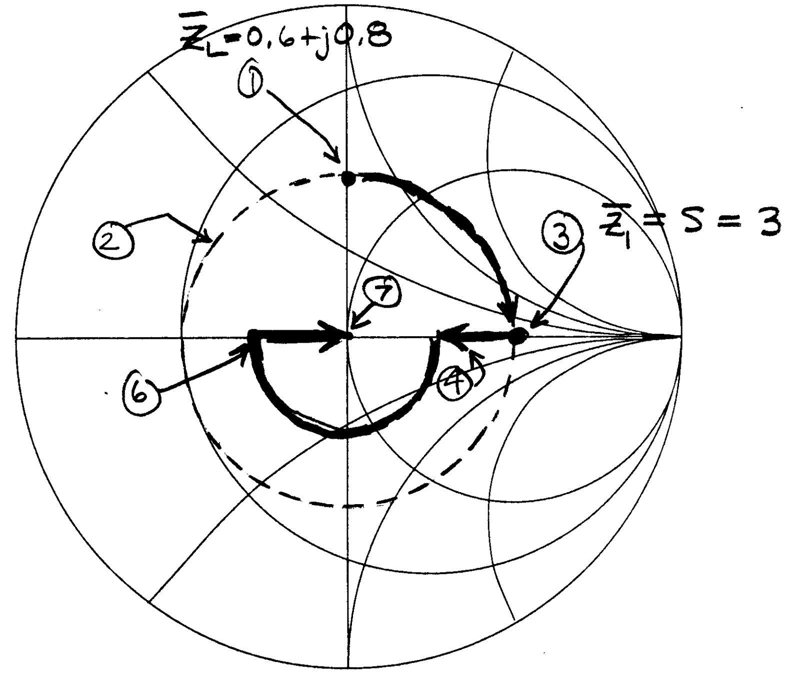
④ Determine $Y_{in} = 1 + j\overline{B}$ by moving on constant ρ circle towards source (clockwise) until you reach $\overline{G} = 1$ ($\overline{R} = 1$) circle.
The amount of rotation determines l which is $0.162 - 0.04 = 0.122\lambda$
The intersection is at $\overline{Y} = 1 + j1$ so we require stub with $\overline{Y}_S = -j1$

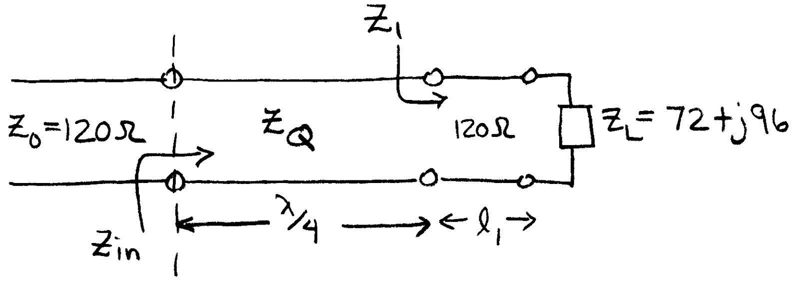
⑤ A short circuit is the right most point on the admittance chart.

⑥ move on circle of constant ρ, i.e. outer edge of chart until you reach $\overline{B} = 1$. This is what you wanted and gives $\lambda = 0.125\lambda$.

⑦ A second solution exists at $\overline{Y} = 1 - j1$. This solution is at $l = .338 - .040 = .298\lambda$
This requires $\overline{Y}_S = +1$ and requires a much longer stub, i.e., $l = .375\lambda$

3,25 Given a transmission line with a characteristic impedance $Z_0 = 120\Omega$ and line impedance $Z_0 = 120\Omega$ and load impedance $Z_L = 72 + j96\Omega$, match the line to the given load using a quarter-wave transformer.





This is different than stub matching where we find ℓ_1 such that $\bar{Y}_1$ would be $1 + j\bar{B}$. With quarter-wave transformer find ℓ_1 where $\bar{Z}_1$ is entirely real, i.e., $\bar{X} = 0$

1. $\bar{Z}_L = \dfrac{72 + j96}{120} = 0.6 + j0.8$
2. Draw constant ρ circle thru $\bar{Z}_L$
3. Move along circle away from load (clockwise) to intersection with horizontal axis, i.e. $\bar{X} = 0$.
   At this point $\bar{Z}_1 = S_1 = 3$.
4. Quarter-wave transformer gives $Z_Q = \sqrt{R_1 R_2} = \sqrt{(Z_0)(3Z_0)}$
   $Z_Q = 1.732 \times 120 = 207.8$. This moves us from circle of $S = 3$ to circle of $S = 1.732$, i.e. $3/\sqrt{3} = \sqrt{3}$
5. $\lambda/4$ corresponds to a $180°$ rotation clockwise (towards source).
6. $\bar{Z}_{in} = 0.577$. This is referred to $Z_Q = 207.8$. Converting back to $Z_0 = 120$ we get $Z_{in} = 0.577\left(\dfrac{207.8}{120}\right) = 1$ so we are matched.