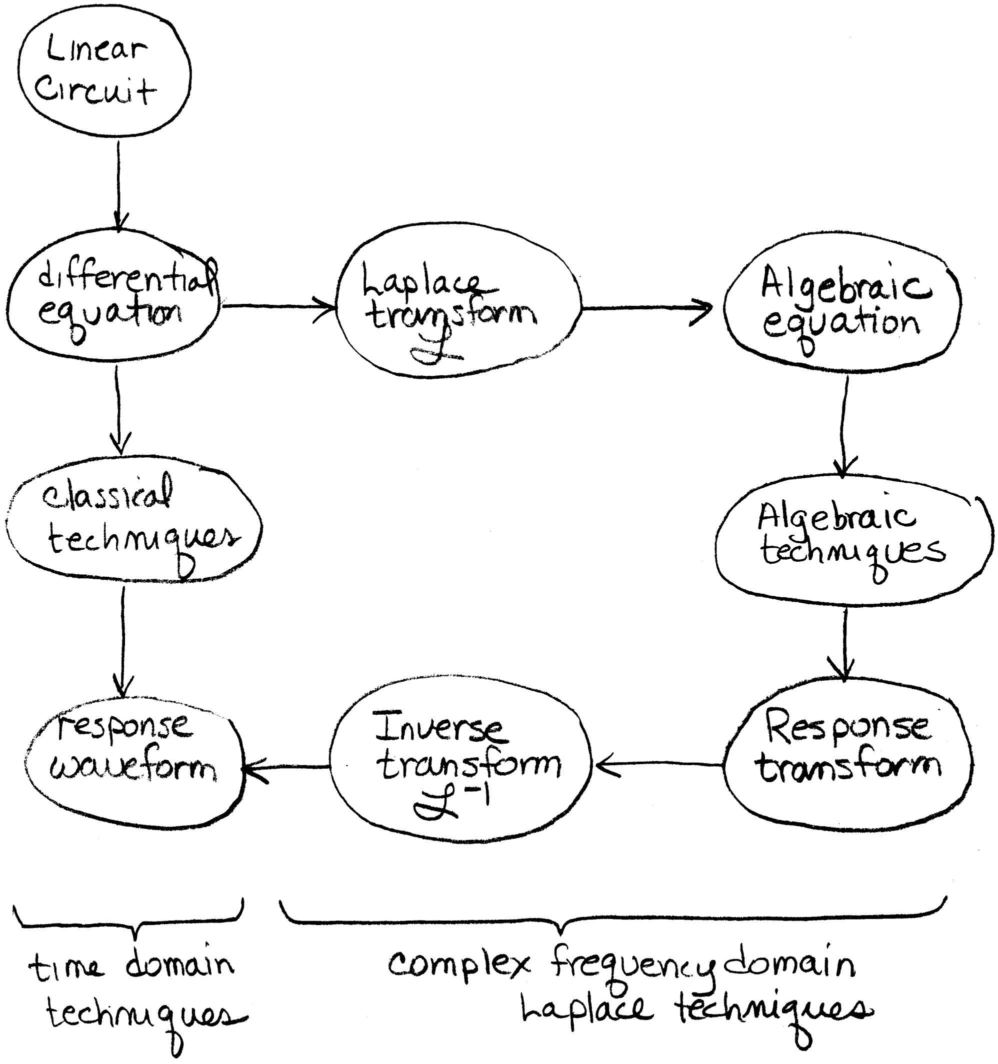
Laplace Transforms (Read Chapter 9)



Basic Laplace transformation properties

Properties	Time Domain	Frequency Domain
Independent variable	t	s
Signal representation	$f(t)$	$F(s)$
Uniqueness	$\mathcal{L}^{-1}\{F(s)\} = f(t)u(t)$	$\mathcal{L}\{f(t)\} = F(s)$
Linearity	$A_1 f_1(t) + A_2 f_2(t)$	$A_1 F_1(s) + A_2 F_2(s)$
Integration	$\int_0^t f(\tau)\,d\tau$	$\frac{F(s)}{s}$
Differentiation	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f(t)}{dt^2}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
s-Domain translation	$e^{-\alpha t}f(t)$	$F(s+\alpha)$
t-Domain translation	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Basic Laplace transform pairs

signal	waveform $f(t)$	transform $F(s)$
Impulse	$\delta(t)$	1
Step function	$u(t)$	$\frac{1}{s}$
Ramp	$t\,u(t)$	$\frac{1}{s^2}$
Exponential	$e^{-at}\,u(t)$	$\frac{1}{s+a}$
Damped Ramp	$te^{-at}\,u(t)$	$\frac{1}{(s+a)^2}$
Sine	$\sin\beta t\,u(t)$	$\frac{\beta}{s^2+\beta^2}$
Cosine	$\cos\beta t\,u(t)$	$\frac{s}{s^2+\beta^2}$
Damped sine	$e^{-at}\sin\beta t\,u(t)$	$\frac{\beta}{(s+\alpha)^2+\beta^2}$
Damped cosine	$e^{-at}\cos\beta t\,u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\beta^2}$

circuit analysis

$$\text{forcing function} \longrightarrow \boxed{\text{circuit}} \longrightarrow \text{output}$$

DC	$V_0 e^0$
transient	$V_0 e^{kt}$
sinusoids	$V_0 e^{j\theta} e^{j\omega t}$

In general the response will be $V_0 e^{j\theta} \underbrace{e^{\overbrace{(\sigma+j\omega)}^{s} t}}_{\text{complex number}}$

DC	$s=0$
transient (exponential)	$s=\sigma$
sinusoid	$s=j\omega$
exponential sinusoid	$s=\sigma+j\omega$

$$v(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} e^{st} V(s)\, ds = \mathcal{L}^{-1}[V(s)]$$

bilateral Laplace transform

$$V(s) = \int_{-\infty}^{\infty} e^{-st} v(t)\, u(t)\, dt = \int_{0}^{\infty} e^{-st} v(t)\, dt = \mathcal{L}[v(t)]$$

one-sided Laplace transform

$$v(t) \Longleftrightarrow V(s)$$

Some sample transforms:

$$\mathcal{L}[u(t)] = \int_0^\infty e^{-st} u(t)\,dt = \int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s}$$

unit step

$$\mathcal{L}[\delta(t-t_0)] = \int_0^\infty e^{-st}\,\delta(t-t_0)\,dt = e^{-st_0}$$

delayed impulse
(time shifted)

$$\therefore\ \delta(t-t_0) \Longleftrightarrow e^{-st_0}$$

$$\delta(t) \Longleftrightarrow 1$$

$$\mathcal{L}[e^{-\alpha t}u(t)] = \int_0^\infty e^{-\alpha t}e^{-st}\,dt = -\frac{1}{s+\alpha}e^{-(s+\alpha)t}\Big|_0^\infty = \frac{1}{s+\alpha}$$

exponential

$$e^{-\alpha t}u(t) \Longleftrightarrow \frac{1}{s+a}$$

What about initial conditions

Suppose $v(t) \Longleftrightarrow V(s)$

then $$\mathcal{L}\left\{\frac{dv}{dt}\right\} = \int_0^\infty e^{-st}\left(\frac{dv}{dt}\right)dt = \int_0^\infty e^{-st}\,dv$$

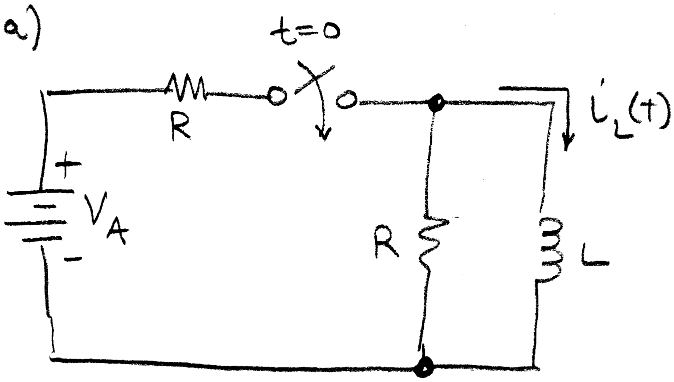
integrate by parts

$$= e^{-st}v\Big|_{0^-}^\infty - \int_{0^-}^\infty \underbrace{\left(-se^{-st}\right)}_{\text{this is the derivative of } e^{-st}} v(t)\,dt$$

$$= 0 - v(0^-) + s\underbrace{\int_{0^-}^\infty e^{-st}v(t)\,dt}_{V(s)}$$

$$= sV(s) - v(0^-)$$

9-33 (a)

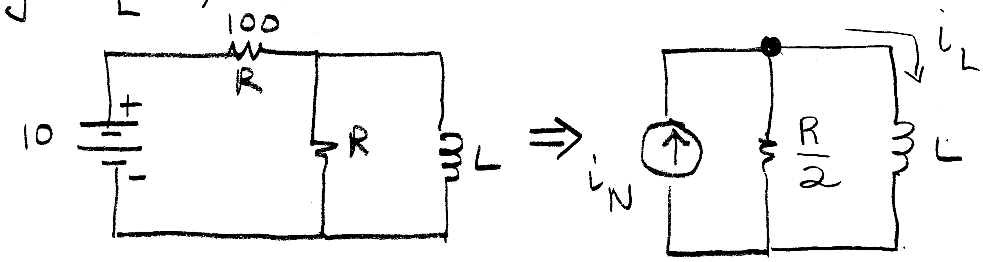


$R = 100\Omega$
$L = 100mH$
$V_A = 10V.$

(a) Find the D.E. and the initial condition for $i_L(t)$

initially $i_L(0^-) = 0$

$t>0$



where $i_N = \frac{10}{100} = 0.1A.$

do KCL at node $\sum_{+in} i = 0$

$$+i_N - \frac{v_L}{R/2} - i_L = 0$$

$$i_N - \frac{L\frac{di_L}{dt}}{R/2} - i_L = 0$$

$$i_N - \frac{2L}{R}\frac{di_L}{dt} - i_L = 0$$

$$\frac{2(\cancel{100}\times10^{-3})}{\cancel{100}}\frac{di_L}{dt} + i_L = +i_N = 0.1\,u(t)$$

$$\underbrace{2\times10^{-3}\frac{di_L}{dt} + i_L}_{\text{homogeneous}} = \overbrace{0.1\,u(t)}^{\text{forced.}}$$

(b) Solve for $i_L(t)$ using Laplace transforms.

Laplace transforming $2\times10^{-3}\left[sI_L(s) - \cancelto{0}{i_L(0^-)}\right] + I_L(s) = \frac{0.1}{s}$

$$I_L(s)\left[.002s+1\right] = \frac{0.1}{s}$$

$$I_L(s) = \frac{0.1}{s(.002s+1)} \cdot \frac{500}{500} = \frac{50}{s(s+500)}$$

$$= \frac{a}{s} + \frac{b}{s+500} = \frac{50}{s(s+500)}$$

$$\therefore \quad bs + as + a500 = 50$$

$$bs + as = 0 \qquad a500 = 50$$

$$\Rightarrow a = \frac{50}{500} = \frac{1}{10}$$

$$b = -a \Rightarrow b = -\frac{1}{10}.$$

$$I_L(s) = \frac{0.1}{s} - \frac{0.1}{s+500}$$

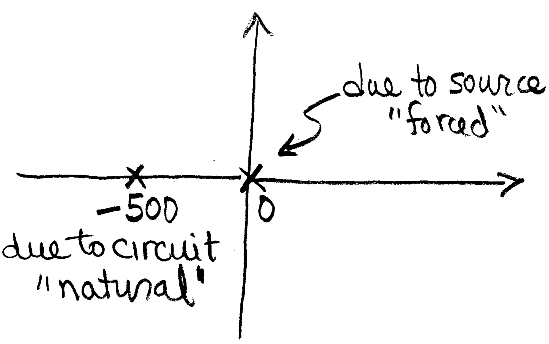
Now look up in Laplace transform table.

$$\frac{1}{s+\alpha} \longleftrightarrow e^{-\alpha t}u(t) \quad ; \quad \frac{1}{s} \longleftrightarrow u(t)$$

$$\therefore \quad i_L(t) = 0.1\,u(t) - 0.1\,e^{-500t}u(t)$$

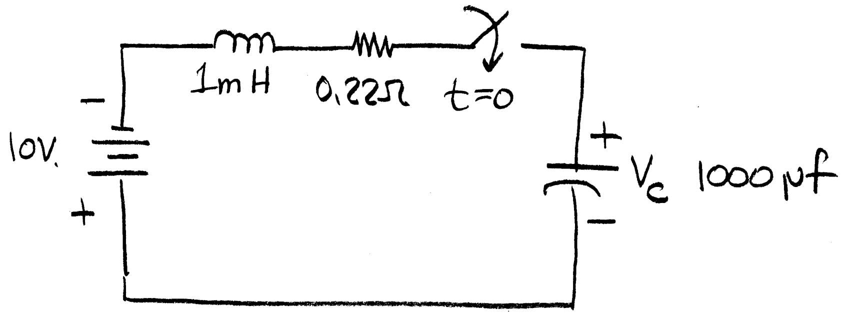
(c) Identify the forced and natural response terms.

forced natural

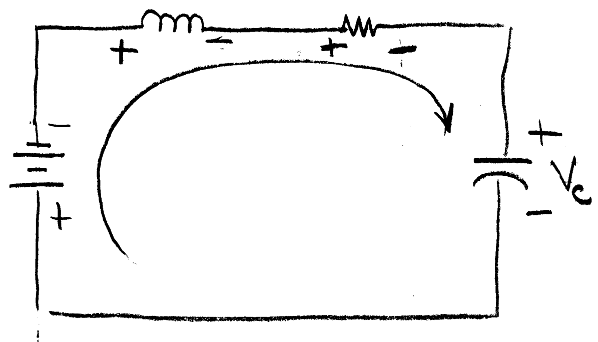


pole-zero diagram

Detailed example



For switch closed



solve for $i(t)$

using KVL $\quad +10 + L\dfrac{di}{dt} + iR + V_c = 0.$

where $V_c(t) = \frac{1}{C}\int_0^t i(x)\,dx$

How do we get Laplace transform of $V_c(t)$?

$$\mathcal{L}\left[\int_0^t i(x)\,dx\right] = \int_0^\infty e^{-st}\left[\int_0^t i(x)\,dx\right]dt$$

Integrate by parts: $\displaystyle\int_0^\infty u\,dw = uw\Big|_0^\infty - \int_0^\infty w\,du$

identify $\displaystyle u = \int_0^t i(x)\,dx \qquad dw = e^{-st}\,dt$

then $du = i(t)\,dt$ and $w = \dfrac{e^{-st}}{-s} = -\dfrac{1}{s}e^{-st}$

Then $$\mathcal{L}\left[\int_0^t i(x)\,dx\right] = \left[\int_0^t i(x)\,dx\right]\left[-\frac{1}{s}e^{-st}\right]\Bigg|_0^\infty + \int_0^\infty \frac{e^{-st}}{s}\,i(t)\,dt$$

$$= 0 + \frac{1}{s}I(s)$$

Then $$10 + L\frac{di}{dt} + iR + \frac{1}{C}\int_0^t i(x)dx = 0$$

becomes $$10 + L\left[sI(s) - i(0^-)\right] + I(s)R + \frac{1}{C}\left[\frac{1}{s}I(s)\right] = 0$$

Before the switch closed $i(0^-) = 0$ since there was no current through the capacitor.

$$10 + sL\,I(s) + R\,I(s) + \frac{1}{sC}I(s) = 0$$

Solving this algebraically

$$I(s) = \frac{-10}{\frac{1}{sC} + R + sL} = \frac{-10sC}{1 + sRC + s^2LC}$$

Putting in the numbers

$$I(s) = \frac{-10s\,(1000 \times 10^{-6})}{1 + s(0.2)(1000 \times 10^{-6}) + s^2(1 \times 10^{-3})(1000 \times 10^{-6})}$$

$$= \frac{-10s}{10^3 + 0.2s + 10^{-3}s^2} = \frac{-10{,}000s}{10^6 + 200s + s^2}$$

$$= \frac{-10{,}000s}{s^2 + 200s + 10^6}$$

We want to express denominator as product of terms. Use quadratic formula

$$s = \frac{-200 \pm \sqrt{(200)^2 - 4(1)(10^6)}}{2} = -100 \pm j995$$

$$I(s) = \frac{-10{,}000s}{(s+100+j995)(s+100-j995)}$$

Now use partial fraction expansion to separate terms

$$= \frac{a}{s+100-j995} + \frac{b}{s+100+j995}$$

Cross-multiply and solve for a and b

$$a(s+100+j995) + b(s+100-j995) = -10{,}000s$$

separating terms

$$as + bs = -10{,}000s$$

$$a+b = -10{,}000 \qquad (1)$$

$$a(100+j995) + b(100-j995) = 0$$

$$a = -b\,\frac{(100-j995)}{(100+j995)}$$

substituting this result back into (1) we get

$$-b\,\frac{(100-j995)}{(100+j995)} + b = -10{,}000$$

Solving for b

$$b = \frac{-10{,}000}{1 - \frac{(100-j995)}{(100+j995)}} = \frac{-10000}{\frac{(100+j995)-(100-j995)}{100+j995}}$$

$$= \frac{-10{,}000\,(100+j995)}{j1990}$$

$$= -5000 + j503$$

$$a = -10{,}000 - b$$

$$= -10{,}000 - (-5000 + j503)$$

$$a = -5000 - j503$$

$$I(s) = \frac{-5000 - j503}{s + 100 - j995} + \frac{-5000 + j503}{s + 100 + j995}$$

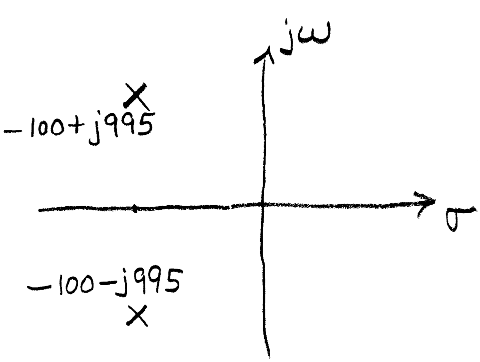
$$i(t) = \mathcal{L}^{-1}[I(s)] = \left[(-5000 - j503)e^{-(100 - j995)t} + (-5000 + j503)e^{-(100 + j995)t}\right]u(t)$$

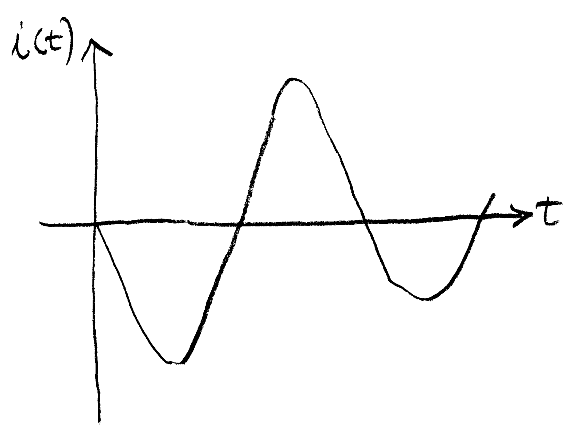
$$= e^{-100t}\left[(-5000 - j503)e^{+j995t} + (-5000 + j503)e^{-j995t}\right]u(t)$$

$$= e^{-100t}\left[-5000\left(e^{j995t} + e^{-j995t}\right) - j503\left(e^{j995t} - e^{-j995t}\right)\right]u(t)$$

$$= e^{-100t}\left[-5000\,(2\cos 995t) - j503\,(2j\sin 995t)\right]u(t)$$

$$= e^{-100t}\left[-10000\cos 995t + 1006\sin 995t\right]u(t)$$





Network functions

Definition of network function

For linear circuits the output of a circuit $y(t)$ is proportional to the input $x(t)$

In the s-domain this proportionality factor is a rational function called a network function.

$$Y(s) = T(s)\,X(s)$$

More formally

$$\text{Network function} \triangleq \frac{\text{Zero-state Response Transform}}{\text{Input Signal Transform}} \quad \frac{\text{(output)}}{\text{(input)}}$$

$$X(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s)$$

In general,

$$Y(s) = \underbrace{\sum_{j=1}^{N} \frac{k_j}{s-p_j}}_{\text{natural poles}} + \underbrace{\sum_{\ell=1}^{M} \frac{k_\ell}{s-p_\ell}}_{\text{forced poles}}$$

and

$$y(t) = \underbrace{\sum_{j=1}^{N} k_j e^{p_j t}}_{\text{natural response}} + \underbrace{\sum_{\ell=1}^{M} k_\ell e^{p_\ell t}}_{\text{forced response}}$$

in a stable circuit all natural poles are in LHP and all exponential terms in natural response decay to zero.

the poles of $X(s)$ cause the forced response. The elements that do NOT decay to zero are called the steady-state response.

$$F(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

The Cover-Up Algorithm (p. 448)

For the case of proper rational functions ($n > m$) and no repeated poles we can find the inverse transform.

Consider $F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}$

a partial fraction expansion of $F(s)$

The k's are called residues.

Inverse laplace transform this to get

$$f(t) = \left[k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_n e^{p_n t}\right] u(t)$$

If we know the residues the transform is easy to determine.

Example:

$$F(s) = 2\frac{s+3}{s(s+1)(s+2)}$$

Do a partial fraction expansion

$$F(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$(s+1)F(s) = \cancel{(s+1)}\, 2\frac{(s+3)}{s\cancel{(s+1)}(s+2)} = (s+1)\frac{k_1}{s} + (s+1)\frac{k_2}{s+1} + (s+1)\frac{k_3}{s+2}$$

If we evaluate this at $s=-1$ a lot of terms vanish to give k_2

$$(s+1)F(s)\Big|_{s=-1} = \frac{2(-1+3)}{(-1)(-1+2)} = \overbrace{\cancel{(-1+1)}}^{0}\frac{k_1}{s} + k_2 + \overbrace{\cancel{(-1+1)}}^{0}\frac{k_3}{s+2}$$

or

$$(s+1)F(s)\Big|_{s=-1} = -4 = k_2$$

Continuing

$$k_1 = sF(s)\Big|_{s=0} = \frac{2(s+3)}{(s+1)(s+2)}\Bigg|_{s=0} = \frac{2(3)}{(1)(2)} = 3$$

$$k_3 = (s+2)F(s)\Big|_{s=-2} = \frac{2(s+3)}{s(s+1)}\Bigg|_{s=-2} = \frac{2(-2+3)}{(-2)(-2+1)} = 1$$

$$\therefore \quad f(t) = \frac{3}{s} + \frac{-4}{s+1} + \frac{1}{s+2}$$

There are two other techniques we need to know.

I. What if $F(s)$ is an improper fraction, $m \geq n$?

We need to divide the denominator into the numerator to put $F(s)$ into the form of a proper fraction.

See Problem 9-26a.

II. What if $F(s)$ has a multiple pole?

Factor out one of the multiple poles and do partial fraction expansion of what is left.

See Example 9.14

Problem 9.26 (a)

Find the inverse transform of $F(s) = \dfrac{(s+50)^2}{(s+10)(s+100)}$

This is an improper fraction. Multiply out to get

$$F(s) = \frac{s^2 + 100s + 2500}{s^2 + 110s + 1000}$$

Divide to get a proper fraction remainder:

$$\begin{array}{r|l} & \quad 1 \\ \hline s^2+110s+1000 & s^2+100s+2500 \\ & s^2+110s+1000 \\ \cline{2-2} & \quad -10s+1500 \end{array}$$

Rewrite F(s) as

$$F(s) = 1 + \underbrace{\frac{-10s+1500}{(s+10)(s+100)}}_{F'(s)}$$

$$= 1 + \frac{k_1}{s+10} + \frac{k_2}{s+100}$$

Using the cover-up algorithm

$$k_1 = (s+10)F'(s)\Big|_{s=-10} = \frac{-10(-10)+1500}{(-10+100)} = \frac{100+1500}{90} = \frac{160}{9}$$

$$k_2 = (s+100)F'(s)\Big|_{s=-100} = \frac{-10(-100)+1500}{(-100+10)} = \frac{2500}{-90} = -\frac{250}{9}$$

Then $F(s) = 1 + \dfrac{\frac{160}{9}}{s+10} - \dfrac{\frac{250}{9}}{s+100}$

$$f(t) = \delta(t) + \frac{160}{9}e^{-10t}u(t) - \frac{250}{9}e^{-100t}u(t)$$

Example 9-14

Inverse transform $F(s) = \dfrac{s}{(s+1)(s+2)^2}$

This has a multiple root. Factor out one multiple root and do a partial fraction expansion.

$$F(s) = \frac{1}{s+2}\left[\underbrace{\frac{s}{(s+1)(s+2)}}_{F'(s)}\right] = \frac{1}{s+2}\left[\frac{k_1}{s+1} + \frac{k_2}{s+2}\right].$$

$$k_1 = (s+1)F'(s)\Big|_{s=-1} = \frac{-1}{-1+2} = -1$$

$$k_2 = (s+2)\,F'(s)\Big|_{s=-2} = \frac{-2}{-2+1} = 2$$

$$\therefore\ F(s) = \frac{1}{s+2}\left[\frac{-1}{s+1} + \frac{2}{s+2}\right] = \underbrace{\frac{-1}{(s+1)(s+2)}}_{\text{expand this as } F''(s)} + \frac{2}{(s+2)^2}$$

$$F(s) = \frac{k_3}{s+1} + \frac{k_4}{s+2} + \frac{2}{(s+2)^2}$$

$$k_3 = (s+1)\,F''(s)\Big|_{s=-1} = \frac{-1}{-1+2} = -1$$

$$k_4 = (s+2)\,F''(s)\Big|_{s=-2} = \frac{-1}{-2+1} = 1$$

$$F(s) = \frac{-1}{s+1} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$f(t) = \left[-e^{-t} + e^{-2t} + 2te^{-2t}\right]u(t)$$

Superposition in the s-domain

For a linear circuit $y = k_1 x_1 + k_2 x_2 + k_3 x_3 + \ldots$
where y is the output and the x_i are inputs
We can do this in the s-domain
but the k's become rational functions in s.

Two types of s-domain inputs:

(1) voltage & current sources representing the external driving inputs for $t \geq 0$

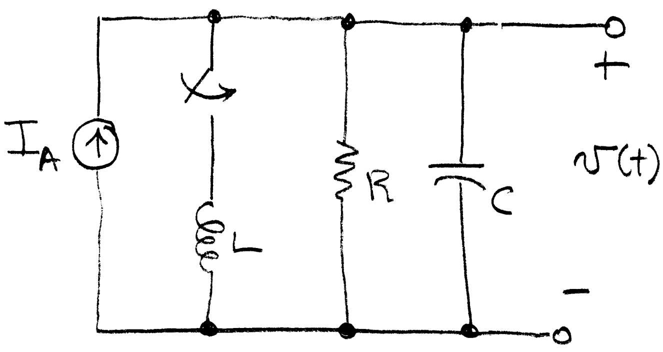
(2) initial condition voltage & current sources representing energy stored at $t = 0$.

$$\text{Then} \quad V(s) = \underbrace{V_{zs}(s)}_{\text{zero-state}} + \underbrace{V_{zi}(s)}_{\text{zero-input}}$$

zero-state: caused by external inputs with initial conditions turned off

zero-input: caused by initial conditions with external inputs turned off.

Example 10-6



(a) Transform into the s-domain

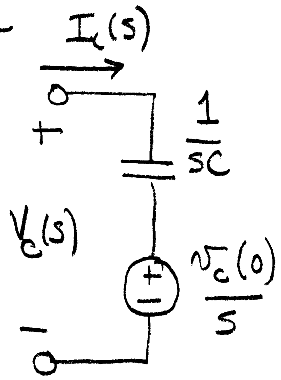
find initial conditions

$i_L(0^-) = 0$ amps since switch is open

capacitor charges to $V_c = I_A R$ since no current through C.

Better to use current source to represent $v_c(0^-)$

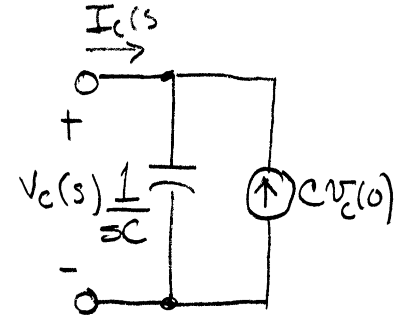
See Ch. 10, p. 485-6 Impedance and admittance



$$v_c(t) = \frac{1}{C}\int_0^t i_c(\tau)\,d\tau + v_c(0)$$

$$V_c(s) = \frac{1}{sC} I_c(s) + \underbrace{\frac{v_c(0)}{s}}_{\text{initial condition}}$$

However, this can also be written as

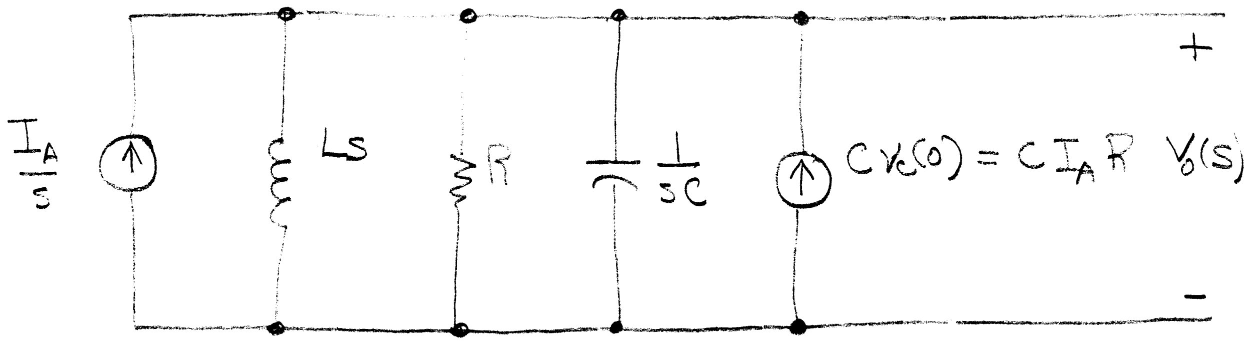


$$sC\left[V_c(s) = \frac{1}{sC} I_c(s) + \frac{v_c(0)}{s}\right]$$

$$sC\,V_c(s) = I_c(s) + C\,v_c(0)$$

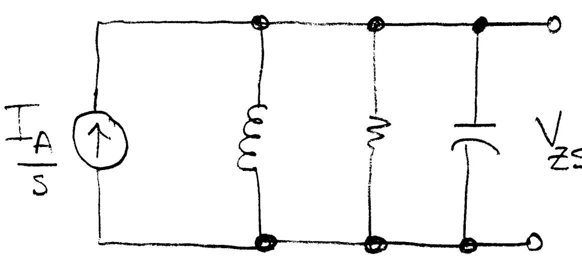
$$I_c(s) = sC\,V_c(s) - C\,v_c(0)$$

Transformed circuit

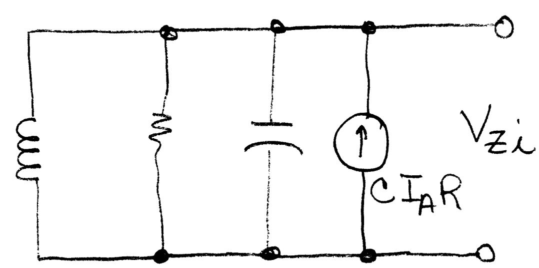


put switch here

(b) Find the zero-state and zero-input components of V(s)



zero-state
(replaced initial condition by open)



zero-input
(replaced external input by open)

combine R, L, C.

$$Z_{EQ} = \frac{1}{Y_{EQ}} = \frac{1}{\frac{1}{sL} + \frac{1}{R} + sC} = \frac{sLR}{R + sL + RLCs^2}$$

$$V_{ZS} = Z_{EQ}(s)\,\frac{I_A}{s} = \frac{RL\cancel{s}}{RLCs^2 + Ls + R} \cdot \frac{I_A}{\cancel{s}} = \frac{I_A/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

$$V_{ZI} = Z_{EQ}(s)\left[CRI_A\right] = \frac{RLs}{RLCs^2 + Ls + R}\,RCI_A = \frac{RI_A s}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

(c) Find $v(t)$ if

$I_A = 1\,mA$, $L = 2H$, $R = 1.5k\Omega$, $C = \frac{1}{6}\mu F$

$$V_{zs}(s) = \frac{10^{-3}/\frac{1}{6}\times10^{-6}}{s^2 + \frac{s}{(1500)(\frac{1}{6}\times10^{-6})} + \frac{1}{(2)(\frac{1}{6}\times10^{-6})}} = \frac{6000}{s^2 + 4000s + 3000000}$$

$$= \frac{6000}{(s+1000)(s+3000)} = \frac{k_1}{s+1000} + \frac{k_2}{s+3000}$$

$$k_1 = (s+1000)V_{zs}(s)\Big|_{s=-1000} = \frac{6000}{(-1000+3000)} = 3$$

$$k_2 = (s+3000)V_{zs}(s)\Big|_{s=-3000} = \frac{6000}{(-3000+1000)} = -3$$

$$V_{zs}(s) = \frac{3}{s+1000} - \frac{3}{s+3000}$$

$$V_{zi}(s) = \frac{(1500)(10^{-3})s}{s^2 + \frac{s}{(1500)(\frac{1}{6}\times10^{-6})} + \frac{1}{2(\frac{1}{6}\times10^{-6})}} = \frac{1.5s}{(s+1000)(s+3000)}$$

$$= \frac{k_3}{s+1000} + \frac{k_4}{s+3000}$$

$$k_3 = (s+1000)V_{zi}(s)\Big|_{s=-1000} = \frac{1.5(-1000)}{(-1000+3000)} = -0.75$$

$$k_4 = (s+3000)V_{zi}(s)\Big|_{s=-3000} = \frac{1.5(-3000)}{(-3000+1000)} = +2.25$$

$$V_{zi}(s) = \frac{-0.75}{s+1000} + \frac{2.25}{s+3000}$$

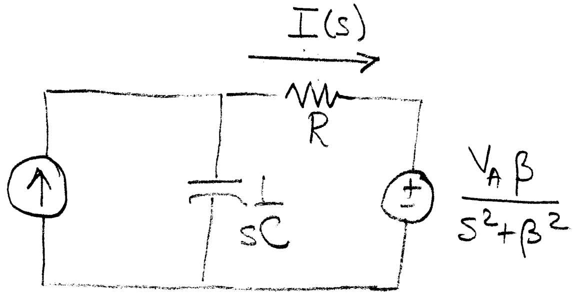
$$v_{zs}(t) = \left[3e^{-1000t} - 3e^{-3000t}\right]u(t)$$

$$v_{zi}(t) = \left[-0.75e^{-1000t} + 2.25e^{-3000t}\right]u(t).$$

No steady state. L acts like a short in time domain.
No forced response since forced pole canceled by zero.

Example 10-7

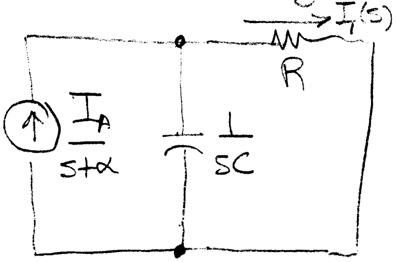
$e^{-\alpha t} u(t)$



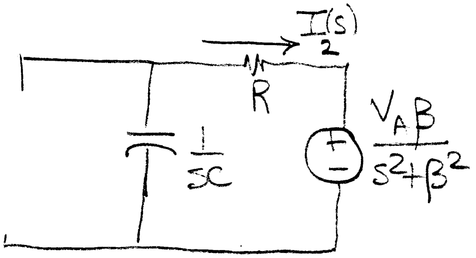
$\sin\beta t \, u(t)$

Find the zero state component of $I(s)$.

Easiest to find by superposition.



$$I_1(s) = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} \cdot \frac{I_A}{(s+\alpha)} = \frac{I_A}{(1+sRC)(s+\alpha)}$$



$$I_2(s) = -\frac{\frac{V_A\beta}{s^2+\beta^2}}{R + \frac{1}{sC}}$$

$$= -\frac{V_A \beta s C}{(1+sRC)(s^2+\beta^2)}$$

$$I(s) = I_1(s) + I_2(s) = \frac{I_A}{(1+sRC)(s+\alpha)} - \frac{V_A \beta C s}{(1+sRC)(s^2+\beta^2)}$$

natural pole comes from circuit $\quad s = -\frac{1}{RC}$

forced poles come from sources $\quad s = -\alpha \, , \; s = \pm j\beta$

$$\underbrace{K_1 e^{-t/RC}}_{\text{transient due to natural pole}} + \underbrace{K_2 e^{-\alpha t}}_{\text{forced transient due to forced pole}} + \underbrace{K_3 \cos\beta t + K_4 \sin\beta t}_{\text{forced component due to forced poles } \pm j\beta}$$

11-5 Steady State Frequency Response

Consider general sinusoidal input

$$x(t) = X_A \cos(\omega t + \phi) = X_A(\cos\omega t \cos\phi - \sin\omega t \sin\phi)$$

Transform this to get

$$X(s) = X_A \cos\phi \frac{s}{s^2+\omega^2} - X_A \sin\phi \frac{\omega}{s^2+\omega^2}$$

$$X(s) = X_A \frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$$

Consider the output of a system described by $T(s)$

$$Y(s) = X_A \left[\frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}\right] T(s) = X_A \frac{s\cos\phi - \omega\sin\phi}{(s-j\omega)(s+j\omega)} T(s)$$

Expanding by partial fractions

$$Y(s) = \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\text{forced poles}} + \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \dots + \frac{k_N}{s-p_N}}_{\text{natural poles of transfer function}}$$

Inverse transforming

$$y(t) = \underbrace{k e^{j\omega t} + k^* e^{-j\omega t}}_{\substack{\text{forced response} \\ \text{is a sinusoidal} \\ \text{steady-state}}} + \underbrace{k_1 e^{p_1 t} + k_2 e^{p_2 t} + \dots + k_N e^{p_2 t}}_{\substack{\text{natural response} \\ \text{decays to zero!}}}$$

To find k use cover-up algorithm

$$k = (s-j\omega) X_A \frac{s\cos\phi - \omega\sin\phi}{(s-j\omega)(s+j\omega)} T(s) \Big|_{s=j\omega}$$

$$k = X_A \frac{j\omega\cos\phi - \omega\sin\phi}{2j\omega} T(j\omega) = X_A \frac{\cos\phi + j\sin\phi}{2} T(j\omega)$$

$$k = \frac{1}{2} X_A e^{j\phi} T(j\omega) = \frac{1}{2} X_A e^{j\phi} |T(j\omega)| e^{j\theta}$$

$$k = \frac{1}{2} X_A |T(j\omega)| e^{j(\phi+\theta)}$$

$$y(t) = \frac{1}{2} X_A |T(j\omega)| e^{j(\theta+\phi)} e^{j\omega t} + \frac{1}{2} X_A |T(j\omega)| e^{-j(\theta+\phi)} e^{-j\omega t}$$

$$y(t) = \underset{\substack{\uparrow \\ \text{input amplitude}}}{X_A} |T(j\omega)| \cos(\omega t + \theta + \phi)$$

(1) Output frequency = Input frequency = ω

(2) Output amplitude = Input Amplitude $*$ $|T(j\omega)|$

(3) Output phase = Input phase + $\angle T(j\omega)$

Exercise 11-8

The transfer function of a linear circuit is

$$T(s) = \frac{5(s+100)}{(s+500)}$$

Find the steady state output for (a) $x(t) = 3\cos 100t$

$$|T(j100)| = \left| \frac{5(j100+100)}{(j100+500)} \right| = \left| \frac{5(100+j100)}{500+j100} \right|$$

$$= \left| 1.1538 + j0.7692 \right| = \left| 1.3868\, e^{j33.7^\circ} \right| = 1.387$$

$$\angle T(j100) = 33.7^\circ$$

$$y_{ss}(t) = (1.387)(3)\cos(100t + 33.7^\circ)$$

$$y_{ss}(t) = 4.16\cos(100t + 33.7^\circ)$$

You will often see the terms step response and impulse response.

The step response is the zero-state response when the driving function (input) is a unit step applied at $t=0$.

The impulse response is the zero state response when the driving function (input) is a unit impulse applied at $t=0$.

11-49 Step response of a linear circuit is

$$g(t) = 10e^{-1000t} \sin(1000t)$$

$$G(s) = \frac{10\,000}{(s+1000)^2 + 1000000}$$

Since $G(s) = T(s)\frac{1}{s}$ multiply by s to get $T(s)$

$$T(s) = \frac{10\,000s}{(s+1000)^2 + 1000000}$$

Exercise 11-6 The impulse response of a circuit is

$$h(t) = 0.1\,\delta(t) + 90e^{-100t} u(t)$$

$$H(s) = 0.1 + \frac{90}{s+100}$$

Since $H(s) = T(s)\cdot 1$ we have $T(s)$

$$T(s) = H(s) = 0.1 + \frac{90}{s+100}$$

If you present $T(j\omega)$ as a function of frequency in terms of $|T(j\omega)|$ and $\angle T(j\omega)$ this is called a Bode diagram.

In general,

$$T(j\omega) = A \frac{(j\omega + z_1)(j\omega + z_2)\ldots(j\omega + z_k)}{(j\omega)^n (j\omega + p_1)(j\omega + p_2)\cdots(j\omega + p_m)}$$

Rewrite each term as

$$(j\omega + z_i) = z_i \left(1 + j\frac{\omega}{z_i}\right)$$

$$(j\omega + p_j) = p_j \left(1 + j\frac{\omega}{p_j}\right)$$

$$20\log|T(j\omega)| = 20\log K + \sum_{i=1}^{k} 20\log\left|1 + j\frac{\omega}{z_i}\right| - 20n\log\omega - \sum_{j=1}^{m} 20\log\left|1 + j\frac{\omega}{p_j}\right|$$

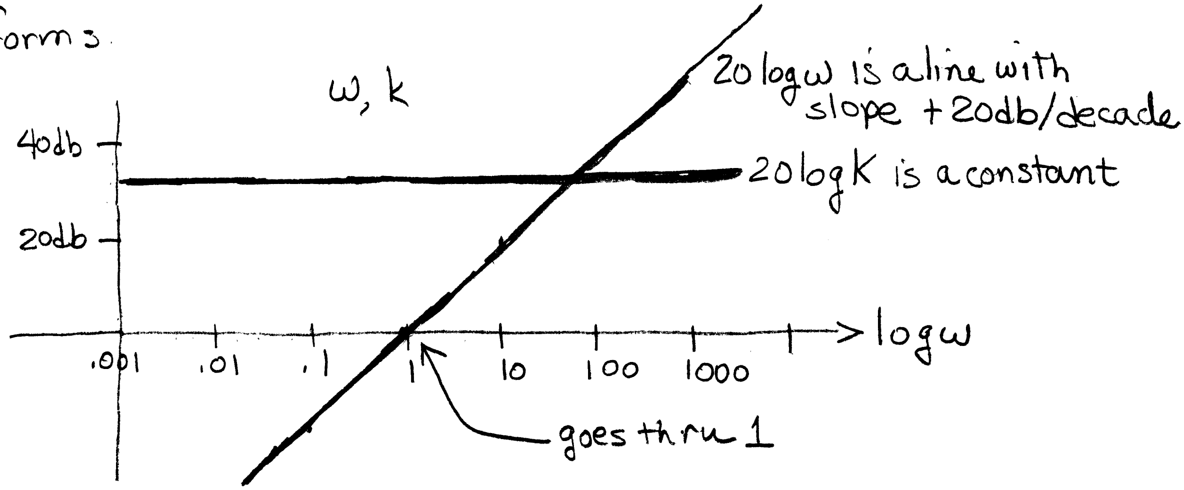
where $K = A\dfrac{\prod_{i=1}^{k} z_i}{\prod_{j=1}^{m} p_j}$

and

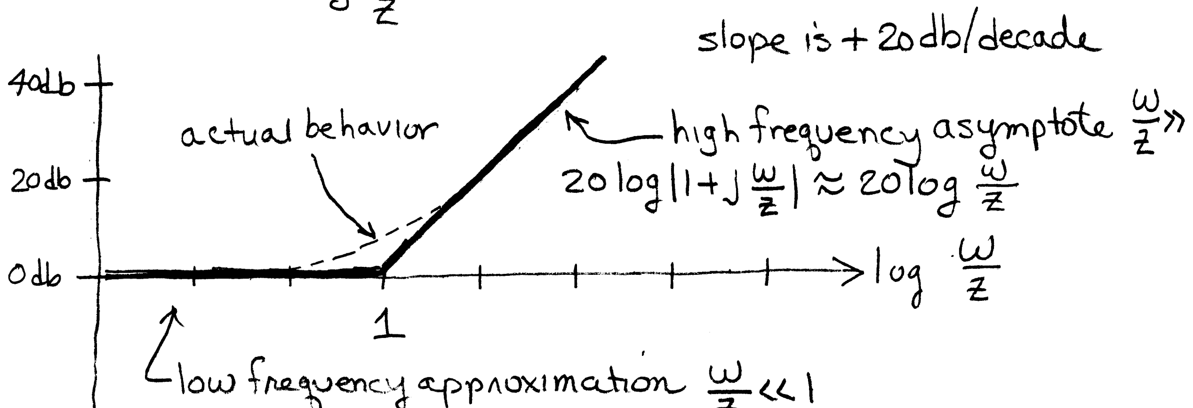
$$\angle T(j\omega) = \underbrace{\sum_{i=1}^{k} \operatorname{Tan}^{-1}\left(\frac{\omega}{z_i}\right)}_{\substack{\angle(z+j\omega) \\ = \operatorname{Tan}^{-1}\left(\frac{\omega}{z}\right)}} \underbrace{- 90^\circ n}_{\text{from } (j\omega)^n} - \sum_{j=1}^{m} \operatorname{Tan}^{-1}\left(\frac{\omega}{p_j}\right)$$

basic forms:

ω, k

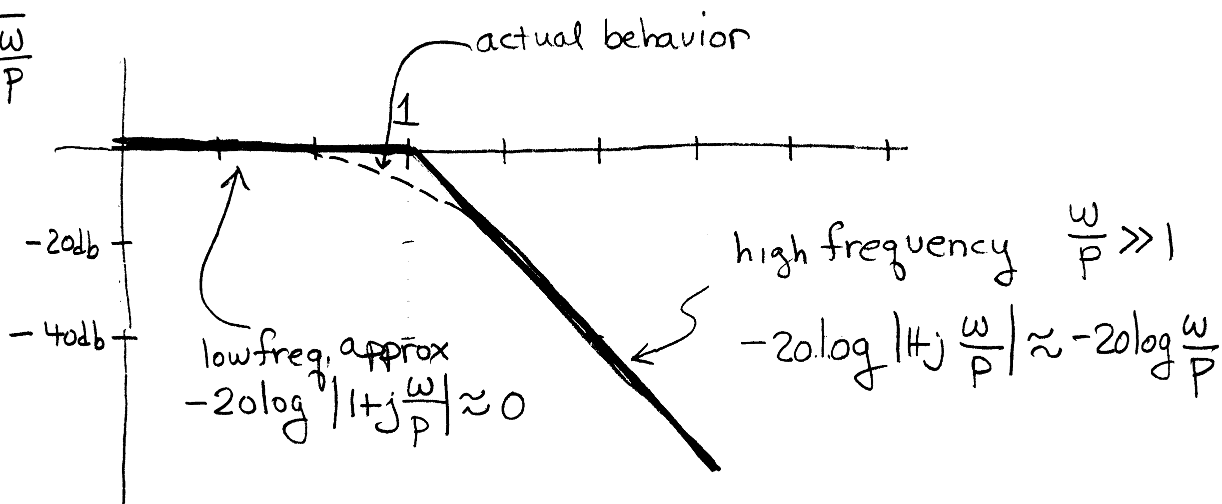


$1 + j\frac{\omega}{z}$

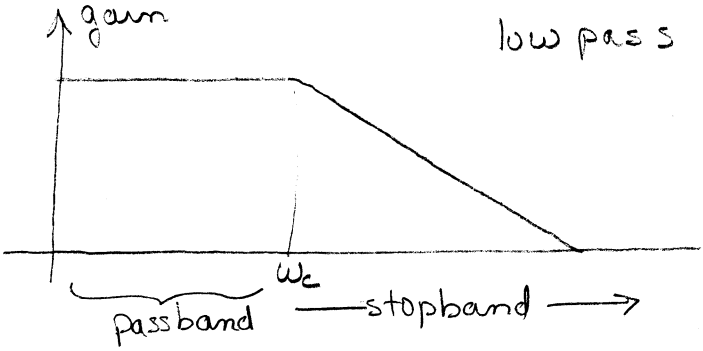


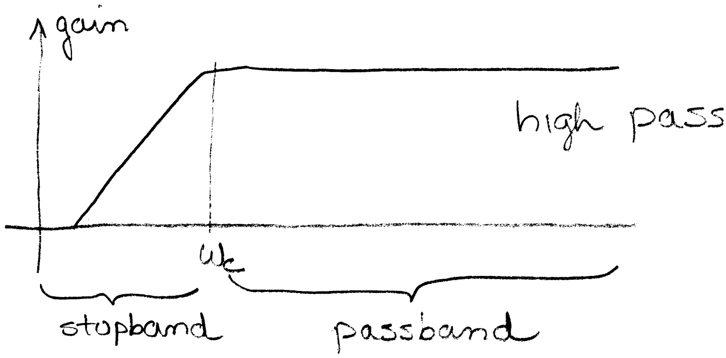
$20\log\left|1 + j\frac{\omega}{z}\right| \approx 20\log(1) = 0$

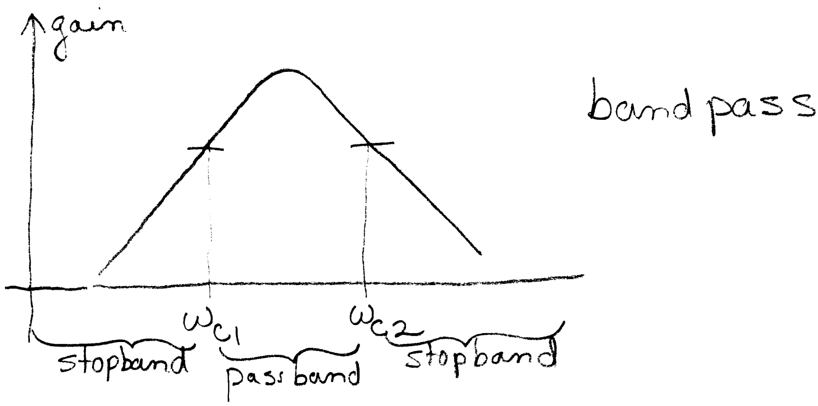
$\frac{1}{1+j\frac{\omega}{p}}$

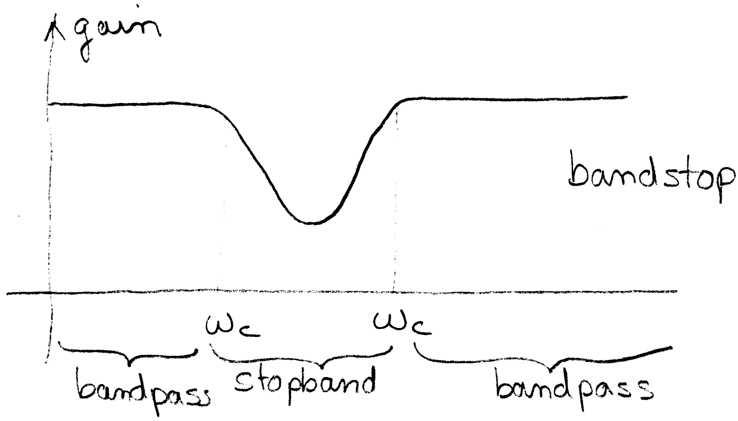


Prototype gain responses









Example of how pole-zero placement affects response

Consider $G_1(j\omega) = K \dfrac{(1+j\frac{\omega}{z})}{(1-j\frac{\omega}{p})}$

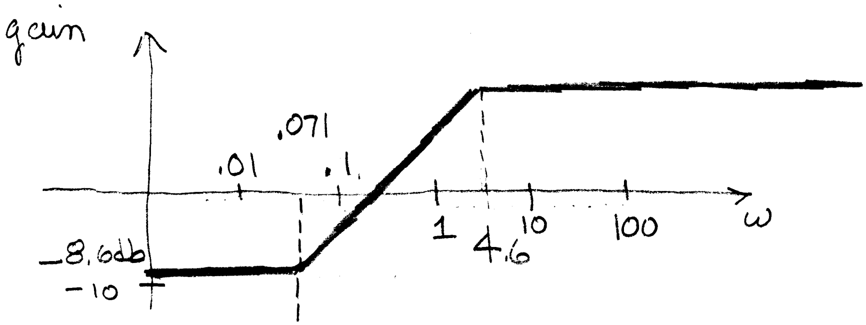
Taking logs

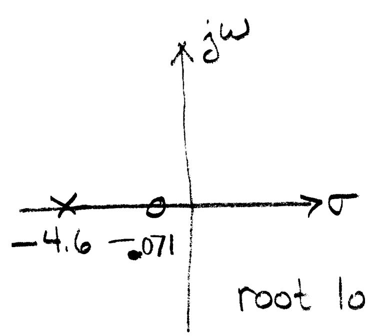
$$|G(j\omega)|_{db} = 20\log K + 20\log\left|1+j\frac{\omega}{z}\right| - 20\log\left|1+j\frac{\omega}{p}\right|$$

Case I $z<p$

Let $k=0.37$, $z=0.071$, $p=4.6$

$$|G(j\omega)|_{db} = \underbrace{20\log|0.37|}_{-8.6db} + 20\log\left|1+j\frac{\omega}{.071}\right| - 20\log\left|1+j\frac{\omega}{4.6}\right|$$





root locus

for small ω only have first term at $-8.6db$

since $z<p$, the zero comes in first adding

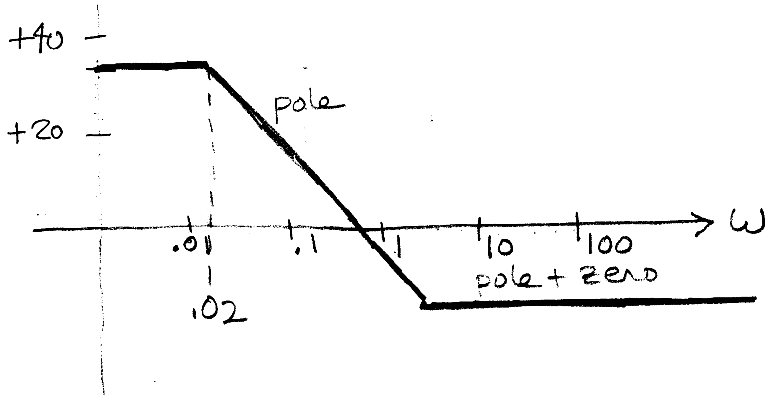
$+20\log\left|1+j\frac{\omega}{.071}\right|$ @ $\omega=.071$ this term starts to add $+20db/decade$

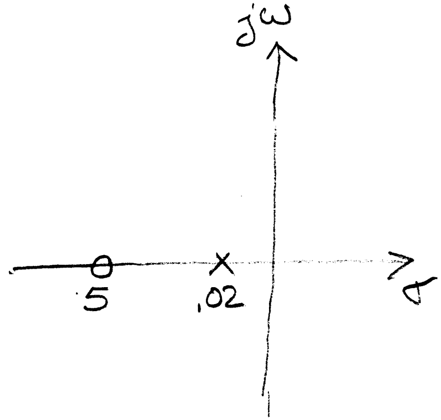
pole comes in at $\omega=4.6$

$-20\log\left|1+j\frac{\omega}{4.6}\right|$ @ $\omega=4.6$ this term begins adding $-20db/decade$

Case II. Same transfer function but $p < z$

let $K = 40$, $z = 5$, $p = .02$





$$|G_1(j\omega)|_{db} = 20 \log 40 + 20 \log \left|1 + j\frac{\omega}{5}\right| - 20 \log \left|1 + j\frac{\omega}{.02}\right|$$

at low frequencies $20 \log 40 = +32$ db

as ω increases pole at $\omega = .02$ comes in at -20 db/decade

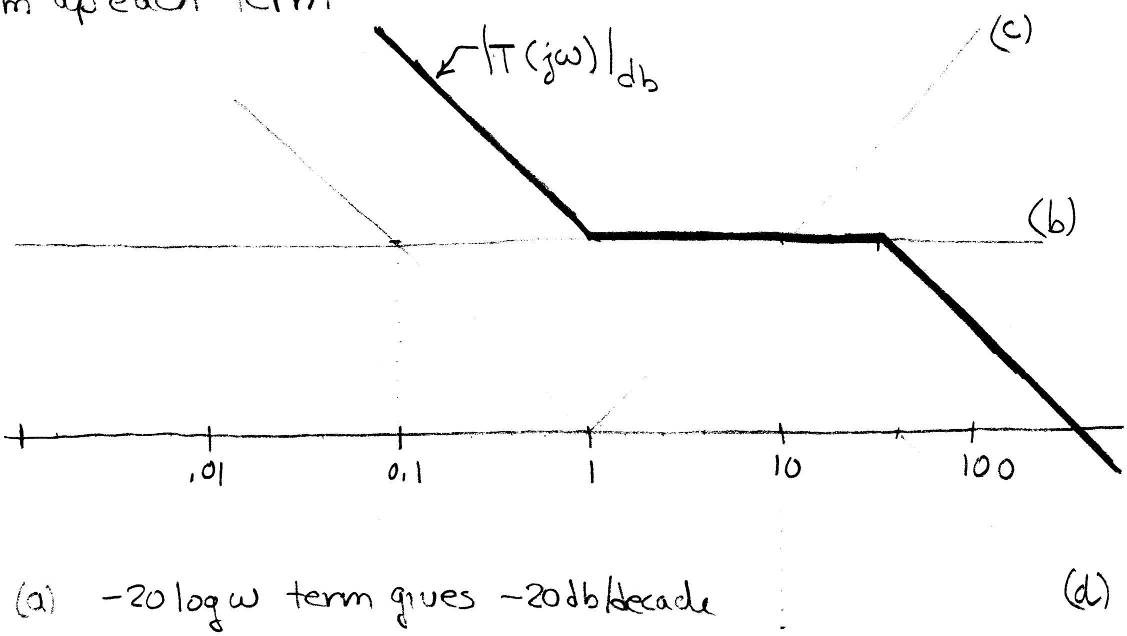
as ω continues to increase zero at $\omega = 5$ comes in giving $+20$db/decade canceling out effect of pole.

You can do more complex functions

Consider
$$T(j\omega) = \sqrt{10}\ \frac{1 + j\frac{\omega}{1}}{\omega\left(1 + j\frac{\omega}{50}\right)}$$

Then
$$|T(j\omega)|_{db} = 20\log\sqrt{10} + 20\log\left|1 + j\frac{\omega}{1}\right| - 20\log\omega - 20\log\left|1 + j\frac{\omega}{50}\right|$$

Sum up each term



(a) $-20\log\omega$ term gives -20 db/decade

(b) $20\log\sqrt{10}$ term gives $+10$ db.

(c) @ $\omega = 1$ $+20$ db/decade

(d) @ $\omega = 50$ -20 db/decade.

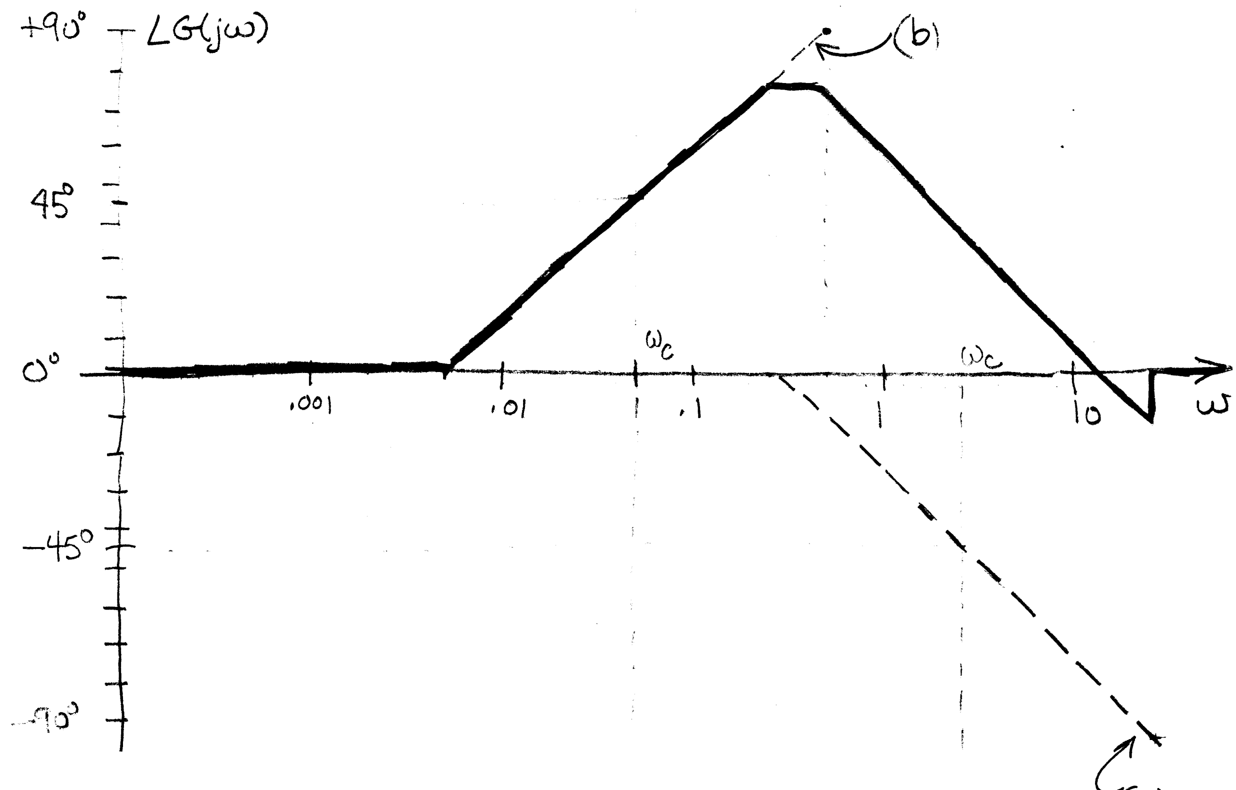
Let's do phase response of 1st order systems.

Phase gets plotted a little differently. It is $\pm 45°$ at ω_c and goes between 0 and ± 90

Example

$$|G(j\omega)|_{db} = 20\log K + 20\log\left|1+j\frac{\omega}{z}\right| - 20\log\left|1+j\frac{\omega}{p}\right|$$

$$\angle G(j\omega) = \angle K + \angle\left(1+j\frac{\omega}{.071}\right) - \angle\left(1+j\frac{\omega}{4.6}\right)$$



(a) $\angle(.37) = 0°$

(b) $\angle\left(1+j\frac{\omega}{.071}\right)$

(c) $-\angle\left(1+j\frac{\omega}{4.6}\right)$