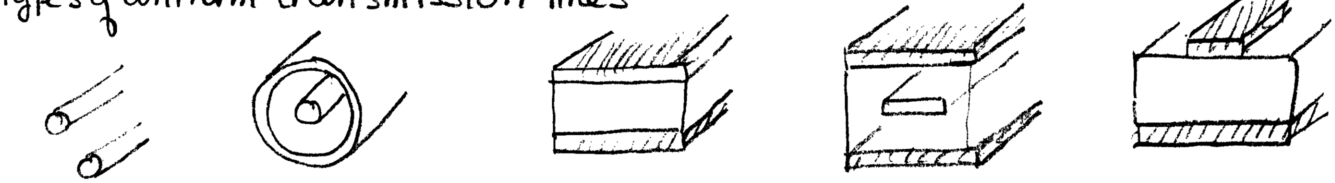
Inan & Inan, Ch. 2

Types of uniform transmission lines



parallel two-wire | coaxial cable | parallel plate | stripline | microstrip

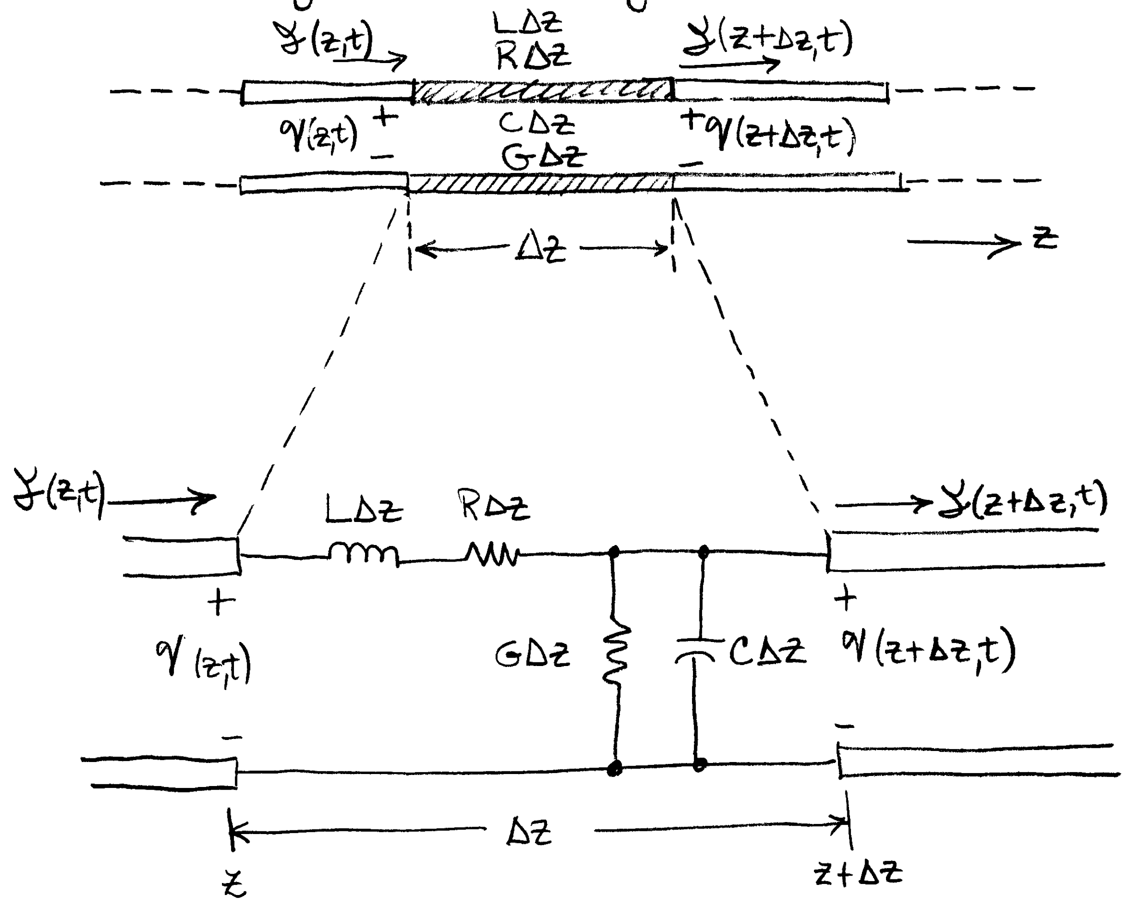
metal

Parallel two-wire and coaxial cable: flexible up to 10 GHz, Rigid up to 35 GHz

Parallel plate, stripline and microstrip: usually fabricated with PCB technology

Transmission lines have two conductors.

Distributed circuit model of two-conductor transmission line.
(Many models are possible. Calculate L & C from definitions and electromagnetic field configurations)




This is basically the simplest model.

Relate input and output voltages (KVL)

$$-v(z,t) + L\Delta z \frac{\partial i(z,t)}{\partial t} + R\Delta z\, i(z,t) + v(z+\Delta z, t) = 0$$

Rearranging

$$v(z+\Delta z,t) - v(z,t) = -R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t}$$

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R\, i(z,t) - L \frac{\partial i(z,t)}{\partial t}$$

$$\frac{\partial v(z,t)}{\partial z} = -R\, i(z,t) - L \frac{\partial i(z,t)}{\partial t} \qquad (1)$$

Do KCL at output node $\left( \sum_{+in} i = 0 \right)$

$$+ i(z,t) - G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0$$

Rearranging

$$i(z+\Delta z,t) - i(z,t) = -G\Delta z\, v(z+\Delta z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t}$$

Slightly more complex
expand $v(z+\Delta z,t) = v(z,t) + \frac{\partial v(z,t)}{\partial t}\Delta z + \frac{\partial^2 v(z,t)}{\partial t^2}\Delta z^2 + \ldots$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G\, v(z,t) - C \frac{\partial v(z,t)}{\partial t} - \lim_{\Delta z \to 0} \left\{ \begin{matrix} \text{higher order} \\ \text{terms} \end{matrix} \right\}$$

and in the limit

$$\frac{\partial i(z,t)}{\partial z} = -G\, v(z,t) - C \frac{\partial v(z,t)}{\partial t} \qquad (2)$$

2.2.2. Lossless lines

$$R = G = 0$$

(1) becomes $\dfrac{\partial v}{\partial z} = -L\dfrac{\partial i}{\partial t}$

(2) becomes $\dfrac{\partial i}{\partial z} = -C\dfrac{\partial v}{\partial t}$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right) = \frac{\partial}{\partial z}\left(-L\frac{\partial i}{\partial t}\right) = -L\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right) = -L\frac{\partial}{\partial t}\left(-C\frac{\partial v}{\partial t}\right)$$

$$\frac{\partial^2 v}{\partial z^2} = +LC\frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial^2 i}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial i}{\partial z}\right) = \frac{\partial}{\partial z}\left(-C\frac{\partial v}{\partial t}\right) = \underbrace{-C\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial z}\right) = -C\frac{\partial}{\partial t}\left(-L\frac{\partial i}{\partial t}\right)}_{\text{reverse order of differentiation}}$$

$$\frac{\partial^2 i}{\partial z^2} = +LC\frac{\partial^2 i}{\partial t^2}$$

These are the wave equations for voltage & current.

Consider voltage equation

$$\frac{\partial^2 v}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 v}{\partial t^2} \qquad \text{where } v_p = \frac{1}{\sqrt{LC}}$$

$v_p$ is a function of the electrical & magnetic properties of the media and NOT the geometry.

General solution is

$$v(z,t) = f\left(t - \frac{z}{v_p}\right) = f(\xi) \qquad \text{where } \xi = t - \frac{z}{v_p}$$

This can be easily confirmed.

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial v}{\partial \xi}$$

differentiating again

$$\frac{\partial^2 v}{\partial t^2} = \underbrace{\frac{\partial}{\partial t}\frac{\partial v}{\partial \xi}}_{\text{reversing order}} = \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial^2 v}{\partial \xi^2} \qquad \text{since } \frac{\partial \xi}{\partial t} = 1$$

and

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial z} \qquad \text{where} \qquad \partial\xi = -\frac{1}{v_p}\partial z$$

$$\frac{\partial v}{\partial z} = -\frac{1}{v_p}\frac{\partial v}{\partial \xi} \qquad\qquad \frac{\partial \xi}{\partial z} = -\frac{1}{v_p}$$

differentiating again

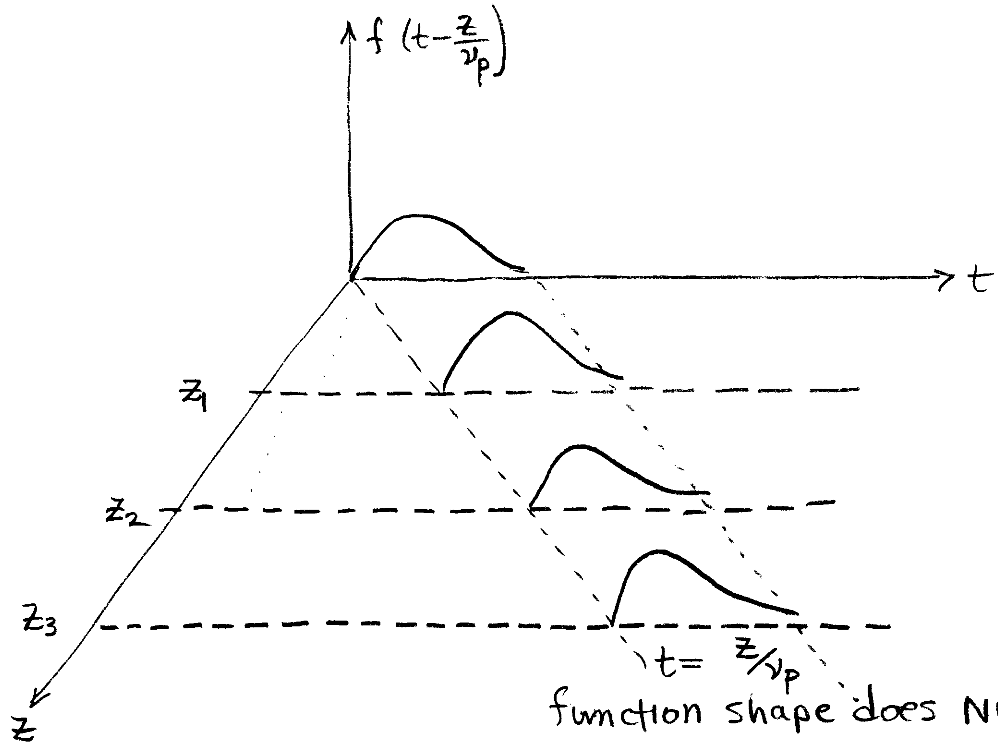
$$\frac{\partial^2 v}{\partial z^2} = -\frac{1}{v_p}\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial \xi}\right) = -\frac{1}{v_p}\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial z}\right) = -\frac{1}{v_p}\frac{\partial}{\partial \xi}\left(-\frac{1}{v_p}\frac{\partial v}{\partial \xi}\right)$$

$$\frac{\partial^2 v}{\partial z^2} = +\frac{1}{v_p^2}\frac{\partial^2 v}{\partial \xi^2}$$

$$\Rightarrow \quad \frac{\partial^2 v}{\partial t^2} = v_p^2\frac{\partial^2 v}{\partial z^2} \quad \blacksquare$$

Actually two solutions

$$v(z,t) = f^+\left(t - \frac{z}{v_p}\right) + f^-\left(t + \frac{z}{v_p}\right)$$




function shape does NOT change.
(lossless line)

moves at velocity $v_p$

Table 2.1 Propagation speeds in some materials

| | |
|---|---|
| Air | 30 cm/ns |
| Glass | 3–15 cm/ns |
| Polyethylene | 20.0 cm/ns |
| Teflon | 20.7 cm/ns |

Consider current equation

$$\frac{\partial^2 \mathcal{I}}{\partial z^2} = LC \frac{\partial^2 \mathcal{I}}{\partial t^2}$$

Since this is the same as the previous equation the solution is of the form

$$\mathcal{I}(z,t) = g\left(t - \frac{z}{v_p}\right) = g(\xi) \quad \text{where } v_p = \frac{1}{\sqrt{LC}}$$

To relate $\mathcal{V}(z,t)$ and $\mathcal{I}(z,t)$

use $$\frac{\partial \mathcal{V}}{\partial z} = -L \frac{\partial \mathcal{I}}{\partial t}$$

$$\frac{\partial \mathcal{V}}{\partial z} = -\frac{1}{v_p}\frac{\partial \mathcal{V}}{\partial \xi} = -\frac{1}{v_p}\frac{\partial f}{\partial \xi} = -\sqrt{LC}\frac{\partial f}{\partial \xi}$$

$$\frac{\partial \mathcal{I}}{\partial t} = \frac{\partial g}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial g}{\partial \xi}$$

$$\therefore \quad -\sqrt{LC}\frac{\partial f}{\partial \xi} = -L\frac{\partial g}{\partial \xi}$$

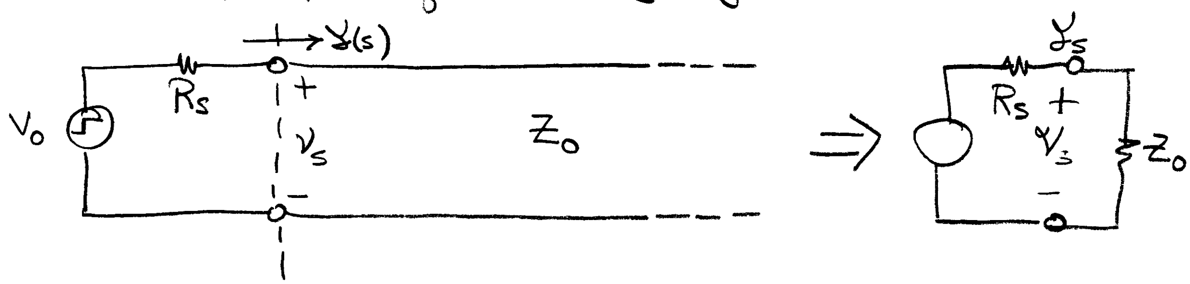
$$\frac{\partial f}{\partial \xi} = \sqrt{\frac{L}{C}}\frac{\partial g}{\partial \xi}$$

$$\therefore \quad g = \frac{1}{\sqrt{\frac{L}{C}}} f = \frac{1}{Z_0} f$$

where $Z_0 \equiv \sqrt{\frac{L}{C}}$ the characteristic impedance of the transmission line.

$Z_0$ = the ratio of voltage to current for a single wave propagating in $+z$ direction

Example 2-1 Step response of an infinitely long lossless line.



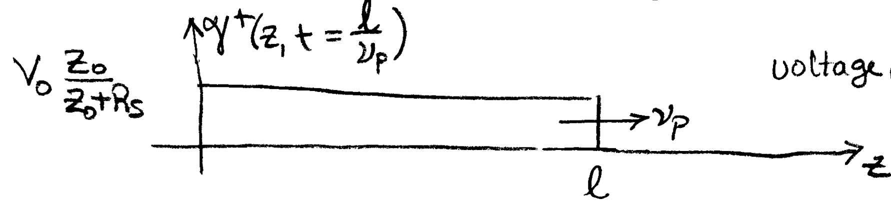

The initial voltage at the source end (no reflection) so just $\mathcal{V}^+$

$$\mathcal{V}_s(t) = \mathcal{V}^+(z=0,t) = V_0 \frac{Z_0}{Z_0 + R_s}$$

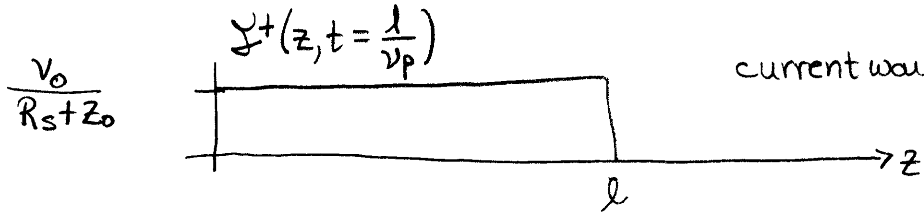
The line looks like $Z_0$ to the $+z$ traveling wave. i.e. $\mathcal{V}_s$ and $\mathcal{I}_s$ at input

since $Z_0$ is the resistance that the line appears to the source initially

$$\mathcal{I}_s(t) = \mathcal{I}^+(z=0,t) = \frac{V_0}{Z_0 + R_s}$$

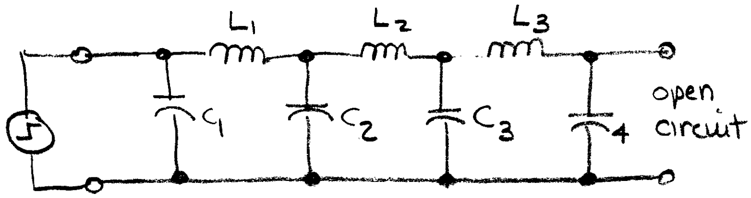



voltage waveform




current waveform

2. Reflection from an open end




There is an orderly progression of voltages down the line.

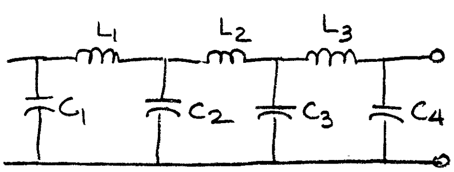
Remember the I.C.'s for L & C { Capacitors look like shorts
Inductors look like opens

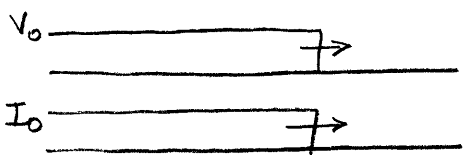
Initially, there is no voltage across $C_1$ (a short) and $L_1$ blocks a voltage to $C_2$.

However, $C_1$ charges up and a voltage appears across it. $L_1$ starts to allow current through to charge $C_2$.

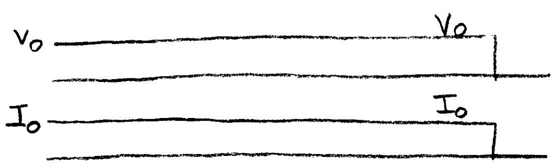
$C_2$ is initially a short and $L_2$ is open. But $C_2$ starts to charge up and current passes through $L_2$.

If the inductance of the line is small, the line looks like a single capacitor

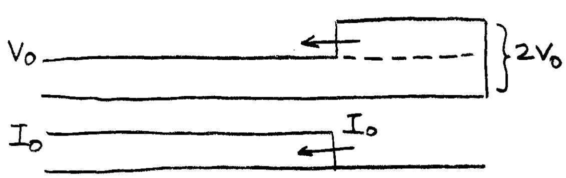



} voltage & current flow down line in phase




} wave reaches end of line. Current charges up capacitor $C_4$ to $2V_0$ — this is a result of the telegraph equation

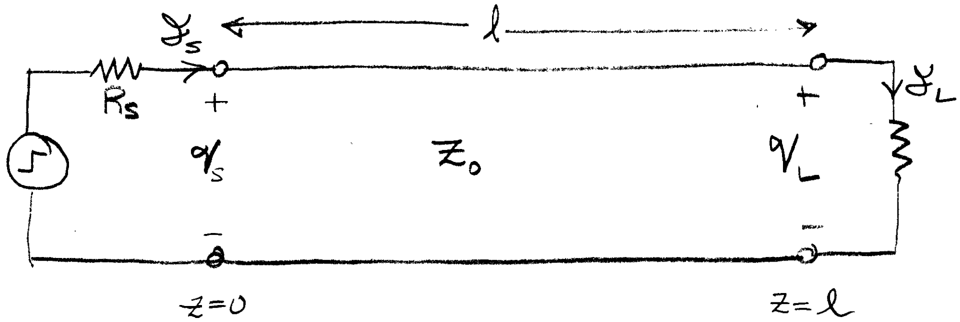



The voltage is a result of $\frac{1}{C}\int I\,dt$

The current is exactly canceled out.

There MUST be zero current at the end of an open line.

## 2.3 Reflection at discontinuities




$$\xrightarrow{t>0} \mathcal{V}_1^+(z,t)$$

$$t_d = \frac{\ell}{v_p}$$

$$\xleftarrow[t>t_d]{\mathcal{V}_1^-(z,t)}$$

the amplitude of the reflected wave is determined by the boundary conditions

write total $\mathcal{V}$ & $\mathcal{I}$ at load

$$\begin{cases} \mathcal{V}_L(t) = \mathcal{V}_1^+(\ell,t) + \mathcal{V}_1^-(\ell,t) \\ \mathcal{I}_L(t) = \dfrac{\mathcal{V}_1^+(\ell,t)}{Z_0} - \underbrace{\dfrac{\mathcal{V}_1^-(\ell,t)}{Z_0}}_{\text{current is flowing towards source}} \end{cases}$$

boundary condition comes from $R_L$ at load side

$$\mathcal{V}_L(t) = \mathcal{I}_L(t) R_L$$

$$\therefore \mathcal{I}_L(t) = \frac{\mathcal{V}_L(t)}{R_L}$$

$$\frac{\mathcal{V}_1^+(\ell,t)}{Z_0} - \frac{\mathcal{V}_1^-(\ell,t)}{Z_0} = \frac{\mathcal{V}_1^+(\ell,t)}{R_L} + \frac{\mathcal{V}_1^-(\ell,t)}{R_L}$$

$$\left(\frac{1}{Z_0} - \frac{1}{R_L}\right)\mathcal{V}_1^+(\ell,t) = \left(\frac{1}{R_L} + \frac{1}{Z_0}\right)\mathcal{V}_1^-(\ell,t)$$

$$\Gamma_L \equiv \frac{\mathcal{V}_1^-(\ell,t)}{\mathcal{V}_1^+(\ell,t)} = \frac{\frac{1}{Z_0} - \frac{1}{R_L}}{\frac{1}{Z_0} + \frac{1}{R_L}} = \frac{R_L - Z_0}{R_L + Z_0}$$

$$\xrightarrow{t>2t_d} \mathcal{V}_2^+(z,t)$$

this wave will get back to source where similar process occurs.

$t > 2t_d$

$$V_s(t) = \underbrace{V_1^+(0,t)}_{\text{initial source is still on}} + \underbrace{V_1^-(0,t)}_{\text{reflected wave from load}} + \underbrace{V_2^+(0,t)}_{\text{newly generated wave at load/line interface}}$$

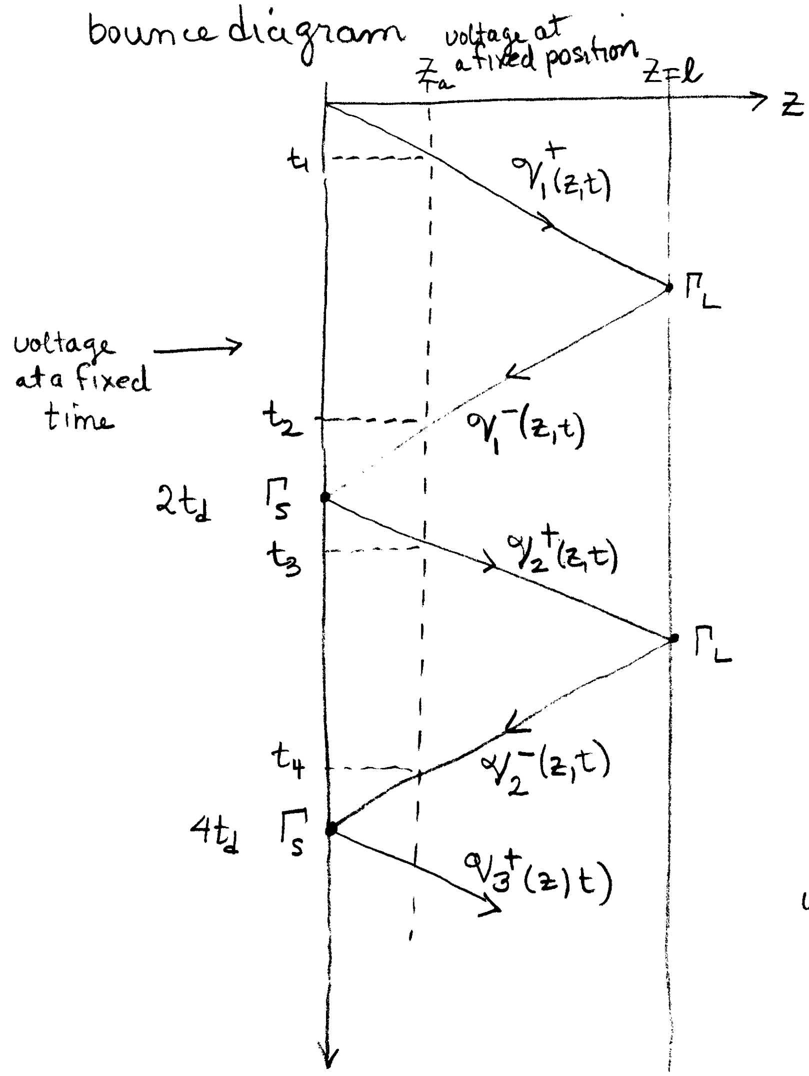
$$I_s(t) = I_1^+(0,t) + I_1^-(0,t) + I_2^+(0,t)$$

Note that $I_1^-(0,t) = -\frac{1}{Z_0} V_1^-(0,t)$

Just as $\Gamma_L = \dfrac{R_L - Z_0}{R_L + Z_0}$ occurred at load

$\Gamma_s = \dfrac{R_s - Z_0}{R_s + Z_0}$ occurs at source
reflects reflection back to load.

bounce diagram




to find $V(z_a, t)$

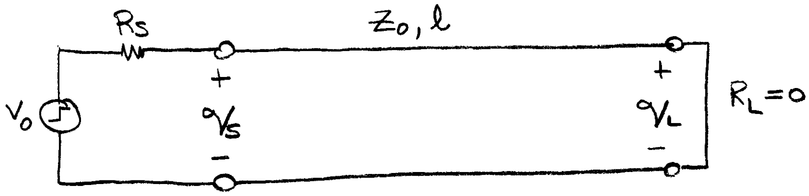
$t_1 = \frac{v_p}{z_a}$

$$V(z_a,t) = \begin{cases} 0 & 0 < t < t_1 \\ V_1^+(z_a,t) & t_1 < t < t_2 \\ V_1^+(z_a,t)(1+\Gamma_L) & t_2 < t < t_3 \\ V_1^+(z_a,t)(1+\Gamma_L+\Gamma_L\Gamma_s) & t_3 < t < t_4 \\ \text{etc.} & \end{cases}$$

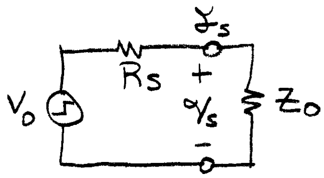
wave evolution

## 2.2.3 Open & Short Circuited Lines

Example 2-2 Short Circuited Lossless Line

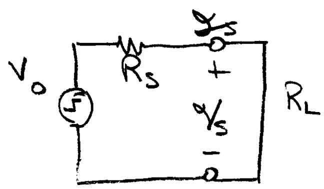



Initially



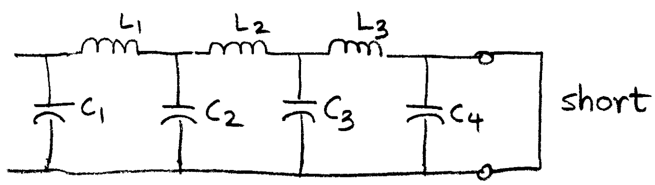

Initially the line looks like the characteristic impedance.

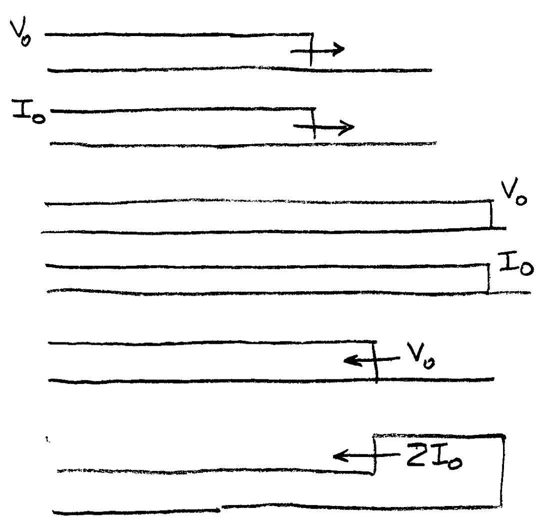
Finally




Finally the line looks like the short that it is.

The short circuit behaves differently than the open.




$C_3$ is changed to $V_o$, but $C_4$ is shorted and cannot charge.

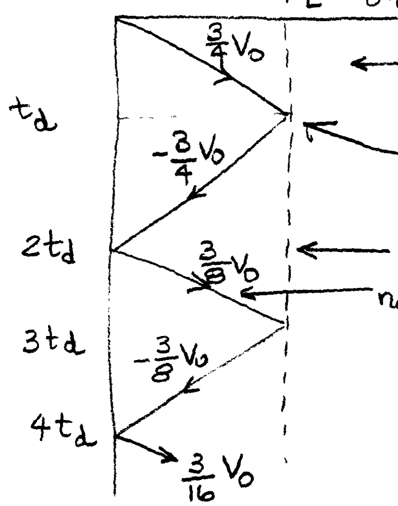
The short dischanges $C_3$ through $L_3$ doubling the current through $L_3$ and propagating down the line.

Analyze using a bounce diagram and $\Gamma_L$.

$R_s$ also has an effect

(a) $R_s = Z_0/3$

$$\Gamma_L = \frac{0 - Z_0}{0 + Z_0} = -1$$




etc.

← wave launched is due to voltage divider $\dfrac{Z_0}{Z_0 + \frac{Z_0}{3}} V_0 = \dfrac{3}{3+1} V_0 = \dfrac{3}{4} V_0$

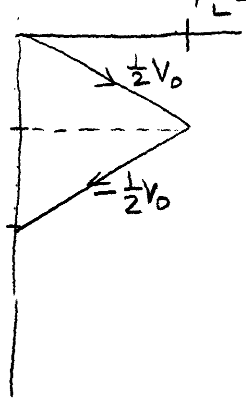
← reflection is $-1$, Note sign change since $V^- = -V^+$ (short)

← at source line sees $\Gamma_s = \dfrac{R_s - Z_0}{R_s + Z_0} = \dfrac{Z_0/3 - Z_0}{Z_0/3 + Z_0} = \dfrac{-2/3}{4/3} = -\dfrac{1}{2}$

← now reflection in $+z$ $\;-\frac{1}{2} \cdot -\frac{3}{4} V_0 = +\frac{3}{8} V_0$

(b) $R_s = Z_0$

$\Gamma_L = -1$



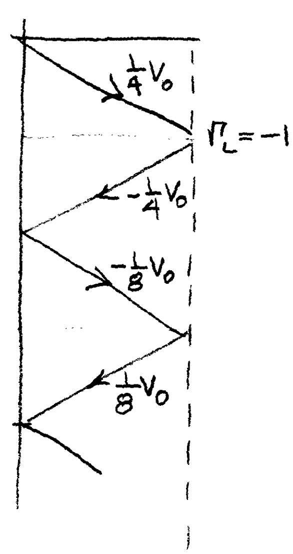

$$\frac{Z_0}{Z_0 + Z_0} V_0 = \frac{1}{2} V_0$$

wave is perfectly reflected

– at source $\Gamma_s = \dfrac{R_s - Z_0}{R_s + Z_0} = 0 \Rightarrow$ no reflection!

wave stops after one bounce.
It is absorbed by $R_s$.

(c) $R_s = 3Z_0$

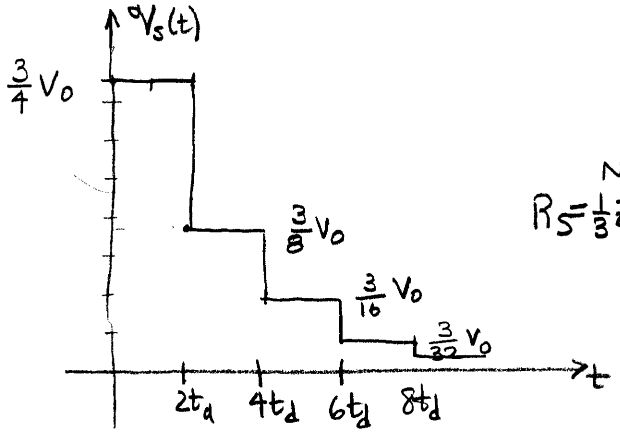



$$\frac{Z_0}{Z_0 + 3Z_0} V_0 = \frac{1}{4} V_0$$

$$\Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} = \frac{3Z_0 - Z_0}{3Z_0 + Z_0} = \frac{2Z_0}{4Z_0} = \frac{1}{2}$$

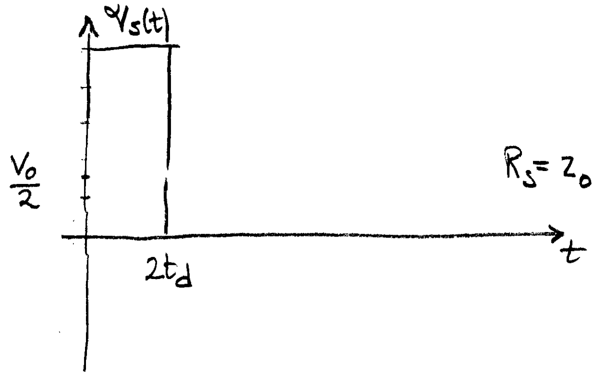
$$\frac{1}{2} \cdot -\frac{1}{4} V_0 = -\frac{1}{8} V_0$$

What's interesting is to plot these voltages as a function of time.

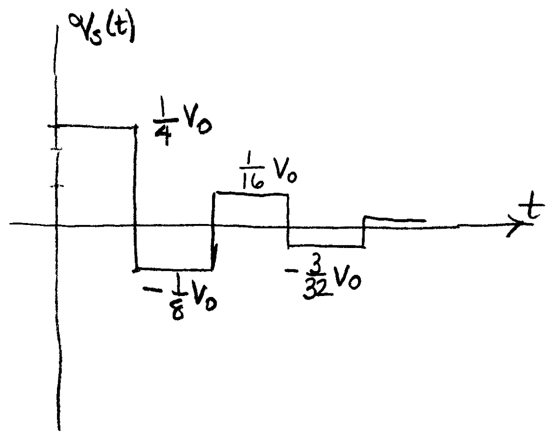



Note how wave decays to zero.

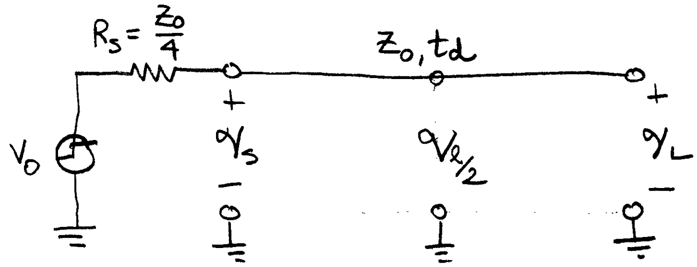
$R_S = \frac{1}{3}Z_0$ overdamped.



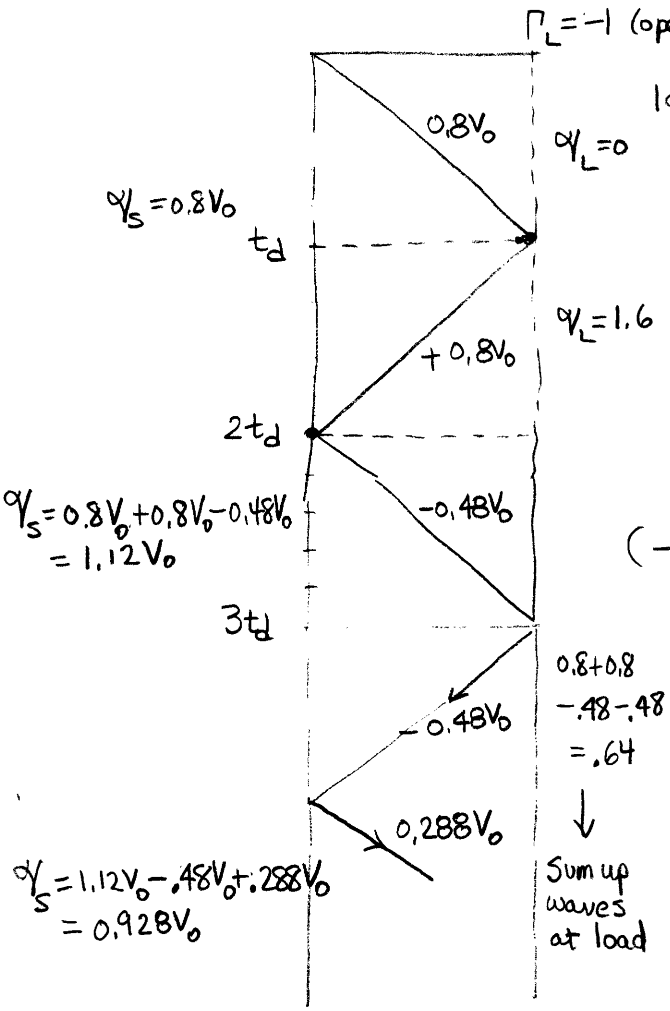

$R_s = Z_0$, critically damped




$R_s = 3Z_0$, underdamped This is ringing.



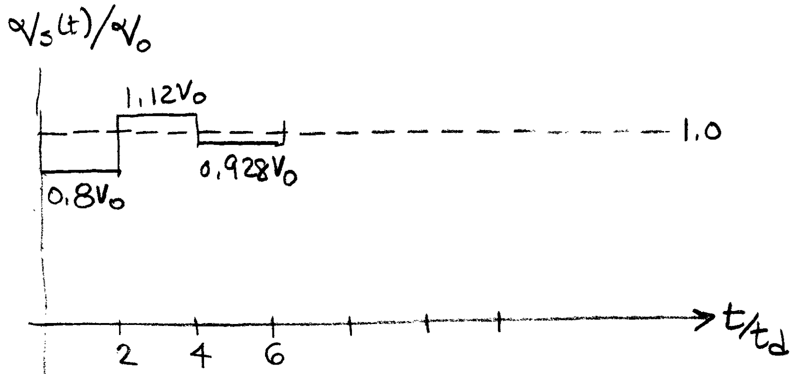

open $\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{\infty - Z_0}{\infty + Z_0} = +1$




launch wave $\frac{Z_0}{Z_0 + \frac{1}{4}Z_0} V_0 = \frac{4}{4+1} V_0 = 0.8V_0$

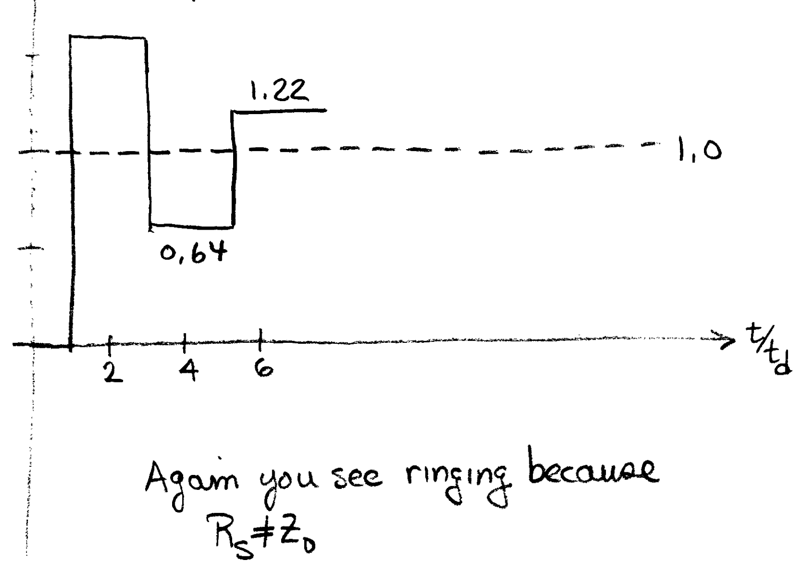
$$\Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} = \frac{\frac{1}{4}Z_0 - Z_0}{\frac{1}{4}Z_0 + Z_0} = -\frac{3}{5} = -0.6$$

$$(-0.6)(+0.8V_0) = -0.48V_0$$



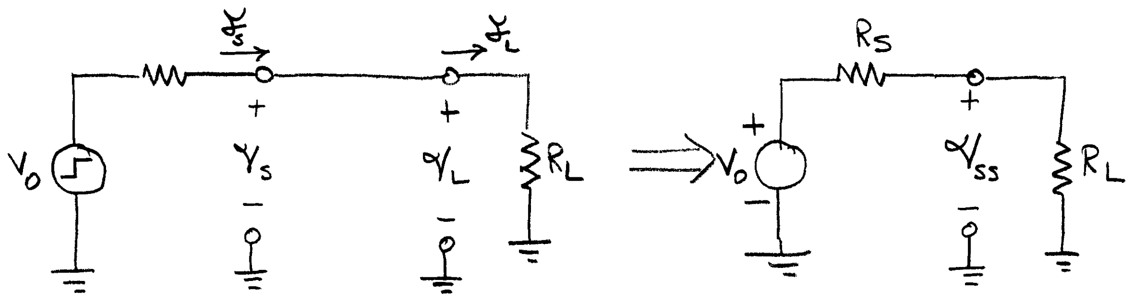

You have to sum **all** the waves at source.

this way.




Again you see ringing because $R_s \neq Z_0$

## 2.4 Transient response / resistive terminations




steady-state equivalent circuit

The voltage at any position $z$ on the line is given by. (steady state)

$$\mathcal{V}(z,t=\infty) = \underset{\substack{+z\\ \text{launched}\\ \text{pulse}}}{\mathcal{V}_1^+(z,\infty)} + \underset{\substack{-z\\ \text{reflected}\\ \text{pulse at}\\ \text{load}}}{\mathcal{V}_1^-(z,\infty)} + \underset{\substack{+z\\ \text{reflected}\\ \text{pulse at source}}}{\mathcal{V}_2^+(z,\infty)} + \mathcal{V}_2^-(z,\infty) + \mathcal{V}_3^+(z,\infty) + \dots$$

using source and load reflectances we can re-write this as

$$\mathcal{V}(z,\infty) = \mathcal{V}_1^+(z,\infty) + \Gamma_L\mathcal{V}_1^+(z,\infty) + \Gamma_S\Gamma_L\mathcal{V}_1^+(z,\infty) + \Gamma_L\Gamma_S\Gamma_L\mathcal{V}_1^+(z,\infty) + \Gamma_S\Gamma_L\Gamma_S\Gamma_L\mathcal{V}_1^+(z,\infty) + \dots$$

$$= \mathcal{V}_1^+\underbrace{\left[1 + (\Gamma_S\Gamma_L) + (\Gamma_S\Gamma_L)^2 + (\Gamma_S\Gamma_L)^3 + \dots\right]}_{\text{these are reflections at source end}} + \mathcal{V}_1^+\underbrace{\left[\Gamma_L + \Gamma_S\Gamma_L^2 + \Gamma_S^2\Gamma_L^3 + \dots\right]}_{\text{these are reflections at load end}}$$

$$= \mathcal{V}_1^+\left[\frac{1}{1-\Gamma_S\Gamma_L}\right] + \mathcal{V}_1^+\Gamma_L\left[\frac{1}{1-\Gamma_S\Gamma_L}\right]$$

$$\mathcal{V}(z,\infty) = \mathcal{V}_1^+(z,\infty)\,\frac{1+\Gamma_L}{1-\Gamma_S\Gamma_L}$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}, \quad \Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}, \quad \mathcal{V}_1^+(z,\infty) = \frac{Z_0}{R_S + Z_0}V_0$$

Substituting,

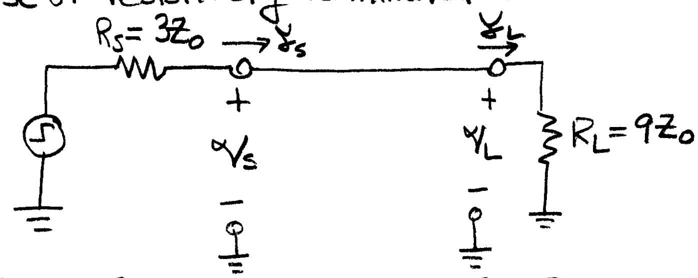
$$\mathcal{V}(z,\infty) = \frac{Z_0}{R_S+Z_0}V_0\,\frac{1+\frac{R_L-Z_0}{R_L+Z_0}}{1-\left(\frac{R_S-Z_0}{R_S+Z_0}\right)\left(\frac{R_L-Z_0}{R_L+Z_0}\right)} = \frac{Z_0V_0}{\cancel{R_S+Z_0}}\,\frac{\cancel{(R_S+Z_0)}(R_L+Z_0)+(R_L-Z_0)\cancel{(R_S+Z_0)}}{(R_S+Z_0)(R_L+Z_0)-(R_S-Z_0)(R_L-Z_0)}$$

$$\mathcal{V}(z,\infty) = Z_0V_0\,\frac{2R_L}{\cancel{R_SR_L}+Z_0R_L+Z_0R_S+\cancel{Z_0^2}-\cancel{R_SR_L}+Z_0R_L+Z_0R_S-\cancel{Z_0^2}} = \frac{2R_LZ_0V_0}{Z_0\,2(R_L+R_S)}$$

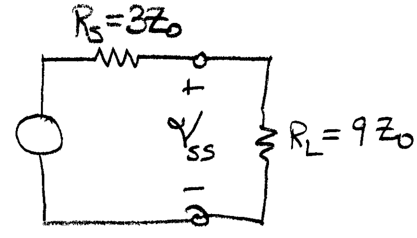
$$\mathcal{V}(z,\infty) = \frac{R_L}{R_L+R_S}V_0 \quad \blacksquare$$

as if transmission line is not there.

Step response of resistively terminated lossless line.

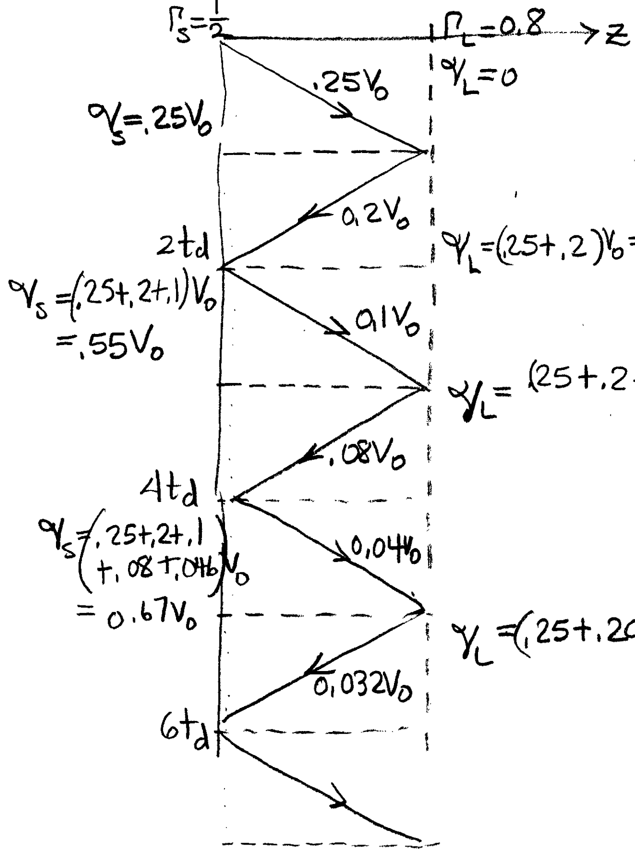



$\Rightarrow$




$$v_{SS} = \frac{9Z_0}{9Z_0+3Z_0}V_0 = 0.75V_0$$

$$v_1^+ = \frac{Z_0}{Z_0+3Z_0}V_0 = .25V_0 \qquad \Gamma_S = \frac{3Z_0-Z_0}{3Z_0+Z_0} = \frac{1}{2} \qquad \Gamma_L = \frac{9Z_0-Z_0}{9Z_0+Z_0} = 0.8$$

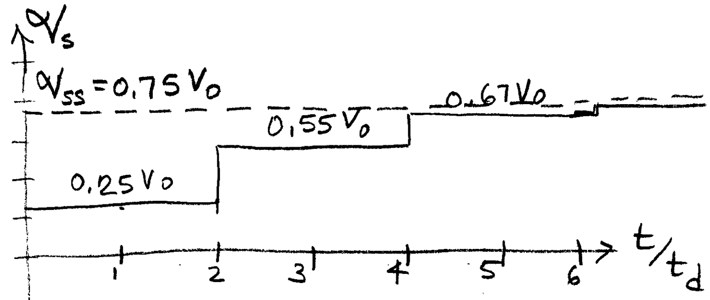


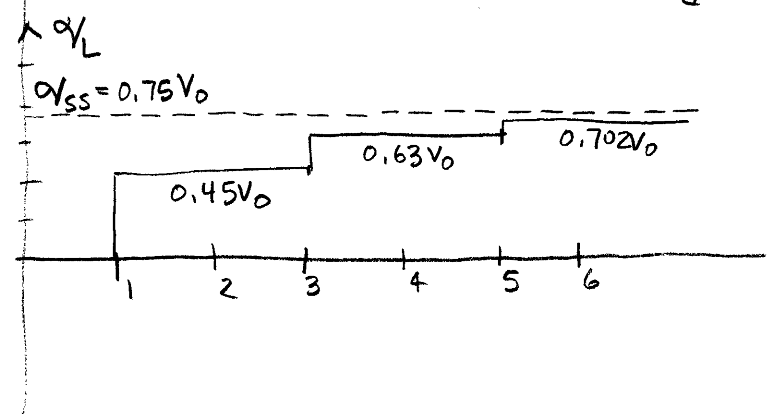

$.25(0.8)V_0 = 0.2V_0$

$(0.5)(0.2)V_0 = 0.1V_0$

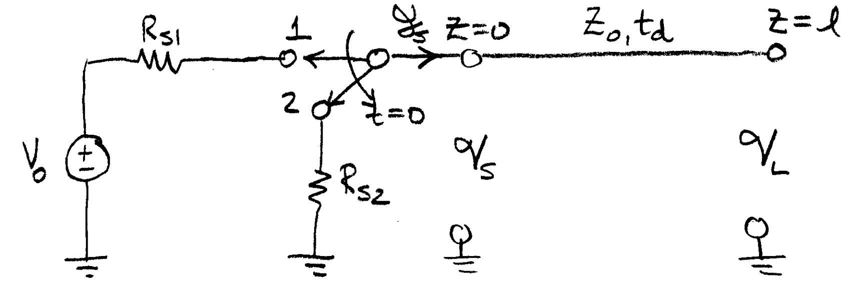
$(0.1V_0)(0.8) = 0.08V_0$

$(0.5)(0.08V_0) = 0.04V_0$



Example 2-5 A Charged Line Connected To a Resistor




At $t=0$ the switch is moved from position 1 to position 2

(a) Assume $R_{S2} = \dfrac{Z_0}{3}$

The line is initially charged to $V_0$ volts. $\mathcal{V}_s(0^-) = \mathcal{V}_L(0^-) = V_0$

The new wave launched on the line is the difference between $\mathcal{V}_s$ at $t=0^-$ and $\mathcal{V}_s$ at $t=0^+$.

$$\mathcal{V}_1^+(0,0) = \mathcal{V}_s(0^+) - \mathcal{V}_s(0^-) = \mathcal{V}_s(0^+) - V_0$$

where $\mathcal{V}_s(0^+) = - R_{S2}\, \mathcal{I}_s(0^+)$

– sign because current will be going to ground and negative wrt $\mathcal{I}_s$

$$\mathcal{I}_s(0^+) = \mathcal{I}_1(0,0) = \frac{\mathcal{V}_1^+(0,0)}{Z_0}$$

Substituting gives

$$\mathcal{V}_1^+(0,0) = - R_{S2} \frac{\mathcal{V}_1^+(0,0)}{Z_0} - V_0$$

solving for $\mathcal{V}_1^+(0,0)$

$$\mathcal{V}_1^+(0,0)\, Z_0 = - R_{S2} \mathcal{V}_1^+(0,0) - Z_0 V_0$$

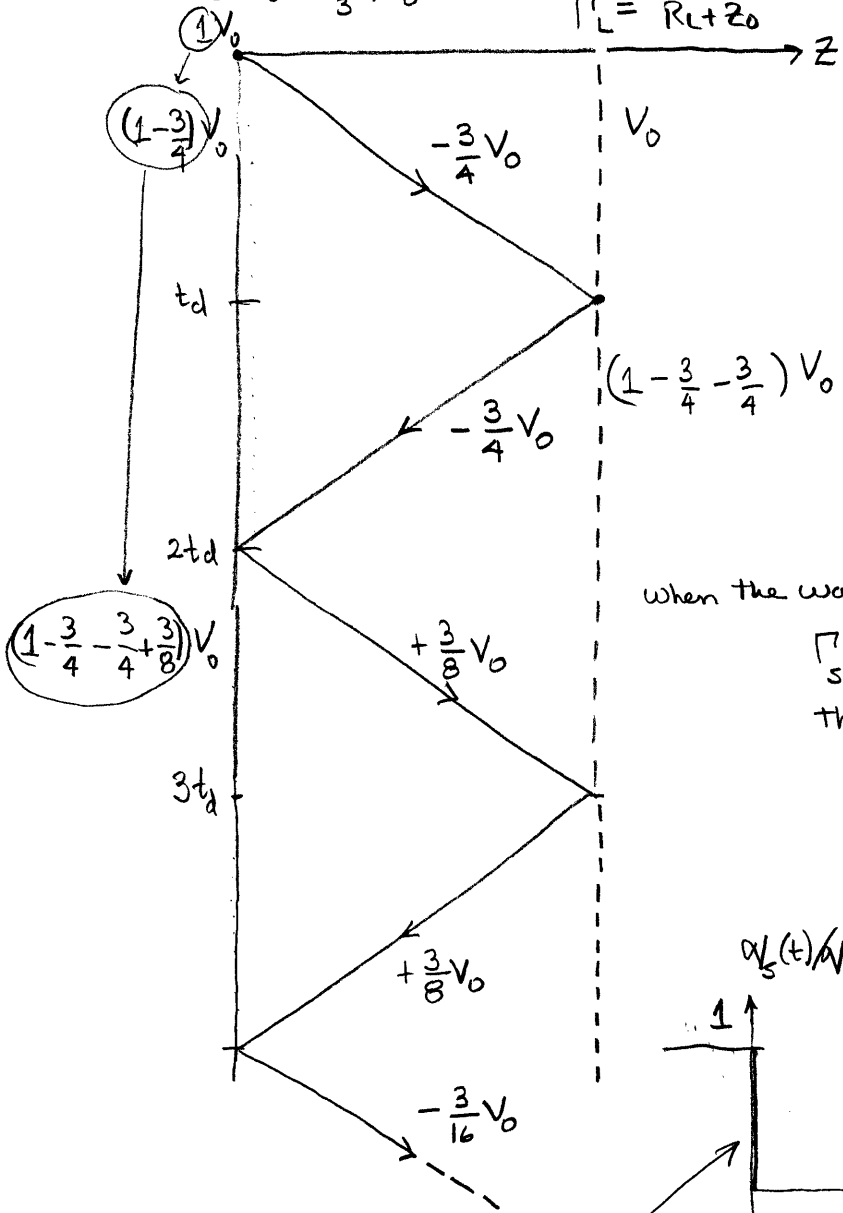
$$\mathcal{V}_1^+(0,0) = \frac{- Z_0 V_0}{Z_0 + R_{S2}}$$

This is a bit like a voltage divider except it is $Z_0$ and not $R_{S2}$ in the numerator since this is the wave in the line.

given this you can construct a bounce diagram

$$\Gamma_{S2} = \frac{R_S - Z_0}{R_L + Z_0} = \frac{\frac{Z_0}{3} - Z_0}{\frac{Z_0}{3} + Z_0} = -\frac{1}{2}$$

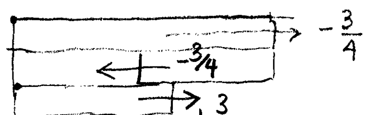
$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = 1 \text{ as } R_L \to \infty$$



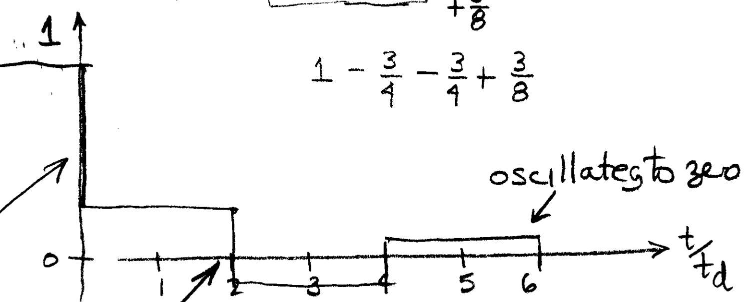

wave launched is $-\frac{Z_0}{Z_0+R_{S2}} V_0$

$$v^+(0,0) = -\frac{Z_0}{Z_0+\frac{1}{3}Z_0} V_0 = -\frac{3}{4}V_0$$

when the wave gets to the source end it sees $\Gamma_{S2} = -\frac{1}{2}$. The reflected wave is then $-\frac{1}{2}\left(-\frac{3}{4}V_0\right) = +\frac{3}{8}V_0$




$1 - \frac{3}{4} - \frac{3}{4} + \frac{3}{8}$



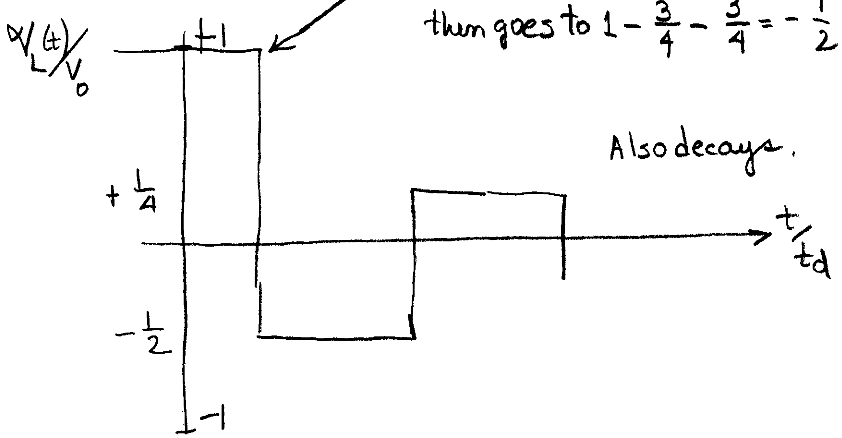

$v_S = V_0 - v^+$
$= 1 - 3/4$

$v_S = V_0 - v^+$
$= 1 - \frac{3}{4} - \frac{3}{4} + \frac{3}{8} = -\frac{1}{8}$

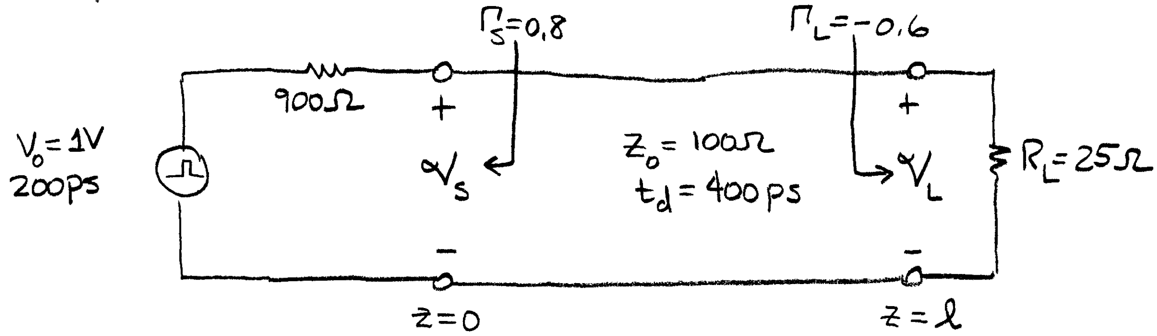
at $V_0$ until pulse gets there

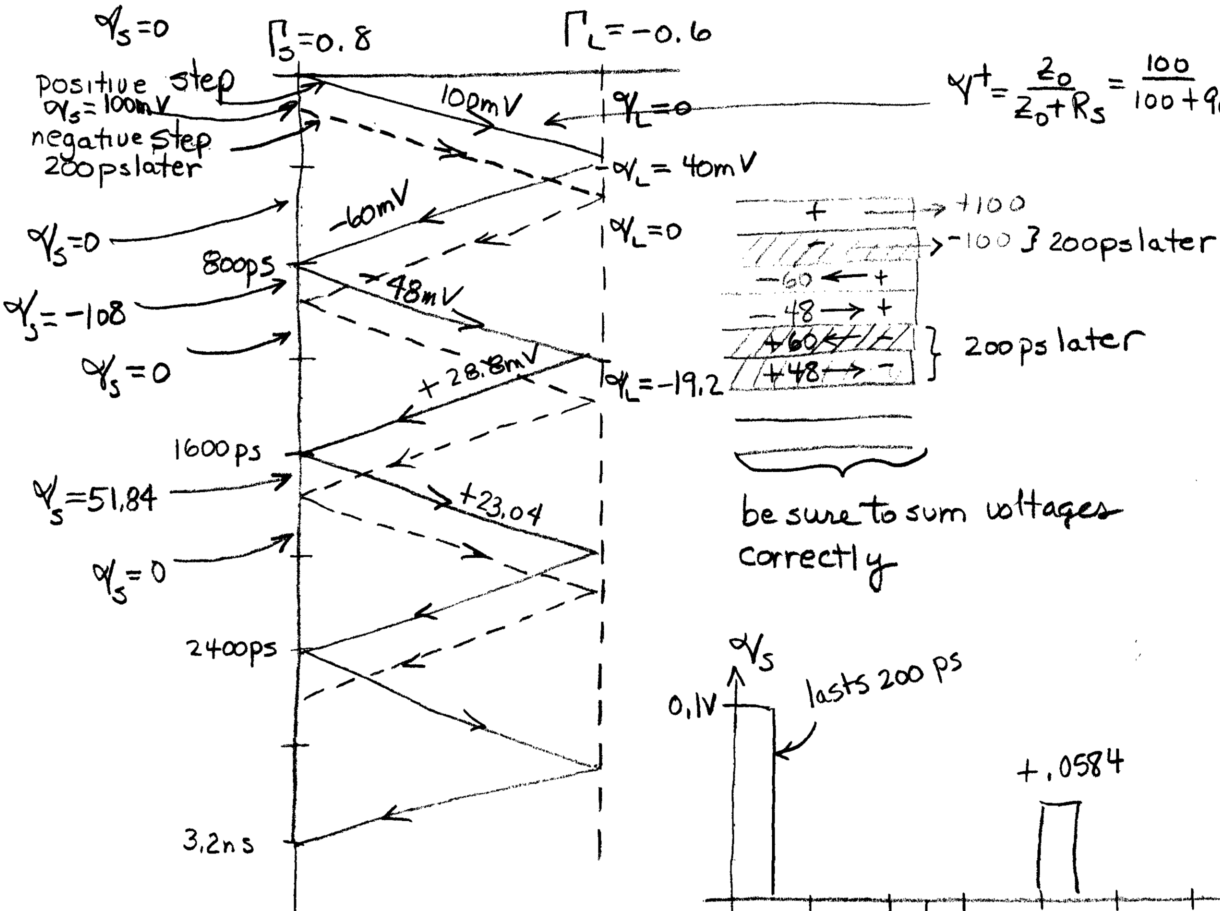
then goes to $1 - \frac{3}{4} - \frac{3}{4} = -\frac{1}{2}$

Also decays.



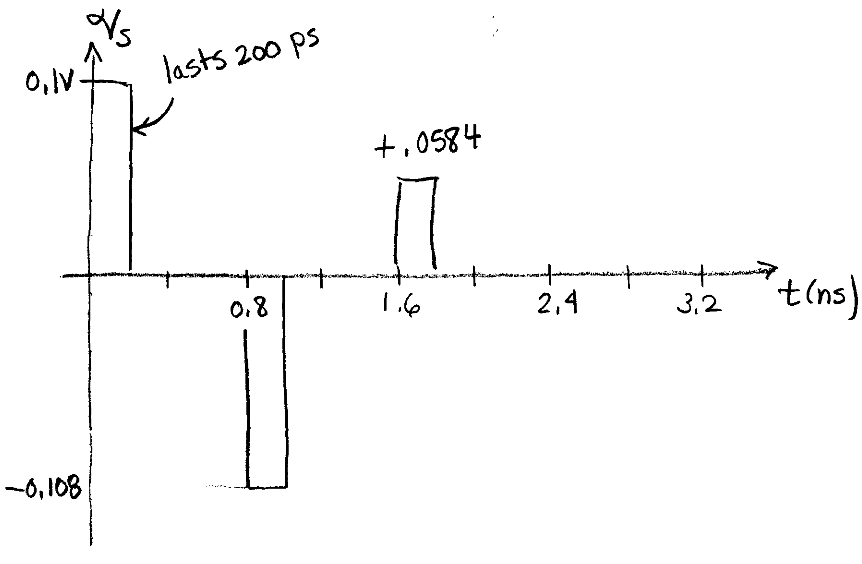
# Example 2-6 Pulse Excitation of a Transmission line



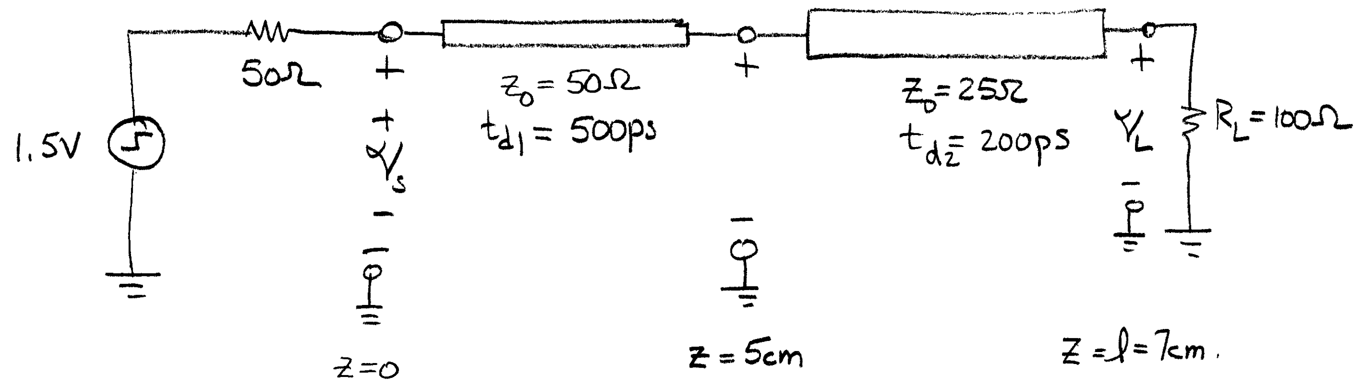




$$V^+ = \frac{Z_0}{Z_0 + R_s} = \frac{100}{100+900} \cdot 1V = 100\,mV$$

be sure to sum voltages correctly



# Example 2.8 Cascaded Transmission Lines




compute the reflection coefficients

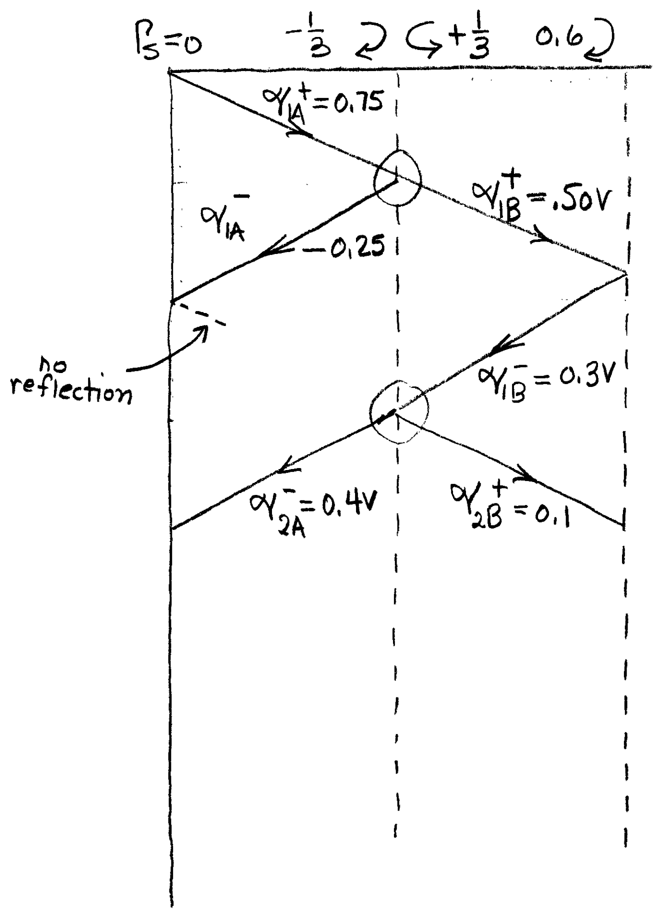
$$\Gamma = \frac{R_L - Z_0}{R_L + Z_0}$$

$$\Gamma_s = \frac{50-50}{50+50} = 0$$

$$\Gamma_L = \frac{100-25}{100+25} = 0.6$$

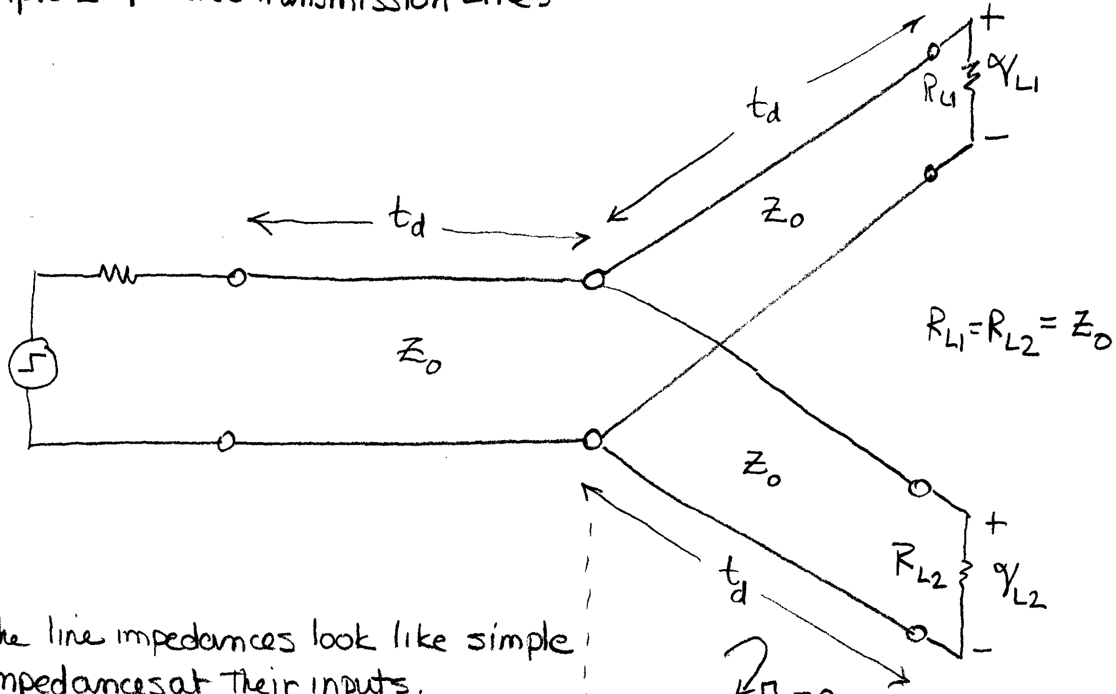
$$\Gamma_{AB} = \frac{25-50}{25+50} = -\frac{1}{3} \qquad \Gamma_{BA} = \frac{50-25}{50+25} = +\frac{1}{3}$$

we treat the second transmission line as a simple input impedance




Note boundary conditions at junction of transmission lines

Example 2-9 Three Transmission Lines




$R_{L1} = R_{L2} = Z_0$

The line impedances look like simple impedances at their inputs.

$\Gamma_L = 0$

$-\frac{1}{3}$

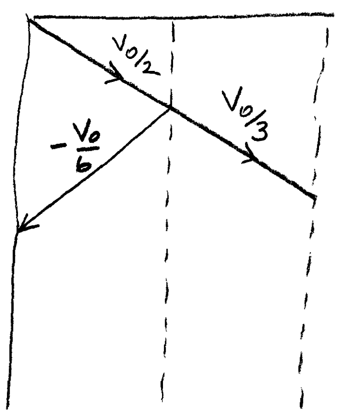
$\Gamma = 0$

$\Gamma = \frac{25-50}{25+50} = -\frac{1}{3}$ $\quad -\frac{1}{3}$

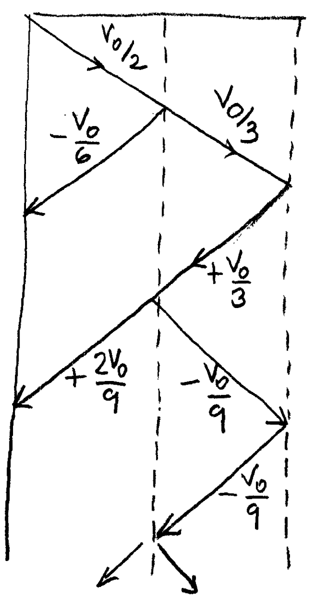
since $50 \| 50 = 25\Omega$.

$\Gamma_L = 0$

Simple if $R_{L1} = R_{L2} = Z_0$




Suppose $R_{L1} = Z_0$, $R_{L2} = \infty$



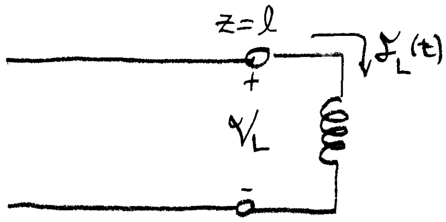

for $R_{L2} = \infty$ $\Gamma = +1$

## 2.5 Reactive & Non-Linear Terminations

Examples: capacitive loading by busses
inductive loading due to bonding wire inductances
pins on IC packages, PCB vias, and variations in line width create inductances and capacitances

The solution is similar to our previous termination solutions but will involve integrals or derivatives

Analysis:




For the load $\quad \mathcal{V}_L(t) = L\dfrac{d\mathcal{I}_L(t)}{dt}$

For the transmission line

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(\ell,t) + \mathcal{V}_1^-(\ell,t)$$

$$\mathcal{I}_L(t) = \mathcal{I}^+(\ell,t) + \mathcal{I}^-(\ell,t) = \frac{\mathcal{V}_1^+(\ell,t)}{Z_0} - \frac{\mathcal{V}_1^-(\ell,t)}{Z_0}$$

Combining these

$$\mathcal{V}_1^+(\ell,t) + \mathcal{V}_1^-(\ell,t) = L\frac{d}{dt}\left(\frac{\mathcal{V}_1^+(\ell,t)}{Z_0} - \frac{\mathcal{V}_1^-(\ell,t)}{Z_0}\right)$$

Combining the + and − traveling waves gives

$$\frac{L}{Z_0}\frac{d\mathcal{V}_1^-(\ell,t)}{dt} + \mathcal{V}_1^-(\ell,t) = \frac{L}{Z_0}\frac{d\mathcal{V}_1^+(\ell,t)}{dt} - \mathcal{V}_1^+(\ell,t)$$

$$\underbrace{\frac{d\mathcal{V}_1^-(\ell,t)}{dt} + \frac{Z_0}{L}\mathcal{V}_1^-(\ell,t)}_{\text{this is simply a first order D.E. for the reflection.}} = \underbrace{\frac{d}{dt}\mathcal{V}_1^+(\ell,t) - \frac{Z_0}{L}\mathcal{V}_1^+(\ell,t)}_{\text{we assume this is known since it is coming from the source}}$$

Let's assume the incident wave is a step as we have been doing

$$\mathcal{V}_1^+(l,t) = V_0$$

The D.E. at the load is then

$$\frac{d\mathcal{V}_1^-(l,t)}{dt} + \frac{Z_0}{L}\mathcal{V}_1^-(l,t) = -\frac{Z_0}{L}V_0$$

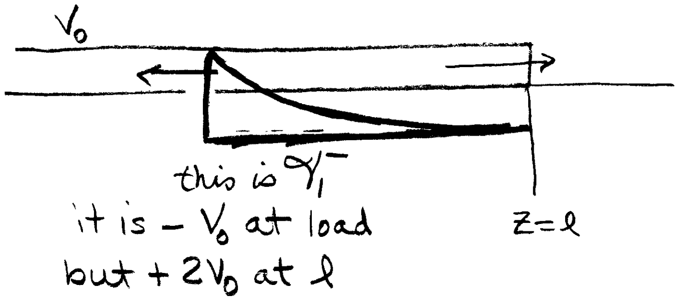
The solution is then the sum of the homogeneous & transient solution

$$\mathcal{V}_1^-(l,t) = -V_0 + Ke^{-\frac{Z_0}{L}t}$$

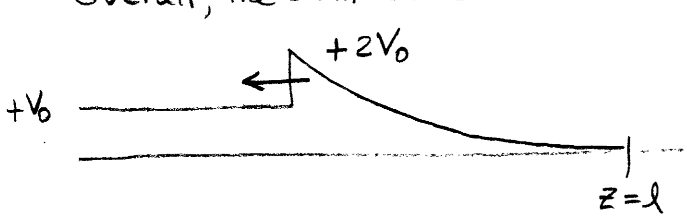
To determine K we note that at $t=0$ $\mathcal{I}_L(t=0)=0$. This is equivalent to an open circuit and we know that KCL gives $\mathcal{V}_1^-(l,t=0) = V_0$, i.e., its completely reflected.

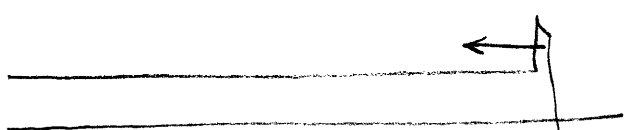
For $\mathcal{V}_1^-(l,t=0)=0$ $\quad K = 2V_0$

$$\therefore \quad \mathcal{V}_1^-(l,t=0) = -V_0 + 2V_0 e^{-\frac{Z_0}{L}t}$$



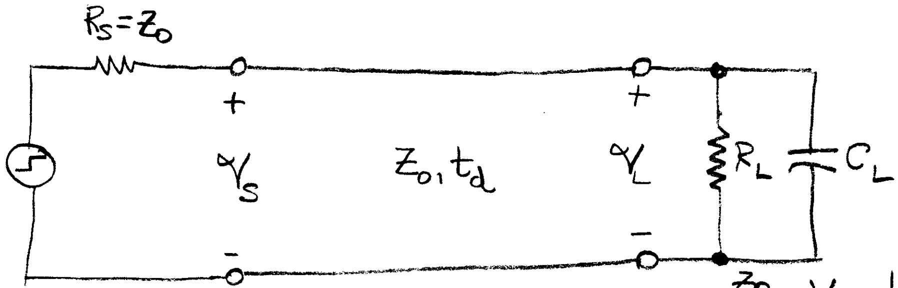

Overall, the sum looks like this



This is the same sum at $t=0^+$

Example 2-10. Lossy capacitative load




The wave launched on the line is $\mathcal{V}_1^+(0,0) = \frac{Z_0}{Z_0+Z_0} V_0 = \frac{1}{2} V_0$

We write the voltage and current equations at the load.

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(l,t) + \mathcal{V}_1^-(l,t)$$

$$\mathcal{I}_L(t) = \mathcal{I}_1^+(l,t) + \mathcal{I}_1^-(l,t) = \frac{\mathcal{V}_1^+(l,t)}{Z_0} - \frac{\mathcal{V}_1^-(l,t)}{Z_0}$$

The load equation is given by KCL as

$$\mathcal{I}_L(t) = \frac{\mathcal{V}_L(t)}{R_L} + C_L \frac{d\mathcal{V}_L(t)}{dt}$$

We do algebra to get the D.E. at the load.

$$\frac{\mathcal{V}_1^+(l,t)}{Z_0} - \frac{\mathcal{V}_1^-(l,t)}{Z_0} = \frac{\mathcal{V}_1^+(l,t)}{R_L} + \frac{\mathcal{V}_1^-(l,t)}{R_L} + C_L \frac{d}{dt}\left[\mathcal{V}_1^+(l,t) + \mathcal{V}_1^-(l,t)\right]$$

$$\frac{1}{C_L}\left[\frac{\mathcal{V}_1^-(l,t)}{Z_0} + \frac{\mathcal{V}_1^-(l,t)}{R_L} + C_L \frac{d\mathcal{V}_1^-(l,t)}{dt} = \frac{\mathcal{V}_1^+(l,t)}{Z_0} - \frac{\mathcal{V}_1^+(l,t)}{R_L} - C_L \frac{d\mathcal{V}_1^+(l,t)}{dt}\right]$$

$$\frac{d\mathcal{V}_1^-(l,t)}{dt} + \left[\frac{1}{C_L Z_0} + \frac{1}{C_L R_L}\right] \mathcal{V}_1^-(l,t) = -\frac{d\mathcal{V}_1^+(l,t)}{dt} + \left[\frac{1}{C_L Z_0} - \frac{1}{C_L R_L}\right] \mathcal{V}_1^+(l,t)$$

As before we have a step, this time of magnitude $\frac{1}{2} V_0$

Under these conditions it becomes

$$\frac{d\mathcal{V}_1^-(l,t)}{dt} + \left(\frac{R_L + Z_0}{R_L Z_0}\right)\frac{1}{C_L}\mathcal{V}_1^-(l,t) = \frac{R_L - Z_0}{R_L Z_0}\frac{1}{C_L}\frac{V_0}{2}$$

the $\frac{d\mathcal{V}_1^+(l,t)}{dt}$ term disappears because the incident voltage is constant in time

The general solution is $\mathcal{V}_1^-(l,t) = K_1 + K_2 e^{-\frac{R_L+Z_0}{R_L Z_0}\frac{1}{C_L}(t - t_d)}$

since the wave started at $z = 0$ at $t = 0$

The capacitor is initially a short circuit giving $V_1^-(l,t_d) + V_1^+(l,t_d) = 0$

or $\quad V_1^-(l,t_d) = -V_1^+(l,t_d) = -\frac{V_0}{2}$

After a long time the capacitor is fully changed and $V^+$ sees only the resistor. This is a resistive termination $R_L$ and

$$V_1^-(l,\infty) = \Gamma_L V_1^+(l,\infty) = \frac{R_L - Z_0}{R_L + Z_0} V_1^+(l,\infty) = \frac{R_L - Z_0}{R_L + Z_0}\frac{V_0}{2}$$

We can use these initial conditions to write the general solution

$$V_1^-(l,\infty) = \frac{R_L - Z_0}{R_L + Z_0}\frac{V_0}{2} = K_1$$

$$V_1^-(l,t_d) = K_1 + K_2 = -\frac{V_0}{2}$$

$$K_2 = -\frac{V_0}{2} - K_1 = -\frac{V_0}{2} - \frac{R_L - Z_0}{R_L + Z_0}\frac{V_0}{2}$$

$$= -\frac{V_0}{2}\left[1 + \frac{R_L - Z_0}{R_L + Z_0}\right] = -\frac{V_0}{2}\left[\frac{R_L + \cancel{Z_0} + R_L - \cancel{Z_0}}{R_L + Z_0}\right] = -\frac{V_0 R_L}{R_L + Z_0}$$

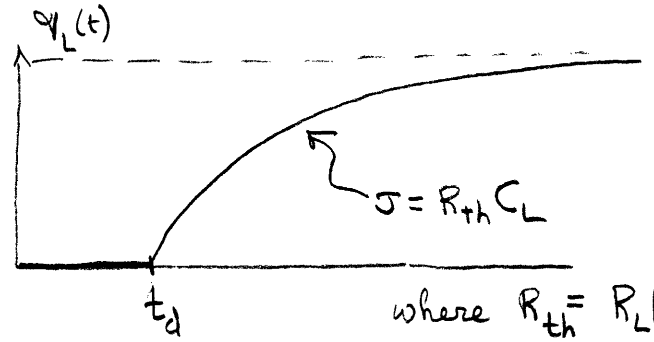
$$V_1^-(l,t) = \frac{R_L - Z_0}{R_L + Z_0}\frac{V_0}{2} - \frac{V_0 R_L}{R_L + Z_0} e^{-\frac{R_L + Z_0}{R_L Z_0}\frac{1}{C_L}(t - t_d)}$$

However, this is more useful to write in terms of $V^+ = \frac{V_0}{2}$

$$V_1^-(l,t) = V_1^+(l,t)\left[\frac{R_L - Z_0}{R_L + Z_0} - \frac{2R_L}{R_L + Z_0} e^{-\frac{R_L + Z_0}{R_L Z_0}\frac{1}{C_L}(t - t_d)}\right]$$

It will look like this

Our book likes to plot both $v_L(t)$ and $v_s(t)$



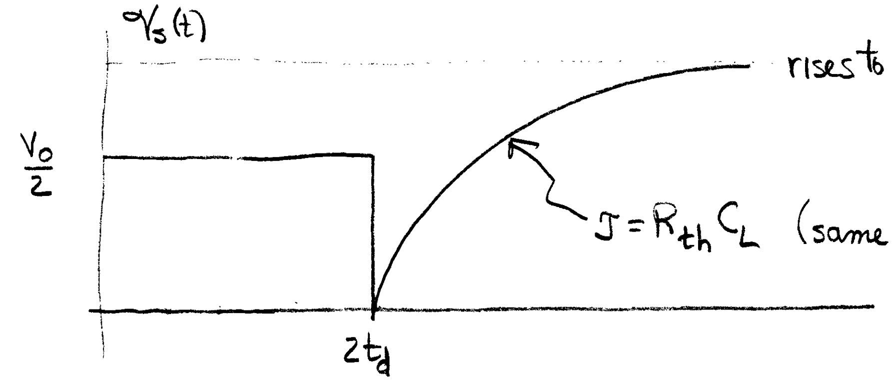

rises to $\frac{R_L}{R_L + Z_0} V_0$

DC voltage divider at $t = \infty$

where $R_{th} = R_L \| Z_0$

parallel combination of line and load

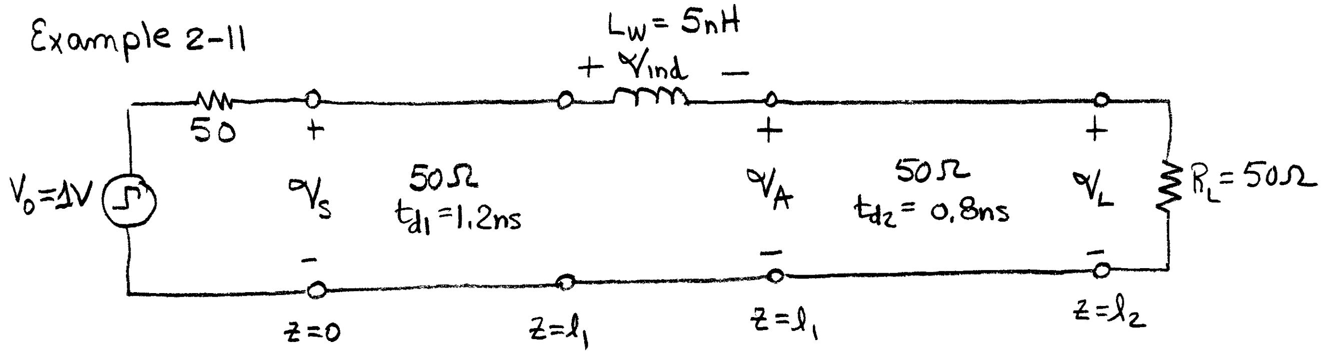
voltage is zero until wave gets to load.




rises to $\frac{R_L}{R_L + Z_0} V_0$ (same final value)

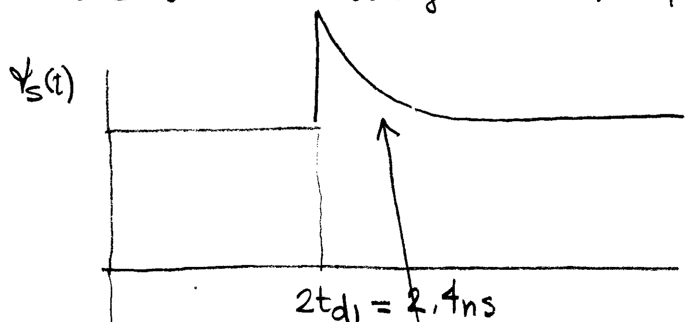
$\tau = R_{th} C_L$ (same time constant)

# Example 2-11



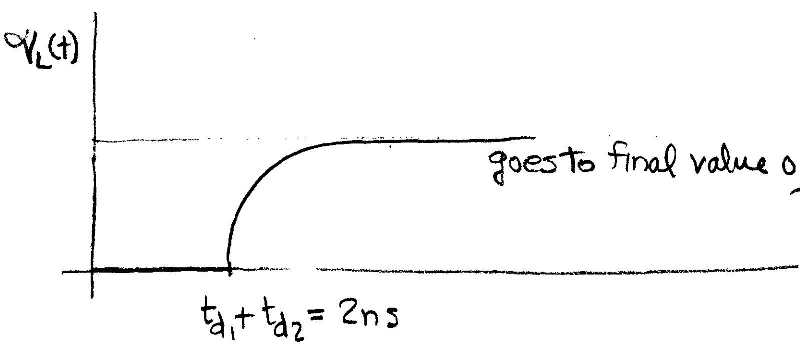

This is an example of how transmission lines might be connected together by a jumper.

Because $Z_0 = 50\Omega$ a voltage wave $V^+(z,t) = 0.5V$ is launched.




remains at 0.5V until reflection from L comes back.

initially looks like a open with reflection +1 which decays back to 0.5V

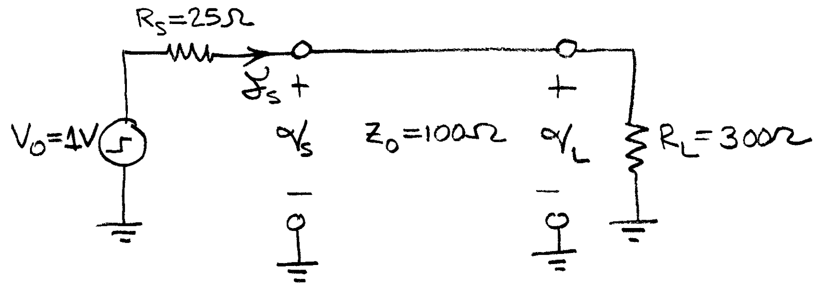



goes to final value of $\frac{50}{50+50} \cdot 1V = 0.5$ volts.

## 2.52 Non-linear Terminations

Bergeron graphical technique

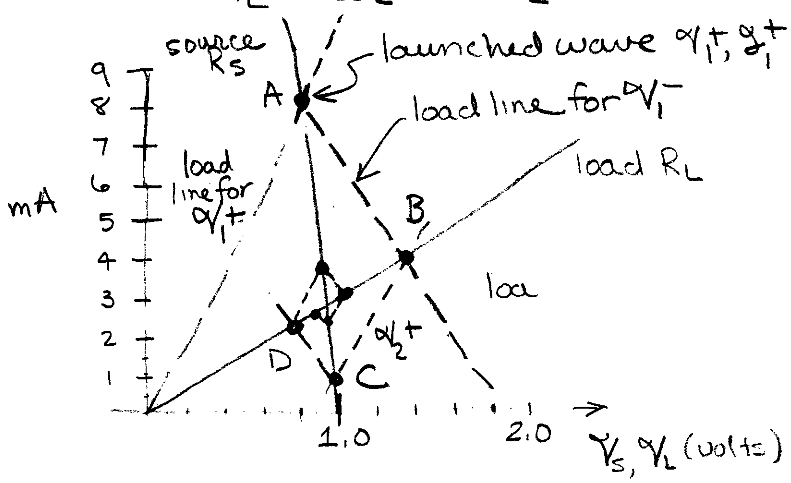
Example 2-13 Graphical solution of resistively terminated line




① Plot i-v characteristics of source and load

@ source $\quad V_S = V_0 - R_S I_S = 1 - 25 I_S \qquad I_S = -40 V_S + 40 \,(mA)$

@ load $\quad V_L = R_L I_L = 300 I_L \qquad I_L = \frac{10}{3} V_L \,(mA)$




A: $I_S, V_S$ @ $t=0$ (0.8V, 8mA)
B: $V_L, I_L$ @ $t=t_d$ (0.4V, 4mA)
C: $I_S, V_S$ @ $t=2t_d$ (0.96V, 1.6mA)

② launch $V^+$. Since it is in line $\quad I^+ = \frac{V_1^+}{Z_0} = \frac{V_1^+}{100} = 10 V_1 \,(mA)$

③ intersection of $V^+$ and source load lines is wave launched.
This gives what we previously got by voltage divider.
Intersection is $V_1^+ = 0.8\,v$, $I_1^+ = 8\,mA$.

④ Add in load line for $V_1^-$ generated at $t = t_d$.
This load line has slope $-\frac{1}{Z_0} = -\frac{1}{100}$. Negative since - direction.

⑤ Consider reflection back to source. Reflection reaches source at $t = 2t_d$ generating $V_2^+$, $I_2^+$

Total voltage $V_S = \underbrace{V_1^+ + V_1^-}_{\text{This is the voltage } V_B \text{ at the load}} + V_2^+ = V_B + V_2^+$

total current $\mathcal{I}_s = \underbrace{\mathcal{I}_1^+ + \mathcal{I}_1^-}_{\text{This is } I_B \text{ from load}} + \mathcal{I}_2^+ \;\; = I_B + \mathcal{I}_2^+$

At the line input $\quad \mathcal{I}_2^+ = \dfrac{\mathcal{V}_2^+}{Z_0}$

Substituting results gives

$$\mathcal{I}_S - I_B = \frac{\mathcal{V}_S - V_B}{Z_0}$$

This is the equation of a line passing through $(V_B, I_B)$ with slope $\frac{1}{Z_0}$ and described $\mathcal{V}_2^+$, $\mathcal{I}_2^+$. This load line intersects the source load line at C, which is about $\mathcal{V}_S = 0.96V$, $\mathcal{I}_S = 1.6mA$

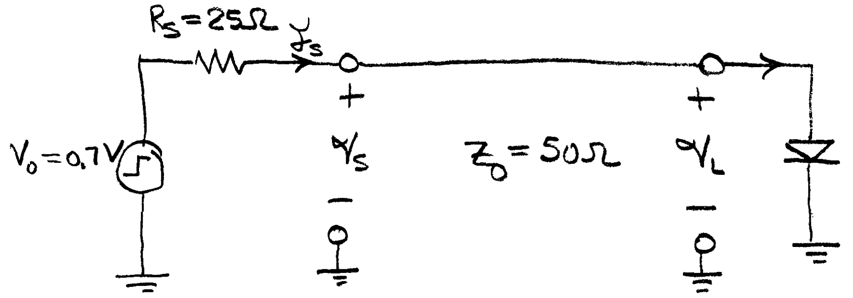
(6) Reflection at source gives $\mathcal{V}_2^-$ back towards load with slope $-\frac{1}{Z_0} = -\frac{1}{100}$
This gives D.

(7) Repeat process until converges.

- Each dashed line is the load line for a + or − z directed traveling wave with slope $\pm\frac{1}{Z_0}$.
- Intersection with source or load load lines defines launch of new wave by reflection.
- Converges to steady state values.

NOTE: You would have to calculate $\mathcal{V}_n^+$, $\mathcal{V}_n^-$ from sum of voltages at each reflection

## Example 2-14 Nonlinear termination

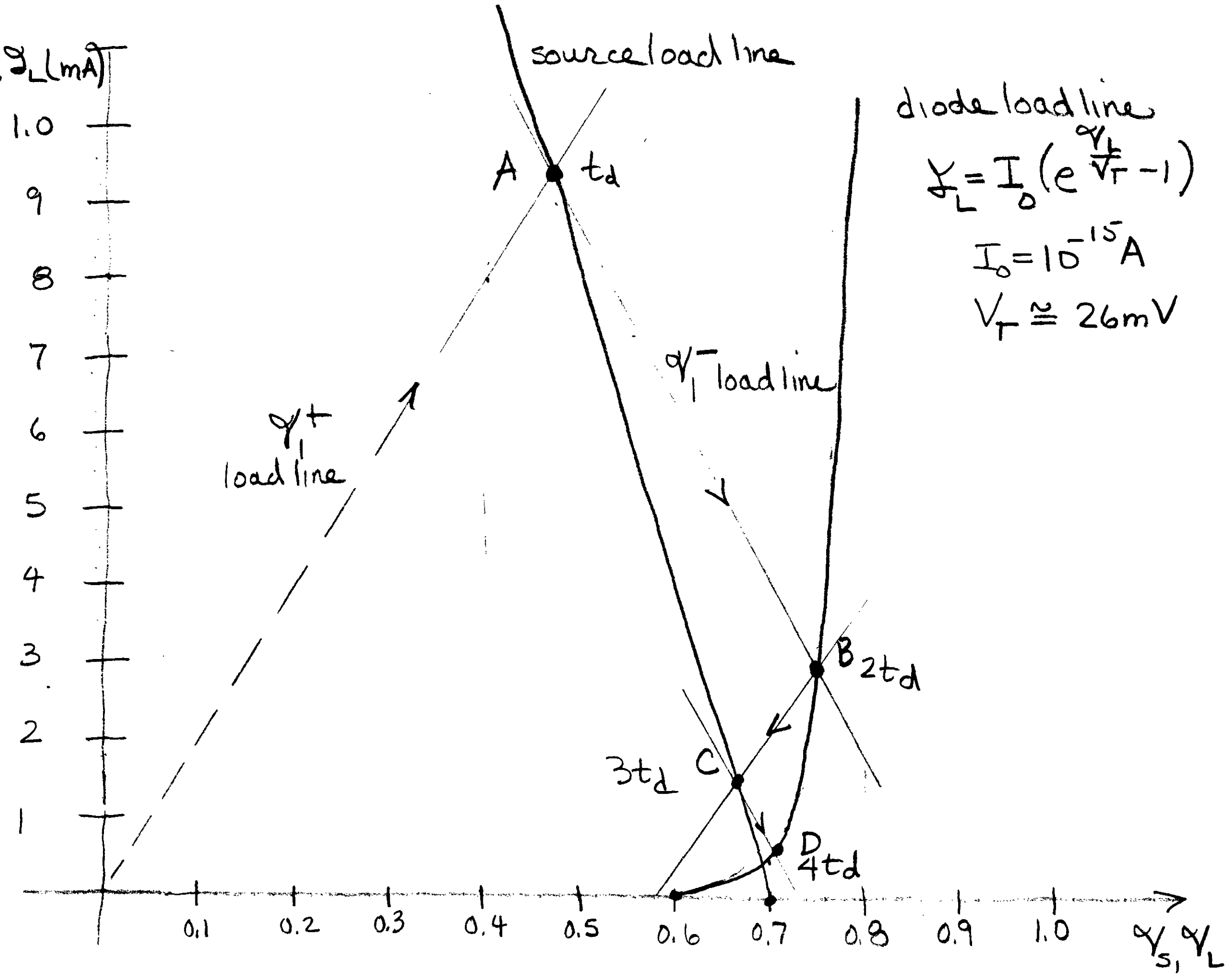



source load line

$V_o - 25\,\mathcal{I}_s = V_s$

$\mathcal{I}_s = \dfrac{V_s - V_o}{-25}$

$\mathcal{I}_s = -40V_s + 28$ (mA)




diode load line

$\mathcal{I}_L = I_o\left(e^{\frac{V_L}{V_T}} - 1\right)$

$I_o = 10^{-15}$ A

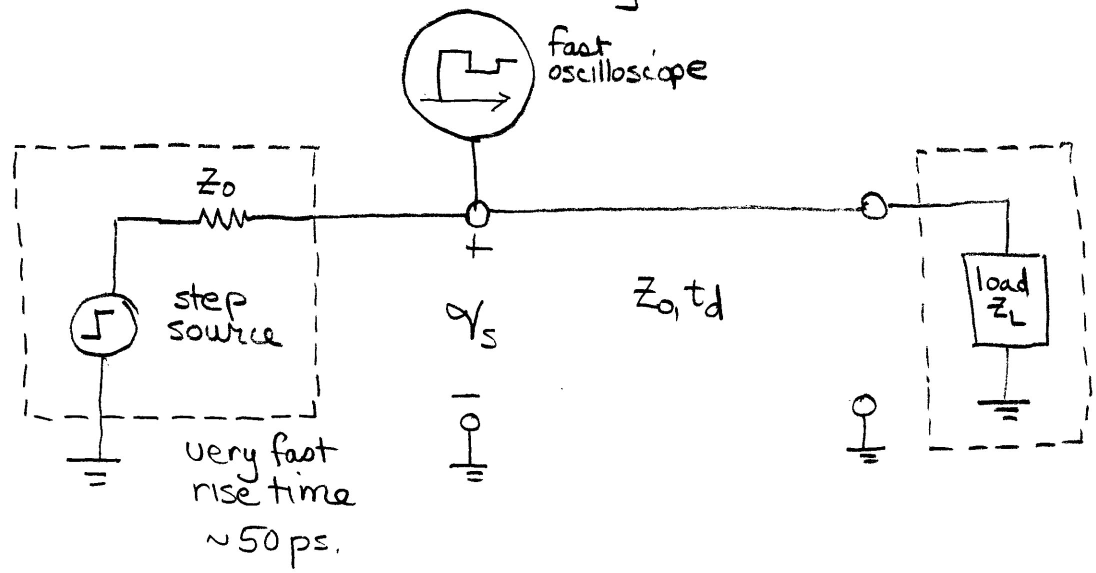
$V_T \cong 26$ mV

Easy to draw keeping slopes the same $\pm \dfrac{1}{Z_o} = \pm \dfrac{1}{50}$

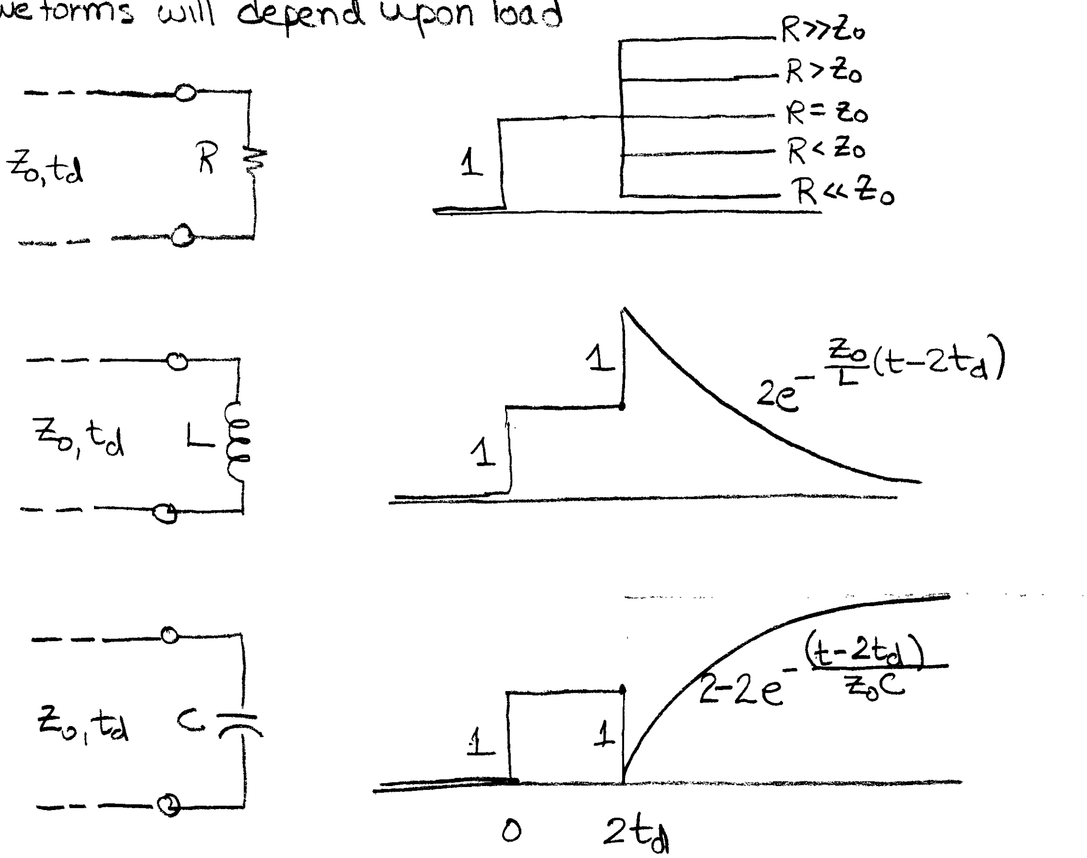
Converges to steady state values $V_s = V_L = 0.6912$ Volts and $\mathcal{I}_s = \mathcal{I}_L \simeq 0.35$ mA in about $4t_d$.

Neglected capacitance of diodes.

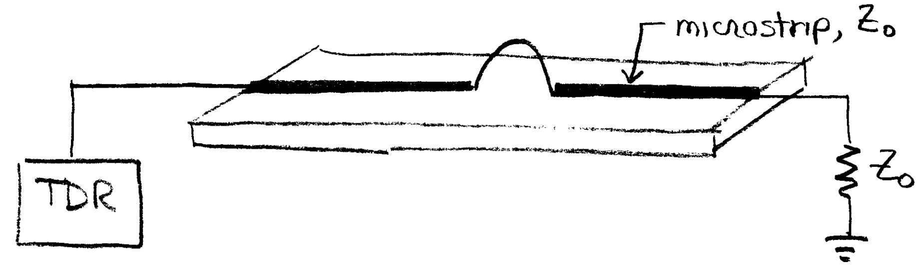
## 2.6.1. Time Domain Reflectometry

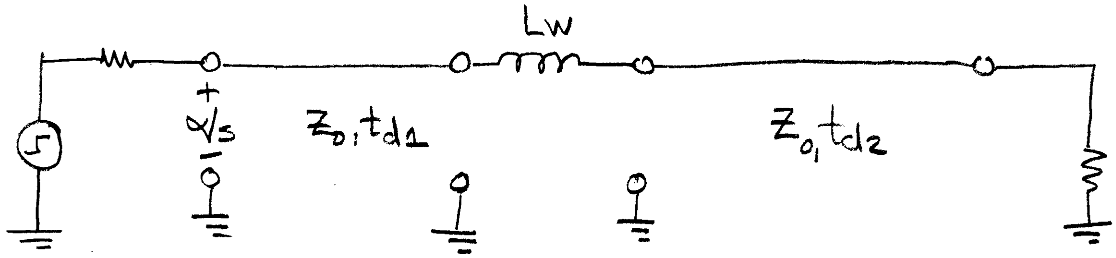



Wave forms will depend upon load

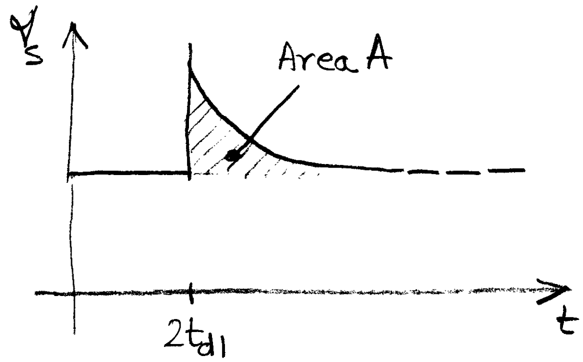


Example 2-16 TDR measurement of bonding wire inductance




TDR measures




You would immediately think of fitting exponential to waveform.

Already know $V_s$ from Example 2-11

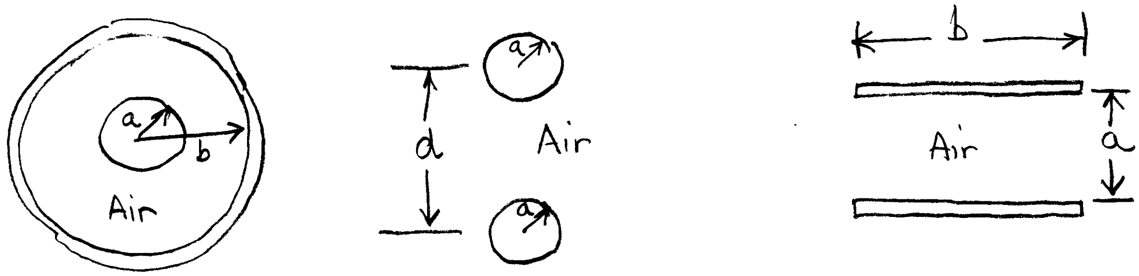
$$V_s(t) = \begin{cases} \frac{V_0}{2} & 0 < t < 2t_{d1} \\ \frac{V_0}{2}\left[1 + e^{-2\frac{Z_0}{L_w}(t-2t_{d1})}\right] & t \geq 2t_{d1} \end{cases}$$

More accurate to integrate Area under curve to get A

$$A = \int_{2t_{d1}}^{\infty} \frac{V_0}{2} e^{-2\frac{Z_0}{L_w}(t-2t_{d1})} dt = \underbrace{\frac{V_0}{2}\int_0^{\infty} e^{-\frac{2Z_0}{L_w}t'} dt'}_{\text{use } t' = t-2t_{d1}} = -\frac{L_w V_0}{4Z_0}\Bigg|_0^{\infty}$$

$$A = \frac{L_w V_0}{4Z_0} \Rightarrow L_w = \frac{4Z_0 A}{V_0}$$

## 2.7 Transmission Line Parameters




coaxial  two-line  Parallel Plate*

| | | | |
|---|---|---|---|
| $L(\mu H/m)$ | $0.2 \ln\left(\frac{b}{a}\right)$ | $0.4 \ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]$ | $\frac{1.26a}{b}$ |
| $c\,(pF/m)$ | $\frac{55.6}{\ln\left(\frac{b}{a}\right)}$ | $\frac{27.8}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$ | $\frac{8.85b}{a}$ |
| $R(\Omega/m)$ | $\frac{4.15\times10^{-8}(a+b)\sqrt{f}}{ab}$ | $\frac{8.3\times10^{-8}\sqrt{f}}{a}$ | $\frac{5.22\times10^{-7}\sqrt{f}}{b}$ |
| $G^{**}(S/m)$ | $\frac{7.35\times10^{-4}}{\ln(b/a)}$ | $\frac{3.67\times10^{-4}}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$ | $\frac{1.17\times10^{-4}b}{a}$ |
| $Z_0(\Omega)$ | $60 \ln\left(\frac{b}{a}\right)$ | $120 \ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]$ | $\frac{377a}{b}$ |

* Valid for $b \gg a$

** For polyethylene at 3GHz

Example 2-18 TV antenna twin-lead

$a \sim 1\,mm$

$d \sim 0.7\,cm$

Assume dielectric constant of plastic approximates air

$f = 200\ MHz$

This is a two-wire line.

$$L = 0.4 \ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] = 0.4 \ln\left[\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 - 1}\right] \cong 1.05\,\mu H/m$$

$$C = \frac{27.8}{\ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right]} = \frac{27.8}{\ln\left[\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 - 1}\right]} \cong 10.6\,pf/m$$

$$R = \frac{8.3\times10^{-8}\sqrt{f}}{a} = \frac{8.3\times10^{-8}\sqrt{200\times10^{6}}}{(1\times10^{-3})} \cong 2.35\,\Omega/m$$

$$Z_0 = 120 \ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] = 120 \ln\left[\frac{7}{2} + \sqrt{\left(\frac{7}{2}\right)^2 - 1}\right] \cong 316\,\Omega$$

This is often called 300 Ω twin lead even though it is actually 316 Ω.