Evaluation of Scattered Field

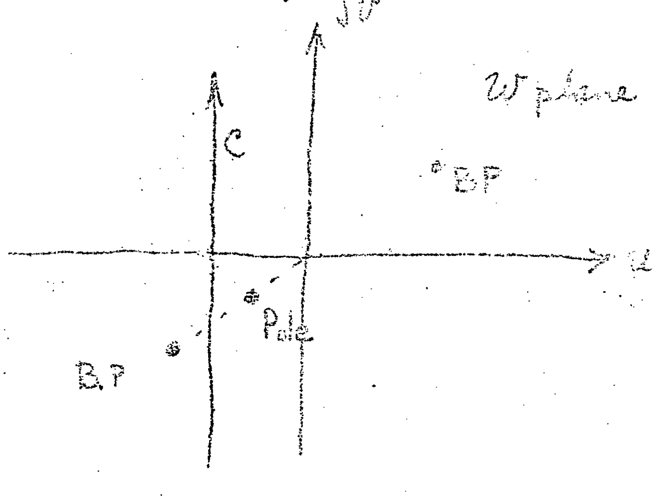
The scattered field is given by

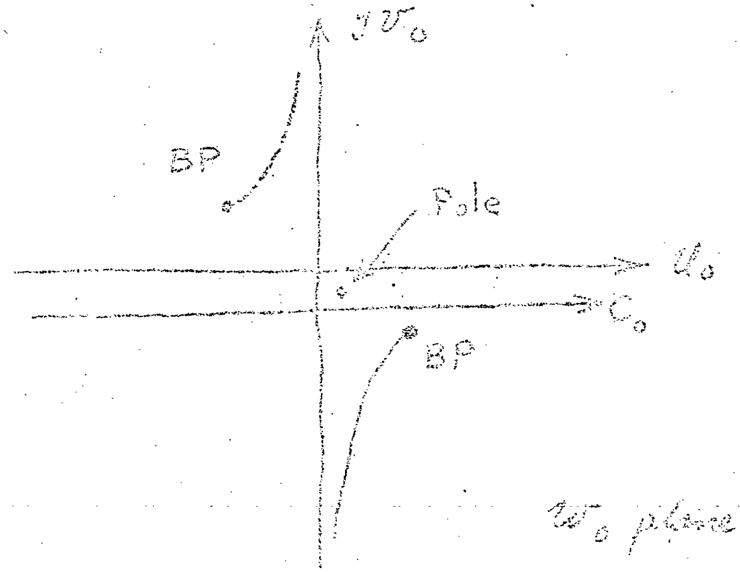
$$\phi_s = \frac{1}{2\pi j}\int_C A\sqrt{jk_0 - jk_0\sin\theta}\;\frac{e^{wx - j\sqrt{w^2+k_0^2}\,|y|}}{(w+jk_0\sin\theta)\sqrt{w+jk_0}}\,dw$$

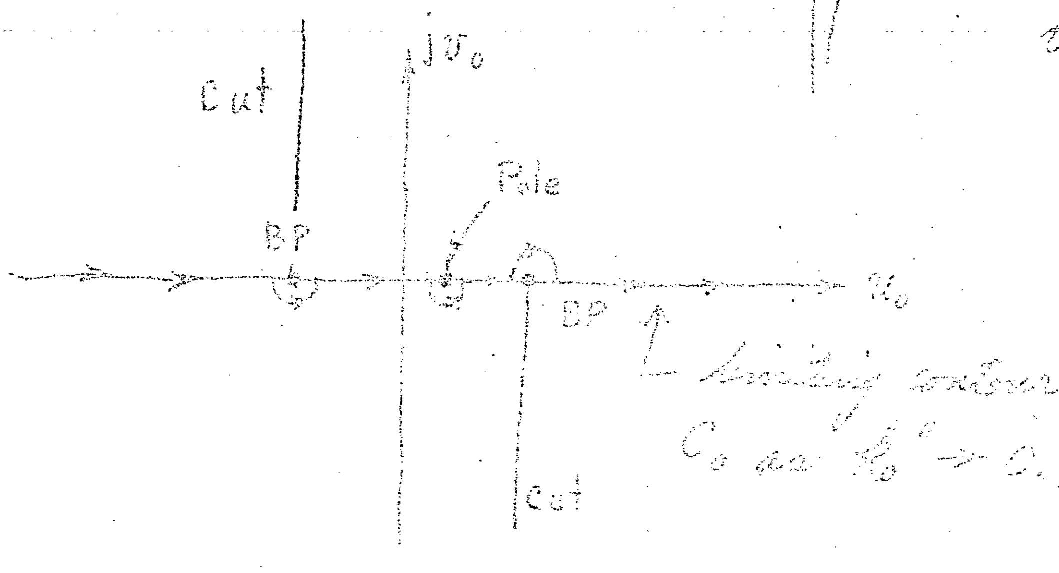
To conform with notation used in Appendix A put $w = -jw_0$ or $w_0 = jw$ to get

$$\phi_s = -\frac{A}{2\pi j}\sqrt{k_0(1-\sin\theta)}\int_{C_0}\frac{e^{-jw_0 x - j\sqrt{k_0^2-w_0^2}\,|y|}}{(w_0 - k_0\sin\theta)\sqrt{k_0-w_0}}\,dw_0$$

This is equivalent to rotating w plane 90° anti-clockwise.





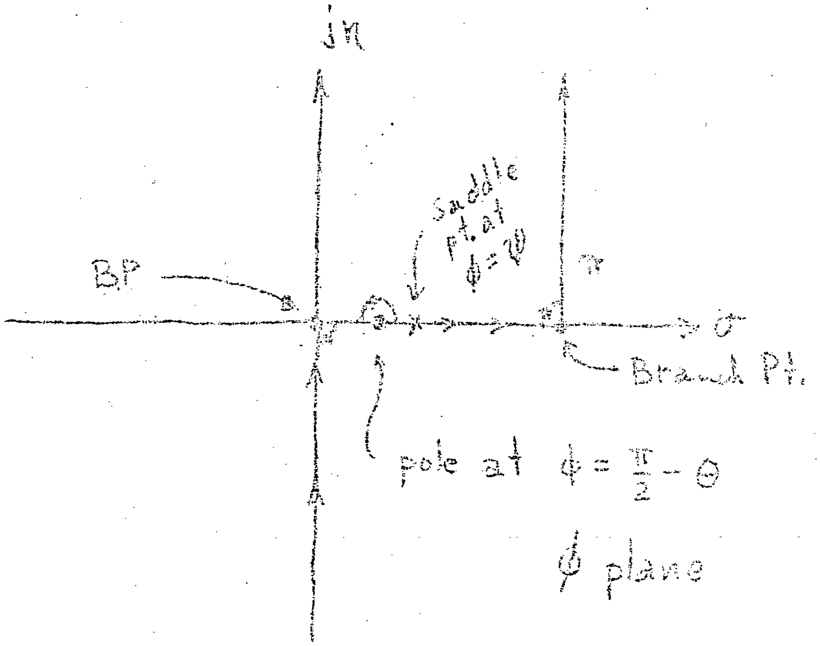


We will now transform to polar coordinates. Then $w_0 = k_0 \cos \phi$, $\beta = k_0 \sin\phi$, $x = r\cos\psi$, $y = r\sin\psi$, as per [illegible]

We then obtain $$\phi_s = \frac{A\sqrt{k_0(1-\sin\theta)}}{-2\pi j\sqrt{k_0}} \int_C \frac{e^{-jk_0 r\cos(\phi-\psi)} \sin\phi \, d\phi}{(\cos\phi - \sin\theta)\sqrt{1-\cos\phi}}$$

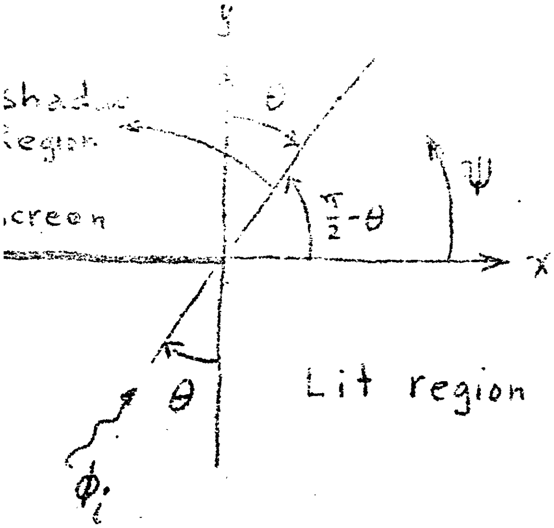
$$= \frac{A\sqrt{1-\sin\theta}}{-2\pi j}\sqrt{2}\, I \quad \text{where } I = \int_C \frac{e^{-jk_0 r\cos(\phi-\psi)} \cos\frac{\phi}{2}\, d\phi}{(\cos\phi - \sin\theta)}$$

since $1-\cos\phi = 2\sin^2 \phi/2$, $\sin\phi = 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}$.



The contour C in the ϕ plane is indented around the branch points at $\phi = 0, \pi$ and the pole at $\phi = \frac{\pi}{2} - \theta$ to conform with the indentations in the w_0 plane. Note, however, that indentations at $\phi = 0, \pi$ may be removed because the integrand does not have branch points in the ϕ plane.

The saddle point occurs at $\phi = \psi$. If we deform C into a steepest descent contour SDC passing through ψ at 45°. the pole at θ is crossed whenever $\psi > \frac{\pi}{2} - \theta$. The pole gives rise to a modified diffraction effect at the boundary separating the lit region and shadow region in physical space. The pole is crossed when the observation angle ψ carries us across the shadow boundary at $\psi = \frac{\pi}{2} - \theta$.



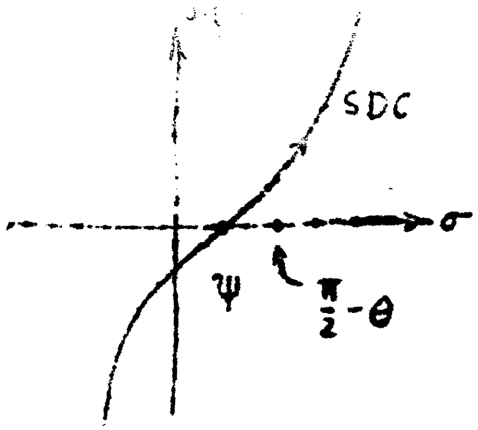
Case I, $\psi \neq \frac{\pi}{2} - \theta$

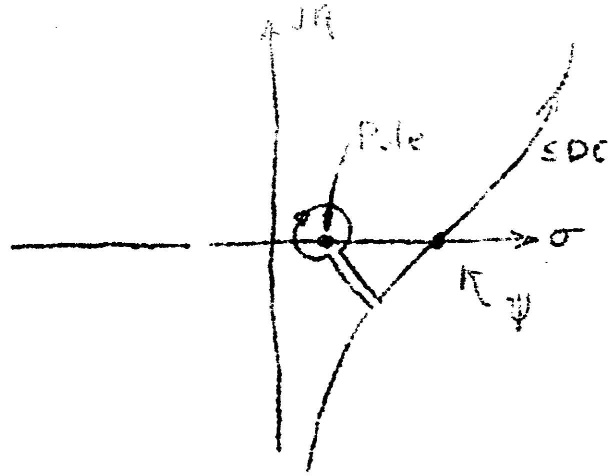
If we consider ψ different from $\frac{\pi}{2} - \theta$ the standard saddle point method may be used to give (pg. A4)

$$I = 2\sqrt{\frac{\pi}{2k_0 r}}\, e^{-jk_0 r + j\frac{\pi}{4}} \left\{ \frac{\cos \psi/2}{\cos\psi - \sin\theta} \right\}$$

$$\phi_s = -\frac{A\sqrt{1-\sin\theta}}{\pi j} \sqrt{\frac{\pi}{k_0 r}}\, e^{-jk_0 r + j\frac{\pi}{4}} \left\{ \frac{\cos \psi/2}{\cos\psi - \sin\theta} \right\}$$

when the pole is not crossed, i.e. $0 < \psi < \frac{\pi}{2} - \theta$. When $\psi > \frac{\pi}{2} - \theta$ the pole is crossed and we get an additional residue contribution. Thus we have





$$\phi_s = +Aj\sqrt{\frac{1}{k_0 r \pi}}\, e^{-jk_0 r + j\pi/4} \; \frac{\sqrt{1-\sin\theta}\;\cos \psi/2}{\cos\psi - \sin\theta}$$

$$- \frac{A\sqrt{1-\sin\theta}}{\cos\theta}\,\sqrt{2}\,\cos\frac{\pi-2\theta}{4}\; e^{-jk_0 r\sin(\theta+\psi)}\; U(\psi - \tfrac{\pi}{2} + \theta)$$

where $U(x) = 1,\ x > 0$ and is the unit step function.
$\qquad\qquad = 0,\ x < 0$

Note that the residue of $(\cos\phi - \sin\theta)^{-1}$ at the pole is given by $\left[\frac{d}{d\phi}(\cos\phi - \sin\theta) \Big|_{\frac{\pi}{2}-\theta} \right]^{-1} = \frac{-1}{\cos\theta}$.

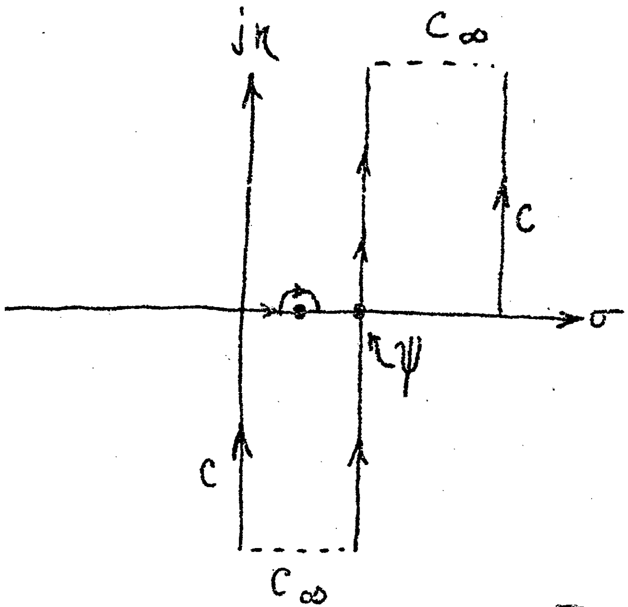
Since the original integral for ϕ_s involves $|y|$ the scattered field for $y<0$ is the same as for $y>0$, thus

$\phi_s(r, \psi)$ for $0 \le \psi \le \pi$

$\phi_s(r, -\psi)$ for $-\pi \le \psi \le 0$

Case 2, ψ near or equal to $\frac{\pi}{2}-\theta$

When ψ approaches the pole at $\frac{\pi}{2}-\theta$ the modified saddle point method must be used. A simple expression for ϕ_s is obtained only after considerable manipulation. First consider a deformation of the contour C into a contour along a line parallel to the η axis at $\sigma=\psi$ in the ϕ plane. Along C_∞, the real part of



$-jk_0 r\cos(\sigma+j\eta-\psi)$ is negative so the integral along C_∞ vanishes and the shift in contour can be made to give (note that $\phi=\psi+j\eta$, $d\phi=j\,d\eta$ along new contour)

$$I=\int_{-\infty}^{\infty}\frac{e^{-jk_0 r\cos j\eta}\cos\frac{\psi+j\eta}{2}}{\cos(\psi+j\eta)-\sin\theta}\,j\,d\eta$$

The range in η is a symmetrical one so only the even part of the integrand gives a non-vanishing result. We will use this property later. If the pole is crossed ($\psi>\frac{\pi}{2}-\theta$) a residue term must be added to the above expression for I.

If we use the identity $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$,
put $\alpha = \frac{\pi}{2} - \theta$, $\sin\theta = \cos\alpha$ we obtain

$$\frac{1}{\cos(\psi + j\eta) - \cos\alpha} = \frac{-1}{2 \sin \frac{\psi + j\eta + \alpha}{2} \sin \frac{\psi + j\eta - \alpha}{2}}$$

We now use a partial fraction type expansion to write the above in the form $\dfrac{B_1}{\sin \frac{\psi + j\eta + \alpha}{2}} + \dfrac{B_2}{\sin \frac{\psi + j\eta - \alpha}{2}}$

and find that

$$\frac{\cos \frac{\psi + j\eta}{2}}{\cos(\psi + j\eta) - \cos\alpha} = \frac{1}{4 \sin \frac{\alpha}{2}} \left\{ \frac{1}{\sin \frac{j\eta + \psi + \alpha}{2}} - \frac{1}{\sin \frac{j\eta + \psi - \alpha}{2}} \right\}$$

The even part of this is

$$\frac{1}{8 \sin \frac{\alpha}{2}} \left\{ \frac{1}{\sin \frac{j\eta + \psi + \alpha}{2}} + \frac{1}{\sin \frac{-j\eta + \psi + \alpha}{2}} - \left(\begin{array}{c} \text{same expression with} \\ \alpha \text{ replaced by } -\alpha \end{array} \right) \right\}$$

$$= \frac{1}{4 \sin \frac{\alpha}{2}} \left\{ \frac{\sin \frac{\psi + \alpha}{2} \cos j\frac{\eta}{2}}{\sin \frac{j\eta + \psi + \alpha}{2} \sin \frac{\psi + \alpha - j\eta}{2}} - (-\alpha) \right\}$$

$$= \frac{1}{2 \sin \frac{\alpha}{2}} \left\{ \frac{\sin \frac{\psi + \alpha}{2} \cos j\frac{\eta}{2}}{\cos(\psi + \alpha) - \cos j\eta} - (-\alpha) \right\}$$

after using the identity involving $\cos x - \cos y$ again.

Our integral I becomes $I = I_1(\alpha) + I_1(-\alpha) + U(\psi - \frac{\pi}{2} + \theta)$ × (residue term) where

$$I_1(\alpha) = \frac{\sin \frac{\psi+\alpha}{2}}{2 \sin \frac{\alpha}{2}} \int_{-\infty}^{\infty} \frac{e^{-jk_0 r \cos j\eta} \cos j\frac{\eta}{2}}{\cos(\psi+\alpha) - \cos j\eta} \, j\,d\eta$$

If we go over to the SDC in the ϕ plane ($\phi = \psi + j\eta$ or $j\eta = \phi - \psi$) we get

$$I_1(\alpha) = \frac{\sin \frac{\psi+\alpha}{2}}{2 \sin \frac{\alpha}{2}} \int_{SDC} \frac{e^{-jk_0 r \cos(\phi-\psi)} \cos \frac{\phi-\psi}{2}}{\cos(\psi+\alpha) - \cos(\phi-\psi)} \, d\phi$$

As in Appendix A let $s^2 = 2j(\cos(\phi-\psi) - 1)$ or $s = 2e^{-j\frac{\pi}{4}} \sin \frac{\phi-\psi}{2}$ so that SDC becomes real axis in s plane. Since $ds = e^{-j\frac{\pi}{4}} \cos \frac{\phi-\psi}{2} \, d\phi$ we get

$$I_1(\alpha) = \frac{\sin \frac{\psi+\alpha}{2}}{2 \sin \frac{\alpha}{2}} \int_{-\infty}^{\infty} \frac{e^{-jk_0 r + j\frac{\pi}{4}} \, e^{-k_0 r \frac{s^2}{2}}}{\cos(\psi+\alpha) - 1 + j\frac{s^2}{2}} \, ds$$

In the Appendix it was noted that

$$I_p = \int_{-\infty}^{\infty} \frac{e^{-k_0 r s^2/2}}{s - s_0} \, ds = j\pi e^{-\frac{k_0 r s_0^2}{2}} \left[1 + \operatorname{erf}\left(j\sqrt{\tfrac{k_0 r}{2}}\, s_0\right)\right], \quad \operatorname{Im} s_0 > 0.$$

But integration is over a symmetrical range in S so only the even part of $(S-S_0)^{-1}$ will contribute. The latter is

$$\frac{1}{2}\left\{\frac{1}{S-S_0}+\frac{1}{-S-S_0}\right\}=\frac{-S_0}{-S^2+S_0^2} \quad \text{so} \quad I_p = S_0\int_{-\infty}^{\infty}\frac{e^{-k_0 r S^2/2}}{S^2-S_0^2}\,dS \quad \text{also.}$$

It is now apparent that I_1 can be evaluated exactly in terms of error functions and an asymptotic expansion is not required when ψ lies near the pole at $\frac{\pi}{2}-\theta$ or for other values of ψ either. Thus

$$I_1(\alpha)=\frac{S_0 \sin\frac{\psi+\alpha}{2}\, e^{-jk_0 r + j\frac{\pi}{4}}}{j\sin\frac{\alpha}{2}}\int_{-\infty}^{\infty}\frac{e^{-k_0 r S^2/2}\,dS}{S^2-\left[\sqrt{2j(1-\cos\overline{\psi+\alpha})}\right]^2}$$

where $S_0 = \sqrt{-2j(1-\cos\overline{\psi+\alpha})}$ and

$\operatorname{Im} S_0 < 0$ for all ψ except when $\psi+\alpha=0$

i.e $\psi = \theta-\frac{\pi}{2}$ (angle for specular reflection) and $S_0=0$.

$$\therefore\ I_1(\alpha)=\frac{S_0\sin\frac{\psi+\alpha}{2}}{\sin\frac{\alpha}{2}}\, e^{-jk_0 r + j\frac{\pi}{4}}\left\{-\pi e^{+k_0 r\, 2j(1-\cos\overline{\psi+\alpha})}\right.$$

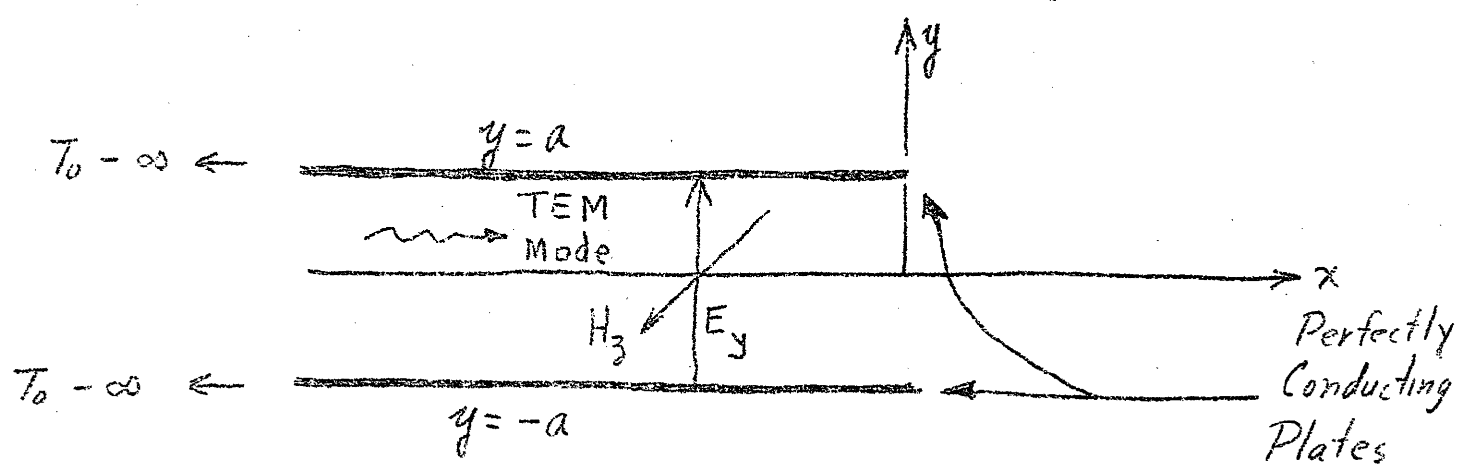
$$\left.\operatorname{erfc}\left[j\sqrt{\frac{k_0 r}{2}}\sqrt{-2j(1-\cos\overline{\psi+\alpha})}\right]\right\}$$

$I_1(\alpha)$ can also be expressed in terms of Fresnel integrals. We have $\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ $= 1 - \operatorname{erf} x = 1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$. If $x = j^{1/2}y$ we have $\operatorname{erfc}\sqrt{j}\,y = \frac{2}{\sqrt{\pi}}\int_{\sqrt{j}\,y}^\infty e^{-t^2}\,dt$. Now put $t = \sqrt{j}\,\lambda$ to get $\operatorname{erfc}\sqrt{j}\,y = \frac{2\sqrt{j}}{\sqrt{\pi}}\int_y^\infty e^{-j\lambda^2}\,d\lambda$. The latter integral is the standard Fresnel integral.

Radiation From A Parallel Plate Waveguide

The incident wave in the parallel plate waveguide is taken to be a TEM mode. Since there is no variation of the fields with z the only field components present are E_x, E_y, and H_z. If we let $H_z = \psi$ then $j\omega\epsilon_0 E_x = \frac{\partial \psi}{\partial y}$ and $j\omega\epsilon_0 E_y = -\frac{\partial \psi}{\partial x}$. ψ is a solution of $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k_0^2 \psi =$

By symmetry considerations we can conclude that ψ is an even function of y about $y = 0$ because of the symmetrical excitation. The boundary conditions require that $E_x = 0$ on the plates and hence $\frac{\partial \psi}{\partial y} = 0$ at $y = a$, $-\infty \le x \le 0$. Also we have $\frac{\partial \psi}{\partial y}$ is continuous for all x across the line $y = a$. The current on the upper plate is in the x direction and is given by $J_-(x) = \psi(y = a+) - \psi(y = a-)$ where $J_-(x) \equiv 0$ for $x > 0$. From the edge condition we know that $J_-(x) = O\, x^{1/2}$ as $x \to 0$. A general solution for ψ that has $\frac{\partial \psi}{\partial y}$ continuous at $y = a$ for all x may be represented by the Fourier transform integral



$$\psi = \int_{-\infty}^{\infty} e^{-jwx} f(w) \overset{\sin}{\cos} \lambda y \, dw, \quad 0 \leq y \leq a \qquad (1a)$$

$$= \int_{-\infty}^{\infty} -j e^{-jwx} e^{-j\lambda(y-a)} \cos? \sin \lambda a \, f(w) \, dw, \quad y > a \qquad (1b)$$

SEE BACK OF LAST PG

where $\lambda = \sqrt{k_0^2 - w^2}$, $k_0 = k_0' - jk_0''$ and $f(w)$ is an unknown function of w which can be viewed as an amplitude function. If we can determine $f(w)$ then we will know ψ. For any finite value of y greater than 'a' we know that $\psi = O\, e^{-jk_0|x|} = O\, e^{-k_0''|x|}$ as $|x| \to \infty$. Therefore $e^{-j\lambda(y-a)} \overset{\cos}{\sin} \lambda a \, f(w)$ is analytic in the strip $-k_0'' < \underset{\mathrm{Im}\{w\}}{v} < k_0''$ where $w = u + jv$. For $-\infty \leq x \leq 0$ and $0 \leq y \leq a$ the field is an incident $\overset{TM}{TEM}$ mode, a reflected $\overset{TM}{TEM}$ mode, plus higher order reflected waveguide modes. Thus we have

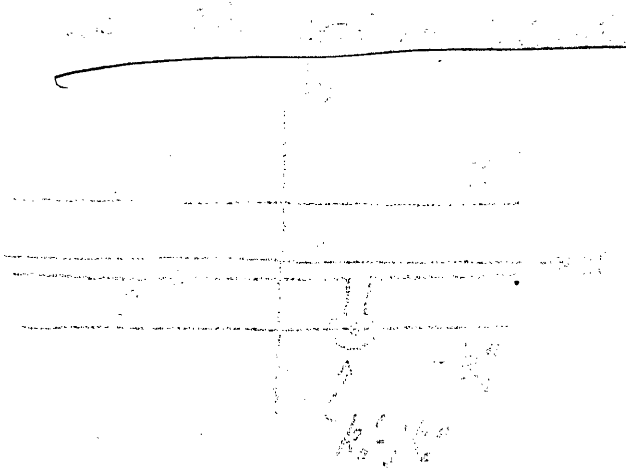
$$\psi = A e^{-jk_0 x} + RA e^{jk_0 x} + \sum_{n=1}^{\infty} A_n \overset{\sin}{\cos} \frac{n\pi y}{a} e^{\gamma_n x}, \quad 0 \leq y \leq a, \ -\infty \leq x \leq 0$$

where $\gamma_n = \sqrt{\left(\frac{n\pi}{a}\right)^2 - k_0^2}$. We will assume that $\frac{\pi}{a} > k_0$ so that all γ_n, $n = 1, 2, 3, --$, are real when k_0 is real. R is the reflection coefficient. The incident field $\psi_i = A e^{-jk_0 x}$ in the region $-\infty \leq x \leq 0$ can be

OK(?)

DEFIN. OF C

[illegible] contour C



For $x > 0$ we can close C in lower half plane (Jordan's lemma) since the path does not enclose any pole, and get zero. For $x < 0$ we can close C in upper half plane (Jordan) and by residue theorem we get $A e^{-jk_0 x}$.

For $0 \le y \le a$, $-\infty \le x \le 0$, the scattered field $\psi_s = \psi - \psi_i = O\left(e^{-k_0''|x|}\right)$ as $x \to -\infty$. For $0 \le y \le a$, $x > 0$ we have $\psi_i = 0$, $\psi = \psi_s = O\left(e^{-k_0'' x}\right)$ as $x \to \infty$.

Therefore for $0 \le y \le a$, all x, we can write

$$\psi_s = \int_{-\infty}^{\infty} e^{-j w x} f(w) \overset{\sin}{\cos} \lambda y \, dw - \frac{A}{2\pi j} \int_C \frac{e^{-jwx}}{w - k_0} dw$$

$$= \int_C e^{-jwx} \left\{ f(w) \overset{\sin}{\cos} \lambda y - \frac{A}{2\pi j (w - k_0)} \right\} dw$$

Because of the behavior of ψ_s as $|x| \to \infty$ we know $f(w) \overset{\sin}{\cos} \lambda y - \frac{A}{2\pi j (w - k_0)}$ is analytic in the strip $-k_0'' < \operatorname{Im} w < k_0''$. [illegible]

ψ arises from the inversion of $f(w) \overset{\sin}{\cos} \lambda y$ this function must have a pole at $w = k_0$. It must also have poles at $w = -k_0$, $w = j\gamma_n$ in order to give rise to the reflected waves in the waveguide. These poles lie in the uhp and are enclosed when the contour is closed in the upper half plane, which can be done for $x < 0$. We now see that $f(w) \overset{\sin}{\cos} \lambda y$ will be analytic in the strip $-k_0'' < v < k_0''$ but that in order to get the incident wave included the integral $\int_{-\infty}^{\infty} (\ \ )$ must be warped around and below the pole at $w = k_0$, i.e. be taken along the contour C shown on the previous page.

In order to obtain further properties of $f(w)$ we impose the boundary conditions at $y = a$, $-\infty \leq x \leq 0$. Thus since $\frac{\partial \psi}{\partial y} = 0$ along this boundary

$$\int_C \lambda \overset{\cos}{\sin} \lambda a \ f(w) e^{-jwx} dw = 0, \ -\infty \leq x \leq 0$$

Re{W}>0 FOR x<0

For $x < 0$ we can close C in the uhp. For the above integral to vanish we must then have

$\lambda \overset{\cos}{\sin} \lambda a \; f(w)$ analytic in u.h.p. above C and also tending toward zero as $|w| \to \infty$ in u.h.p. Note that $\lambda \overset{\cos}{\sin} \lambda a = 0$ for $w = \pm k_0, j\gamma_n$ so that the postulated poles of $f(w)$ are cancelled by $\lambda \sin \lambda a$ in u.h.p. The integral

$\lambda \equiv (k_0^2 - w^2)^{1/2}, \; \gamma_n \equiv \sqrt{\left(\frac{n\pi}{a}\right)^2 - k_0^2}$?

$$\int_C \lambda \overset{\cos}{\sin} \lambda a \; f(w) e^{-jwx} dw \text{ gives } -jw\epsilon_0 E_x(x, y=a).$$

Eq. (1a)

From the edge condition $E_x(x, y=a) = \mathcal{O}\, x^{-1/2}$ as $x \to 0$ and therefore by the final value theorem $\lambda \overset{\cos}{\sin} \lambda a \; f(w) = \mathcal{O}\, w^{-1/2}$ as $|w| \to \infty$.

The current J_1 flowing on the inside of the plate at $y = a$, $-\infty \leq x \leq 0$ is related to $H_z = \psi$ on the interior surface and is given by

Eq. (1a)

$$J_1(x) = -\int_C e^{-jwx} f(w) \overset{\sin}{\cos} \lambda a \; dw, \quad -\infty \leq x \leq 0$$

Since $x < 0$ we can close C in the upper half plane and must then have an $f(w) \cos \lambda a$ with poles at $w = \pm k_0, j\gamma_n$, and no other singularities.

The current $J_2(x)$ on the outside of the plate is given by $J_2(x) = \int_C -j e^{-jwx} \sin\lambda a \; f(w)\,dw, \quad -\infty \le x \le 0$

Since we can close C in the upper half plane and must get a finite non zero value for J_2 the function $\sin\lambda a\, f(w)$, must have a branch cut singularity in the upper half plane. The total current $J = J_1 + J_2$ is given by

$$-\int_C e^{-jwx} f(w)\,(\cos\lambda a + j\sin\lambda a)\,dw$$

$$= -\int_C e^{-jwx} e^{j\lambda a} f(w)\,dw = J(x), \quad -\infty \le x \le 0$$

$$= 0, \quad x > 0$$

For $x > 0$ we can close C in the lhp and in order for the integral to vanish $f(w)\,e^{j\lambda a}$ must be analytic in the lhp below C. Since $J(x) = \mathcal{O}\,x^{1/2}$ as $x \to 0$, $f(w)\,e^{j\lambda a} = \mathcal{O}\,w^{-3/2}$ as $|w| \to \infty$.

We now have the following conditions that must be satisfied by $f(w)$:

(a) $e^{j\lambda a} f(w) = \mathcal{O}\,w^{-3/2}$ as $|w| \to \infty$ in lhp pg 49

(b) $\lambda \sin\lambda a\; f(w)$ analytic in uhp above C pg 45

(c) $f(w)e^{j\lambda a}$ analytic in lhp below C   pg 49 (MIDDLE)

(d) $f(w)$ has simple poles at $w = \pm k_0,\ j\gamma_n$

(e) $\lambda \sin \lambda a\ f(w) \sim w^{-1/2}$ as $|w| \to \infty$ (this condition includes condition (a) since $\lambda \to jw$ as $w \to \infty$)

To give decaying waves at infinity $\sqrt{k_0^2 - w^2} = \lambda = -j\sqrt{w^2 - k_0^2}$, i.e. the branch with negative imaginary part is chosen (or positive real part when λ is real).

From (a) we can write $\underline{e^{j\lambda a} f = L_-(w)}$ FROM COND (a) where L_- is an unknown function that is analytic in a lhp and has poles at $w = \pm k_0, j\gamma_n$ and is of order $w^{-3/2}$ as $w \to \infty$. From (b) we now get $\underline{\lambda \overset{\cos}{\sin} \lambda a\ e^{-j\lambda a} L_-(w) = L_+(w)}$ where L_+ is a function analytic in an upper half plane. We can express $\lambda \overset{\cos}{\sin} \lambda a$ as an infinite product in the form

$$\lambda^2 a \frac{\sin \lambda a}{\lambda a} = \lambda^2 a \frac{\sin k_0 a}{\cancel{k_0 a}} \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{w}{j\gamma_n}\right) e^{w/j\gamma_n}$$

where the prime means omission of the $n = 0$ term. Now assume that we can factor $e^{-j\lambda a}$ into the form $S_+(w) S_-(w)$. Then

$$L_-(w) = \frac{L_+}{S_+ S_-}\left[\lambda^2 a \prod_{n=-\infty}^{\infty}{}' (1 - w/j\gamma_n) e^{w/j\gamma_n}\right]^{-1} \frac{\cancel{k_0 a}}{\sin k_0 a}.$$

For L_- to

[illegible]

[illegible] in the [illegible]

and in addition gives L_- algebraic growth according to $L_- = O(w^{-\frac{1}{2}})$ as $|w| \to \infty$. Thus we set

$$L_+ = S_+ \prod_{n=1}^{\infty} (1 + w/j\gamma_n)\, e^{-w/j\gamma_n} \; I(w)$$

where $I(w)$ is to be chosen to give L_- the correct algebraic behavior at infinity. Note that I must be an entire function in the LHP to give L_- this property. The constant to be associated with I is fixed by the requirement that the incident wave arising from the residue at $w = k_0$ must equal A. For later convenience we will choose L_- of the following form

$$L_- = \frac{I(w)}{S_-(w)\,(w^2 - k_0^2) \prod_{n=1}^{\infty} \left(1 - \frac{w}{j\gamma_n}\right) e^{wa/n\pi j}}$$

which is permissible since we have not specified I as yet. Note that $\gamma_n \to n\pi/a$ as $n \to \infty$ and so we can choose $e^{wa/jn\pi}$ as convergence factor in the infinite product. This introduces a factor which can be absorbed in $I(w)$.

[illegible]

with the form f_+ [illegible] $f_+ f_- =$ [illegible]

[illegible]

$g(\omega)$ in the form $g(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{d\zeta}{(\zeta-\omega)\sqrt{k_0^2-\zeta^2}}$.

(see The Wiener-Hopf Technique by Noble, Pergamon Press 1958, Sec. 1.3). The integral can be evaluated to give

$$g(\omega) = \frac{1}{\pi\sqrt{\omega^2-k_0^2}} \ln \frac{-\omega+\sqrt{\omega^2-k_0^2}}{\omega+\sqrt{\omega^2-k_0^2}}.$$

We now get

$$f_+ = \frac{1}{2\sqrt{k_0^2-\omega^2}} - \frac{1}{2\pi\sqrt{\omega^2-k_0^2}} \ln \frac{-\omega+\sqrt{\omega^2-k_0^2}}{\omega+\sqrt{\omega^2-k_0^2}}$$

and

$$\ln S_+ = \frac{i\sqrt{\omega^2-k_0^2}}{2\pi}\left[i\pi - \ln \frac{-\omega+\sqrt{\omega^2-k_0^2}}{\omega+\sqrt{\omega^2-k_0^2}}\right]$$

and

$$S_+ = e^{-\sqrt{\omega^2-k_0^2}\,\frac{1}{2}} \; e^{-i\sqrt{\omega^2-k_0^2}\,\frac{1}{2\pi}\ln\frac{-\omega+\sqrt{\omega^2-k_0^2}}{\omega+\sqrt{\omega^2-k_0^2}}}$$

[illegible]

$\ln \dfrac{[illegible]}{w+\sqrt{}} = \ln \dfrac{w-\sqrt{}}{w+\sqrt{}} + \ln(-1)$ and $\ln(-1) = \pi j$

so that $S_- = e^{-j\sqrt{w^2-k_0^2}\,\frac{a}{2\pi}\ln\frac{w-\sqrt{w^2-k_0^2}}{w+\sqrt{w^2-k_0^2}}}$ also.

If we replace $\sqrt{}$ by $-\sqrt{}$, S_- does not change value which must be the case since S_- is analytic in the l.h.p.

The term $\dfrac{1}{\prod_1^\infty\left(1+\frac{jw}{\gamma_n}\right)e^{-jwa/n\pi}}$ differs from

$$\left[\prod_1^\infty\left(1+\frac{jwa}{n\pi}\right)e^{-jwa/n\pi}\right] = \frac{jwa}{\pi}\, e^{j\gamma wa/\pi}\, \Gamma(jwa/\pi)$$

where $\gamma = 0.57722$ is Euler's constant, by a constant only when $w \to \infty$. Therefore it's asymptotic value can be found using the result

$$\Gamma(z) \to \sqrt{2\pi}\, e^{z\ln z}\, z^{-1/2}\, e^{-z+1} \quad \text{as } |z| \to \infty,$$

Apart from a constant we thus find that

$$L_-(w) \sim \frac{I(w)\; w\; e^{j\gamma wa/\pi}}{w^2\, e^{j\frac{wa}{\pi}\ln 2w/k_0}}\; e^{j\frac{wa}{\pi}\ln j\frac{wa}{\pi}}\; w^{-1/2}\, e^{-jwa/\pi}$$

as $w \to \infty$

For L_- to be of order $w^{-3/2}$ we must choose $I(w)$ equal to $C\, e^{-j(\gamma-1)wa/\pi}\, e^{j\frac{wa}{\pi}\left[\ln\left(\frac{2w}{k_0}\right) - \ln jwa/\pi\right]}$

$$I(w) = C\, e^{[illegible]}\, e^{[illegible]} = C\, e^{j(1-\gamma)\frac{wa}{\pi}}\, e^{\frac{wa}{2}}\, e^{-j\frac{wa}{\pi}\ln\frac{k_0 a}{2\pi}}$$

where C is a constant. We now have

$$L_- = e^{j\lambda a} f(w) = \frac{C e^{j(1-\gamma)\frac{wa}{\pi} + \frac{wa}{2} - j\frac{wa}{\pi}\ln\frac{k_0 a}{2\pi}}}{(w^2 - k_0^2)\prod_1^\infty\left(1 + \frac{jw}{\gamma_n}\right)e^{-\frac{jwa}{n\pi}}\, e^{-j\sqrt{w^2-k_0^2}\,\frac{a}{2\pi}\ln\frac{w-v}{w+v}}}$$

From pg. 48 we see that $J_1(x) = -\int_C e^{-jwx} L_-(w) e^{-j\lambda a} \,dw$

The residue at $w = k_0$ gives the incident wave while that at $w = -k_0$ gives the reflected wave. Hence the reflection coefficient is given by

$$R = \frac{\lim\limits_{w\to -k_0} L_-(w)(w+k_0)}{\lim\limits_{w\to k_0} L_-(w)(w-k_0)} = -\frac{e^{-j(1-\gamma)\frac{k_0 a}{\pi}}\, e^{-k_0 a/2}}{e^{j(1-\gamma)\frac{k_0 a}{\pi}}\, e^{k_0 a/2}}$$

$$\frac{e^{+j\frac{k_0 a}{\pi}\ln\frac{k_0 a}{2\pi}}\displaystyle\prod_1^\infty\left(1 + \frac{jk_0}{\gamma_n}\right)e^{-jk_0 a/n\pi}}{e^{-j\frac{k_0 a}{\pi}\ln\frac{k_0 a}{2\pi}}\displaystyle\prod_1^\infty\left(1 - \frac{jk_0}{\gamma_n}\right)e^{jk_0 a/n\pi}}$$

$$= -\,e^{-k_0 a}\, e^{+j\frac{2k_0 a}{\pi}\left(-1+\gamma+\ln\frac{k_0 a}{2\pi}\right)}\prod_1^\infty \frac{\gamma_n + jk_0}{\gamma_n - jk_0}\, e^{-j\,2k_0 a/n\pi}$$

[illegible]

[illegible]

[illegible] $\gamma_n \to \frac{n\pi}{a}$ for n large

and [illegible] $\to$ [illegible] $=$ [illegible].

Other field quantities of interest may also be evaluated now since $f(w)$ is known.

In the integrand for $J_1(x)$ we can write $L_- e^{-j\lambda a} = L_- S_+ S_- = \frac{L_+}{P(w)}$ and since $e^{j\lambda a} L_- \cos \lambda a$ is analytic in the upper half plane except for simple poles at the zeros of $P(w) = \lambda^2 a \prod_{-\infty}^{\infty}{}'\left(1 - \frac{w}{j\gamma_n}\right) e^{w/j\gamma_n}$ (see p. 50).

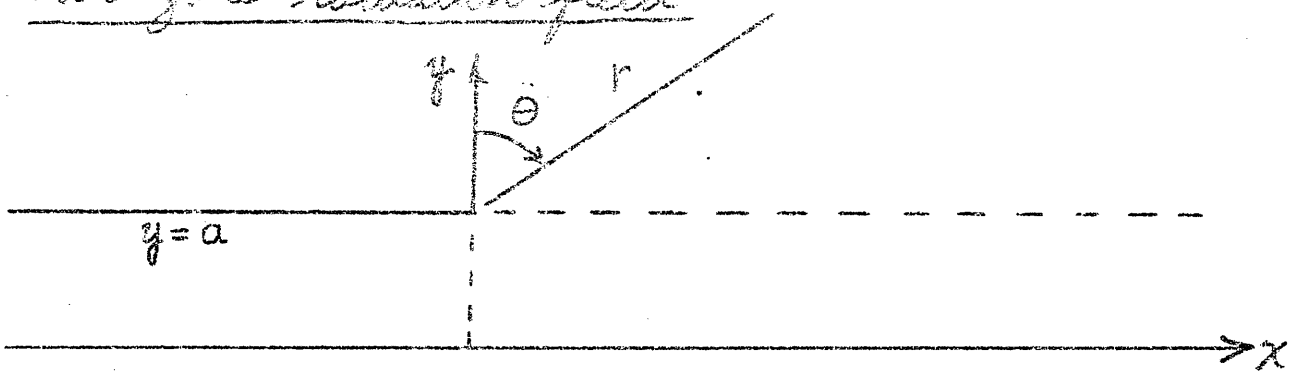
Then the contour C can be closed in the upper half plane and J_1 obtained as a residue expansion.

However, [illegible]

[illegible]

[illegible]

Far zone radiation field



The far zone radiation field may be found from a knowledge of E_x along the surface $y=a$. For $x<0$, $E_x(x,a)=0$ while for $x>0$, $j\omega\epsilon_0 E_x = \frac{\partial\psi}{\partial y}$. Thus

$$j\omega\epsilon_0 E_x = \int_{-\infty}^{\infty} \pm e^{-jwx} f(w)\, \lambda \overset{\cos}{\sin} \lambda a\, e^{-j\lambda(y-a)}\, dw$$

EQ. 1b

for $y \geq a$. This expression may be viewed as the Fourier transform solution for the radiated field in terms of the Fourier transform of the aperture field E_x along $y=a$. If we put $x = r\sin\theta$, $y-a = r\cos\theta$, $w = k_0 \sin\phi$, $\lambda = k_0\cos\phi$, we obtain

$$j\omega\epsilon_0 E_x = \int_{C_\phi} - f(k_0 \sin\phi)(k_0\cos\phi)\overset{2\cos}{\sin}(k_0 a\cos\phi)\, e^{-jk_0 r\cos(\phi-\theta)}\, d\phi$$

where C_ϕ is the mapping of the contour in the w plane into the corresponding contour in the ϕ plane.

By means of the standard saddle point (the saddle point is at $\phi = \theta$) we obtain

$$j\omega\epsilon_0 E_x \sim -\sqrt{\frac{2\pi}{k_0 r}}\, e^{-jk_0 r + j\frac{\pi}{4}}\, f(k_0 \sin\theta)(k_0\cos\theta)^2 \overset{\cos}{\sin}(k_0 a\cos\theta)$$

We can find $j\omega\epsilon_0 E_y$ from the condition $\nabla\cdot\vec{E} = 0$ or by using $E_x = E_\theta \cos\theta$, $E_y = -E_\theta \sin\theta = -E_x \tan\theta$, which holds in the radiation zone. Hence

$$j\omega\epsilon_0 E_\theta \sim -\sqrt{\frac{2\pi}{k_0 r}}\, e^{-jk_0 r + j\pi/4}\, k_0^2 f(k_0\sin\theta)\cos\theta\, \overset{\cos}{\sin}(k_0 a\cos\theta).$$

Apart from constant factors the radiation pattern is described by the normalized field

$$\frac{E_\theta(\theta)}{E_\theta(0)} = \frac{f(k_0\sin\theta)\cos\theta\, \overset{\cos}{\sin}(k_0 a\cos\theta)}{f(0)\sin k_0 a}$$

Now $f(w) = e^{-j\lambda a} L_-(w)$ and $|f(k_0\sin\theta)|$

$$= C\, e^{(k_0 a/2)\sin\theta} \Big/ \left\{ \left[k_0^2\cos^2\theta\right]\left[\prod_{n=1}^{\infty}\left(1 + \frac{k_0^2\sin^2\theta}{\gamma_n^2}\right)\right]^{1/2} \right\}$$

But $$\frac{\sin\sqrt{k_0^2-w^2}\,a}{\sqrt{k_0^2-w^2}\,a} = \prod_{n=1}\left(1+\frac{w^2}{\gamma_n^2}\right)\frac{\sin k_0 a}{k_0 a}$$ so

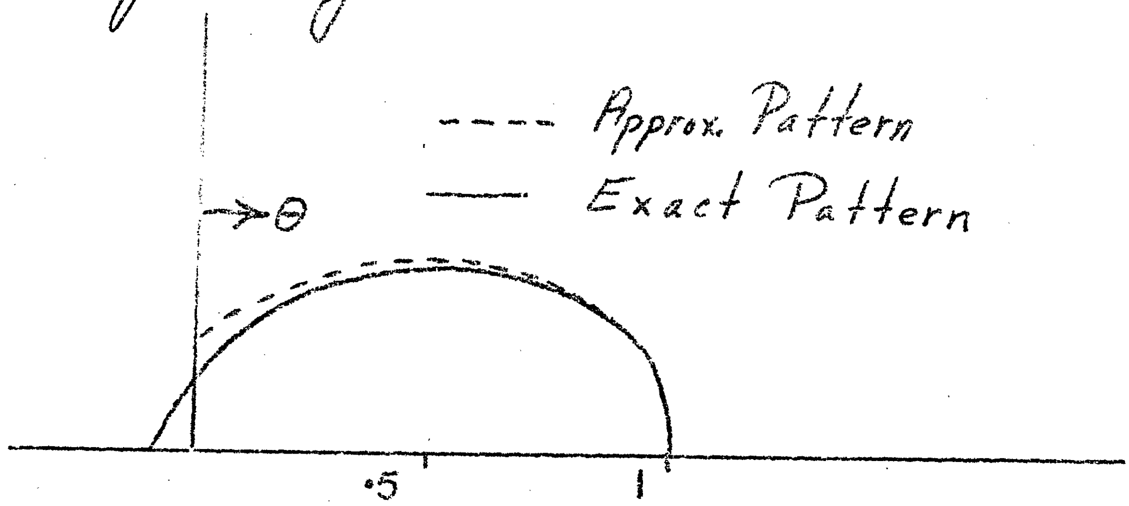
$$|E_\theta(\theta)| = e^{(k_0 a/2)(\sin\theta - 1)}\left[\frac{\sin(k_0 a\cos\theta)}{k_0 a\cos\theta}\right]^{1/2}, \quad 2a < \lambda_0$$

which we have normalized to unity at $\theta = \pi/2$.

If we assumed that the field in the waveguide opening was a constant along $-a<y<a$, $x=0$, and computed the radiation field from this we would obtain a pattern given by

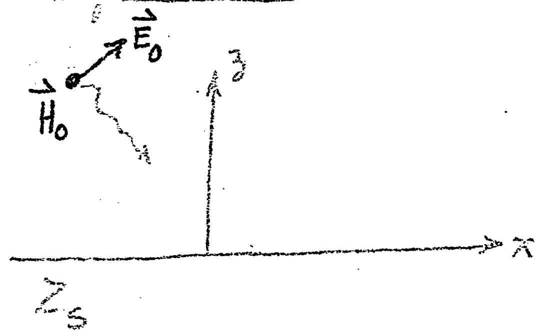
$$|E_\theta(\theta)| = \frac{\sin(k_0 a\cos\theta)}{k_0 a\cos\theta}$$

The two patterns are sketched below and are seen to agree very closely.



ELECTROMAGNETIC SURFACE WAVES

Surface Waves



Consider incident $(TM)_z$ wave.

$$H_y = H_0 e^{-\gamma_x x + \gamma_z z}, \quad \gamma_i = \alpha_i + j\beta_i .$$

From Helmholtz equation, $\gamma_x^2 + \gamma_z^2 + k_0^2 = 0.$

Imag. part of equation gives $\alpha_x \beta_x + \alpha_z \beta_z = 0$. $\therefore$ if $\vec{\alpha} = \alpha_x \vec{a}_x + \alpha_z \vec{a}_z$ and $\vec{\beta} = \beta_x \vec{a}_x + \beta_z \vec{a}_z$, $\vec{\alpha}\cdot\vec{\beta} = 0$. Since $H_y = H_0 e^{-\vec{\alpha}\cdot\vec{r}_1 - j\vec{\beta}\cdot\vec{r}_1}$ where $\vec{r}_1 = \vec{a}_x x - \vec{a}_z z$ for the incident wave it is seen that the constant phase and constant amplitude planes are perpendicular. $\Rightarrow$ AN INHOMOGENEOUS PLANE WAVE, I.E., A CONSTANT PHASE SURFACE OF A PLANE WAVE HAS A NON-CONSTANT (VARYING) AMPLITUDE.

$$j\omega\epsilon_0 \vec{E}_0 e^{-\gamma_x x + \gamma_z z} = \nabla\times\vec{H} = (-\gamma_x \vec{a}_x + \gamma_z \vec{a}_z)\times \vec{a}_y H_0 e^{-(\gamma_x x)} \cdots e^{\gamma_z z}$$

$$\therefore \quad E_{0x} = \frac{j\gamma_z}{k_0} Z_0 H_0, \quad \text{Wave impedance } Z_w = -\frac{E_{0x}}{H_y} = \frac{-j\gamma_z}{k_0} Z_0$$

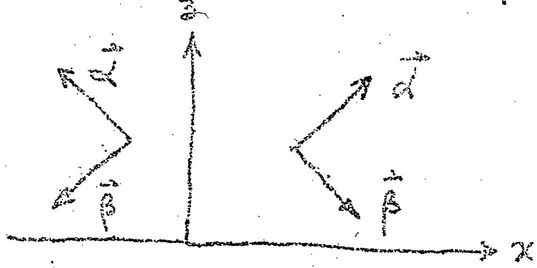
Reflection coeff. $\Gamma = \dfrac{Z_s - Z_w}{Z_s + Z_w}$. If $Z_s = Z_w$, $\Gamma = 0$ and incident wave satisfies boundary condition at $z = 0$ by itself. If $Z_s + Z_w = 0$ (transverse resonance) then Γ has a pole and reflected wave can exist by itself, i.e. amplitude of incident wave can be reduced to zero and still have a finite amplitude reflected wave.

The reflected transverse fields are $\Gamma H_0 e^{-\gamma_x x - \gamma_z z}$, $\Gamma E_{0x} e^{-\gamma_x x - \gamma_z z}$ with wave impedance $Z_w = \frac{j\gamma_z}{k_0} Z_0$ looking in the $-z$ direction. If we let $H_0 \to 0$ when $\Gamma \to \infty$ such that ΓH_0 remains finite only the reflected wave is present. Let $Z_s = R_s + jX_s$. Then $j\gamma_z = j\alpha_z - \beta_z = \frac{R_s + jX_s}{Z_0} k_0$ for a Γ pole.

The four following possibilities are of interest:

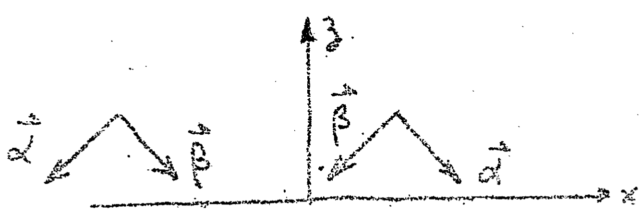
(1) $R_s +$, $X_s +$; $\alpha_z +$, $\beta_z -$; then $\alpha_x \pm$, $\beta_x \pm$

Reflected wave has a propagation factor $e^{-(\vec{\alpha} + j\vec{\beta})\cdot\vec{r}}$, $\vec{r} = \vec{a}_x x + \vec{a}_z z$



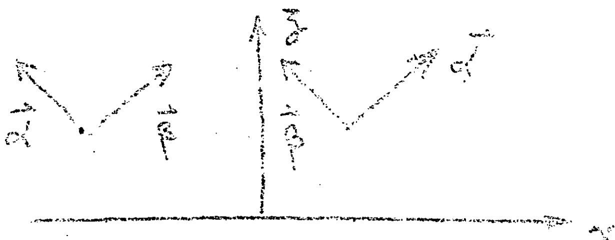
$\vec{\alpha}$ and $\vec{\beta}$ have the orientations shown. This is a complex pole non-radiating wave ($\vec{\beta}$ directed towards surface) which attenuates away from surface and is called a TRUE "surface" wave. It is a physically real wave and can exist.

(2) $R_s +$, $X_s -$; $\alpha_z -$, $\beta_z -$; $\alpha_x \pm$, $\beta_x \mp$



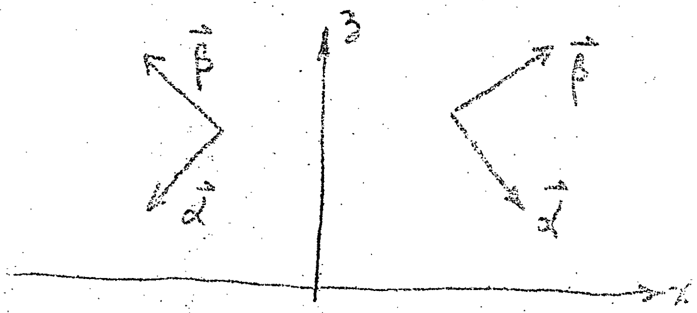
This wave is a complex pole non-radiating wave also but increases exponentially away from the surface and hence is a non-physical solution.

(3) $R_s -$, $X_s +$; $\alpha_z +$, $\beta_z +$; $\alpha_x \pm$, $\beta_x \mp$



This is a complex pole radiating wave and could exist only if there are sources on the $z=0$ plane.

(4) $R_s -$, $X_s -$; $\alpha_z -$, $\beta_z +$; $\alpha_x \pm$, $\beta_x \pm$

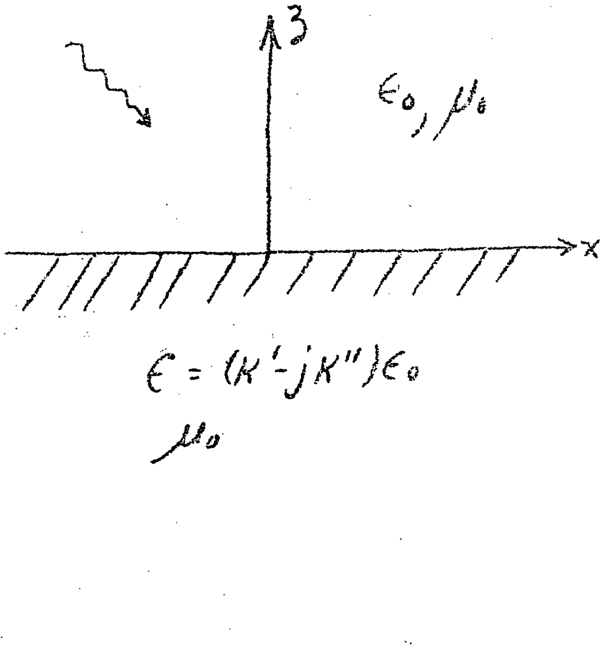


This is a complex pole radiating wave that increases exponentially away from the surface. It is a non-physical wave called a "leaky" wave.

The above four wave types satisfy the boundary conditions at $z=0$. The wave is viewed as a reflected wave arising by letting the incident wave amplitude vanish when Γ has a pole such that ΓH_0 remains finite. An identical field is obtained by using an alternative viewpoint, namely, considering an incident wave with an impedance $Z_w = Z_s$ so that $\Gamma = 0$, i.e. no reflected wave. In this case the wave is given by

$H_0 e^{-\gamma_x x + \gamma_z z}$ and the condition $\frac{-j\gamma_z}{k_0} Z_0 = Z_s$. This leads to the same solution as using the condition $Z_w = -\frac{j\gamma_z}{k_0} Z_0 = -Z_s$ and taking the wave as a reflected wave $\Gamma H_0 e^{-\gamma_x x - \gamma_z z}$ since it amounts to a change in the sign in front of γ_z in the propagation factor and also in the expression relating Z_w to Z_s. When a source is located above the plane then the "pole wave" solutions are the natural ones

Zenneck Surface Wave



For $z > 0$, $H_y = H_0 e^{-(\gamma_x x) + (\gamma_z z)} - \Gamma H_0 e^{-\gamma_x x - \gamma_z z}$

For $z < 0$,

$$H_y = T H_0 e^{-\gamma_{x1} x + \gamma_{z1} z}$$

where $\gamma_{x1} = \gamma_x$, $\gamma_{x1}^2 + \gamma_{z1}^2 + K k_0^2 = 0$.

Wave impedance of dielectric is

$$Z_s = \frac{-j\gamma_{z1}}{\sqrt{K}\, k_0} \sqrt{\frac{\mu_0}{K \epsilon_0}} = \frac{-j\gamma_{z1}}{K k_0} Z_0$$

$$= -j \frac{\sqrt{\gamma_z^2 - (K-1) k_0^2}}{K k_0} Z_0$$

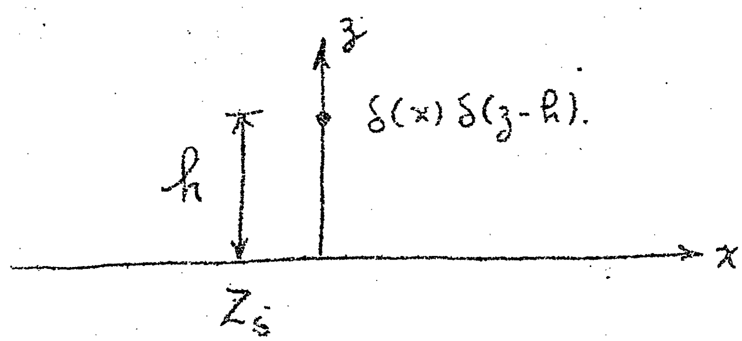
The reflected wave vanishes if $\Gamma = 0$, i.e. $Z_w = Z_s$ or $-\frac{j\gamma_z}{k_0} Z_0 = -j \left\{ \frac{\sqrt{\gamma_z^2 - (K-1) k_0^2}}{K k_0} Z_0 \right\}$. Solving for γ_z gives

$\gamma_z = \sqrt{\dfrac{-k_0^2}{K+1}} \approx \pm \dfrac{jk_0}{\sqrt{K'+1}} \mp \dfrac{k_0 K''}{2(K'+1)^{3/2}}$ for $K'' \ll K'$. But for a physical solution $\gamma_{z1} = \alpha_{z1} + j\beta_{z1}$ with α_{z1} and β_{z1} positive so from $\gamma_z = \gamma_{z1}/K$ the solution we want is $\gamma_z = \dfrac{jk_0}{\sqrt{K'+1}} - \dfrac{k_0 K''}{2(K'+1)^{3/2}}$

The field above the interface is thus $H_0\, e^{-(j\beta_x + \alpha_x)x}\, e^{(j\beta_z - \alpha_z)z}$ with $\alpha_x, \beta_x, \alpha_z, \beta_z$ all positive. This is a surface wave.

If we look for the pole wave then we consider the reflected wave $-\Gamma H_0\, e^{-\gamma_x x - \gamma_z z}$ above the interface and enforce the condition $Z_w = -Z_s$, i.e. $\gamma_z = -\gamma_{z1}/K$. This procedure will give the same field solution in both regions $z \gtrless 0$.

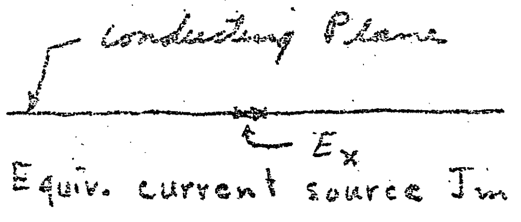
Excitation of Surface Waves by a Magnetic Line Source



$H_y = \psi$,

$$\nabla^2 \psi + k_0^2 \psi = j\omega\epsilon_0 J_m$$
$$= j\omega\epsilon_0\, \delta(x)\,\delta(z-h)$$

where J_m is equivalent magnetic current directed in y direction. J_m can be produced by a narrow slot with electric field across it.



ψ can be represented as follows:

$$\psi = \int_{-\infty}^{\infty} A(w)\, e^{-jwx - j\sqrt{k_0^2 - w^2}\,(z-h)}\, dw \quad \text{for } z \geq h$$

$$= \int_{-\infty}^{\infty} B(w)\, e^{-jwx} \left\{ e^{j\sqrt{k_0^2 - w^2}\, z} - \Gamma\, e^{-j\sqrt{k_0^2-w^2}\, z} \right\} dw, \quad 0 \leq z \leq h$$

where
$$\Gamma = \frac{Z_s - Z_w}{Z_s + Z_w} = \frac{Z_s + \sqrt{k_0^2 - w^2}}{Z_s - \sqrt{k_0^2 - w^2}}$$

Cont. at $z = h$ gives $A = B\left(e^{j\sqrt{\;}\,h} - \Gamma e^{-j\sqrt{\;}\,h}\right)$

Also
$$\left.\frac{\partial \psi}{\partial z}\right|_{h-}^{h+} = j\omega\epsilon_0 = -j\sqrt{\;}\,A - B(j\sqrt{\;})\left(e^{j\sqrt{\;}\,h} + \Gamma e^{-j\sqrt{\;}\,h}\right)$$

Thus we find that $A = \frac{-\omega\epsilon_0}{2\sqrt{\;}}\left(1 - \Gamma e^{-2j\sqrt{\;}\,h}\right)$,

$$B = -\frac{\omega\epsilon_0}{2\sqrt{\;}}\, e^{-j\sqrt{\;}\,h} \quad \text{and}$$

$$\psi = -\frac{\omega\epsilon_0}{2}\int_{-\infty}^{\infty} \frac{e^{-j\sqrt{\;}\,(z_> - z_<)}}{\sqrt{k_0^2 - w^2}}\left[1 - \Gamma e^{-2j\sqrt{\;}\, z_<}\right] e^{-jwx}\, dw$$

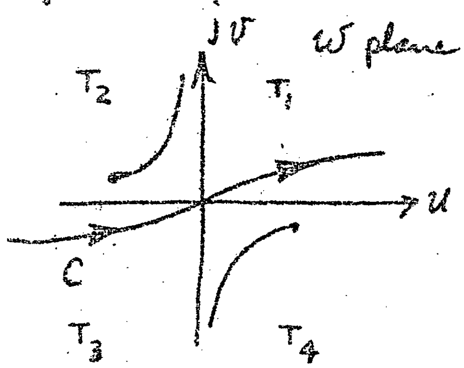
where $z_> = z$, $z_< = h$ for $z > h$ and $z_> = h$, $z_< = z$ for $z < h$.

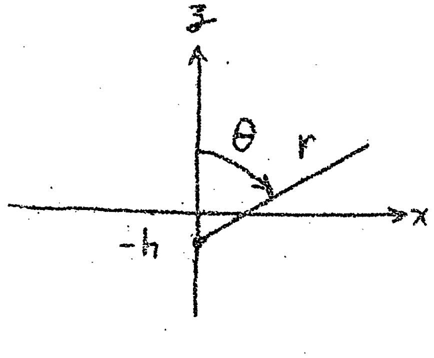
The solution gives ψ as a superposition of plane waves radiated by the source and reflected from the impedance surface (Z_s).

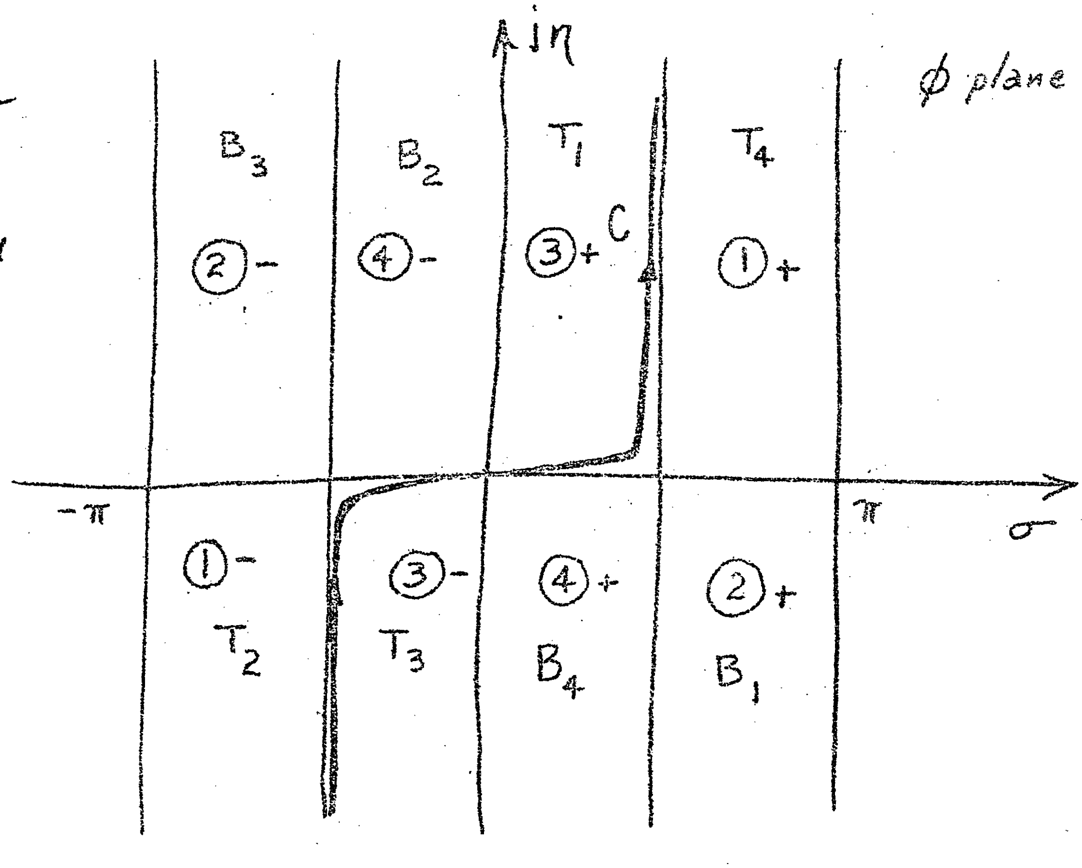
Let $w = k_0 \sin\phi$, $\phi = \sigma + j\eta$; $\sqrt{k_0^2 - w^2} = k_0 \cos\phi$, $x = r\sin\theta$, $z_> + z_< = r\cos\theta$. We then get

$$\psi_s = \frac{\omega\epsilon_0}{2} \int_C \Gamma(\phi)\, e^{-jk_0 \cos(\phi-\theta)}\, d\phi$$

for the reflected field. The other term gives the directly radiated field and is proportional to a Hankel function $H_0^2(k_0\sqrt{x^2 + (z_> - z_<)^2})$. The contour C is illustrated below. The correct branch of $\sqrt{k_0^2 - w^2}$ is the one with a negative imaginary part.







The quadrants labelled T_i are those on the proper (top) Riemann surface while those labelled B_i are on the improper (bottom) Riemann surface.

Four types of boundary waves were introduced on pages 126-127. For these we have

(1) $Re\, w > 0$, $Imag.\, w < 0$, $Re\, \sqrt{\ } < 0$, $Imag.\, \sqrt{\ } < 0$, (surface wave)

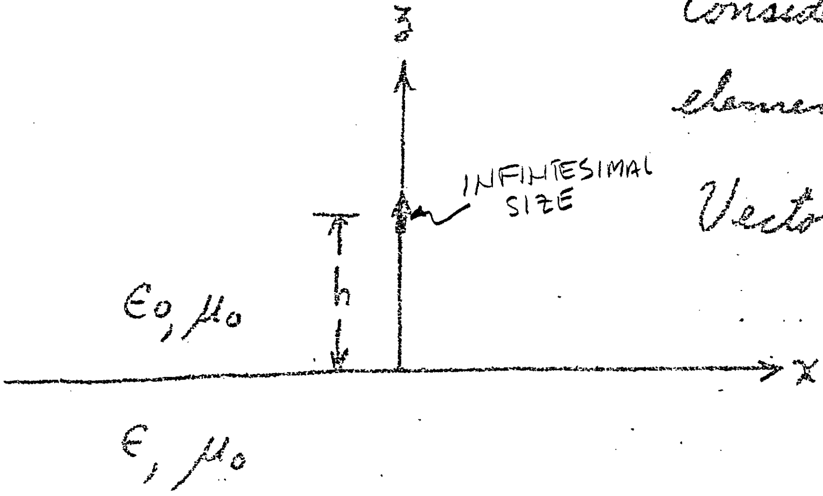
(2) $Re\, w > 0$, $Imag.\, w > 0$, $Re\, \sqrt{\ } < 0$, $Imag.\, \sqrt{\ } > 0$ (complex pole non-radiating wave)

(3) $Re\, w > 0$, $Imag.\, w > 0$, $Re\, \sqrt{\ } > 0$, $Imag.\, \sqrt{\ } < 0$ (complex pole radiating wave)

(4) $Re\, w > 0$, $Imag.\, w < 0$, $Re\, \sqrt{\ } > 0$, $Imag.\, \sqrt{\ } > 0$ (Leaky wave)

The quadrants in which these wave types can occur are marked on the ϕ plane on page 65. The signs ($\pm$) following the encircled number indicates propagation in the $+x$ and $-x$ directions respectively. The angle θ ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ to cover physical space above the impedance plane. The saddle point is located at $\phi = \theta$ i.e. along the line $-\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2}$, $\eta = 0$ in the ϕ plane. It is now seen that when C is deformed into a steepest descent contour the contour may pass into the following quadrants: T_2, T_3, B_4, B_2, T_1 and T_4. Thus only the wave types ①, ③ and ④ will occur in the representation for the field. Only types ① and ③ are physical waves. The leaky wave ④ is a non-physical wave since its pole lies on the wrong Riemann surface. Wave type ③ would occur only if radiating sources were distributed over the impedance plane.

Radiation from a Vertical Dipole over a Flat Earth



Consider a vertical current element at $z = h$, $x = 0$. Vector potential $A_z = \psi$ is a solution of

$$\nabla^2 \psi + k_0^2 \psi = -\mu_0 \delta(x)\delta(z-h)\delta(y)$$

for $z > 0$,

$$\nabla^2 \psi + k^2 \psi = 0, \quad z \le 0$$

where $k^2 = \omega^2\mu_0\epsilon = \omega^2\mu_0\epsilon_0 K = K k_0^2$. In cylindrical coord's $\rho = \sqrt{x^2+y^2}$, θ, z field will be independent of θ (circular symmetry). However, it will be instructive to solve the problem in rectangular coordinates first.

Let $$\psi(\beta_x, \beta_y, z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \psi(x,y,z)\, e^{j\beta_x x + j\beta_y y}\, dx\,dy.$$

Taking a double Fourier transform of the equations for $\psi(x,y,z)$ gives

$$\left(\frac{\partial^2}{\partial z^2} + k_0^2 - \beta^2\right)\psi(\beta_x,\beta_y,z) = -\frac{\mu_0}{4\pi^2}\,\delta(z-h), \quad z \geq 0$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2 - \beta^2\right)\psi(\beta_x,\beta_y,z) = 0, \quad z \leq 0$$

Continuity of the tangential electric and magnetic fields at $z=0$ requires that $\psi(z+) = \psi(z-)$ and $\left.\frac{1}{\epsilon}\frac{\partial\psi}{\partial z}\right|_{z-} = \left.\frac{1}{\epsilon_0}\frac{\partial\psi}{\partial z}\right|_{z+}$. We may choose

$$\psi = \psi_1 = A\, e^{-j\sqrt{k_0^2-\beta^2}\,(z-h)}, \quad z \geq h, \quad \beta^2 = \beta_x^2 + \beta_y^2$$

$$\psi = \psi_2 = \frac{A\left[e^{j\sqrt{k_0^2-\beta^2}\,z} - \Gamma e^{-j\sqrt{k_0^2-\beta^2}\,z}\right]}{e^{j\sqrt{\;}\,h}\left[1 - \Gamma e^{-j2\sqrt{\;}\,h}\right]}, \quad 0 \leq z \leq h$$

which is the sum of a downward and upward propagating wave with constants A and Γ chosen so that $\psi_1 = \psi_2$ at $z = h$. Γ can be interpreted as a reflection coefficient. For $z \leq 0$ let $\psi = \psi_3$ and choose ψ_3 so that it equals ψ_2 at $z = 0$ and is a downward propagating wave, thus

$$\psi_3 = \frac{A(1-\Gamma) \qquad e^{j\sqrt{k^2-\beta^2}\, z}}{e^{j\sqrt{k_0^2-\beta^2}\, h}\left(1-\Gamma e^{-j2\sqrt{k_0^2-\beta^2}\, h}\right)}$$

A and Γ can be found from the other boundary condition at $z=0$ and from the discontinuity condition $\left.\frac{\partial \psi}{\partial z}\right|_{h-}^{h+} = -\frac{\mu_0}{4\pi^2}$ at $z = h$.

We find that $\Gamma = \dfrac{\gamma - \kappa \gamma_0}{\gamma + \kappa \gamma_0}$ where $\kappa = \epsilon/\epsilon_0$

and $\gamma = \sqrt{k^2 - \beta^2}$, $\gamma_0 = \sqrt{k_0^2 - \beta^2}$. Also

$$A = \frac{-\mu_0/4\pi^2}{-j\gamma_0\left[1 + \dfrac{1+\Gamma e^{-2j\gamma_0 h}}{1-\Gamma e^{-2j\gamma_0 h}}\right]}$$

For $z \geq h$ we now get

$$\psi = \psi_1 = \frac{-j\mu_0}{8\pi^2} \iint_{-\infty}^{\infty} \frac{1}{\gamma_0}\left[1 - e^{-2j\gamma_0 h}\frac{\gamma - K\gamma_0}{\gamma + K\gamma_0}\right] e^{-j\beta_x x - j\beta_y y - j\gamma_0(z-h)} \, d\beta_x \, d\beta_y$$

This represents the solution as a spectrum of plane waves radiated directly from the source plus plane waves reflected from the interface and appearing to come from the image point at $z = -h$.

To transform to cylindrical coordinates let $\beta_x = w\cos\phi$, $\beta_y = w\sin\phi$, $x = \rho\cos\theta$, $y = \rho\sin\theta$. We then get

$$\psi_1 = \frac{-j\mu_0}{8\pi^2}\int_0^{2\pi}\int_0^{\infty}\frac{1}{\gamma_0}\left[1 - e^{-2j\gamma_0 h}\frac{\gamma - K\gamma_0}{\gamma + K\gamma_0}\right]e^{-jw\rho\cos(\phi-\theta)}$$

$$\cdots e^{-j\gamma_0(z-h)} \, w \, dw \, d\phi$$

where now $\gamma_0 = \sqrt{k_0^2 - w^2}$, $\gamma = \sqrt{k^2 - w^2}$

Using the result $\int_0^{2\pi} e^{-jw\rho\cos(\phi-\theta)} \, d\phi = 2\pi J_0(w\rho)$

where J_0 is the zero order Bessel function we get

Sommerfeld's classical result

$$\psi_1 = -\frac{j\mu_0}{4\pi}\int_0^\infty \frac{1}{\gamma_0}\left[1-\Gamma e^{-2j\gamma_0 h}\right] J_0(w\rho)\, e^{-j\gamma_0(z-h)}\, w\,dw$$

For $\Gamma = 0$ we must obtain the well known result $\psi_1 = \frac{\mu_0}{4\pi}\frac{e^{-jk_0\sqrt{\rho^2+(z-h)^2}}}{\sqrt{\rho^2+(z-h)^2}}$ for radiation from a current element in free space. Therefore we see that

$$\int_0^\infty \frac{w\,J_0(w\rho)}{\sqrt{k_0^2-w^2}}\, e^{-j\sqrt{k_0^2-w^2}\,(z-h)}\,dw = j\,\frac{e^{-jk_0R_1}}{R_1}$$

where $R_1^2 = \rho^2+(z-h)^2$. We are now able to express ψ_1 in the form

$$\psi_1 = \frac{\mu_0}{4\pi}\left[\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2} + 2K(-j)\int_0^\infty \frac{e^{-j\gamma_0(z+h)}\,J_0(w\rho)\,w\,dw}{\gamma+K\gamma_0}\right]$$

where $R_2^2 = \rho^2+(z+h)^2$. For infinite conductivity K becomes infinite and the integral gives twice the term in R_2 above to yield

$$\psi_1 = \frac{\mu_0}{4\pi}\left[\frac{e^{-jk_0R_1}}{R_1} + \frac{e^{-jk_0R_2}}{R_2}\right], \quad K\to\infty.$$

To examine the problem further note that $J_0(w\rho) = \frac{1}{2}\left[H_0^2(w\rho) + H_0^1(w\rho)\right]$. In the part of the integral involving H_0^1 we can put $w = -\xi$ to get

$$\int_0^{-\infty} \frac{e^{-j\gamma_0(z+h)} H_0^1(-\xi\rho)(-\xi)(-d\xi)}{\gamma + K\gamma_0}$$

and since γ, γ_0, are even functions of ξ and $H_0^1(-\xi\rho) = -H_0^2(\xi\rho)$ we get

$$\int_{-\infty}^{0} \frac{e^{-j\gamma_0(z+h)} H_0^2(\xi\rho)\,\xi\, d\xi}{\gamma + K\gamma_0}$$

Using this result enables us to express ψ_1 as

$$\psi_1 = \frac{\mu_0}{4\pi}\left[\frac{e^{-jk_0R_1}}{R_1} - \frac{e^{-jk_0R_2}}{R_2} + 2KI\right] \text{ where}$$

$$I = -j\int_{-\infty}^{\infty} \frac{w\, H_0^2(w\rho)\, e^{-j\gamma_0(z+h)}}{2(\gamma + K\gamma_0)}\, dw$$

Note that there is a pole when $\gamma + K\gamma_0 = 0$ which is the Zenneck surface wave discussed earlier. There has been considerable controversy in the past as regards whether or not the Zenneck surface wave is excited or not. A discussion of this aspect of the problem is given below.[†]

The integrand for I has branch points at $w = \pm k_0$ and $\pm k$ and is a four valued function. It may be considered single valued on a four sheeted Riemann surface. The correct sheet of the Riemann surface to carry out the integration on is the one for which Imag. $\sqrt{k^2 - w^2} \le 0$ and Imag. $\sqrt{k_0^2 - w^2} \le 0$ in order to get decaying waves at infinity in physical space. The proper branch cuts are the lines which give Imag. γ = Imag. $\gamma_0 = 0$ and are illustrated below for k_0 real. The surface wave pole is the root of $\sqrt{k^2 - w^2} + K\sqrt{k_0^2 - w^2} = 0$ which may be solved for $w = w_p$ to give $w_p = \pm\sqrt{\frac{K}{K+1}}\, k_0$.

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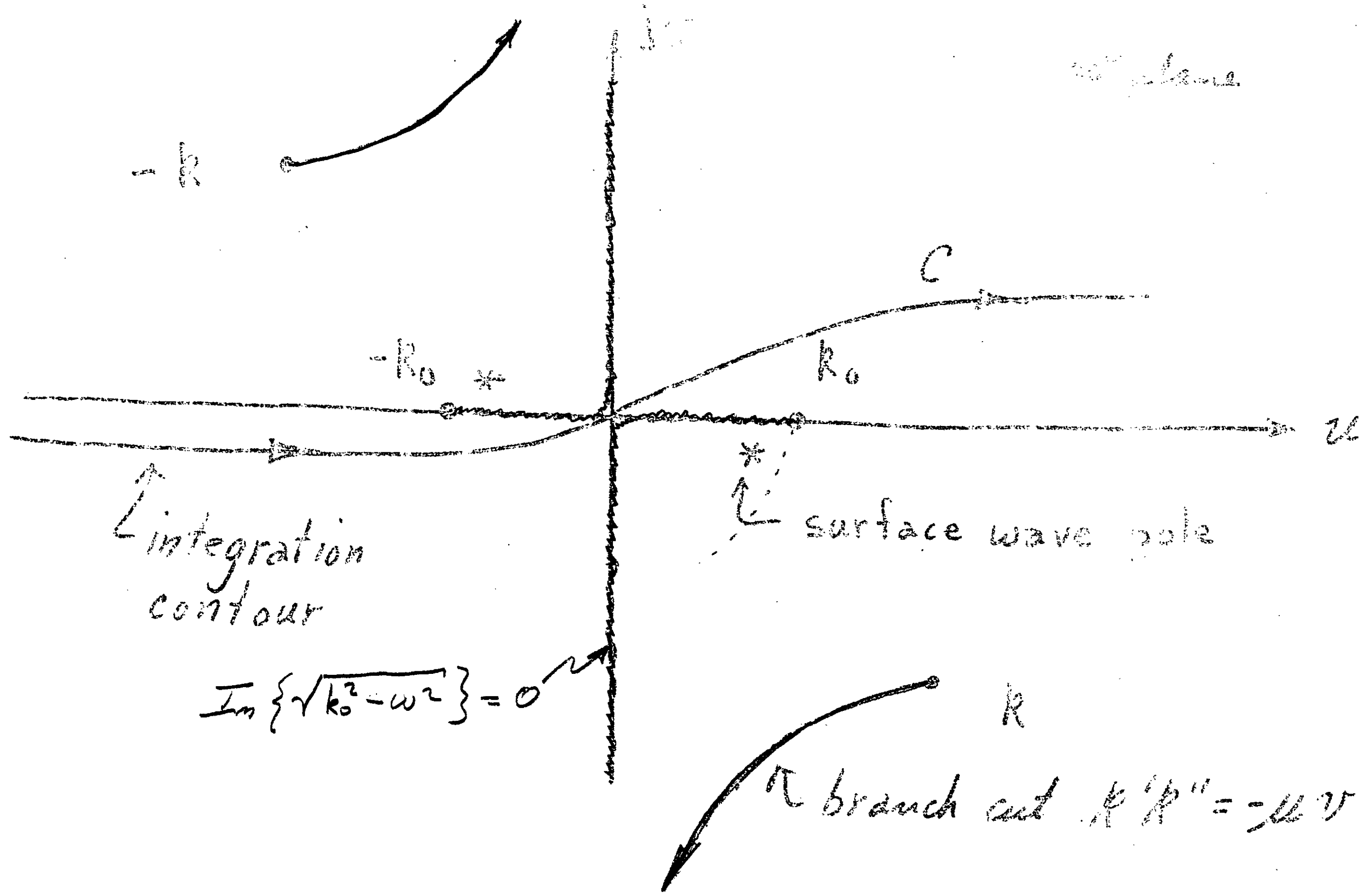
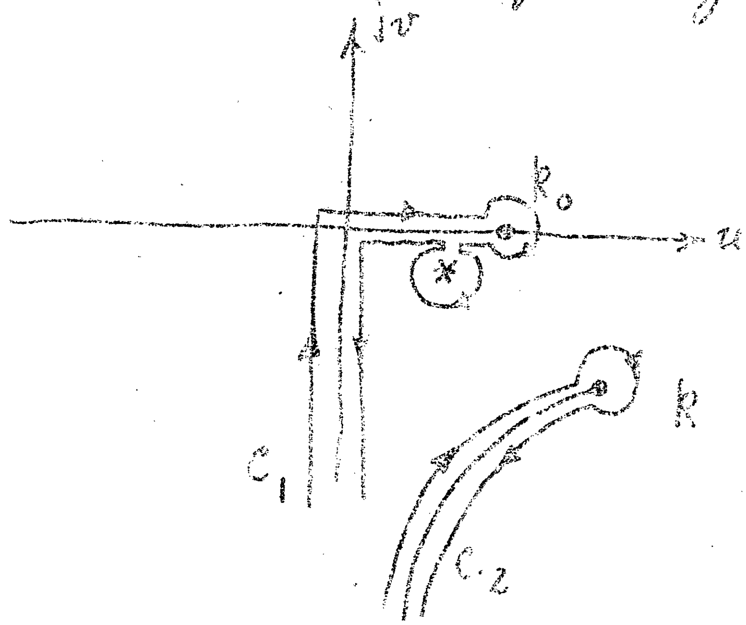
† See for example: I. Kahan, G. Eckart, On the existence of a surface wave in dipole radiation over a plane earth. Proc. IRE, vol. 38, pp 807-812, July 1950.



Note that $|w_p| = k_0\left[\frac{K'^2+K''^2}{(K'+1)^2+K''^2}\right]^{1/2} < k_0$ since $K = K' - jK''$ for a lossy earth. The phase angle of w_p is $-\frac{1}{2}\left[\tan^{-1}\frac{K''}{K'} - \tan^{-1}\frac{K''}{K'+1}\right]$ in the 4'th quadrant and hence the pole is located within the k_0 circle and slightly below the real axis since for a typical ground K'' and K' are quite large compared with unity.

For $z+h$ greater than unity we can deform the orginal contour C into the lowerhalf [illegible] I. At [illegible] infinity the integrand vanishes since $H_0^{(2)}(w\rho) \sim e^{-iw\rho} = C e^{v\rho}$ and vanishes for v negative. I is thus given by two branch cut integrals plus a residue term (surface wave) from the pole encirclement since the pole lies on the proper Riemann surface with branch cuts chosen as in the earlier figure. The new contour of integration is shown below.



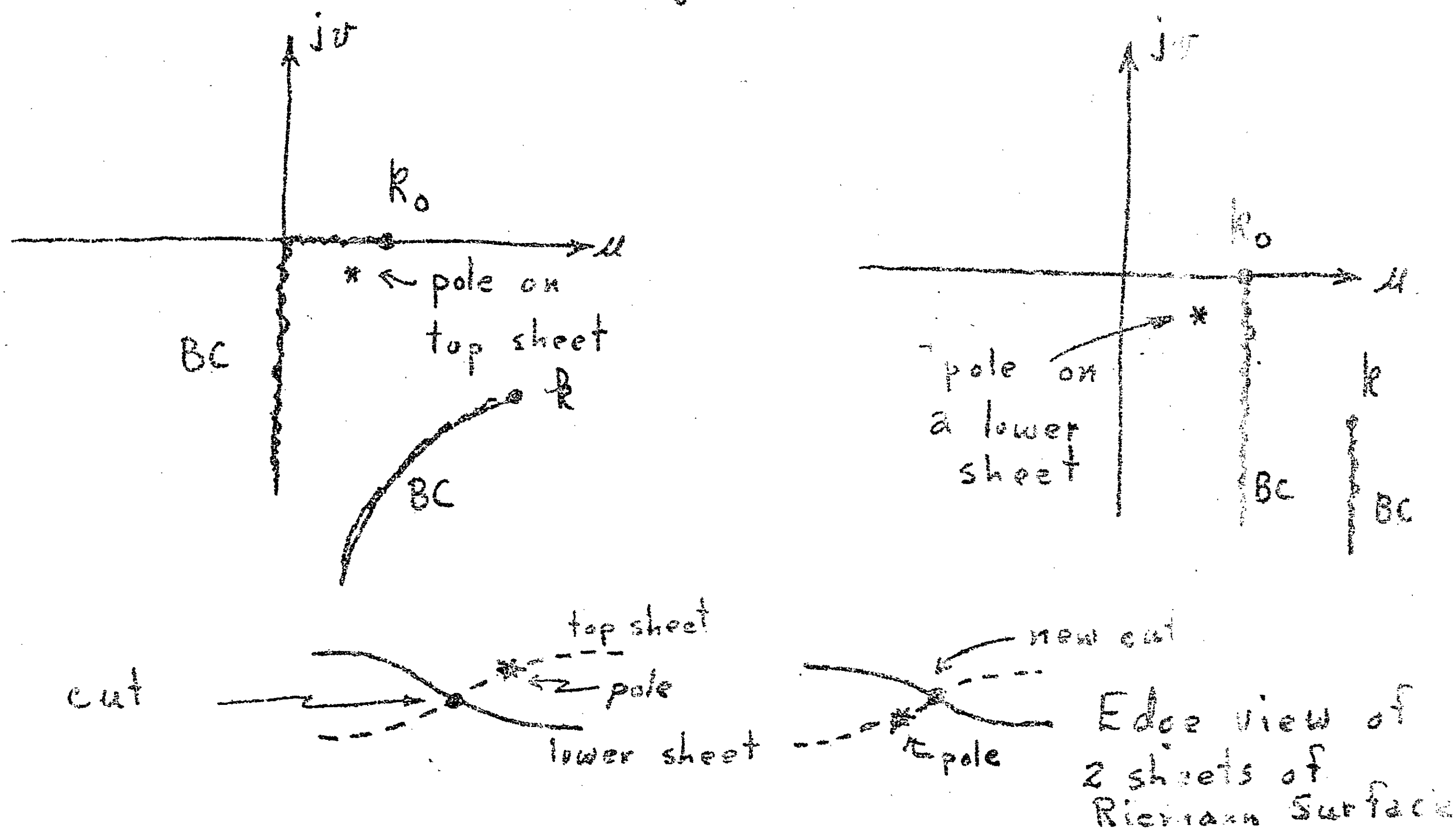
Since the waves arising from the branch cut integral around k are much more highly damped than those arising from the integration around k_0 only the latter integral needs to be retained in many practical situations where Imag. $k \ll k_0$. Thus $I \cong \int_{C_1} (\quad)\, dw$. The Zenneck surface wave pole is enclosed and the integration around this pole gives the surface wave contribution

$$\frac{-2\pi j(-j)\, w_p\, H_0^2(w_p \rho)\, e^{-j\sqrt{k_0^2 - w_p^2}\,(z+h)}}{2\frac{d}{dw}\left[\sqrt{k^2-w^2} + K\sqrt{k_0^2-w^2}\right]\Big|_{w=w_p}}$$

to I. In this sense the surface wave can be said to be excited. However, in evaluating the rest of the integral along C_1 by the method of steepest descent the modified saddle point technique must be employed since the surface wave pole is very close to the branch point at k_0 which becomes the saddle point for $\theta = \pi/2$ if we transform over to the ϕ plane via $w = k_0 \sin\phi$. It will then be found that the above surface wave contribution will be cancelled out. We examine this aspect of the problem below but from a slightly different point of view than that just outlined.

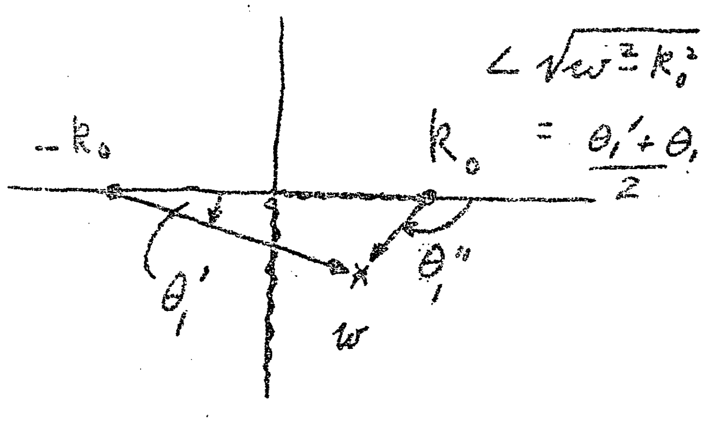
We may choose the branch cuts to run vertically downwards (and upwards) from the branch points as long as we terminate these cuts parallel to the imaginary axis at infinity so as to make $e^{-j\gamma_0(z+1)}$ and $e^{j\gamma z}$ exponentially decaying on the semi-circle joining the branch cuts in the lower half plane.

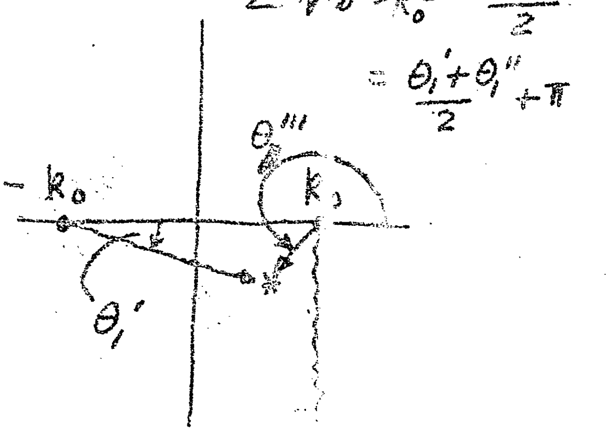
With this choice of branch cuts the surface wave pole moves from the proper or top sheet of the Riemann surface to a lower sheet as we pull the branch cut from its original position to its new position (see Figure below)



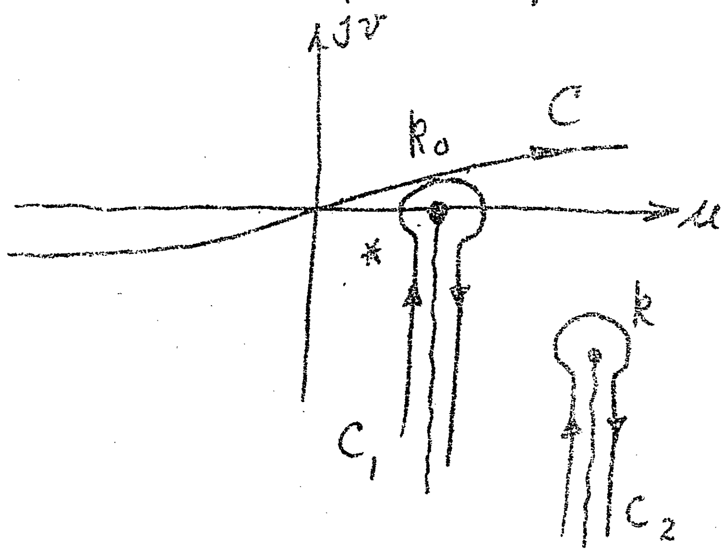
That the pole is no longer on the surface on which the integration is being performed is readily verified. Let $\angle -\frac{\sqrt{w^2-k^2}}{k}$ be θ_2. The pole occurs when

$$-\frac{\sqrt{w^2-k^2}}{k} = +\sqrt{w^2-k_0^2}$$ and requires that $\theta_2 = \theta_1$ where $\theta_1 = \angle\sqrt{w^2-k_0^2}$ for a solution. If the phase angle of $\sqrt{w^2-k_0^2}$ is compared for $w = w_p$ for the two positions of the branch cut then it is seen that if w_p was a solution for the case with the cut in the original position it is no longer a solution with the branch cut moved (to obtain a solution in the latter case requires choosing a different sign for $\sqrt{w^2-k_0^2}$ which is equivalent to going over to another sheet of the Riemann surface).

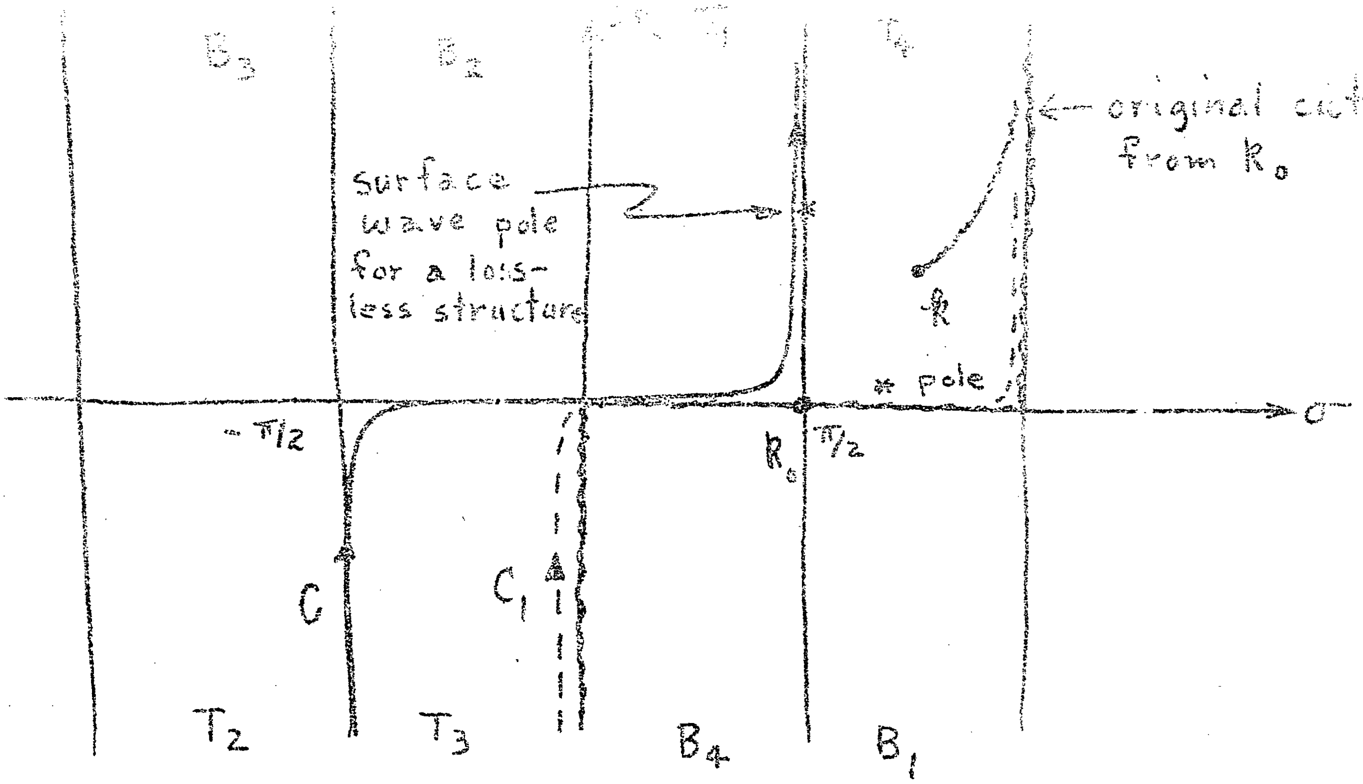




If we deform the contour C into the lower half plane as before we get I as the sum of two branch cut integrals but there is now no pole encirclement. In this case we would say that the Zenneck wave is not excited although the pole will still influence the integral along C_1.



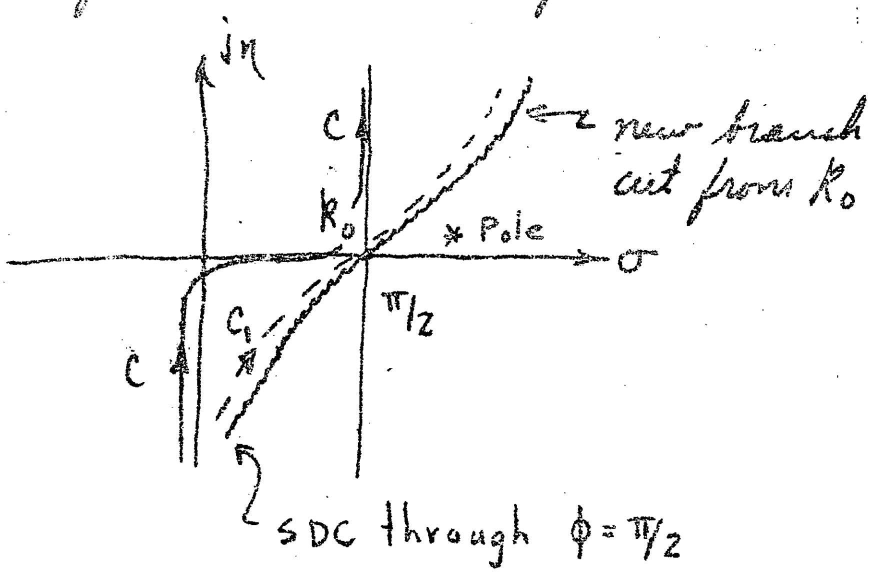
If we map w into the ϕ plane via $w = k_0 \sin\phi$, $\phi = \sigma + j\eta$ the original contour and branch cuts map as shown in the figure on the next page. Note that the branch cut from k_0 separates quadrants T_4, B_3; T_4, B_1; T_1, B_4; and T_3, B_4. The pole has the location shown and lies just slightly above the $\phi = \sigma$ axis. For a lossless structure the surface wave poles always lie on the $\phi = \pi/2$ axis, $\eta > 0$.



When C is deformed into C_1 plus C_2 the contour C_1 will coincide with the branch cut through k_0. In the deformation the pole is passed over and gives a residue contribution.

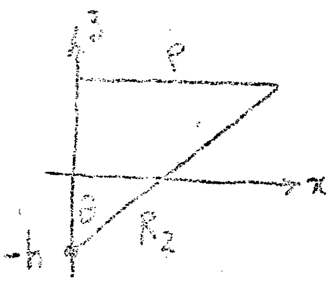
Consider now the modified branch cut from k_0 for which $w = k_0 + jv, -\infty < v \leq 0$. In the ϕ plane this corresponds to $\sin\sigma \cosh\eta = 1$, $k_0 \cos\sigma \sinh\eta = jv$. The steepest descent contour is given by $\cos(\sigma-\theta)\cosh\eta = 1 = \sin\sigma\cosh\eta$ for $\theta = \pi/2$. Therefore the new branch cut is the steepest descent contour passing through the point

$\phi = \pi/2$. But since this contour is inclined at an angle of $\pi/4$ or greater to the σ axis it passes above the pole. Thus deforming C into a contour along the new branch cut from k_0 is equivalent to deforming it into a steepest descent contour through $\phi = \pi/2$. In this deformation the pole is not passed over.



If we approximate I by the contour integral along C_1 i.e. the branch cut from k_0, we can evaluate this integral by the modified steepest descent method and choose the steepest descent contour passing through $\phi = \theta$ in the ϕ plane, or integrate along the cut from k_0 in the w plane.

We will begin with the integral along the SDC contour in the ϕ plane. Since $|w| \neq 0$ along this contour we can choose ρ so large that $|w\rho| \gg 1$ and then $H_0^2(w\rho) \sim \sqrt{\frac{2}{\pi w\rho}}\, e^{-j(w\rho - \frac{\pi}{4})}$



When we put $w = k_0 \sin\phi$, $\rho = R_2 \sin\theta$, $z + h = R_2 \cos\theta$ we get

$$I = -\frac{j}{2}\sqrt{\frac{2k_0}{\pi\rho}}\, e^{j\pi/4} \int_{SDC} e^{-jk_0 R_2 \cos(\phi-\theta)} \frac{\cos\phi\sqrt{\sin\phi}}{\sqrt{K - \sin^2\phi} + K\cos\phi}\, d\phi$$

For θ not close to $\pi/2$ the pole is not close to SDC and the standard saddle point method gives

$$I \sim -\frac{j}{2}\sqrt{\frac{2k_0}{\pi R_2 \sin\theta}}\, e^{j\pi/4} \frac{\cos\theta\sqrt{\sin\theta}}{\sqrt{K - \sin^2\theta} + K\cos\theta} \int_{SDC} e^{-jk_0 R_2 \cos(\phi-\theta)}\, d\phi$$

$$= \frac{\cos\theta}{\sqrt{K - \sin^2\theta} + K\cos\theta} \frac{e^{-jk_0 R_2}}{R_2}$$

The field is thus given by

$$\psi = \psi_1 \sim \frac{\mu_0}{4\pi}\left\{ \frac{e^{-jk_0 R_1}}{R_1} - \frac{\sqrt{K - \sin^2\theta} - K\cos\theta}{\sqrt{K - \sin^2\theta} + K\cos\theta}\, e^{-jk_0 R_2} \right\}, \quad \theta \neq \pi/2$$

and is the sum of a direct wave and a reflected wave. The above is called the space wave and vanishes for $\theta = \pi/2$.

However, for $\theta = \pi/2$ the pole is close to the SDC so ψ_1 is not well approximated by the above. For θ near $\pi/2$ we must use the modified steepest descent method.

Let $\zeta = 2e^{-j\pi/4} \sin \frac{\phi-\theta}{2}$, $d\zeta = e^{-j\pi/4} \cos \frac{\phi-\theta}{2}\, d\phi$

In ϕ plane surface wave pole is at ϕ_p where $\sin\phi_p = \sqrt{\frac{K}{K+1}}$. In ζ plane pole is at ζ_p. Consider ϕ a function of ζ. Then we can write

$$I \doteq -\frac{j}{2} e^{j\pi/4} \sqrt{\frac{2k_0}{\pi\rho}} \int_{-\infty}^{\infty} e^{-jk_0R_2}\, e^{-k_0R_2\zeta^2/2}\, G(\zeta)\, d\zeta$$

where $$G(\zeta)\, d\zeta = \frac{\sqrt{\sin\phi}\,\cos\phi}{\sqrt{K-\sin^2\phi} + K\cos\phi} \frac{d\phi}{d\zeta}\, d\zeta$$

Let $G_1(\zeta) = \sqrt{\sin\phi}\,\cos\phi\, d\phi/d\zeta$, $G_2(\zeta) = \sqrt{K-\sin^2\phi} + K\cos\phi$.

We now remove the pole at ζ_p and expand the remainder in a Taylor series about $\zeta = 0$ (the transformed saddle point). Thus $G(\zeta) = \frac{A}{\zeta-\zeta_p} + \frac{G(\zeta)(\zeta-\zeta_p) - A}{\zeta-\zeta_p}$

$$= \frac{A}{\zeta-\zeta_p} + G(0) + \frac{A}{\zeta_p} + \text{higher order terms in } \zeta.$$

The constant A is given by the residue at ζ_p which is $\frac{G_1(\zeta_p)}{dG_2/d\zeta_p}$

$= G_1(\zeta_p)/\left[\frac{dG_2}{d\phi}\frac{d\phi}{d\zeta_p}\right]_{\zeta_p} = \left[\sqrt{\sin\phi_p}\,\cos\phi_p\right]/\left[-\sin\phi_p\cos\phi_p(K-\sin^2\phi_p)^{-1/2} - K\sin\phi_p\right]$

while $G(0)$ is given by (put $\phi=\theta$)

$$G(0)=\sqrt{\sin\theta}\ \cos\theta\ /\left[\sqrt{K-\sin^2\theta}+K\cos\theta\right]$$

We now have $I=-\frac{j}{2}e^{j\pi/4}\sqrt{\frac{2k_0}{\pi R_2\sin\theta}}\ e^{-jk_0R_2}$

$$\left\{\int_{-\infty}^{\infty}\frac{A}{\zeta-\zeta_p}e^{-k_0R_2\zeta^2/2}d\zeta+\left[G(0)+\frac{A}{\zeta_p}\right]\int_{-\infty}^{\infty}e^{-k_0R_2\zeta^2/2}d\zeta\right\}$$

The second integral is equal to $[2\pi/k_0R_2]^{1/2}$ while the first integral equals $-j\pi\ e^{-k_0R_2\zeta_p^2/2}\operatorname{erfc}\left(j\sqrt{\frac{k_0R_2}{2}}\zeta_p\right)$ since Imag. $\zeta_p<0$. The phase of ζ_p when K is real is $\angle 2e^{-j\pi/4}\sin(\phi_p-\pi/2)/2$ for $\theta=\pi/2$ and is $-\pi/4$ so Imag. $\zeta_p<0$ since $\phi_p>\pi/2$. Since the pole is not crossed Imag. ζ_p remains less than zero.

If θ is not near $\pi/2$ then $\zeta_p=2e^{-j\pi/4}\sin\frac{\phi_p-\theta}{2}$ is not small. Hence $x=j\sqrt{\frac{k_0R_2}{2}}\zeta_p$ is large for k_0R_2 large. The first term in the asymptotic expansion of erfc x may then be used to approximate of x, i.e. $\operatorname{erfc} x \sim e^{-x^2}/(x\sqrt{\pi})$. In this case we find that the two terms involving A cancel each other to order R_2^{-1} and I takes on the same value as found earlier (pg. 82)

In the general case for all θ we obtain

$$\psi_1 = \frac{\mu_0}{4\pi}\underbrace{\left\{\frac{e^{-jk_0R_1}}{R_1} - \frac{\sqrt{K-\sin^2\theta} - K\cos\theta}{\sqrt{K-\sin^2\theta} + K\cos\theta}\,\frac{e^{-jk_0R_2}}{R_2}\right\}}_{\text{space wave}}$$

$$+\ \frac{\mu_0 K}{4\pi}\underbrace{\left[j e^{j\pi/4}\sqrt{\frac{2k_0}{\pi R_2\sin\theta}}\,e^{-jk_0R_2}\right]\left[j\pi e^{-k_0R_2\zeta_p^2/2}\operatorname{erfc}\left(j\sqrt{\frac{k_0R_2}{2}}\,\zeta_p\right)\right.}_{\text{Norton surface wave}}$$

$$\left.-\ \underline{\frac{1}{\zeta_p}\sqrt{\frac{2\pi}{k_0R_2}}}\ \right]\left[\frac{-1}{\sqrt{\sin\phi_p}}\ \frac{\cos\phi_p\sqrt{K-\sin^2\phi_0}}{\cos\phi_p + K\sqrt{K-\sin^2\phi_p}}\right]$$

The last part is called the Norton surface wave since this is the only part of the field which does not vanish at the surface $\theta = \pi/2$

If the expression for $\psi = \psi_2$ in the region $0 \le z \le h$ is examined it will be seen to be given by the analytic continuation of ψ_1 into the region $0 \le z \le h$, i.e. the expression we have obtained for ψ_1 gives ψ for all values of $z \geq 0$.

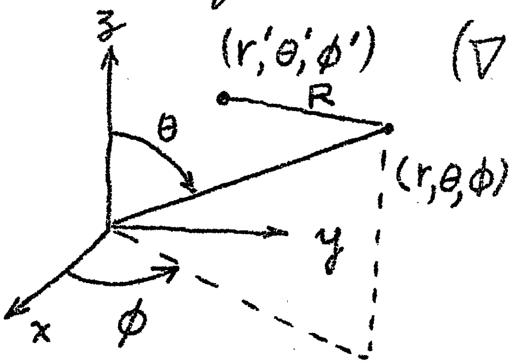
As regards the Zenneck surface wave there is no term in the expression for ψ that decays as $R_2^{-1/2}$ and hence this surface wave is really not excited. The Norton surface wave is not a true surface wave arising from a pole in Γ but represents the residual field at the surface. When θ is not close to $\pi/2$ it vanishes

- Chapter 5 -

Wave Solutions in Spherical Coordinates

1. Scalar Wave Solutions

The scalar Helmholtz equation for a unit point source at r', θ', ϕ' gives the free space Green's function G. The equation satisfied by G is



$$(\nabla^2 + k_0^2) G = - \frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r'^2 \sin\theta'} \qquad (1)$$

It is known that G can be expressed in the form

$$G = \frac{e^{-jk_0 R}}{4\pi R} \qquad (2)$$

By solving (1) in spherical coordinates we will obtain an alternative form for G which consists of an expansion in terms of elementary spherical waves.

Consider first the homogeneous equation

$$(\nabla^2 + k_0^2) G = \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} + k_0^2 \right] G = 0.$$

If we assume $G = f(r) g(\theta) h(\phi)$ we obtain

$$gh\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f}{\partial r}+k_0^2 f\right]+\frac{fh}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial g}{\partial\theta}\right]$$

$$+\frac{fg}{r^2}\left[\frac{1}{\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2}\right]=0.$$ If we divide by fgh and multiply by r^2 we get one group of terms which depend on r only and a second group which depends on θ, ϕ only. This leads to the conclusion that

$$\frac{1}{f}\frac{\partial}{\partial r}r^2\frac{\partial f}{\partial r}+k_0^2 r^2 = \lambda_r = \text{constant} \qquad (3a)$$

$$\frac{1}{g\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial g}{\partial\theta}+\frac{1}{h\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2}=-\lambda_r \qquad (3b)$$

If we now multiply (3b) by $\sin^2\theta$ we conclude that

$$\frac{1}{h}\frac{\partial^2 h}{\partial\phi^2}=-m^2 \qquad (4a)$$

and

$$\frac{\sin\theta}{g}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial g}{\partial\theta}+\lambda_r\sin^2\theta-m^2=0 \qquad (4b)$$

The solutions of (4a) are

$$h=\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix} \text{ or } e^{\pm jm\phi} \qquad (5a)$$

If the problem includes the range $\phi = 0$ to 2π then m must be an integer so that $h(2\pi) = h(0)$.

The solutions to (4b) are Legendre functions and only if we choose $\lambda_r = n(n+1)$, n an integer, do we get solutions which are finite at $\theta = 0, \pi$. These solutions are designated $P_n^m(\cos\theta)$ and satisfy

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial P_n^m}{\partial\theta} + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]P_n^m = 0 \quad (5b)$$

A second solution of (4b) is Q_n^m which is the Legendre function of the second kind. This function is singular at $\theta = 0, \pi$ so if the polar axis is included in the region of interest than the Q_n^m functions are not used in a problem where the field must be finite at $\theta = 0, \pi$.

The equation for $f(r)$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f}{\partial r} + k_0^2 f - \frac{n(n+1)}{r^2}f = 0 \qquad (5c)$$

This equation is closely related to Bessel's equation.

The solutions are called spherical Bessel functions and are written in the form $j_n(k_0 r)$, $y_n(k_0 r)$, $h_n^{1,2}(k_0 r)$ where

$$z_n(k_0 r) = \sqrt{\frac{\pi}{2k_0 r}}\, Z_{n+\frac{1}{2}}(k_0 r) \qquad (5d)$$

where z_n is a spherical Bessel function and $Z_{n+1/2}$ is the corresponding cylindrical Bessel function.

The functions $e^{\pm jm\phi}$, $\cos m\phi$, $\sin m\phi$ and P_n^m are orthogonal sets of functions so we can assume that a general solution of (1) has the form

$$G = \sum_{n=0}^{\infty} \sum_{m=0}^{n} b_{nm} \cos m(\phi-\phi')\, P_n^m(\cos\theta)\, f_n(r) \quad (6)$$

where the b_{nm} are amplitude coefficients and we have used the property $P_n^m = 0$ for $m > n$. The unknown radial functions $f_n(r)$ are still to be determined. We require G to be finite at $r=0$, to be continuous at $r=r'$, and to represent outward propagating waves at infinity. Hence we can choose $f_n(r)$ in the form

$$f_n(r) = \begin{cases} j_n(k_0 r)\, h_n^2(k_0 r'), & r \le r' \\ j_n(k_0 r')\, h_n^2(k_0 r), & r \ge r' \end{cases}$$

We can expand the unit source function in the following form

$$\frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{r'^2 \sin\theta'} = \frac{\delta(r-r')}{r'^2 \sin\theta'} \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{nm} P_n^m \cos m(\phi-\phi')$$

If we multiply by $\cos l(\phi-\phi')\, P_s^l \sin\theta$ and integrate we get $P_s^l(\cos\theta') \sin\theta' \cos l(\phi'-\phi') = \sin\theta' P_s^l(\cos\theta')$

$$= \int_0^{2\pi}\int_0^{\pi} \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_{nm} P_n^m P_s^l \sin\theta\, d\theta \cos m(\phi-\phi') \cos l(\phi-\phi')\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} C_{sl} \left(P_s^l\right)^2 \sin\theta\, d\theta \cos^2 l(\phi-\phi')\, d\phi$$

$$= C_{sl}\, \epsilon_{0l}\, \pi \frac{2}{2s+1} \frac{(s+l)!}{(s-l)!}, \quad \epsilon_{0l} = \begin{cases} 2, & l=0 \\ 1, & l>0 \end{cases}$$

Hence upon solving for C_{sl} we obtain

$$\delta(\vec{r}-\vec{r}') = \frac{\delta(r-r')}{r'^2} \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{2n+1}{2\pi\epsilon_{0m}} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta)$$

$$\cdots P_n^m(\cos\theta') \cos m(\phi-\phi') \qquad (7)$$

When we substitute (6) into (1) and use (7) we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} b_{nm}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f_n}{\partial r}+\left(k_o^2-\frac{n(n+1)}{r^2}\right)f_n\right]P_n^m(\cos\theta)$$

$$\cos m(\phi-\phi') = -\frac{\delta(r-r')}{r'^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2n+1}{2\pi\epsilon_{om}}\frac{(n-m)!}{(n+m)!}$$

$$P_n^m(\cos\theta)\,P_n^m(\cos\theta')\cos m(\phi-\phi') \qquad (8)$$

In view of the orthogonal properties of the functions $P_n^m \cos m(\phi-\phi')$ we must have

$$b_{nm}\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f_n}{\partial r}+\left(k_o^2-\frac{n(n+1)}{r^2}\right)f_n\right]$$

$$= -\frac{2n+1}{2\pi\epsilon_{om}}\frac{(n-m)!}{(n+m)!}\frac{\delta(r-r')}{r'^2} \qquad (9)$$

We now multiply by r^2 and integrate between r'_- and r'_+ to obtain

$$b_{nm}r^2\frac{\partial f_n}{\partial r}\Big|_{r'_-}^{r'_+} = -\frac{2n+1}{2\pi\epsilon_{om}}\frac{(n-m)!}{(n+m)!} \qquad (10)$$

since f_n is continuous at $r = r'$. The Wronskian determinant for spherical Bessel functions equals a constant divided by r^2 and is readily evaluated by using the asymptotic forms for the spherical Bessel functions. Thus we obtain

$$b_{nm} r'^2 \left[j_n(k_0 r') \frac{\partial h_n^2(k_0 r)}{\partial r} - h_n^2(k_0 r') \frac{\partial j_n(k_0 r)}{\partial r} \right]\Bigg|_{r=r'}$$

$= -j b_{nm}/k_0$. Our final solution for G is now found to be given by

$$G = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{-jk_0}{2\pi \epsilon_{om}} \frac{(2n+1)(n-m)!}{(n+m)!} P_n^m(\cos\theta)$$

$$P_n^m(\cos\theta') \cos m(\phi - \phi') \, j_n(k_0 r_<) h_n^2(k_0 r_>) \quad (11)$$

This solution must also be equal to the closed form solution given by (2).

Plane Wave Expansion in Terms of Spherical Wave Functions

If we let $r' \to \infty$ such that $r \ll r'$ then

$$R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} \approx r' - \frac{\vec{r}\cdot\vec{r}'}{\vec{r}'}$$ and

$$G = \frac{e^{-jk_o R}}{4\pi R} \sim \frac{e^{-jk_o r'}}{4\pi r'} \, e^{jk_o \vec{a}_{r_o}\cdot\vec{r}} \qquad (12)$$

which may be viewed as a plane wave with amplitude $(e^{-jk_o r'})/(4\pi r')$ and propagating in the direction $-\vec{a}_r' = -\vec{r}'/r'$.

In (11) we now use the asymptotic form $h_n^2(k_o r') \sim \frac{j^{n+1}}{k_o r'} e^{-jk_o r'}$ for $r' \to \infty$

so that for $r \ll r'$, $r' \to \infty$,

$$G \sim \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{-jk_o}{2\pi \epsilon_{om}} \frac{(2n+1)(n-m)!}{(n+m)!} \cos m(\phi-\phi')$$

$$P_n^m(\cos\theta) P_n^m(\cos\theta') \, j_n(k_o r) \frac{j^{n+1}}{k_o r'} e^{-jk_o r'} \qquad (13)$$

Since (12) and (13) are equal we obtain the following expansion for a plane wave

$$e^{jk_0 \vec{a}_{r'} \cdot \vec{r}} = e^{jk_0(x \sin\theta' \cos\phi' + y \sin\theta' \sin\phi' + z\cos\theta')}$$

$$= + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2j^n}{\epsilon_{om}} \frac{(2n+1)(n-m)!}{(n+m)!} \cos m(\phi - \phi')$$

$$P_n^m(\cos\theta)\, P_n^m(\cos\theta')\, j_n(k_0 r) \qquad (14)$$

<u>Scattering of a Scalar Plane Wave by a Sphere</u>

The expansion (14) can be used to solve the problem of scattering of a plane scalar wave by a sphere. If we let the incident plane wave be

$$\psi_i = e^{jk_0(x \sin\theta' \cos\phi' + y \sin\theta' \sin\phi' + z\cos\theta')} \qquad (15a)$$

the scattered field will have the form

$$\psi_s = \sum_{n=0}^{\infty} \sum_{m=0}^{n} C_{nm} P_n^m(\cos\theta) \cos m(\phi - \phi')\, h_n^2(k_0 r) \qquad (15b)$$

If the boundary conditions on the sphere of radius 'a' is $K_1 \psi + K_2\, d\psi/dr = 0$ where $\psi = \psi_i + \psi_s$ and K_1 and K_2 are given constants then we must have $K_1 \psi_s + K_2\, d\psi_s/dr = -(K_1\psi_i + K_2\, d\psi_i/dr)$. This relation must hold for each value of n and m. Hence by using (14) and (15b) we obtain

$$-\frac{2j^n(2n+1)(n-m)!}{\epsilon_{om}\,(n+m)!} P_n^m(\cos\theta')\left[K_1\, j_n(k_0 a) + K_2 \left.\frac{dj_n(k_0 r)}{dr}\right|_a\right]$$

$$= C_{nm}\left[K_1\, h_n^2(k_0 a) + K_2 \left.\frac{dh_n^2(k_0 r)}{dr}\right|_a\right] \qquad (16)$$

which determines the C_{nm}. The special cases $\psi = 0$ or $d\psi/dr = 0$ on $r = a$ are obtained by putting K_2 and K_1, respectively, equal to zero.

The above solution is applicable to acoustic wave scattering by a sphere, but not valid for vector problems such as the electromagnetic case.

Expansion of Spherical Hankel Function

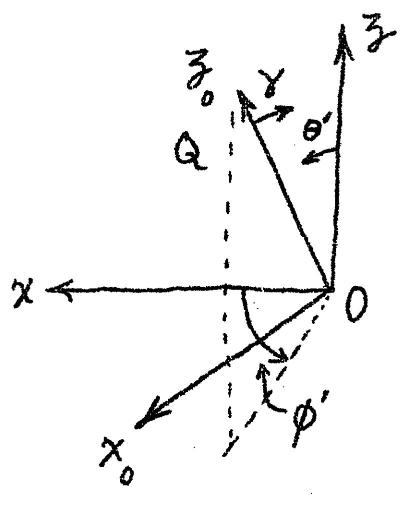
The field from a point source at $\vec{r}'$ is given by $e^{-jk_0R}(4\pi R)^{-1}$ where $R = |\vec{r}-\vec{r}'|$ and can also be expressed as $-(jk_0/4\pi)h_0^2(k_0|\vec{r}-\vec{r}'|)$. We may equate this to (11) and thus obtain

$$h_0^2(k_0|\vec{r}-\vec{r}'|) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{2}{\epsilon_{0m}} \frac{(2n+1)(n-m)!}{(n+m)!} P_n^m(\cos\theta)$$

$$P_n^m(\cos\theta')\cos m(\phi-\phi')\, j_n(k_0r_<)h_n^2(k_0r_>) \quad (17)$$

Addition Formula for Legendre Polynomials

A plane wave incident from the direction θ', ϕ' can be expanded relative to a new polar axis OQ shown in the figure. For simplicity we will choose the incident plane wave to be



$$e^{jk_0z_0} = e^{jk_0(x\sin\theta' + z\cos\theta')}$$

In the plane wave expansion given by (14) we can choose $\phi' = \theta' = 0$ to obtain

$$e^{jk_0 z} = \sum_{n=0}^{\infty} (2n+1) j^n P_n(\cos\theta) j_n(k_0 r) \qquad (18)$$

since $P_n^m(1) = 1$ for $m=0$ and equals zero for $m>0$.
By analogy we have

$$e^{jk_0 z_0} = \sum_{n=0}^{\infty} (2n+1) j^n P_n(\cos\gamma) j_n(k_0 r)$$

which also must be equal to (14). Hence we have

$$\sum_{n=0}^{\infty} (2n+1) j^n P_n(\cos\gamma) j_n(k_0 r) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2 j^n}{\epsilon_{om}} \frac{(2n+1)(n-m)!}{(n+m)!}$$

$$P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi') j_n(k_0 r)$$

For the plane wave specified earlier $\phi'=0$. If $\phi' \neq 0$ then the incident wave is given by $e^{jk_0(x\cos\phi' + y\sin\phi')\sin\theta' + jk_0 z\cos\theta'}$
The above relation can hold for all values of r only if the coefficients of $j_n(k_0 r)$ are equal. By equating the coefficients of $j_n(k_0 r)$ we obtain the desired addition formula which is

$$P_n(\cos\gamma) = \sum_{m=0}^{n} \frac{2}{\epsilon_{om}} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi-\phi') \qquad (19)$$

The new polar axis, with polar angle γ, is in the direction specified by the angles θ', ϕ' relative to the original polar axis.

Alternative Representations for Scalar Green's Function in Spherical Coordinates

G is a solution of $\nabla^2 G + k_0^2 G = -\delta(\vec{r}-\vec{r}_0)$ which may be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial G}{\partial r} + k_0^2 G + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta} \sin\theta \frac{\partial G}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 G}{\partial \phi^2}$$

$$= - \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{r_0^{\,2} \sin\theta_0} = - \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{r^2 \sin\theta}$$

Define the following operators:

$$\mathcal{L}_r = \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + k_0^2 r^2 , \quad \mathcal{L}_\theta = \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} , \quad \mathcal{L}_\phi = \frac{\partial^2}{\partial \phi^2} .$$

Thus we have

$$\left(\mathcal{L}_r + \frac{1}{\sin\theta}\mathcal{L}_\theta + \frac{1}{\sin^2\theta}\mathcal{L}_\phi\right) G = - \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{\sin\theta}$$

Associated with this equation is a one dimensional Green's function problem $(\mathcal{L}_r + \lambda)\, G_r = -\delta(r-r_0)$ and a two dimensional Green's function problem

$$\left(\mathcal{L}_\theta + \frac{1}{\sin\theta}\mathcal{L}_\phi + \lambda_1 \sin\theta\right) G_{\theta\phi} = -\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)$$

For later convenience we will replace λ_1 by $\nu(\nu+1)$. Associated with the two dimensional Green's function problem are two additional one dimensional problems in each can be chosen as

$$\left(\mathcal{L}_\theta + \nu(\nu+1)\sin\theta + \frac{\lambda_2}{\sin\theta} \right) G_\theta = -\delta(\theta-\theta_0)$$

$$\left(\mathcal{L}_\phi + \lambda_3 \right) G_\phi = -\delta(\phi-\phi_0)$$

The solution to a Green's function problem of the form $\left(\mathcal{L} + \sigma(x)\lambda \right) G_x(x,\lambda) = -\delta(x-x_0)$ has the property

$$\frac{1}{2\pi j} \oint_C G(x,\lambda)\, d\lambda = -\frac{\delta(x-x_0)}{\sigma(x)}$$

where C is a contour enclosing all of the singularities of $G(x,\lambda)$ in the λ eigenvalue plane. Thus G_ϕ has the property $\frac{1}{2\pi j} \oint_{C_\phi} G_\phi(\lambda_3)\, d\lambda_3 = -\delta(\phi-\phi_0)$.

The solution for $G_{\theta\phi}$ may be expressed as

$$G_{\theta\phi} = -\frac{1}{2\pi j} \oint_{C_\phi} G_\phi(\lambda_3)\, G_\theta(-\lambda_3)\, d\lambda_3$$ where C_ϕ

is a contour enclosing the singularities of G_ϕ but excluding all of the singularities of G_θ.

Proof. We must show that $g_{\theta\phi}$ as defined satisfy the required equation. We have (we have interchanged the order of integration and differentiation below)

$$\frac{-1}{2\pi j}\oint_{C_\phi}\left(\mathcal{L}_\theta + \frac{1}{\sin\theta}\mathcal{L}_\phi + \lambda_1 \sin\theta\right) G_\theta G_\phi \, d\lambda_3$$

$$= -\frac{1}{2\pi j}\oint_{C_\phi}\left[\frac{\lambda_3}{\sin\theta} G_\theta G_\phi - G_\phi \delta(\theta-\theta_0) - \frac{\lambda_3}{\sin\theta} G_\theta G_\phi - G_\theta \frac{\delta(\phi-\phi_0)}{\sin\theta}\right] d\lambda_3$$

$$= \frac{1}{2\pi j}\oint_{C_\phi}\left[G_\phi\,\delta(\theta-\theta_0) + G_\theta\,\frac{\delta(\phi-\phi_0)}{\sin\theta}\right] d\lambda_3$$

$$= -\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)$$ since $(\mathcal{L}_\theta + \lambda_1 \sin\theta) G_\theta G_\phi$

$= G_\phi \left[\frac{\lambda_3}{\sin\theta} G_\theta - \delta(\theta-\theta_0)\right]$ and $\frac{\mathcal{L}_\phi}{\sin\theta} G_\theta G_\phi$

$= \frac{G_\theta}{\sin\theta}\left[-\lambda_3 G_\phi - \delta(\phi-\phi_0)\right]$ and the integral involving G_θ vanishes since no singularities are included within C_ϕ while the integral involving G_ϕ gives $-\delta(\phi-\phi_0)\,\delta(\theta-\theta_0)$. Therefore $g_{\theta\phi}$ as defined is a solution of the two dimensional Green's function problem.

The solution for $G_\theta(\theta|\theta_0, \lambda_2)$ has the property

$$\frac{1}{2\pi j}\oint_{C_\theta} G_\theta(\lambda_2)\, d\lambda_2 = -\sin\theta\,\delta(\theta-\theta_0)$$ since λ_2 is

multiplied by $(\sin\theta)^{-1}$ in the equation for G_θ. C_θ is a contour enclosing all of the singularities of G_θ. We can now show that another solution for $G_{\theta\phi}$ is

$$-\frac{1}{2\pi j}\oint_{C_\theta} G_\theta(\lambda_2)\, G_\phi(-\lambda_2)\, d\lambda_2$$

If we follow the same steps as before we find that we are led to the following integral to consider:

$$\frac{1}{2\pi j}\oint_{C_\theta}\left[G_\phi\,\delta(\theta-\theta_0) + G_\theta\,\frac{\delta(\phi-\phi_0)}{\sin\theta}\right] d\lambda_2$$ which is

equal to $-\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)$ since C_θ encloses the singularities of $G_\theta(\lambda_2)$ but excludes the singularities of $G_\phi(-\lambda_2)$. Thus $G_{\theta\phi}$ can be expressed as an integral over the spectrum of either G_θ or G_ϕ.

Before giving the solution for the 3 dimensional Green's function G we need to show that $G_{\theta\phi}(\lambda_1)$ has the property $\frac{1}{2\pi j}\oint_C G_{\theta\phi}(\lambda_1)\,d\lambda_1 = -\frac{\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{\sin\theta_0}$

which is an extension of the corresponding property for a one dimensional Green's function. The equation for $G_{\theta\phi}$ is $\left(\mathcal{L}_\theta + \frac{\mathcal{L}_\phi}{\sin\theta} + \lambda_1 \sin\theta\right)G_{\theta\phi} = -\delta(\theta-\theta_0)\delta(\phi-\phi_0)$

The homogeneous equation can be separated into the two equations $\left(\mathcal{L}_\theta - \frac{\lambda_2}{\sin\theta} + \lambda_1 \sin\theta\right)\Psi = 0$ and

$(\mathcal{L}_\phi + \lambda_2)\Phi = 0$ and defines two eigenvalue problems.

Let Φ_m be an eigenfunction with λ_{2m} an eigenvalue, i.e.

$(\mathcal{L}_\phi + \lambda_{2m})\Phi_m = 0$. Similarly let Ψ_{nm} with eigenvalues λ_{2m}, λ_{1n} be a solution of

$\left(\mathcal{L}_\theta - \frac{\lambda_{2m}}{\sin\theta} + \lambda_{1n}\sin\theta\right)\Psi_{nm} = 0$. In terms of these eigenfunctions we can express $G_{\theta\phi}$ in the form

$G_{\theta\phi} = \sum_{n,m} C_{nm}\Psi_{nm}\Phi_m$. The Ψ_{nm} and Φ_m satisfy the orthogonality properties [illegible] $\int \psi_{nm} \psi_{sm} \sin\theta \, d\theta = 0, \; n \neq s$

so we may determine the unknown coefficients by a Fourier series type analysis. Substituting the expansion for $G_{\theta\phi}$ into the governing equation gives

$$\sum_{n,m} C_{nm} (\lambda_1 - \lambda_{1n}) \sin\theta \; \Phi_m \psi_{nm} = -\delta(\theta - \theta_0)\delta(\phi - \phi_0)$$

By Fourier analysis we get $C_{nm} = -\dfrac{\Phi_m(\phi_0)\psi_{nm}(\theta_0)}{N_{nm}(\lambda_1 - \lambda_{1n})}$

where $N_{nm} = \iint \Phi_m^2 \psi_{nm}^2 \sin\theta \, d\theta \, d\phi$ and is a normalization constant. Thus we get

$$G_{\theta\phi} = -\sum_{n,m} \frac{\Phi_m(\phi)\,\Phi_m(\phi_0)\,\psi_{nm}(\theta)\,\psi_{nm}(\theta_0)}{N_{nm}(\lambda_1 - \lambda_{1n})}$$

In a similar manner we find that the expansion of $\delta(\theta - \theta_0)\delta(\phi - \phi_0)$ is given by

$$\delta(\theta - \theta_0)\,\delta(\phi - \phi_0) = \sin\theta_0 \sum_{n,m} \frac{\Phi_m(\phi)\Phi_m(\phi_0)\psi_{nm}(\theta)\psi_{nm}(\theta_0)}{N_{nm}}$$

Using the expansion for $G_{\theta\phi}$ we readily find that

$$\frac{1}{2\pi j} \oint_C G_{\theta\phi}(\lambda_1)\, d\lambda_1 = -\sum_{n,m} \frac{\Phi_m(\phi)\,\Phi_m(\phi_0)\,\psi_{nm}(\theta)\,\psi_{nm}(\theta_0)}{N_{nm}} =$$

[illegible]

which proves the stated result.

We can now show that the three dimensional Green's function G is given by

$$G = \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_r} \oint_{C_\phi} G_r(\lambda)\, G_\theta(-\lambda, -\lambda_3)\, G_\phi(\lambda_3)\, d\lambda\, d\lambda_3$$

$$= \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_r} \oint_{C_{\theta 2}} G_r(\lambda)\, G_\theta(-\lambda, \lambda_2)\, G_\phi(-\lambda_2)\, d\lambda\, d\lambda_3$$

$$= \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_\phi} \oint_{C_{\theta 1}} G_r(-\lambda_1)\, G_\theta(\lambda_1, -\lambda_3)\, G_\phi(\lambda_3)\, d\lambda_1\, d\lambda_3$$

$$= \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_{\theta 1}} \oint_{C_{\theta 2}} G_r(-\lambda_1)\, G_\phi(-\lambda_2)\, G_\theta(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2$$

where the contours are chosen so that

C_r encloses only the singularities of $G_r(\lambda)$ in the λ plane,

C_ϕ " " " " " $G_\phi(\lambda_3)$ " " λ_3 " ,

$C_{\theta 2}$ " " " " " $G_\theta(-\lambda, \lambda_2)$ " " λ_2 " ,

$C_{\theta 1}$ " " " " " $G_\theta(\lambda_1, \lambda_2)$ " " λ_1 " .

[illegible]

$$\cdots \, d\lambda \, d\lambda_3$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} \int_{\lambda} \int_{\lambda_3} \Big[\cdots \, G_\theta G_\phi - \frac{\delta(\theta-\theta_0)}{\sin\theta} G_r G_\theta$$

$$- \frac{\delta(\phi-\phi_0)}{\sin^2\theta} G_r G_\phi \Big] \, d\lambda \, d\lambda_3$$

In view of the way G_ρ and G_ϕ have been chosen [illegible] the properties of G_ρ and G_ϕ, the integral involving $G_\theta G_\phi$ and $G_r G_\theta$ vanish [illegible] we get $(\nabla^2 - k_0^2)G =$

$$\left(\frac{1}{2\pi}\right)^{3/2} \int_{\lambda} \int_{\lambda_3} - \frac{\delta(\theta-\theta_0)}{\sin\theta} G_r G_\phi \, d\lambda \, d\lambda_3 = - \frac{\delta(\theta-\theta_0)\,\delta(r-r_0)\,\delta(\phi-\phi_0)}{\sin\theta}$$

[illegible]

... planes then we get $(\nabla^2 - k^2)G =$

$$\left(-\frac{1}{2\pi j}\right)^2 \oint_{C_{\theta 1}} \oint_{C_{\theta 2}} \Big[-\delta(r-r_0)\, G_\theta(\lambda_1, \lambda_2)\, G_\phi(-\lambda_2)$$

$$- \frac{\delta(\theta-\theta_0)}{\sin\theta} G_r(-\lambda_1) G_\phi(-\lambda_2) - \frac{\delta(\phi-\phi_0)}{\sin^2\theta} G_r(-\lambda_1) G_\theta(\lambda_1,\lambda_2) \Big]$$

$$\cdots d\lambda_1\, d\lambda_2$$

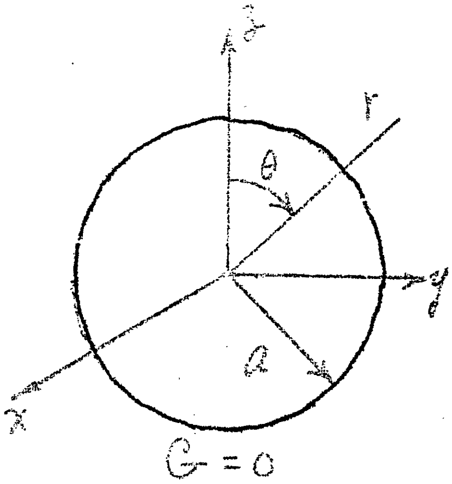
$$= \frac{-1}{2\pi j} \oint_{C_{\theta 1}} \Big[-\delta(r-r_0)\, G_{\theta\phi}(\lambda_1) - \frac{\delta(\phi-\phi_0)\,\delta(\theta-\theta_0)}{\sin\theta} G_r(-\lambda_1) \Big] d\lambda_1$$

$$= -\frac{\delta(r-r_0)\,\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{\sin\theta}$$

since the term involving $G_r G_\phi$ integrates to zero, the term in $G_\theta G_\phi$ gives $G_{\theta\phi}$ and the term in $G_r G_\theta$ integrates to zero in the λ_1 plane. We have also used the result that

$$-\frac{1}{2\pi j} \oint_{C_{\theta 1}} G_{\theta\phi}(\lambda_1)\, d\lambda_1 = -\frac{\delta(\theta-\theta_0)\,\delta(\phi-\phi_0)}{\sin\theta_0}$$

Green's Function for a Sphere (Dirichlet Boundary Conditions)



$$\nabla^2 G + k_0^2 G = -\delta(\vec{r} - \vec{r}_0)$$

$G = 0$ at $r = a$,

G is an outward propagating wave at $r = \infty$.

G will be obtained in terms of G_r, G_θ, and G_ϕ.

Solution for G_r

$$(\mathcal{L}_r + \lambda) G_r = -\delta(r - r_0) = \frac{d}{dr} r^2 \frac{dG_r}{dr} - \nu(\nu+1) G_r + k_0^2 r^2 G_r$$

where $\lambda = -\nu(\nu+1)$. For $r > r_0$ let $G_r = A h_\nu^2(k_0 r)$ and for $r < r_0$ let $G_r = B[h_\nu^2(k_0 a) j_\nu(k_0 r) - h_\nu^2(k_0 r) j_\nu(k_0 a)]$ which is chosen so that $G_r = 0$ at $r = a$. At $r = r_0$, G_r must be continuous and dG_r/dr must have a discontinuity equal to $-r_0^{-2}$. Thus we get

$$A h_\nu^2(k_0 r_0) = B[h_\nu^2(k_0 a) j_\nu(k_0 r_0) - h_\nu^2(k_0 r_0) j_\nu(k_0 a)]$$

$$A h_\nu^{2\prime} - B[h_\nu^2(k_0 a) j_\nu' - j_\nu(k_0 a) h_\nu^{2\prime}] = -\frac{1}{k_0 r_0^2}$$

The Wronskian determinant for the above equations equals a constant divided by r_0^2. For r_0 very large we have

$$j_n(k_0 r_0) = \frac{1}{k_0 r_0} \cos\left(k_0 r_0 - \frac{n+1}{2}\pi\right), \quad h_n^2(k_0 r_0) = \frac{j^{n+1}}{k_0 r_0} e^{-jk_0 r_0}$$

and hence $W = h_\nu^{2\,\prime}(k_0 r_0)\left[h_\nu^2(k_0 a) j_\nu(k_0 r_0) - h_\nu^2(k_0 r_0) j_\nu(k_0 a)\right]$

$- h_\nu^2(k_0 r_0)\left[h_\nu^2(k_0 a) j_\nu'(k_0 r_0) - h_\nu^{2\,\prime}(k_0 r_0) j_\nu(k_0 a)\right]$

is found to equal $-j h_\nu^2(k_0 a)/k_0^2 r_0^2$. We thus find that G_r is given by

$$G_r = \frac{-j k_0}{h_\nu^2(k_0 a)} \left\{ h_\nu^2(k_0 a) j_\nu(k_0 r_<) - h_\nu^2(k_0 r_<) j_\nu(k_0 a) \right\} h_\nu^2(k_0 r_>)$$

where $\begin{Bmatrix} r_< = r_0 \\ r_> = r \end{Bmatrix}$ for $r > r_0$ and $\begin{Bmatrix} r_< = r \\ r_> = r_0 \end{Bmatrix}$ for $r < r_0$.

A solution for G_r in terms of an eigenfunction expansion is also possible. G_r has simple poles whenever $h_\nu^2(k_0 a)$ vanishes. Thus there exists a set of eigenvalues ν_i and eigenfunctions $h_{\nu_i}^2(k_0 r)$ in which G_r can be expanded. From differential equation it is seen that

$$\int_a^\infty h_{\nu_i}^2(k_0 r)\, h_{\nu_j}^2(k_0 r)\, dr = 0 \text{ for } \nu_i \neq \nu_j$$

If we let the normalization factor be $N_{\nu_i} = \int_a^\infty \left[h_{\nu_i}^2(k_0 r) \right]^2 dr$ and now assume $G_r = \sum_i C_i h_{\nu_i}^2(k_0 r)$ we readily find that

$$G_r = \sum_i \frac{h_{\nu_i}^2(k_0 r_0)\, h_{\nu_i}^2(k_0 r)}{N_{\nu_i} \left[\nu(\nu+1) - \nu_i(\nu_i+1) \right]}.$$

<u>Solution For G_ϕ</u>

$$\frac{d^2 G_\phi}{d\phi^2} + \lambda_3 G_\phi = -\delta(\phi - \phi_0), \quad G_\phi(2\pi) = G_\phi(0)$$

The solutions are the same as in the cylindrical problem. From our earlier notes we thus have

$$G_\phi = -\sum_{m=0}^{\infty} \frac{\cos m(\phi - \phi_0)}{\epsilon_{0m}\, \pi\, (\lambda_3 - m^2)} = -\sum_{m=-\infty}^{\infty} \frac{e^{j\sqrt{\lambda_3}\,|\phi - \phi_0 - 2m\pi|}}{2j\sqrt{\lambda_3}}$$

$$= -\frac{\cos\sqrt{\lambda_3}\,(\pi - \phi_> + \phi_<)}{2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi}$$

<u>Solution For G_θ</u>

A solution for G_θ as an expansion in terms of the eigenfunctions $P_n^m(\cos\theta)$ is readily found. G_θ is a solution of $\left\{ \frac{d}{d\theta} \sin\theta \frac{d}{d\theta} + \lambda_1 \sin\theta - \frac{m^2}{\sin\theta} \right\} G_\theta = -\delta(\theta - \theta_0)$

and must remain bounded in the range $0 \le \theta \le \pi$.

If we let $G_\theta = \sum_{n=0}^{\infty} P_n^m(\cos\theta)\, C_n$ we obtain

$$\sum_{n=0}^{\infty} C_n \left[\lambda_1 - n(n+1)\right] P_n^m \sin\theta = -\delta(\theta-\theta_0)$$

Since $\int_0^{\pi} P_n^m(\cos\theta)\, P_s^m(\cos\theta) \sin\theta \, d\theta = \frac{2\delta_{ns}}{2n+1} \frac{(n+m)!}{(n-m)!}$

we find that

$$G_\theta = -\sum_{n=0}^{\infty} \frac{P_n^m(\cos\theta_0)\, P_n^m(\cos\theta)}{2 \quad [\lambda_1 - n(n+1)]} \frac{(2n+1)(n-m)!}{(n+m)!}$$

For a second solution we note that $P_\nu^m(\cos\theta)$ is bounded at $\cos\theta = 1$ for all values of ν, but not at $\cos\theta = -1$. The second kind of Legendre function $Q_\nu^m(\cos\theta)$ is singular at both $\theta = 0$ and π and must be excluded for the sphere problem. For G_θ we may choose $G_\theta = A P_\nu^m(\cos\theta)$ for $\theta < \theta_0$ and $G_\theta = B P_\nu^m(-\cos\theta)$ for $\theta > \theta_0$. Continuity at $\theta = \theta_0$ gives $A P_\nu^m(\cos\theta_0) - B P_\nu^m(-\cos\theta_0) = 0$. From differential equation we find that

$\left.\dfrac{d\Theta}{d\theta}\right|_{\theta_0^-}^{\theta_0^+} = \dfrac{1}{\sin\theta_0}$ and hence $+B\sin\theta_0\, P_\nu^{m\prime}(-\cos\theta_0)$

$- A\sin\theta_0\, P_\nu^{m\prime}(\cos\theta_0) = -1/\sin\theta_0$ where the prime means differentiation with respect to $-\cos\theta_0$ or $\cos\theta_0$, i.e. with respect to the arguement. In the above expressions we have replaced λ_1 by $\nu(\nu+1)$. Using the Wronskian relation $P_\nu^m(\cos\theta_0)\, P_\nu^{m\prime}(-\cos\theta_0) - P_\nu^{m\prime}(\cos\theta_0)\, P_\nu^m(-\cos\theta_0)$

$$= -\frac{2\sin(\nu+m)\pi}{\pi\sin^2\theta}\,\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}$$ we obtain

$$G_\theta = -\frac{\pi}{2}\,\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\,\frac{P_\nu^m(\cos\theta_<)\,P_\nu^m(-\cos\theta_>)}{\sin(\nu+m)\pi}$$

Now $\Gamma(\nu-m+1)$ has simple poles when $\nu-m+1 = 0, -1, \cdots = -S$, $S = 0, +1, +2, \cdots$. $\Gamma(\nu+m+1)$ has poles when $\nu+m+1 = -S$, $S = 0, 1, 2$, etc.† But $\sin(\nu+m)\pi$ has zeroes when $\nu+m = \pm S$, $S = 0, 1, 2, \ldots$. Thus the poles of $\Gamma(\nu+m+1)$ at $\nu+m = -1, -2, -3, -4 \cdots$ are cancelled by the zeroes of $\sin(\nu+m)\pi$ at the same points. Therefore the poles are located at $\nu = m-1, m-2, m-3, \cdots$ and $\nu = -m, -m+1, -m+2, \cdots$.

† Note that $\Gamma(z) = \dfrac{e^{-\gamma z}}{z}\left[\prod_{n=1}^{\infty}(1+z/n)\,e^{-z/n}\right]^{-1}$

It is noted that some of the poles overlap to give double poles in the ν plane unless m is a negative integer. Since $m^2 = \lambda_2$ and $m = \pm j\sqrt{\lambda_2}$ we can choose either branch. It is convenient to choose that branch for which m is negative when λ_2 is **negative** and real. If we then write $-m$ for our previous m and consider $-m$ to be a negative number for λ_2 **negative** and real we have

$$G_\theta = -\frac{\pi}{2}\,\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\;\frac{P_\nu^{-m}(\cos\theta_<)\,P_\nu^{-m}(-\cos\theta_>)}{\sin(\nu-m)\pi}$$

The poles are now located at $\nu = m, m+1, m+2, ---, -m-1, -m-2, -m-3, ---,$ etc. in the ν plane and do not overlap for m a positive integer. There is no loss in generality in choosing the above branch of $\sqrt{\lambda_2}$ since $(-1)^m P_\nu^{-m}(z) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^{m}(z)$ so P_ν^{-m} and P_ν^{m} are linearly dependent solutions.

In the λ_1 plane, where $\lambda_1 = \nu(\nu+1)$ there are two values of ν for each λ_1, i.e. $\nu = \mu$ gives $\lambda_1 = \mu(\mu+1)$ and $\nu = -\mu-1$ gives $\lambda_1 = (-\mu-1)(-\mu-1+1) = \mu(\mu+1)$ which is the same. In addition $P_\nu^{-m} = P_{-\nu-1}^{-m}$ so only one set of values of λ_{1i} and ν_i will be needed.

transformation from the λ_1 plane to the ν plane is given by $\nu = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+\lambda_1}$. If we choose the branch $\nu = -\frac{1}{2}+\sqrt{\frac{1}{4}+\lambda_1}$ and $-\pi \leq$ phase $\lambda_1 \leq \pi$ then the λ_1 plane maps into the right half ν plane $Re\,\nu \geq 0$. In this case only the poles at $\nu = m, m+1, m+2, \cdots$ are to be considered. If instead we choose $\nu = -\frac{1}{2}-\sqrt{\frac{1}{4}+\lambda_1}$ then the poles at $\nu = -m-1, -m-2, \cdots$ in the ν plane are of interest. The first choice leads to the use of the functions P_{m+s}^{-m} while the second choice leads to the use of the functions P_{-m-s-1}^{-m} which are equal to the P_{m+s}^{-m} and therefore give the same result.

<u>Three Dimensional Green's Function</u>

1. <u>Classical Eigenfunction Form</u>

$$G = \left(-\frac{1}{2\pi j}\right)^2 \oint_{C_\phi} \oint_{C_{\theta 1}} G_r(-\lambda_1) G_\theta(\lambda_1, -\lambda_3) G_\phi(\lambda_3)\, d\lambda_1\, d\lambda_3$$

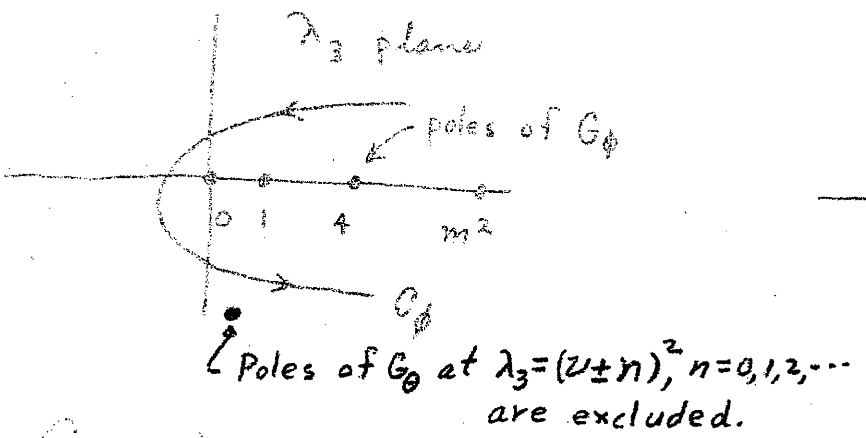
$$= \left(-\frac{1}{2\pi j}\right)^2 \oint_{C_\phi} \oint_{C_{\theta 1}} G_r(\nu) G_\theta(\nu, -\lambda_3) G_\phi(\lambda_3)\, d\lambda_3\, d\nu(\nu+1)$$

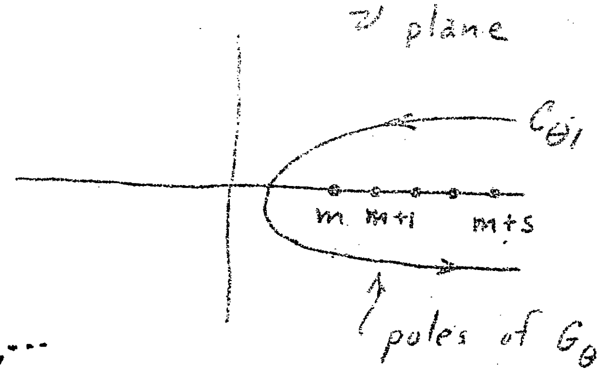
$$\left(\frac{1}{2\pi j}\right)^2 \int_{C_\phi} \int_{C_{\theta 1}} \frac{-jk_0}{h_\nu^2(k_0 a)} \left\{ h_\nu^2(k_0 a)\, j_\nu(k_0 r_<) - h_\nu^2(k_0 r_<)\, j_\nu(k_0 a) \right\}$$

$$h_\nu^2(k_0 r_>) \left(\frac{-\pi}{2}\right) \frac{\Gamma(\nu+\sqrt{\lambda_3}+1)}{\Gamma(\nu-\sqrt{\lambda_3}+1)} \; \frac{P_\nu^{-\sqrt{\lambda_3}}(\cos\theta_<)\, P_\nu^{-\sqrt{\lambda_3}}(-\cos\theta_>)}{\sin(\nu-\sqrt{\lambda_3})\pi}$$

$$\left\{ - \sum_{m=0}^{\infty} \frac{\cos m(\phi-\phi_0)}{\epsilon_{om}\pi(\lambda_3-m^2)} \right\} d\lambda_3 \; d\nu(\nu+1)$$

where C_ϕ and $C_{\theta 1}$ are chosen as illustrated in the λ_3 and ν planes.





A residue evaluation gives

$$G = -\sum_{m=0}^{\infty} \sum_{s=0}^{\infty} \frac{jk_0}{h_{m+s}^2(k_0 a)} \left\{ h_{m+s}^2(k_0 a)\, j_{m+s}(k_0 r_<) \right.$$

$$\left. - h_{m+s}^2(k_0 r_<)\, j_{m+s}(k_0 a) \right\} h_{m+s}^2(k_0 r_>) \; \frac{2m+2s+1}{2} \; \frac{\Gamma(2m+s+1)}{\Gamma(s+1)(-1)^s}$$

$$P_{m+s}^{-m}(\cos\theta_<)\, P_{m+s}^{-m}(-\cos\theta_>) \; \frac{\cos m(\phi-\phi_0)}{\epsilon_{om}\pi}$$

If we put $m+s=n$ then $\Gamma(2m+s+1) = (n+m)!$, $\Gamma(s+1) = (n-m)!$. Also if we use the relations $P^{-m}_{m+s} = (-1)^m \dfrac{\Gamma(s+1)}{\Gamma(2m+s+1)} P^{m}_{m+s}$ and $(-1)^s P^{m}_{m+s}(-\cos\theta_>) = P^{m}_{m+s}(\cos\theta_>)$ and note that $\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}(\ ) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}(\ ) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(\ )$ we finally get the standard form

$$G = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{j k_0}{h_n^2(k_0 a)} \left\{ h_n^2(k_0 a)\, j_n(k_0 r_<) - h_n^2(k_0 r_<)\, j_n(k_0 a) \right\} h_n^2(k_0 r_>)\, \frac{2n+1}{2}\, \frac{(n-m)!}{(n+m)!}\, P_n^m(\cos\theta_<)\, P_n^m(\cos\theta_>)\, \frac{\cos m(\phi-\phi_0)}{\epsilon_{0m}\pi}$$

The part of G arising from the first term $h_n^2(k_0 a)\, j_n(k_0 r_<)$ in braces $\{\ \}$ is readily seen to be the free space Green's function derived earlier. The remainder represents the field scattered from the sphere.

[illegible]

When the Green's function problem $(\mathcal{L} + \lambda \sigma) G = -\delta(x - x_0)$ is solved in the form

$$G = -\sum_n \frac{\psi_n(x_0)\,\psi_n(x_0)}{\lambda - \lambda_n}$$

by expanding G as a series $\sum_n a_n \psi_n(x)$ involving the eigenfunctions $\psi_n(x)$ which are a solution of $(\mathcal{L} + \lambda_n \sigma)\psi_n = 0$ it is important to know whether or not a complete set of eigenfunctions have been used. We will illustrate a technique due to Titchmarsh for proving the completeness property of a set of eigenfunctions. The method involves constructing G in the form $G = -\dfrac{\psi_1(x_<)\,\psi_2(x_>)}{W}$ first and then considering $(\mathcal{L} + \lambda\sigma) F(x) = -f(x)$ where $f(x)$ satisfies the same boundary conditions as G. The solution for $F(x)$ is $\int_{x_0} G(x|x_0) f(x_0)\, dx_0$ by using the superposition theorem. Next a suitable contour integral in the complex λ plane is considered, i.e

$$\frac{1}{2\pi j}\oint_C F(x)\, d\lambda = \frac{1}{2\pi j}\int_{x_0}\oint_C f(x_0)\, G(x|x_0,\lambda)\, d\lambda\, dx_0 .$$

[illegible] of F. For the case where G has only simple poles a residue evaluation leads to an eigenfunction expansion of $-f(x)$ in the form $f(x) = \sum_n C_n \Psi_n(x)$. If it can also be shown directly that $\frac{1}{2\pi j} \oint_C F(x)\, d\lambda = -f(x)$ then it will have been proved that $f(x)$ has an eigenfunction expansion in terms of the eigenfunctions $\Psi_n(x)$. If $f(x)$ is arbitrary it is then concluded that the Ψ_n form a complete set. The case of multiple poles and branch type singularities in G can be treated in a similar way. A particular case is when $f(x) = \delta(x-x_0)$ in which case $F \equiv G$ and $\frac{1}{2\pi j} \oint_C G\, d\lambda$

$$= \frac{1}{2\pi j} \oint_C \int_{x_0} \delta(x-x_0)\, G(x|x_0,\lambda)\, dx_0\, d\lambda$$

Evaluation of the latter integral gives an eigenfunction expansion of $-\delta(x-x_0)$ and if it can be shown directly that $\frac{1}{2\pi j} \oint_C G\, d\lambda = -\delta(x-x_0)$ then this relation is a completeness relation for the eigenfunction set that serves for the expansion of $\delta(x-x_0)$. The above concepts will

[illegible]

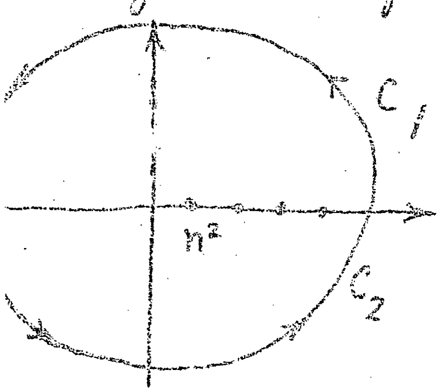
Consider $G_\phi = -\dfrac{\cos\sqrt{\lambda_3}\,(\pi-\phi_0+\phi)}{2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi} = -\sum_{n=0}^{\infty}\dfrac{\cos n(\phi-\phi_0)}{\epsilon_{0n}\pi(\lambda_3-n^2)}$

for $-\pi \le \phi \le \pi$. We wish to show that the normalized functions $\dfrac{\begin{Bmatrix}\sin n(\phi)\\ \cos n(\phi)\end{Bmatrix}}{\sqrt{\epsilon_{0n}\pi}}$ form a complete set.

If we consider $\left(\dfrac{d^2}{d\phi^2}+\lambda_3\right)F = -f(\phi)$ we have as a solution $F = \int_{-\pi}^{\pi} G_\phi(\phi|\phi_0)\,f(\phi_0)\,d\phi_0$

$$= -\int_{-\pi}^{\phi}\frac{\cos\sqrt{\lambda_3}\,(\pi-\phi+\phi_0)}{2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi}f(\phi_0)\,d\phi_0 - \int_{\phi}^{\pi}\frac{\cos\sqrt{\lambda_3}(\pi-\phi_0+\phi)}{2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi}f(\phi_0)\,d\phi_0$$

The illustrated contour $C = C_1 + C_2$ encloses all the singularities of G_ϕ and hence since $2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi$



approaches $\pi\epsilon_{0n}\cos n\pi\,(\lambda_3 - n^2)$ near a pole at $\lambda_3 = n^2$ a residue evaluation of $I = \dfrac{1}{2\pi j}\oint_C F\,d\lambda_3$ gives

$$I = -\sum_{n=0}^{\infty}\left\{\int_{-\pi}^{\phi}\frac{\cos n(\pi-\phi+\phi_0)}{\pi\epsilon_{0n}\cos n\pi}f(\phi_0)\,d\phi_0\right.$$

$$\left. + \int_{\phi}^{\pi}\frac{\cos n(\pi-\phi_0+\phi)}{\pi\epsilon_{0n}\cos n\pi}f(\phi_0)\,d\phi_0\right\}$$

$$\sum_{n=0} \Big\{ \int_{-\pi} \frac{\cos n(\phi_0 - \phi)}{\pi \epsilon_{0n}} f(\phi_0)\, d\phi_0 + \int_{\phi}^{\pi} \frac{\cos n(\phi - \phi_0)}{\pi \epsilon_{0n}} f(\phi_0)\, d\phi_0 \Big\}$$

$$= - \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{f(\phi_0) \cos n(\phi_0 - \phi)}{\pi \epsilon_{0n}}\, d\phi_0$$

$$= - \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \frac{f(\phi_0) \cos n\phi_0}{\sqrt{\pi \epsilon_{0n}}}\, d\phi_0 \right\} \frac{\cos n\phi}{\sqrt{\pi \epsilon_{0n}}}$$

$$- \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{\pi} \frac{f(\phi_0) \sin n\phi_0}{\sqrt{\pi \epsilon_{0n}}}\, d\phi_0 \right\} \frac{\sin n\phi}{\sqrt{\pi \epsilon_{0n}}}$$

Note that if we assume that $-f(\phi) = -\sum_{n=0}^{\infty} \left\{ a_n \frac{\cos n\phi}{\sqrt{\pi \epsilon_{0n}}} + b_n \frac{\sin n\phi}{\sqrt{\pi \epsilon_{0n}}} \right\}$ we will get the same expansion as was obtained for $I = \frac{1}{2\pi j} \oint_C F\, d\lambda_3$. If we can show also that $I \equiv -f(\phi)$ then we will have shown that the eigenfunction set $\left\{ \begin{matrix} \sin n\phi \\ \cos n\phi \end{matrix} \right\} / \sqrt{\pi \epsilon_{0n}}$ is a complete set.

On the semi-circle in the upper half plane $\delta \leq \theta \leq \pi - \delta$ where δ is a small positive number $\sqrt{\lambda_3}$ has a positive imaginary part and G_ϕ is asymptotic to

$$G_\phi = -\frac{\cos\sqrt{\lambda_3}\,\pi\,\cos\sqrt{\lambda_3}\,(\phi_> - \phi_<) + \sin\sqrt{\lambda_3}\,\pi\,\sin\sqrt{\lambda_3}\,(\phi_> - \phi_<)}{2\sqrt{\lambda_3}\,\sin\sqrt{\lambda_3}\,\pi}$$

$$\to -\left\{\frac{e^{-j\sqrt{\lambda_3}\,\pi}\cos\sqrt{\lambda_3}\,(\phi_> - \phi_<)}{2\sqrt{\lambda_3}\;\frac{e^{-j\sqrt{\lambda_3}\,\pi}}{-j}} + \frac{\sin\sqrt{\lambda_3}\,(\phi_> - \phi_<)}{2\sqrt{\lambda_3}}\right\}$$

$$= \frac{j}{2\sqrt{\lambda_3}}\,e^{j\sqrt{\lambda_3}\,(\phi_> - \phi_<)}$$

. On the semi-circle in the lower half plane G_ϕ is asymptotic to

$$\frac{-j}{2\sqrt{\lambda_3}}\,e^{-j\sqrt{\lambda_3}\,(\phi_> - \phi_<)}$$

. We should now integrate from δ to $\pi - \delta$ and take the limit as δ goes to zero on the upper half circle and similarly on the lower half circle. This will show that in effect we can use the above forms for G_ϕ for all θ and hence, in order to simplify the discussion we put $\delta = 0$ at this point. Then

$$\frac{1}{2\pi j}\oint_C G_\phi\, d\lambda_3 = \frac{1}{2\pi j}\int_0^{\pi}\frac{j}{2\sqrt{\lambda_3}}\,e^{j\sqrt{\lambda_3}\,(\phi_> - \phi_<)}\,d\lambda_3$$

$$+ \frac{1}{2\pi j}\int_{-\pi}^{0}\frac{-j}{2\sqrt{\lambda_3}}\,e^{-j\sqrt{\lambda_3}\,(\phi_> - \phi_<)}\,d\lambda_3 \quad =$$

$$\to \frac{1}{\pi} \frac{\sin \sqrt{\rho}\,(\phi_> - \phi_<)}{\phi_> - \phi_<} \quad \text{since } \sqrt{\lambda_3} = \mp i\sqrt{\rho} \text{ at } \mp\pi$$

and equals $\sqrt{\rho}$ at 0 where $\rho = |\lambda_3|$ = radius of circle C. We now have

$$I = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi_0) \frac{\sin\sqrt{\rho}\,(\phi_0 - \phi)}{\phi_0 - \phi}\, d\phi_0$$

If we put $\sqrt{\rho}\,(\phi_0 - \phi) = x$, then $\phi_0 = \mp\pi$ corresponds to $x = -\sqrt{\rho}\,(\pi + \phi)$ and $\sqrt{\rho}\,(\pi - \phi)$ and

$$I = -\frac{1}{\pi} \int_{-\sqrt{\rho}(\pi+\phi)}^{\sqrt{\rho}(\pi-\phi)} f\left(\phi + \frac{x}{\sqrt{\rho}}\right) \frac{\sin x}{x}\, dx$$

When we take the limit $\rho \to \infty$ we get

$$I = -\frac{1}{\pi} f(\phi) \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = -f(\phi)$$

which shows that the eigenfunction expansion obtained from a residue evaluation of I is equal to $-f(\phi)$.

Spherical Bessel Functions

Differential equation: $\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr} z_n(k_0 r) + \left[k_0^2 - \frac{n(n+1)}{r^2}\right] z_n(k_0 r) = 0$

Solutions are of the form $r^{-1/2} Z_{n+\frac{1}{2}}(k_0 r)$ where $Z_{n+1/2}$ is a cylindrical Bessel functions. Thus the basic solutions are

$$j_n(u) = \sqrt{\frac{\pi}{2u}}\; J_{n+1/2}(u)\,, \quad y_n(u) = \sqrt{\frac{\pi}{2u}}\; Y_{n+\frac{1}{2}}(u)$$

$$h_n^{1,2}(u) = \sqrt{\frac{\pi}{2u}}\; H^{1,2}_{n+1/2}(u) = j_n(u) \pm j\, y_n(u)$$

These functions are polynomials in $1/u$ multiplied by the trigonometric functions. Thus

$$j_0(u) = \frac{\sin u}{u}\,, \quad y_0(u) = -\frac{\cos u}{u}\,, \quad h_0^2(u) = j\,\frac{e^{-ju}}{u}$$

or in general

$$j_n(u) = \frac{1}{u}\left[P_{n+1/2}(u)\cos\left(u - \frac{n+1}{2}\pi\right) - Q_{n+\frac{1}{2}}(u)\sin\left(u - \frac{n+1}{2}\pi\right)\right]$$

$$y_n(u) = \frac{1}{u}\left[P_{n+\frac{1}{2}}(u)\sin\left(u - \frac{n+1}{2}\pi\right) + Q_{n+\frac{1}{2}}(u)\cos\left(u - \frac{n+1}{2}\pi\right)\right]$$

where

$$P_{n+\frac{1}{2}}(u) = 1 - \frac{n(n^2-1)(n+2)}{2^2\, 2!\, u^2} + \frac{n(n^2-1)(n^2-4)(n^2-9)(n+4)}{2^4\, 4!\, u^4} - \cdots$$

$$Q_{n+\frac{1}{2}}(u) = \frac{n(n+1)}{2 \cdot 1!\, u} - \frac{n(n^2-1)(n^2-4)(n+3)}{2^3\, 3!\, u^3} + \cdots$$

For $u \to \infty$, $j_n(u) = \frac{1}{u} \cos\left(u - \frac{n+1}{2}\pi\right)$,

$y_n(u) = \frac{1}{u} \sin\left(u - \frac{n+1}{2}\pi\right)$, $h_n^{\,2}(u) = \frac{j^{n+1}}{u} e^{-ju}$

Recurrence Relations

$$z_{n-1} + z_{n+1} = \frac{2n+1}{u} z_n$$

$$\frac{dz_n}{du} = z_n' = \frac{1}{2n+1}\left[n z_{n-1} - (n+1) z_{n+1}\right]$$

$$\frac{d}{du}\left(u^{n+1} z_n\right) = u^{n+1} z_{n-1}$$

$$\frac{d}{du}\left(u^{-n} z_n\right) = -u^{-n} z_{n+1}$$

Wronskean

Let $z_n(u)$ and $\bar{z}_n(u)$ be two linearly independent solutions. Then

$$z_n \bar{z}_n' - z_n' \bar{z}_n = \frac{\text{Constant}}{u^2}$$

The constant is readily found by using the asymptotic forms for z_n and $\bar{z}_n$.

Legendre Functions

Differential Equation: $\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dL_n^m}{d\theta}+\left[n(n+1)-\frac{m^2}{\sin^2\theta}\right]L_n^m=0$

L_n^m = Legendre function

If we put $\cos\theta = u$ we obtain the standard form

$$(1-u^2)\frac{d^2L_n^m}{du^2}-2u\frac{dL_n^m}{du}+\left[n(n+1)-\frac{m^2}{1-u^2}\right]L_n^m=0$$

For $m=0$, n an integer, one solution is a polynomial $P_n(u)$ called the Legendre polynomial of order n. The P_n may be obtained from Rodrigues formula

$$P_n(u)=\frac{1}{2^n n!}\frac{d^n}{du^n}(u^2-1)^n$$

A second solution is $Q_n(u)$ which is an infinite series and is called the Legendre function of the second kind. It is given by

$$Q_n(u)=\frac{1}{2^n n!}\frac{d^n}{du^n}\left[(u^2-1)^n\ln\frac{u+1}{u-1}\right]-\frac{1}{2}P_n(u)\ln\frac{u+1}{u-1}$$

for all u except $-1\le u\le 1$. For the latter range $Q_n(u)=\frac{1}{2}P_n(u)\ln\frac{1+u}{1-u}-W_{n-1}$ where

$$W_{n-1}=\frac{2n-1}{n}P_{n-1}(u)+\frac{2n-5}{3(n-1)}P_{n-3}+\frac{2n-9}{5(n-2)}P_{n-5}+\cdots$$

Note that the Q_n have a logarithmic singularity at $u = \pm 1$ ($\cos\theta = \pm 1$, $\theta = 0, \pi$).

Associated Legendre Functions, $m \neq 0$

$$P_n^m(u) = \frac{(1-u^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{du^{n+m}} (u^2-1)^n = (1-u^2)^{m/2} \frac{d^m P_n}{du^m}$$

$$Q_n^m(u) = (1-u^2)^{m/2} \frac{d^m Q_n}{du^m}$$

Recurrence Relations

$L_n^m = P_n^m, Q_n^m$, or a linear combination of P_n^m and Q_n^m.

$$(n-m+1) L_{n+1}^m - (2n+1) u L_n^m + (n+m) L_{n-1}^m = 0$$

$$L_{n-1}^m = u L_n^m - (n-m+1)(1-u^2)^{1/2} L_n^{m-1}$$

$$L_{n+1}^m = u L_n^m + (n+m)(1-u^2)^{1/2} L_n^{m-1}$$

$$(1-u^2)^{1/2} L_n^{m+1} = (n+m+1) u L_n^m - (n-m+1) L_{n+1}^m$$

$$(1-u^2)^{1/2} L_n^{m+1} = 2 m u L_n^m - (n+m)(n-m+1)(1-u^2)^{1/2} L_n^{m-1}$$

$$(1-u^2)^{1/2} L_n^m = \frac{1}{2n+1} \left[L_{n+1}^{m+1} - L_{n-1}^{m+1} \right]$$

$$\frac{m}{(1-u^2)^{1/2}} L_n^m = \frac{1}{2} u \left[(n-m+1)(n+m) L_n^{m-1} + L_n^{m+1} \right] + m(1-u^2)^{1/2} L_n^m$$

Differential Relations

$$(1-u^2)\frac{dL_n^m}{du} = (n+1)u\,L_n^m - (n-m+1)L_{n+1}^m$$

$$= (n+m)L_{n-1}^m - n\,u\,L_n^m$$

$$\frac{dL_n^m}{d\theta} = -(1-u^2)^{1/2}\frac{dL_n^m}{du} = \frac{1}{2}\left[(n-m+1)(n+m)L_n^{m-1} - L_n^{m+1}\right]$$

Orthogonality Relations

$$\int_{-1}^{1} L_n^m L_\ell^m\,du = 0,\ \ell \neq n;\quad \int_{-1}^{1} L_n^m L_n^\ell\,\frac{du}{1-u^2} = 0,\ m \neq \ell$$

$$\int_0^\pi L_n^m(\cos\theta)\,L_\ell^m(\cos\theta)\sin\theta\,d\theta = 0,\ \ell \neq n$$

$$\int_0^\pi L_n^m L_n^\ell\,\frac{d\theta}{\sin\theta} = 0,\ \ell \neq m$$

Normalization

$$\int_{-1}^{1}(P_n^m)^2\,du = \frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!} = \int_0^\pi (P_n^m)^2\sin\theta\,d\theta$$

$$\int_{-1}^{1}(P_n^m)^2\,\frac{du}{1-u^2} = \int_0^\pi (P_n^m)^2\,\frac{d\theta}{\sin\theta} = \frac{1}{m}\,\frac{(n+m)!}{(n-m)!}$$

Miscellaneous Relations

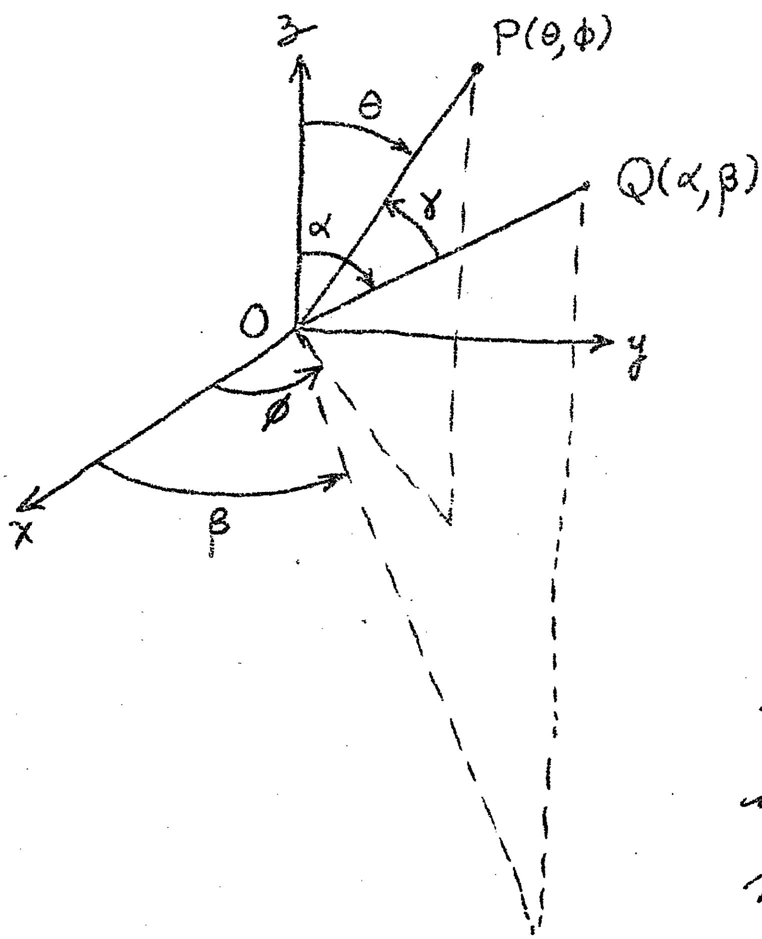
$$P_{m+2\ell+1}^m(0) = 0,\quad P_{m+2\ell}^m(0) = (-1)^\ell\,\frac{(2m+2\ell)!}{2^{m+2\ell}\,\ell!\,(m+\ell)!}$$

$$\int P_n(u)\,du = \frac{P_{n+1} - P_{n-1}}{2n+1},\ \text{for } n=0 \text{ replace } P_{n-1} \text{ by } 1.$$

Wronskian

$$W = P_\alpha \frac{dQ_\alpha}{du} - Q_\alpha \frac{dP_\alpha}{du} = \frac{1}{1-u^2}$$

Addition Formula



$$P_n(\cos\gamma) = P_n(\cos\alpha)P_n(\cos\theta) + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) P_n^m(\cos\theta)\cos m(\phi-\beta)$$

This formula expresses zonal harmonics at P in terms of coordinates with respect to OQ as the new polar axis.

Vector Wave Solutions in Spherical Coordinates

The vector Helmholtz equation is

$$(\nabla^2 + k_0^2)\vec{F} = (\nabla\nabla\cdot - \nabla\times\nabla\times + k_0^2)\vec{F} = 0.$$

Hansen showed that if ψ is a solution of $(\nabla^2+k_0^2)\psi=0$ in spherical coordinates than appropiate solutions for $\vec{F}$ are of the form $\vec{M} = \nabla\times(\vec{r}\,\psi)$, $\vec{N} = k_0^{-1}\,\nabla\times\vec{M} = k_0^{-1}\,\nabla\times\nabla\times(\vec{r}\,\psi)$, and $\vec{L} = \nabla\psi$.

By direct substitution these can be easily verified to be solutions of the vector Helmholtz equation; i.e

$$(\nabla\nabla\cdot - \nabla\times\nabla\times + k_0^2)\begin{Bmatrix}\vec{M}\\ \vec{N}\\ \vec{L}\end{Bmatrix} = 0$$

Note that $\nabla\cdot\vec{M} = \nabla\cdot\vec{N} = 0$ so these are solenoidal wave functions while $\nabla\times\vec{L} = \nabla\times\nabla\psi = 0$ so $\vec{L}$ is an irrotational (LOGITUDINAL, HENCE $\vec{L}$) wave function.

Thus $$(\nabla\times\nabla\times - k_0^2)\begin{Bmatrix}\vec{M}\\ \vec{N}\end{Bmatrix} = 0,$$

$$\nabla\nabla\cdot\vec{L} + k_0^2\vec{L} = (\nabla^2+k_0^2)\,\nabla\psi = 0$$

We have defined $\vec{N}$ such that $\nabla \times \vec{N} = k_0^{-1} \nabla\times\nabla\times\vec{M}$ $= k_0 \vec{M}$, from equation for $\vec{M}$, so that we have

$$k_0 \vec{N} = \nabla \times \vec{M}, \quad k_0 \vec{M} = \nabla \times \vec{N}$$

Since $\vec{r}\,\psi$ occurs in the theory it is convenient to use the modified spherical Bessel functions as introduced by Schelkunoff, i.e. use $k_0 r\, z_n(k_0 r) = \hat{z}_n(k_0 r)$ where z_n is a suitable spherical Bessel function. Thus for $k_0 r \psi$ we choose solutions of the form $P_n^{|m|}(\cos\theta)\, e^{-jm\phi}\, \hat{z}_n(k_0 r)$

where $n = 0, 1, 2, ---;$ $m = 0, \pm 1, \pm 2, ---$. Note that since

$$P_n^{-|m|} = (-1)^{|m|} \frac{(n-|m|)!}{(n+|m|)!} P_n^{|m|}$$

we can use $P_n^{|m|}$ instead of P_n^m for negative integer values of m.

In terms of ψ the $\vec{M}$ and $\vec{N}$ functions are given by

$$\vec{M} = \nabla \times \vec{a}_r (k_o r \psi) = \frac{\vec{a}_\theta}{r \sin\theta} \frac{\partial (k_o r \psi)}{\partial \phi} - \frac{\vec{a}_\phi}{r} \frac{\partial (k_o r \psi)}{\partial \theta}$$

$$\vec{N} = k_o^{-1} \nabla \times \vec{M} = - \frac{\vec{a}_r}{k_o r^2 \sin\theta} \left[\frac{\partial}{\partial \theta} \sin\theta \frac{\partial (k_o r \psi)}{\partial \theta} + \frac{1}{\sin\theta} \frac{\partial^2 (k_o r \psi)}{\partial \phi^2} \right]$$

$$+ \frac{\vec{a}_\theta}{k_o r} \frac{\partial^2 (k_o r \psi)}{\partial r \partial \theta} + \frac{\vec{a}_\phi}{k_o r \sin\theta} \frac{\partial^2 (k_o r \psi)}{\partial r \partial \phi}$$

Note that when ψ contains the factor $P_n^m e^{-jm\phi}$ that $\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right) P_n^m e^{-jm\phi}$

$= - n(n+1) P_n^m e^{-jm\phi}$ which may be used to simplify the expression for $\vec{N}$.

Solution for Electric Field

The electric field is a solution of

$$\nabla \times \nabla \times \vec{E} - k_o^2 \vec{E} = -j\omega\mu_o \vec{J}$$

The divergence of this equation gives

$$\nabla \cdot (k_o^2 \vec{E}) = k_o^2 \nabla \cdot \vec{E} = j\omega\mu_o \nabla \cdot \vec{J} = k_o^2 \rho / \epsilon_o$$

If we expand $\vec{E}$ in the form

$$\vec{E} = \sum_n (a_n \vec{M}_n + b_n \vec{N}_n + c_n \vec{L}_n)$$

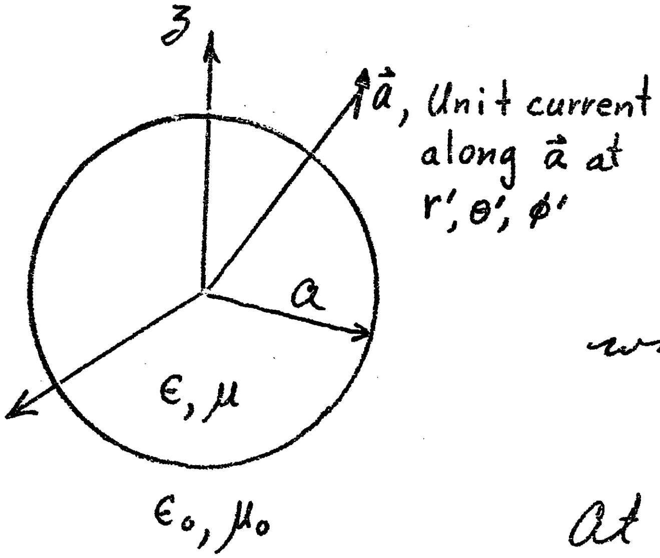
then we obtain $\nabla\cdot\vec{E} = \sum_n c_n \nabla\cdot\vec{L}_n = \sum_n c_n \nabla^2 \psi_n$ $= -k_0^2 \sum_n c_n \psi_n = \rho/\epsilon_0$. Hence the part of $\vec{E}$ which has non-zero divergence is described by the $\vec{L}_n$ functions (n is a general summation index over all possible eigenfunctions of the given type) and these add up to zero outside the source region. (~~Therefore, outside the source region we can describe~~ ~~$\vec{E}$ in terms of the $\vec{M}_n$ and $\vec{N}_n$ functions alone~~.) NOT SO; SEE NEXT NOTE SET. The expansion coefficients are readily found by using the Lorentz reciprocity theorem. The procedure is best described by means of an example which we give in the next section.

The solution for $\vec{H}$ is found from $-j\omega\mu_0 \vec{H}$

$$= \nabla\times\vec{E} = \sum_n (a_n \nabla\times\vec{M}_n + b_n \nabla\times\vec{N}_n)$$

$$= k_0 \sum_n (a_n \vec{N}_n + b_n \vec{M}_n)$$

Diffraction by a Sphere



The field is a solution of

$$\nabla \times \nabla \times \vec{E} - k_0^2 \vec{E} = -j\omega\mu_0 \vec{J}, \quad r > a$$

$$\nabla \times \nabla \times \vec{E} - k^2 \vec{E} = 0, \quad r < a$$

where $\vec{J} = \dfrac{\vec{a}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{r'^2 \sin\theta'}$

At $r = a$, $\vec{a}_r \times \vec{E}$ and $\vec{a}_r \times \vec{H}$ must be continuous.

For points $\vec{r} \neq \vec{r}'$ we can construct a vector function $\vec{M}^{TE}$ for $\vec{E}$ corresponding to transverse electric waves (no E_r component). From $\vec{M}^{TE}$ we obtain $\vec{N}^{TE} = k_i^{-1} \nabla \times \vec{M}^{TE}$, $k_i = k_0$ or k, which will describe the related magnetic field. $\vec{M}^{TE}$ can be constructed to make both $\vec{a}_r \times \vec{E}^{TE}$ and $\vec{a}_r \times \vec{H}^{TE}$ continuous at $r=a$.

We can also construct a function $\vec{M}^{TM}$ to describe $\vec{H}$ for transverse magnetic waves and then $\vec{N}^{TM} = k_i^{-1} \nabla \times \vec{M}^{TM}$ describes the related electric field. The total field is a superposition of the TE and TM waves with amplitudes determined by the source.

TE Waves

For $\vec{M}_{nm}^{TE}$ we choose $k_i r \psi_{nm}$ in the form

$$A_{nm} P_n^{|m|} e^{-jm\phi} \hat{J}_n(kr) \quad , \; r<a$$

$$B_{nm} P_n^{|m|} e^{-jm\phi} \left(C_n \hat{h}_n^2 + \hat{h}_n^1\right) , \quad a<r<r'$$

$$D_{nm} P_n^{|m|} e^{-jm\phi} \hat{h}_n^2 \quad , \quad r>r'$$

with the $\hat{h}_n^{1,2}$ having arguements $k_o r$. To make $\vec{a}_r \times \vec{E}_{nm}^{TE} = \vec{a}_r \times \vec{M}_{nm}^{TE}$ continuous at $r=a$ we see from Pg. 45 that $k_i r \psi_{nm}$ should be continuous. To make $\vec{a}_r \times \vec{H}_{nm}^{TE} = \dfrac{\vec{a}_r \times \nabla \times \vec{E}_{nm}^{TE}}{-j\omega\mu_i} = -\dfrac{k_i}{j\omega\mu_i} \vec{a}_r \times \vec{N}_{nm}^{TE}$

continuous at $r=a$ requires that $\mu_i^{-1} \dfrac{\partial(k_i r \psi_{nm})}{\partial r}$ be continuous (note that μ_i, k_i equal μ, k for $r<a$ and μ_o, k_o for $r>a$). Using the above conditions gives $A_{nm} \hat{J}_n(ka) = B_{nm}\left[C_n \hat{h}_n^2(k_o a) + \hat{h}_n^1(k_o a)\right]$

$$\frac{k}{\mu} A_{nm} \hat{j}_n' = \frac{k_0}{\mu_0} B_{nm} \left[C_n \hat{h}_n^{2\,\prime} + \hat{h}_n^{1\,\prime} \right]$$

where the prime means a derivative with respect to $k_0 r$ or kr and evaluated at $r = a$. From the above we obtain

$$C_n = - \left. \frac{\mu k_0 \hat{j}_n \hat{h}_n^{1\,\prime} - \mu_0 k \hat{j}_n' \hat{h}_n^1}{\mu k_0 \hat{j}_n \hat{h}_n^{2\,\prime} - \mu_0 k \hat{j}_n' \hat{h}_n^2} \right|_{r=a}$$

$$A_{nm} = \frac{C_n \hat{h}_n^2(k_0 a) + \hat{h}_n^1(k_0 a)}{\hat{j}_n(ka)} B_{nm}$$

The electric field is given by (for TE waves)

$$\vec{E}^{TE} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \nabla \times k_i r \psi_{nm} \vec{a}_r$$

TM Waves

For TM waves we choose $k_i r \overline{\psi}_{nm}$ of the form

$$\bar{A}_{nm} P_n^{|m|} e^{-jm\phi} \hat{j}_n(kr) \quad , \; r < a$$

$$\bar{B}_{nm} P_n^{|m|} e^{-jm\phi} \left[\bar{C}_n \hat{h}_n^2(k_0 r) + \hat{h}_n^1(k_0 r) \right] , \quad a < r < r'$$

$$\bar{D}_{nm} P_n^{|m|} e^{-jm\phi} \hat{h}_n^2(k_0 r) \quad , r > r'$$

The electric field, for TM waves, is given by

$$\vec{E}^{TM} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{k_i}{j\omega\epsilon_i} \vec{N}_{nm}^{TM}$$

since we choose

$$\vec{H}^{TM} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \nabla \times k_i r \bar{\Psi}_{nm} \vec{a}_r$$

where $k_i \vec{N}_{nm}^{TM} = \nabla \times \vec{M}_{nm}^{TM} = \nabla \times \nabla \times (k_i r \vec{a}_r \bar{\Psi}_{nm})$.

To make $\vec{a}_r \times \vec{E}^{TM}$ and $\vec{a}_r \times \vec{H}^{TM}$ continuous at $r = a$ we must make $k_i r \bar{\Psi}_{nm}$ and $\frac{k_i}{\epsilon_i} \frac{\partial(k_i r \bar{\Psi}_{nm})}{\partial(k_i r)}$ continuous. Thus we find that

$$\bar{C}_n = - \left. \frac{\epsilon k_0 \hat{J}_n \hat{h}_n^{1\prime} - \epsilon_0 k \hat{J}_n' \hat{h}_n^1}{\epsilon k_0 \hat{J}_n \hat{h}_n^{2\prime} - \epsilon_0 k \hat{J}_n' \hat{h}_n^2} \right|_{r=a}$$

$$\bar{A}_{nm} = \frac{\bar{C}_n \hat{h}_n^2(k_0 a) + \hat{h}_n^1(k_0 a)}{\hat{J}_n(ka)} \bar{B}_{nm}$$

Excitation Amplitudes

Let $\vec{E}, \vec{H}$ be the total field radiated by $\vec{J}$ and let $\underset{\sim}{\vec{E}}, \underset{\sim}{\vec{H}}$ be a source free field. The Lorentz reciprocity theorem then gives

$$\oint_S (\vec{E} \times \underset{\sim}{\vec{H}} - \underset{\sim}{\vec{E}} \times \vec{H}) \cdot \vec{n}\, dS = \int_V \underset{\sim}{\vec{E}} \cdot \vec{J}\, dV$$

For S we choose spherical surfaces at $r = r_1, r_2$ with $a < r_1 < r'$, $r_2 > r'$. If $\underset{\sim}{\vec{E}}, \underset{\sim}{\vec{H}}$ corresponds to a TE mode then the only terms in the surface integral which do not vanish are those involving the TE modes (i.e. the TE and TM modes are orthogonal with orthogonality defined as the vanishing of the above surface integral).

Let us choose $\underset{\sim}{\vec{E}} = \nabla \times \vec{a}_r \hat{h}_n^2(k_0 r) P_n^{|m|} e^{jm\phi}$, $\underset{\sim}{\vec{H}} = (-j\omega\mu_0)^{-1} \nabla \times \underset{\sim}{\vec{E}}$. Since $dS = r^2 \sin\theta\, d\theta\, d\phi$ and the $P_n^{|m|} e^{-jm\phi}$ functions are orthogonal with $\sin\theta$ as a weighting factor we obtain

$$\frac{k_0 B_{nm}}{-j\omega\mu_0}\int_0^{2\pi}\int_0^{\pi}\left(\hat{h}_n^2\,\hat{h}_n^{1\prime}-\hat{h}_n^{2\prime}\,\hat{h}_n^{1}\right)\left[\frac{m^2\left(P_n^{|m|}\right)^2}{\sin\theta}\right.$$

$$\left.+\left(\frac{\partial P_n^{|m|}}{\partial\theta}\right)^2\sin\theta\right]_{r=r_1} d\theta\, d\phi$$

The integrand vanishes identically for $r=r_2$ and the part at $r=r_1$ involving $C_n\hat{h}_n^2$ is also zero since the part of the $\vec{E}_{nm}^{TE}$ field involving the $\hat{h}_n^2$ functions are proportional to $\underset{\sim}{\vec{E}}$. The Wronskian $\hat{h}_n^2\hat{h}_n^{1\prime}-\hat{h}_n^{2\prime}\hat{h}_n^{1}$ equals $2j$. The term $\int_0^{\pi}\left(\frac{\partial P_n^{|m|}}{\partial\theta}\right)^2\sin\theta\, d\theta$

$$= P_n^{|m|}\sin\theta\frac{\partial P_n^{|m|}}{\partial\theta}\Bigg|_0^{\pi}-\int_0^{\pi}P_n^{|m|}\frac{\partial}{\partial\theta}\left(\sin\theta\, P_n^{|m|}\right)d\theta$$

$$= -\int_0^{\pi}P_n^{|m|}\left[\frac{m^2}{\sin\theta}-n(n+1)\sin\theta\right]P_n^{|m|}\,d\theta$$ by using the differential equation for $P_n^{|m|}$. Hence we get

$$\frac{k_0 B_{nm}}{-j\omega\mu_0} \, 2\pi \int_0^{\pi} (2j) \; n(n+1) \left(P_n^{|m|}\right)^2 \sin\theta \, d\theta$$

$$= - \frac{8\pi k_0 B_{nm}}{\omega\mu_0} \; \frac{n(n+1)}{2n+1} \; \frac{(n+|m|)!}{(n-|m|)!} = \int_V \underset{\sim}{\vec{E}} \cdot \vec{J} \, dV$$

$$= \frac{jm}{r' \sin\theta'} \hat{h}_n^2(k_0 r') \, P_n^{|m|}(\cos\theta') \, e^{jm\phi'} \, \vec{a}_\theta \cdot \vec{a}$$

$$- \frac{1}{r'} \hat{h}_n^2(k_0 r') \frac{dP_n^{|m|}(\cos\theta')}{d\theta'} \, e^{jm\phi'} \, \vec{a}_\phi \cdot \vec{a}$$

which can be solved for B_{nm}.

Although the other coefficients can be found in a similar way the whole procedure can be simplified by introducing the following vector wave functions

$$\vec{m}_{nm}^{1,2} = \nabla \times \left(\vec{a}_r \, P_n^{|m|} \, e^{-jm\phi} \, \hat{h}_n^{1,2}(k_0 r) \right)$$

$$\vec{n}_{nm}^{1,2} = \frac{1}{k_0} \nabla \times \vec{m}_{nm}^{1,2}$$

These functions satisfy the following orthogonality and normalization relations:

$$\int_0^{\pi}\int_0^{2\pi}[\vec{m}^{\,i}_{nm}\times\vec{n}^{\,l}_{uv}-\vec{m}^{\,l}_{uv}\times\vec{n}^{\,i}_{nm}]\cdot\vec{a}_r\, r^2\sin\theta\, d\theta\, d\phi=0$$

unless $i\neq l, -m=v, u=+n$. In the latter case

$$\int_0^{\pi}\int_0^{2\pi}[\vec{m}^{\,i}_{nm}\times\vec{n}^{\,l}_{n(-m)}-\vec{m}^{\,l}_{n(-m)}\times\vec{n}^{\,i}_{nm}]\cdot\vec{a}_r\, r^2\sin\theta\, d\theta\, d\phi$$

$$= 8\pi j\,\frac{n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!}(-1)^i = (-1)^i Q_{nm}$$

The result may be proved by direct evaluation.

In the region $a<r<r'$ we can express the total field as

$$\vec{E}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big\{B_{nm}[C_n\vec{m}^{\,2}_{nm}+\vec{m}^{\,1}_{nm}]$$

$$+\frac{k_0}{j\omega\epsilon_0}[\bar{B}_{nm}(\bar{C}_n\vec{n}^{\,2}_{nm}+\vec{n}^{\,1}_{nm})]\Big\}$$

$$\vec{H}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big\{-\frac{k_0}{j\omega\mu_0}[B_{nm}(C_n\vec{n}^{\,2}_{nm}+\vec{n}^{\,1}_{nm})]$$

$$+\bar{B}_{nm}[\bar{C}_n\vec{m}^{\,2}_{nm}+\vec{m}^{\,1}_{nm}]\Big\}$$

while in the region $r>r'$ we have

$$\vec{E} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ D_{nm} \vec{m}_{nm}^{\,2} + \frac{k_0}{j\omega\epsilon_0} \bar{D}_{nm} \vec{n}_{nm}^{\,2} \right\}$$

$$\vec{H} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left\{ \bar{D}_{nm} \vec{m}_{nm}^{\,2} - \frac{k_0}{j\omega\mu_0} D_{nm} \vec{n}_{nm}^{\,2} \right\}$$

If we now choose $\underset{\sim}{E} = \vec{m}_{n(-m)}^{\,2}$ and $\underset{\sim}{H} = -\frac{k_0}{j\omega\mu_0} \vec{n}_{n(-m)}^{\,2}$ we obtain for the integral over the sphere $r = r_2$ the result

$$\int_0^{\pi} \int_0^{2\pi} \left\{ \left[D_{nm} \vec{m}_{nm}^{\,2} + \frac{k_0}{j\omega\epsilon_0} \bar{D}_{nm} \vec{n}_{nm}^{\,2} \right] \times \left(-\frac{k_0}{j\omega\mu_0} \vec{n}_{n(-m)}^{\,2} \right) \right.$$

$$\left. - \vec{m}_{n(-m)}^{\,2} \times \left[\bar{D}_{nm} \vec{m}_{nm}^{\,2} - \frac{k_0}{j\omega\mu_0} D_{nm} \vec{n}_{nm}^{\,2} \right] \right\} \cdot \vec{a}_r \, r^2 \sin\theta \, d\theta \, d\phi$$

$= 0$ since no functions with upper superscript 1 are involved.

If we choose $\underset{\sim}{E} = \vec{m}_{n(-m)}^{\,1}$, $\underset{\sim}{H} = -\frac{k_0}{j\omega\mu_0} \vec{n}_{n(-m)}^{\,1}$ we obtain

$$-\frac{k_0}{j\omega\mu_0} D_{nm} \int_0^{\pi} \int_0^{2\pi} \left[\vec{m}_{nm}^{\,2} \times \vec{n}_{n(-m)}^{\,1} - \vec{m}_{n(-m)}^{\,1} \times \vec{n}_{nm}^{\,2} \right] \cdot \vec{a}_r \, dS$$

$$= -\frac{k_0}{j\omega\mu_0} D_{nm} \frac{8\pi j\, n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!}$$

since $\vec{n}^{\,2}_{nm} \times \vec{n}^{\,1}_{n(-m)} = 0 = \vec{m}^{\,1}_{n(-m)} \times \vec{m}^{\,2}_{nm}$.

For the above two choices of $\underset{\sim}{\vec{E}}, \underset{\sim}{\vec{H}}$, the value of the integral over the sphere $r = r_1$ is

$$(-1)\int_0^{\pi}\int_0^{2\pi}\left[B_{nm}\,\vec{m}^{\,1}_{nm} \times \left(-\frac{k_0}{j\omega\mu_0}\,\vec{n}^{\,2}_{n(-m)}\right) - \vec{m}^{\,2}_{n(-m)} \times \left(-\frac{k_0}{j\omega\mu_0} B_{nm}\vec{n}^{\,1}_{nm}\right)\right]$$

$$\cdot\,\vec{a}_r\, dS = \frac{+k_0}{j\omega\mu_0} B_{nm} \frac{8\pi j\, n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!}(-1)^1$$

and

$$(-1)\int_0^{\pi}\int_0^{2\pi}\left[B_{nm}C_n\,\vec{m}^{\,2}_{nm} \times \left(-\frac{k_0}{j\omega\mu_0}\,\vec{n}^{\,1}_{n(-m)}\right) - \vec{m}^{\,1}_{n(-m)} \times\right.$$

$$\left.\left(-\frac{k_0}{j\omega\mu_0} B_{nm}C_n\,\vec{n}^{\,2}_{nm}\right)\right]\cdot\vec{a}_r\, dS$$

$$= \frac{k_0}{j\omega\mu_0} B_{nm}C_n \frac{8\pi j\, n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!}$$

From these two choices of $\underset{\sim}{\vec{E}}, \underset{\sim}{\vec{H}}$ we thus get

$$-\frac{k_0 B_{nm}}{j\omega\mu_0} \frac{8\pi j\, n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!} = \vec{a} \cdot \vec{m}^{\,2}_{n(-m)}(r', \theta', \phi')$$

$$= -\frac{k_0 B_{nm} Q_{nm}}{j\omega\mu_0}$$

and

$$-\frac{k_0}{j\omega\mu_0}(D_{nm} - B_{nm} C_n) Q_{nm} = \vec{a} \cdot \vec{m}^{\,1}_{n(-m)}(r', \theta', \phi')$$

If we choose $\underset{\sim}{\vec{E}} = \vec{n}^{\,1}_{n(-m)}$, $\underset{\sim}{\vec{H}} = -\frac{k_0}{j\omega\mu_0} \vec{m}^{\,1}_{n(-m)}$

we obtain

$$(\bar{D}_{nm} - \bar{B}_{nm}\bar{C}_n) Q_{nm} = \vec{a} \cdot \vec{n}^{\,1}_{n(-m)}(r', \theta', \phi')$$

while for $\underset{\sim}{\vec{E}} = \vec{n}^{\,2}_{n(-m)}$, $\underset{\sim}{\vec{H}} = -\frac{k_0}{j\omega\mu_0} \vec{m}^{\,2}_{n(-m)}$ gives

$$\bar{B}_{nm} Q_{nm} = \vec{a} \cdot \vec{n}^{\,2}_{n(-m)}(r', \theta', \phi')$$

From the above we see that for a radial current element $\vec{a} = \vec{a}_r$ and $B_{nm} = D_{nm} = 0$ so only TM waves are excited. If the radial current is placed along the z axis then the coefficients for $m \neq 0$ are zero (azimuthal symmetry)

The explicit expressions for the vector wave functions are (from Pg. 45 and Pg. 53):

$$\vec{m}^{1,2}_{n,m} = \frac{\vec{a}_\theta}{r\sin\theta}(-jm)P_n^{|m|}e^{-jm\phi}\hat{h}_n^{1,2}(k_0 r)$$

$$+ \frac{\vec{a}_\phi}{r}P_n^{|m|'}e^{-jm\phi}\hat{h}_n^{1,2}\sin\theta$$

$$\vec{n}^{1,2}_{n,m} = \frac{\vec{a}_r}{k_0 r^2}n(n+1)P_n^{|m|}e^{-jm\phi}\hat{h}_n^{1,2}$$

$$-\frac{\vec{a}_\theta \sin\theta}{r}P_n^{|m|'}e^{-jm\phi}\hat{h}_n^{1,2\,'} - \frac{\vec{a}_\phi jm}{r\sin\theta}P_n^{|m|}e^{-jm\phi}\hat{h}_n^{1,2\,'}$$

where the prime means differentiation with respect to the arguement $\cos\theta$ or $k_0 r$.

Horizontal Current Element on z Axis

As a special case consider an x directed current element $I_0\vec{a}_x\,\delta(x)\delta(y)\delta(z-z')$ located on the z axis. For this case we require the values of $\vec{m}^{1,2}_{n-m}(\vec{r}\,')$ and $\vec{n}^{1,2}_{n-m}(\vec{r}\,')$ on the polar axis where $\theta = 0$.

We use the result $\frac{\partial}{\partial\theta} P_n^m(\cos\theta) = \frac{1}{2}[(n-m+1)(n+m)P_n^{m-1} - P_n^{m+1}]$ and the property that $P_n^m(1) = 0$ for $|m| > 0$ and $P_n^0(1) = 1$ for all n.

Thus $P_n^{|m|'} \sin\theta = -\frac{\partial P_n^{|m|}}{\partial\theta} = -\frac{1}{2}n(n+1)$ for $m = \pm 1$ and $\theta = 0$. For $|m| \neq 1$, $P_n^{|m|'} \sin\theta = 0$ for $\theta = 0$. The limit of $P_n^{|m|}/\sin\theta$ as $\theta \to 0$ is $(\partial P_n^{|m|}/\partial\theta)/\cos\theta = \frac{1}{2}n(n+1)$. We thus have

$$\vec{a}_x \cdot \vec{m}_{n,\mp 1}^{1,2} = \pm \frac{\cos\phi'}{r'} j e^{\pm j\phi'} \frac{n(n+1)}{2} \hat{h}_n^{1,2}(k_0 z')$$

$$+ \frac{\sin\phi'}{r'} e^{\pm j\phi'} \hat{h}_n^{1,2} \frac{n(n+1)}{2} = \frac{n(n+1)}{2r'}(\pm j)\hat{h}_n^{1,2}(k_0 z')$$

For $|m| \neq 1$, $\vec{a}_x \cdot \vec{m}_{n-m}^{1,2} = 0$. Also we have

$$\vec{a}_x \cdot \vec{n}_{n,\mp 1}^{1,2} = \frac{\cos\phi'}{r'} \frac{n(n+1)}{2} e^{\pm j\phi'} \hat{h}_n^{1,2'}$$

$$\mp \frac{j\sin\phi'}{r'} \frac{n(n+1)}{2} e^{\pm j\phi'} \hat{h}_n^{1,2'} = \frac{n(n+1)}{2r'} \hat{h}_n^{1,2'}(k_0 z')$$

The expansion coefficients for the field are given by

$$B_{nm} = \frac{-j\omega\mu_0}{k_0 Q_{nm}} I_0 \vec{a}_x \cdot \vec{m}^{2}_{n,-m}(\vec{r}\,')$$

$$D_{nm} = \frac{-j\omega\mu_0}{k_0 Q_{nm}} I_0 \left[\vec{a}_x \cdot \vec{m}^{1}_{n,-m}(\vec{r}\,') + C_n \vec{a}_x \cdot \vec{m}^{2}_{n,-m}(\vec{r}\,') \right]$$

$$\bar{B}_{nm} = \frac{I_0}{Q_{nm}} \vec{a}_x \cdot \vec{n}^{2}_{n,-m}(\vec{r}\,')$$

$$\bar{D}_{nm} = \frac{I_0}{Q_{nm}} \left[a_x \cdot \vec{n}^{1}_{n,-m}(\vec{r}\,') + \vec{a}_x \cdot \vec{n}^{2}_{n,-m}(\vec{r}\,') \bar{C}_n \right]$$

where $Q_{nm} = 8\pi j \dfrac{n(n+1)(n+|m|)!}{(2n+1)(n-|m|)!}$

For the present problem only the coefficients for $m = \pm 1$ are non-zero. Thus (note that $r' = z'$)

$$B_{n,\pm 1} = \frac{-j\omega\mu_0}{k_0 Q_{n1}} I_0 \frac{n(n+1)}{2r'} (\pm j) \hat{h}^{2}_{n}(k_0 z')$$

$$D_{n,\pm 1} = \frac{-j\omega\mu_0 I_0}{k_0 Q_{n1}} \frac{n(n+1)}{2r'} (\pm j) \left[\hat{h}^{1}_{n}(k_0 z') + C_n \hat{h}^{2}_{n}(k_0 z') \right]$$

$$\bar{B}_{n,\pm 1} = \frac{I_0}{Q_{n1}} \frac{n(n+1)}{2r'} \hat{h}^{2\,\prime}_{n}(k_0 z')$$

$$\bar{D}_{n,\pm 1} = \frac{I_0}{Q_{n1}} \frac{n(n+1)}{2r'} \left[\hat{h}_n^{1\prime}(k_0 z') + \bar{C}_n \hat{h}_n^{2\prime}(k_0 z') \right]$$

The coefficients C_n and $\bar{C}_n$ are given on Pgs. 49 and 50.

<u>Expansion of a Plane Wave</u>

If $\mu = \mu_0$, $\epsilon = \epsilon_0$, so that we do not have a dielectric sphere present then $C_n = \bar{C}_n = +1$ and $\hat{h}_n^1 + C_n \hat{h}_n^2 = 2\hat{j}_n$ and $\hat{h}_n^{1\prime} + \bar{C}_n \hat{h}_n^{2\prime} = 2\hat{j}_n^{\prime}$. Thus the electric field in the region $0 < r < r'$ is given by (note that $A_{nm} = 2B_{nm}$, $\bar{A}_{nm} = 2\bar{B}_{nm}$)

$$\vec{E}(\vec{r}) = \sum_{n=1}^{\infty} \sum_{m=-1,1} \left\{ \frac{\omega\mu_0 I_0}{k_0 Q_{n1}} \frac{n(n+1)m}{r'} \hat{h}_n^2(k_0 r') \vec{m}_{nm} \right.$$

$$\left. + \frac{k_0}{j\omega\epsilon_0} \frac{I_0}{Q_{n1}} \frac{n(n+1)}{r'} \hat{h}_n^{2\prime}(k_0 r') \vec{n}_{nm} \right\}, \quad r' = z'$$

where $\vec{m}_{nm}$ and $\vec{n}_{nm}$ are the same as $\vec{m}_{nm}'$, $\vec{n}_{nm}'$ with $\hat{h}_n^1$ replaced by $\hat{j}_n(k_0 r)$.

If we now let $z' \to \infty$ we can replace $\hat{h}_n^2(k_0 z')$ by $(k_0 z') j^{n+1} e^{-jk_0 z'}/k_0 z' = j^{n+1} e^{-jk_0 z'}$.

Thus since the electric field from the current element is also given by

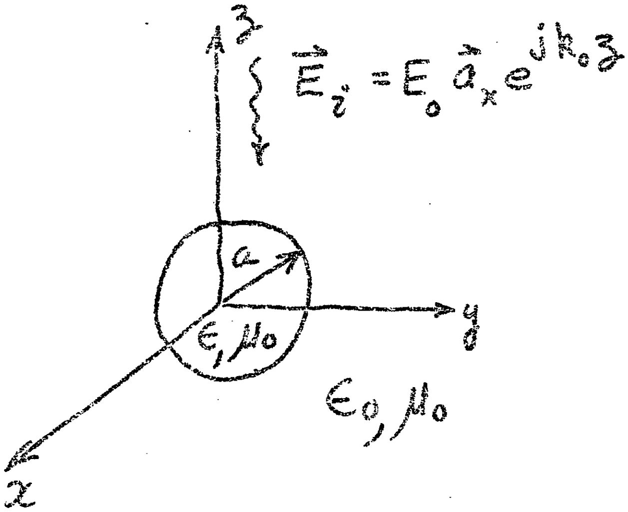
$$\vec{E} = \frac{-j\omega\mu_0 I_0}{4\pi z'} e^{-jk_0 r' + jk_0 z} (\vec{a}_x)$$

we find that the expansion of the plane wave $I_0 e^{jk_0 z} \vec{a}_x$ is given by

$$I_0 e^{jk_0 z} \vec{a}_x = -4\pi I_0 \sum_{n=1}^{\infty} \sum_{m=-1,1} \left\{ \frac{n(n+1)m}{k_0 Q_{n1}} j^n \vec{m}_{nm} - \frac{n(n+1)}{k_0 Q_{n1}} j^n \vec{n}_{nm} \right\} = \frac{j I_0}{k_0} \sum_{n=1}^{\infty} \sum_{m=-1,1} j^n \frac{2n+1}{2n(n+1)} \left\{ m \vec{m}_{nm}(k_0 r, \theta, \phi) - \vec{n}_{nm}(k_0 r, \theta, \phi) \right\}$$

upon substituting for Q_{n1}. The $n=0$ term is absent because $P_n^{|m|} = 0$ for $|m| > n$.

Diffraction of a Plane Wave by a Dielectric Sphere



For the incident field we choose $\vec{E}_i = E_0 \vec{a}_x e^{jk z}$

$$= \sum_{n=1}^{\infty} \sum_{m=1,-1} \frac{j E_0}{k_0} j^{n} \frac{2n+1}{2n(n+1)} \left\{ m \vec{m}_{nm}(k_0 r, \theta, \phi) - \vec{n}_{nm}(k_0 r, \theta, \phi) \right\}.$$

The scattered field will involve only the $m = \pm 1$ modes. Thus for $r < a$ let the <u>total</u> field be

$$\vec{E}_T = \sum_{n=1}^{\infty} \sum_{m=-1,1} \left[a_{nm} \vec{m}_{nm}(kr, \theta, \phi) + \frac{k}{j\omega\epsilon} b_{nm} \vec{n}_{nm}(kr, \theta, \phi) \right]$$

$$\vec{H}_T = \sum_{n=1}^{\infty} \sum_{m=-1,1} \left[\frac{-k}{j\omega\mu} a_{nm} \vec{n}_{nm}(kr, \theta, \phi) + b_{nm} \vec{m}_{nm}(kr, \theta, \phi) \right]$$

while for $r > a$ let the scattered field be

$$\vec{E}_s = \sum_{n=1}^{\infty} \sum_{m=1,-1} \left[c_{nm} \vec{m}^{2}_{nm}(k_0 r, \theta, \phi) + \frac{k_0}{j\omega\epsilon_0} d_{nm} \vec{n}^{2}_{nm}(k_0 r, \theta, \phi) \right]$$

$$\vec{H}_s = \sum_{n=1}^{\infty} \sum_{m=1,-1} \left[\frac{-k_0}{j\omega\mu_0} c_{nm} \vec{n}^{2}_{nm}(k_0 r, \theta, \phi) + d_{nm} \vec{m}^{2}_{nm}(k_0 r, \theta, \phi) \right]$$

$\hat{a}_r \times \vec{E}_t = \hat{a}_r \times (E_i + E_s)$

Continuity of the total tangential field components at $r = a$ gives

$E_i \quad \vec{M} \qquad E_s$

$$\frac{jE_0 j^n}{k_0}\frac{2n+1}{2n(n+1)} m\, \hat{J}_n(k_0 a) + c_{nm} \hat{h}_n^2(k_0 a) = a_{nm} \hat{J}_n(ka) \qquad a, c$$

$E_i \quad \vec{N} \qquad E_s$

$$-\frac{jE_0 j^n}{k_0}\frac{2n+1}{2n(n+1)} \hat{J}_n'(k_0 a) + \frac{k_0}{j\omega\epsilon_0} d_{nm} \hat{h}_n^{2\prime}(k_0 a) = \frac{k}{j\omega\epsilon} b_{nm} \hat{J}_n'(ka) \qquad b, d$$

$B_i \quad \vec{M}$

$$\vec{H} \quad -\frac{k_0}{j\omega\mu_0}\frac{jE_0 j^n}{k_0}\frac{(2n+1)m}{2n(n+1)} \hat{J}_n'(k_0 a) - \frac{k_0}{j\omega\mu_0} c_{nm} \hat{h}_n^{2\prime}(k_0 a) = -\frac{k}{j\omega\mu} a_{nm} \hat{J}_n'(ka) \qquad a, c$$

$B_i \quad \vec{N}$

$$\vec{H} \quad \frac{k_0}{j\omega\mu_0}\frac{jE_0 j^n}{k_0}\frac{2n+1}{2n(n+1)} \hat{J}_n(k_0 a) + d_{nm} \hat{h}_n^2(k_0 a) = b_{nm} \hat{J}_n(ka) \qquad b, d$$

These equations are readily solved to give

$$c_{nm} = m K_n \frac{\eta \hat{J}_n(k_0 a) \hat{J}_n'(ka) - \hat{J}_n(ka) \hat{J}_n'(k_0 a)}{\hat{J}_n(ka) \hat{h}_n^{2\prime}(k_0 a) - \eta \hat{J}_n'(ka) \hat{h}_n^2(k_0 a)}$$

$$d_{nm} = \frac{k_0 K_n}{j\omega\mu_0} \frac{\hat{J}_n(k_0 a) \hat{J}_n'(ka) - \eta \hat{J}_n(ka) \hat{J}_n'(k_0 a)}{\eta \hat{J}_n(ka) \hat{h}_n^{2\prime}(k_0 a) - \hat{J}_n'(ka) \hat{h}_n^2(k_0 a)}$$

where $\eta = \sqrt{\epsilon/\epsilon_0}$, $K_n = j^{n+1} E_0 (2n+1)/k_0 2n(n+1)$

When $k_0 a$ and ka are very small the coefficients may be expanded in power series in $k_0 a$ by using

$\hat{J}_n(x) \sim \frac{x^2}{3} - \frac{x^4}{30}$, $\hat{J}_n'(x) = \frac{2}{3}x - \frac{2}{15}x^3$,

$\hat{h}_n^2(x) \sim \frac{j}{x} + \frac{jx}{2} + \frac{x^2}{3}$, $\hat{h}_n^{2\prime}(x) = \frac{-j}{x^2} + \frac{j}{2} + \frac{2}{3}x$

Note that the above expressions are valid for $n=1$ only.

This gives (the formulas in Stratton, p 571 are in error)

$$c_{1m} \sim -m K_1 \, j \, \frac{\eta^2-1}{45} (k_0 a)^5$$

$$d_{1m} \sim \frac{-K_1 k_0}{j\omega\mu_0} \frac{2j}{3(\eta^2+2)} (k_0 a)^3 \left[(\eta^2-1) + \frac{3}{5} \frac{(\eta^2-1)(\eta^2-2)}{\eta^2+2} (k_0 a)^2 - j\frac{2}{3} \frac{(\eta^2-1)^2}{\eta^2+2} (k_0 a)^3 \right]$$

All the other coefficients are of order $(k_0 a)^5$ or less. By using only the coefficients d_{1-1} and d_{11} to order $(k_0 a)^3$ we get the quasi-static result equivalent to that obtained in Chap. 2 from the dipole approximation. This approximation gave $\sigma_s = \frac{8\pi}{3}(k_0 a)^6 \left(\frac{\eta^2-1}{\eta^2+2}\right)^2 k_0^{-2}$ for the total scattering cross section. This same result for the total scattering cross section is obtained from the forward scatter theorem Eq. (2.57) provided the expressions for d_{1-1}, d_{11} to order $(k_0 a)^6$ are used since this is the first term in phase quadrature with the incident field that arises in the expansion of the coefficients, i.e. the term in $(k_0 a)^6$ in d_{1-1}, d_{11} gives the cross section to order $(k_0 a)^6$.

On the $\theta = \pi$ axis the x component of the scattered field is $j\frac{\hat{h}_1^{2}(k_0 r)}{r}(c_{1-1} - c_{11}) + \frac{k_0 \hat{h}_1^{2\prime}(k_0 r)}{j\omega\epsilon_0 r}(d_{11} + d_{1-1})$

But $\hat{h}_1^{2} \sim -e^{-jk_0 r}$, $\hat{h}_1^{2\prime} \sim j e^{-jk_0 r}$ and since we write $\vec{F}\,\frac{e^{-jk_0 r}}{r}$ for the scattered field we have

$$\vec{F}\cdot\vec{a}_x = +j(c_{1-1} - c_{11}) + \frac{k_0}{\omega\epsilon_0}(d_{11} + d_{1-1}) \quad \text{and}$$

$$\sigma_s = \frac{-2\pi Y_0}{k_0 \frac{Y_0}{2}|E_0|^2}\,\text{Imag}\,\vec{E}_0^{*}\cdot\vec{F} = -\frac{4\pi}{E_0 k_0}\,\text{Imag}\,\vec{a}_x\cdot\vec{F}$$

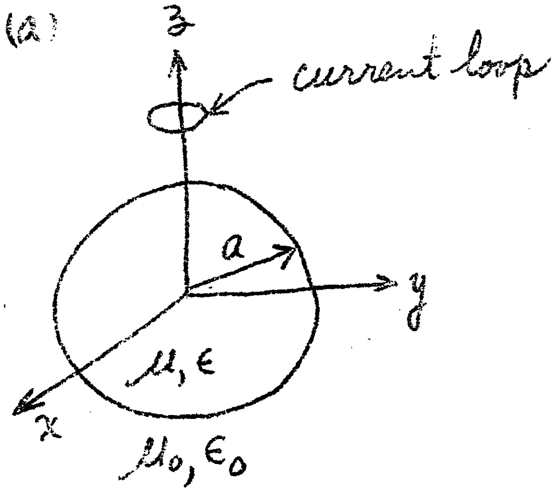
The c_{1-1}, c_{11} are pure imaginary so the only term which contributes is the $(k_0 a)^6$ term from d_{1-1} and d_{11}. Thus

$$\sigma_s = \frac{4\pi}{\cancel{k_0}}\,\frac{\cancel{k_0}}{\omega\epsilon_0}\left(\frac{+3}{4\cancel{k_0}}\right)\frac{\cancel{k_0}}{\omega\mu_0}\,\frac{2\cdot 4}{9}\,\frac{(\eta^2-1)^2}{(\eta^2+2)^2}\,(k_0 a)^6$$

$$= \frac{8}{3}\pi\left(\frac{\eta^2-1}{\eta^2+2}\right)^2 k_0^4 a^6$$

Problems

1. (a)



Find the field radiated by a small current loop, located on the z axis, in the presence of a sphere of radius 'a' with parameters μ, ϵ. The current loop may be treated as a magnetic dipole $\vec{M} = I A \vec{a}_z \, \delta(\vec{r}-\vec{r}\,')$ where A is the area and I is the loop current. Note that only TE waves are excited and the field expansions given in the notes may be used. To find the amplitudes use an appropiate form of the Lorentz reciprocity theorem involving magnetic dipoles as a source term.

(b) Repeat for a perfectly conducting sphere.

2. Find the field radiated by a z directed current element located at $y = z = 0$, $x = x'$, in terms of vector wave functions in spherical coordinates.

Let x' tend to infinity and thus obtain the expansion for a vector plane wave in terms of spherical vector wave functions.

For this problem use the $\vec{m}^{1,2}_{nm}$, $\vec{n}^{1,2}_{nm}$ functions.

3. A plane wave $E_0 \vec{a}_x e^{jk_0 z}$ is incident on a perfectly conducting $\varepsilon \to \infty$, $\mu \to 0$ sphere of radius a. Find the scattered field. For $k_0 a$ find the scattered field to order $(k_0 a)^3$ and calculate the total scattering cross section. Also evaluate the coefficients to order $(k_0 a)^6$ and find the cross section by using the forward scatter theorem.