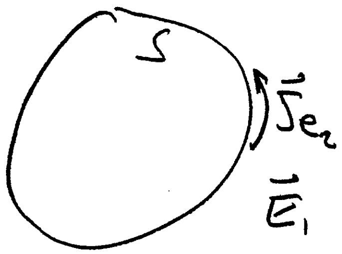
BOB MANNING
EEAP 563

PROB 1.4

$$\int_V \vec{E}_2 \cdot \vec{J}_{e_1}\, dv = \int_V \vec{E}_1 \cdot \vec{J}_{e_2}\, dv$$, FROM EQ.(48) OF CHAPT. 1



$\uparrow \vec{J}_{e_1} = J_{e_1}\hat{x}$

$\vec{E}_2$

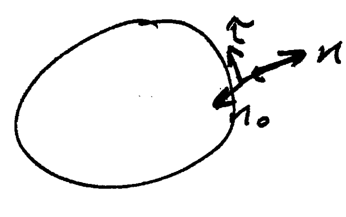
$$\int \vec{E}_2 \vec{J}_{e_1}\, dv = \vec{E}_2 \cdot \hat{x}\, I_0 \Delta x$$, WHERE IT WAS ASSUMED THAT $\vec{E}_2$ DOES NOT CHANGE OVER THE SIZE OF THE CURRENT DISTRIBUTION.

IF S IS A PERFECT CONDUCTOR, ONE HAS THAT $\hat{n} \times \vec{E}_1 = 0$. HENCE, ONE HAS FOR THE SECOND INTEGRAL,

$$\int_V \vec{E}_1 \cdot \vec{J}_{e_2}\, dv = \int \vec{E}_1 \cdot \hat{\tau}\, J_{e_2}\, dv,$$ $\hat{\tau} \equiv$ SURFACE TANGENT VECTOR

$$= \int \vec{E}_1 \cdot (\hat{n}_0 \times \hat{n}) J_{e_2}\, dv$$ $\hat{\tau} = \hat{n}_0 \times \hat{n}$

$$= \int \hat{n}_0 \cdot (\hat{n} \times \vec{E}_1) J_{e_2}\, dv$$



$$= \hat{n}_0 \cdot \int (\hat{n} \times \vec{E}_1) J_{e_2}\, dv$$

$= 0$, SINCE $J_{e_2} = J_{e_S}$ IS A SURFACE CURRENT DEFINED ON THE SURFACE S AND $\hat{n} \times \vec{E}_1$ VANISHES ON THAT SURFACE.

$$\therefore \quad \vec{E}_2 \cdot \hat{x}\, I_o \Delta x = 0$$

SINCE THE INDIVIDUAL DIRECTIONS $\bar{E}_2$ AND $\hat{x}$ ARE ARBITRARY, AND I_o + Δx ARE GENERALLY NON ZERO, ONE MUST HAVE $\bar{E}_2 = 0$.

THEREFORE, A SURFACE CURRENT DISTRIBUTION ON A PERFECT CONDUCTOR DOES NOT RADIATE AN ELECTRIC FIELD.

A

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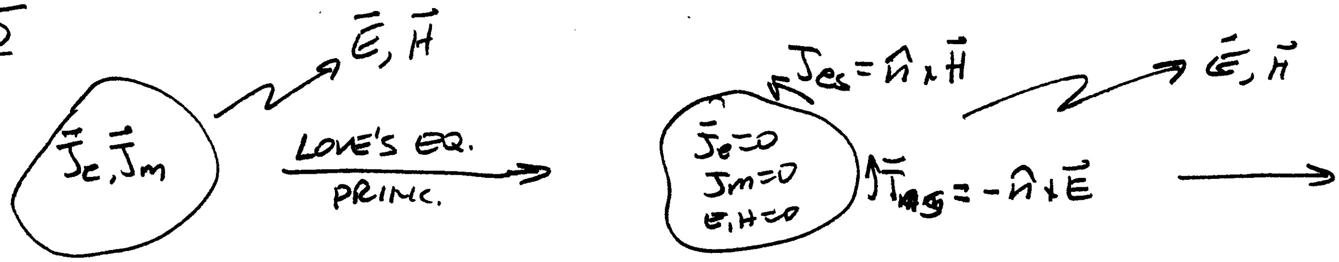
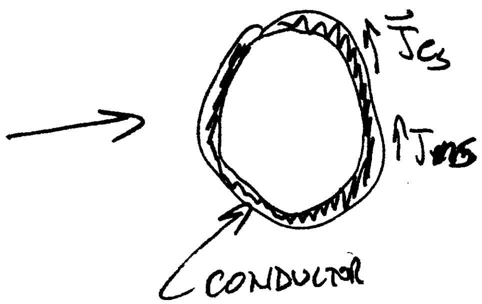
EEAP 563

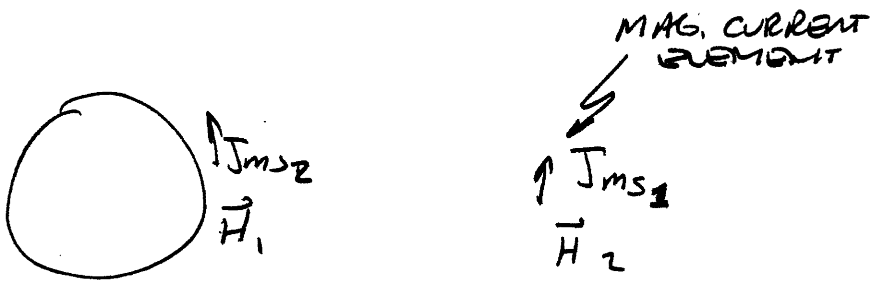
1.5





BUT FROM PROB. 1.4, AN ELECTRIC SURFACE CURRENT DOES NOT RADIATE. So ~~...~~ $\vec{J}_{ms}$ MUST CONTRIBUTE TO THE RADIATION.

$\therefore$ CONSIDER THE FOLLOWING



✓ A

"MAGNETIC" CURRENT

RECIP. $\Rightarrow$ $\int_V \vec{H}_1 \cdot \vec{J}_{ms_2}\, dV = \int_V \vec{H}_2 \cdot \vec{J}_{ms_1}\, dV = \vec{H}_2 \cdot \hat{x}\, I_{om} \Delta x$

BUT $\vec{J}_{ms_2} = -\hat{n} \times \vec{E}$ SO $\int_V \vec{H}_1 \cdot \vec{J}_{ms_2}\, dV = \int \vec{H}_1 \cdot (-\hat{n} \times \vec{E})\, dV$

$$= -\int \hat{n} \cdot (\vec{E} \times \vec{H}_1)\, dV$$

$\therefore \int \hat{n} \cdot (\vec{E} \times \vec{H}_1)\, dV = -\vec{H}_2 \cdot \hat{x}\, I_{om} \Delta x \neq 0$ $\quad\therefore$ RADIATION DOES INDEED EXIST WITH A $\vec{J}_{ms}$ DISTRIB. ON THE SURFACE

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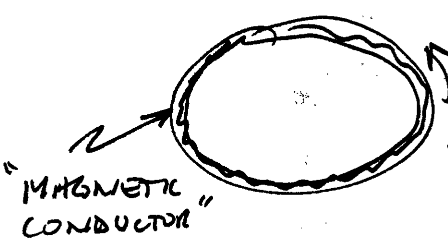
1.6

AS IN PROBLEM 1.5, ONE HAS THE FOLLOWING RECIPROCAL SITUATION.

FROM LOVE'S E.P.



$\vec{J}_{es_2} = \hat{n} \times \vec{H}$ $\qquad$ $\vec{J}_{es_1}$

$\vec{E}_1$ $\qquad$ $\vec{E}_2$

$$\int_v \vec{E}_1 \cdot \vec{J}_{es_2}\, dv = \int_v \vec{E}_2 \cdot \vec{J}_{es_1}\, dv$$

$$= \vec{E}_2 \cdot \hat{x}\, I_0 \Delta x$$

BUT $\int \vec{E}_1 \cdot \vec{J}_{es_2}\, dv = \int \vec{E}_1 \cdot (\hat{n} \times \vec{H})\, dv$

$$= \int \hat{n} \cdot (\vec{H} \times \hat{E}_1)\, dv$$

$$= -\int \hat{n} \cdot (\vec{E}_1 \times \vec{H})\, dv$$

A

$$\therefore \int \hat{n} \cdot (\vec{E}_1 \times \vec{H})\, dv = -\vec{E}_2 \cdot \hat{x}\, I_0 \Delta x \neq 0$$

AND THE ELECTRIC CURRENT DISTRIBUTION ON A "MAGNETIC CONDUCTOR" RADIATES A FIELD.

1.7

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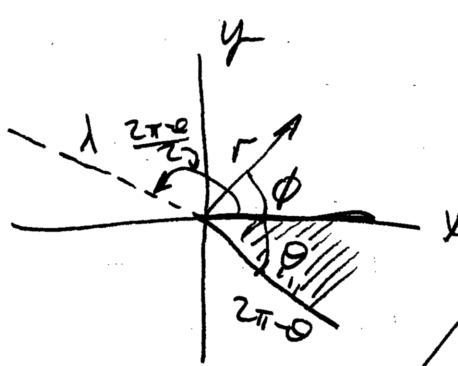
EEAP 563

CONFORMAL MAP: USE TRANSFORM

$$u + iv = (x+iy)^{\pi/(2\pi-\theta)}$$

LET $x+iy = re^{i\phi}$ WHERE $r = (x^2+y^2)^{1/2}$ AND ϕ IS DEFINED AS IN THE FIGURE.



$$\therefore\ u+iv = \left(re^{i\phi}\right)^{\pi/(2\pi-\theta)}$$

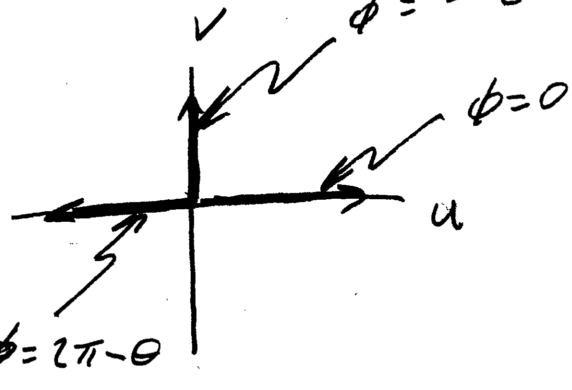
$$u = \mathrm{Re}\left\{\left(re^{i\phi}\right)^{\pi/(2\pi-\theta)}\right\}$$

$$= r^{\pi/(2\pi-\theta)} \cos\left(\frac{\pi\phi}{2\pi-\theta}\right)$$

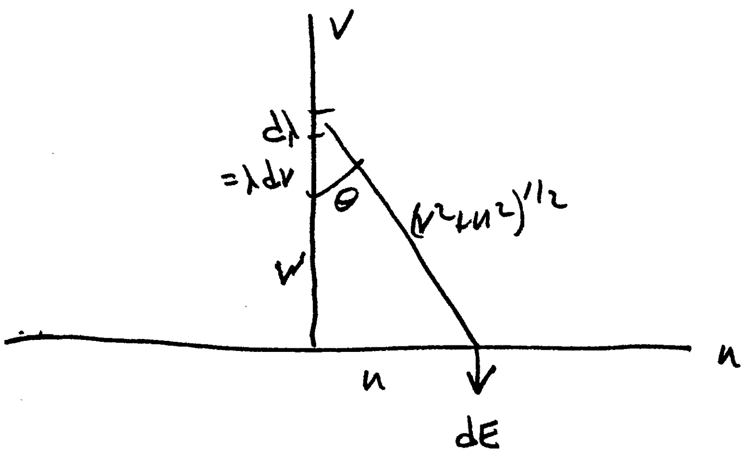
$$v = \mathrm{Im}\left\{\left(re^{i\phi}\right)^{\pi/(2\pi-\theta)}\right\}$$

$$= r^{\pi/(2\pi-\theta)} \sin\left(\frac{\pi\phi}{2\pi-\theta}\right)$$

$$u = r^{\pi/(2\pi-\theta)} \cos\left(\frac{\pi\phi}{2\pi-\theta}\right)$$
$$v = r^{\pi/(2\pi-\theta)} \sin\left(\frac{\pi\phi}{2\pi-\theta}\right) \Longrightarrow$$



$\therefore$ THE PROBLEM REDUCES TO HAVING TO FIND AN INDUCED SURFACE CHARGE DENSITY ON A PLANE DUE TO A VERTICAL CHARGE DISTRIBUTION.



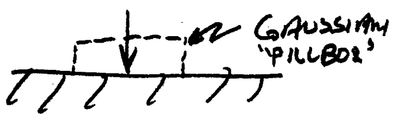
$$E = \int_0^\infty \frac{d\lambda}{(v^2+u^2)^{1/2}} \cos\theta = \int_0^\infty \frac{\lambda\, dv}{[v^2+u^2]^{1/2}} \left(\frac{v}{(v^2+u^2)}\right)$$

$$= \lambda \int_0^\infty \frac{v\, dv}{(v^2+u^2)^{3/2}}$$

$$= \lambda \left(-\left(\frac{1}{v^2+u^2}\right)^{1/2} \Bigg|_0^\infty \right)$$

$$= \frac{\lambda}{u}$$

$$E = \frac{\lambda}{u}$$



GAUSS'S LAW $\Rightarrow \oint \vec{E} \cdot d\vec{S} = q = \rho_s S$, ρ_s = INDUCED SURFACE CHARGE DENS.

$$\therefore \rho_s = \frac{-\lambda}{u}$$

USING THE EXPRESSION FOR u FROM THE 1st PAGE GIVES FOR THE INDUCED SURFACE CHARGE DENSITY AS A FUNCTION OF r,

$$\rho_s = -\left.\frac{\lambda r^{-\frac{\pi}{(2\pi-\theta)}}}{\cos\left(\frac{\pi\phi}{2\pi-\theta}\right)}\right|_{\phi=0}$$

$$= \lambda r^{-\frac{\pi}{(2\pi-\theta)}}$$

$$\therefore \rho_s \sim r^{-\alpha}, \quad \alpha \equiv \frac{\pi}{2\pi-\theta}$$

2.1

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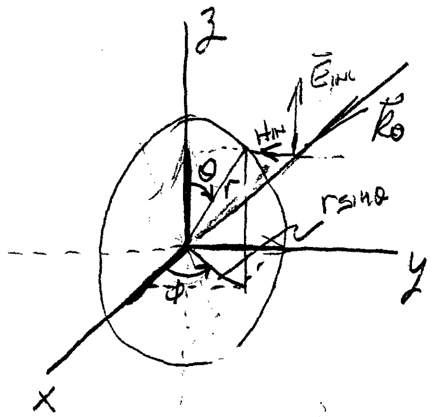
EEAP 563



LET $\vec{k}_0$ LIE IN THE X-y PLANE

$$\therefore\ \vec{E}_{INC} = \hat{z} E_0 e^{-i\vec{k}_0\cdot\vec{r}}$$

$$\vec{k}_0\cdot\vec{r} = k_0\hat{x}\cdot(x\hat{x}+z\hat{z})$$

$$= k_0 x$$

HENCE,

$$\boxed{\vec{E}_{INC} = \hat{z} E_0 e^{-ik_0 x}}$$

PLANE WAVE $\Rightarrow$ $\vec{H}_{INC} = Y_0\,\hat{k}_0 \times \vec{E}_{INC}$

SINCE $\vec{k}_0 = k_0\hat{x}$, $\hat{k}_0 = \hat{x}$

SO
$$\vec{H}_{INC} = Y_0(\hat{x}\times\hat{z})E_0 e^{-ik_0 x} = -Y_0\,\hat{y}\,E_0 e^{-ik_0 x}$$

$$\boxed{\vec{H}_{INC} = -Y_0\,\hat{y}\,E_0 e^{-ik_0 x}}$$

ONE MUST NOW SLIGHTLY MODIFY THE EQUATIONS DEVELOPED IN THE NOTES DUE TO THE OPPOSITE DIRECTION THAT $\vec{M}$ LIES HERE AS COMPARED TO THAT IN THE NOTES. HENCE, ONE MUST MAKE THE TRANSCRIPTION $M \rightarrow -M$ WHEN USING THE

EQUATIONS IN THE NOTES.

THUS, USING EQS (2.21) + (2.22) IN THE NOTES AND LETTING $M \to -M$, ONE HAS FOR THE DIFFERENTIAL SCATTERING CROSS SECTION

$$\sigma(\theta,\phi) = \left(\frac{1}{16\pi^2}\right) \frac{\left[(M Z_0 k_0^2)^2 (\cos^2\phi + \cos^2\theta \sin^2\phi) + \left(\frac{k_0^2 p}{\varepsilon_0}\right)^2 \sin^2\theta + \frac{2 M Z_0 k_0^4 p}{\varepsilon_0} \cdot \cos\phi \sin\theta \right]}{|\bar{E}_{INC}|^2}$$

WITH $p = |\bar{p}| = \varepsilon_0 |\alpha_e \bar{E}_{INC}| = \alpha_e \varepsilon_0 |\bar{E}_{INC}|$

AND $M = |\bar{M}| = |\alpha_m \bar{H}_{INC}| = \alpha_m Y_0 |\bar{E}_{INC}|$

$$\therefore \sigma(\theta,\phi) = \left(\frac{1}{16\pi^2}\right) \Big[(\alpha_m \cancel{Y_0 Z_0} k_0^2)^2 \cancel{|\bar{E}_{INC}|^2} (\cos^2\phi + \cos^2\theta \sin^2\phi) + $$
$$+ \left(\frac{\cancel{\varepsilon_0} k_0^2 \alpha_e}{\cancel{\varepsilon_0}}\right)^2 \cancel{|\bar{E}_{INC}|^2} \sin^2\theta + \frac{\cancel{\varepsilon_0} 2 \alpha_m \cancel{Y_0 Z_0} k_0^4 \alpha_e}{\cancel{\varepsilon_0}} \cancel{|\bar{E}_{INC}|^2} \cos\phi \sin\theta \Big] \cdot \frac{1}{\cancel{|\bar{E}_{INC}|^2}}$$

$$\boxed{\sigma(\theta,\phi) = \left(\frac{k_0^4}{16\pi^2}\right) \left[\alpha_m^2 (\cos^2\phi + \cos^2\theta \sin^2\phi) + \alpha_e^2 \sin^2\theta + (2\alpha_m \alpha_e) \cos\phi \sin\theta \right]} \quad 1)$$

THE TOTAL SCATTERING CROSS SECTION IS

$$\sigma_s = \int_0^{2\pi}\int_0^{\pi} \sigma(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

$$\sigma_s = \left(\frac{k_0^4}{16\pi^2}\right)\left[\alpha_m^2\left\{\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^2\phi\sin\theta\,d\theta\,d\phi\right\} + \right.$$

$$\left. + (\alpha_e)^2\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,d\theta\,d\phi + (2\alpha_e\alpha_m)\int_0^{2\pi}\int_0^{\pi}\cos\phi\sin^2\theta\,d\theta\,d\phi\Big\}\right]$$

$$\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\,d\theta\,d\phi = 2\int_0^{2\pi}\cos^2\phi\,d\phi = 2\pi \quad ✓$$

$$\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin\theta\sin^2\phi\,d\theta\,d\phi = \frac{2}{3}\int_0^{2\pi}\sin^2\phi\,d\phi = \frac{2\pi}{3} \quad ✓$$

$$\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\,d\theta\,d\phi = \int_0^{2\pi}\left(\frac{4}{3}\right)d\phi = \frac{8\pi}{3} \quad ✓$$

$$\int_0^{2\pi}\int_0^{\pi}\cos\phi\sin^2\theta\,d\theta\,d\phi = 0 \quad —$$

$$\therefore \quad \sigma_s = \left(\frac{k_0^4}{16\pi^2}\right)\left[\alpha_m^2\left(2\pi + \frac{2\pi}{3}\right) + \alpha_e^2\,\frac{8\pi}{3}\right]$$

$$= \left(\frac{k_0^4}{\cancel{16}\pi^{\cancel{2}}}\right)\left[\alpha_m^2\left(\frac{8\pi}{3}\right) + (\alpha_e)^2\,\frac{8\pi}{3}\right]$$

$$\sigma_s = \frac{k_0^4}{6\pi}\left[\alpha_m^2 + \alpha_e^2\right] \qquad 2)$$

THUS, FOR A CONDUCTING SPHERE,

$$\alpha_e = 4\pi a^3, \quad \alpha_m = -2\pi a^3$$

FROM EQ (1)

$$\sigma(\theta,\phi) = \left(\frac{k_0^4}{\cancel{16\pi^2}}\right)\left[\cancel{4\pi^2} a^6(\cos^2\phi + \cos^2\theta \sin^2\phi) + \cancel{16\pi^2 a^6}\sin^2\theta - \right.$$

$$\left. - \left(\cancel{16\pi^2 a^6}\right)\cos\phi \sin\theta\right]$$

$$= \frac{k_0^4 a^6}{4}\left[\cos^2\phi + \cos^2\theta \sin^2\phi + 4\sin^2\theta - \right.$$

$$\left. - 4\cos\phi \sin\theta\right]$$

$$= \frac{k_0^4 a^6}{4}\left[\cos^2\phi + \cos^2\theta \sin^2\phi + 4\sin^2\theta \pm 4\cos\phi \sin\theta\right]$$

$$\sigma(\theta,\phi) = \left(\frac{k_0^4 a^6}{4}\right)\left[(2\sin\theta + \cos\phi)^2 + \cos^2\theta \sin^2\phi\right]$$

FOR A CONDUCTING SPHERE

FROM EQ. (2),

$$\sigma_s = \frac{k_0^4}{\cancel{6\pi}}\left[\cancel{4\pi^2} a^6 + \cancel{16}\pi^2 a^6\right]$$

$$= \frac{k_0^4 a^6}{3}\left[2\pi + 8\pi\right]$$

$$= \frac{2\pi k_0^4 a^6}{3}\left[1 + 4\right]$$

$$\sigma_s = \frac{2\pi k_o^4 a^6}{3}\left[5\right]$$

$$\sigma_s = \frac{10\pi k_o^4 a^6}{3}$$

FOR A CONDUCTING SPHERE

FOR A DIELECTRIC SPHERE,

$$\alpha_m = 0, \quad \alpha_e = 4\pi a^3\left(\frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}\right)$$

FROM EQ. (1)

$$\sigma(\theta,\phi) = \left(\frac{k_o^4}{\cancel{16\pi^2}}\right)\left[\cancel{16\pi^2} a^6\left(\frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}\right)^2 \sin^2\theta\right]$$

$$\sigma(\theta,\phi) = k_o^4 a^6\left(\frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}\right)^2 \sin^2\theta$$

FOR A DIELECTRIC SPHERE

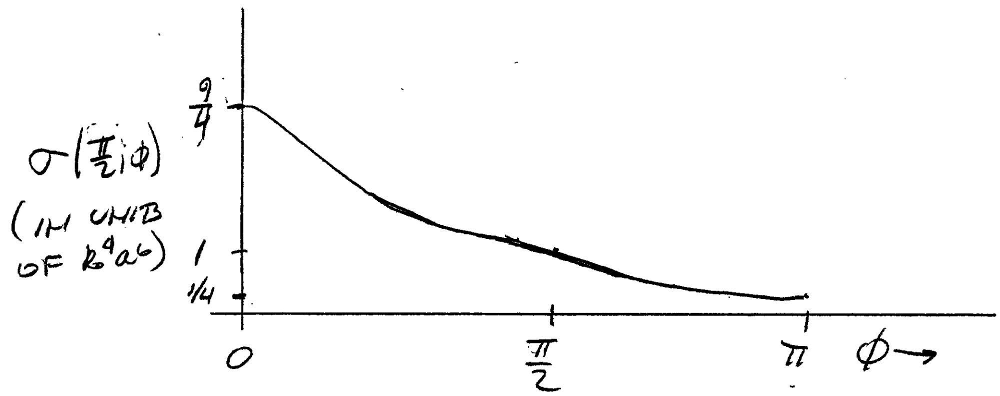
FROM EQ. (2),

$$\sigma_s = \frac{k_o^4}{\cancel{6\pi}_{3}}\left(\overset{8}{\cancel{16}}\pi^2 a^6\right)\left(\frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}\right)^2$$

$$\sigma_s = \frac{8\pi k_o^4 a^6}{3}\left(\frac{\varepsilon-\varepsilon_o}{\varepsilon+2\varepsilon_o}\right)^2$$

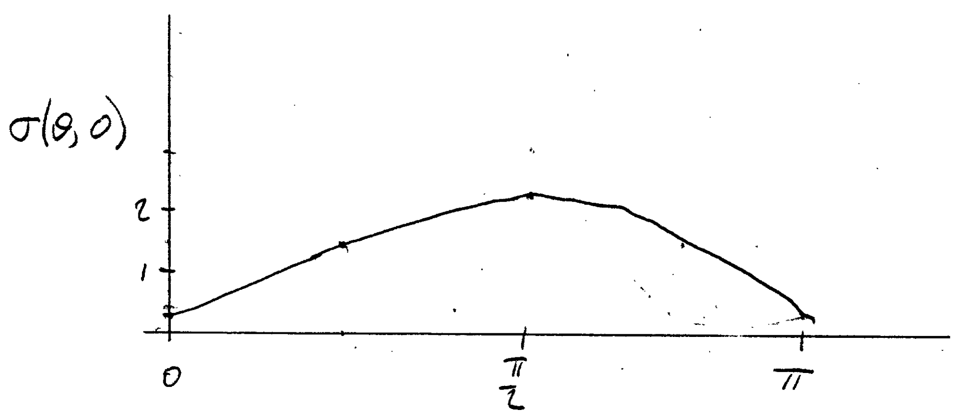
FOR A DIELECTRIC SPHERE

$\sigma\left(\frac{\pi}{2},\phi\right)$ vs ϕ



$$\sigma\left(\frac{\pi}{2},\phi\right)=\left(\frac{k_0{}^4a^6}{4}\right)\left[(2+\cos\phi)^2\right]$$

$\sigma(\theta,0)$ vs θ



$$\sigma(\theta,0)=\left(\frac{k_0{}^4a^6}{4}\right)\left[(2\sin\theta+1)^2\right]$$

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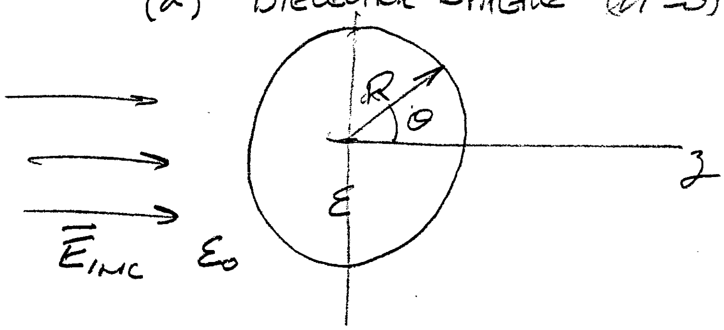
EEAP 563

PROB. 2.2

(a) DIELECTRIC SPHERE ($\mu=0$)



LAPLACE'S EQUATION: $\nabla^2\varphi = 0$

SPHERICAL COORD. SOLUTION (GENERAL):

OUTSIDE: $\varphi(r,\theta) = \sum_{\ell=0}^{\infty}\left[a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right] P_\ell(\cos\theta)$

INSIDE: $\varphi(r,\theta) = \sum_{\ell=0}^{\infty} c_\ell r^\ell P_\ell(\cos\theta)$

OUTSIDE THE SPHERE, THE SOLUTION IS OF THE GENERAL FORM GIVEN ABOVE. HOWEVER, THE BOUNDARY CONDITION AT INFINITY DEMANDS THAT

$$\varphi(z) = -E_{INC}z \rightarrow \varphi(r,\theta) = -E_{INC}r\cos\theta$$

AS $r\to\infty$; HENCE,

$$-E_0 r\cos\theta = \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) \Rightarrow a_1 = -E_0,\ \underline{a_0, a_2, a_3 \cdots = 0}$$

SINCE THE INTERIOR OF THE SPHERE CONTAINS THE ORIGIN, THE SOLUTION INSIDE THE SPHERE IS OF THE FORM

$$\varphi(r,\theta) = \sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta)$$

THE EXTERIOR + INTERIOR SOLUTIONS ARE CONNECTED VIA THE BOUNDARY CONDITIONS AT THE SURFACE OF THE SPHERE, i.e,

$$\left.\frac{\partial \varphi_{out}}{\partial \theta}\right|_{r=R} = \left.\frac{\partial \varphi_{int}}{\partial \theta}\right|_{r=R}, \qquad \varepsilon_0 \left.\frac{\partial \varphi_{out}}{\partial r}\right|_{r=R} = \varepsilon \left.\frac{\partial \varphi_{int}}{\partial r}\right|_{r=R}$$

(TANGENTIAL $\vec{E}$) (NORMAL $\vec{D}$)

THE 1ST B.C. GIVES

$$\sum_{l=0}^{\infty}\left[a_l R^l + \frac{b_l}{R^{l+1}}\right]\sin\theta\, P_l'(\cos\theta) = \sum_{l=0}^{\infty} c_l R^l \sin\theta\, P_l'(\cos\theta)$$

THIS GIVES FOR $l=1$

$$a_1 R + \frac{b_1}{R^2} = c_1 R$$

BUT $a_1 = -E_{inc}$ SO

$$-E_{inc}R + \frac{b_1}{R^2} = c_1 R$$

or $c_1 = -E_{inc} + \frac{b_1}{R^3}$ ✓

FOR ALL OTHER l, $l \neq 1$,

$$\frac{b_l}{R^{l+1}} = c_l R^l \Rightarrow c_l = \frac{b_l}{R^{2l+1}}$$

SINCE $a_l = 0$, AS FOUND EARLIER, FOR $l \neq 1$.

THE 2ND B.C. GIVES

$$\varepsilon_0 \sum_{l=0}^{\infty} \left[l a_l R^{l-1} - (l+1) \frac{b_l}{R^{l+2}} \right] P_l(\cos\theta) = \varepsilon \sum_{l=0}^{\infty} l c_l R^{l-1} P_l(\cos\theta)$$

AGAIN, FOR $l=1$,

$$\varepsilon_0 (a_1 - 2 b_1 R) = \varepsilon C_1$$ BUT $a_1 = -E_{INC}$ SO

$$-\varepsilon_0 \left(E_{INC} + 2 \frac{b_1}{R^3} \right) = \varepsilon C_1$$

$$C_1 = -\frac{\varepsilon_0 \left(E_{INC} + 2 \frac{b_1}{R^3} \right)}{\varepsilon}$$

FOR $l \neq 1$

$$\varepsilon_0 \left[-(l+1) \frac{b_l}{R^{l+2}} \right] = \varepsilon l c_l R^{l-1}$$

$$\Rightarrow C_l = -\frac{\varepsilon_0 (l+1) b_l}{\varepsilon l R^{l+2-l+1}}$$

$$C_l = -\frac{(l+1) b_l \varepsilon_0}{l R^3 \varepsilon}$$

FROM THESE FOUR CIRCLED EQUATIONS, ONE SEES THAT FOR THESE TO BE SIMULTANEOUSLY, ONE MUST HAVE $C_l = b_l = 0$ FOR $l \neq 0$; FOR $l=1$, ONE HAS

$$-E_{INC} + \frac{b_1}{R^3} = -\frac{\varepsilon_0\left(E_{INC} + 2\frac{b_1}{R^3}\right)}{\varepsilon}$$

SOLVING FOR b_1:

$$-\varepsilon E_{INC} + \varepsilon \frac{b_1}{R^3} = -\varepsilon_0 E_{INC} - \varepsilon_0 2 \frac{b_1}{R^3}$$

$$\varepsilon \frac{b_1}{R^3} + 2\varepsilon_0 \frac{b_1}{R^3} = E_{INC}(\varepsilon - \varepsilon_0)$$

$$b_1 \left(\frac{1}{R^3}\right)(\varepsilon + 2\varepsilon_0) = E_{INC}(\varepsilon - \varepsilon_0)$$

$$b_1 = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) R^3 E_{INC}$$

$$C_1 = -E_{INC} + \frac{b_1}{R^3} = -E_{INC} + \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_{INC}$$

$$= E_{INC}\left(-1 + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$$

$$= E_{INC}\left(\frac{-\cancel{\varepsilon} - 2\varepsilon_0 + \cancel{\varepsilon} - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$$

$$C_1 = -E_{INC}\left(\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$$

HENCE, OUTSIDE THE SPHERE, ONE HAS FOR THE TOTAL POTENTIAL:

$$\varphi(r,\theta) = \left(-E_{INC} r + \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\frac{R^3}{r^2} E_{INC}\right)\cos\theta$$

DUE TO APPLIED FIELD

DUE TO DIPOLE MOMENT SITUATED AT ORIGIN

ONE CAN CONSIDER THE 2ND TERM IN THIS EXPRESSION FOR THE POTENTIAL TO BE DUE TO A DIPOLE WITH DIPOLE MOMENT P

$$\vec{P} = 4\pi\varepsilon_0 \vec{A} = 4\pi\varepsilon_0 \left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right) R^3 \vec{E}_{INC}$$

$$= 4\pi R^3 \left(\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0}\right) \left(\varepsilon_0 \vec{E}_{INC}\right)$$

$$\lim_{\varepsilon\to\infty} P = 4\pi R^3 \left(\frac{\varepsilon}{\varepsilon}\right)\left(\varepsilon_0 E_{INC}\right) = 4\pi R^3 \left(\varepsilon_0 E_{INC}\right)$$

$$= \alpha_e \, \varepsilon_0 E_{INC}$$

$$\alpha_e \equiv 4\pi R^3$$

(b) TO OBTAIN A SOLUTION FOR THE MAGNETIC DIPOLE MOMENT, ONE COULD REPEAT THE ABOVE ANALYSIS FOR A SPHERE OF PERMEABILITY μ IN A HOMOGENEOUS $\vec{H}$ FIELD BUT ONE CAN ALSO APPLY DUALITY TO THE SOLUTION OBTAINED IN PART a). USING THE DUALITY TRANSCRIPTIONS

$$\vec{E}_{INC} \to -\left(\frac{\mu_0}{\varepsilon_0}\right)^{1/2} \vec{H}_{INC}, \quad \vec{P}_e \to -(\mu_0\varepsilon_0)^{1/2} \vec{P}_m, \quad \varepsilon \to \mu, \quad \varepsilon_0 \to \mu_0$$

ONE HAS $\varepsilon_0 \vec{E}_0 = -(\varepsilon_0 \mu_0)^{1/2} \vec{H}$ AND HENCE

$$-\cancel{(\mu_0 \varepsilon_0)}^{1/2} \vec{P}_m = -4\pi R^3 \left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right) \cancel{(\varepsilon_0 \mu_0)} \vec{H}_{INC}$$

$$\vec{P}_m = +4\pi R^3 \left(\frac{\mu - \mu_0}{\mu + 2\mu_0}\right) \vec{H}_{INC}$$

$$\lim_{\mu \to 0} P_m = 4\pi R^3 \left(-\frac{\mu_0}{2\mu_0}\right) H_{INC}$$

$$= -2\pi R^3 H_{INC}$$

$$= \alpha_m H_{INC}, \quad \alpha_m \equiv -2\pi R^3$$

A

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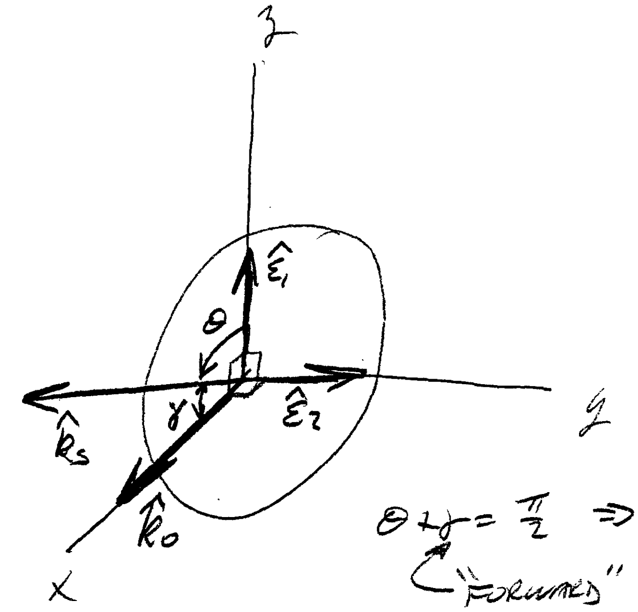
PROB 2.3



$\theta + \gamma = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} - \gamma$

"FORWARD" SCATT. ANGLE

(i) DIFF CROSS-SECTION FOR INCIDENT POLARIZATION $\hat{\varepsilon}_1$:

$$\sigma_{\hat{\varepsilon}_1}(\theta) = k_0^4 a^6 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)^2 \sin^2\theta$$

IN TERMS OF γ, $\sin^2\theta = \sin^2\left(\frac{\pi}{2} - \gamma\right) = \cos^2\gamma$ SO

$$\sigma_{\hat{\varepsilon}_1}(\gamma) = k_0^4 a^6 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)^2 \cos^2\gamma \qquad 1)$$

(ii) DIFF. CROSS-SECTION FOR INCIDENT POLARIZATION $\hat{\varepsilon}_2$

$$\sigma_{\hat{\varepsilon}_2}(90^\circ) = k_0^4 a^6 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)^2 \qquad 2)$$

SINCE, BY THE GEOMETRY OF THE SITUATION, $\hat{\varepsilon}_2 \perp \hat{k}_s$

NOW DEFINE THE TOTAL POLARIZATION OF THE SCATTERED RADIATION AS

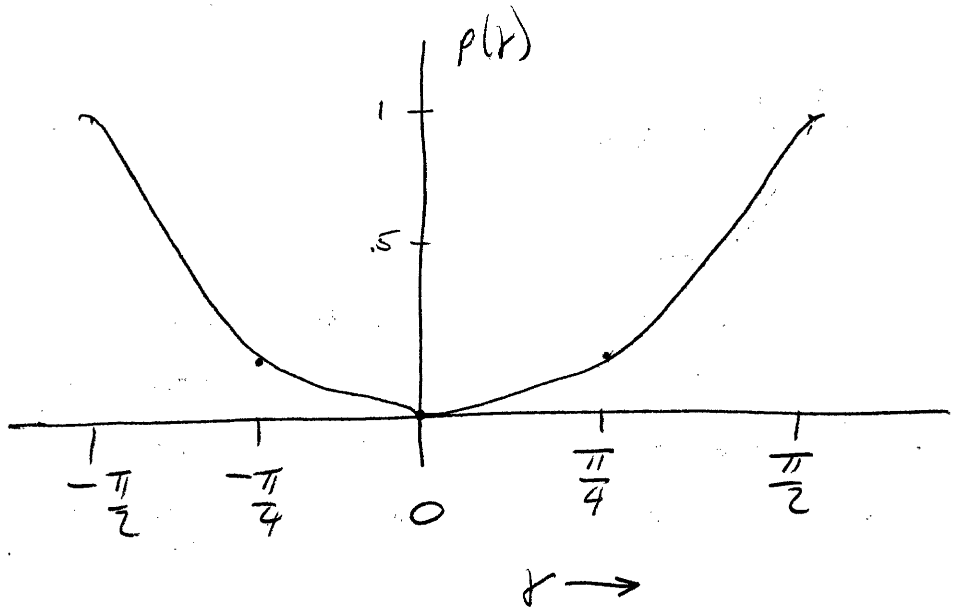
$$\rho(\gamma) \equiv \left| \frac{\sigma_{\hat{\varepsilon}_1}(\gamma) - \sigma_{\hat{\varepsilon}_2}(90^\circ)}{\sigma_{\hat{\varepsilon}_1}(\gamma) + \sigma_{\hat{\varepsilon}_2}(90^\circ)} \right|$$

FROM EQS (1) + (2), ONE HAS

$$\rho(\gamma) = \left| \frac{\cos^2\gamma - 1}{\cos^2\gamma + 1} \right|$$

$$= \left| \frac{-\sin^2\gamma}{1+\cos^2\gamma} \right|$$

$$= \frac{\sin^2\gamma}{1+\cos^2\gamma}$$



HENCE, THERE IS MAXIMUM POLARIZATION AT $\gamma = \pm \frac{\pi}{2}$, I.E., IN A DIRECTION PERPENDICULAR TO THE DIRECTION OF THE INCIDENT RADIATION.

THIS EFFECT CAN BE OBSERVED IN THE POLARIZATION OF SCATTERED LIGHT FROM THE SKY; PERPENDICULAR TO THE INCIDENT LIGHT, THE SCATTERED LIGHT IS VERTICALLY POLARIZED W.R.T. THE GROUND.

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PROB. 2,6

THE SCATTERING OF ELECTROMAGNETIC RADIATION AT WAVELENGTHS LARGE COMPARED TO THE SCATTER-ER'S SIZE HAS $\left(\frac{1}{\lambda}\right)^4$ DEPENDENCE. HENCE, IN THE VISIBLE SPECTRUM OF WAVELENGTHS, RED LIGHT ($\lambda \sim 6500 Å$) IS SCATTERED LEAST AND BLUE-VIOLET ($\lambda \sim 4000 Å$) IS SCATTERED MOST, RELATIVLY SPEAKING.

THUS, DURING THE DAY, MOST LIGHT REFLECTED AWAY FROM THE ~~[illegible]~~ INCIDENT DIRECTION IS TOWARD THE BLUE END (HIGH FREQUENCY, LOW WAVELENGTH) OF THE SPECTRUM. AT SUNRISE AND SUNSET, HOWEVER, THE TRANS-MITTED BEAM IS WEIGHTED TOWARD THE RED END OF THE SPECTRUM; ITS INTENSITY BECOMES SMALLER ALSO.

A

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PROB 3.1

$$G(x,x') = \sum_n a_n \varphi_n \,, \quad \varphi_n = \sqrt{\frac{2}{a}} \sin\sqrt{\lambda_n}\, x \,, \quad \lambda_n \equiv \left(\frac{n\pi}{a}\right)^2$$

$$G(x,x') = \sum_n a_n \sqrt{\frac{2}{a}} \sin(\sqrt{\lambda_n}\, x) = \frac{\sin(\sqrt{\lambda} x_<) \sin(\sqrt{\lambda}\{a - x_>\})}{\lambda \sin(\lambda a)}$$

$$= \begin{cases} \dfrac{\sin(\sqrt{\lambda} x) \sin(\sqrt{\lambda}\{a - x'\})}{\sqrt{\lambda} \sin(\sqrt{\lambda} a)} & 0 \le x \le x' \\[2ex] \dfrac{\sin(\sqrt{\lambda} x') \sin(\sqrt{\lambda}\{a - x\})}{\sqrt{\lambda} \sin(\sqrt{\lambda} a)} & x' \le x < a \end{cases}$$

THE APPROACH HERE WILL BE TO FIND THE FOURIER EXPANSION COEFFICIENTS IN THE LIMIT $\sqrt{\lambda} \to \lambda_n$.

$$\sum_n a_n \sqrt{\frac{2}{a}} \int_0^a \sin(\sqrt{\lambda_m}\, x) \sin(\sqrt{\lambda_n}\, x)\, dx = \sqrt{\frac{a}{2}}\, a_m$$

$$= \lim_{\sqrt{\lambda} \to \sqrt{\lambda_m}} \left\{ \frac{\sin(\sqrt{\lambda}\{a - x'\})}{\sqrt{\lambda} \sin(\sqrt{\lambda} a)} \int_0^{x'} \sin(\sqrt{\lambda}\, x) \sin(\sqrt{\lambda_n}\, x)\, dx \; + \right.$$

$$\left. + \frac{\sin(\sqrt{\lambda} x')}{\sqrt{\lambda} \sin\sqrt{\lambda} a} \int_{x'}^{a} \sin(\sqrt{\lambda}\{a - x\}) \sin(\sqrt{\lambda_n}\, x)\, dx \right\} \quad (1)$$

$$I_1 \equiv \int_0^{x'} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda_m}x)\,dx = \frac{\sin(\sqrt{\lambda}-\sqrt{\lambda_m})x'}{2(\sqrt{\lambda}-\sqrt{\lambda_m})} - \frac{\sin(\sqrt{\lambda}+\sqrt{\lambda_m})x'}{2(\sqrt{\lambda}+\sqrt{\lambda_m})}$$

$$\lim_{\sqrt{\lambda}\to\sqrt{\lambda_n}} I_1 = \frac{x'}{2} - \frac{\sin 2\sqrt{\lambda_n}\,x'}{4\sqrt{\lambda_n}} \qquad (2)$$

$$I_2 \equiv \int_{x'}^{a} \sin(\sqrt{\lambda}\{a-x\})\sin(\sqrt{\lambda_n}x)\,dx = \int_{x'}^{a} \{\sin(\sqrt{\lambda}a)\cos(\sqrt{\lambda}a) -$$

$$- \cos(\sqrt{\lambda}a)\sin(\sqrt{\lambda}x)\}\sin(\sqrt{\lambda_n}x)\,dx$$

$$= \sin(\sqrt{\lambda}a)\int_{x'}^{a} \cos(\sqrt{\lambda}x)\sin(\sqrt{\lambda_n}x)\,dx -$$

$$- \cos(\sqrt{\lambda}a)\int_{x'}^{a} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda_n}x)\,dx$$

$$= \sin(\sqrt{\lambda}a)\left(-\frac{1}{2}\right)\left(\frac{\cos(\sqrt{\lambda_n}-\sqrt{\lambda})a}{\sqrt{\lambda_n}-\sqrt{\lambda}} + \frac{\cos(\sqrt{\lambda_n}+\sqrt{\lambda})a}{\sqrt{\lambda_n}+\sqrt{\lambda}} -\right.$$

$$\left. - \frac{\cos(\sqrt{\lambda_n}-\sqrt{\lambda})x'}{\sqrt{\lambda_n}-\sqrt{\lambda}} - \frac{\cos(\sqrt{\lambda_n}+\sqrt{\lambda})x'}{\sqrt{\lambda_n}+\sqrt{\lambda}}\right) -$$

$$- \cos(\sqrt{\lambda}a)\left(\frac{1}{2}\right)\left(\frac{\sin(\sqrt{\lambda}-\sqrt{\lambda_n})a}{\sqrt{\lambda}-\sqrt{\lambda_n}} - \frac{\sin(\sqrt{\lambda}+\sqrt{\lambda_n})a}{\sqrt{\lambda}+\sqrt{\lambda_n}} -\right.$$

$$\left. - \frac{\sin(\sqrt{\lambda}-\sqrt{\lambda_n})x'}{\sqrt{\lambda}-\sqrt{\lambda_n}} + \frac{\sin(\sqrt{\lambda}+\sqrt{\lambda_n})x'}{\sqrt{\lambda}+\sqrt{\lambda_n}}\right)$$

$$\lim_{\sqrt{\lambda}\to\sqrt{\lambda_n}} I_2 = \sin(\sqrt{\lambda_n}a)\left(-\frac{1}{2}\right)\left(\cancel{\frac{1}{\sqrt{\lambda_n}-\sqrt{\lambda}}} + \cancel{\frac{\cos(2\sqrt{\lambda_n}a)}{2\sqrt{\lambda_n}}} - \right.$$

$$\left. - \cancel{\frac{1}{\sqrt{\lambda_n}-\sqrt{\lambda}}} - \cancel{\frac{\cos(2\sqrt{\lambda_n}a)}{2\sqrt{\lambda_n}}}\right) -$$

$$-\cos(\sqrt{\lambda_n}a)\left(\frac{1}{2}\right)\left(a - \frac{\sin 2\sqrt{\lambda_n}a}{2\sqrt{\lambda_n}} - \right.$$

$$\left. - x' + \frac{\sin(2\sqrt{\lambda_n}x')}{2\sqrt{\lambda_n}}\right)$$

$$\lim_{\sqrt{\lambda}\to\sqrt{\lambda_n}} I_2 = -\cos(\sqrt{\lambda_n}a)\left(\frac{a}{2} - \frac{x'}{2} + \frac{\sin(2\sqrt{\lambda_n}x')}{4\sqrt{\lambda_n}} - \frac{\sin(2\sqrt{\lambda_n}a)}{4\sqrt{\lambda_n}}\right) \quad (3)$$

THEREFORE SUBSTITUTING EQS. (2) + (3) INTO EQ. (1) YIELDS

$$\sqrt{\frac{a}{2}}\,a_m = \lim_{\sqrt{\lambda}\to\sqrt{\lambda_m}}\left\{\frac{\sin(\sqrt{\lambda}\{a-x'\})}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\left(\frac{x'}{2} - \frac{\sin 2\sqrt{\lambda_n}x'}{4\sqrt{\lambda_n}}\right) - \right.$$

$$- \frac{\sin(\sqrt{\lambda}x')}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\cos(\sqrt{\lambda_n}a)\left(\frac{a}{2} - \frac{\sin(2\sqrt{\lambda_n}a)}{4\sqrt{\lambda_n}} - \right.$$

$$\left.\left. - \frac{x'}{2} + \frac{\sin(2\sqrt{\lambda_n}x')}{4\sqrt{\lambda_n}}\right)\right\} \quad (4)$$

EXPANDING $\sin(\sqrt{\lambda}\{a-x'\}) = \sin(\sqrt{\lambda}a)\cos(\sqrt{\lambda}x') -$

$- \cos\sqrt{\lambda}a \sin\sqrt{\lambda}x'$,

TAKING THE LIMIT $\sqrt{\lambda} \to \sqrt{\lambda_m}$ AND NOTING THAT $\sin(\sqrt{\lambda_m}a) = \sin\left(\frac{m\pi}{a}a\right) = \sin m\pi = 0$ AND $\cos(\sqrt{\lambda_m}a) = \cos\left(\frac{m\pi}{a}a\right) = \cos(m\pi) = (-1)^m$ FOR ALL m, EQUATION (4) REDUCES TO

$$\sqrt{\frac{a}{2}}\,a_m = \lim_{\sqrt{\lambda}\to\sqrt{\lambda_m}} \left\{ \cancel{-\frac{(-1)^m \sin\sqrt{\lambda_m}x'}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\left(\frac{x'}{2} - \frac{\sin 2\sqrt{\lambda_m}x'}{4\sqrt{\lambda_m}}\right)} - \right.$$

$$- \frac{(-1)^m \sin\sqrt{\lambda_m}x'}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\left(\frac{a}{2} - \underbrace{\frac{\sin(2\sqrt{\lambda_m}a)}{4\sqrt{\lambda_m}}}_{=0}\right) +$$

$$\left. + \cancel{\frac{(-1)^m \sin(\sqrt{\lambda_m}x')}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\left(\frac{x'}{2} - \frac{\sin(2\sqrt{\lambda_m}x')}{4\sqrt{\lambda_m}}\right)} \right\}$$

$$= \lim_{\sqrt{\lambda}\to\sqrt{\lambda_m}} \left\{ -\frac{(-1)^m \sin\sqrt{\lambda_m}x'}{\sqrt{\lambda}\sin\sqrt{\lambda}a}\left(\frac{a}{2}\right) \right\}$$

TAKING THE FINAL LIMIT IN THE DENOMINATOR, ONE HAS USING THE HANDY PERSCRIPTION PROVIDED IN THE NOTES, VIZ,

$$\lim_{\sqrt{\lambda}\to\sqrt{\lambda_m}} \frac{1}{\sin\sqrt{\lambda}\,a} = \frac{2\sqrt{\lambda_m}}{(\sqrt{\lambda}-\sqrt{\lambda_m})\,a\cos(n\pi)}$$

ONE FINALLY HAS

$$\sqrt{\frac{a}{2}}\,a_m = -\left(\cancel{\frac{a}{2}}\right)\frac{\cancel{(-1)^m}\sin(\sqrt{\lambda_m}\,x')\cdot\cancel{2\sqrt{\lambda_m}}}{\left((\sqrt{\lambda})^2-(\sqrt{\lambda_m})^2\right)\cancel{a\cos(n\pi)}\;\cancel{\sqrt{\lambda_m}}}$$

$$= -\frac{\sin\sqrt{\lambda_m}\,x'}{\lambda-\lambda_m}$$

$$a_m = -\sqrt{\frac{2}{a}}\left(\frac{\sin\sqrt{\lambda_m}\,x'}{\lambda-\lambda_m}\right)$$

THUS,

$$G(x,x') = \sum a_n\sqrt{\frac{2}{a}}\sin(\sqrt{\lambda_n}\,x)$$

$$= -\sum \frac{\sqrt{\frac{2}{a}}\sin(\sqrt{\lambda_n}\,x')\sqrt{\frac{2}{a}}\sin(\sqrt{\lambda_n}\,x)}{\lambda-\lambda_n}$$

$$G(x,x') = -\sum_{n=1}^{\infty}\frac{\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x'\right)\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)}{\lambda-\frac{n^2\pi^2}{a^2}}$$

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EEAP 563

PROB 3.2

$$\frac{d^2\psi}{dx^2} + \lambda\psi = 0 \;, \quad 0 \le x \le \infty$$

$$\psi + 2\frac{d\psi}{dx} = 0 \;, \quad x = 0$$

$$\psi = 0 \;, \quad x = a$$

$$\psi = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

B.C. @ $x=0$:

$$A + 2(B\sqrt{\lambda}) = 0 \Rightarrow A = -2B\sqrt{\lambda} \quad (1)$$

B.C. @ $x=a$:

$$0 = A\cos\sqrt{\lambda}a + B\sin\sqrt{\lambda}a \quad (2)$$

$$\therefore \; -2\cancel{B}\sqrt{\lambda}\cos\sqrt{\lambda}a + \cancel{B}\sin\sqrt{\lambda}a = 0$$

$$2\sqrt{\lambda}\cos\sqrt{\lambda}a = \sin\sqrt{\lambda}a$$

$$\Rightarrow 2\sqrt{\lambda} = \tan\sqrt{\lambda}a$$

THIS DETERMINES THE EIGENVALUES $\lambda = \lambda_n$

THE EIGENFUNCTIONS ARE

$$\psi = -2B\sqrt{\lambda}\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$\psi = \sin\sqrt{\lambda}x - 2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

WHERE B WAS ARBITRARILY SET EQUAL TO 1 SINCE THIS IS AN UNNORMALIZED SOLUTION.

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EE/AP 563

PROB 3.3

$$\frac{\partial V}{\partial z} = -j\omega L I, \quad \frac{\partial I}{\partial z} = -j\omega C V + I_g\,\delta(z-z')$$

B

$$\therefore \quad \frac{\partial^2 V}{\partial z^2} = -j\omega L \frac{\partial I}{\partial z}$$

$$= -j\omega L\left(-j\omega C V + I_g\,\delta(z-z')\right)$$

$$= -\omega^2 LCV - j\omega L I_g\,\delta(z-z')$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 V}{\partial z^2} + \omega^2 LCV = -j\omega L I_g\,\delta(z-z')}$$

THE BOUNDARY CONDITIONS ON THIS PROBLEM ARE

$V = 0 \;@\; z = 0$

$\frac{\partial V}{\partial z}\left(\frac{X}{\omega L}\right) + V = 0 \;@\; z = a$

THE HOMOGENEOUS PROBLEM IS

$$\frac{\partial^2 V}{\partial z^2} + \omega^2 LCV = 0 \;\Rightarrow\; \frac{\partial^2 V}{\partial z^2} + \lambda^2 V = 0$$

$V = 0 \;, \quad z = 0$

$\frac{\partial V}{\partial z}\left(\frac{X}{\omega L}\right) + V = 0 \;, \quad z = a$

$\therefore \quad V = A\cos(\lambda z) + B\sin(\lambda z) \qquad \lambda = \omega\sqrt{LC}$

$V=0 \;@ z=0 \Rightarrow A=0$

$\frac{\partial V}{\partial z}\left(\frac{X}{\omega L}\right) + V = 0 \;@ z=a \Rightarrow \left(\frac{X}{\omega L}\right)\lambda B\cos(\lambda a) +$

$+ B\sin(\lambda a) = 0$

OR $\left(\frac{X}{\omega L}\right)\lambda\cos(\lambda a) + \sin(\lambda a) = 0$

$\left(\frac{X}{\omega L}\right)\lambda = -\tan(\lambda a)$

$$\boxed{\left(\frac{X}{\omega L}\right)\lambda_n = -\tan(\lambda_n a)}, \quad \lambda = \lambda_n$$

(ONE SHOULD ALSO CHECK TO SEE IF $\lambda=0$ IS A POSSIBLE SOLUTION $\Rightarrow V = Az+B$; $@z=0, V=0 \Rightarrow B=0$ + $@z=a$ $A\left(\frac{X}{\omega L}\right) + Aa = 0 \Rightarrow a = -\frac{X}{\omega L}$ $\therefore$ $\lambda=0$ IS NOT A POSSIBLE EIGENVALUE)

THUS, THE EIGENVALUES ARE DETERMINED BY THIS RELATIONSHIP. THE EIGENFUNCTIONS ARE

$\Psi_n = B\sin(\lambda_n z)$

NORMALIZING GIVES

$$1 = B^2\int_0^a \sin^2(\lambda_n z)\,dz = \left(\frac{1}{\lambda_n}\right)B^2\int_0^{\lambda_n a}\sin^2(z')\,dz'$$

$$= \frac{B^2}{\lambda_n}\left[\frac{\lambda_n a}{2} - \frac{\sin 2\lambda_n a}{4}\right]$$

$$\therefore B_n = \left(\frac{\lambda_n}{\frac{\lambda_n a}{2} - \frac{\sin(2\lambda_n a)}{4}}\right)^{1/2}$$

NOW, USING METHOD I TO OBTAIN A SOLUTION FOR THE PROBLEM, ONE WRITES

$$V = \sum_{n=1}^{\infty} a_n B_n \sin(\lambda_n z)$$

PUTTING THIS INTO

$$\frac{\partial^2 V}{\partial z^2} + \lambda^2 V = -j\omega L I_g \, \delta(z-z')$$

GIVES

$$-\sum_{n=1}^{\infty} a_n B_n \lambda_n^2 \sin(\lambda_n z) + \sum_{n=1}^{\infty} a_n B_n \lambda^2 \sin(\lambda_n z) = -j\omega L I_g \, \delta(z-z')$$

$$\sum_{n=1}^{\infty} a_n B_n \left(\lambda^2 - \lambda_n^2\right) \sin(\lambda_n z) = -j\omega L I_g \, \delta(z-z')$$

MULTIPLYING THROUGH BY $B_m \sin \lambda_m z$ AND INTEGRATING FROM 0 TO a YIELDS

$$\sum_{n=1}^{\infty} a_n B_n B_m \left(\lambda^2 - \lambda_n^2\right) \int_0^a \sin(\lambda_m z) \sin(\lambda_n z)\, dz = -j\omega L I_g B_m \int_0^a \sin(\lambda_m z)\, \delta(z-z')\, dz$$

$$a_m\left(\lambda^2 - \lambda_m^2\right) = -j\omega L I_g B_m \sin \lambda_m z'$$

$$\therefore \; V = -j\omega L I_g \sum_{n=1}^{\infty} \frac{B_n^2 \sin(\lambda_n z) \sin(\lambda_n z')}{\lambda^2 - \lambda_n^2}$$

THUS, SUMMING UP, THE SOLUTION VIA METHOD I IS

$$V = -j\omega L I_g \sum_{n=1}^{\infty} \frac{B_n^2 \sin(\lambda_n z)\sin(\lambda_n z')}{\lambda^2 - \lambda_n^2}$$

WHERE THE EIGENVALUES ARE DETERMINED BY

$$\left(\frac{X}{\omega L}\right)\lambda_n = -\tan(\lambda_n a)$$

AND THE NORMALIZATION COEFFICIENTS ARE GIVEN BY

$$B_n^2 = \left(\frac{\lambda_n}{\frac{\lambda_n a}{2} - \frac{\sin(2\lambda_n a)}{4}}\right) \quad \checkmark$$

EMPLOYING THE SECOND METHOD TO FIND THE SOLUTION, ONE HAS FOR A SOLUTION SATISFYING THE B.C. @ $z=0$

$$\phi_1 = \sin(\lambda z), \quad z \le z'$$

AND FOR THE SOLUTION SATISFYING THE B.C. @ $z=a$,

$$\phi_2 = \sin(\lambda_n z), \quad z' \le z$$ This doesn't make much sense.

WHERE THE λ_n IS DETERMINED FROM the

$$\left(\frac{X}{\omega L}\right)\lambda_n = -\tan(\lambda_n a)$$

[THE B.C. AT $z=a$, viz

$$\frac{X}{\omega L}\frac{\partial V}{\partial z} + V = 0, \quad z=a$$

USING THE FORM FOR ϕ_2 GIVES THIS RELATION.]

$\Phi_2 = A\cos\lambda z + B\sin\lambda z$

$\lambda = \omega\sqrt{LC}$, Can use $A=1$ for Φ_2

$A\cos\lambda a + B\sin\lambda a + \frac{X}{\omega L}\lambda\left[-A\sin\lambda a + B\cos\lambda a\right] = 0$

$B\left[\sin\lambda a + \frac{X\lambda}{\omega L}\cos\lambda a\right] + A\left[\cos\lambda a - \frac{X\lambda}{\omega L}\sin\lambda a\right] = 0$

Solve for B in terms of A

$$W(z) = \phi_1(z)\phi_2'(z) - \phi_1'(z)\phi_2(z)$$

$$= \sin(\lambda z)\lambda_n\cos(\lambda_n z) - \lambda\cos(\lambda z)\sin(\lambda_n z)$$

$$\therefore \quad G = -\frac{\sin(\lambda z_<)\sin(\lambda_n z_>)}{W(z')}$$

$$= \frac{\sin(\lambda z_<)\sin(\lambda_n z_>)}{\lambda\cos(\lambda z)\sin(\lambda_n z) - \lambda_n\sin(\lambda z)\cos(\lambda_n z)}$$

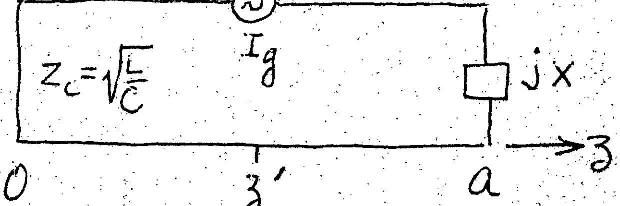
SINCE $V = j\omega L I_g G$ FOR THIS PROBLEM, ONE FINALLY HAS FOR THE SOLUTION VIA METHOD II

$$\boxed{V = \frac{j\omega L I_g \sin(\lambda z_<)\sin(\lambda_n z_>)}{\lambda\cos(\lambda z)\sin(\lambda_n z) - \lambda_n\sin(\lambda z)\cos(\lambda_n z)}}$$

WHERE λ_n IS DETERMINED FROM

$$\left(\frac{X}{\omega L}\right)\lambda_n = -\tan(\lambda_n a)$$

Solution For Prob. 3.3



$$\frac{d^2V}{dz^2} + k_0^2 V = -j\omega L I_g \, \delta(z-z')$$

$$k_0^2 = \omega^2 LC$$

$$V(0)=0, \quad V(a) + \frac{X}{\omega L}\frac{dV}{dz}\bigg|_a = 0$$

Method 1

Let $V = \Phi_1(z) = \sin k_0 z, \quad z < z'$

$\quad = \Phi_2(z) = \sin k_0(a-z) + A\cos k_0(a-z), \quad z' < z < a$

At $z=a$ we require $A + \frac{X}{\omega L}(-k_0) = 0$

so $A = k_0 X/\omega L = X\sqrt{C/L} = X/Z_c$

By a standard method it is then found that

$$V = j\omega L I_g \frac{\Phi_1(z_<)\,\Phi_2(z_>)}{W} = jZ_c I_g \frac{\sin k_0 z_< \left[XY_c \cos k_0(a-z_>) + \sin k_0(a-z_>)\right]}{\sin k_0 a + XY_c \cos k_0 a}$$

Method 2

Find eigenfunctions for $\frac{d^2\psi}{dz^2} + \lambda\psi = 0$,

$$\psi(0)=0, \quad \psi(a) + \frac{X}{\omega L}\frac{d\psi}{dz}\bigg|_a = 0$$

$\psi = \sin\sqrt{\lambda}\, z$ to satisfy conditions at $z=0$.

$\sin\sqrt{\lambda}\, a + \frac{X}{\omega L}\sqrt{\lambda}\cos\sqrt{\lambda}\, a = 0$, condition at $z=a$

Hence

$$\frac{\tan\sqrt{\lambda}\, a}{\sqrt{\lambda}} = -\frac{X}{\omega L}.$$ The λ are

functions of ω. The roots λ_n form a discrete set. The functions ψ_n are easily shown to be orthogonal. The ψ_n satisfy the same boundary conditions as V does for a given value of ω. The resonant frequencies for the transmission line resonator are found from $k_n = \omega_n \sqrt{LC} = \sqrt{\lambda_n}$ with

$$\frac{\tan \sqrt{\lambda_n}\, a}{\sqrt{\lambda_n}\, a} = -\frac{X}{\omega_n L a}$$ since λ_n is a function of ω.

The normalization is given by

$$\int_0^a \sin^2 \sqrt{\lambda_n}\, z \; dz = \frac{1}{2}\left[a - \frac{\sin 2\sqrt{\lambda_n}\, a}{2\sqrt{\lambda_n}}\right]$$

Let $V = \sum_n a_n \psi_n$. By standard procedures

$$V = +\sum_n \frac{j\omega L I_g \sin \sqrt{\lambda_n}\, z_< \sin \sqrt{\lambda_n}\, z_>}{(\lambda_n - \omega^2 LC)\, \frac{1}{2}\left(a - \frac{\sin 2\sqrt{\lambda_n}\, a}{2\sqrt{\lambda_n}}\right)}$$

<u>Method 3</u>

This solution was obtained by Mr. C-C Hew. If $\omega^2 LC$ is called λ, then $\omega = \sqrt{\lambda / LC}$. The boundary condition is then

$$\frac{\tan \sqrt{\lambda}\, a}{\sqrt{\lambda}} = -\frac{X}{\omega L} = -\frac{X}{\sqrt{\lambda}\, Z_c}$$ or $\tan \sqrt{\lambda}\, a = -\frac{X}{Z_c}$

The eigenvalue is now part of the boundary condition so the corresponding eigenfunctions are no longer orthogonal. To overcome this difficulty it is necessary to extend the definition of the operator and its domain so as to incorporate the boundary conditions. Let $\vec{U}$ denote the 2 element column matrix

$$\vec{U} = \begin{bmatrix} \psi(z) \\ \Phi(z) \end{bmatrix}$$

and define $\mathcal{L}\vec{U}$ to be $\mathcal{L}\vec{U} = \begin{bmatrix} \Phi' \\ -\psi' \end{bmatrix}$

where $\Phi' \equiv d\Phi/dz$, etc. The system

$$\frac{d^2\psi}{dz^2} + k_0^2\psi = 0, \quad \psi(0) = 0, \quad \psi'(a) = -\frac{k_0 Z_c}{X}\psi(a)$$

is equivalent to $\mathcal{L}\vec{U} = k_0\vec{U}$ since

$$\mathcal{L}\vec{U} = \begin{bmatrix} \Phi' \\ -\psi' \end{bmatrix} = k_0\begin{bmatrix} \psi \\ \Phi \end{bmatrix} \quad \text{or} \quad \frac{d\Phi}{dz} = k_0\psi, \quad \frac{d\psi}{dz} = -k_0\Phi$$

so $\dfrac{d^2\psi}{dz^2} = -k_0\dfrac{d\Phi}{dz} = -k_0^2\psi$. The boundary condition

at $z=a$ is $\psi'(a) = -\dfrac{k_0 Z_c}{X}\psi(a) = -k_0\Phi(a)$

or $\psi(a) = \dfrac{X}{Z_c}\Phi(a)$. At $z=0$ the boundary condition is $\psi(0) = 0 = d\Phi/dz\big|_0$.

Consider 2 solutions $\vec{U}_n$, $\vec{U}_m$ with eigenvalues k_n, k_m. The scalar product is

$$\int_0^a (\psi_n\psi_m + \Phi_n\Phi_m)\,dz$$

With $\psi_n = \sin k_n z$, $\quad \Phi_n = -\frac{1}{k_n}\frac{d\psi_n}{dz} = -\cos k_n z$, etc

we get $\displaystyle\int_0^a (\sin k_n z \sin k_m z + \cos k_n z \cos k_m z)\,dz$

$$= \int_0^a \cos(k_n - k_m)z\,dz = \frac{\sin(k_n-k_m)a}{(k_n-k_m)}$$

$$= \frac{\sin k_n a \cos k_m a - \sin k_m a \cos k_n a}{k_n - k_m} = 0 \text{ upon}$$

using $\sin k_n a = -\frac{X}{Z_c}\cos k_n a$ etc.

Hence $\vec{U}_n$, $\vec{U}_m$ are orthogonal.

For $n=m$, $\displaystyle\int_0^a (\psi_n^2 + \Phi_n^2)\,dz = \int_0^a dz = a$

which is normalization constant. It is easy to show that

$$\int_0^a \vec{V}\,\mathcal{L}\,\vec{U}\,dz = \int_0^a \vec{U}\,\mathcal{L}\,\vec{V}\,dz$$

when $\vec{V}$ satisfies the same boundary conditions as $\vec{U}$. Hence $\mathcal{L}$ is a self-adjoint operator.

$\dfrac{d^2V}{dz^2} + k_0^2 V = -j\omega L I_g \delta(z-z')$ is equivalent to

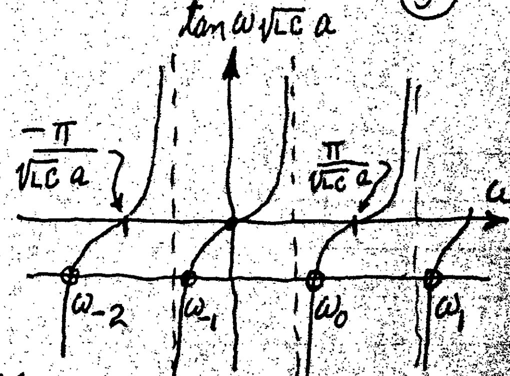
$$\mathcal{L}\begin{bmatrix} V \\ \Phi \end{bmatrix} - k_0\begin{bmatrix} V \\ \Phi \end{bmatrix} = \begin{bmatrix} jZ_c I_g \delta(z-z') \\ 0 \end{bmatrix}$$

Let $$\begin{bmatrix} V \\ \Phi \end{bmatrix} = \sum_n C_n \begin{bmatrix} \sin k_n z \\ -\cos k_n z \end{bmatrix}$$

By using $\mathcal{L}\begin{bmatrix} \sin k_n z \\ -\cos k_n z \end{bmatrix} = k_n \begin{bmatrix} \sin k_n z \\ -\cos k_n z \end{bmatrix}$, the

orthogonality, and normalization, it is readily found that

$$V(z) = -\sum_{n=-\infty}^{\infty} \frac{Z_c\, j\, I_g \sin k_n z' \sin k_n z}{(\omega-\omega_n)\sqrt{LC}\; a},$$



Equivalence Between 3 Solutions

We can show that all 3 solutions are the same. Consider

$$\frac{1}{2\pi j}\oint_C \frac{j Z_c I_g\, \omega\sqrt{LC}\, \sin w z_< \left[\frac{wX}{\omega L}\cos w(a-z_>) + \sin w(a-z_>)\right]}{(w^2-\omega^2 LC)\left(\sin wa + \frac{wX}{\omega L}\cos wa\right)}\, dw$$

$= 0$. C is circle with infinite radius. Integrand is asymptotic to

$$\frac{e^{|w|(z_< + a - z_> - a)}}{w} = \frac{e^{-|w|(z_> - z_<)}}{w}$$

so integral around circle at infinity is zero. The poles are at $w = \pm\omega\sqrt{LC} = \pm k_0$ and

$$\frac{\tan wa}{w} = -\frac{X}{\omega L}, \text{ i.e. at } w = \pm\sqrt{\lambda_n} = \pm w_n$$

Hence $\sum$ Residues at $\pm k_0$ $+ \sum$ Residues at $\pm\sqrt{\lambda_n}$ $= 0$. The residues at $\pm k_0$ are the same and give the solution described under method 1. The residues at $\pm\sqrt{\lambda_n}$ give the negative of the solution described in Method 2. To show this requires several algebraic steps which [illegible]

The contribution to the residue at $w = w_n = \sqrt{\lambda_n}$ from the denominator pole factor is

$$\frac{d}{dw}\left[\sin wa + \frac{wX}{\omega L}\cos wa\right]\Bigg|_{w_n} = a\left[\cos w_n a - \frac{w_n X}{\omega L}\sin w_n a\right.$$

$$\left. + \frac{X}{\omega L a}\cos w_n a\right] = a\,\frac{\left[\cos^2 w_n a + \sin^2 w_n a - \dfrac{\cos w_n a \sin w_n a}{w_n a}\right]}{\cos w_n a}$$

$$= a\left[1 - \frac{\sin 2w_n a}{2w_n a}\right]\frac{1}{\cos w_n a}$$ upon using the

eigenvalue equation $\frac{X}{\omega L} = -\frac{\tan w_n a}{w_n}$. This factor is proportional to the normalization constant for the ψ_n used in method 2. The numerator factor

$\frac{w_n X}{\omega L}\cos w_n(a - z_>) + \sin w_n(a - z_>)$ upon expansion and use of the eigenvalue equation becomes $-\sin w_n z_> / \cos w_n a$. With these the residue contributions at $w = w_n$ become the negative of the solution given in method 2. This verifies that the method 1 and 2 solutions are equivalent.

In order to show that the method 1 and 3 solutions are equivalent the integrand in the contour integral is chosen to be

$$\frac{jZ_c I_g \sin w z_< \left[\dfrac{X}{Z_c}\cos w(a - z_>) + \sin w(a - z_>)\right]}{(w - \omega\sqrt{\epsilon_c})\left[\sin wa + \dfrac{X}{Z_c}\cos wa\right]}$$

The contour integral gives a residue contribution at $w = \omega\sqrt{LC}$ which gives the method 1 solution. The sum of the residues at $w = w_n$ where $\tan w_n a = -\frac{X}{Z_c}$ gives the negative of the method 3 solution and hence establishes the equivalence between these two solutions. Thus since solution 1 is equivalent to solutions 2 and 3 the latter two are also equivalent even though they involve different eigenvalues and eigenfunctions.

The method 3 solution is described in

Friedman, B., Principles and Techniques of Applied Mathematics, John Wiley & Sons, 1956.

BOB MANNING
EEAP 563

PROB. 3.4

$$G_x = \frac{\sin\sqrt{\lambda_x}\, x_< \sin\sqrt{\lambda_x}\,(a - x_>)}{\sqrt{\lambda_x}\sin\sqrt{\lambda_x}\, a}$$

$$G_y(y, y'; \lambda_y) = -\sum_{m=1}^{\infty} \frac{\frac{2}{b}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{\lambda_y - (m\pi/b)^2}$$

$$G(x, x', y, y') = -\frac{1}{2\pi j}\oint_{C_x} G_x(x, x'; \lambda_x)\, G_y(y, y'; -\lambda_x)\, d\lambda_x$$

$$= +\left(\frac{1}{2\pi j}\right)\oint_{C_x} \frac{\sin(\sqrt{\lambda_x}\, x_<)\sin(\sqrt{\lambda_x}(a - x_>))}{\sqrt{\lambda_x}\sin\sqrt{\lambda_x}\, a} \sum_{m=1}^{\infty} \frac{\frac{2}{b}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{-\lambda_x - (m\pi/b)^2}\, d\lambda_x$$

THIS HAS POLES ALONG THE λ_x AXIS LOCATED AT $\left(\frac{n\pi}{a}\right)^2 = \lambda_x$. AS SHOWN IN THE NOTES, THE RESIDUE FORM OF THE TERM $\frac{1}{\sqrt{\lambda_x}\sin\sqrt{\lambda_x}\, a}$ IS $\frac{2}{a}\left(\frac{1}{\cos n\pi}\right) = (-1)^n\left(\frac{2}{a}\right)$. THUS, PERFORMING THE CONTOUR INTEGRATION INDICATED ABOVE YIELDS

$$G = -\frac{2\pi j}{2\pi j}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n\left(\frac{2}{a}\right)\frac{\sin\left(\frac{n\pi}{a}x_<\right)\sin\left(\frac{n\pi}{a}(a - x_>)\right)\frac{2}{b}\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{m\pi y'}{b}\right)}{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

WRITING $\sin\left(\frac{n\pi}{a}(a - x_>)\right) = -\cos n\pi \sin\frac{n\pi x_>}{a} = -(-1)^n \sin\left(\frac{n\pi x_>}{a}\right)$

AND LETTING $x_< = x$ AND $x_> = x'$, ONE HAS

$$G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\left(\frac{2}{a}\right) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi x'}{a}\right) \left(\frac{2}{b}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right)}{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$$G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left(\frac{4}{ab}\right) \frac{\sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{m\pi y'}{b}\right)}{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

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EEAP 563

PROB. 3.5

$$\frac{d^2V}{dz^2} + k_0^2 V = -j\omega L I_g \delta(z - z') \qquad -\infty < z < \infty \qquad (1)$$

$k_0 = k_0' - jk_0''$

LET $V = -j\omega L I_g G$; THEN ONE HAS

$$\frac{d^2G}{dz^2} + k_0^2 G = -\delta(z - z') \qquad (2)$$

NOW DEFINE THE FOURIER TRANSFORM OF $G(z)$ TO BE $\hat{G}(\beta)$ GIVEN BY THE DIRECT TRANSFORM

$$\hat{G}(\beta) = \int_{-\infty}^{\infty} G(z)\, e^{i\beta z}\, dz \qquad (3)$$

THEN BY THE INVERSE TRANSFORM, ONE HAS

$$G(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(\beta)\, e^{-i\beta z}\, d\beta \qquad (4)$$

TAKING THE FOURIER TRANSFORM OF EQ. (2) YIELDS

$$\int_{-\infty}^{\infty}\left(\frac{d^2G}{dz^2}\right) e^{i\beta z}\, dz + k_0^2 \int_{-\infty}^{\infty} G\, e^{i\beta z}\, dz = -\int_{-\infty}^{\infty} \delta(z - z')\, e^{i\beta z}\, dz \qquad (5)$$

THE 1ST TERM INTEGRATES BY PARTS TO YIELD

$$\int_{-\infty}^{\infty}\underbrace{\left(\frac{d^2G}{dz^2}\right)}_{dv}\underbrace{e^{i\beta z}}_{u}dz = e^{i\beta z}\frac{dG}{dz}\Big|_{-\infty}^{\infty} - i\beta\int_{-\infty}^{\infty}\underbrace{\frac{dG}{dz}}_{dv}\underbrace{e^{i\beta z}}_{u}dz$$

$$= e^{i\beta z}\frac{dG}{dz}\Big|_{-\infty}^{\infty} - i\beta\left\{e^{i\beta z}G\Big|_{-\infty}^{\infty} - i\beta\int_{-\infty}^{\infty}e^{i\beta z}G\,dz\right\}$$

$$= e^{i\beta z}\frac{dG}{dz}\Big|_{-\infty}^{\infty} - i\beta e^{i\beta z}G\Big|_{-\infty}^{\infty} - \beta^2\int_{-\infty}^{\infty}G\,e^{i\beta z}dz$$

BASED ON CONSIDERATIONS OF BOUNDEDNESS OF THE FUNCTIONS $e^{i\beta z}$, $\frac{dG}{dz}$, AND G AT $\pm\infty$, ONE HAS

$$\int_{-\infty}^{\infty}\left(\frac{d^2G}{dz^2}\right)e^{i\beta z}dz = -\beta^2\int_{-\infty}^{\infty}G\,e^{i\beta z}dz$$

AND EQ. (5) BECOMES

$$-\beta^2\int_{-\infty}^{\infty}G\,e^{i\beta z}dz + k_0^2\int_{-\infty}^{\infty}G\,e^{i\beta z}dz = -e^{i\beta z'}$$

$$\Rightarrow\quad -\beta^2\hat{G}(\beta) + k_0^2\hat{G}(\beta) = -e^{i\beta z'}$$

$$\therefore\quad \hat{G}(\beta) = -\frac{e^{i\beta z'}}{k_0^2-\beta^2}$$

$$= \frac{e^{i\beta z'}}{\beta^2-k_0^2}$$

$$= \frac{e^{i\beta z'}}{(\beta+k_0)(\beta-k_0)}$$

FROM EQ. (4) ONE THEN HAS

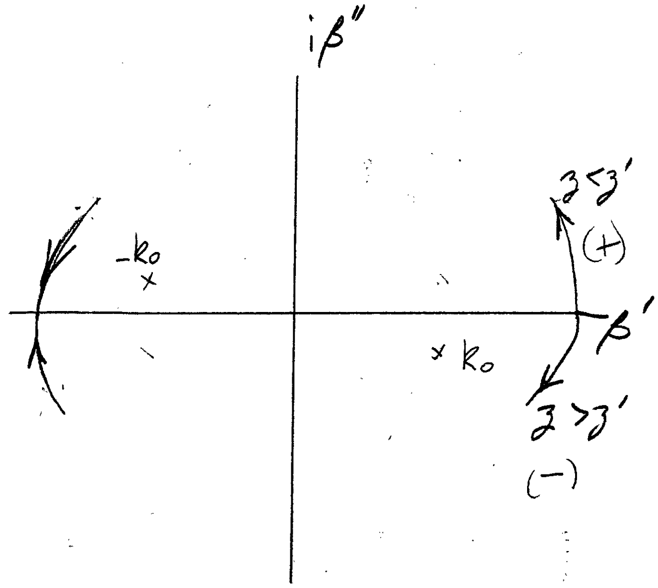
$$G(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\beta(z-z')}}{(\beta+k_0)(\beta-k_0)}\, d\beta \quad , \quad \beta = \beta' + i\beta''$$

FOR THIS PARTICULAR PROBLEM, ONE HAS THAT

$$k_0 = k_0' - i\, k_0''$$

WHERE

$$k_0' \equiv \omega^2 LC, \quad k_0'' \equiv i G\omega L$$



THE NUMERATOR OF THE INTEGRAND IS

$$e^{-i\beta(z-z')} = e^{-i(\beta'+i\beta'')(z-z')}$$
$$= e^{-i\beta'(z-z') + \beta''(z-z')}$$

THUS FOR $z>z'$ ONE MUST INTEGRATE IN THE LHP. FOR $z<z'$ ONE IS IN THE UHP.

HENCE, FOR $z>z'$

$$\int_{-\infty}^{\infty} \frac{e^{-i\beta(z-z')}}{(\beta+k_0)(\beta-k_0)}\, d\beta = \oint_{C} \frac{e^{-i\beta(z-z')}}{(\beta+k_0)(\beta-k_0)}\, d\beta$$

$$= -2\pi i\, \frac{e^{-ik_0(z-z')}}{2k_0}$$

$$= -i\pi \frac{e^{-ik_0(z-z')}}{k_0}$$

FOR $z < z'$

$$\int_{-\infty}^{\infty} \frac{e^{i\beta(z-z')}}{(\beta+k_0)(\beta-k_0)}\, d\beta = \int_{C_I} \frac{e^{-i\beta(z-z')}}{(\beta+k_0)(\beta-k_0)}\, d\beta$$

$$= +2\pi i \, \frac{e^{+ik_0(z-z')}}{-2k_0}$$

$$= -i\pi \, \frac{e^{ik_0(z-z')}}{k_0}$$

THUS

$$G(z) = \begin{cases} -\frac{i}{2k_0} e^{-ik_0(z-z')} , & z > z' \\ -\frac{i}{2k_0} e^{ik_0(z-z')} , & z < z' \end{cases}$$

$$\Rightarrow \; G(z) = -\frac{i}{2k_0} e^{-ik_0|z-z'|} , \quad -\infty < z < \infty$$

AND

$$v(z) = -i\omega L I_g \, G(z)$$

$$\boxed{V(z) = -\frac{\omega L I_g}{2k_0} e^{-ik_0|z-z'|} , \quad -\infty < z < \infty}$$

PROB. 3.6

BOB MANNING
EEAP 563

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x-x')\delta(y-y'), \quad G=0, \ 0<x<a, \quad 0<y<b$$

THE ASSOCIATED HOMOGENEOUS PROBLEM IS

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad 0 \le x \le a, \ 0 \le y \le b$$

$$\psi = X(x)\,Y(y)$$

$$\Rightarrow Y(y)\frac{\partial^2 X}{\partial x^2} + X(x)\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\underbrace{\frac{1}{X}\left(\frac{\partial^2 X}{\partial x^2}\right)}_{\equiv -\lambda_x^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{\equiv -\lambda_y^2} = 0, \quad X \neq 0, \ Y \neq 0$$

$$\therefore \frac{\partial^2 X}{\partial x^2} + \lambda_x^2 X = 0 \Rightarrow X = A\cos(\lambda_x x) + B\sin(\lambda_x x)$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda_y^2 Y = 0 \Rightarrow Y = C\cos(\lambda_y x) + D\sin(\lambda_y x)$$

B.C.s:

$$0 \le x \le a \Rightarrow A = 0, \quad \lambda_x a = \pi n, \ n = 1, 2, \cdots$$

$$0 \le y \le b \Rightarrow C = 0, \quad \lambda_y b = \pi m, \ m = 1, 2, \cdots$$

$$\therefore \psi = \psi_{nm} = \sin\left(\frac{\pi n}{a}x\right)\sin\left(\frac{\pi m}{b}y\right)$$

THEREFORE, LET

$$G = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin\left(\frac{\pi n}{a}x\right)\sin\left(\frac{\pi m}{b}y\right)$$

SUBSTITUTING THIS INTO THE ORIGINAL EQUATION GIVES

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\left[-\left(\frac{\pi n}{a}\right)^2-\left(\frac{\pi m}{b}\right)^2\right]\sin\left(\frac{\pi n}{a}x\right)\sin\left(\frac{\pi m}{b}y\right) =$$

$$= -\delta(x-x')\delta(y-y')$$

WHERE, IN ORDER TO INTRODUCE THE OPERATIONS OF DIFFERENTIATION UNDER THE SUMMATIONS, THE FOURIER EXPANSION IS ASSUMED TO CONVERGE.

$$\int_0^a\int_0^b \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\left[\left(\frac{\pi n}{a}\right)^2+\left(\frac{\pi m}{b}\right)^2\right]\sin\left(\frac{\pi n}{a}x\right)\sin\left(\frac{\pi m}{b}y\right)\sin\left(\frac{\pi k}{a}x\right)\sin\left(\frac{\pi l}{b}y\right)dy\,dx =$$

$$= \int_0^a\int_0^b \delta(x-x')\delta(y-y')\sin\left(\frac{\pi k}{a}x\right)\sin\left(\frac{\pi l}{b}y\right)dy\,dx$$

$$a_{kl}\left[\left(\frac{\pi k}{a}\right)^2+\left(\frac{\pi l}{b}\right)^2\right]\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \sin\left(\frac{\pi k}{a}x'\right)\sin\left(\frac{\pi l}{b}y'\right)$$

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$$a_{kl} = \left(\frac{4}{ab}\right)\frac{\sin\left(\frac{\pi k}{a}x'\right)\sin\left(\frac{\pi l}{b}y'\right)}{\left[\left(\frac{\pi k}{a}\right)^2+\left(\frac{\pi l}{b}\right)^2\right]}$$

$$G = \left(\frac{4}{ab}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{\pi n}{a}x\right)\sin\left(\frac{\pi n}{a}x'\right)\sin\left(\frac{\pi m}{b}y\right)\sin\left(\frac{\pi m}{b}y'\right)}{\left(\frac{\pi n}{a}\right)^2+\left(\frac{\pi m}{b}\right)^2}$$

BOB MANNING
EEAP 563

PROB. 3.7

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x-x')\delta(y-y') \quad , G=0 \quad 0\le x\le a, \; 0\le y\le b$$

$$G = \sum_{n=1}^{\infty} f_n(y)\sin\left(\frac{n\pi x}{a}\right)$$

$$\therefore \sum_{n=1}^{\infty}\left[-\left(\frac{n\pi}{a}\right)^2 f_n(y) + \frac{\partial^2 f_n}{\partial y^2}\right]\sin\left(\frac{n\pi x}{a}\right) = -\delta(x-x')\delta(y-y')$$

$$\int_0^a \sum_{n=1}^{\infty}\left[\frac{\partial^2 f_n}{\partial y^2} - \left(\frac{n\pi}{a}\right)^2 f_n\right]\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

$$= -\int_0^a \delta(x-x')\delta(y-y')\sin\left(\frac{m\pi x}{a}\right)dx$$

$$\Rightarrow \left(\frac{a}{2}\right)\left[\frac{\partial^2 f_n}{\partial y^2} - \left(\frac{n\pi}{a}\right)^2 f_n\right] = -\delta(y-y')\sin\left(\frac{n\pi x'}{a}\right)$$

OR, SINCE f_n IS ONLY A FUNCTION OF y,

$$\frac{d^2 f_n}{dy^2} - \left(\frac{n\pi}{a}\right)^2 f_n = -\left(\frac{2}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\delta(y-y')$$

SOLVING THIS EQUATION VIA THE "2ND METHOD", ONE HAS

$$\phi_1(y) \sim A e^{\left(\frac{n\pi}{a}\right)y} + B e^{-\left(\frac{n\pi}{a}\right)y} \quad , \quad \phi_1(0)=0 \Rightarrow A+B=0$$

$$\text{or } A=-B\equiv A$$

$$= C\left(e^{\left(\frac{n\pi}{a}\right)y} - e^{-\left(\frac{n\pi}{a}\right)y}\right) = 2C\sinh\left(\frac{n\pi}{a}y\right)$$

$$\therefore \quad f_n(y) = \left(\frac{2}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\left[\frac{\sinh\left(\frac{n\pi}{a}y_<\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi}{a}b\right)}\right]$$

$$\therefore G = \sum_{n=1}^{\infty}\left(\frac{2}{\cancel{a}}\right)\left(\frac{\cancel{a}}{n\pi}\right)\left(\frac{\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{\sinh\left(\frac{n\pi}{a}b\right)}\right)\sin\left(\frac{n\pi x}{a}\right)$$

$$G = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\right)\frac{\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x'}{a}\right)\sinh\left(\frac{n\pi y_<}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

BOB MANNING
EEAP 563

PROB. 3.8

$$\frac{\partial^2 G_x}{\partial x^2} + \lambda_x G_x = -\delta(x-x')$$

$$\frac{\partial^2 G_y}{\partial y^2} + \lambda_y G_y = -\delta(y-y')$$

CONSIDER THE 1ST EQUATION FOR G_x; SOLVING IT VIA THE SECOND METHOD (METHOD II), ONE HAS

$$\frac{\partial^2 G_x}{\partial x^2} + \lambda_x G_x = 0$$

$$\Rightarrow \quad G_x = A\sin(\sqrt{\lambda_x}\, x) + B\cos(\sqrt{\lambda_x}\, x)$$

$G = 0$ @ $x = 0 \Rightarrow B = 0$

$G = 0$ @ $x = a \Rightarrow \sin(\sqrt{\lambda_x}\, a) = 0$ or $\sqrt{\lambda_x}\, a = n\pi \quad \lambda_x = \left(\frac{n\pi}{a}\right)^2$

$\therefore \quad \phi_1(x) = \sin(\sqrt{\lambda_x}\, x) \qquad 0 \le x \le x'$

$\phi_2(x) = \sin\left(\sqrt{\lambda_x}(a-x)\right)$ WHICH IS LINEARLY INDEPENDENT OF THE 1ST

THE WRONSKIAN FOLLOWS FROM AN ANALOGOUS CALCULATION DONE IN PROB. 3.7;

$$W(x) = -\sqrt{\lambda_x}\, \sin(\sqrt{\lambda_x}\, a)$$

THUS,

$$G_x = -\frac{\phi_1(x_<)\,\phi_2(x_>)}{p(x')\,W(x')}$$

$$= \frac{\sin(\sqrt{\lambda_x}\,x_<)\sin(\sqrt{\lambda_x}\,(a-x_>))}{\sqrt{\lambda_x}\,\sin(\sqrt{\lambda_x}\,a)} \qquad 1)$$

SIMILARLY,

$$G_y = \frac{\sin(\sqrt{\lambda_y}\,y_<)\sin(\sqrt{\lambda_y}\,(b-y_>))}{\sqrt{\lambda_y}\,\sin(\sqrt{\lambda_y}\,b)} \qquad 2)$$

NOW, FOR THE TWO DIMENSIONAL GREEN FUNCTION, ONE HAS

$$G = -\frac{1}{2\pi i}\int_{C_x} G_x(\lambda_x)\,G_y(-\lambda_x)\,d\lambda_x \qquad 3)$$

USING THE λ_x AS THE INTEGRATION VARIABLE.

SUBSTITUTING EQS. (1) AND (2) INTO EQ (3) GIVES

$$G = -\left(\frac{1}{2\pi i}\right)\int_{C_x} \frac{\sin(\sqrt{\lambda_x}\,x_<)\sin(\sqrt{\lambda_x}\,(a-x_>))\sin(i\sqrt{\lambda_x}\,y_<)\sin(i\sqrt{\lambda_x}\,(b-y_>))}{\sqrt{\lambda_x}\,\sin(\sqrt{\lambda_x}\,a)\;i\sqrt{\lambda_x}\,\sin(i\sqrt{\lambda_x}\,b)}\,d\lambda_x$$

SINCE THIS INTEGRAL IS TO BE PERFORMED AROUND THE POLES OF THE +X AXIS, ONE HAS

$$G = -\left(\frac{1}{2\pi i}\right)\frac{\sin(i\sqrt{\lambda_x}\,y_<)\sin(i\sqrt{\lambda_x}\,(b-y_>))}{i\sqrt{\lambda_x}\,\sin(i\sqrt{\lambda_x}\,b)}\underbrace{\int_{C_x}\frac{\sin(\sqrt{\lambda_x}\,x_<)\sin(\sqrt{\lambda_x}\,(a-x_>))}{\sqrt{\lambda_x}\,\sin(\sqrt{\lambda_x}\,a)}\,d\lambda_x}_{\cancel{(2\pi i)}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{a}x_<\right)\sin\left(\frac{n\pi}{a}(a-x_>)\right)\frac{2}{a\cos(n\pi)}}$$

$$= i\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{a}x_<\right)\sin\left(\frac{n\pi}{a}(a-x_>)\right)\sin\left(i\left(\frac{n\pi}{a}\right)y_<\right)\sin\left(i\left(\frac{n\pi}{a}\right)(b-y_>)\right)}{a\left(\frac{n\pi}{a}\right)\sin\left(i\left(\frac{n\pi}{a}\right)b\right)\cos n\pi}$$

EXPANDING

$$\sin\left(\frac{n\pi}{a}(a-x_>)\right) = \cancelto{0}{\sin(n\pi)}\cos\left(\frac{n\pi}{a}x_>\right) - \cos(n\pi)\sin\left(\frac{n\pi x_>}{a}\right)$$

AND USING

$$\sin\left(i\left(\frac{n\pi}{a}\right)b\right) = i\sinh\left(\frac{n\pi}{a}b\right)$$

GIVES

$$G = \cancel{-i}\sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{a}x_<\right)\cancel{\cos(n\pi)}\sin\left(\frac{n\pi x_>}{a}\right)\cancel{(i)}\sinh\left(\frac{n\pi}{a}y_<\right)\cancel{(i)}\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{n\pi\cancel{(i)}\sinh\left(\frac{n\pi}{a}b\right)\cancel{\cos(n\pi)}}$$

$$G = \sum_{n=1}^{\infty}\frac{2\sin\left(\frac{n\pi}{a}x_<\right)\sin\left(\frac{n\pi x_>}{a}\right)\sinh\left(\frac{n\pi}{a}y_<\right)\sinh\left(\frac{n\pi}{a}(b-y_>)\right)}{(n\pi)\sinh\left(\frac{n\pi}{a}b\right)}$$

THIS IS THE SAME RELATION AS DERIVED IN PROBLEM 3.7. ALSO, VIA THE ANALOGY PROVIDED BY PROBLEM 3.1, IT IS THE FOURIER TRANSFORM OF THE RESULT DERIVED IN PROB. 3.6.

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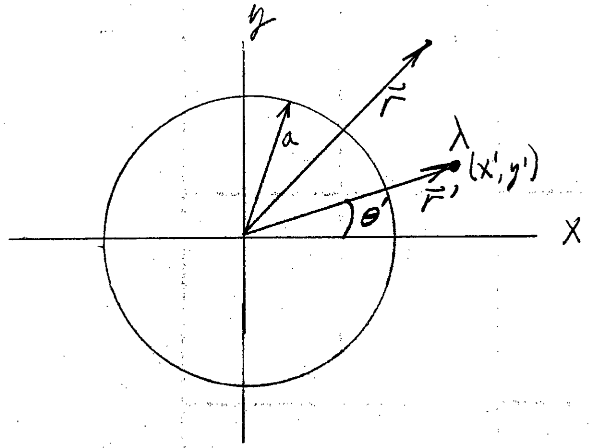
PROB 3.9

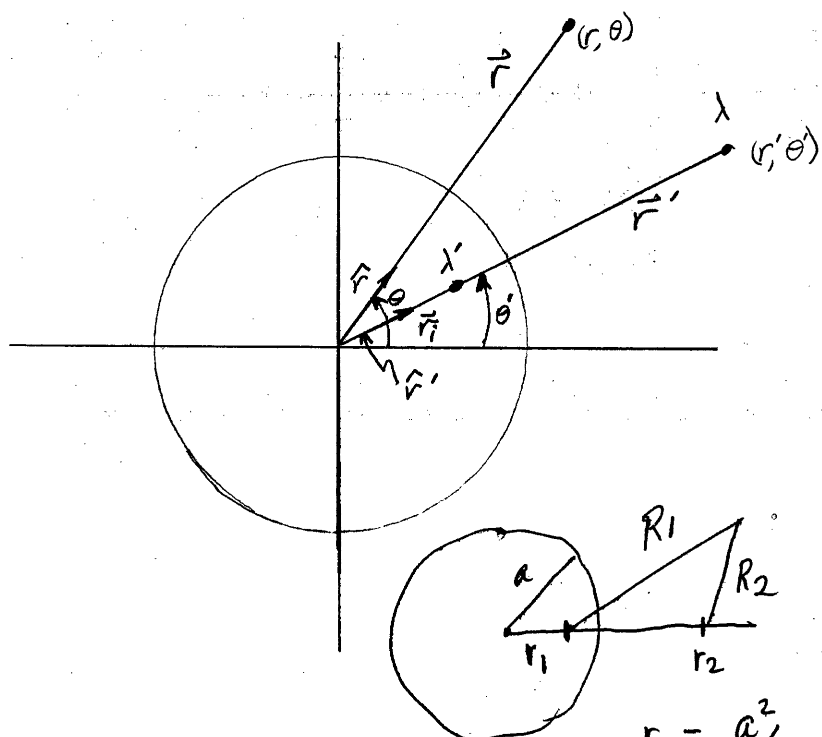
a) SOLUTION VIA METHOD OF IMAGES B



TAKE THE CENTER OF THE CYLINDER TO BE LOCATED AT THE ORIGIN OF COORDINATES OF A PLANE POLAR COORDINATE SYSTEM; THE CYLINDER, ESSENTIALLY INFINITE IN LENGTH, HAS A RADIUS a.

LET THE LINE SOURCE, A CHARGED LINE TAKEN TO HAVE UNIT LINEAR CH. DENS. BE LOCATED AT A DISTANCE r' FROM THE ORIGIN DISPLACED AT AN ANGLE θ' WITH RESPECT TO THE POLAR AXIS, $r_1' > a$.

EMPLOYING THE METHOD OF IMAGES, LET THERE EXIST AN IMAGE UNIT SOURCE λ' LOCATED AT $\vec{r}_1$ WHICH LIES ALONG THE SAME RAY AS $\hat{r}'$ DOES. THE TOTAL POTENTIAL AT $\vec{r}$ DUE TO THESE CHARGES IS GIVEN BY



$$r_1 = a^2/r_2$$

$\bullet \rightarrow E_r \, 2\pi r = \rho_\ell / \epsilon_0$

$G(\vec{r})$

$E_r = \frac{\rho_\ell}{2\pi\epsilon_0 r} = -\frac{\partial \Phi}{\partial r}$ so $\Phi = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln r + C$

$$G(\vec{r}) = \frac{\lambda}{|\vec{r}-\vec{r}'|} + \frac{\lambda'}{|\vec{r}-\vec{r}_1|} \qquad (A.1)$$

BUT $\vec{r} = r\hat{r}$, $\vec{r}' = r'\hat{r}'$, AND $\vec{r}_1 = r_1\hat{r}'$ SO EQ. (A.1) BECOMES

$$G(\vec{r}) = \frac{\lambda}{|r\hat{r} - r'\hat{r}'|} + \frac{\lambda'}{|r\hat{r} - r_1\hat{r}'|} + \frac{\rho_\ell}{2\pi\epsilon_0}\left(\ln R_1/R_2\right)$$

as image solution

$$= \frac{\lambda}{r\left|\hat{r} - \frac{r'}{r}\hat{r}'\right|} + \frac{\lambda'}{r_1\left|\frac{r}{r_1}\hat{r} - \hat{r}'\right|} \qquad (A.2)$$

NOTING THAT ONE CAN WRITE $\frac{\rho_\ell}{2\pi\epsilon_0} \ln\left[\frac{y^2 + (x - a^2/r_2)^2}{y^2 + (x - r_2)^2}\right]^{1/2}$

$$\left|\frac{r}{r_1}\hat{r} - \hat{r}'\right| = \left|\hat{r} - \frac{r}{r_1}\hat{r}'\right|$$

IN THE LAST TERM OF EQ. (A.2) GIVES

$$G(\vec{r}) = \frac{\lambda}{r\left|\hat{r} - \frac{r'}{r}\hat{r}'\right|} + \frac{\lambda'}{r_1\left|\hat{r} - \frac{r}{r_1}\hat{r}'\right|} \qquad (A.3)$$

$\rightarrow \frac{\rho_\ell}{4\pi\epsilon_0} \ln\left[\frac{\cancel{(x^2+y^2)}\left(1 - \frac{2xa^2}{r_2^2(x^2+y^2)}\right)}{\left(1 - \frac{2xr_2}{(x^2+y^2)}\right)}\right]$

THE PARAMETERS λ' AND r_1 ARE DETERMINED BY THE PREVAILING BOUNDARY CONDITIONS; IN THIS CASE THEY ARE GIVEN BY

$$G(\vec{r})\Big|_{|\vec{r}| = a} = 0 \rightarrow \frac{\rho_\ell}{4\pi\epsilon_0}\left[\frac{2xa^2}{r_2^2(x^2+y^2)} - \frac{2xr_2}{x^2+y^2}\right] \rightarrow 0$$

HENCE, FROM EQ. (A.3) THIS GIVES

$$G(\vec{r})\Big|_{|\vec{r}|=a} = 0 = \frac{\lambda}{a\left|\hat{r} - \frac{r'}{a}\hat{r}'\right|} + \frac{\lambda'}{r_1\left|\hat{r} - \frac{a}{r_1}\hat{r}'\right|}$$

WHICH CAN ONLY HOLD IF

$$\frac{\lambda}{a} = -\frac{\lambda'}{r_1} \quad \underline{\text{AND}} \quad \frac{r'}{a} = \frac{a}{r_1}.$$

SOLVING THESE RELATIONS FOR λ' AND r_1 GIVES

$$\lambda' = -\frac{\lambda a}{r'} \qquad (A.4)$$

AND

$$r_1 = \frac{a^2}{r'} \qquad (A.5)$$

EXPANDING THE DENOMINATORS IN EQ. (A.3) AND SUBSTITUTING EQS (A.4) AND (A.5) GIVES

$$G(\vec{r}) = G(r,\theta) = \frac{\lambda}{\left(r^2 - 2r'r\cos(\theta-\theta') + r'^2\right)^{1/2}} - \frac{\lambda}{\frac{r'}{a}\left(r^2 - 2r\left(\frac{a^2}{r'}\right)\cos(\theta-\theta') + \left(\frac{a^2}{r'}\right)^2\right)^{1/2}}$$

FOR THE GREEN FUNCTION OF THE PROBLEM.
(FOR A UNIT SOURCE, $\lambda = 1$)

b) EXPANSION IN EIGENFUNCTIONS.

RETURNING TO THE FIGURE ON THE 1ST PAGE, ONE CAN WRITE FOR LAPLACE'S EQUATION IN PLANE POLAR COORDINATES

$$\nabla^2 V = -\delta(x')\delta(y')$$

$$= -C\,\delta(r-r')\,\delta(\theta-\theta') \qquad (B.1)$$

THE NORMALIZATION CONSTANT C MUST BE SUCH THAT

$$\int_0^{2\pi}\int_0^{\infty} C\,\delta(r-r')\,\delta(\theta-\theta')\,r\,dr\,d\theta = 1$$

PERFORMING THE INTEGRATION YIELDS

$$Cr' = 1 \quad \Rightarrow \quad C = \frac{1}{r'}$$

THUS EQ. (B.1) BECOMES

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = -\frac{\delta(r-r')\,\delta(\theta-\theta')}{r'} \qquad (B.2)$$

THE SEPARABILITY OF THE ANGULAR AND RADIAL δ-FUNCTIONS SUGGESTS THAT ONE CAN EMPLOY EITHER THE SIN or COSINE EIGENFUNCTIONS OF THE ANGULAR DERIVATIVE OPERATOR.

THUS, WRITING

$$G(r,\theta) = \sum_{n=0}^{\infty} R(r) \cos(n\{\theta - \theta'\})$$

AND DEFINING $\theta'' \equiv \theta - \theta'$, ONE HAS FROM EQ. (B.2)

$$\sum_{n=0}^{\infty} \left[\cos(n\theta'') \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right) + \frac{n^2}{r^2} R \cos(n\theta'') \right] = -\frac{\delta(r-r')\,\delta(\theta'')}{r'}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial R}{\partial r}\right) - \frac{n^2}{r^2} R \right] \cos(n\theta'') = -\frac{\delta(r-r')\,\delta(\theta'')}{r'} \tag{B.3}$$

EMPLOYING THE ORTHOGONALITY OF $\cos(n\theta'')$ YIELDS

$$\int_0^{2\pi} \sum_n [\quad] \cos(n\theta'')\cos(m\theta'')\, d\theta'' = -\int_0^{2\pi} \frac{\delta(r-r')\,\delta(\theta'')}{r'} \cos(m\theta'')\, d\theta''$$

$$\Rightarrow \int_0^{2\pi} [\quad] \cos^2(m\theta'')\, d\theta'' = -\frac{\delta(r-r')}{r'} \int_0^{2\pi} \delta(\theta'') \cos(m\theta'')\, d\theta'' \tag{B.4}$$

THE LEFT SIDE OF THIS EQUATION IS READILY EVALUATED:

FOR $m \neq 0$, ONE HAS

$$\int_0^{2\pi} \cos^2(m\theta'')\,d\theta'' = \frac{1}{m}\int_0^{2\pi m} \cos^2\phi\,d\phi$$

$$= \frac{1}{2m}\int_0^{2\pi m} \left[1 + \cos(2\phi)\right] d\phi$$

$$= \frac{1}{2m}\int_0^{2\pi m} d\theta + \frac{1}{2m}\int_0^{2\pi m} \cos(2\phi)\,d\phi$$

$$= \pi + \underbrace{\frac{1}{4\pi}\int_0^{4\pi m} \cos\phi'\,d\phi'}$$

$$\rightarrow = \frac{1}{4m} \sin\phi' \Big|_0^{4\pi m}$$

$$= \frac{1}{4m}\sin(4\pi m)$$

$= 0$ SINCE m IS AN INTEGER

FOR $m = 0$

$$\int_0^{2\pi} \cos^2(m\theta'')\,d\theta'' = \int_0^{2\pi} d\theta'' = 2\pi$$

THE INTEGRAL ON THE RIGHT SIDE OF EQ. (B.4) BECOMES FOR $m \neq 0$

$$\int_0^{2\pi} \delta(\theta'')\cos(m\theta'')\,d\theta'' = 1$$

AND FOR $m = 0$

$$\int_0^{2\pi} \delta(\theta'')\cos(m\theta'')\,d\theta'' = \int_0^{2\pi} \delta(\theta'')\,d\theta'' = 1.$$

HENCE, EQ. (B.4) BECOMES

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{n^2}{r^2}R = -\frac{\delta(r-r')}{r'} \cdot \begin{cases} \frac{1}{2\pi}, & n=0 \\ \frac{1}{\pi}, & n\neq 0 \end{cases}$$

$$\Rightarrow \boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{n^2}{r^2}R = -\frac{\delta(r-r')}{\varepsilon_n \pi r'}, \quad \varepsilon_n = \begin{cases} 2, & n=0 \\ 1, & n>0 \end{cases}} \tag{B.5}$$

IN THE REGIONS $r \lessgtr r'$, EQ. (B.5) BECOMES

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \frac{n^2}{r^2}R = 0 \tag{B.6}$$

WRITING $R(r) = r^\lambda$, ONE HAS FROM EQ. (B.6)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}(r^\lambda)\right) - \frac{n^2}{r^2}r^\lambda =$$

$$= \frac{1}{r}\frac{d}{dr}\left(\lambda r^\lambda\right) - n^2 r^{\lambda-2}$$

$$= \frac{1}{r}\left(\lambda^2 r^{\lambda-1}\right) - n^2 r^{\lambda-2}$$

$$= \lambda^2 r^{\lambda-2} - n^2 r^{\lambda-2} = 0$$

$$\Rightarrow \lambda^2 = n^2$$

$$\Rightarrow \lambda = \pm n$$

THEREFORE, FOR $n \neq 0$, THE SOLUTION TO EQ. (B.6) IS

$$\boxed{R_n(r) = A_n r^n + B_n r^{-n}} \quad n > 0 \qquad (B.7)$$

IN THE EVENT THAT $n=0$ EQ (B.6) REDUCES TO

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = 0$$

WHICH DIRECTLY INTEGRATES TO

$$\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = 0$$

$$r\frac{dR}{dr} = C$$

$$\Rightarrow \frac{dR}{C} = \frac{dr}{r}$$

$$\frac{R}{C} = \ln r + C'$$

$$\Rightarrow \boxed{R_0(r) = A_0 \ln(r) + B_0 \quad \text{FOR } n=0} \qquad (B.8)$$

Expansion of $\delta(\theta')$ has a $n=0$ term.

$A_0 \ln r + B_0, \; r<r'$

$C_0 \ln r + D_0, \; r>r'$

$A_0 \ln a + B_0 = 0,$ (1)

$A_0 \ln r' + B_0 = C_0 \ln r' + D_0$ (2)

$\frac{C_0}{r'} - \frac{A_0}{r'} = -\frac{1}{2\pi}$ (3) (4)

$C_0 = 0$

$D_0 \neq 0$

$A_0 \ln r/a$ in $a<r<r'$

$-\frac{A_0}{r'} = -\frac{1}{2\pi}$, $D_0 = A_0 \ln r'/a$

D_n can be zero if cyl. is restated ... + potential

IN THE REGION $r>r'$, ONE MUST HAVE BOUNDED SOLUTIONS SO THIS DEMANDS THAT $A_n=0$ IN EQ. (B.7). ALSO, TO HAVE THAT $V \to 0$ AS $r \to \infty$, $A_0 = B_0 = 0$. FOR $r<r'$, ONE NEEDS TO HAVE $R_n(r)$ VANISH AT $r=a$. THUS, FOR THIS REGION ONE MUST HAVE $A_0 + B_0 = 0$ AND CONSTANT COEFFICIENTS C_n SUCH THAT

$$R_n(r) = C_n\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right] \qquad (B.9)$$

WHICH VANISHES AT $r=a$.

REQUIRING CONTINUITY AT $r = r'$ YIELDS THE FACT THAT

$$\frac{B_n}{(r')^n} = C_n\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] \qquad (B.10)$$

TO OBTAIN ANOTHER EXPRESSION INVOLVING THE COEFFICIENTS B_n AND C_n, CONSIDER THE FIRST INTEGRAL OF EQ. (B.5); ONE HAS THAT

$$\lim_{\varepsilon \to 0} \int_{r'-\varepsilon}^{r'+\varepsilon} \left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r^2}R\right] r\,dr = -\lim_{\varepsilon \to 0} \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{\delta(r-r')}{\varepsilon_n \pi r'}\, r\,dr$$

$\to 0$ SINCE NOT BOUNDED

$$\Rightarrow \lim_{\varepsilon \to 0} \int_{r'-\varepsilon}^{r'+\varepsilon} \cancel{\frac{1}{r}}\frac{d}{dr}\left(r\frac{dR}{dr}\right) \cancel{r}\,dr - \cancel{\lim_{\varepsilon \to 0} \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{n^2}{r^2} R\, r\,dr} = -\frac{1}{\varepsilon_n \pi}$$

$$\Rightarrow r'\left[\left.\frac{dR}{dr}\right|_{r'+} - \left.\frac{dR}{dr}\right|_{r'-}\right] = -\frac{1}{\varepsilon_n \pi} \qquad (B.11)$$

(NOTING THAT THERE ARE NO TERMS IN THE SOLUTIONS FOR $n = 0$, ONE CAN SET $\varepsilon_n = 1$ HEREAFTER.) FROM EQS. (B.7) AND (B.9)

$$\left.\frac{dR}{dr}\right|_{r'+} = \left.\frac{d}{dr} B_n (r)^{-n}\right|_{r=r'} = \left.-B_n n\, r^{-n-1}\right|_{r=r'} = -\frac{n B_n}{(r')^{n+1}}$$

$$\left.\frac{dR}{dr}\right|_{r'-} = \left.\frac{d}{dr}\left\{C_n\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right]\right\}\right|_{r=r'} = C_n\left[n\left(\frac{1}{a}\right)^n (r')^{n-1} + n\, a^n \left(\frac{1}{r'}\right)^{n+1}\right]$$

THUS, FROM EQ. (B.11)

$$r'\left[-\frac{n B_n}{(r')^{n+1}} - n C_n \left\{\frac{(r')^{n-1}}{a^n} + a^n\left(\frac{1}{r'}\right)^{n+1}\right\}\right] = -\frac{1}{\pi}$$

$$-\frac{n B_n}{(r')^n} - n C_n \left\{\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right\} = -\frac{1}{\pi}$$

$$\frac{B_n}{(r')^n} + C_n \left\{\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right\} = \frac{1}{n\pi} \qquad \text{(B.12)}$$

HENCE, ONE MUST SOLVE THE SYSTEM OF EQUATIONS GIVEN BY EQS (B.10) AND (B.12) TO OBTAIN B_n AND C_n:

$$\left.\begin{aligned} \frac{B_n}{(r')^n} - C_n\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] &= 0 \\ \frac{B_n}{(r')^n} + C_n\left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] &= \frac{1}{n\pi} \end{aligned}\right\} \qquad \text{(B.13)}$$

EMPLOYING CRAMER'S RULE,

$$B_n = \frac{\begin{vmatrix} 0 & -\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] \\ \frac{1}{n\pi} & \left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{r'}\right)^n & -\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] \\ \left(\frac{1}{r'}\right)^n & \left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] \end{vmatrix}} \qquad \text{(B.14)}$$

$$C_n = \frac{\begin{vmatrix} \left(\frac{1}{r'}\right)^n & 0 \\ \left(\frac{1}{r'}\right)^n & \frac{1}{n\pi} \end{vmatrix}}{\begin{vmatrix} \left(\frac{1}{r'}\right)^n & -\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] \\ \left(\frac{1}{r'}\right)^n & \left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] \end{vmatrix}} \qquad (B.15)$$

THE DENOMINATOR OF EQS (B.14) AND (B.15) WHICH IS ESSENTIALLY THE WRONSKI DETERMINANT OF THE SOLUTIONS OF EQ. (B.6), IS

$$\text{DEN} \equiv \begin{vmatrix} \left(\frac{1}{r'}\right)^n & -\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right] \\ \left(\frac{1}{r'}\right)^n & \left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] \end{vmatrix}$$

$$= \left(\frac{1}{r'}\right)^n \left[\left(\frac{r'}{a}\right)^n + \left(\frac{a}{r'}\right)^n\right] + \left(\frac{1}{r'}\right)^n \left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right]$$

$$= \left(\frac{1}{a}\right)^n + \cancel{\frac{a^n}{(r')^{2n}}} + \left(\frac{1}{a}\right)^n - \cancel{\frac{a^n}{(r')^{2n}}}.$$

$$= 2\left(\frac{1}{a}\right)^n$$

HENCE, USING EQ. (B.14), ONE HAS FOR B_n

$$\boxed{B_n = \frac{\frac{1}{n\pi}\left[\left(\frac{r'}{a}\right)^n - \left(\frac{a}{r'}\right)^n\right]}{2\left(\frac{1}{a}\right)^n} = \frac{1}{2n\pi}\left[\left(\cdot r'\right)^n - \left(\frac{a^2}{r'}\right)^n\right]} \qquad (B.16)$$

AND FROM EQ. (B.15), C_n IS

$$\boxed{C_n = \frac{\frac{1}{n\pi}\left(\frac{1}{r'}\right)^n}{2\left(\frac{1}{a}\right)^n} = \frac{1}{2n\pi}\left(\frac{a}{r'}\right)^n} \qquad (B.17)$$

THUS, FOR $r > r'$, ONE HAS USING EQ. (B.7) REMEMBERING THAT $A_n = 0$,

$$R_n(r) = \frac{B_n}{r^n} = \left(\frac{1}{2\pi n}\right)\left[\left(\frac{r'}{r}\right)^n - \left(\frac{a^2}{rr'}\right)^n\right], \quad r > r'$$

AND FROM EQ. (B.9) FOR $r < r'$,

$$R_n(r) = C_n\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right] = \left(\frac{1}{2\pi n}\right)\left[\left(\frac{r}{r'}\right)^n - \left(\frac{a^2}{rr'}\right)^n\right]$$

$r < r'$

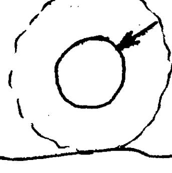
THESE CAN COLLECTIVELY BE WRITTEN

$$R_n(r) = \left(\frac{1}{2\pi n}\right)\left[\left(\frac{r_<}{r_>}\right)^n - \left(\frac{a^2}{rr_>}\right)^n\right]$$ WHERE $r_> \equiv$ LARGER OF r, r'; $r_< \equiv$ SMALLER OF r, r'

There is a $n=0$ term

FINALLY, GOING BACK TO THE ORIGINAL EIGEN-FUNCTION EXPANSION

$$G(r,\theta) = \sum_n R_n(r)\cos(n(\theta-\theta'))$$



$n=0$ term in expansion of $\delta(\theta)$ is a uniform cyl. of charge. E_r exists in ...

$$\boxed{G(r,\theta) = \left(\frac{1}{2\pi}\right)\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\left[\left(\frac{r_<}{r_>}\right)^n - \left(\frac{a^2}{rr_>}\right)^n\right]\cos(n(\theta-\theta'))}$$

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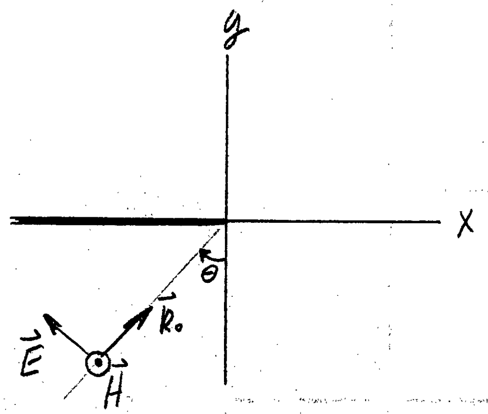
DIFFRACTION BY
A HALF PLANE

(A)



THE CONDUCTING PLANE IS TAKEN TO EXTEND TO ±∞ IN THE z DIRECTION ∴ A TWO DIMENSIONAL PROBLEM
A PLANE WAVE WITH $\vec{H}$ GIVEN BY

$$\vec{H}_i(x,y) = H_0 \hat{z} e^{-ik_0(x\sin\theta + y\cos\theta)}$$

RELATIVE TO THE COORDINATE SYSTEM SHOWN HAS WITH IT AN ELECTRIC FIELD GIVEN BY

$$\vec{E}_i(x,y) = -\frac{i}{\omega\varepsilon} \vec{\nabla} \times \vec{H}_i(x,y)$$

$$= -\frac{i}{\omega\varepsilon} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & H \end{vmatrix}$$

$$= -\frac{i}{\omega\varepsilon}\Big[\hat{x} H_0(-ik_0\cos\theta) e^{-ik_0(x\sin\theta + y\cos\theta)} +$$

$$- \hat{y} H_0(-ik_0\sin\theta) e^{-ik_0(x\sin\theta + y\cos\theta)} \Big]$$

$$= \frac{H_0 k_0}{\omega\varepsilon}\left[\hat{y}\sin\theta - \hat{x}\cos\theta\right] e^{-ik_0(x\sin\theta + y\cos\theta)}$$

$$\therefore \quad \vec{E}_i(x,y) = \frac{H_0 k_0}{\omega\varepsilon}\left(\hat{y}\sin\theta - \hat{x}\cos\theta\right) e^{-ik_0(x\sin\theta + y\cos\theta)}$$

THUS, THERE ARE x + y COMPONENTS OF THE ELECTRIC FIELD. THE x-COMPONENT GIVES RISE TO CURRENT FLOW WITHIN THE CONDUCTOR SO AS TO ESTABLISH A ZERO FIELD AT ITS INTERFACE. THE CURRENT THUS FLOWS IN THE $-\hat{x}$ DIRECTION AND LINES OF CONSTANT CURRENT DENSITY ARE PARALLEL TO THE z AXIS. (THIS IS ALL CONSISTANT WITH THE B.C. THAT REQUIRES $\vec{J} = \hat{n} \times \vec{H}$ WHERE $\hat{n}$ IS THE UNIT NORMAL TO THE BOTTOM SURFACE OF THE CONDUCTOR.) THE y-COMPONENT GIVES RISE TO A BUILDUP OF CHARGE ON THE CONDUCTOR. (THIS IS ALSO CONSISTENT WITH THE FACT THAT THE CURRENT DENSITY IS NOT CONSTANT ALONG THE CONDUCTOR SURFACE AND CONTINUITY OF CHARGE / CURRENT MUST HOLD.

A TYPICAL DIFFERENTIAL "CONSTANT CURRENT STRIP" IN THE $\hat{z}$-DIRECTION OF DIFFERENTIAL THICKNESS dx' AT AN ARBITRARY POINT x' CREATES A DIFFERENTIAL VECTOR POTENTIAL $dA_x(x,y;x')$ AT (x,y) DUE TO A CURRENT DENSITY $J_x(x')$ AT THE POINT x' GIVEN BY

$$(\nabla^2 + k_0^2)\, dA_x = -\mu_0 J_x(x')\, dx'\, \delta(x-x')\, \delta(y) \quad (1)$$

THIS EQUATION CAN BE REWRITTEN AS

$$(\nabla^2 + k_0^2)\frac{dA_x}{\mu_0 J_x(x')dx'} = -\delta(x-x')\delta(y)$$

OR

$$(\nabla^2 + k_0^2)\, G_x = -\delta(x-x')\delta(y), \quad G_x \equiv \frac{dA_x}{\mu_0 J_x(x')dx'} \qquad (3)$$

THIS EQUATION HAS BEEN THE SUBJECT OF AN EARLIER STUDY AND HAS AS A SOLUTION

$$G_x(x,y;x') = -\left(\frac{i}{4}\right) H_0^{(2)}\left(k_0\{(x-x')^2 + y^2\}^{1/2}\right) \qquad (4)$$

THUS

$$dA_x(x,y;x') = -\left(\frac{i}{4}\right)\mu_0 H_0^{(2)}\left(k_0\{(x-x')^2+y^2\}^{1/2}\right) J_x(x')\,dx' \qquad (5)$$

THE TOTAL VECTOR POTENTIAL IS OBTAINED BY INTEGRATING OVER THE TOTAL EXTENT OF THE CONDUCTING PLANE, $-\infty < x' \leq 0$. THUS, INTEGRATING EQ. (5) GIVES

$$A_x(x,y) = \int_{-\infty}^{0} dA(x,y;x')\,dx' = -\left(\frac{i\mu_0}{4}\right)\int_{-\infty}^{0} H_0^{(2)}\left(k_0\{(x-x')^2+y^2\}^{1/2}\right) J_x(x')\,dx' \qquad (6)$$

THE SCATTERED ELECTRIC FIELD $E_s(x,y)$ IS THEN GIVEN BY

$$\vec{E}_s(x,y) = -i\omega A_x(x,y)\hat{x} + \frac{\vec{\nabla}(\partial_x A_x)}{i\omega\varepsilon_0\mu_0}$$

$$\cong \left(-i\omega A_x(x,y) + \frac{\partial_x^2 A_x}{i\omega\varepsilon_0\mu_0}\right)\hat{x} \qquad (7)$$

$$= \left(\left(\frac{-1}{i\omega\varepsilon_0\mu_0}\right)\left(+\omega^2\varepsilon_0\mu_0 A_x + \partial_x^2 A_x\right)\right)\hat{x}$$

USING THE DEFINITION $k_0^2 \equiv \omega^2\varepsilon_0\mu_0$ AND SUBSTITUTING EQ. (6) INTO THE ABOVE RELATION GIVES FOR THE X-DIRECTED SCATTERED FIELD

$$E_{s_x}(x,y) = \left(\frac{-i\mu_0}{4}\right)\left(\frac{1}{i\omega\varepsilon_0\mu_0}\right)\left[k_0^2+\partial_x^2\right]\int_{-\infty}^{0} H_0^{(2)}\left(k_0\{(x-x')^2+y^2\}^{1/2}\right)J_x(x')\,dx'$$

$$\boxed{E_{s_x}(x,y) = \left(\frac{-1}{4\omega\varepsilon_0}\right)\left[k_0^2+\partial_x^2\right]\int_{-\infty}^{0} H_0^{(2)}\left(k_0\{(x-x')^2+y^2\}^{1/2}\right)J_x(x')\,dx'} \qquad (8)$$

AT THIS POINT, ONE NEEDS TO KNOW WHAT $J_x(x')$ IS IN ORDER TO EVALUATE EQ. (8). AS MENTIONED EARLIER $J_x(x')$ MUST BE SUCH THAT THE TOTAL TANGENTIAL (INCIDENT + SCATTERED) FIELD VANISH AT THE SURFACE OF THE CONDUCTOR. LETTING E_{T_x} BE THE TOTAL TANGENTIAL FIELD, ONE HAS

$$E_{T_x}(x,y) \equiv E_{i_x}(x,y) + E_{s_x}(x,y) \tag{9}$$

AND, AT THE CONDUCTOR SURFACE,

$$E_{T_x}(x,0) = 0 \tag{10}$$

THUS, FROM THE EQUATION FOR THE INITIAL E-FIELD ON pg 1 AND FROM EQS. (8), (9) AND (10), ONE HAS

$$-\left(\frac{H_0 k_0}{\omega\varepsilon_0}\right)\cos\theta\, e^{-ik_0 x\sin\theta} + \left(\frac{-1}{4\omega\varepsilon_0}\right)\left[k_0^2+\partial_x^2\right]\int_{-\infty}^{0} H_0^{(2)}\left(k_0\{(x-x')^2+y^2\}^{1/2}\right)\cdot J_x(x')\,dx'$$

or

$$\boxed{K e^{-ik_0 x\sin\theta} = \left[k_0^2+\partial_x^2\right]\int_{-\infty}^{0} H_0^{(2)}(k_0|x-x'|)J_x(x')\,dx'} \tag{11}$$

WHERE $K \equiv -4 H_0 k_0 \cos\theta$

EQUATION (11) DETERMINES $J_x(x')$. THIS EQUATION MUST BE SOLVED FOR $J_x(x')$ WHICH CAN THEN BE INTRODUCED INTO EQ. (8) TO DETERMINE $E_{s_x}(x,y)$

A. SOLUTION OF EQUATION (11) VIA THE WIENER-HOPF TECHNIQUE

EQUATION (11) WILL TAKE THE FORM OF A CONVOLUTION INTEGRAL, WHICH CAN THEN BE EASILY INVERTED TO SOLVE FOR $J_x(x)$, PROVIDED IT CAN BE EXTENDED IN THE INTERVAL $0 \leq x < \infty$. TO THIS END, CONSIDER THE FUNCTION $f_-(x)$ DEFINED AS

$$f_-(x) \equiv \begin{cases} Ke^{-ik_0 x \sin\theta} & , -\infty < x \leq 0 \\ 0 & , x > 0 \end{cases}$$

AND HYPOTHESIZE THE EXISTANCE OF A FUNCTION $f_+(x)$ SUCH THAT ONE CAN WRITE

$$f_-(x) + f_+(x) = \left[k_0^2 + \partial_x^2\right] \int_{-\infty}^{\infty} H_0^{(2)}(k_0 |x - x'|) J_-(x')\, dx' \qquad (12)$$

WHERE

$$J_-(x') \equiv \begin{cases} J_x(x') & , -\infty < x' \leq 0 \\ 0 & , x' > 0 \end{cases}$$

TAKING THE LAPLACE TRANSFORM OF EQ. (12) GIVES

$$\mathcal{L}\{f_-(x) + f_+(x)\} = \mathcal{L}\left\{\left[k_0^2 + \partial_x^2\right]\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)\,J_-(x')\,dx'\right\}$$

$$\boxed{\mathcal{L}\{f_-(x)\} + \mathcal{L}\{f_+(x)\} = \mathcal{L}\left\{\left[k_0^2 + \partial_x^2\right]\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)\,J_-(x')\,dx'\right\}} \tag{13}$$

PROVIDED THE TRANSFORMS EXIST. THE FIRST TERM ON THE LHS OF EQ. (13) IS

$$\mathcal{L}\{f_-(x)\} = \int_{-\infty}^{\infty} f_-(x)\,e^{-wx}\,dx$$

$$= K \int_{-\infty}^{0} e^{-(w + ik_0\sin\theta)x}\,dx$$

LETTING $k_0 \equiv k_0' - i k_0''$, THE INTEGRAND IN THE ABOVE EXPRESSION BECOMES

$$e^{-(w+ik_0\sin\theta)x} = e^{-[w + i(k_0' - ik_0'')\sin\theta]x}$$

$$= e^{-[w + (ik_0' + k_0'')\sin\theta]x}$$

$$= e^{-(w + k_0''\sin\theta)x}\,e^{-ik_0'\sin\theta\,x}$$

HENCE, THE TRANSFORM IN EQ. (13) EXISTS PROVIDED

$$Re\{w\} + k_o'' \sin\theta < 0$$

$$\Rightarrow \quad Re\{w\} < -k_o'' \sin\theta$$

AND IS

$$\mathcal{L}\{f_-(x)\} = K \left[-\frac{e^{-(w+ik_o \sin\theta)x}}{(w+ik_o\sin\theta)} \right]_{-\infty}^{0}$$

$$= -K\left(\frac{1}{w+ik_o \sin\theta}\right) \qquad (14)$$

TO ESTABLISH THE EXISTANCE OF $\mathcal{L}\{f_+(x)\}$,

$$\mathcal{L}\{f_+(x)\} \equiv \int_0^{\infty} f_+(x)\, e^{-wx}\, dx$$

ONE NEED ONLY CONSIDER THE FIELDS AT $y=0$ AS $x \to \infty$. SINCE THE SCATTERED FIELD IS ASSUMED TO SATISFY THE RADIATION CONDITION AT $x \to \infty$, THE ONLY FIELD THAT CAN EXIST THERE IS THE INCIDENT FIELD. HENCE

$$f_+(x) \sim e^{-ik_o x}$$

scattered field varies like $e^{-ik_o x}$ also.
f_+ is scattered field

AND

$$\mathcal{L}\{f_+(x)\} \sim \int_0^\infty e^{-(w+ik_o)x}\,dx$$

$$\sim \int_0^\infty e^{-[w+i(k_o'-ik_o'')]x}\,dx$$

$$\sim \int_0^\infty e^{-(w+k_o'')x}\, e^{-ik_o' x}\,dx$$

AND THUS EXISTS PROVIDED $Re\{w\} + k_o'' > 0$, i.e, $Re\{w\} > -k_o''$.

GOING TO THE OTHER SIDE OF EQ. (13), ONE HAS

$$\mathcal{L}\left\{[k_o^2 + \partial_x^2]\int_{-\infty}^{\infty} H_o^{(2)}(k_o|x-x'|)\,J_-(x')\,dx'\right\} =$$

$$= k_o^2 \mathcal{L}\left\{\int_{-\infty}^{\infty} H_o^{(2)}(k_o|x-x'|)\,J_-(x')\,dx'\right\} +$$

$$+ \mathcal{L}\left\{\partial_x^2 \int_{-\infty}^{\infty} H_o^{(2)}(k_o|x-x'|)\,J_-(x')\,dx'\right\} \qquad (15)$$

THE LAST TERM ON THE RHS OF EQ. (15) BECOMES

$$\mathcal{L}\left\{\partial_x^2\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)J_-(x')\,dx'\right\} =$$

$$= \int_{-\infty}^{\infty} \partial_x^2 \underbrace{\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)J_-(x')\,dx'}_{\equiv I(x)} \, e^{-wx}\,dx$$

$$= \cancel{\partial_x I(x) e^{-wx}\Big|_{-\infty}^{\infty}} + w\int_{-\infty}^{\infty} \partial_x I(x)\, e^{-wx}\,dx$$

$$= w\left[\cancel{I(x) e^{-wx}\Big|_{-\infty}^{\infty}} + w\int_{-\infty}^{\infty} I(x)\, e^{-wx}\,dx\right]$$

Sorry for my mess here!

$$= w^2\int_{-\infty}^{\infty} I(x)\, e^{-wx}\,dx$$

$I(x) \sim e^{-jk_0\sin\theta\, x}$

From J_- due to incident wave

Scattered field must cancel E_{inc} as $x \to -\infty$

$$= w^2\,\mathcal{L}\left\{\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)J_-(x')\,dx'\right\}$$

$e^{k_0''(1-\sin\theta)x}$ as $x \to -\infty$

$I(x) \sim e^{-jk_0 x} \sim e^{-k_0'' x}$ as $x \to \infty$

UPON INTEGRATING BY PARTS TWICE AND NOTING THAT THE "SURFACE" TERMS VANISH PROVIDED

$$\lim_{|x| \to \infty} I(x)\, e^{-wx} \sim e^{-ik_0 x} e^{-wx} = e^{(-ik_0' - k_0'' - w)x}$$

$$= e^{-(k_0''+w)x}\, e^{-ik_0' x}$$

$\to 0$

$$\Rightarrow \quad k_0'' + \mathrm{Re}\{w\} > 0 \quad \text{or} \quad k_0'' > {}^{\text{sign}}\!\!-\mathrm{Re}\{w\}$$

$\mathrm{Re}\,w > -k''$

AND SIMILARLY

as $x \to -\infty$ $I(x)$ behaves like $e^{-ik_0 \sin\theta\, x}$ because of incident field on screen

$$\lim_{x \to -\infty} I(x)\, e^{-wx} \sim e^{ik_0 x + wx} = e^{(ik_0' + k_0'' + w)x}$$

$\mathrm{Re}\, w < k_0'' \sin\theta$

$$e^{-ik_0 \sin\theta\, x - wx}$$

$$= e^{(k_0'' + w)x}\, e^{ik_0' x}$$

$$\to 0$$

$$\Rightarrow \quad k_0'' + \mathrm{Re}\{w\} > 0 \quad \text{or} \quad \mathrm{Re}\{w\} > -k_0''$$

THUS EQ. (15) BECOMES

$$\mathcal{L}\{\quad\} = (k_0^2 + w^2)\, \mathcal{L}\left\{\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x-x'|)\, J(x')\, dx'\right\}$$

$$= (k_0^2 + w^2)\, \mathcal{L}\{H_0^{(2)}(k_0|x|)\}\, \mathcal{L}\{J_-(x)\}$$

valid in strip $-k_0'' \sin\theta > \mathrm{Re}\, w > -k_0''$

THE 1ST TRANSFORM IS

$$\mathcal{L}\{H_0^{(2)}(k_0|x|)\} = \int_{-\infty}^{\infty} H_0^{(2)}(k_0|x|)\, e^{-wx}\, dx$$

$$\sim \int_{-\infty}^{\infty} e^{-ik_0|x|}\, e^{-wx}\, dx$$

$$= \int_{-\infty}^{\infty} e^{k_0''|x| - wx}\, e^{-ik_0'|x|}\, dx$$

$$= \int_0^{\infty} e^{-(k_0'' + w)x}\, e^{-ik_0' x}\, dx - \int_0^{\infty} e^{-(k_0'' - w)x}\, e^{-ik_0' x}\, dx$$

THIS PARTICULAR TRANSFORM EXISTS PROVIDED

$$Re\{w\} + k_0'' > 0 \quad \underline{\text{AND}} \quad -Re\{w\} + k_0'' > 0$$

$$\left.\begin{array}{l} Re\{w\} + k_0'' > 0 \\ \quad \Rightarrow Re\{w\} > -k_0'' \\ -Re\{w\} + k_0'' > 0 \\ \quad \Rightarrow -Re\{w\} > -k_0'' \\ \quad Re\{w\} < k_0'' \end{array}\right\} \quad -k_0'' < Re\{w\} < k_0''$$

FINALLY, FOR THE LAST TRANSFORM,

$$\mathcal{L}\{J_-(x)\} = \int_{-\infty}^{\infty} J_-(x)\, e^{-wx}\, dx$$

$$= \int_{-\infty}^{0} J_x(x)\, e^{-wx}\, dx$$

ONE NOTES THAT, FOR THE ELECTRIC FIELD INDUCING THE CURRENT DENSITY $E_{INC} \sim e^{-ik_0 x \sin\theta}$ SO

$$\mathcal{L}\{J_-(x)\} \sim \int_{-\infty}^{0} e^{-(w + ik_0 \sin\theta)x}\, dx$$

$$= \int_{-\infty}^{0} e^{-(w + i(k_0' - ik_0'')\sin\theta)x}\, dx$$

$$= \int_{-\infty}^{0} e^{-(w + k_0''\sin\theta)x}\, e^{-ik_0' x}\, dx$$

THEREFORE, ONE MUST HAVE

$$Re\{W\} + k_0'' \sin\theta \; \cancel{\neq 0} < 0 \text{ OK} \quad \text{since } x \to -\infty$$

Re w

$$\Rightarrow Re\{W\} < -k_0'' \sin\theta$$

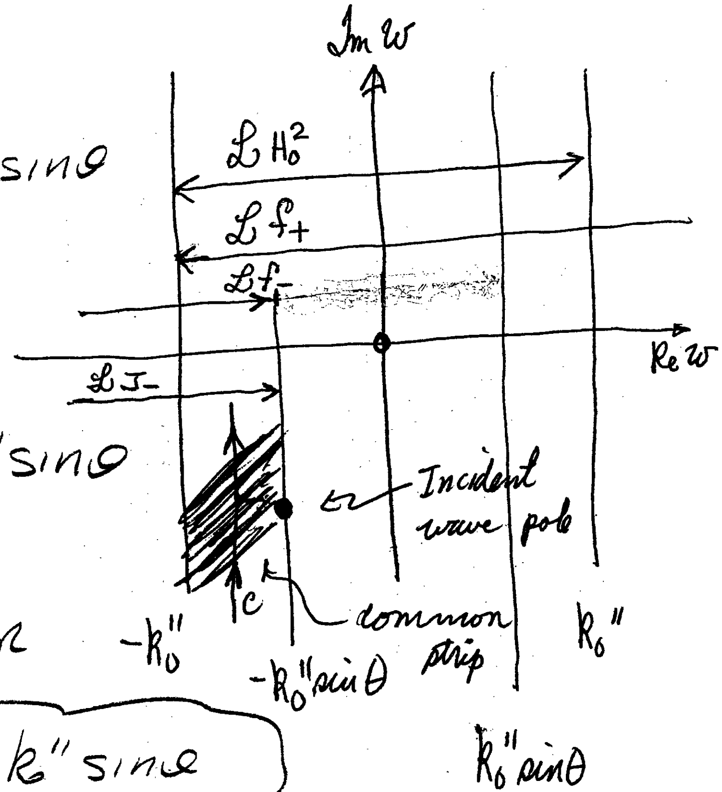
THUS, SUMMING UP, FOR THE TRANSFORMS TO EXIST IN EQ. (13) ONE MUST SIMULTANEOUSLY SATISFY

$$Re\{W\} < -k_0'' \sin\theta$$

$$-k_0'' < Re\{W\}$$

$$-k_0'' < Re\{W\} < k_0''$$

$$Re\{W\} < -k_0'' \sin\theta$$



$\therefore$ EQ. (1) HOLDS FOR

$$\boxed{-k_0'' < Re\{W\} < -k_0'' \sin\theta}$$

DEFINING

$$F_-(W) \equiv \mathcal{L}\{f_-(x)\}, \quad F_+(W) \equiv \mathcal{L}\{f_+(x)\}$$

$$G(W) \equiv (k_0^2 + W^2)\,\mathcal{L}\{H_0^{(2)}(k_0|x|)\}, \quad \hat{J}_-(W) \equiv \mathcal{L}\{J_-(x)\}$$

EQ. (13) BECOMES

$$F_-(w) + F_+(w) = G(w)\,\hat{J}_-(w) \qquad (16)$$

IN ORDER TO FACTOR THIS EQUATION INTO TWO EQUATIONS THAT SEPARATLY HOLD FOR $Re\{w\} > -k_o''$ AND $Re\{w\} < -k_o'' \sin\theta$, ONE MUST POSTULATE THE EXISTANCE OF $G_+(w)$ AND $G_-(w)$ SUCH THAT

$$G(w) = \frac{G_-(w)}{G_+(w)} \qquad (17)$$

GIVING, FROM EQ. (16)

$$F_-(w)\,G_+(w) + F_+(w)\,G_+(w) = G_-(w)\,\hat{J}_-(w) \qquad (18)$$

THEN, ASSUMING THE EXISTANCE OF TWO TERMS $S_+(w)$ AND $S_-(w)$ SUCH THAT

$$F_-(w)\,G_+(w) = S_+(w) + S_-(w) \qquad (19)$$

ONE CAN WRITE EQ. (18) AS

$$S_+(w) + S_-(w) + F_+(w)\,G_+(w) = G_-(w)\,\hat{J}_-(w)$$

OR AS

$$S_+(w) + F_+(w)\,G_+(w) = G_-(w)\,\hat{J}_-(w) - S_-(w) \qquad (20)$$

WITH SUCH A RELATION ESTABLISHED, ONE CAN SOLVE FOR $\hat{J}_-(w)$.

FIRST, ONE MUST ESTABLISH $G_-(w)$ AND $G_+(w)$

FOR EQ. (17), ONE MUST FIND $G(w)$.

$$G(w) = (k_0^2 + w^2)\,\mathcal{L}\left\{H_0^{(2)}(k_0|x|)\right\}$$

$$= (k_0^2 + w^2)\int_{-\infty}^{\infty} H_0^{(2)}(k_0|x|)\,e^{-wx}\,dx$$

SUBSTITUTING AN INTEGRAL REPRESENTATION FOR $H_0^{(2)}(k_0|x|)$ INTO THIS EQUATION (EQ. 4.6 FROM CHAPT. 4) GIVES

$G_x = -\frac{i}{4} H_0^2$

$$G(w) = (k_0^2 + w^2)\int_{-\infty}^{\infty}\left(\frac{i}{\pi}\right)\int_C \frac{e^{-i\xi x}\,e^{+wx}}{(\xi^2 - k_0^2)^{1/2}}\,d\xi\,dx \qquad (y=0)$$

THE x INTEGRATION IS READILY PERFORMED AND GIVES

$$G(w) = (k_0^2 + w^2)\left(\frac{i}{\pi}\right)\int_C (2\pi i)\,\frac{\delta(w + i\xi)}{(\xi^2 - k_0^2)^{1/2}}\,d\xi$$

$$G(w) = (k_0^2+w^2)\,2i\int \frac{\delta(w+i\xi)}{(\xi^2-k_0^2)^{1/2}}\, d(i\xi)$$

$$= (k_0^2+w^2)\,2i\,\frac{1}{(-w^2-k_0^2)^{1/2}}$$

$$= \frac{(k_0^2+w^2)\,2\cancel{i}}{\cancel{i}(w^2+k_0^2)^{1/2}}$$

$$= 2(w^2+k_0^2)^{1/2}\left(\frac{-i}{4}\right) \leftarrow$$ put this with G to get G_x

EXPANDING THE RADICAL AND LETTING $k_0 = k_0' - ik_0''$ GIVES

Branch Pt. at $w = -ik_0'' - k_0''$ in lhp

$$G(w) = 2(w+ik_0)^{1/2}(w-ik_0)^{1/2}$$

$$= 2\left(w+k_0''+ik_0'\right)^{1/2}\left(w-k_0''-ik_0'\right)^{1/2}$$

NOTING THAT THE MIDDLE TERM HAS NO BRANCH POINT IN THE "RIGHT-HALF PLANE", $Re\{w\} > -k_0''$,

ONE HAS

$$G_+(w) \equiv \frac{1}{2(w+ik_0)^{1/2}} \qquad (21)$$

THUS MAKING

$$G_-(w) \equiv (w-ik_0)^{1/2} \qquad (22)$$

NEXT, ONE MUST FIND A $S_+(w)$ AND $S_-(w)$ TO SATISFY EQ.(19). TO THIS END, FROM EQS (14) AND (21), ONE HAS

$$F_-(w)G_+(w) = \frac{-K}{2(w+ik_0)^{1/2}(w+ik_0\sin\theta)} \qquad (23)$$

THIS EXPRESSION HAS A POLE AT $w=-ik_0\sin\theta$ WITH A RESIDUE $-K/2(ik_0-ik_0\sin\theta)^{1/2}$. THUS, SUBTRACTING THIS RESIDUE AND ITS ATTENDANT POLE FROM EQ.(23) AND ADDING THEM AGAIN AS A SEPARATE TERM GIVES

$$F_-(w)G_+(w) = -\frac{K}{2}\left[\left\{\frac{1}{(w+ik_0)^{1/2}(w+ik_0\sin\theta)} - \frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\right\} + \right.$$

$$\left. + \frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\right]$$

THE TWO TERMS WITHIN THE CURLEY BRACKETS IS NOW ANALYTIC FOR $Re\{w\} > -k_0''$, I.E., IN THE RIGHT HALF PLANE AND THE LAST TERM IS ANALYTIC FOR $Re\{w\} < -k_0''\sin\theta$. THUS, ONE CAN WRITE

$$S_+ \equiv -\frac{K}{2}\left[\frac{1}{(w+ik_0)^{1/2}(w+ik_0\sin\theta)} - \frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\right] \quad (24)$$

$$S_- \equiv -\frac{K}{2}\left[\frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\right] \quad (25)$$

USING EQS (22) AND (25) IN EQ. (20) YIELDS

$$(w-ik_0)^{1/2}\hat{J}_-(w) + \frac{K}{2}\left[\frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\right] =$$

$$= S_+(w) + F_+(w)G_+(w) \equiv H(w) \quad (26)$$

SOLVING EQ. (26) FOR $\hat{J}_-(w)$ GIVES

$$\hat{J}_-(w) = \frac{H(w)}{(w-ik_0)^{1/2}} - \frac{K}{2}\left[\frac{1}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)(w-ik_0)^{1/2}}\right] \quad (27)$$

THE FUNCTION $H(w)$, AN ENTIRE FUNCTION ON THE COMPLEX PLANE, MUST NOW BE DETERMINED VIA BOUNDARY CONDITIONS. THE ONLY PREVAILING BOUNDARY CONDITION IS THE "EDGE" CONDITION THAT THE CURRENT

MUST SATISFY AT $x \to 0$. IN THIS CASE ONE HAS $\lim_{x \to 0} J_+(x) \sim x^{1/2}$. VIA THE FINAL

VALUE THEOREM, WHICH STATES THAT AS A FUNCTION $F(x) \sim x^{\alpha}$ AS $x \to 0$, ITS SPECTRAL TRANSFORM $\hat{F}(w) \sim w^{-\alpha-1}$ AS $w \to \infty$, ONE HAS THAT $\hat{J}_-(w) \sim w^{-3/2}$. IN THE LIMIT $w \to \infty$, THE SECOND TERM IN EQ. (27) GOES AS $w^{-3/2}$; THUS ONE CAN LET $H(w) = 0$. THEREFORE, THE LAPLACE TRANSFORM OF THE SOUGHT AFTER CURRENT DENSITY IS

$$\hat{J}_-(w) = -\frac{K}{2}\left[\frac{i \checkmark \text{ sign error } inK}{(ik_0 - ik_0 \sin\theta)^{1/2}(w + ik_0 \sin\theta)(w - ik_0)^{1/2}}\right] \qquad (28)$$

$-2iK[\,]$

$K = -4H_0 k_0 \cos\theta$

B. SOLUTION OF EQ. (11)

RETURNING TO EQ (8), ONE CAN REWRITE IT USING THE DEFINITION OF $J_-(x)$ GIVEN ON THE BOTTOM OF Pg 6 AS

$$E_s(x,y) = \left(\frac{-1}{4w\varepsilon_0}\right)\left[k_0^2 + \partial_x^2\right]\int_{-\infty}^{\infty} H_0^{(2)}\left(k_0\{(x-x')^2 + y^2\}^{1/2}\right) J_-(x')\,dx'$$

LAPLACE TRANSFORMING THIS EXPRESSION, REMEMBERING THAT THE DERIVATIVE OPERATOR CAN BE INTEGRATED BY PARTS, YIELDS

$$\mathcal{L}\{E_s(x,y)\} = \left(\frac{-1}{4\omega\varepsilon_0}\right)(k_0^2+w^2)\,\mathcal{L}\left\{\int_{-\infty}^{\infty} H_0^{(2)}\left(k_0\{x^2+y^2\}^{1/2}\right)\right\}\hat{J}_-(w) \tag{29}$$

SINCE

$$H_0^{(2)}\left(k_0\{x^2+y^2\}^{1/2}\right) = \frac{1}{2\pi}\int_C \frac{e^{-i\xi x-(\xi^2-k_0^2)^{1/2}|y|}}{2(\xi^2-k_0^2)^{1/2}}\,d\xi$$

FROM EQ (4.6) OF CHAPT. 4, ONE HAS

$$\mathcal{L}\left\{\int_{-\infty}^{\infty} H_0^{(2)}\left(k_0\{x^2+y^2\}^{1/2}\right)\right\} =$$

ex for $-\frac{j}{4}H_0^2$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_C \frac{e^{-(\xi^2-k_0^2)^{1/2}|y|}\,e^{-(i\xi+w)x}}{2(\xi^2-k_0^2)^{1/2}}\,d\xi\,dw$$

$$= \frac{1}{2\pi}\,2\pi i\int_C \frac{e^{-(\xi^2-k_0^2)^{1/2}|y|}}{2(\xi^2-k_0^2)^{1/2}}\,\delta(i\xi+w)\,d\xi$$

$$= \frac{e^{-(-w^2-k_0^2)^{1/2}|y|}}{2(-w^2-k_0^2)^{1/2}} = \frac{e^{-j\sqrt{}\,|y|}}{2j\sqrt{}}$$

$$= \frac{e^{-i(w^2+k_0^2)^{1/2}|y|}}{2i(w^2+k_0^2)^{1/2}}$$ *Transform of* $\frac{-j}{4}H_0^2 = \frac{H_0^2}{4j}$

SUBSTITUTING THIS EXPRESSION AND EQ.(28) INTO EQ.(29) GIVES

← *sign error in K will absorb this negative sign*

$$\mathcal{L}\{E_s(x,y)\} = \left(\frac{-1}{4\omega\varepsilon_0}\right)\left(-\frac{K}{2}\right)\frac{(w^2+k_0^2)^{1/2}e^{-i(w^2+k_0^2)^{1/2}|y|}}{2i\cancel{(w^2+k_0^2)^{1/2}}(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)(w-ik_0)^{1/2}} \times 4j$$

$$= \left(\frac{iK}{16\omega\varepsilon_0}\right)\frac{(w+ik_0)^{1/2}\cancel{(w-ik_0)^{1/2}}e^{-i(w^2+k_0^2)^{1/2}|y|}}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)\cancel{(w-ik_0)^{1/2}}}$$

$$= \left(\frac{iK}{16\omega\varepsilon_0}\right)\frac{(w+ik_0)^{1/2}e^{-i(w^2+k_0^2)^{1/2}|y|}}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)} \times 4j$$

TAKING THE INDIRECT LAPLACE TRANSFORM OF THIS EXPRESSION FINALLY GIVES FOR THE SCATTERED FIELD

$$E_s(x,y) = \frac{K}{32\pi\omega\varepsilon_0}\int_C \frac{(w+ik_0)^{1/2}e^{-i(w^2+k_0^2)^{1/2}|y|+wx}}{(ik_0-ik_0\sin\theta)^{1/2}(w+ik_0\sin\theta)}\,dw \times 4j \quad (30)$$

WHERE C IS THAT LOCUS OF POINTS II SUCH THAT

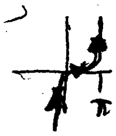
$$\mathrm{II} \in \{u \mid -k_0'' < u < -k_0''\sin\theta\}$$

USING THE CHANGE OF VARIABLES

$w = -ik_0\cos\phi$, $\left.\begin{matrix}x\\ y\end{matrix}\right\} = r\left\{\begin{matrix}\cos\psi\\ \sin\psi\end{matrix}\right.$

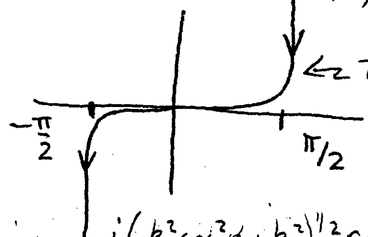
$\phi=\pi,\ w = ik_0$

$\phi=0,\ w=-ik_0$

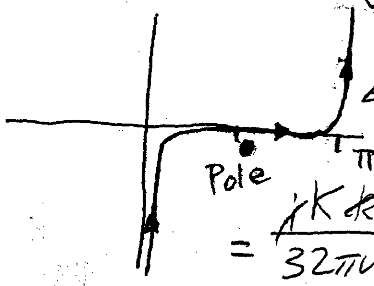


$\frac{\pi}{2}-\theta=\psi$

EQ. (30) BECOMES



$$E_s(x,y) = \left(\frac{iKk_0}{32\pi\omega\varepsilon_0}\right)\int_C \frac{(ik_0 - ik_0\cos\phi)^{1/2}\, e^{-i(-k_0^2\cos^2\phi+k_0^2)^{1/2} r\sin\psi - ik_0 r\cos\phi\cos\psi}}{(ik_0-ik_0\sin\theta)^{1/2}(-ik_0\cos\phi+ik_0\sin\theta)}\sin\phi\,d\phi$$



$$= \frac{\cancel{i}K\cancel{k_0}}{32\pi\omega\varepsilon_0}\int_C \frac{(1-\cos\phi)^{1/2}\, e^{-irk_0(\sin\phi\sin\psi-\cos\phi\cos\psi)}}{(1-\sin\theta)^{1/2}\,\cancel{ik_0}(\sin\theta-\cos\phi)}\sin\phi\,d\phi$$

$$=\frac{K}{32\pi\omega\varepsilon(1-\sin\theta)^{1/2}}\int_C\frac{(2)^{1/2}\sin(\frac{\phi}{2})\, e^{-ik_0 r\cos(\phi-\psi)}}{(\sin\theta-\cos\phi)}\sin\phi\,d\phi$$

$$=\frac{(2)^{1/2}K}{32\pi\omega\varepsilon_0(1-\sin\theta)^{1/2}}\int\frac{e^{-ik_0 r\cos(\phi-\psi)}}{(\sin\theta-\cos\phi)}\sin(\tfrac{\phi}{2})\sin\phi\,d\phi$$

BUT SINCE $\sin(\frac{\phi}{2})\sin\phi = \frac{1}{2}\left[\cos(\frac{\phi}{2})-\cos(\frac{3\phi}{2})\right]$

THE LAST LINE BECOMES

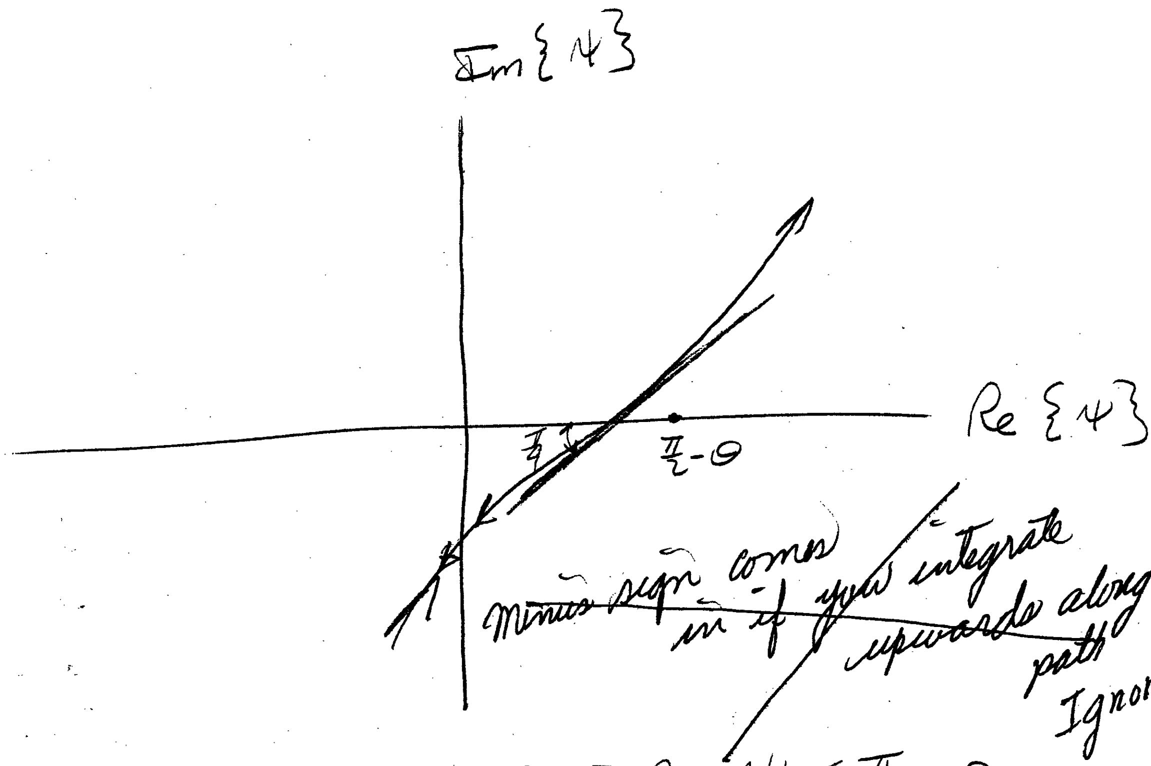
$$E_s(x,y) = \frac{(2)^{1/2} K}{64\pi\omega\varepsilon_0 (1-\sin\theta)^{1/2}} \left[\int_C^{-\frac{\pi}{2}+\theta} \frac{e^{-ik_0 r\cos(\phi-\psi)}}{(\sin\theta - \cos\phi)} \cos\left(\frac{\phi}{2}\right) d\phi - \right.$$

$$\left. - \int_C \frac{e^{-ik_0 r\cos(\phi-\psi)}}{(\sin\theta - \cos\phi)} \cos\left(\frac{3\phi}{2}\right) d\phi \right] \tag{31}$$

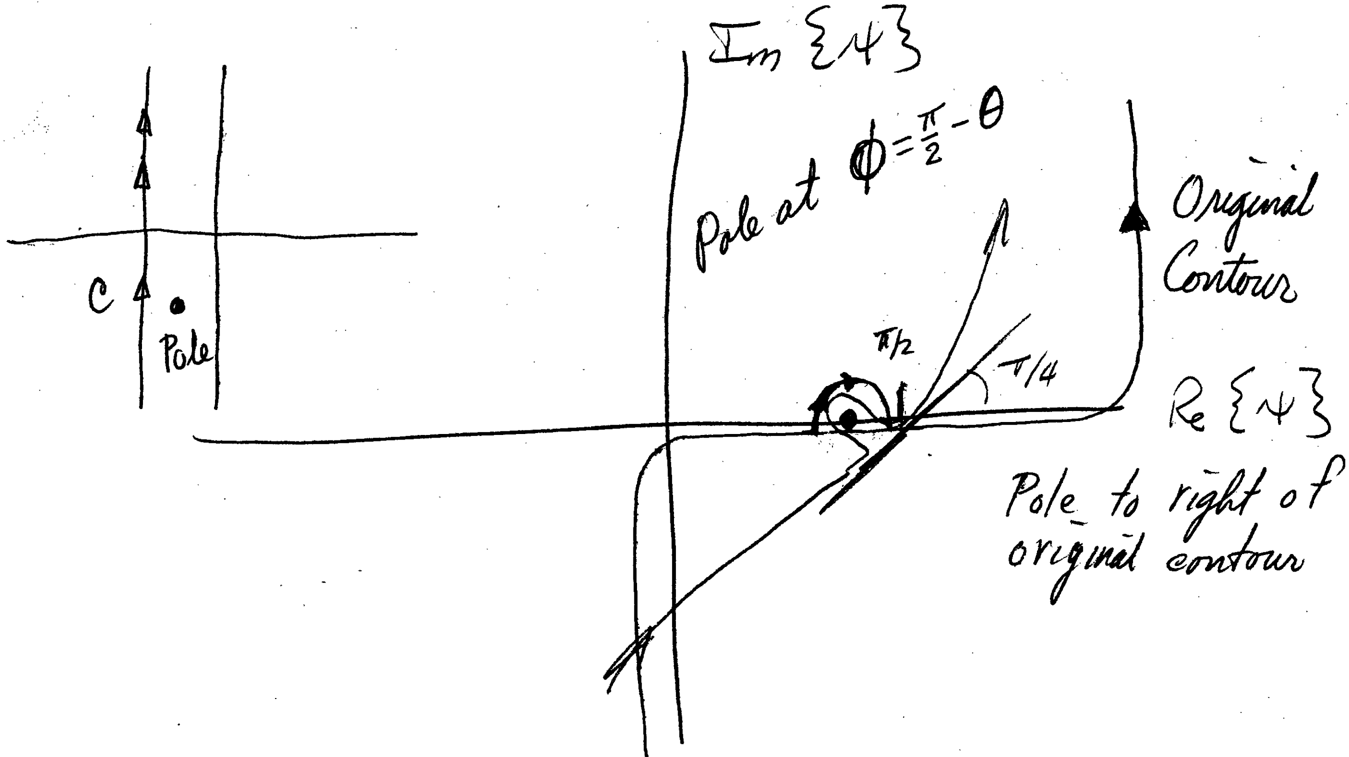
BOTH TERMS WITHIN THE BRACKETS HAVE A SADDLE POINT AT $\phi = \psi$ AND A POLE AT $\phi = \frac{\pi}{2} - \theta$. THERE ARE THUS TWO REGIONS OF SOLUTION: ONE FOR $\psi < \frac{\pi}{2} - \theta$ WHERE THE POLE IS NOT CROSSED AND ONE FOR $\psi > \frac{\pi}{2} - \theta$ WHERE THE POLE IS CROSSED AND AN EXTRA RESIDUE TERM IS PICKED UP. (SEE NEXT PAGE)

1.) <u>SOLUTION OF EQ. (31) FOR $\psi < \frac{\pi}{2} - \theta$</u>

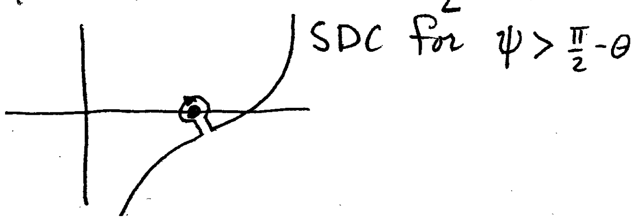
APPLYING THE SADDLE POINT METHOD TO THE EVALUATION OF EQ. (31) GIVES, BY INSPECTION



SOLUTION CONTOUR FOR $\psi < \frac{\pi}{2} - \theta$



SOLUTION CONTOUR FOR $\psi > \frac{\pi}{2} - \theta$



$$E_s(x,y) = \frac{(2)^{1/2} K}{\cancel{64}_{32}\, \pi \omega \varepsilon_0 (1-\sin\theta)^{1/2}} \cancel{(2)} \left(\frac{\pi}{k_0 r}\right)^{1/2} e^{-i(k_0 r - \frac{\pi}{4})} \left[\frac{\cos\left(\frac{\psi}{2}\right)}{\sin\theta - \cos\psi} - \right.$$

$$\left. - \frac{\cos\left(\frac{3\psi}{2}\right)}{\sin\theta - \cos\psi}\right]$$

$$= \left(\frac{K e^{-i(k_0 r - \frac{\pi}{4})}}{16\sqrt{2\pi}\, \omega \varepsilon_0 (1-\sin\theta)^{1/2}}\right)\left(\frac{1}{k_0 r(\sin\theta - \cos\psi)}\right) \cdot$$

$$\left(\cos\left(\frac{\psi}{2}\right) - \cos\left(\frac{3\psi}{2}\right)\right)$$

If $\psi = \pi/2$
$\cos \pi/4 - \cos \frac{3\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$
$2 \sin^2 \frac{\pi}{2} = 2$, But $2 \sin \frac{\pi}{4} \sin \frac{\pi}{2} = \sqrt{2}$

$= 2 \sin \frac{\psi}{2} \sin \psi$ from Pg 22

BUT SINCE $\cos\left(\frac{\psi}{2}\right) - \cos\left(\frac{3\psi}{2}\right) = -2\sin(\psi)\sin(-\psi) =$
$= 2\sin^2\psi$ ONE FINALLY GETS, UPON EMPLOYING THE DEFINITION FOR THE PARAMETER K (Pg 5),

$$\boxed{E_s(x,y) = \left(\frac{H_0 k_0 (4j)}{2\sqrt{2\pi}\, \omega \varepsilon_0}\right)\left(\frac{e^{-i(k_0 r - \frac{\pi}{4})}}{k_0 r}\right)\left(\frac{\sin^2\psi}{\sin\theta - \cos\psi}\right)\left(\frac{\cos\theta}{(1-\sin\theta)^{1/2}}\right)}$$

FOR $\psi < \frac{\pi}{2} - \theta$

2.) <u>SOLUTION OF EQ (31) FOR $\psi > \frac{\pi}{2} - \theta$</u>

IN THIS PARTICULAR CASE, THE POLE AT $\frac{\pi}{2} - \theta$ IS CROSSED AS SHOWN IN THE SECOND FIGURE ON Pg 24. EVALUATION OF EQ. (31) PICKS UP

THE POLE AND, AGAIN USING THE SADDLE POINT METHOD RESULTS IN

$$E_s(x,y) = \frac{(2)^{1/2} K}{\underset{32}{\cancel{64}} \pi \omega \varepsilon_0 (1-\sin\theta)^{1/2}} \cancel{(2)} \left(\frac{\pi}{k_0 r}\right)^{1/2} \Bigg\{ e^{-i(k_0 r - \frac{\pi}{4})} \left[\frac{\cos\left(\frac{\psi}{2}\right)}{\sin\theta - \cos\psi} - \frac{\cos\left(\frac{3\psi}{2}\right)}{\sin\theta - \cos\psi} \right] -$$

$$- \frac{e^{-ik_0 r \sin(\theta+\psi)}}{\cos\theta} \left[\cos\left[\tfrac{1}{2}\left(\tfrac{\pi}{2} - \theta\right)\right] - \cos\left[\tfrac{3}{2}\left(\tfrac{\pi}{2} - \theta\right)\right] \right] \Bigg\}$$

$$= \left(\frac{K}{16\sqrt{2\pi}\, \omega \varepsilon_0 (1-\sin\theta)^{1/2}}\right) \left(\frac{e^{-i(k_0 r - \frac{\pi}{4})}}{k_0 r}\right) \Bigg\{ \frac{\cos\left(\frac{\psi}{2}\right) - \cos\left(\frac{3\psi}{2}\right)}{\sin\theta - \cos\psi} -$$

$$- k_0 r \frac{e^{-ik_0 r \sin(\theta+\psi) + i(k_0 r - \frac{\pi}{4})}}{\cos\theta} \left(\cos\left[\tfrac{1}{2}\left(\tfrac{\pi}{2} - \theta\right)\right] - \cos\left[\tfrac{3}{2}\left(\tfrac{\pi}{2} - \theta\right)\right] \right) \Bigg\}$$

USING THE FACT THAT $\cos\left(\frac{\psi}{2}\right) - \cos\left(\frac{3\psi}{2}\right) = 2\sin^2\psi$ AND $\cos\left[\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right] - \cos\left[\frac{3}{2}\left(\frac{\pi}{2} - \theta\right)\right] = -2\sin\left[\frac{1}{2}\left\{\frac{1}{2}\left(\frac{\pi}{2} - \theta\right) + \frac{3}{2}\left(\frac{\pi}{2} - \theta\right)\right\}\right] \sin\left[\frac{1}{2}\left\{\frac{1}{2}\left(\frac{\pi}{2} - \theta\right) - \frac{3}{2}\left(\frac{\pi}{2} - \theta\right)\right\}\right] = 2\sin^2\left(\frac{\pi}{2} - \theta\right)$

AND THE DEFINITION FOR K GIVES

$$\boxed{E_s(x,y) = \left(\frac{H_0 k_0}{2\sqrt{2\pi}\,\omega\varepsilon_0}\right)\left(\frac{e^{-i\left(k_0 r-\frac{\pi}{4}\right)}}{k_0 r}\right)\left(\frac{\cos\theta}{(1-\sin\theta)^{1/2}}\right)\cdot\left\{\frac{\sin^2\psi}{\sin\theta-\cos\psi} - \frac{e^{-ik_0 r(\sin(\theta+\psi)-1)}\,e^{-i\frac{\pi}{4}}\,k_0 r}{\cos\theta}\,\sin^2\left(\frac{\pi}{2}-\theta\right)\right\}}$$

FOR $\psi > \frac{\pi}{2}-\theta$

Pole term is $-2\pi j\,\dfrac{4\,H_0 k_0\cos\theta\sqrt{2}}{64\pi\omega\epsilon_0\sqrt{1-\sin\theta}}\quad e^{-ik_0 r\cos\left(\frac{\pi}{2}-\theta-\psi\right)}$ $\quad e^{-jk_0 x\sin\theta - jk_0 y\cos\theta}$

$$\frac{\left(\cos\frac{\phi}{2}-\cos\frac{3\phi}{2}\right)\Big|_{\phi=\frac{\pi}{2}-\theta}}{\sin\left(\frac{\pi}{2}-\theta\right)}$$

$$-2\pi j\,\frac{H_0 k_0\sqrt{2}\cos\theta}{16\pi\omega\epsilon_0\sqrt{1-\sin\theta}}\;\frac{\overset{2\sin\left(\frac{\pi}{2}-\theta\right)\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\cancel{2\sin\left(\frac{\pi}{2}-\theta\right)}}}{\sin\left(\frac{\pi}{2}-\theta\right)} = -\frac{H_0 k_0\sqrt{2}\cos^{\cancel{2}}\theta}{8\pi\omega\epsilon_0\sqrt{1-\sin\theta}}\;2\pi j\,\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= \frac{-jH_0 k_0\sqrt{2}\,\cos^2\theta\;\cancel{\sin}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{4\,\omega\epsilon_0\sqrt{1-\sin\theta}\;\cos\theta}\,,\qquad \sqrt{1-\sin\theta} = \sqrt{1-\cos\left(\frac{\pi}{2}-\theta\right)} = \sqrt{2}\,\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$= -\frac{jH_0 k_0\cos\theta}{4\omega\epsilon_0}(\times 4j) = +\frac{H_0 k_0\cos\theta}{\omega\epsilon_0}$$

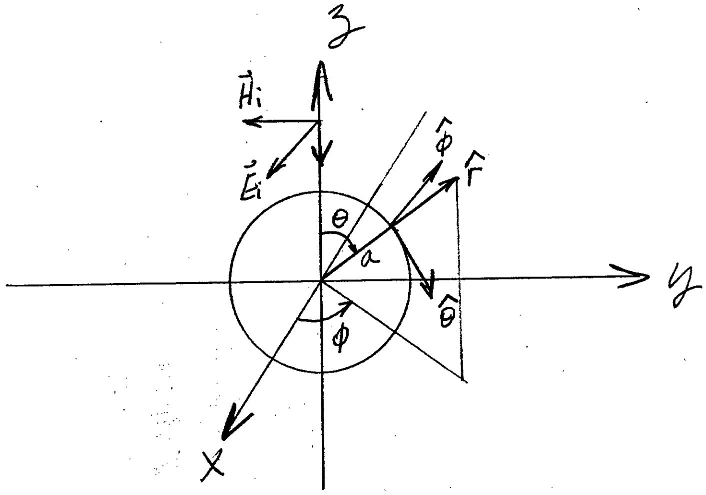
$$E_{xi} = -\frac{H_0 k_0\cos\theta}{\omega\epsilon_0}\,e^{-ik_0(x\sin\theta+y\cos\theta)}$$

and should be cancelled in shadow zone

This is a good check on algebra! With corrections as shown it checks out.

BOB MANNING
EEAP 563

SCATTERING FROM A CONDUCTING SPHERE



B

CONSIDER A PLANE WAVE INCIDENT FROM THE $-\hat{z}$ DIRECTION, SCATTERING FROM A CONDUCTING SPHERE OF RADIUS a AS SHOWN ON THE LEFT.

EXPANDING THE PLANE WAVE IN SPHERICAL COORDINATES USING THE $\vec{M}$ AND $\vec{N}$ VECTORS GIVES

$$\vec{E}_i = \hat{x} E_0 e^{ik_0 z} = \sum_{n=1}^{\infty} \sum_{m=1,-1} \left(\frac{iE_0}{k_0}\right) (i)^n \frac{2n+1}{2n(n+1)} \left\{ m \vec{M}_{nm}^{(0)} + \vec{N}_{nm}^{(0)} \right\} \quad (1)$$

WHERE

$$\vec{M}_{nm}^{(0)} \equiv \frac{-im P_n^{|m|} e^{-im\phi}}{r \sin\theta} \hat{j}_n(k_0 r) \hat{\theta} + \frac{P_n^{|m|'} \sin\theta \, e^{-im\phi}}{r} \hat{j}_n(k_0 r) \hat{\phi} \quad (2)$$

$$\vec{N}_{nm}^{(0)} = \frac{n(n+1) P_n^{|m|} e^{-im\phi}}{k_0 r^2} \hat{j}_n(k_0 r) \hat{r} - \frac{P_n^{|m|'} \sin\theta \, e^{-im\phi}}{r} \hat{j}_n'(k_0 r) \hat{\theta} - \frac{im P_n^{|m|} e^{-im\phi}}{r \sin\theta} \hat{j}_n'(k_0 r) \hat{\phi} \quad (3)$$

THE PRIMES IN EQS (2) + (3) DENOTE DIFFERENTIATION WITH RESPECT TO THE ARGUMENT ($k_0 r$ FOR j_n AND $\cos\theta$ FOR $P_n^{|m|}$).

THE SCATTERED ELECTRIC FIELD CAN BE GIVEN BY

$$\vec{E}_s = \sum_{n=1}^{\infty} \sum_{m=1,-1} \left\{ C_{nm} \vec{M}_{nm}^{(2)} + \frac{k_0}{i\omega\varepsilon_0} d_{nm} \vec{N}_{nm}^{(2)} \right\} \qquad (4)$$

WHERE $\vec{M}_{nm}^{(2)}$ AND $N_{nm}^{(2)}$ ARE THE SAME AS THOSE GIVEN IN EQS (2) AND (3) WITH THE j_n REPLACED BY $h_n^{(2)}$.

THE BOUNDARY CONDITION AT THE SURFACE OF THE CONDUCTING SPHERE IS THAT THE TOTAL TANGENTIAL ELECTRIC FIELD MUST VANISH:

$$\hat{r} \times (\vec{E}_i + \vec{E}_s) = 0 \qquad (5)$$

USING EQS (1) + (5), THIS GIVES

$$\sum_{n=1}^{\infty} \sum_{m=1,-1} \left(\frac{iE_0}{k_0}\right) (i)^n \frac{2n+1}{2n(n+1)} \left\{ m\, \hat{r} \times \vec{M}_{nm}^{(0)} + \hat{r} \times \vec{N}_{nm}^{(0)} \right\} =$$

$$= - \sum_{n=1}^{\infty} \sum_{m=1,-1} \left\{ C_{nm}\, \hat{r} \times \vec{M}_{nm}^{(2)} + \frac{k_0}{i\omega\varepsilon_0} d_{nm}\, \hat{r} \times \vec{N}_{nm}^{(2)} \right\}$$

EQUATING COEFFICIENTS AND COMPONENTS GIVES

$$K_n \, m \, \hat{r} \times \vec{M}_{nm}^{(0)} = - C_{nm} \, \hat{r} \times \vec{M}_{nm}^{(2)} \qquad (6)$$

$$K_n \, \hat{r} \times \vec{N}_{nm}^{(0)} = - \frac{k_0}{i\omega\varepsilon_0} d_{nm} \, \hat{r} \times N_{nm}^{(2)} \qquad (6)$$

WHERE $K_n \equiv \frac{E_0}{k_0} (i)^{n+1} \frac{2n+1}{2n(n+1)}$. EMPLOYING EQS (2) + (3) AND USING THE THE RELATIONS $\hat{r} \times \hat{\theta} = \hat{\phi}$ AND $\hat{r} \times \hat{\phi} = -\hat{\theta}$ GIVES

$$K_n m \left[\frac{\cancel{-im P_n^{|m|} e^{-im\phi}}}{\cancel{r \sin\theta}} j_n \hat{\phi} + \frac{P_n^{|m|'} \sin\theta \, e^{-im\phi}}{r} j_n \hat{\theta} \right] =$$

$$= - C_{nm} \left[\frac{\cancel{-im P_n^{|m|} e^{-im\phi}}}{\cancel{r \sin\theta}} h_n^{(2)} \hat{\phi} - \frac{P_n^{|m|'} \sin\theta \, e^{-im\phi}}{r} h_n^{2} \hat{\theta} \right] \qquad (8)$$

EQUATING VECTOR COMPONENTS GIVES, IN BOTH CASES

$$K_n \, m \, j_n = - C_{nm} h_n^{(2)}$$

$$\boxed{C_{nm} = - K_n m \frac{j_n(k_0 a)}{h_n^{(2)}(k_0 a)}} \qquad (9)$$

SIMILARLY, FROM EQ. (7)

$$K_n\left[-\frac{P_n^{|m|}\sin\theta\, e^{-im\phi}}{r}\, j_n'\,\hat{\phi} + \frac{im P_n^{|m|} e^{-im\phi}}{r\sin\theta}\, j_n'\,\hat{\theta}\right] =$$

$$= -\frac{k_0}{i\omega\varepsilon_0}\, d_{nm}\left[-\frac{P_n^{|m|}\sin\theta\, e^{-im\phi}}{r}\, h_n^{(2)\prime}\hat{\phi} + \frac{im P_n^{|m|} e^{-im\phi}}{r\sin\theta}\, h_n^{(2)\prime}\,\hat{\theta}\right]$$

AGAIN, EQUATING THE VECTOR COMPONENTS FIELDS

$$K_n\, j_n'(k_0a) = -\frac{k_0}{i\omega\varepsilon_0}\, d_{nm}\, h_n^{(2)\prime}(k_0a)$$

$$\boxed{d_{nm} = -\left(\frac{i\omega\varepsilon_0}{k_0}\right) K_n\, \frac{j_n'(k_0a)}{h_n^{(2)\prime}(k_0a)}} \qquad (10)$$

HENCE, ONE HAS FOR THE SCATTERED ELECTRIC FIELD, EQ. (4),

$$\boxed{\begin{gathered}\vec{E}_s = -\sum_{n=1}^{\infty}\sum_{m=1,-1} K_n\left\{ m\,\frac{j_n(k_0a)}{h_n^{(2)}(k_0a)}\,\vec{M}_{nm}^{(2)} + \frac{j_n'(k_0a)}{h_n^{(2)\prime}(k_0a)}\,\vec{N}_{nm}^{(2)}\right\}\\ K_n \equiv \frac{(i)^{n+1}E_0}{k_0}\left(\frac{2n+1}{2n(n+1)}\right)\end{gathered}}$$

(11)

RETURNING TO EQ. (4) AND CONSIDERING THE $n=1$ TERM, ONE HAS

$$\vec{E}_s = C_{1-1}\vec{M}_{1-1}^{(2)} + C_{11}\vec{M}_{11}^{(2)} + \frac{k_0}{i\omega\epsilon_0}\left[d_{1-1}\vec{N}_{1-1}^{(2)} + d_{11}\vec{N}_{11}^{(2)}\right] \quad (12)$$

$$\vec{H}_s = \frac{j}{k_0 Z_0}\nabla\times\vec{E}_s = \frac{j}{k_0 Z_0}\left[C_{1-1}k_0\vec{N}_{1-1} + C_{11}k_0\vec{N}_{11} \cdots\right]$$

GOING IMMEDIATELY TO THE FAR FIELD, ONE CAN NEGLECT THE $\hat{r}$ TERM IN EQ. (3). ALSO, TAKING $k_0 a$ TO BE SMALL, ONE CAN WRITE

$$\left.\begin{aligned} j_1(k_0 a) &\cong \frac{(k_0 a)^2}{3} - \frac{(k_0 a)^4}{30}, \\ j_1'(k_0 a) &\cong \frac{2}{3}(k_0 a) - \frac{2}{15}(k_0 a)^3 \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} h_1^2(k_0 a) &\cong \frac{i}{k_0 a} + \frac{i(k_0 a)}{2} + {}^{?}\frac{(k_0 a)^2}{3} \\ h_1^{2\prime}(k_0 a) &\cong \frac{-i}{(k_0 a)^2} + \frac{i}{2} + \frac{2}{3}(k_0 a) \end{aligned}\right\} \quad (14)$$

THUS, TO ORDER $(k_0 a)^3$, ONE HAS FROM EQ (9)

$$C_{1m} = -K_1 m i \frac{(k_0 a)^3}{3}$$

$$d_{1m} = -\frac{i\omega\epsilon_0}{k_0}K_n \frac{\left(\frac{2}{3}x - \frac{2}{15}x^3 \cdots\right)}{-\frac{i}{x^2} + \frac{i}{2} + \frac{2}{3}x}$$

$$= -\frac{i\omega\epsilon_0 K_n}{k_0}\left(\frac{2}{3}x - \frac{2}{15}x^3 \cdots\right) i x^2 \left(1 - \frac{x^2}{2} + \frac{2}{3}ix^3 \cdots\right)^{-1}$$

$$= \frac{\omega\epsilon_0 K_n}{k_0} x^2\left(\frac{2}{3}x - \frac{2}{15}x^3 \cdots\right)\left(1 + \frac{x^2}{2} - \frac{2}{3}ix^3 + \cdots\right)$$

$$= \frac{\omega\epsilon_0 K_n}{k_0} \left(\frac{2}{3}x^3 - \frac{2}{15}x^5 + \frac{x^5}{3} + \cdots\right)$$

AND FROM EQ. (10)

$$d_{1m} = -\frac{i\omega\epsilon_0}{k_0}K_1\, i\left(\frac{2}{3}\right)(k_0 a)^3$$

$$= \frac{\omega\epsilon_0}{k_0}K_1\,\frac{2}{3}(k_0 a)^3$$

AT THIS POINT, IT IS ALSO USEFUL TO NOTE THAT

$$P_1(x) = x$$

SO

$$P_1'(x) = (1-x^2)^{1/2}$$

AND

$$P_1''(x) = -\frac{x}{(1-x^2)^{1/2}}$$

THEREFORE,

$$P_1'(\cos\theta) = (1-\cos^2\theta)^{1/2} = \sin\theta$$

AND

$$P_1''(\cos\theta) = -\frac{\cos\theta}{\sin\theta}$$

USING WHAT HAS BEEN SAID AND DEVELOPED ABOVE, ONE CAN, BY INSPECTION, SUBSTITUTE EQS. (2) AND (3) INTO EQ. (12) TO OBTAIN

It is easier to find scattered ~~field~~ power
use $\vec{E}_s \times \vec{H}_s^*$ and orthogonality & normalization properties of $\vec{M}$ & $\vec{N}$ modes

$$\vec{E}_s = -\frac{iK_1(k_0a)^3}{3}\left[\left(\frac{1}{r\sin\theta}\right)(\sin\theta)\,e^{i\phi}\,h_1^{(2)}\,\hat{\theta} + \frac{\cos\theta}{r}e^{i\phi}h_1^{(2)}\hat{\phi}\right]$$

$$-\frac{iK_1(k_0a)^3}{3}\left[-\left(\frac{1}{r\sin\theta}\right)\sin\theta\,e^{-i\phi}h_1^{(2)}\hat{\theta} - \frac{\cos}{r}e^{-i\phi}h_1^{(2)}\hat{\phi}\right]$$

$$+\frac{1}{i}\left(\frac{2}{3}\right)(k_0a)^3\left[+\frac{i\cos\theta\, e^{i\phi}h_1^{(2)'}}{r}\hat{\theta} + \frac{\cos\theta\, e^{-i\phi}h_1^{(2)'}}{r}\hat{\theta} +\right.$$

$$\left. + \frac{i e^{i\phi}h_1^{(2)'}}{r}\hat{\phi} - \frac{i e^{-i\phi}h_1^{(2)}}{r}\hat{\phi}\right]$$

USING THE APPROXIMATIONS

$$h_1^{(2)}(k_0r) \sim -e^{-ik_0r}$$

AND

$$h_1^{(2)'} \sim i\,e^{-ik_0r}$$

ONE GETS

$$\vec{E}_s = -\frac{iK_1(k_0a)^3}{3}\left[\left(-\left(\frac{i}{r}e^{-ik_0r}e^{i\phi} - \frac{i}{r}e^{-ik_0r}e^{-i\phi}\right)\right)\hat{\theta} -\right.$$

$$\left. + \frac{\cos\theta}{r}e^{-ik_0r}\left(e^{i\phi}+e^{-i\phi}\right)\hat{\phi}\right] +$$

$$\cancel{\left(\frac{1}{i}\right)}K_1\left(\frac{2}{3}\right)(k_0a)^3\left[+\frac{\cos\theta}{r}\cancel{i}\,e^{-ik_0r}\left(e^{i\phi}+e^{-i\phi}\right)\hat{\theta} +\right.$$

$$\left. + \frac{(\cancel{i})\,i\,e^{-ik_0r}}{r}\left(e^{i\phi} - e^{-i\phi}\right)\hat{\phi}\right]$$

NOTING THAT $e^{i\phi} - e^{-i\phi} = 2i\sin\phi$ AND $e^{i\phi}+e^{-i\phi} = 2\cos\phi$, THIS BECOMES

$$\vec{E}_s = \frac{iK_1(k_0a)^3}{3} e^{-ik_0r}\Big[\frac{i}{r} 2i\sin\phi\,\hat{\theta} - \frac{\cos\theta}{r} 2\cos\phi\,\hat{\phi} -$$

$$-(i)2\Big[+\frac{i\cos\theta}{r} 2\cos\phi\,\hat{\theta} - \frac{2\sin\phi}{r}\hat{\phi}\Big]\Big]$$

$$= \frac{iK_1(k_0a)^3}{3}\frac{e^{-ik_0r}}{r}\Big[-2\sin\phi\,\hat{\theta} - 2i\cos\theta\cos\phi\,\hat{\phi} +$$

$$+(2)\big[2\cos\theta\cos\phi\,\hat{\theta} - 2\sin\phi\,\hat{\phi}\big]\Big]$$

TAKING THE COMPLEX CONJUGATE DOT PRODUCT OF TH EXPRESSION GIVES

$$|\vec{E}_s|^2 = \frac{|K_1|^2(k_0a)^6}{9}\left(\frac{1}{r^2}\right)(4)^2\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]$$

You have some missing terms

NOW ONE HAS THAT

$$|K_1|^2 = \left|\frac{E_0}{k_0} i^2 \frac{3}{4}\right|^2 = \frac{|E_0|^2}{k_0^2}\frac{9}{16}$$

$$\therefore |\vec{E}_s|^2 = \frac{|E_0|^2}{k_0^2}\left(\cancel{\frac{9}{16}}\right)\left(\cancel{\frac{16}{9}}\right)(k_0a)^6\frac{1}{r^2}\Big[\qquad\Big]$$

$$= \frac{|E_0|^2(k_0a)^6}{k_0^2r^2}\left[\cos^2\theta\cos^2\phi + \sin^2\phi\right]$$

ONE CAN NOW OBTAIN THE DIFFERENTIAL CROSS SECTION

$$\sigma(\theta,\phi) = \frac{r^2 |E_s|^2}{|E_0|^2}$$

$$= \frac{(k_0 a)^6}{k_0^2}\left[\cos^2\theta \cos^2\phi + \sin^2\phi\right]$$

THE TOTAL CROSS SECTION THEN BECOMES

$$\sigma_T = \int_0^{2\pi}\int_0^{\pi} \sigma(\theta,\phi) \sin\theta \, d\theta \, d\phi$$

$$= \frac{(k_0 a)^6}{k_0^2}\left\{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \cos^2\phi \, d\theta \, d\phi + \right.$$

$$\left. + \int_0^{2\pi}\int_0^{\pi} \sin^2\phi \sin\theta \, d\theta \, d\phi\right\}$$

$$= \frac{(k_0 a)^6}{k_0^2}\left\{\pi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta + \right.$$

$$\left. + \int_0^{\pi} \sin\theta \, d\theta\right\}$$

$$= \frac{(k_0 a)^6}{k_0^2}\left\{\pi\left(-\frac{\cos^3\theta}{3}\Big|_0^{\pi} + (-\cos\theta\Big|_0^{\pi}\right)\right\}$$

$$= \frac{(k_0 a)^6}{k_0^2}\pi\left\{-\frac{1}{3}(-1-1) - (-1-1)\right\}$$

$$= \frac{(k_0 a)^6}{k_0^2}\pi\left\{\underbrace{\frac{2}{3}+2}_{\frac{8}{3}}\right\}$$

THUS

$$\sigma_T = \frac{(k_0 a)^6}{k_0^2} \pi\left(\frac{8}{3}\right)$$

$$= \frac{8\pi}{3} k_0^4 a^6$$

You should have gotten $\frac{10}{3}$ factor

TO BE ABLE TO EMPLOY THE FORWARD SCATTERING THEOREM TO CALCULATE σ_T, ONE MUST OBTAIN AN EXPRESSION, FROM THE ORIGINAL SCATTERING EQUATION ~~TO~~ THAT DESCRIBES THE ELECTRIC FIELD (ALONG THE $\hat{x}$ AXIS) IN THE $\theta = \pi$ DIRECTION. SINCE THIS INVOLVES MORE ALGEBRA AND THE EXPRESSION EXISTS IN THE NOTES, IT WILL BE EMPLOYED. HOWEVER, IT REQUIRES AN IMAGINARY PART OF THE SCATTERED FIELD THAT ONLY EXISTS IF THE d_{lm} COEFFICIENTS ARE TAKEN TO $(k_0 a)^6$. TO THIS END, FROM EQS. (13) + (14) ONE HAS THAT

$$d_{lm} = + K_1 \frac{i\omega\varepsilon_0}{k_0}\left(\frac{2}{3}\right)(k_0 a)^3\, i\left[1 - i\,\frac{2}{3}(k_0 a)^3\right]$$

ONE CAN NOW EVALUATE THE SCATTERING CROSS SECTION VIA THE FORWARD SCATT. THM.

$$\sigma_T = -\frac{4\pi}{\varepsilon_0 k_0} \operatorname{Im}\{\vec{a}_x \cdot \vec{F}\} \qquad \text{(AS DERIVED IN NOTES)}$$

WHERE

$$\operatorname{Im}\{\vec{a}_x \cdot \vec{F}\} = \left(\frac{k_0}{\omega\varepsilon_0}\right) \operatorname{Im}\{d_{11} + d_{1-1}\}$$

But

$$d_{11} + d_{1-1} = 2\,\frac{i\omega\varepsilon_0}{k_0} K_1 \left(\frac{2}{3}\right)(k_0 a)^3 i\left[1 - i\left(\frac{2}{3}\right)(k_0 a)^3\right]$$

$$= -\frac{2\omega\varepsilon_0}{k_0} K_1 \left(\frac{2}{3}\right)(k_0 a)^3 \left[1 - i\left(\frac{2}{3}\right)(k_0 a)^3\right]$$

$$Im\left\{d_{11} + d_{1-1}\right\} = \frac{2\omega\varepsilon_0}{k_0} K_1 \left(\frac{2}{3}\right)^2 (k_0 a)^6$$

$$\sigma_T = -\left(\frac{4\pi}{E_0 k_0}\right)\left(\frac{2\omega\varepsilon_0}{k_0}\right) K_1 \left(\frac{2}{3}\right)^2 (k_0 a)^6 \left(\frac{k_0}{\omega\varepsilon_0}\right)$$

But $K_1 = \frac{(i)^2 E_0}{k_0}\left(\frac{3}{4}\right)$ so

$$\sigma_T = \left(\frac{4\pi}{\cancel{E_0} k_0}\right)\left(\frac{2\cancel{\omega\varepsilon_0}}{\cancel{k_0}}\right)\left(\frac{\cancel{k_0}}{\cancel{\omega\varepsilon_0}}\right)\left(\frac{\cancel{E_0}}{k_0}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{2}{\cancel{3}}\right)\left(\frac{2}{3}\right)(k_0 a)^6$$

$$= \frac{\pi(2)(4)}{3k_0^2}(k_0 a)^6$$

$$\boxed{\sigma_T = \frac{8\pi}{3} k_0^4 a^6}$$

limit for dielectric sphere with $K \to \infty$

There is also a contribution from C_{1m}, Re $C_{1m} = -m \frac{(k_0 a)^6 E_0}{12 k_0}$

Conducting sphere requires $\vec{n} \cdot \vec{H} = 0$ on surface, a magnetic dipole moment is produced which changes σ_T to $\frac{10\pi}{3} k_0^4 a^6$