11.7 Line Charge Distribution : The integral equation and the method of moments.
(Antennas, 3le, 14.2)

The fundamental idea behind the method of moments is to convert the integral equations which arise in many antenna problems (charge distribution, radar cross section, etc.) into a set of simultaneous equations which can be solved by computer techniques.

14.10 Integral Equations and the Method of Moments in Electrostatics

Consider the electric potential V at a point P due to a charge Q.

$$V = \frac{Q}{4\pi\epsilon\, r}$$

For a line of charge of density ρ_L then the potential V at P is the integral of the potential from the point charges over the line.

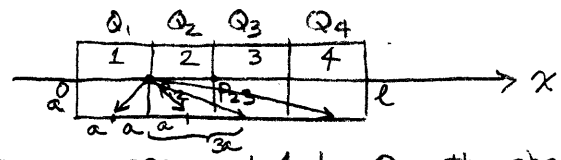$$V = \frac{1}{4\pi\epsilon} \int_0^{\ell} \frac{\rho_L(x)}{r}\, dx$$

If $\rho_L(x)$ is not known this represents an integral equation

Example 14-10.1 Charge Distribution on wire

Let the line be an isolated conducting rod or wire of radius a and length $l = 8a$ on which a total charge $+Q$ has been placed. Since like charges repel, it may be anticipated that the charge will tend to separate and pile up near the ends of the rod, making the charge distribution $\rho_L(x)$ using an incremental numerical technique or moment method as an introduction to integral equations.

Solution.

Divide the rod of length l into 4 segments with each segment of length $2a$ as shown below so that $l = 8a$.

Let the total charge on segment 1 be Q_1; the charge on segment 2 be Q_2. By symmetry $Q_3 = Q_2$ and $Q_4 = Q_1$.

<u>Assume</u> that all of the charge in each segment is concentrated in a circle on the surface of the segment at its middle. Further <u>assume</u> that we will observe (measure) the potential at various points on the axis of the rod.

The potential at point P_{12}

$$V(P_{12}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{\sqrt{a^2+a^2}} + \frac{Q_2}{\sqrt{a^2+a^2}} + \frac{Q_2}{\sqrt{a^2+9a^2}} + \frac{Q_1}{\sqrt{a^2+25a^2}}\right]$$

$$V(P_{23}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{\sqrt{a^2+9a^2}} + \frac{Q_2}{\sqrt{a^2+a^2}} + \frac{Q_2}{\sqrt{a^2+a^2}} + \frac{Q_1}{\sqrt{a^2+9a^2}}\right]$$

This is a boundary value problem where $V(P_{12}) = V(P_{23})$ since the potential must be constant along the rod.

We can then equate these expressions

$$\frac{Q_1}{\sqrt{2}\,a} + \frac{Q_2}{\sqrt{2}\,a} + \frac{Q_2}{\sqrt{10}\,a} + \frac{Q_1}{\sqrt{26}\,a} = \frac{Q_1}{\sqrt{10}\,a} + \frac{Q_2}{\sqrt{2}\,a} + \frac{Q_2}{\sqrt{2}\,a} + \frac{Q_1}{\sqrt{10}\,a}$$
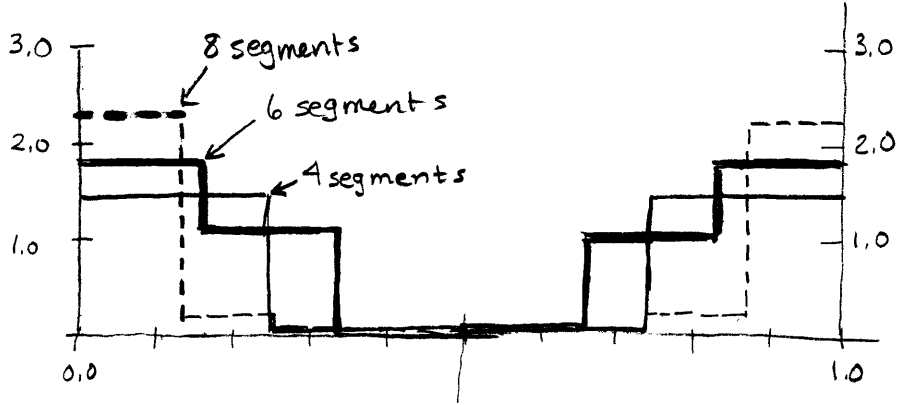
$$Q_1\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{26}} - \frac{1}{\sqrt{10}} - \frac{1}{\sqrt{10}}\right] = Q_2\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{10}}\right]$$

$$Q_1\left[0.27077\right] = Q_2\left[0.39088\right]$$

$$Q_1 = 1.4436\, Q_2$$

You can continue this process dividing the rod into 6 and 8 segments.

The results from these calculations are shown below.

As we see the change moves towards the ends of the rod.

This can be done more formally for N segments of length Δx

$$V = \frac{1}{4\pi\epsilon} \int_0^\ell \frac{\rho_L(x)}{r} \, dx \qquad\qquad (1)$$

where the charge on each segment is $Q_n = \overline{\rho_L(x)_n} \, \Delta x_n \qquad n = 1, \ldots, N$

and the total charge on the wire is
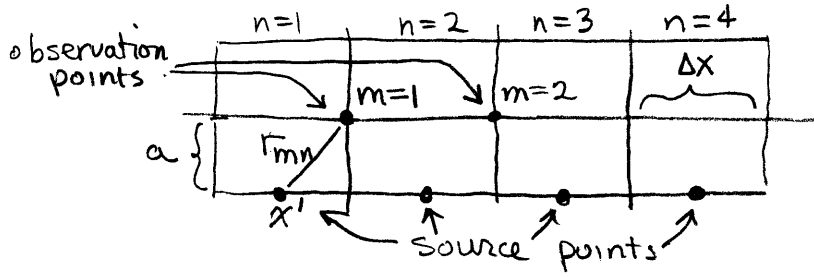
$$Q = \sum_{n=1}^{N} Q_n$$

The original integral equation (1) can now be written

$$V_m = \sum_{n=1}^{N} \ell_{mn} Q_n$$

where $\ell_{mn} = \frac{1}{4\pi\epsilon \, r_{mn}}$ and $r_{mn} = \sqrt{a^2 + (x-x')^2}$

x = axial distance of observation point m

x' = axial distance of source point at middle of segment n

This can now be written in matrix notation

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \cdots & \ell_{1N} \\ \ell_{21} & \ell_{22} & \ell_{23} & \cdots & \ell_{2N} \\ \ell_{31} & \ell_{32} & \ell_{33} & \cdots & \ell_{3N} \\ \vdots & \vdots & \vdots & & \vdots \\ \ell_{M1} & \ell_{M2} & \ell_{M3} & & \ell_{MN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \\ Q_N \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_M \end{bmatrix}$$

$$\begin{bmatrix} \ell_{mn} \end{bmatrix} \begin{bmatrix} Q_n \end{bmatrix} = \begin{bmatrix} V_m \end{bmatrix}$$

This is now N equations. ℓ_{mn} are known from the geometry. V_m are known boundary conditions. Q_n are the unknowns.
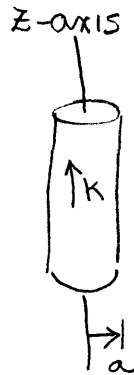
For the charged rod symmetry says that $M = \frac{N}{2}$ and $V_1 = V_2 = V_3 = \ldots$

For 4 segments as we started with this reduces to

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} & \ell_{14} \\ \ell_{21} & \ell_{22} & \ell_{23} & \ell_{24} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = V$$
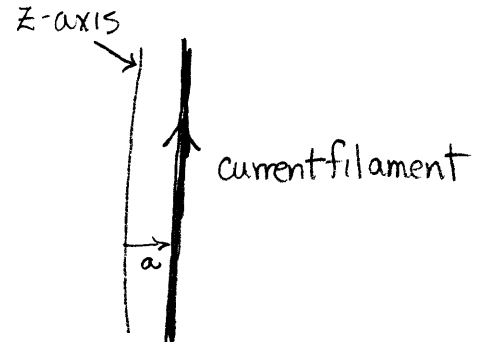
14-11 The moment method (mm) and its application to a wire antenna

Consider a short cylinder of radius a isolated in free space. Let $\sigma = \infty$ so the r.f. current is entirely on the surface.

z-axis

$$I(z') = K(z')\, 2\pi a$$

where $I(z')$ is the total current at z'

z-axis

current filament

Replace it by a thin filament parallel to the z-axis

$$\underline{E} = -j\omega\mu_0 \underline{A} - \nabla V \qquad (1)$$

scalar potential

vector potential

For current only in z-direction

$$E_z = -j\omega\mu_0 A_z - \frac{\partial V}{\partial z} \qquad (2)$$

The divergence of A is specified by the Lorentz condition

$$\nabla \cdot \underline{A} = -j\omega\epsilon_0 \, V$$

giving

$$\frac{\partial A_z}{\partial z} = -j\omega\epsilon_0 \, V$$

Taking the derivative with respect to z

$$\frac{\partial^2 A}{\partial z^2} = -j\omega\epsilon_0 \frac{\partial V}{\partial z}$$

$$-\frac{\partial V}{\partial z} = \frac{1}{j\omega\epsilon_0} - \frac{\partial^2 A}{\partial z^2} \qquad (3)$$

Substituting (3) into (2)

$$E_z = \frac{1}{j\omega\epsilon_0} \frac{\partial^2 A}{\partial z^2} - j\omega\mu_0 A_z$$

$$E_z = \frac{1}{j\omega\epsilon_0}\left(\frac{\partial^2 A_z}{\partial z^2} + \omega^2\mu_0\epsilon_0 A_z\right)$$

$$E_z = \frac{1}{j\omega\epsilon_0}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta^2 A_z\right) \tag{4}$$

For a current element dz the vector potential is

$$dA_z = \frac{I(z')e^{-j\beta r}\,dz}{4\pi r} \tag{5}$$

where $\dfrac{e^{-j\beta r}}{r} = G_{zz'}$ (the free space Green's function)

$$r = \sqrt{(z-z')^2 + a^2}$$
z = observation point
z' = source point.

The electric field from this current element is

$$dE_z = \frac{1}{j\omega\epsilon_0}\frac{I(z')}{4\pi}\left(\frac{\partial^2 G_{zz'}}{\partial z^2} + \beta^2 G_{zz'}\right)dz' \tag{6}$$

For a conductor of length L, the total field is given by integrating (6) to give Pocklington's equation

$$E_z = \frac{1}{4\pi j\omega\epsilon_0}\int_{-\frac{L}{2}}^{+\frac{L}{2}}\left(\frac{\partial^2 G_{zz'}}{\partial z^2} + \beta^2 G_{zz'}\right)I(z')\,dz' \tag{7}$$

This is the radiated field due to $I(z')$ which is due to a source field $E_{z'}$ which is typically due to a voltage applied to the antenna terminals

On and inside the conductor there can be no E field so

$$E_z = -E_{z'}$$

Equation (7) was derived by Pocklington in 1897. However, it is not appropriate for numerical work.

Richmond (1965) has differentiated and re-arranged (7) into a form more suitable for numerical computation.

$$-E_z = \frac{\lambda Z_0}{8\pi^2 j}\int_{-\frac{L}{2}}^{+\frac{L}{2}}\frac{e^{-j\beta r}}{r^5}\left[(1+j\beta r)(2r^2-3a^2)+\beta^2 a^2 r^2\right]I(z')\,dz' \tag{8}$$

where $r = \sqrt{(z'-z)^2+a^2}$ the distance between source and observation points

$$Z_0 = 377\ \Omega$$

In dimensionless form this equation can be re-written

$$-V = -\Delta z\, E(z')$$

$$-V = -j\frac{Z_0\,\Delta z_\lambda}{8\pi^2}\int_{-\frac{L}{2}}^{+\frac{L}{2}} \frac{e^{-j2\pi r_\lambda}}{r_\lambda^3}\left\{1+j2\pi r_\lambda\left[2-3\left(\frac{a}{r}\right)^2\right]+4\pi^2 a_\lambda^2\right\} I(z')\,dz_\lambda$$

where

$$r_\lambda = \frac{r}{\lambda}$$

$$V = \text{voltage developed by } E(z') \text{ over } \Delta z_\lambda$$

Re-writing

$$-E(z') = \int_{-\frac{L}{2}}^{+\frac{L}{2}} I(z')\, G(r_{mn})\, dz' \qquad\qquad (9)$$

where

$$G(r_{mn}) = \frac{-j Z_0}{8\pi^2 \lambda^2}\,\frac{e^{-j2\pi r_\lambda}}{r_\lambda^3}\left\{(1+j2\pi r_\lambda)\left[2-3\left(\frac{a}{r}\right)^2\right]+4\pi^2 a_\lambda^2\right\}\frac{\Omega}{m^2} \quad (10)$$

$$r = r_{mn}$$
$$m = \text{observation point}$$
$$n = \text{source point}$$

Now approximate $I(z')$ by a series expansion

$$I(z') = \sum_{n=1}^{N} I_n F_n(z') \qquad \text{where } F_n(z') \text{ is a rect function} \\ \text{for incremental segments } \Delta z_n'$$

Then, for segment m

$$-E_{z'}(z_m) = \sum_{n=1}^{N} I_n \int_{\Delta z_n'} G(r_{mn})\, dz' \qquad \frac{V}{m} \qquad (11)$$

The electric field
at z_m due to
the source points

Considering this integral of G define

$$G_{mn} = \int_{\Delta z_n'} G(r_{mn})\,dz' \cong G(r_{mn})\,\Delta z_n' \qquad \text{if } \Delta z_n' \text{ sufficiently (12)} \atop \text{small.}$$

and use this result to re-write (11) as

$$-E_{z'}(z_m) = I_1 G_{m1} + I_2 G_{m2} + \cdots + I_N G_{mN}$$

If we repeat this for each of the N segments we get a system of N equations

$$I_1 G_{11} + I_2 G_{12} + \cdots + I_N G_{1N} = -E_{z'}(z_1)$$
$$I_1 G_{21} + I_2 G_{22} + \cdots + I_N G_{2N} = -E_{z'}(z_2)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$I_1 G_{N1} + I_2 G_{N2} + \cdots + I_N G_{NN} = -E_{z'}(z_N)$$

Which can be written in matrix form as

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & & & \\ G_{N1} & G_{N2} & & G_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \\ I_N \end{bmatrix} = \begin{bmatrix} -E_{z'}(z_1) \\ -E_{z'}(z_2) \\ \vdots \\ -E_{z'}(z_N) \end{bmatrix} \qquad (13)$$
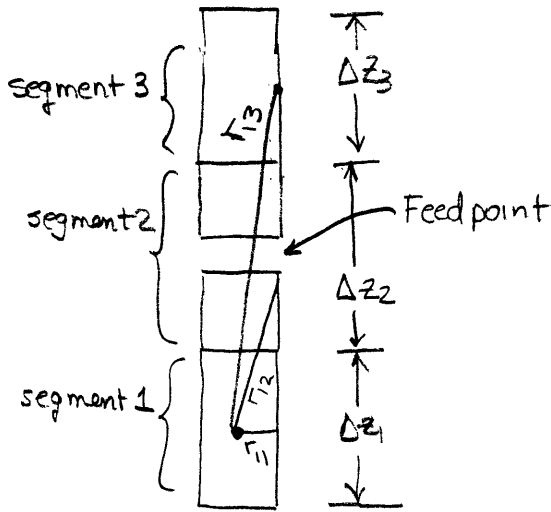
or $$\quad [G_{mn}][I_n] = -[E_m]$$

We can multiply both sides by Δz to get

$$\underbrace{\Delta z\,[G_{mn}]}_{\text{an impedance}}[I_n] = -\underbrace{\Delta z\,[E_m]}_{\text{a voltage.}}$$

$$[Z_{mn}][I_n] = -[V_m]$$

Example 14-11.1 Current Distribution and Impedance of a Short Dipole by the Moment Method.

Use Equations (10), (12) and (13) to calculate the current distribution and input impedance of a perfectly conducting center-fed cylindrical dipole 0.1λ long with a radius of 0.001λ. Assume the current distribution is symmetrical.

each segment is $\frac{0.1\lambda}{3} = 0.033\lambda$ long.

$a = 0.001\lambda$

$r_{12} = \sqrt{(.033\lambda)^2 + (.001\lambda)^2} = .033\lambda$

$r_{13} = \sqrt{(.066\lambda)^2 + (.001\lambda)^2} = .066\lambda$

$N = 3$

$I_1 G_{11} + I_2 G_{12} + I_3 G_{13} = -E(z_1')$

$I_1 G_{21} + I_2 G_{22} + I_3 G_{23} = -E(z_2')$

$I_1 G_{31} + I_2 G_{32} + I_3 G_{33} = -E(z_3')$

Because of symmetry

$r_{12} = r_{21} = r_{23} = r_{32}$

$r_{13} = r_{31}$

$r_{11} = r_{22} = r_{33}$

So we only need to evaluate for three G's

$$G(r_{mn}) = \frac{-jZ_0}{8\pi^2\lambda^2} \frac{e^{-j2\pi r_\lambda}}{r_\lambda^3} \left\{ (1+j2\pi r_\lambda)\left[2-3\left(\frac{a}{r}\right)^2\right] + 4\pi^2 a_\lambda^2 \right\} \qquad (1)$$

$$G_{13} = \frac{-j(377)}{8\pi^2} \frac{\{\cos 2\pi(.066) - j\sin 2\pi(.066)\}}{(.066)^3}$$

$$\times \left\{ (1+j2\pi(.066))\left[2-3\left(\frac{.001}{.066}\right)^2\right] + 4\pi^2(.001)^2 \right\}$$
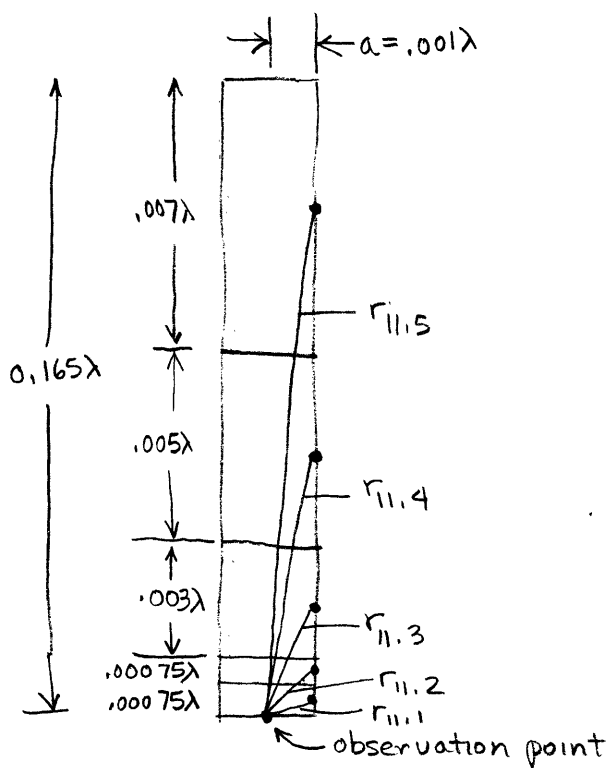
$$G_{13} = -25.8 - j1184 \quad \frac{\mho}{\lambda}$$

Similarly,

$$G_{11} = -20 + j\,52700 \quad \frac{\Omega}{\lambda}$$

$$G_{12} = -25.6 - j\,12800 \quad \frac{\Omega}{\lambda}$$

However, since (1) is very sensitive to small changes in r we actually break G_{11} into five subsegments and compute

$$G_{11} = \int_{\Delta z'} G(r_{mn})\,dz' = 2\left(G_{11,1} + G_{11,2} + G_{11,3} + G_{11,4} + G_{11,5}\right)$$

Once you get the G's and multiply by $\Delta z = .033\lambda$ we get

$$I_1\,(0.66 - j1739) + I_2\,(0.85 + j422) + I_3\,(0.85 + j39) = V_1 \quad (a)$$

$$I_1\,(0.85 + j422) + I_2\,(0.66 - j1739) + I_3\,(0.85 + j422) = V_2 \quad (b)$$

$$I_1\,(0.85 + j39) + I_2\,(0.85 + j422) + I_3\,(0.66 - j1739) = V_3 \quad (c)$$

By symmetry $I_1 = I_3$

For a center-fed dipole $V_1 = V_3 = 0$ so (a) and (c) are identical.

From (a) $I_1 (0.66 - j1739) + I_2 (0.85 + j422) + I_1 (0.85 + j39) = 0$

∴ $\dfrac{I_1}{I_2} = 0.25 + j.0002$

Substituting this into (b), dividing by I_2, and setting $V_2 = 1$ volt gives

$Z = \dfrac{V_2}{I_2} = R + jX = 1.1 - j1528$

11-4 · Repetitive Laplace Solution (finite difference method)
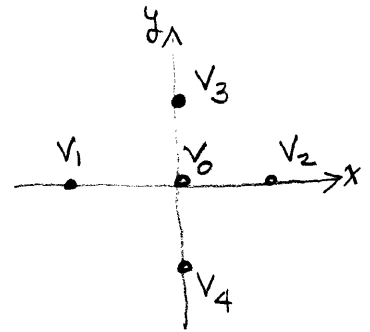
In rectangular coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If the potential is independent of Z

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = - \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right)$$

$$\frac{\dfrac{V_2 - V_0}{\Delta x} - \dfrac{V_0 - V_1}{\Delta x}}{\Delta x} = - \frac{\dfrac{V_3 - V_0}{\Delta y} - \dfrac{V_0 - V_4}{\Delta y}}{\Delta y}$$
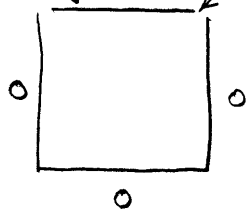
Now let $\Delta x = \Delta y = 1$ to get

$$V_1 + V_2 + V_3 + V_4 - 4 V_0 \approx 0$$
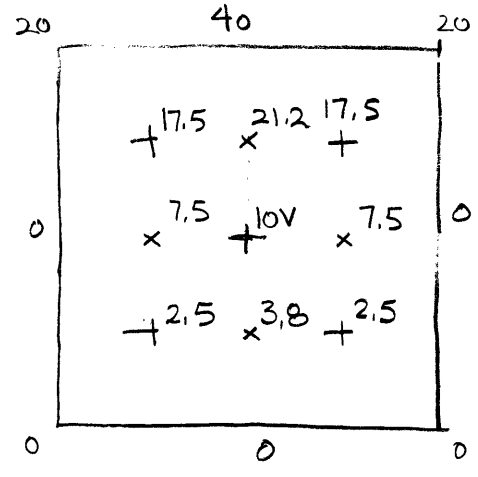
$$V_0 = \frac{1}{4}\left(V_1 + V_2 + V_3 + V_4\right)$$

Example 11-2 The infinite square trough with insulated lid.

insulated lid → 40V   gap   Find the potential distribution.

0   0

0

Find the potential at the center as $\dfrac{40+0+0+0}{4} = 10$ volts.

20   40   20

+17.5 ×21.2 17.5 +

×7.5 +10V ×7.5   0

+2.5 ×3.8 +2.5

0   0   0

Now find the potential for each of the four sub-boxes. Assume the voltage at the trough-lid gap is the average (i.e $\dfrac{40+0}{2} = 20$)

Then, $\dfrac{20+40+0+10}{4} = +17.5$

And, $\dfrac{10+0+0+0}{4} = +2.5$

This process can be repeated to give additional points

$$\dfrac{40+17.5+17.5+10}{4} = 21.2$$

$$\dfrac{17.5+10+2.5+0}{4} = 7.5$$

$$\dfrac{10+2.5+2.5+0}{4} = 3.8$$

You can use this to generate constant potential contours.

30V
20V +
10V
5V
+
1V