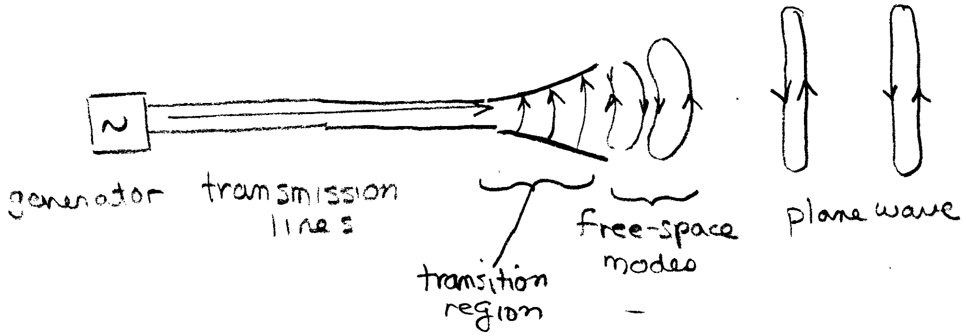
Kraus Ch. 5 Antennas, Radiation and Wireless Systems

Basic radiation equation

$$\underbrace{\underset{\substack{\uparrow\\ \text{time changing current}}}{\frac{dI}{dt}}\;\underset{\substack{\uparrow\\ \text{length of current element}}}{L}}_{\text{usually we analyze this}} = \underset{\substack{\uparrow\\ \text{charge}}}{Q}\;\underset{\substack{\uparrow\\ \text{accelerating charge}}}{\frac{dv}{dt}}$$

An antenna is a transformer between a guided wave (such as in a waveguide or transmission line) and free-space (TEM) waves.

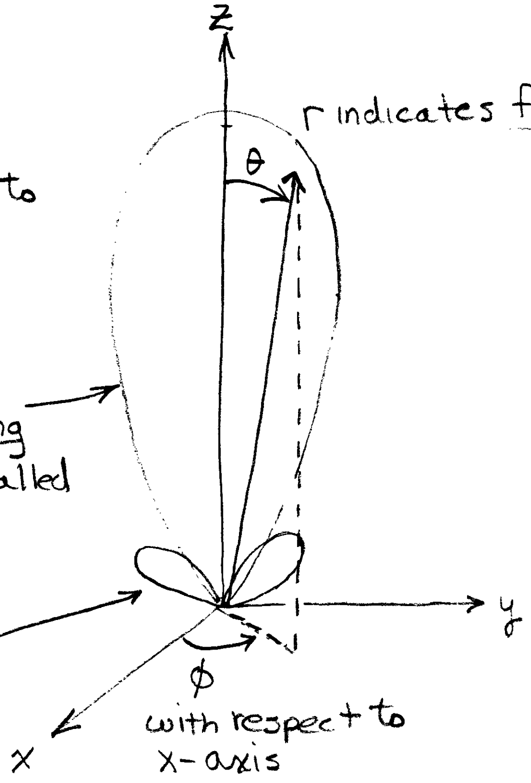


Antenna appears as a radiation resistance R_r which represents the losses to radiation in the antenna.

A receiving antenna also has a radiation resistance R_r. but the background resistance of objects or other antennas raises the temperature of R_r.

An antenna is characterized by electric field strength (or power) as a function of θ and ϕ.

1. $|E_\theta \ (\theta,\phi)|$
2. $|E_\phi \ (\theta,\phi)|$
3. the phase of these fields $\delta_\theta(\theta,\phi)$ or $\delta_\phi(\theta,\phi)$



r indicates field strength

θ with respect to z-axis

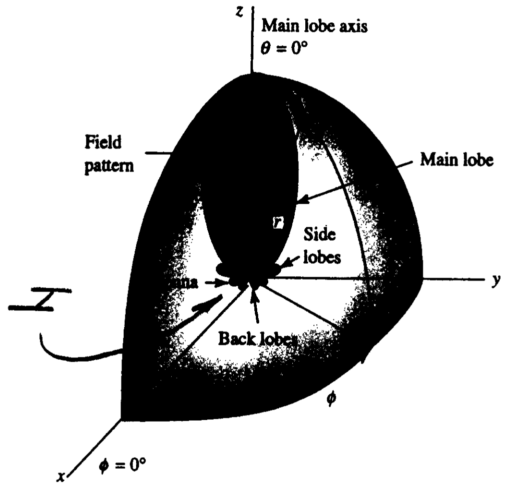
We use r, θ, ϕ to display radiation patterns

Antennas typically have a single strong radiation pattern called the main lobe.

There are often many smaller patterns called minor lobes.

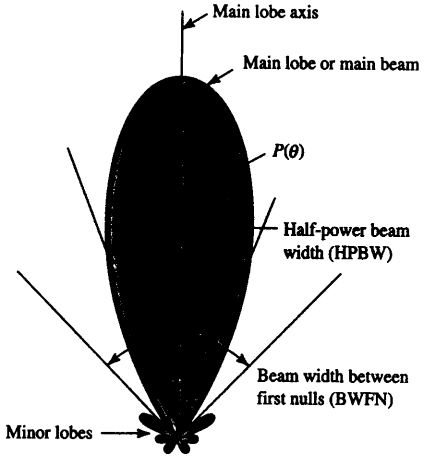
ϕ with respect to x-axis

antenna located at (0,0,0)



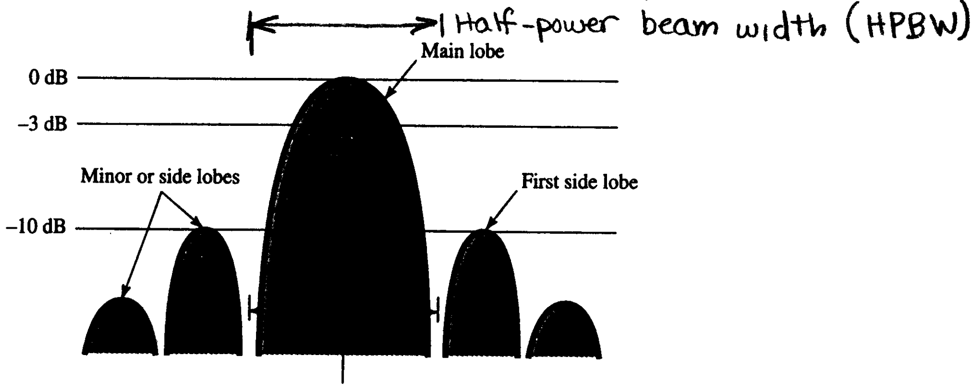
(*a*) Antenna field pattern with coordinate system.

It's usually easier to visualize antenna patterns in 2-D. If the pattern is symmetric about z then we can use a transverse slice of this pattern, i.e., in r and θ.



(*b*) Antenna power pattern in polar coordinates (linear scale).

However, it is often better to simply plot θ on a linear scale and P(θ) on a log scale.



(*c*) Antenna pattern in rectangular coordinates and decibel (logarithmic) scale. Patterns (*b*) and (*c*) are the same.

Because antenna patterns can be complex there are several scalar quantities which are often used

- beam area Ω_A
- directivity D (or gain G)
- effective aperture A_e

Before defining these quantities we need to define the radiation intensity
The radiation intensity is the power/solid angle

$$P(\theta,\phi) = \frac{E_\theta^2(\theta,\phi) + E_\phi^2(\theta,\phi)}{Z_0} r^2 = S_r(\theta,\phi) r^2$$

$\frac{E_\theta^2}{Z_0}, \frac{E_\phi^2}{Z_0}$ are the Poynting vector components, $\frac{V^2}{m^2}$

$(\bar{S} = \bar{E} \times \bar{E})$

A sphere has 4π steradians.

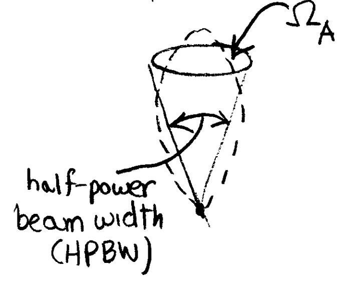
$$P_n(\theta,\phi) = \frac{P(\theta,\phi)}{P(\theta,\phi)\big|_{max}}$$ is the normalized power pattern

For constant total power
what would be the solid angle if $P_n(\theta,\phi)$ was 1 over Ω_A and zero elsewhere.

- The beam area is the normalized power integrated over a sphere.

$$\Omega_A = \int_0^{2\pi}\int_0^{\pi} P_n(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

The total power radiated $= P(\theta,\phi)\big|_{max} \Omega_A$



A good approximation is the product of the half-power beamwidths

$$\Omega_A = \theta_{HP} \phi_{HP} \qquad \text{(both in radians)}$$

You can compute the beam area for the main lobe Ω_A and the minor lobes Ω_M

The main beam efficiency is $\varepsilon_M = \frac{\Omega_M}{\Omega_A}$

- Directivity is the ratio of max. power density to average power density.

$$D = \frac{S(\theta,\phi)_{max}}{S(\theta,\phi)_{avg}} = \frac{P(\theta,\phi)_{max}}{P(\theta,\phi)_{avg}}$$

$$P(\theta,\phi)_{avg} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} P(\theta,\phi)\sin\theta\, d\theta\, d\phi = \frac{1}{4\pi}\iint P(\theta,\phi)\,d\Omega$$

$$D = \frac{P(\theta,\phi)_{max}}{\frac{1}{4\pi}\iint P(\theta,\phi)\,d\Omega} = \frac{4\pi}{\iint \frac{P(\theta,\phi)}{P(\theta,\phi)_{max}}\,d\Omega} = \frac{4\pi}{\iint P_n(\theta,\phi)\,d\Omega} = \frac{4\pi}{\Omega_A}$$

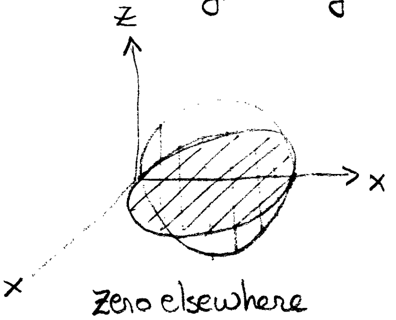
Since $\Omega_A \le 4\pi$ $D \geq 1$

A common approximation is $D \simeq \frac{4\pi}{\theta_{HP}\,\phi_{HP}} \cong \frac{41{,}000}{\theta^\circ_{HP}\,\phi^\circ_{HP}}$ (actually 41253)

D is often expressed in dBi (dB above isotropic) $10\log_{10} D = D_{dBi}$

If the antenna is lossless $G = D$, otherwise $G = kD$
where k is a efficiency factor.

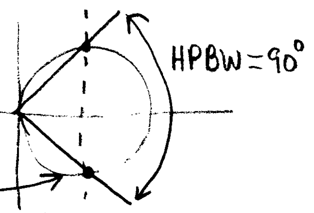
Example 5-1. Suppose the normalized field pattern of an antenna is given by $E_n = \sin\theta\sin\phi$.



zero elsewhere

$$D = \frac{4\pi}{\int_0^{\pi}\int_0^{\pi} \underbrace{\sin^2\theta\sin^2\phi}_{E_n^2} \sin\theta\, d\theta\, d\phi} = \frac{4\pi}{2\pi/3} = 6$$

The



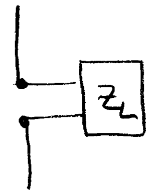
at 45° down to $\frac{1}{\sqrt{2}}$ when squared is $\frac{1}{2}$

half-power

$$D \cong \frac{41000}{(90)(90)} \cong 5.1$$

- The <u>aperture</u> is the area over which the antenna extracts power from a passing wave.

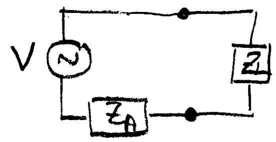
Poynting vector of the incident wave $\left(\frac{W}{m^2}\right)$



power delivered to receiver is P(watts)

$$A = \frac{P\ (\text{watts})}{S\ (\text{watts}/m^2)}$$ is the aperture over which the antenna extracts power from the wave.

The antenna can be modeled as



$Z_L = R_L + jX_L$

$$Z_A = \underset{\substack{\text{antenna}\\ \text{radiation}\\ \text{resistance}}}{\underset{\uparrow}{R_r}} + \underset{\substack{\text{antenna}\\ \text{loss resistance}}}{\underset{\uparrow}{R_\ell}} + j\underset{\substack{\text{antenna}\\ \text{reactance}}}{\underset{\uparrow}{X_A}}$$

$$I = \frac{V}{\sqrt{(R_r + R_\ell + R_L)^2 + (X_A + X_L)^2}}$$

$$P = I^2 R_L = \frac{V^2 R_L}{\sqrt{(R_r + R_\ell + R_L)^2 + (X_A + X_L)^2}}$$

lossless antenna $R_\ell = 0$

conjugate matching $R_r = R_L$
$X_L = -X_A$

under these conditions $P = \frac{V^2}{4R_r}$ and $A = \frac{P}{S} = \frac{V^2}{4SR_r}$

This is called the <u>maximum effective aperture</u>.

When A is less than this maximum it is called the effective aperture A_e.

Consider an antenna with an effective aperture A_e and a beam solid angle Ω_A

Assuming the field E_a is uniform over the aperture

$$P = \frac{E_a^2}{Z_0} A_e$$

Assume the field is E_r for a given r. The radiated power is then

$$P = \frac{E_r^2}{Z_0} \underbrace{r^2\Omega_A}_{\text{the area of the aperture at } r}$$

Using $E_r = \frac{E_a A_e}{r\lambda}$ (which I cannot prove to you at this time)

Substituting $P = \left(\frac{E_a A_e}{r\,\lambda}\right)^2 \frac{1}{Z_0} r^2\Omega_A = \frac{E_a^2 A_e^2}{\lambda^2}\frac{\Omega_A}{Z_0}$

Equating $\frac{\cancel{E_a^2} A_e^{\cancel{2}}}{\lambda^2}\frac{\Omega_A}{\cancel{Z_0}} = \frac{\cancel{E_a^2}}{\cancel{Z_0}}\cancel{A_e}$

$$\lambda^2 = A_e\Omega_A$$

This is a useful result. We can also use it to re-write the directivity in terms of the aperture

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi A_e}{\lambda^2}$$

which is also a very useful result.

Problem 5-2-3 An antenna has a uniform field $E = 2\frac{V}{m}$ at a distance of 100 meters for zenith angles θ between 30° and 60° and azimuth angle ϕ between 0 and 90° with $E = 0$ elsewhere. The antenna terminal current is 3A (rms).

Find

(a) directivity

The beam area Ω_A is given by

$$\Omega_A = \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=\pi/6}^{\pi/3} \sin\theta \, d\theta \, d\phi = \left(-\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}\right)\left(\frac{\pi}{2}\right) = (-.5+.866)\left(\frac{\pi}{2}\right) = .575 \text{ sr}^2$$

The directivity is then

$$D = \frac{4\pi}{\Omega_A} = \frac{4\pi}{.575} = 21.85$$

(b) effective aperture.

$$D = 4\pi \frac{Ae}{\lambda^2}$$

$$A_e = \frac{D}{4\pi}\lambda^2 = \frac{21.85}{4\pi}\lambda^2 = 1.74 \lambda^2$$ (leave in terms of λ)

(c) radiation resistance

radiation intensity $P = \frac{E^2 r^2}{Z_0} = \frac{(2)^2}{377}(100)^2 = 106.1 \frac{w}{sr^2}$

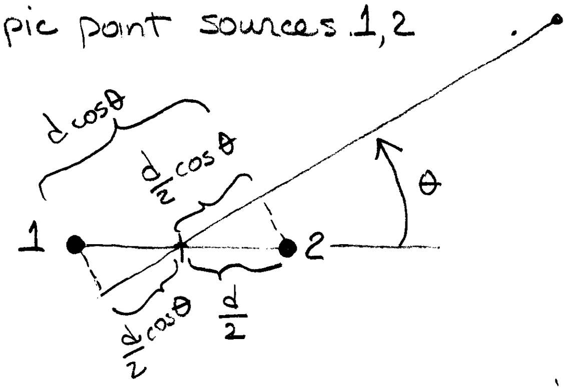
Power $= P \cdot \Omega_A = \left(106.1 \frac{w}{sr^2}\right)(.575 sr^2) = 61.0$ watts

the radiation resistance (assuming a lossless antenna)

$$R = \frac{\text{Power}}{I^2} = \frac{61.0}{(3)^2} = 6.78\Omega$$

5.3 Arrays

Two isotropic point sources 1, 2



Note: use center as reference.

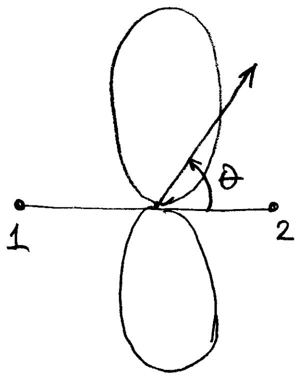
The far field electric field is $E = E_2 e^{j\frac{\psi}{2}} + E_1 e^{-j\frac{\psi}{2}}$

where $\beta z = \beta(d\cos\theta) = \psi$ the phase difference in the θ direction

If $E_1 = E_2$ $\quad E = 2E_1 \dfrac{e^{j\frac{\psi}{2}} + e^{-j\frac{\psi}{2}}}{2} = 2E_1\cos\left(\frac{\psi}{2}\right)$

for $d = \frac{\lambda}{2}$ $\quad \psi = \frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\cancel{2}}\cos\theta = \pi\cos\theta$

the electric field pattern looks like

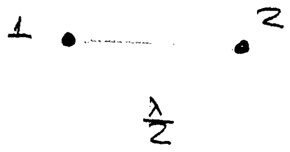


This is known as a broadside pattern.

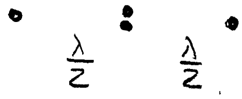
Pattern Multiplication

$$E(\text{total}) = \underbrace{E(\text{source})}_{\substack{\text{pattern of each} \\ \text{source (assumed} \\ \text{to be identical)}}} \times \underbrace{E(\text{isotropic})}_{\substack{\text{the field pattern from the} \\ \text{array of isotropic sources.}}}$$

Binomial array



$$E = \cos\left(\underbrace{\frac{2\pi}{\lambda}\frac{\lambda}{2}\cos\theta}_{\psi\text{ for } d=\frac{\lambda}{2}}\right) = \cos\left(\frac{\pi}{2}\cos\theta\right)$$

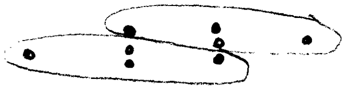


using the pattern multiplication principal this array pattern can be regarded as a 2-element array where each element is given by $\cos(\frac{\pi}{2}\cos\theta)$

$$E(\text{total}) = E(\text{source}) \times E(\underset{\text{isotropic}}{\text{pattern}})$$

$$= \cos\left(\frac{\pi}{2}\cos\theta\right) \cdot \cos\left(\frac{\pi}{2}\cos\theta\right)$$

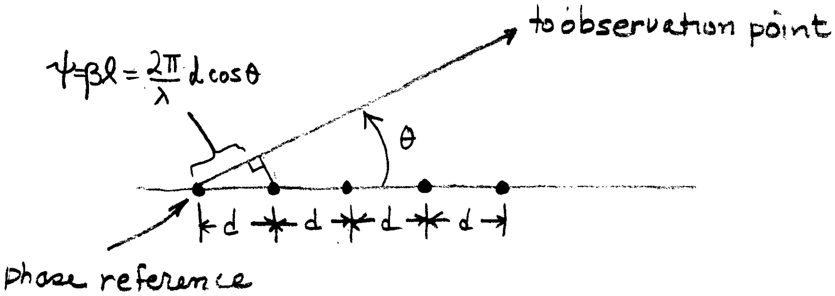
$$E(\text{total}) = \cos^2\left(\frac{\pi}{2}\cos\theta\right)$$



For this pattern

$$E(\text{total}) = \underbrace{E(\text{source})}_{\substack{\text{this is the dipole} \\ \text{pattern } \cos^2(\frac{\pi}{2}\cos\theta) \\ \text{for the 4 sources}}} \times \underbrace{E(\underset{\text{isotropic}}{\text{pattern}})}_{\substack{\text{this is the dipole} \\ \text{pattern — two} \\ \text{patterns like a} \\ \text{dipole.}}}$$

Linear arrays of n isotropic point sources of equal amplitude and spacing



The total field at the observation point (much farther than d) will be the sum of the fields from each source.

$$E = 1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \ldots + e^{j(n-1)\psi} \qquad (1)$$

where ψ is the phase difference between adjacent sources. $\psi = \frac{2\pi}{\lambda} d\cos\theta + \delta$ and I took source #1 as my reference so ψ represents a phase advance. NOTE: δ is the phase difference between sources.

multiply (1) by $e^{j\psi}$

$$E e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + e^{j4\psi} + \ldots + e^{jn\psi} \qquad (2)$$

Subtracting (2) from (1)

$$E\left(1 - e^{j\psi}\right) = 1 - e^{jn\psi}$$ (all the other terms cancel).

$$E = \frac{1 - e^{jn\psi}}{1 - e^{j\psi}}$$

Now let's re-write this using complex exponential representations of trig functions

$$E = \frac{-e^{jn\frac{\psi}{2}}\left(e^{jn\frac{\psi}{2}} - e^{-jn\frac{\psi}{2}}\right)}{-e^{j\frac{\psi}{2}}\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)} = e^{j(n-1)\frac{\psi}{2}} \frac{\sin\left(\frac{n\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$$

$$E = \frac{\sin\left(\frac{n\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} \angle \frac{n-1}{2}\psi$$

↑

this is the average phase seen by the distant observer.

If we would pick our phase reference at the center of the array then $\frac{n-1}{2}\psi = 0$

and $$E = \frac{\sin\left(n\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$$ is the array pattern.

The maximum value that E can attain is at $\psi=0$. At

$\psi=0$

$$\lim_{\psi\to 0} E = \lim_{\psi\to 0} \frac{\sin\left(n\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)} = 1.$$

The normalized array pattern is then

$$E_n = \frac{1}{n}\,\frac{\sin\left(n\frac{\psi}{2}\right)}{\sin\left(\frac{\psi}{2}\right)}$$

For a center referenced array the phase factor is different

$$E = e^{-j(\frac{n-1}{2})\psi} + e^{-j(\frac{n-3}{2})\psi} + \ldots + 1 + \ldots e^{+j(\frac{n-3}{2})\psi} + e^{+j(\frac{n-1}{2})\psi} \quad (1)$$

multiply by $e^{+j\psi}$

$$E e^{+j\psi} = e^{-j(\frac{n-3}{2})\psi} + e^{-j(\frac{n-5}{2})\psi} + \ldots + e^{+j(\frac{n-1}{2})\psi} + e^{j(\frac{n+1}{2})\psi} \quad (2)$$

Subtract (2) from (1)

$$E - E e^{+j\psi} = e^{-j(\frac{n-1}{2})\psi} - e^{+j(\frac{n+1}{2})\psi}$$

all other terms cancel

$$E = \frac{e^{-j(\frac{n-1}{2})\psi} - e^{+j(\frac{n+1}{2})\psi}}{1 - e^{+j\psi}} = \frac{e^{j\frac{\psi}{2}}\left(e^{-j\frac{n}{2}\psi} - e^{+j\frac{n}{2}\psi}\right)}{-e^{+j\frac{\psi}{2}}\left(e^{+j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)}$$

$$E = \left(\frac{-e^{+j\frac{\psi}{2}}}{-e^{+j\frac{\psi}{2}}}\right)\frac{\left(e^{j\frac{n}{2}\psi} - e^{-j\frac{n}{2}\psi}\right)}{\left(e^{j\frac{\psi}{2}} - e^{-j\frac{\psi}{2}}\right)}$$

$$E = \frac{\sin(\frac{n\psi}{2})}{\sin(\frac{\psi}{2})} \angle \xi \qquad \text{where } \xi = 0$$

Note: this is considerably different than the end referenced where $\xi = \frac{n-1}{2}\psi$ for the phase but the magnitude remains the same. The only sign (phase) comes from the sign of $\dfrac{\sin(\frac{n\psi}{2})}{\sin(\frac{\psi}{2})}$

Example 5-4 Five isotropic-source end-fire array.

Five sources have equal amplitudes and are spaced $\frac{\lambda}{4}$ apart. The maximum field is to be in line with the sources at $\theta = 0^\circ$. Plot the field pattern of the array in polar coordinates and indicate the phase referred to the center of the array.

The phase difference $\psi = \frac{2\pi}{\lambda} d\cos\theta + \delta = \frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{4} \cos 0^\circ + \delta = \frac{\pi}{2} + \delta$

The maximum is supposed to be in the $\theta = 0^\circ$ direction. This is called an end-fire array.

To get a maximum in the $\theta = 0^\circ$ direction $\psi = 0$ since E_n has a maximum at $\psi = 0$.

$$\therefore \psi = \frac{\pi}{2} + \delta = 0 \quad \text{or} \quad \delta = -90^\circ$$

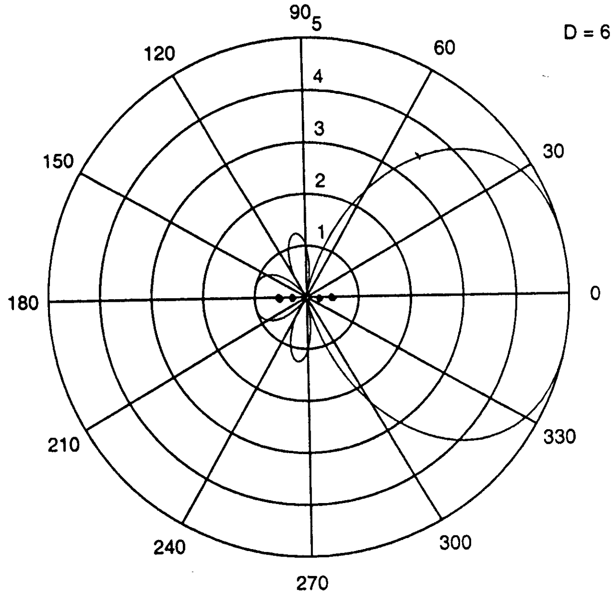
Thus, in general

$$\psi = \frac{\pi}{2}\cos\theta - 90^\circ = 90^\circ(\cos\theta - 1)$$

The field is now given by

$$E_5 = \frac{1}{5} \frac{\sin\left(5 \frac{90^\circ(\cos\theta - 1)}{2}\right)}{\sin\left(\frac{90(\cos\theta - 1)}{2}\right)}$$

A plot of the field is



D = 6.99 dBi from ARRAY PATGAIN

θ is very important to the final direction of the array.

Problem 5-3-1.

(a) Calculate the HPBW for the five-source array of Example 5-4 and, using (5-2-10), its approximate directivity

(b) Compare this with the directivity obtained using ARRAYPATGAIN. Note that since the array is assumed lossless the directivity and gain are equal.

$$E_5 = \frac{1}{5} \frac{\sin\left[5 \cdot \frac{90^\circ}{2} (\cos\theta - 1)\right]}{\sin\left[\frac{90^\circ}{2} (\cos\theta - 1)\right]}$$

$$P_5 = \frac{1}{25} \frac{\sin^2[5x]}{\sin^2[x]} \quad \text{where } x = \frac{90^\circ}{2}(\cos\theta - 1)$$

For the Half-power point set $P_5 = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{25} \frac{\sin^2[5x]}{\sin^2[x]}$$

$$\sin^2[5x] = 12.5 \sin^2[x].$$

I am not aware of an analytical solution so I solved numerically.

$x = \pm 0.283$ radians because of squaring

Using $x = -0.283$ radians (the + root made no sense)

$x = -16.214$ degrees

$$-16.214 = \frac{90}{2}(\cos\theta - 1)$$

and again solving numerically $\theta = 50.232^\circ$

This is the half-angle The HPBW = 100.46° for both θ and φ

$$\text{Using } D \approx \frac{41{,}000}{(100.46)(100.46)} = 4.06$$

$$D = 10 \log 4.06 = 6.09 \text{ dBi}$$

ARRAY PAT GAIN gave D = 6.99

It computes a numerical integral of the beam power and calculates $D = \frac{P(\theta,\phi)_{max}}{P(\theta,\phi)_{av}}$

Example 5-5

If the five isotropic sources of Example 5-4 are replaced by five short dipoles, plot the amplitude pattern and indicate the phase referred to the center of the array.

We have not studied the short dipole yet but it has

$$E_{source} = \cos\phi$$

The array pattern is then

$$E_5 = \frac{1}{5}\,\frac{\sin(5\frac{\psi}{2})}{\sin(\frac{\psi}{2})}$$

where $\psi = 90(\cos\theta - 1)$ as in Example 5-4.

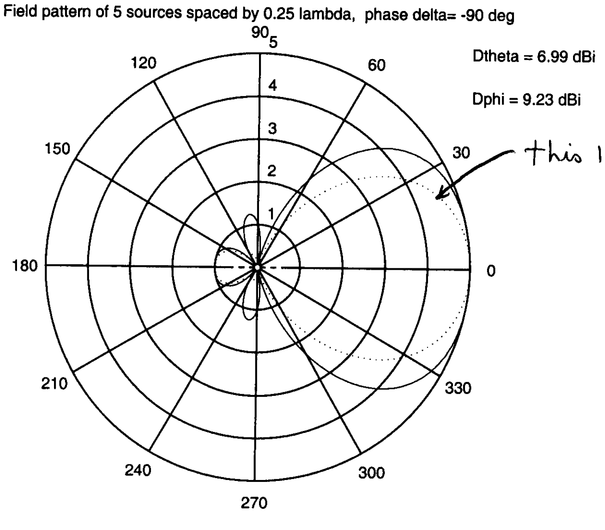
For this array

$$E_{total} = E_{source} \cdot E_5 \qquad \text{(Equation 4.1)}$$

which results in

$$E_{total} = \frac{1}{5}\,\frac{\sin(5\frac{\psi}{2})}{\sin(\frac{\psi}{2})}\cos\phi$$

This modifies the pattern to give different patterns in θ and ϕ.



this is the short dipole array pattern in ϕ making the beam narrower

The overall effect is that the ϕ term is narrower than the θ term. We can estimate HPBW from the plot. $HPBW_\theta$ remains at $100.45°$. $HPBW_\phi$ is approximately $78°$. Then $D_{overall} \approx \frac{41000}{(100.45)(78)} = 5.23$ or 7.2 dBi

Example 5.6 Four isotropic-source broadside array.

Four sources have equal amplitudes, are spaced $\frac{\lambda}{2}$ apart, and are in-phase.

(a) Plot the amplitude in polar coordinates

(b) Plot both amplitude and phase in rectangular coordinates with phase referred to the midpoint of the array and also to source 1.

$$\psi = \frac{2\pi}{\lambda} d \cos\theta + \delta$$

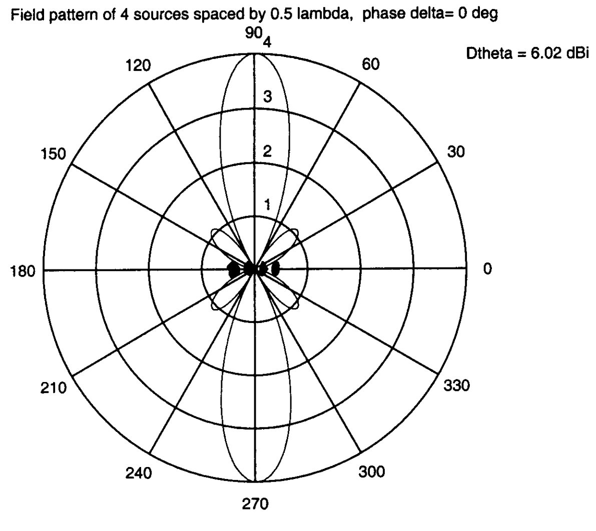
We first need to determine δ for a maximum broadside (say 90°).

Require $\psi = 0$ for a maximum (the fields arrive in phase)

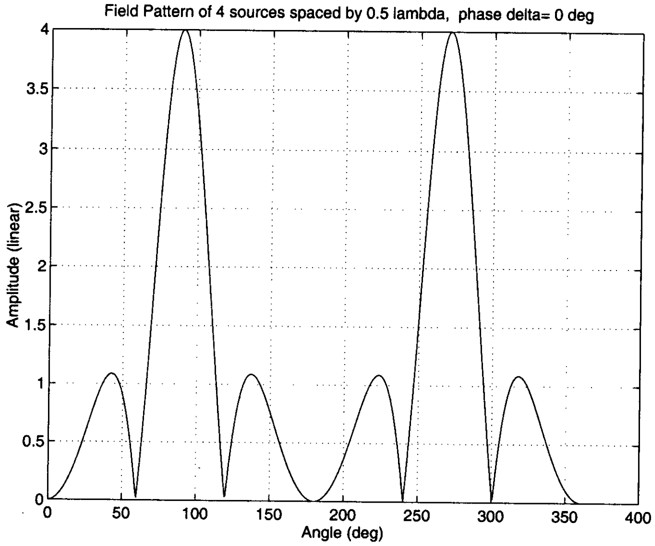
$$0 = \frac{2\pi}{\lambda}\frac{\lambda}{2}\overset{0}{\cancel{\cos 90^\circ}} + \delta$$

$$\therefore \delta = 0 \qquad \text{(This is in-phase)}$$

Note that $\delta=0$ results in this broadside pattern.

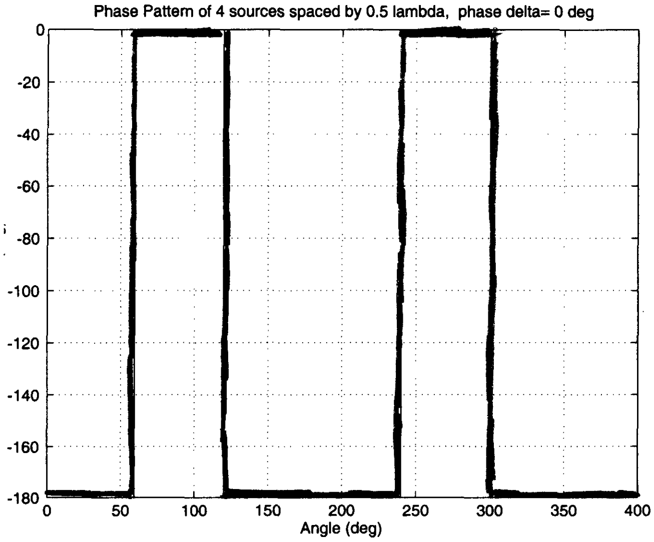

Field pattern of 4 sources spaced by 0.5 lambda, phase delta= 0 deg

four isotropic sources along 0° axis

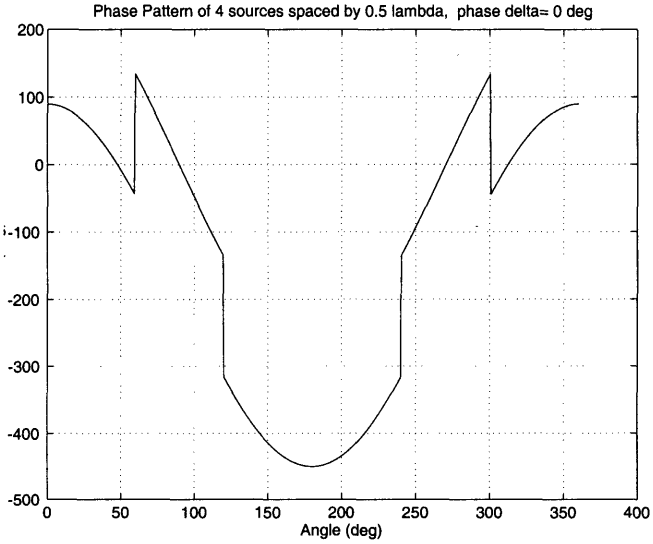

Field Pattern of 4 sources spaced by 0.5 lambda, phase delta= 0 deg

The linear plot is a good way to measure the HPBW

NOTE: Whether the phase is end-referenced or center-referenced does not change the |amplitude| distribution at all



This is the phase distribution referenced to the center of the array.



This is the phase distribution referenced to the end of the array.

All of these amplitude/power/phase plots were produced with a modified version of ARRAY PATGAIN

There is a lot you can do with antenna arrays and phasing.

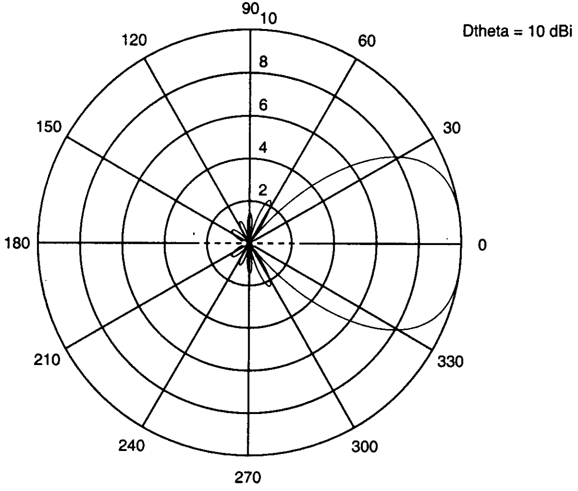
We did a "ordinary" end fire array using $\delta = -90°$. This resulted in The maximum field amplitude in the end-fire direction.

If you change the phasing you can also increase the directivity.
Hansen and Woodyard, Proc. IRE, Vol. 26, p. 333-345 (March 1938)

$$\delta = -\left(\frac{2\pi d}{\lambda} + \frac{\pi}{n}\right)$$

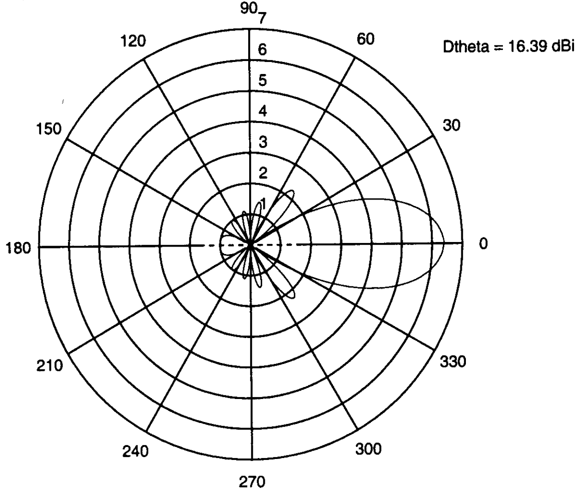
Using This I made the following plots.

Field pattern of 10 sources spaced by 0.25 lambda, phase delta= -90 deg



10 source end fire array
$\delta = -90°$

Field pattern of 10 sources spaced by 0.25 lambda, phase delta= -108 deg



same 10 source endfire array using above expression for δ

Note increase in directivity from 10dBi To 16.4dBi

You can also manipulate the phase to "electrically" steer the directivity

Example 5-10 Four isotropic-source array

Four sources have equal amplitude with $\frac{\lambda}{2}$ spacing.

(a) Find the phase angle δ required to maximize the field in the $\theta = 60°$ direction and using ARRAYPATGAIN plot the field pattern and determine the directivity of the array.

Start with the phase difference expression

$$\psi = \frac{2\pi}{\lambda} d \cos\theta + \delta$$

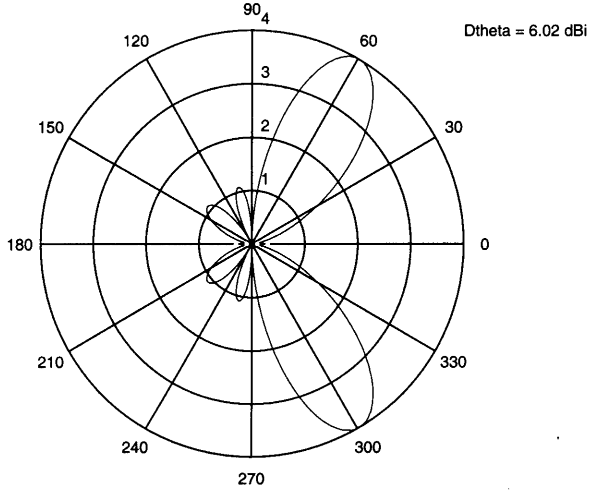
set to zero to get maximum field (ψ)

$d = \frac{\lambda}{2}$ given in problem

set to 60° for desired maximum (θ)

$$0 = \frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{2} \frac{1}{2} + \delta \quad \Rightarrow \delta = -\frac{\pi}{2} = -90°$$

Field pattern of 4 sources spaced by 0.5 lambda, phase delta= -90 deg



(b) Find the phase angle δ required to place a pattern null at $\theta = 60°$ and, using ARRAYPATGAIN, plot the field pattern and determine the directivity of the array.

$$\psi = \frac{2\pi}{\lambda} d \cos\theta + \delta$$

set to 180° for fields out of phase, i.e. they cancel (ψ)

$\frac{\lambda}{2}$ (d)

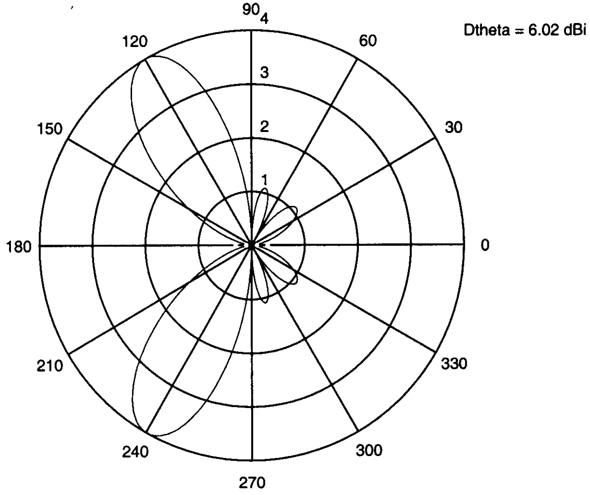
60° (θ)

$$\pi = \frac{2\pi}{\cancel{\lambda}} \cdot \frac{\cancel{\lambda}}{\cancel{2}} \cdot \frac{1}{2} + \delta$$

$$\pi = \frac{\pi}{2} + \delta$$

$$\delta = +90°$$

Field pattern of 4 sources spaced by 0.5 lambda, phase delta= 90 deg



```
%ArrayPatGain - This program is an updated and combined version of the programs
% and "ARRAYPATGAIN" in the 4th edition of Kraus's Electromagnetics.  This prog
% computes and plots the field pattern of a uniform linear array of sources.
clear;
timesrun=0;
while timesrun<1000,
  if timesrun==0,
    SP=0.5;
    PH=0;
    N=5;
    MF=1;
  else,
    SP=input('Enter element spacing in wavelengths: ');
    PH=input('Enter phase difference between elements in degrees: ');
    N=input('Enter number of elements: ');
    MF=input('Enter pattern multiplication factor: ')';
  end;

% This version modified to include isotropic AND short dipole sources
% both of which are displayed on the same polar plot.  This version
% does not do phase plots.
% F. Merat  3/29/03


                                        % Compute fields for plotting
A=0.01:0.01:6.27;                       % Angle over 2pi radians
U=(2*pi*SP*cos(A)+(pi*PH/180))/2;       % compute argument
FP=(sin(N*U)./sin(U));                  % compute isotropic electric field
FPD=(sin(N*U)./sin(U)).*cos(A);         % computer dipole electric field
R=MF.*abs(FP);                          % multiply as appropriate
Rdipole=MF.*abs(FPD);                   % compute for array of short dipoles
pause(0.9);


                                        % compute the beam area in theta
B=0.01:0.01:3.14;                       % B is theta -- the angle from 0 degrees
                                        %    0 <theta (B) < 180 degrees
  W=(2*pi*SP*cos(B)+(pi*PH/180))/2;     % compute psi -- phase shift
  PP=(sin(N*W)./sin(W)).^2;             % compute unnormalized power
  Z=0.01*sin(B).*PP;                    % differential power is PP*sin(theta)*d(t
  SUM=sum(Z);                           % integrate the elements over 180 degrees
```

```
DR=(2*(N^2))/SUM;                       % numerical directivity is Pmax/Pavg.
                                        % N^2 is Pmax
                                        % Not sure of 2; because I only did pi?

DBI=10*log10(DR);                       % express in dBi

                                        % REPEAT FOR DIPOLE
                                        % compute the beam area in theta
B=0.01:0.01:3.14;                       % B is theta -- the angle from 0 degrees
                                        %    0 <theta (B) < 180 degrees
  W=(2*pi*SP*cos(B)+(pi*PH/180))/2;     % compute psi -- phase shift
  PP=(sin(N*W)./sin(W).*cos(B)).^2;            % compute unnormalized power
  Z=0.01*sin(B).*PP;                    % differential power is PP*sin(theta)*d(t
  SUM=sum(Z);                           % integrate the elements over 180 degrees

DRdipole=(2*(N^2))/SUM;                      % numerical directivity is Pmax/Pav
                                        % N^2 is Pmax
                                        % Not sure of 2; because I only did pi?

DBIdipole=10*log10(DRdipole);                    % express in dBi

theta=(0:2*pi/626:2*pi);                % plot antenna pattern in polar coordinat


                                        % plot isotropic pattern

polar(theta,(round(R*100))/100,'r');

hold on;
polar(theta,(round(Rdipole*100))/100,'g:');

title(['Field pattern of ',num2str(N),' sources spaced by ',num2str(SP),' lambd
text(max(R),max(R),['Dtheta = ',num2str((round(DBI*100))/100),' dBi']);
text(max(R),max(R)-1,['Dphi = ',num2str((round(DBIdipole*100))/100),' dBi']);
hold off;

sep=max(R)/20;
xcent=sep*((-(N-1)/2)-1);
for elcount=1:N
```

```
   xcent=xcent+sep;
   ycent=0;
   radius=sep/5;
   xp=xcent+[-radius:radius/10:radius];
   yp=ycent+real(sqrt((radius^2)-(xp-xcent).^2));
   patch(xp,yp,'k');
   patch(xp,-yp,'k');
end;

rp=input('Enter 1 for rectangular plot: ');
if rp==1
   figure(2);
   plot(theta*(180/pi),R);
   grid on;
   zoom on;
   title(['Field Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (linear)');
end;

logp=input('Enter 1 for dB plot: ');
if logp==1
   figure(3);
   plot(theta*(180/pi),10*log10(abs(R)));
   grid on;
   title(['Field Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (dB)');
end;

powerp=input('Enter 1 for power pattern: ');
if powerp==1
   figure(4);
   plot(theta*(180/pi),20*log10(abs(R)));
   grid on;
   title(['Power Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (dB)');
end;
```

```
timesrun=timesrun+1;
done=input('Enter 1 to modify parameters: ');
if done~=1,
   timesrun=1000;
end;
end;
```

```
%ArrayPatGain - This program is an updated and combined version of the programs
% and "ARRAYPATGAIN" in the 4th edition of Kraus's Electromagnetics.  This prog
% computes and plots the field pattern of a uniform linear array of sources.
clear;
timesrun=0;
while timesrun<1000,
   if timesrun==0,
      SP=0.5;
      PH=0;
      N=5;
      MF=1;
   else,
      SP=input('Enter element spacing in wavelengths: ');
      PH=input('Enter phase difference between elements in degrees: ');
      N=input('Enter number of elements: ');
      MF=input('Enter pattern multiplication factor: ')';
   end;

% This version modified to include phase plots
% which are either end referenced or center referenced.
% F. Merat  3/29/03


                                        % Compute fields for plotting
A=0.01:0.01:6.27;                       % Angle over 2pi radians
U=(2*pi*SP*cos(A)+(pi*PH/180))/2;       % compute argument
                                        % Note that /2 is included here rather
                                        %   than in sine(NU)/sine(U) expression

FP=(sin(N*U)./sin(U));                  % compute isotropic electric field
R=MF.*abs(FP);                          % multiply as appropriate

%XI=(180/pi).*((N-1)*U-angle(FP));      % compute phase [end referenced]
                                        % the angle(FP) accounts for the sign
                                        % of Sine(Nu)/Sine(u) in the phase

XI=(180/pi).*(-angle(FP));              % compute phase [center referenced]
                                        % the angle(FP) accounts for the sign
                                        % of Sine(Nu)/Sine(u) in the phase
```

```


pause(0.9);

                                        % compute the beam area in theta
B=0.01:0.01:3.14;                       % B is theta -- the angle from 0 degrees
                                        %    0 <theta (B) < 180 degrees
   W=(2*pi*SP*cos(B)+(pi*PH/180))/2;    % compute psi -- phase shift
   PP=(sin(N*W)./sin(W)).^2;            % compute unnormalized power
   Z=0.01*sin(B).*PP;                   % differential power is PP*sin(theta)*d(t
   SUM=sum(Z);                          % integrate the elements over 180 degrees

DR=(2*(N^2))/SUM;                       % numerical directivity is Pmax/Pavg.
                                        % N^2 is Pmax
                                        % Not sure of 2; because I only did pi?

DBI=10*log10(DR);                       % express in dBi


theta=(0:2*pi/626:2*pi);                % plot antenna pattern in polar coordinat


                                        % plot isotropic pattern

polar(theta,(round(R*100))/100,'r');

title(['Field pattern of ',num2str(N),' sources spaced by ',num2str(SP),' lambd
text(max(R),max(R),['Dtheta = ',num2str((round(DBI*100))/100),' dBi']);

sep=max(R)/20;
xcent=sep*((-(N-1)/2)-1);
for elcount=1:N
   xcent=xcent+sep;
   ycent=0;
   radius=sep/5;
   xp=xcent+[-radius:radius/10:radius];
   yp=ycent+real(sqrt((radius^2)-(xp-xcent).^2));
   patch(xp,yp,'k');
   patch(xp,-yp,'k');
end;
```

```
rp=input('Enter 1 for rectangular plot: ');
if rp==1
   figure(2);
   plot(theta*(180/pi),R);
   grid on;
   zoom on;
   title(['Field Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (linear)');
end;

logp=input('Enter 1 for dB plot: ');
if logp==1
   figure(3);
   plot(theta*(180/pi),10*log10(abs(R)));
   grid on;
   title(['Field Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (dB)');
end;

powerp=input('Enter 1 for power pattern: ');
if powerp==1
   figure(4);
   plot(theta*(180/pi),20*log10(abs(R)));
   grid on;
   title(['Power Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
   ylabel('Amplitude (dB)');
end;


phasep=input('Enter 1 for phase pattern: ');
if phasep==1
   figure(5);
   plot(theta*(180/pi),XI);
   grid on;
   title(['Phase Pattern of ',num2str(N),' sources spaced by ',num2str(SP),' la
   xlabel('Angle (deg)');
```

```
   ylabel('Phase (deg)');
end;

timesrun=timesrun+1;
done=input('Enter 1 to modify parameters: ');
if done~=1,
   timesrun=1000;
end;
end;
```

5-4 Retarded Potentials

Just as propagation time was important for transmission lines, propagation time is important for antennas and radiating systems.

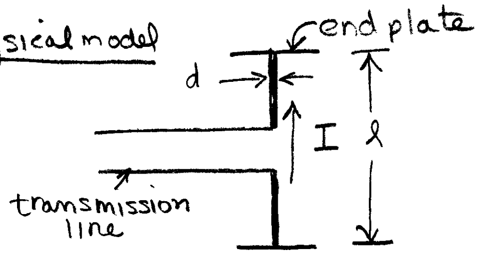
Consequently we write time varying currents in radiating current elements as

$$[I] = I_0 \cos \omega \left(t - \frac{r}{c}\right)$$

where r is the distance from the current element
c is the speed of propagation, the speed of light normally,
and [ ] indicates the current is retarded.

5-5 The short dipole antenna and its radiation resistance

Physical model

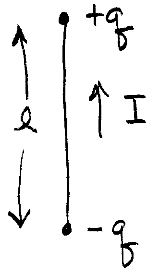


- The length is very short compared to the wavelength. ($l \ll \lambda$)
- Plates may be present at the ends to provide capacitive loading.

Because of these two conditions I is assumed to be uniform along l.

Assumptions
1. The transmission line does not radiate.
2. The diameter d of the dipole is small compared to l and can be neglected

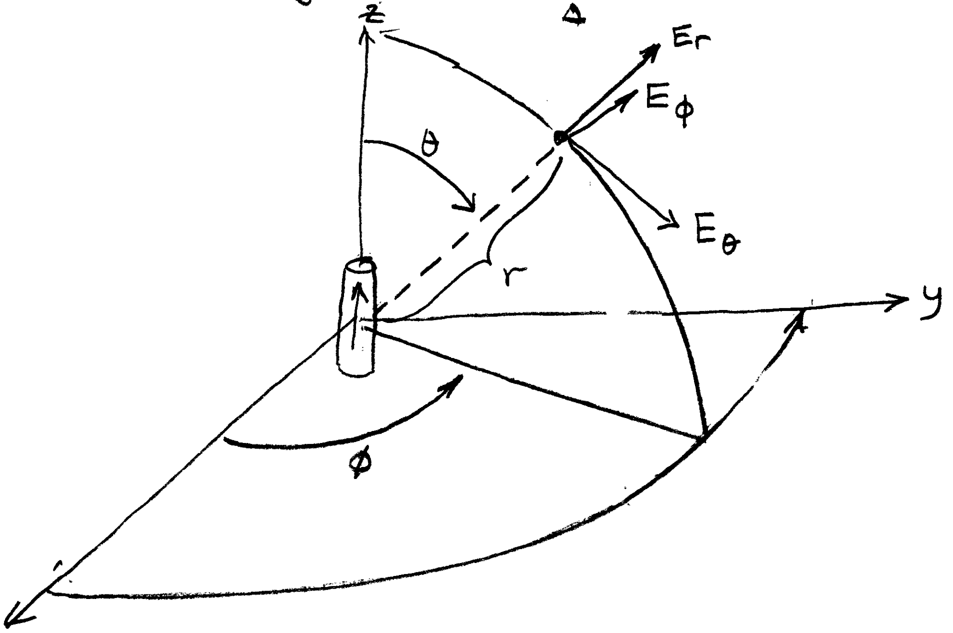
Electrical model



where $\frac{dq}{dt} = I$

Kraus (J.D. Kraus, "Antennas", 2ed, McGraw-Hill, 1988, p.200) has done an exact solution of the fields

As was already noted the dipole has no ϕ dependence so $E_\phi = 0$



For the short dipole given above

$$E_r = \frac{I_0 l \, e^{j(\omega t-\beta r)} \cos\theta}{2\pi\epsilon_0}\left(\frac{1}{cr^2}+\frac{1}{j\omega r^3}\right)$$

$$E_\theta = \frac{I_0 l \, e^{j(\omega t-\beta r)}}{4\pi\epsilon_0}\sin\theta\left(\frac{j\omega}{c^2 r}+\frac{1}{cr^2}+\frac{1}{j\omega r^3}\right)$$

Because the current is time dependent there is also a magnetic term

$$H_\phi = \frac{I_0 l \, e^{j(\omega t-\beta r)}}{4\pi}\sin\theta\left(\frac{j\omega}{cr}+\frac{1}{r^2}\right)$$

We usually deal with antennas in the far-field where $r >> 1$,
In the far-field of this radiating short dipole only the $\frac{1}{r}$ terms survive

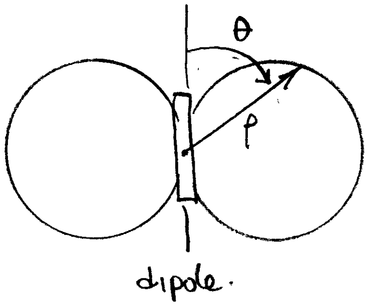
$$E_\theta = \frac{j\omega I_0 l \, e^{j(\omega t-\beta r)} \sin\theta}{4\pi\epsilon_0 c^2 r} = \frac{j\,30 I_0 \beta l}{r} e^{j(\omega t-\beta r)} \sin\theta$$

$$H_\phi = \frac{j\omega I_0 l \, e^{j(\omega t-\beta r)} \sin\theta}{4\pi c r} = j\frac{I_0 \beta l}{4\pi r} e^{j(\omega t-\beta r)} \sin\theta.$$

This defines a TEM wave propagating in the +r direction with characteristic impedance

$$\eta_c = \frac{E_\theta}{H_\phi} = \frac{30}{\frac{1}{4\pi}} = 376.99 \approx 377\Omega$$

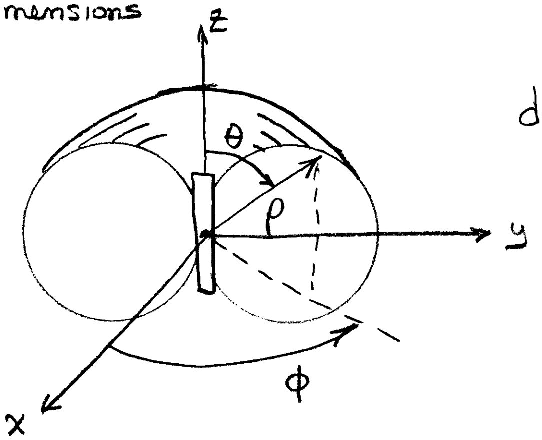
Far-Field of short dipole. (E_θ, H_ϕ)



$$E_\theta = j\,30 I_0 \beta l \,\frac{e^{j(\omega t-\beta r)}}{r} \sin\theta$$

$$H_\phi = j\,\frac{I_0 \beta l}{4\pi}\,\frac{e^{j(\omega t-\beta r)}}{r}\sin\theta$$

In 3-dimensions



doughnut shaped in the far field.

In the far-field E_θ & H_ϕ form a plane traveling wave. since E_θ and H_ϕ are in phase and perdendicular to each other.

In the near-field the r^3 terms dominate and we have E_θ, E_r with imaginary terms. which makes E 90° out of phase with respect to the H_ϕ field. In the near-field the Poynting vector is imaginary and we have a standing wave.

At very low frequencies we have the quasi-stationary case where

$$\left.\begin{aligned} E_r &= \frac{q_0 l \cos\theta}{2\pi\epsilon_0 r^3} \\ E_\theta &= \frac{q_0 l \sin\theta}{4\pi\epsilon_0 r^3} \\ H_\phi &= \frac{I_0 l \sin\theta}{4\pi r^2} \end{aligned}\right\}$$

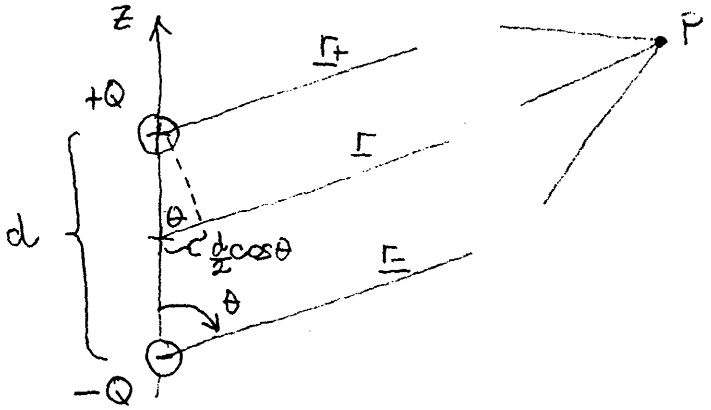
since they vary as $\frac{1}{r^2}$ or $\frac{1}{r^3}$ they are confined to the vicinity of the dipole and there is very little radiation

We can compute the electric fields from this dipole using The electric potential.

Electrostatic Potential resulting from multiple point charges,

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\underline{r} - \underline{r}_k'|}$$

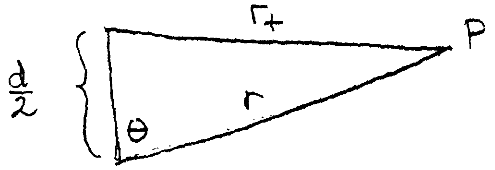
The electric dipole



Summing the potentials

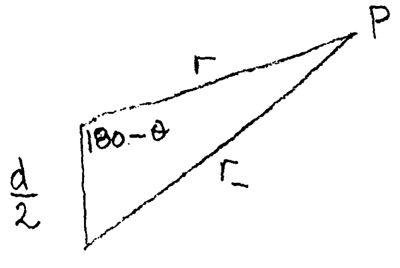
$$\Phi = \frac{+Q}{4\pi\epsilon_0 r_+} + \frac{-Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

Now use law of cosines



$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2(r)\left(\frac{d}{2}\right)\cos\theta$$

$$r_+ = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta}$$



$$r_-^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2(r)\left(\frac{d}{2}\right)\cos(\pi - \theta)$$

$$r_- = \sqrt{r^2 + \left(\frac{d}{2}\right)^2 + rd\cos\theta}$$

In almost every case $r >> d$, i.e. P is far away

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{r^2+(\frac{d}{2})^2-rd\cos\theta}} - \frac{1}{\sqrt{r^2+(\frac{d}{2})^2+rd\cos\theta}}\right)$$

Rewrite denominators and expand as a Taylor series

$$\frac{1}{r\sqrt{1+(\frac{d}{2r})^2-(\frac{d}{r}\cos\theta)}} \cong \frac{1}{r}\left[1-\frac{1}{2}\frac{d}{2r})^2+\frac{d}{2r}\cos\theta+\dots\right]$$

$\uparrow$ neglect this term

$$(1+u)^{-\frac{1}{2}} = 1-\frac{1}{2}u+\dots$$

Then

$$\Phi \cong \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\left[1+\frac{d}{2r}\cos\theta\right] - \frac{1}{r}\left[1-\frac{d}{2r}\cos\theta\right]\right)$$

$$= \frac{Q}{4\pi\epsilon_0 r}\left(\frac{d}{r}\cos\theta\right)$$

$$\Phi = \frac{Qd\cos\theta}{4\pi\epsilon_0 r}$$

$$\underline{E} = -\underline{\nabla}\Phi$$

$$= -\left[\hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right]$$ in spherical coordinates
no ϕ dependence

$$= +\hat{r}\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3} + \hat{\theta}\frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}$$

$$\underline{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right].$$

All antenna fields consist of six factors.

Using the far-field E_θ as an example

$$E_\theta = \underbrace{60\pi}_{\text{magnitude}} \quad \underbrace{I_o}_{\text{current}} \quad \underbrace{\frac{l}{\lambda}}_{\text{length}} \quad \underbrace{\frac{1}{r}}_{\text{distance}} \quad \underbrace{je^{j(\omega t-\beta r)}}_{\text{phase}} \quad \underbrace{\sin\theta}_{\text{pattern}}$$

The pattern factor is what we used when previously constructing arrays of short dipoles.

Table 5-3

Fields of a short dipole

Component	Field	Far-field Radiation	Near-field Quasi-stationary
E_r	$\frac{I_o l e^{j(\omega t-\beta r)}}{2\pi\epsilon_o}\cos\theta\left(\frac{1}{cr^2}+\frac{1}{j\omega r^3}\right)$	0	$\frac{q_o l\cos\theta}{2\pi\epsilon_o r^2}$
E_θ	$\frac{I_o l\, e^{j(\omega t-\beta r)}}{4\pi\epsilon_o}\sin\theta\left(\frac{j\omega}{c^2 r}+\frac{1}{cr^2}+\frac{1}{j\omega r^3}\right)$	$\frac{j60\pi I_o e^{j(\omega t-\beta r)}\sin\theta}{r}\frac{l}{\lambda}$	$\frac{q_o l\sin\theta}{4\pi\epsilon_o r^3}$
H_ϕ	$\frac{I_o l\, e^{j(\omega t-\beta r)}}{4\pi}\sin\theta\left(\frac{j\omega}{cr}+\frac{1}{r^2}\right)$	$\frac{j I_o e^{j(\omega t-\beta r)}\sin\theta}{2r}\frac{l}{\lambda}$	$\frac{I_o l\sin\theta}{4\pi r^2}$

The total power radiated by the antenna is simply

$$P = \int_S \underline{S}_{av} \cdot d\underline{s}$$

Use far-field expressions since they are simpler.

The power radiated by the antenna is equal to the average power delivered to the antenna terminals (assuming no losses). Under these conditions

$$P = \frac{1}{2} I_0^2 \underset{\uparrow \text{ radiation resistance}}{R_r}$$ Note: I_0 is peak value of sine wave.

So, if we know P for a short dipole we can calculate its radiation resistance.

For a short dipole in the far-field

$$P = \int_S \underline{S}_{av} \cdot \underline{ds} = \frac{1}{2} \int Re[\underline{E} \times \underline{H}^*] \cdot d\underline{s}$$

Since only E_θ & H_ϕ are not zero

$$P = \frac{1}{2} \int_S Re[E_\theta H_\phi^*] \hat{r} \cdot d\underline{s} \qquad \text{since } \hat{\theta} \times \hat{\phi} = \hat{r}$$

Since $\hat{r} \cdot d\underline{s} = ds$.

$$P = \frac{1}{2} \int_S Re[E_\theta H_\phi^*] ds = \frac{1}{2} \int_S Re[\underset{\substack{\uparrow \\ E_\theta = H_\phi Z}}{H_\phi} H_\phi^* Z] = \frac{1}{2} \int_S |H_\phi|^2 Re\, Z\, ds$$

For free space $Z = \sqrt{\frac{\mu_0}{\epsilon_0}}$

$$P = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_0^{2\pi} \int_0^{\pi} |H_\phi|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$H_\phi = \frac{I_{av} l e^{j(\omega t - \beta r)}}{4\pi} \sin\theta \; \frac{j\omega}{cr}$$ we use I_{av} since we averaged S in the far-field.

$$|H_\phi| = \frac{\omega I_{av} l \sin\theta}{4\pi c r}$$

$$P = \frac{1}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \int_0^{2\pi} \int_0^{\pi} \frac{\omega^2 I_{av}^2 l^2 \sin^2\theta}{16\pi^2 c^2 r^2} r^2 \sin\theta \, d\theta \, d\phi$$

$$P = \frac{1}{32}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{\beta I_{av} l}{\pi}\right)^2 \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \, d\theta \, d\phi$$

since $\beta = \frac{\omega}{c}$

$$= \frac{1}{32}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{\beta I_{av} l}{\pi}\right)^2 \int_0^{2\pi}\left(-\frac{1}{3}\cos\theta\left(\sin^2\theta + 2\right)\Big|_0^{\pi}\right) d\phi$$

$$= \frac{1}{32}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{\beta I_{av} l}{\pi}\right)^2 \int_0^{2\pi} \underbrace{-\frac{1}{3}\left[(-1)(0+2) - (1)(2)\right]}_{+\frac{4}{3}} d\phi$$

$$= \frac{1}{32}\sqrt{\frac{\mu_0}{\epsilon_0}}\left(\frac{\beta I_{av} l}{\pi}\right)^2 \frac{4}{3} \cdot 2\pi$$

$$P = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{(\beta I_{av} l)^2}{12\pi} \quad \text{watts} \quad \text{far-field, short dipole.}$$

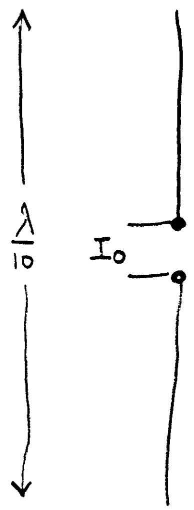
The radiation resistance is then

$$R_r = \frac{P}{\frac{1}{2} I_0^2} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{(\beta I_{av} l)^2}{\frac{1}{2} 12\pi \, I_0^2} = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{(\beta l)^2}{6\pi}\left(\frac{I_{av}}{I_0}\right)^2$$

$$\text{or} \quad R_r = \frac{377}{6\pi}(\beta l)^2\left(\frac{I_{av}}{I_0}\right)^2 = 20.0(\beta l)^2\left(\frac{I_{av}}{I_0}\right)^2$$

Example 5-11

Calculate the radiation resistance of
(a) a center-fed $\frac{\lambda}{10}$ dipole antenna and
(b) half of the same dipole erected vertically over a flat conducting ground plane.



The current must be zero at the ends, and is a maximum at the feed.

If we assume a linear current distribution the average current will be $\frac{1}{2} I_o$

The radiation resistance is then

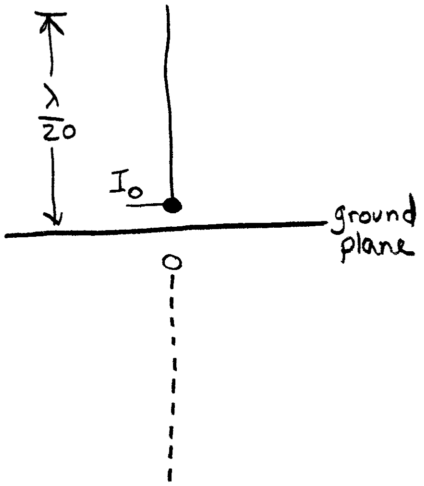
$$R_r = 20(\beta l)\left(\frac{I_{av}}{I_o}\right)^2 = 20\left(\frac{2\pi}{\lambda}\cdot\frac{\lambda}{10}\right)\left(\frac{\frac{1}{2}I_o}{I_o}\right)^2 = 3.14\Omega$$

The antenna will also have a large capacitive reactance of about 1900Ω
(Kraus, Antennas 2/e, McGraw-Hill, 1988, p. 407)

$Z = (3 - j1900)\Omega$

Later we shall see that as the antenna becomes resonant, i.e. $l \approx \frac{\lambda}{2}$

$Z = 73 + j42.5$



The problem is identical except that $l = \frac{\lambda}{20}$ NOT $\frac{\lambda}{10}$ and R_r (and Z) are exactly $\frac{1}{2}$

$R_r \approx 1.5\Omega$

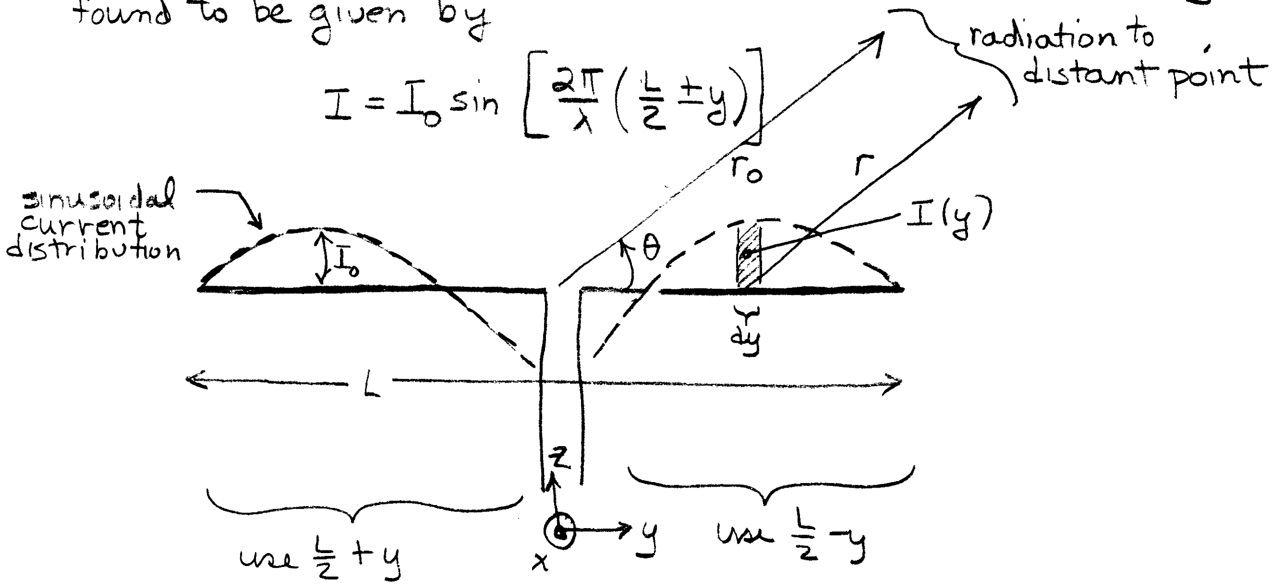
$Z \approx 1.5 - j950$

5-6 Patterns and Radiation Resistance of $\frac{\lambda}{2}$ and $\frac{3\lambda}{2}$ dipoles.

When the length of the antenna increases beyond that of the short dipole we may consider the antenna to be made of of elemental short dipoles of length dy and current I.

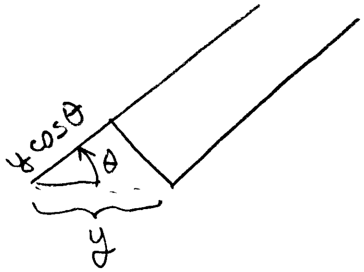
The current on these longer dipole antennas is experimentally found to be given by

$$I = I_0 \sin\left[\frac{2\pi}{\lambda}\left(\frac{L}{2} \pm y\right)\right]$$



The farfield E_θ field from a short dipole is

in general $E_\theta = \frac{j60\pi I_0 e^{j(\omega t - \beta r)}}{r} \sin\theta \, \frac{l}{\lambda}$



for a differential element at y=0 (r_0) where l = dy

$$E_\theta = \underbrace{\frac{j60\pi e^{j(\omega t - \beta r_0)}}{r_0 \lambda}}_{k} I\,dy \sin\theta$$

for a differential element at y (r) where l=dy

$$E_\theta = \underbrace{\frac{j60\pi e^{j(\omega t - \beta r_0)}}{r_0 \lambda}}_{k} I\,dy \sin\theta \, \underbrace{e^{j\beta y\cos\theta}}_{\text{phase shift from differential antenna element at } y}$$

$$E_\theta = k\,I\,dy \sin\theta \, e^{j\beta y\cos\theta}$$

We can now integrate this field contribution over the entire length of the antenna.

$$E_\theta = \int_{-L/2}^{+L/2} kI\sin\theta\, e^{j\beta y\cos\theta}\, dy$$

where $I = I_0 \sin\left[\frac{2\pi}{\lambda}\left(\frac{L}{2} \pm y\right)\right]$

$$E_\theta = k\sin\theta \int_{-\frac{L}{2}}^{+\frac{L}{2}} I_0 \sin\beta\left(\frac{L}{2} \pm y\right) e^{j\beta y\cos\theta}\, dy.$$

The integral is

$$E_\theta = \frac{j60[I_0]}{r_0}\left\{\frac{\cos\left[\frac{\beta L\cos\theta}{2}\right]-\cos\left[\frac{\beta L}{2}\right]}{\sin\theta}\right\}$$

where $[I_0] = I_0 e^{j\omega[t-\frac{r_0}{c}]}$ is the retarded current

this factor determines the far-field shape of the E field

For $L=\frac{\lambda}{2}$

$$\frac{\cos\left[\frac{2\pi}{\lambda}\frac{\lambda}{2}\frac{\cos\theta}{2}\right]-\cos\left[\frac{2\pi}{\lambda}\frac{\lambda}{2}\right]}{\sin\theta} = \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]-\cancelto{0}{\cos\left[\frac{\pi}{2}\right]}}{\sin\theta}$$

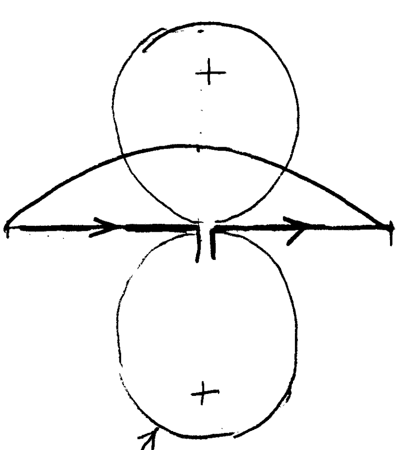
$$= \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta}$$

which is very similar to the $\sin\theta$ pattern of a short dipole

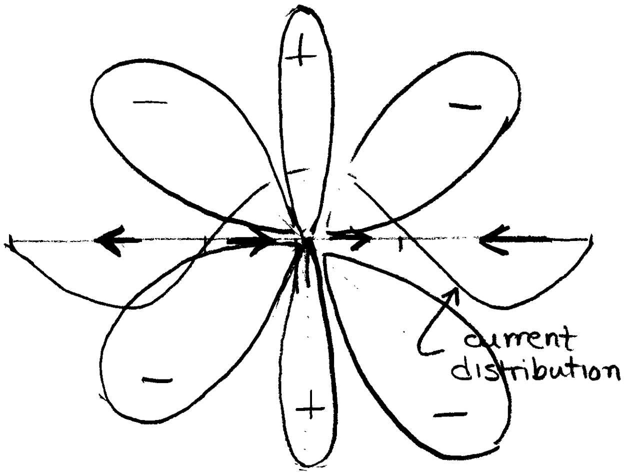
For $L=\frac{3\lambda}{2}$

$$\frac{\cos\left[\frac{2\pi}{\lambda}\frac{3\lambda}{2}\frac{\cos\theta}{2}\right]-\cos\left[\frac{2\pi}{\lambda}\frac{3\lambda}{2}\right]}{\sin\theta} = \frac{\cos\left[\frac{3\pi}{2}\cos\theta\right]-\cancelto{0}{\cos\left[\frac{3\pi}{2}\right]}}{\sin\theta}$$

$$= \frac{\cos\left[\frac{3\pi}{2}\cos\theta\right]}{\sin\theta}$$



$\frac{\lambda}{2}$ antenna pattern
slightly narrower than
the short dipole pattern



$\frac{3\lambda}{2}$ antenna pattern
Note current distribution

Example 5 -12

Find the directivity of a $\frac{\lambda}{2}$ linear dipole.

By definition

$$D = \frac{4\pi}{\iint P_n(\theta,\phi)\,d\Omega} = \frac{4\pi}{\underset{\uparrow}{2\pi}\int_0^{\pi} \frac{\cos^2\left[\frac{\pi}{2}\cos\theta\right]}{\sin^2\theta}\sin\theta\,d\theta} = 1.64$$

Do numerically.

from $d\phi$ integration

Radiation resistance

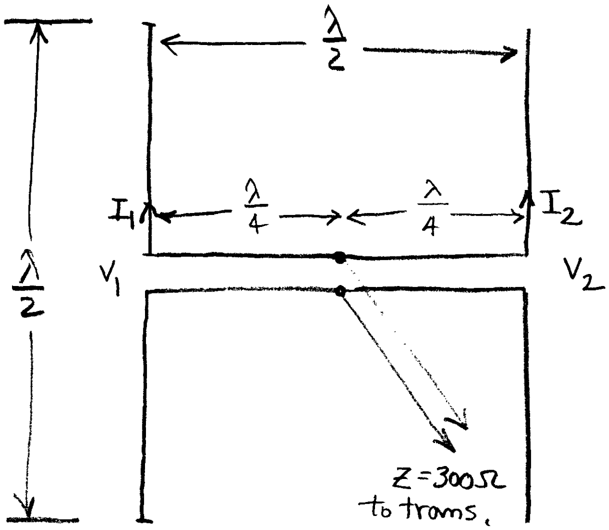
Do as for short dipole integrating to get $P = \frac{1}{2}\int_s \text{Re}\, E_\theta H_\phi^* \, ds$

and using $R_r = \frac{2P}{I_0^2}$

to get

$$R_r = 30 \underbrace{\text{Cin}}_{\text{modified cosine integral}}(2\pi) = 30 \cdot 2.44 = 73\Omega \text{ for a } \frac{\lambda}{2} \text{ dipole.}$$

5.7 Broadside array



dipole 1 dipole 2

$Z_{11} = R_{11} + jX_{11}$ $Z_{22} = R_{22} + jX_{22}$ ← self impedance of the antennas

$Z_{21} = R_{21} + jX_{21}$ $Z_{12} = R_{12} + jX_{12}$

dipole 2 to 1 dipole 1 to dipole 2

from circuit analysis

$$V_1 = I_1 Z_{11} + I_2 Z_{12}$$

$$V_2 = I_2 Z_{22} + I_1 Z_{21}$$

$I_1 = I_2$ since the antennas are driven by a common source

and $Z_{11} = Z_{22}$, $Z_{12} = Z_{21}$ because of symmetry

$$\therefore \quad V_1 = I_1 (Z_{11} + Z_{12}) = V_2$$

$$Z_1 = \frac{V_1}{I_1} = Z_{11} + Z_{12} = Z_2$$ the input impedance of the antenna

Resonate the antennas by tuning with a series reactance at the dipole terminals, Usually done in practice by decreasing the antenna length to make Z_1 and Z_2 real.

$$Z_1 = R_{11} + R_{12} = Z_{12}$$

The input power to the $\frac{\lambda}{2}$ array (at resonance) is given by

$$P = 2 I_1^2 (R_{11} + R_{12})$$

$$I_1 = \sqrt{\frac{P}{2(R_{11}+R_{12})}}$$

The field broadside to the array at some distance $r \gg \lambda$

$$E_{array} = \underset{\substack{\nearrow \\ \text{2 antennas.}}}{2} k I_1 = \underset{\substack{\uparrow \\ \text{dimensionless} \\ \text{function of distance}}}{k} \sqrt{\frac{2P}{R_{11}+R_{12}}}$$

For a single $\frac{\lambda}{2}$ antenna being fed the same power at $r \gg \lambda$

$$E\left(\frac{\lambda}{2}\right) = k I_0 = k \sqrt{\frac{P}{R_{11}}}$$

$$G = \frac{k\sqrt{\frac{2P}{R_{11}+R_{12}}}}{k\sqrt{\frac{P}{R_{11}}}} = \sqrt{\frac{2R_{11}}{R_{11}+R_{12}}}$$

	Directivity	HPBW	A_e, λ^2	R_r, Ω
Isotropic antenna	1.0	—	0.08	—
Short dipole	1.5	90°	0.12	$80\pi^2\left(\frac{l}{\lambda}\right)^2\left(\frac{I_{av}}{I_o}\right)^2$
$\frac{\lambda}{2}$ dipole	1.64	78°	0.13	73

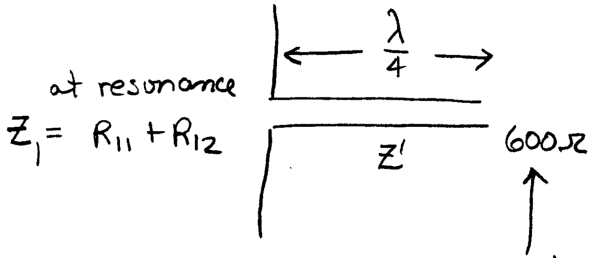
Table 5-4 Dipole Characteristics

Parallel (side by side)		Colinear.	
Spacing, λ	Z(mutual), Ω	Spacing between centers, λ	Z(mutual), Ω
0.5	$-13-j29$	0.5	$26+j18$
1.0	$+4+j18$	1.0	$-4-j2$
1.5	$-2-j12$	1.5	$+2+j0$

Table 5-5 for $\frac{\lambda}{2}$ dipoles

Example 5-13

What is the required impedance of the two $\frac{\lambda}{4}$ feed sections to match the array to a 300Ω two-conductor transmission line?



need 600Ω so that $600 \parallel 600$ from both antennas gives 300Ω to feed line

The spacing between centers is $2\left(\frac{\lambda}{4}\right) = 0.5\lambda$

From Table 5-5 for parallel $\frac{\lambda}{2}$ dipoles $R_{12} = -13\Omega$.

$$\therefore \; Z_1 = 73 - 13 = 60\Omega.$$

For a $\frac{\lambda}{4}$ transformer $Z' = \sqrt{(60)(600)} = 190\Omega.$

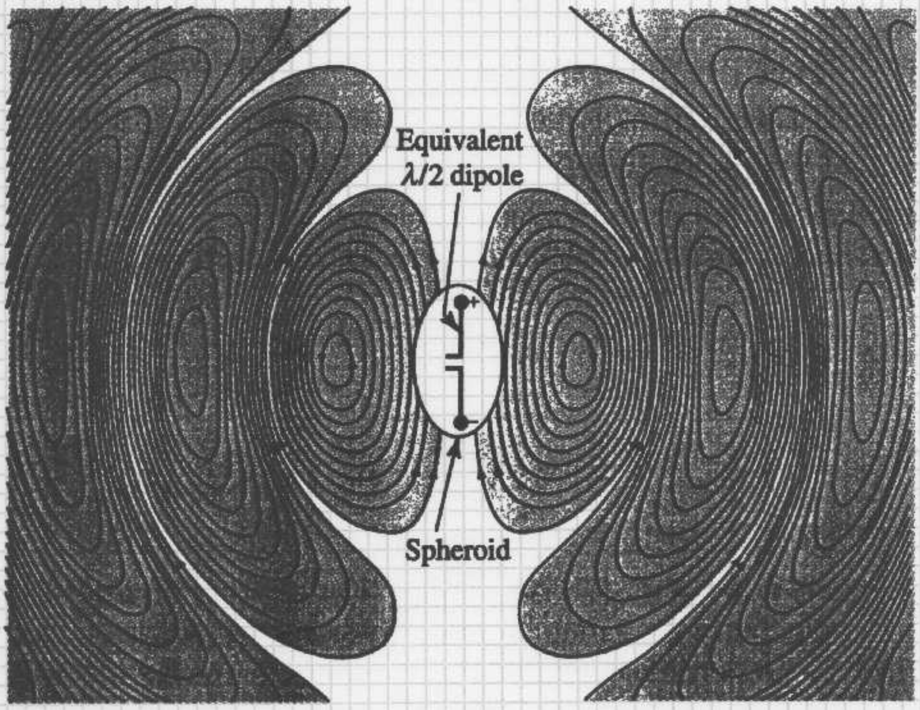


FIGURE 5-26
Electric field configuration for a λ/2 antenna. For the fields in motion see the book's Web site. (*From "A Resonant Spheroid Radiator," produced at the Ohio State University for the National Committee for Electrical Engineering Films; Project Initiator, Prof. Edward M. Kennaugh; diagrams courtesy of Prof. John D. Cowan, Jr.*)