ELECTROMAGNETIC SCATTERING AND DIFFRACTION

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CHAPTER I

Basic Electromagnetic Theory

1.1 Field Equations

In the homogeneous isotropic medium the electromagnetic field produced by electric currents with a volume density $\vec{J}_e$ is, under the assumption of having a sinusoidal time dependence $e^{j\omega t}$, governed by the following equations:

(1a) $\nabla \times \vec{E} = -j\omega\mu\vec{H}$

(1b) $\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J}_e$

(1c) $\nabla \cdot \vec{D} = \rho_e$

(1d) $\nabla \cdot \vec{B} = 0$

(1e) $\nabla \cdot \vec{J}_e = -j\omega\rho_e$

Equations (1c) and (1d) follow from the first two together with (1e) or alternatively the equation of continuity (1e) may be obtained from the divergence of (1b) together with (1c). Each field vector is a phasor quantity with real and imaginary parts in general.

In electromagnetic theory it is often convenient from a mathematical point of view to also introduce a fictitious set of magnetic type sources, namely magnetic currents of density $\vec{J}_m$ and associated magnetic charge ρ_m. These sources do not exist in a physical sense but may, nevertheless, be used as equivalent sources to produce a physical field in a restricted region of space. These magnetic type sources may be introduced in various ways as long as the resultant set of modified Maxwell equations still forms a self-consistent set. The usual scheme is as depicted below:

(2a) $\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{J}_m$

(2b) $\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J}_e$

(2c) $\nabla \cdot \vec{D} = \rho_e$

(2d) $\nabla \cdot \vec{B} = \rho_m$

(2e) $\nabla \cdot \vec{J}_e = -j\omega\rho_e$

(2f) $\nabla \cdot \vec{J}_m = -j\omega\rho_m$

In this modified set of field equations a symmetry between the electric and magnetic fields exists that is absent in the original set. The utility of this generalization will become apparent later on. $\vec{J}_m$ and ρ_m are mathematical entities and are not related to magnetic polarization effects in materials.

1.2 Electric and Magnetic Vector and Scalar Potentials

The set of equations (2) are usually integrated by the introduction of suitable auxiliary potential functions. When the only sources present are electric ones $\vec{J}_e$, ρ_e, we may choose

$$\vec{H} = \frac{1}{\mu} \nabla \times \vec{A}_e \tag{3}$$

where $\vec{A}_e$ is the electric type vector potential. It is now found that

$$\vec{E} = - j\omega\vec{A}_e - \nabla\phi_e \tag{4}$$

where ϕ_e is the electric scalar potential. The potentials are solutions of

$$\nabla^2\vec{A}_e + k^2\vec{A}_e = - \mu\vec{J}_e \tag{5a}$$

$$\nabla^2\phi_e + k^2\phi_e = - \frac{\rho_e}{\varepsilon} \tag{5b}$$

where $k^2 = \omega^2\mu\varepsilon$. In a gauge where $\nabla \cdot \vec{A}_e \neq 0$ the scalar potential may be eliminated (this is possible since ρ_e is not an independent quantity, being, if fact, given by $-\nabla \cdot J_e/j\omega$) by enforcing the Lorentz condition

$$\nabla \cdot \vec{A}_e + j\omega\mu\varepsilon\phi_e = 0 \tag{6}$$

We now have

$$\vec{H} = \mu^{-1} \nabla \times \vec{A}_e \tag{7a}$$

$$\vec{E} = -j\omega\vec{A}_e + (j\omega\mu\varepsilon)^{-1}\nabla\nabla\cdot\vec{A}_e = (j\omega\mu\varepsilon)^{-1}[\nabla \times \nabla \times \vec{A}_e - \mu\vec{J}_e] \tag{7b}$$

by using (5a) and the vector identity $\nabla \times \nabla \times \vec{A}_e = \nabla\nabla \cdot \vec{A}_e - \nabla^2\vec{A}_e$. A solution to (5a) in unbounded space filled with a medium with permittivity ε and permeability μ is*

$$\vec{A}_e(\vec{r}) = \frac{\mu}{4\pi} \int_V \vec{J}_e(\vec{r}\,') \frac{e^{-jkR}}{R} dV' \tag{8}$$

where $\vec{r}\,'$ is the position vector of the source element, $\vec{r}$ is the position vector of the field point and $R = |\vec{r} - \vec{r}\,'|$.

A parallel development shows that the partial field produced by magnetic sources $\vec{J}_m$, ρ_m, is given by

$$\vec{E} = -\varepsilon^{-1}\nabla \times \vec{A}_m \tag{9a}$$

$$\vec{H} = -j\omega\vec{A}_m + \frac{\nabla\nabla \cdot \vec{A}_m}{j\omega\mu\varepsilon} = \frac{1}{j\omega\mu\varepsilon} [\nabla \times \nabla \times \vec{A}_m - \varepsilon\vec{J}_m] \tag{9b}$$

$$\nabla^2\vec{A}_m + k^2\vec{A}_m = -\varepsilon\vec{J}_m \tag{9c}$$

with a solution for $\vec{A}_m$ in unbounded space given by

$$A_m(r) = \frac{\varepsilon}{4\pi} \int_V J_m(r') \frac{e^{-jkR}}{R} dV' \tag{9d}$$

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* For a derivation of the results (6)-(11) see for example, R. Plonsey and R. E. Collin, "Principles and Applications of Electromagnetic Fields," McGraw-Hill Book Co., New York, N. Y., 1961, pp. 321-325.

1.3 Duality

When electric sources only are present Maxwell's curl equations are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}_{o} + J_{e} = j\omega\varepsilon_{o}\vec{E} + (j\omega\vec{P} + \vec{J}_{e})$$

The latter equation is obtained by using the relation $\vec{D} = \varepsilon\vec{E} = \varepsilon_{o}\vec{E} + \vec{P}$ where $\vec{P}$ is the electric dipole polarization in the dielectric. This equation shows that we could equivalently think of $\vec{J}_e$ as being an equivalent dipole polarization per unit volume

$$\vec{J}_e = j\omega\vec{P}_e \qquad (10)$$

When only magnetic sources are present we have

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} - \vec{J}_m = -j\omega\mu_o\vec{H} - j\omega\mu_o\vec{M} - \vec{J}_m$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

since $\vec{B} = \mu\vec{H} = \mu_o(\vec{H} + \vec{M})$. Thus $\vec{J}_m$ can be replaced by an equivalent magnetic dipole polarization source according to the relation

$$\vec{J}_m = j\omega\mu_o\vec{P}_m \qquad (11)$$

where $\vec{P}_m$ is an equivalent magnetic dipole moment per unit volume.

We wish to show how a solution to the problem with magnetic sources can be obtained from the solution for electric sources. This constitutes the duality principle.

Consider an electric dipole $\vec{P}_e$ radiating in free space (if a bounded region was considered we would have to interchange electric and magnetic boundary conditions also) for which the governing equations are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + j\omega\vec{P}_e$$

If we introduce new fields $\vec{E}_d$, $\vec{H}_d$ by the duality relations

$$\vec{E}_d = Z\vec{H} \qquad (12a)$$

$$\vec{H}_d = -Y\vec{E} \qquad (12b)$$

where $Z = Y^{-1} = (\mu/\varepsilon)^{1/2}$ we obtain

$$\nabla \times \vec{E} = -Z\nabla \times \vec{H}_d = -j\omega\mu Y\vec{E}_d$$

$$\nabla \times \vec{H} = Y\nabla \times \vec{E}_d = -j\omega\varepsilon Z\vec{H}_d + j\omega\vec{P}_e$$

or

$$\nabla \times \vec{H}_d = j\omega\mu Y^2\vec{E}_d = j\omega\varepsilon\vec{E}_d$$

$$\nabla \times \vec{E}_d = -j\omega\varepsilon Z^2\vec{H}_d + j\omega Z\vec{P}_e = -j\omega\mu\vec{H}_d + j\omega Z\vec{P}_e$$

But the latter two equations are identical with the equations for the magnetic source case if we introduce for the duality relation between dipole sources

$$j\omega Z\vec{P}_e = -j\omega\mu_o\vec{P}_m$$

or

$$\mu_o\vec{P}_m = -Z\vec{P}_e \qquad (13)$$

The lack of symmetry is due to $\vec{P}_m$ being defined as an $\vec{H}$ like quantity while $\vec{P}_e$ is a $\vec{D}$ like quantity.

The utility of these duality relations may be appreciated by considering the radiation produced by a small electric current loop. But a small (in terms of wavelength) current loop is equivalent to a magnetic dipole of total moment $\vec{M}$ (a solenoidal current has no electric dipole moment), equal to $I\vec{S}$ where $\vec{S}$ is the vector area enclosed by the circulating current I. The field from such a magnetic dipole may be obtained from the solution to the problem of radiation by an electric dipole of total moment $\vec{P}_e$. If the field produced by $\vec{P}_e$ is $\vec{E}$, $\vec{H}$, then the dual solution is

$$\vec{E}_d = Z\vec{H} \qquad (14a)$$

$$\vec{H}_d = -Y\vec{E} \qquad (14b)$$

$$\mu_o\vec{P}_m = \mu_o\vec{M}_d = -Z\vec{P}_e \qquad (14c)$$

where $\vec{M}_d$ is the source which is dual to $\vec{P}_e$. Other duality relations also of interest are

$$\vec{P}_d = \mu_o Y\vec{P}_m \qquad (15a)$$

$$\vec{J}_{ed} = Y\vec{J}_m \qquad (15b)$$

$$\vec{J}_{md} = -Z\vec{J}_e \qquad (15c)$$

hat such a dual field constructed from the dual sources satisfies the set of equations (2) is readily demonstrated by following steps similar to those used to obtain (14a) to (14c)

1.4 Uniqueness and Field Equivalence Principles

A number of field equivalence principles exist that enables one to readily express the electromagnetic field in a given region in terms of equivalent sources placed on the surface enclosing the region of interest. We shall present three of these field equivalence principles. The first one to be discussed will be Love's equivalence principle.* The other two equivalence principles to be considered were first given by Schelkunoff.** The proofs are readily constructed by using the uniqueness theorem.*** The uniqueness theorem states that if a solution to Maxwell's equations in a region V has been found such that the field has the proper singularity at the location of the source, and for which either the tangential electric field or the tangential magnetic field has the specified values on the closed boundary S of V, then this solution is unique, i.e., no other solution exists.

Uniqueness

We will discuss the uniqueness theorem first. One of the results that is obtained by considering conditions which will lead to unique solutions are necessary and sufficient boundary conditions in order for a boundary value problem to be completely specified.

Let $\vec{E}_1$ and $\vec{H}_1$ be a solution to

$$\nabla \times \vec{E}_1 = -\mu \frac{\partial \vec{H}_1}{\partial t} - \vec{J}_m, \quad \nabla \times \vec{H}_1 = \varepsilon \frac{\partial \vec{E}_1}{\partial t} + \vec{J}_e$$

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* A.E.H. Love, The Integration of the Equations of Propagation of Electric Waves, Phil. Trans. Roy. Soc. London, ser. A, Vol. 197, pp. 1-45, 1901.

** S. A. Schelkunoff, Field Equivalence Theorems, Comm. on Pure and Appl. Math., Vol. 4, pp. 43-59, June 1951.

__________, Some Equivalence Theorems of Electromagnetics and their Application to Radiation Problems, Bell System Tech. J., Vol. 15, pp. 92-112.

*** J.A. Stratton, "Electromagnetic Theory," McGrew-Hill Book Company, Inc., New York 1941, pp. 486-488.

in a given region V bounded by a surface S. We are working in the time domain so $\vec{E}_1$, $\vec{H}_1$ $\vec{J}_m$ and $\vec{J}_e$ are real vector functions of time and $\vec{r}$. Let $\vec{E}_2$, $\vec{H}_2$ be another possible solution to the same equations, i.e.,

$$\nabla \times \vec{E}_2 = -\mu \frac{\partial \vec{H}_2}{\partial t} - \vec{J}_m \quad , \quad \nabla \times \vec{H}_2 = \varepsilon \frac{\partial \vec{E}_2}{\partial t} + \vec{J}_e$$

The difference field then satisfies

$$\nabla \times \vec{E}_d = -\mu \frac{\partial \vec{H}_d}{\partial t} , \quad \nabla \times \vec{H}_d = \varepsilon \frac{\partial \vec{E}_d}{\partial t}$$

where

$$\vec{E}_d = \vec{E}_1 - \vec{E}_2 \quad , \quad \vec{H}_d = \vec{H}_1 - \vec{H}_2$$

We wish to establish conditions that will make $\vec{E}_d$ and $\vec{H}_d$ be identically zero. We then arrive at a contradiction to the hypothesis that a second solution $\vec{E}_e$, $\vec{H}_2$ different from the first exists and will then be able to conclude that the first solution is the only solution to the problem.

Uniqueness proofs are usually established by constructing volume integrals of positive quantities involving the difference solution and relating these to surface integrals. By choosing boundary conditions that will make the surface integral vanish we then establish that the difference solution is identically zero. Consider,

$$\begin{aligned} \nabla \cdot (\vec{E}_d \times \vec{H}_d) &= \vec{H}_d \cdot \nabla \times \vec{E}_d - \vec{E}_d \cdot \nabla \times \vec{H}_d \\ &= -\mu \vec{H}_d \cdot \frac{\partial \vec{H}_d}{\partial t} - \varepsilon \vec{E}_d \cdot \frac{\partial \vec{E}_d}{\partial t} \\ &= -\frac{1}{2} \frac{\partial}{\partial t} (\mu |\vec{H}_d|^2 + \varepsilon |\vec{E}_d|^2) \end{aligned}$$

By using the divergence theorem the volume integral can be related to a surface integral, thus

$$\begin{aligned} &\int_V \nabla \cdot (\vec{E}_d \times \vec{H}_d) dV = \oint_S \vec{n} \cdot (\vec{E}_d \times \vec{H}_d) dS \\ &= \oint_S (\vec{n} \times \vec{E}_d) \cdot \vec{H}_d dS = -\oint_S \vec{E}_d \cdot (\vec{n} \times \vec{H}_d) dS \qquad (16) \\ &= -\frac{1}{2} \frac{\partial}{\partial t} \int_V (\mu |\vec{H}_d|^2 + \varepsilon |\vec{E}_d|^2) dV \end{aligned}$$

The surface integral will vanish if we specify as a boundary condition the tangential components of either $\vec{E}$ or $\vec{H}$. That is

$$\vec{n} \times \vec{E}_1 = \vec{n} \times \vec{E}_2 = \vec{n} \times \vec{E}_{bt} \quad \text{on } S \tag{17a}$$

or

$$\vec{n} \times \vec{H}_1 = \vec{n} \times \vec{H}_2 = \vec{n} \times \vec{H}_{bt} \quad \text{on } S \tag{17b}$$

will make either $\vec{n} \times \vec{E}_d$ or $\vec{n} \times \vec{H}_d$ zero on S and hence cause the surface integral to vanish. Here $\vec{n} \times \vec{E}_{bt}$ and $\vec{n} \times \vec{H}_{bt}$ are known given values for the tangential fields on S

When the surface integral vanishes we have

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_V (\mu|\vec{H}_d|^2 + \varepsilon|\vec{E}_d|^2)dV = 0$$

Integrating from $t = 0$ to t gives

$$-\frac{1}{2}\int_V (\mu|\vec{H}_d|^2 + \varepsilon|\vec{E}_d|^2)dV = \text{constant} \tag{18}$$

To make the constant zero we must specify an initial condition at $t = 0$. If we give the value of the field everywhere at $t = 0$ then $\vec{E}_d(\vec{r}, t = 0) = \vec{H}_d(\vec{r}, t = 0) = 0$ because $\vec{E}_1 = \vec{E}_2$ and $\vec{H}_1 = \vec{H}_2$ at $t = 0$. By putting $t = 0$ in (18) we then find that the constant must be zero.

Since the volume integral of a positive quantity such as $\mu|\vec{H}_d|^2 + \varepsilon|\vec{E}_d|^2$ is zero we find that the boundary conditions (17a) or (17b) together with the initial condition

$$\begin{aligned} \vec{E}_1(\vec{r}, t = 0) &= \vec{E}_2(\vec{r}, t = 0) \quad \text{is given} \\ \vec{H}_1(\vec{r}, t = 0) &= \vec{H}_2(\vec{r}, t = 0) \quad \text{is given} \end{aligned} \tag{19}$$

leads to a unique solution.

To examine uniqueness in the frequency domain we consider (*denotes complex conjugate phasor)

$$\nabla \cdot \vec{E}_d \times \vec{H}_d^* = -j\omega\mu\vec{H}_d \cdot \vec{H}_d^* + j\omega\varepsilon^* \vec{E}_d \cdot \vec{E}_d^*$$

We now obtain by similar steps

$$\oint_S (\vec{n} \times \vec{E}_d) \cdot \vec{H}_d^* \, dS = -\oint_S \vec{E}_d \cdot (\vec{n} \times \vec{H}_d^*) dS \tag{20}$$

$$= -j\omega\int_V (\mu \vec{H}_d \cdot \vec{H}_d^* - \varepsilon^* \vec{E}_d \cdot \vec{E}_d^*) dV$$

Note that the application of the divergence theorem to obtain (20) requires that $\vec{E}_d$ and $\vec{H}_d$ be continuous with piece-wise continuous first partial derivatives within V and on S. The surface integral will vanish if (note that $\vec{n}$ points outwards from the volume V)

$$\vec{n} \times \vec{E} \text{ is given on } S \tag{21a}$$

$$\vec{n} \times \vec{H} \text{ is given on } S \tag{21b}$$

It is also of interest to consider an impedance boundary condition such as

$$-Z_s \vec{n} \times \vec{H} = \vec{E}_{tan} \tag{21c}$$

The latter would be appropriate to a lossy metallic surface with $Z_s = (1+j)/\sigma\delta_s$ where σ is the conductivity and δ_s is the skin depth. For an impedance boundary condition it is clear that $\vec{E}_{d,tan} = -Z_s \vec{n} \times \vec{H}_{d,tan}$ also and hence

$$\begin{aligned}
-\vec{n} \times \vec{E}_d \cdot \vec{H}_d^* &= Z_s \vec{n} \times (\vec{n} \times \vec{H}_{d,tan}) \cdot \vec{H}_d^* \\
&= Z_s (\vec{n} \cdot \vec{H}_{d,tan}) \vec{n} \cdot \vec{H}_d^* - Z_s \vec{H}_{d,tan} \cdot \vec{H}_d^* \\
&= (\vec{n} \times \vec{H}_d^*) \cdot \vec{E}_d = -Y_s^* \vec{E}_{d,tan} \cdot \vec{E}_{d,tan}^* \\
&= -Y_s^* |\vec{E}_{d,tan}|^2
\end{aligned} \tag{22}$$

Let $\varepsilon = \varepsilon' - j\varepsilon''$ and $\mu = \mu' - j\mu''$ be complex and assume that an impedance boundary condition is specified. By separating (20) into real and imaginary parts we obtain

$$\oint_S G_s |\vec{E}_{d,tan}|^2 dS + \int_V \omega(\mu'' |\vec{H}_d|^2 + \varepsilon'' |\vec{E}_d|^2) dV = 0 \tag{23a}$$

$$-\oint_S B_s |\vec{E}_{d,tan}|^2 dS + \int_V \omega(\mu' |\vec{H}_d|^2 - \varepsilon' |\vec{E}_d|^2 dV = 0 \tag{23b}$$

where $Y_s = Z_s^{-1} = G_s + jB_s$.

The first is the sum of integrals of positive terms and can vanish if and only if

$$\vec{E}_{d,tan} = 0 \text{ on } S, \ \vec{E}_d = \vec{H}_d = 0 \text{ in } V$$

so an impedance boundary condition is seen to lead to a unique solution if any two of the three quantities G_s, μ'', ε'' are not zero. When any two of G_s, μ'', ε'' are not zero then two of the quantities $\vec{E}_{d,tan}$, $\vec{E}_d$, and $\vec{H}_d$ must vanish from the first equation and the second one will show that the remaining third quantity must vanish. If $G_s = \varepsilon'' = 0$ then (23a) shows that $\vec{H}_d = 0$. Hence $\vec{E}_{d,tan} = 0$ because of (21c) and from (23b) we then get $\vec{E}_d = 0$ also.

If ε'' and μ'' are not zero and the b oundary conditions (21a) or (21b) are imposed the surface integral vanishes and (23) again shows that $\vec{E}_d = \vec{H}_d = 0$ in V. The same conclusion is arrived at if only one of ε'' or μ'' is different from zero.

The two cases $G_s = \varepsilon'' = \mu'' = 0$ for impedance boundary conditions and $\varepsilon'' = \mu'' = 0$ for boundary conditions (21a) or (21b) do not yield unique solutions since they permit difference solutions which satisfy respectively the conditions

$$B_s \oint_S |\vec{E}_{d,tan}|^2 dS = \omega \int_V (\varepsilon' |\vec{E}_d|^2 - \mu' |\vec{H}_d|^2) dV \tag{24a}$$

$$\omega \int_V \mu' |\vec{H}_d|^2 \, dV = \omega \int_V \varepsilon' |\vec{E}_d|^2 dV \tag{24b}$$

Such source free solutions are free resonant modes inside the cavity bounded by the surface S and would be found to exist only for certain discrete values of ω, the resonant frequencies. In any given problem they are easily identified and eliminated from the particular solution of interest and hence the conditions (21a), (21b), or (21c) insures uniqueness for all practical purposes.

If the volume V is bounded by an interior closed surface S_i and the surface S_∞ of a sphere of infinite radius that we must impose a radiation boundary condition on the fields at infinity. This requires the field amplitudes to decay at least as r^{-1} as r tends to infinity. In homogeneous and isotropic media we also require that

$\vec{E} = -Z\,\vec{a}_r \times \vec{H}$ to order $1/r$ as r tends to infinity where $Z = \sqrt{\mu/\varepsilon}$ and $\vec{a}_r$ is a unit vector directed radially outward. In this case (23a) will contain the two terms

$$\frac{1}{Z}\oint_{S_\infty} |\vec{E}_{d,tan}|^2 dS + G_S \oint_{S_1} |\vec{E}_{d,tan}|^2 dS$$

along with the volume integral and thus shows that $\vec{E}_{d,tan}$ vanishes at infinity to an order greater than r^{-1} since S increases as r^2. Hence when the surface at infinity is included the imposition of a radiation boundary condition is needed to ensure uniqueness. The particular radiation condition given above is the requirement that the Poynting vector be radially outward directed at infinity and that the total power flow across S_∞ be finite. The question of uniqueness for sources radiating into unbounded space filled with homogeneous isotropic media is discussed in Prob. 1.1 .

Field Equivalence Principles

All of the field equivalence principles can be understood in terms of the following hypothetical problem. Consider a closed surface S which separates space into two regions V_1 and V_2. Let sources $\vec{J}_{e1}$, $\vec{J}_{m1}$ produce an arbitrary field $\vec{E}_1$, $\vec{H}_1$ in V_1 i.e. we do not specify any boundary conditions on $\vec{E}_1$, $\vec{H}_1$ at S so a variety of solutions to $\nabla \times \vec{E}_1 = -j\omega\mu\vec{H}_1 - \vec{J}_{m1}$, $\nabla \times \vec{H}_1 = j\omega\epsilon\vec{E}_1 + \vec{J}_{e1}$ in V_1 are possible. Similarly let sources $\vec{J}_{e2}$, $\vec{J}_{m2}$ produce an arbitrary field

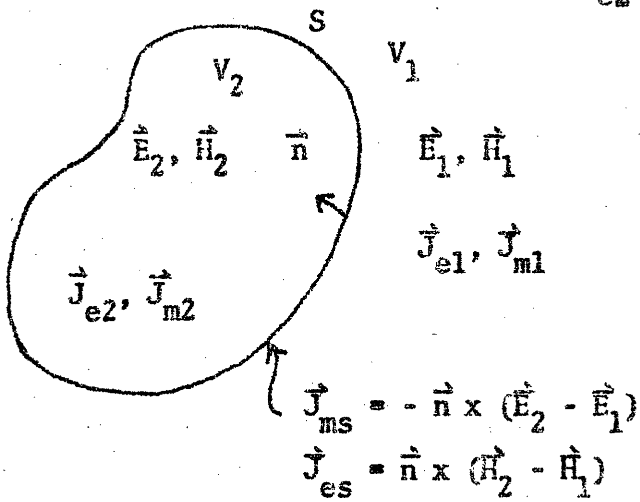


Fig. 1.1, Illustration for joining to arbitrary field solutions across a common boundary.

$\vec{E}_2$, $\vec{H}_2$ in V_2. The total field will be a unique solution to Maxwell's equations only if the two solutions are properly joined across the common boundary S. If we have no surface sources available then we require

$$\vec{n} \times \vec{E}_1 = \vec{n} \times \vec{E}_2 \text{ on S}$$
$$\vec{n} \times \vec{H}_1 = \vec{n} \times \vec{H}_2 \text{ on S}$$

But if we permit arbitrary surface sources $\vec{J}_{es}$, $\vec{J}_{ms}$ then we can keep the fields in V_1 and V_2 arbitrary as long as we choose

$$\vec{J}_{ms} = -\hat{n} \times (\vec{E}_2 - \vec{E}_1) \tag{25a}$$

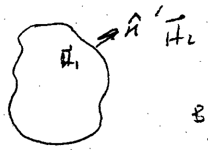
$$\vec{J}_{es} = \hat{n} \times (\vec{H}_2 - \vec{H}_1) \tag{25b}$$

for the surface currents on S as shown in Fig. 1.1. These surface currents will give the proper discontinuous change in the tangential values of the fields as the surface S is crossed. By choosing various possible solutions for $\vec{E}_1$ and $\vec{H}_1$ we can obtain the various field equivalence principles. In other words, we can vary the field in V_1 at will without affecting the field in V_2 as long as we adjust the surface currents according to (25).

Love's Field Equivalence Principle

Consider a closed surface S that separates a homogeneous isotropic medium into two regions as in Fig. 1.2. The exterior region is V_1 and contains all of the sources. Thus the interior region V_2 is taken to be free of all sources. Likewise the surface S is taken as a source free surface. The field equivalence theorem of Love states that the field in V_2 can be obtained from equivalent electric and magnetic currents located on the surface S in place of the original sources. If the original sources produce a field $\vec{E}$, $\vec{H}$, then the equivalent sources to be placed on S are an electric surface current.

$$\vec{J}_{es} = \hat{n} \times \vec{H} \tag{26a}$$



$\hat{n}' \times (\vec{H}_2 - \vec{H}_1) = \vec{J}$
But $\vec{H}_2$ is taken to be zero
$\Rightarrow \vec{J} = -\hat{n}' \times \vec{H}_1 = \hat{n} \times \vec{H}_1$, $\hat{n} = -\hat{n}'$

and a magnetic surface current

$$\vec{J}_{ms} = \vec{E} \times \hat{n} \tag{26b}$$

where $\hat{n}$ is a unit normal directed into V_2. Furthermore, these equivalent surface currents produce a null field in the region V_1 external to S.

To prove the theorem let the field $\vec{E}_1$, $\vec{H}_1$, $\vec{E}_2$, $\vec{H}_2$ be defined as follows:

$$\hat{\vec{E}}_2 = \hat{\vec{E}} \qquad \text{in } V_2$$
$$\hat{\vec{H}}_2 = \hat{\vec{H}}$$

$$\hat{\vec{E}}_1 = \hat{\vec{H}}_1 = 0 \qquad \text{in } V_1$$

The field $\hat{\vec{E}}_2$, $\hat{\vec{H}}_2$, as defined suffers a discontinuity in its tangential components according to the relations

$$\hat{n} \times \hat{\vec{E}}_2 = -\hat{\vec{J}}_{ms} = \hat{n} \times \hat{\vec{E}}$$

$$\hat{n} \times \hat{\vec{H}}_2 = \hat{\vec{J}}_{es} = \hat{n} \times \hat{\vec{H}}$$

as the surface S is crossed, therefore, since the field $\hat{\vec{E}}_2$, $\hat{\vec{H}}_2$, as defined, satisfies the required boundary conditions it is the unique solution to the problem of what field is radiated by the given surface currents. Hence, the equivalent sources given by (25) will produce the same field in V_2 as the original sources do and the field equivalence principle is established. This is a special case of the hypothetical problem introduced earlier with $\hat{\vec{E}}_1 = \hat{\vec{H}}_1 = 0$.

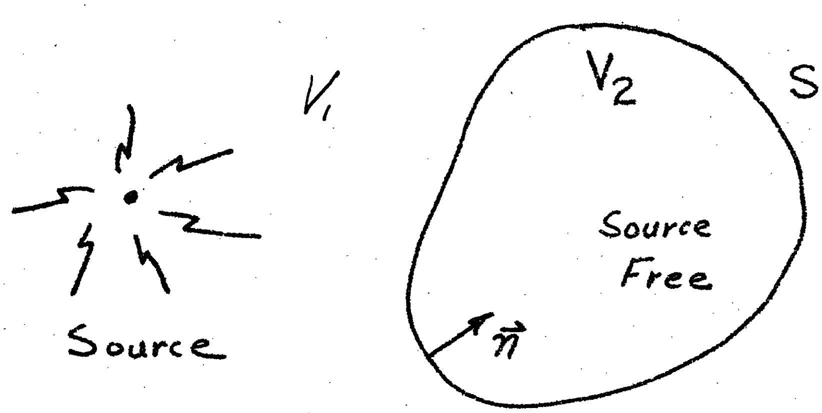


Fig. 1.2 Illustration for field equivalence principles

It may seem a bit strange that the equivalent sources placed on the surface S should radiate a field into the interior volume V_2 and yet not produce a field in the exterior region. A familiar example that illustrates the plausibility of such a result is the electromagnetic cavity. A totally enclosed cavity with perfectly conducting walls will support an electromagnetic field for which $\bar{n} \times \bar{E} = 0$ on the boundary and for which $\bar{n} \times \bar{H} = \bar{J}_{es}$ where $\bar{J}_{es}$ is an electric surface current flowing on the interior surface of the perfectly conducting walls. As long as the current $\bar{J}_{es}$ is maintained the interior field remains unchanged even if the metallic surface is removed. The current $\bar{J}_{es}$ is thus an example of a source distribution that radiates a finite field in the interior volume and a null field outside.

Schelkunoff's Field Equivalence Theorems

Theorem I. With reference to Fig. 1.2 let S again separate the region V_1 containing the sources from the region V_2 interior to S. When the surface S is replaced by a perfectly conducting surface let the original sources in V_1 produce a primary field $\bar{E}_o$, $\bar{H}_o$. On S an electric surface current $-\bar{n} \times \bar{H}_o$ will flow. The theorem states that the original sources radiate a field $\bar{E}_2$, $\bar{H}_2$, into the region V_2 which is identical with that produced by an equivalent electric current $\bar{J}_{es} = \bar{n} \times \bar{H}_o$ on S. In addition the current $\bar{J}_{es}$ will radiate a field $\bar{E}'_1$, $\bar{H}'_1$, into the region V_1 such that $\bar{E}_0 + \bar{E}'_1$, $\bar{H}_0 + \bar{H}'_1$, is identical to the field produced by the original sources. In evaluating the fields $\bar{E}'_1$, $\bar{H}'_1$, and $\bar{E}_2$, $\bar{H}_2$, the surface S is no longer considered as a perfectly conducting surface, i.e., it is a surface on which a current $\bar{n} \times \bar{H}_0$ exists only.

The theorem is readily proved by invoking the uniqueness theorem again. The field $\bar{E}'_1$, $\bar{H}'_1$, is source free in the region V_1 while $\bar{E}_o$, $\bar{H}_o$, provide the correct singularities at the sources. Hence the total field $\bar{E}'_1 + \bar{E}_0$, $\bar{H}'_1 + \bar{H}_0$, satisfies all the boundary conditions in V_1. In addition the field calculated from $\bar{J}_{es}$ on S must be constructed so that $\bar{n} \times \bar{E}'_1 = \bar{n} \times \bar{E}_2$ on S while $\bar{n} \times \bar{H}_2 - \bar{n} \times \bar{H}'_1 = \bar{J}_{es} = \bar{n} \times \bar{H}_0$. Consequently, since $\bar{n} \times \bar{E}_o = 0$ on S while $\bar{n} \times \bar{H}_o = \bar{J}_{es}$ on S the total tangential electric and magnetic fields are continuous across S, i.e. on S

$$\bar{n} \times (\bar{E}'_1 + \bar{E}_0) = \bar{n} \times \bar{E}_2$$

$$\hat{n} \times (\bar{H}'_1 + \bar{H}_0) = \hat{n} \times \bar{H}_2$$

Therefore, the specified field satisfies all the required boundary conditions and, by the uniqueness theorem, is the unique field radiated by the original sources.

Theorem 2. The second field equivalence theorem is essentially the dual of Theorem 1. Consider again Fig. 1.2 and let the sources in V_1 radiate a field $\bar{E}_0$, $\bar{H}_0$, when the surface S is replaced by a perfect magnetic conductor (a fictitious surface which is the dual of a perfect electric conductor and on which $\hat{n} \times \bar{H}$ must vanish). Thus on S a magnetic surface current $-\bar{E}_0 \times \hat{n}$ will flow and $\hat{n} \times \bar{H}_0 = 0$. The original sources will produce a field in V_2 identical with that produced by an equivalent magnetic current $\bar{J}_{ms} = \bar{E}_0 \times \hat{n}$ placed on S (the magnetic conductor is removed in calculating the field produced by $\bar{J}_{ms}$). If $\vec{J}_{ms}$ radiates a field $\bar{E}'_1$, $\bar{H}'_1$, into the region V_1 then $\bar{E}_0 + \bar{E}'_1$, $\bar{H}_0 + \bar{H}'_1$, is identical to the field produced in V_1 by the original sources. The proof is similar to that for theorem 1 and hence will not be reproduced.

The above theorems correspond to choosing $\bar{E}_1 = \bar{E}'_1 + \bar{E}_0$ and $\bar{H}_1 = \bar{H}'_1 + \bar{H}_0$ for the fields in region V_1 for the hypothetical problem discussed earlier and the true correct field is then maintained in V_2 by placing a suitable current on the boundary S.

An alternative viewpoint that helps to clarify the above theorems is as follows: For theorem 1 let the field radiated into V_2 by the original sources be $\vec{E}_2$, $\vec{H}_2$. We now wish to specify an electric current distribution on S that will radiate a field $-\bar{E}_2$, $-\bar{H}_2$ into V_2 so that the combination of the original sources plus this secondary source will produce a null field in V_2. With the field in V_2 reduced to zero the surface just exterior to the electric surface current may be replaced by a perfectly conducting surface without disturbing the field in V_1. But now the situation is the same as that postulated at the beginning of the theorem so clearly the required current source needed to cancel the field in V_2 is $-\hat{n} \times \bar{H}_0$. To restore the original problem we must now place an electric current $\bar{J}_{es} = \hat{n} \times \bar{H}_0$ on S to cancel the current placed on S and used together with the original sources to radiate a null field in V_2. Obviously, if $-\hat{n} \times \bar{H}_0$ radiates a field $-\vec{E}_2$, $-\vec{H}_2$, in V_2 then $\vec{J}_{es} = \hat{n} \times \vec{H}_0$ will radiate a field $\vec{E}_2$, $\vec{H}_2$, identical with that produced by original sources.

The field equivalence principles given above are mathematical statements of Huygen's principle.†

1.5 Further Uniqueness Conditions

In addition to the boundary conditions given in Sec. 1.4 required for unique solutions it is necessary to consider conditions at infinity for unbounded regions and also conditions on the allowed order of singularity in the solutions near sharp corners and edges.

Radiation Conditions

Perfectly general and complete mathematical statements of boundary conditions at infinity are not very easily given because of complications which can arise when complex media such as anisotropic plasma completely fills space. For our purposes it will be sufficient that the solution correspond to outward propagating waves with the Poynting vector directed such that the power flow is radially outward at infinity.

Field Singularities at an Edge of a Conducting Wedge

Figure 1.3 illustrates a two dimensional conducting wedge of internal angle θ.

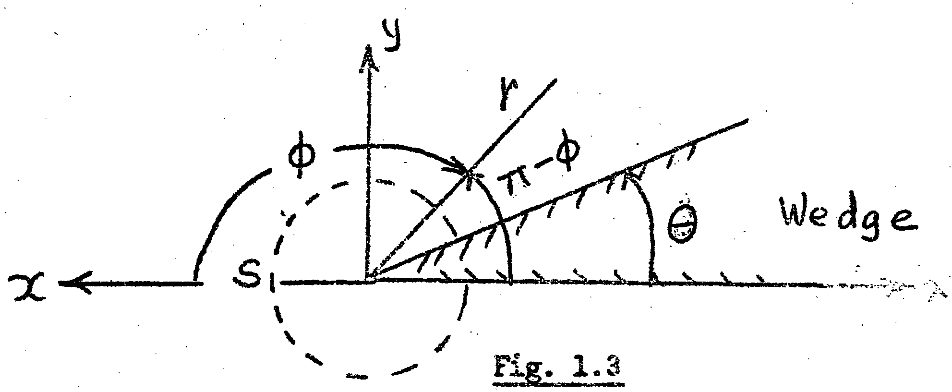


Fig. 1.3

† B. B. Baker, E. T. Copson, "The Mathematical Theory of Huygen's Principle," Oxford Univ. Press, London, 1939.

J.A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, 1941, Sec.'s 8.13-8.15.

For many diffraction problems it is necessary to know the nature of the singularity associated with each field component in the vicinity of the edge $r = 0$ in order to obtain a unique solution. The maximum rate of increase of each field component is limited by the condition that the energy stored in a finite volume around the edge should be finite. Thus we require

$$\int_{-\pi}^{\pi-\theta}\int_0^r (\epsilon\bar{E}\cdot\bar{E}^* + \mu\bar{H}\cdot\bar{H}^*)\,r\,d\phi\,dr$$

to remain finite. Thus each term such as $\bar{E}\cdot\bar{E}^*$, must increase no faster than $r^{-2(1-\eta)}$ where η is a small positive number. Consequently, each field component must increase no faster than $r^{-(1-\eta)}$.

Meixner obtains exact results for the rate of growth of each field component in terms of the wedge angle θ by expanding the fields and Maxwell's equations in a power series in r. We may obtain the same results by using standard solutions for the fields with a radial dependence given in terms of Bessel functions.

For a two dimensional wedge the fields separate into TE and TM fields with respect to the z axis. The r and ϕ components may be found in terms of E_z and H_z. Suitable solutions for E_z and H_z are (the Y_ν or Bessel functions of the second kind would lead to a violation of the energy condition and so are not included; Note that $Y_\nu(x) \propto x^{-|\nu|}$, $Y_0(x) \propto \ln x$, as $x \to 0$)

$$E_z = J_\nu(\sqrt{k^2-\beta^2}\,r)e^{\pm j\beta z}\sin\nu(\phi+\pi)$$

$$H_z = J_\nu(\sqrt{k^2-\beta^2}\,r)e^{\pm j\beta z}\cos\nu(\phi+\pi)$$

where $k^2 = \omega^2\mu\epsilon$ and $\nu = \pm n\pi/(2\pi-\theta)$, $n = 0,1,2,\ldots$ in order for the boundary conditions to hold at $\phi = -\pi$, $\pi-\theta$. For r very small $J_\nu(\sqrt{k^2-\beta^2}\,r)$ is given by $(k^2-\beta^2)^{\nu/2}r^\nu/[2^\nu\Gamma(\nu+1)]$ where Γ is the gamma function. The r and ϕ components of the fields increase by a factor r^{-1} faster as Maxwell's equations show. Thus the field components normal to the edge will increase as $r^{-\xi}$ where

$$\alpha = 1 - \nu = 1 \mp \frac{n\pi}{2\pi - \theta}, \quad n = 0,1,2,\ldots$$

The only values of ν allowed, i.e., integers n, are those that keep α less than $1 - \eta$. Hence, for a 90° wedge ($\theta = \pi/2$) we find that $\alpha = 1/3$ while for a flat infinitely thin plate ($\theta = 0$) we obtain $\alpha = 1/2$. In both cases the smallest value of n allowed is $n = 1$. The field components tangential to the edge (E_z, H_z) always vanish at the edge. Likewise the surface current directed normal to the edge vanishes since this is related to H_z. The surface current parallel to the edge will increase as $r^{-\alpha}$ as the edge is approached. These results also apply to edges occuring in three dimensional problems since the singular behavior of the fields near the edge is determined only by the immediate currents and charges and hence a short segment of a curved edge behaves like the edge of a two dimensional structure.

The edge condition can also be viewed as a requirement that there be no net power radiated from the edge. thus

$$\mathrm{Re} \oint_S \vec{E} \times \vec{H}^* \cdot d\vec{S} = 0$$

where S is a closed surface surrounding the edge as in Fig. 1.3. Obviously just as much power must enter the region enclosed by S as leaves the region because no active sources are included. The radiation condition forces us to eliminate the Hankel functions from the solutions for E_z and H_z and thus leads to the same conclusions as the energy condition does.

1.6 Electric Dipole Radiation

Figure 1.4 illustrates a short current element of length $\Delta z \ll \lambda_0$. It may be viewed as an electric dipole with a total moment $P = -jI_0\Delta z/\omega$. Since any arbitrary current distribution may be built up from elementary current sources of this type it is of interest to evaluate the field radiated by a short linear current element. A general field from an arbitrary current distribution may be obtained by superposition.

CURRENT FLOWING INTO UPPER END = TIME RATE OF INCREASE OF CHARGE AT THE UPPER END, i.e.,

$I = I_0 e^{j\omega t} = \frac{d}{dt} Q e^{j\omega t}$

$= Q j\omega e^{j\omega t}$

$I_0 = Q j\omega$

or $Q = \frac{I_0}{j\omega}$

$P = Q\Delta z = \frac{I_0 \Delta z}{j\omega}$

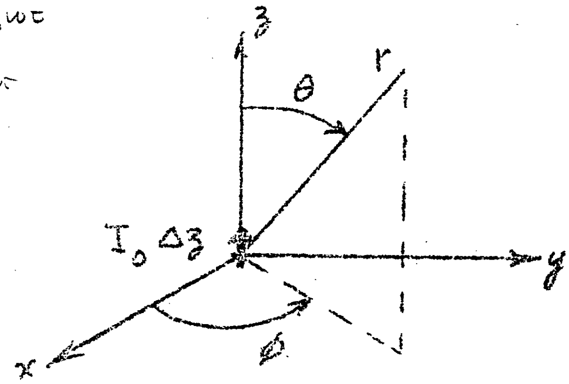


Fig. 1.4. A short linear current radiator

From a short linear current element $I\Delta z = I_0 \Delta z e^{j\omega t}$ the resultant vector potential produced at a point r in free space is

$$\bar{A} = \frac{\mu_0}{4\pi} \int_V \frac{\bar{J}}{R} e^{-jk_0 R} dV'$$

$$\bar{J} = I\, \delta(x-x')\, \delta(y-y')\, \hat{z}$$

$$= \frac{\mu_0}{4\pi} \iiint \frac{I\, \delta(x-x')\, \delta(y-y')\, \Theta(\Delta z - z')}{R} dx' dy' dz'$$

$$= \frac{\mu_0}{4\pi} \frac{I}{R} e^{-jk_0 R} \Delta z\, \hat{z}$$

$$\bar{A}_e = \frac{\mu_0 I_0 \Delta z}{4\pi r} e^{-jk_0 r} \bar{a}_z = A_z \bar{a}_z$$

The field $\bar{B}$ is given by

$$\bar{B} = \nabla \times \bar{A}_e = -\bar{a}_z \times \nabla A_z$$

$$= -(\bar{a}_r \cos\theta - \bar{a}_\theta \sin\theta) \times \bar{a}_r \frac{\partial A_z}{\partial r} = -\bar{a}_\phi \sin\theta \frac{\partial A_z}{\partial r}$$

since A_z is a function of r only. Thus we find

$$\bar{H} = \mu_0^{-1} \bar{B} = \frac{I_0 \Delta z}{4\pi} \bar{a}_\phi \left(\frac{1}{r^2} + \frac{jk_0}{r} \right) \sin\theta\, e^{-jk_0 r} \qquad (28)$$

From the relation $\nabla \times \bar{B} = j\omega\mu_0\epsilon_0 \bar{E}$ we obtain

$$\bar{E} = \bar{a}_r E_r + \bar{a}_\theta E_\theta$$

where

$$E_r = -\frac{jI_0Z_0\Delta z}{2\pi k_0}\cos\theta\left(\frac{1}{r^3}+\frac{jk_0}{r^2}\right)e^{-jk_0r} \tag{29a}$$

$$E_\theta = \frac{jI_0Z_0\Delta z}{4\pi k_0}\sin\theta\left(\frac{k_0^2}{r}-\frac{jk_0}{r^2}-\frac{1}{r^3}\right)e^{-jk_0r} \tag{29b}$$

The terms varying as $1/r$ constitute the far zone or radiation field and account for the power radiated away from the source. The other terms, varying as $1/r^2$ and higher inverse powers of r constitute the near zone or induction field. The induction field results in a storage of reactive energy in the region of space surrounding the current source. The radiation field gives rise to a resistive term in the input impedance of an antenna while the near zone field gives rise to the input reactance in the antenna impedance.

For a short current element the far zone fields are

$$E_\theta = \frac{jI_0Z_0\Delta z k_0}{4\pi r}\sin\theta\, e^{-jk_0r} \tag{30a}$$

$$H_\phi = Y_0E_\theta, \qquad Y_0 = Z_0^{-1} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \tag{30b}$$

The far zone field is always a spherical TEM wave for which

$$E_\theta = Z_0H_\phi, \quad E_\phi = -Z_0H_\theta, \quad E_r = H_r = 0$$

The radiated power is given by

$$P_r = \frac{1}{2}\,\mathrm{Re}\oint_S \bar{E}\times\bar{H}^*\cdot d\bar{S} = \frac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}(E_\theta H_\phi^* - E_\phi H_\theta^*)r^2\sin\theta\, d\theta d\phi$$

in general. For the short current radiator we have

$$P_r = \frac{I_0I_0^*(k_0\Delta z)^2}{12\pi}Z_0 \tag{31}$$

The radiation resistance R_0 for an antenna is defined as the equivalent resistance that would dissipate an amount of power equal to that radiated by the antenna when the current flowing through this resistance is equal to the antenna input current at the antenna terminals. For the short current radiator we have

$$\frac{1}{2} I_0 I_0^* R_0 = P_r$$

and hence the radiation resistance is

$$R_0 = Z_0 \frac{(k_0 \Delta z)^2}{6\pi} = 80\pi^2 \left(\frac{\Delta z}{\lambda_0} \right)^2 \tag{32}$$

When $\Delta z << \lambda_0$ the radiation resistance is very small. Practically this means that the ohmic resistance of a very short antenna would normally be much larger than the radiation resistance and thus the antenna efficiency and gain would be very low. Very short antennas (compared to a wavelength) also exhibit a large reactive term in the input impedance and are, therefore, difficult to match to a power source in order to obtain maximum power transfer. If, for example, $\Delta z = 0.01\lambda_0$ (a six foot long wire at 200 meters or 1500 Kc.) $R_0 \approx 0.079$ ohms.

The power radiated in a given direction per unit solid angle is a measure of the directive properties of an antenna. For a short current element the power density in a direction specified by the coordinates θ, ϕ is 1/2 Re $E_\theta H_\phi^*$ watts per meter^2 . The power density per unit solid angle is

$$1/2 \ \mathrm{Re}\, r^2 E_\theta H_\phi^*$$

The directivity D is given by

$$\text{Directivity} = D(\theta, \phi) = \frac{\text{Power density per unit solid angle in direction } \theta, \phi}{\text{Total radiated power averaged over } 4\pi \text{ sterradians}}$$

$$= 4\pi \frac{\text{Power density per unit solid angle}}{\text{Total radiated power}} \qquad (33)$$

In the present case we have

$$D(\theta, \phi) = D(\theta) = 3/2 \sin^2 \theta \qquad (34)$$

for the directivity of a short linear current element. This is the directivity referred to an isotropic radiator (an antenna radiating P_r watts uniformly in all directions of $P_r/4\pi$ watts per sterradian). The directivity D is sometimes referred to a half-wave length dipole antenna as a reference.

The antenna gain G is defined as

$$G(\theta, \phi) = \frac{\text{Radiated power per unit solid angle in direction } \theta, \phi}{\text{Total input power to antenna } /4\pi}$$

$$= D(\theta, \phi) \quad \text{(antenna efficiency)} \qquad (35)$$

Sometimes the words directivity and gain are used to signify the maximum value of D and G, e.g., the directivity of a short current element is 1.5 relative to an isotropic antenna.

A plot of $D(\theta, \phi)$ gives a three dimensional surface referred to as the radiation pattern. For the short current element the radiation pattern is a figure 8 rotated about the polar axis as in Fig. 1.5. The E plane beamwidth at the half power points is 90° (beamwidth in plane containing the $\vec{E}$ vector). The H plane beamwidth is a full 360°.

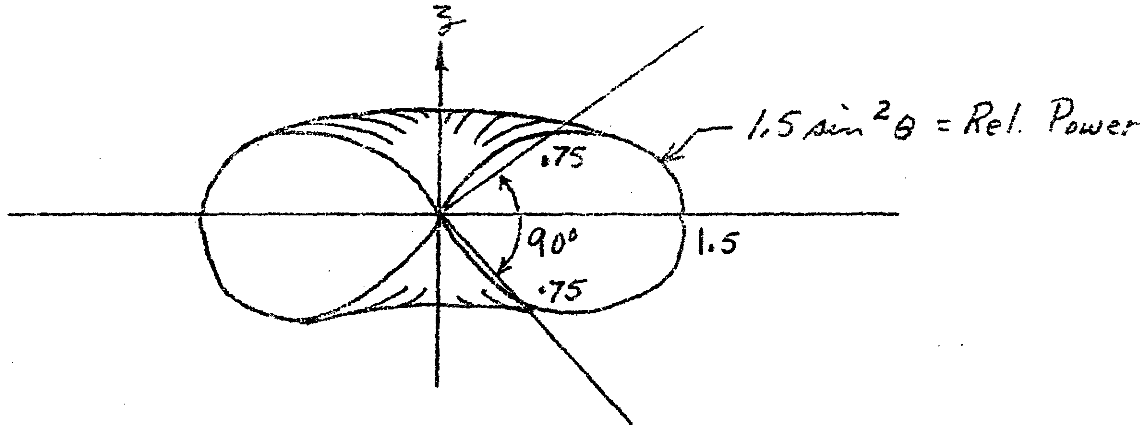


Fig. 1.5. Radiation pattern for a short linear current element

Magnetic Dipole Radiation

A small solenoidal current loop has a total magnetic dipole moment given by $I\vec{S}$. The leading term in the multipole expansion of the field radiated by a solenoidal current loop may be found from a magnetic dipole of total moment given by $I\vec{S}$. When the linear dimensions of the loop are very small compared with the wavelength this is the only significant term in the multipole expansion. Figure 1.6 illustrates a current loop of radius "a" located at the origin. Its magnetic dipole moment is $I_0\pi a^2\vec{a}_z$.

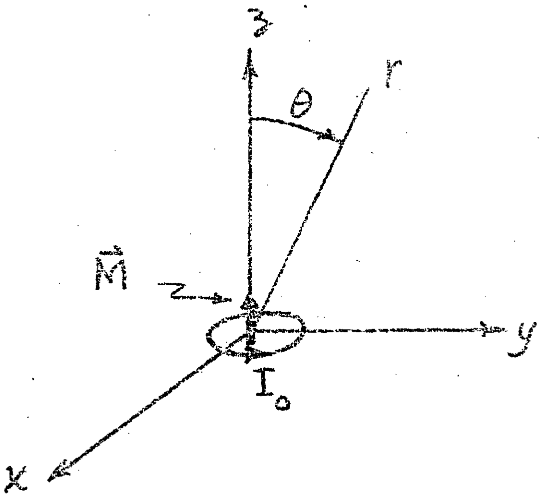


Fig. 1.6. A solenoidal current loop radiator

The field radiated by a magnetic dipole is readily found from the expressions for the fields radiated by an electric dipole by using the duality relations (14). According to (14c) the electric dipole $\vec{P}_e$ that will give a magnetic dipole $\vec{M} = I_0\pi a^2\vec{a}_z$ is $\vec{P}_e = -\epsilon_0\mu_0 I_0\pi a^2\vec{a}_z$. The field radiated by this electric dipole is given by (28) and (29) after multiplying all expressions for the fields by $-j\omega\mu_0 Y_0 \pi a^2/\Delta z$ since the fields given by (28) and (29) where obtained from an electric dipole of moment $-jI_0\Delta z/\omega$. The current I_0 in the loop is taken equal to that in the current filament for convenience. If (14a) and (14b) together with (28) and (29) are used we find that the fields radiated by the magnetic dipole in Fig. 1.6 are given by

$$\vec{E} = -j \frac{k_0 Z_0 a^2 I_0}{4} \vec{a}_\phi \left(\frac{1}{r^2} + \frac{jk_0}{r} \right) \sin\theta \, e^{-jk_0 r} \tag{36a}$$

$$\vec{H} = \frac{a^2 I_0}{4} e^{-jk_0 r} \left[2\vec{a}_r \cos\theta \left(\frac{1}{r^3} + \frac{jk_0}{r^2} \right) - \vec{a}_\theta \sin\theta \left(\frac{k_0^2}{r} - \frac{jk_0}{r^2} - \frac{1}{r^3} \right) \right] \tag{36b}$$

The magnetic dipole field is the dual of the electric dipole field with the role of electric and magnetic fields interchanged. Consequently the directivity and radiation pattern are the same.

1.7 Radiation from Arbitrary Current Distributions

The electromagnetic field radiated by an arbitrary current distribution $\vec{J}_e$ is readily found in terms of the vector potential $\vec{A}_e$. Let $\vec{r}$ specify the field point and $\vec{r}'$ specify the source points and let $R = |\vec{r} - \vec{r}'|$ as in Fig. 1.7. The vector potential $\vec{A}_e(\vec{r})$

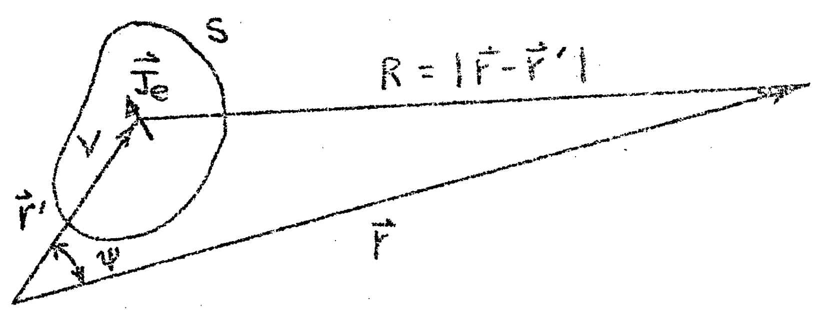


Fig. 1.7. Arbitrary current source

is given by $\vec{A}_e(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \vec{J}_e(\vec{r}\,') \frac{e^{-jk_0R}}{R} dV'$ for a current distribution located in free space. The magnetic field $\vec{H}$ is given by

$$\vec{H}(\vec{r}) = -\int_V \vec{J}_e(\vec{r}\,') \times \nabla \frac{e^{-jk_0R}}{4\pi R} dV' \tag{37}$$

where the curl operator has been brought inside the integral sign. The electric field is given by

$$\vec{E}(\vec{r}) = \frac{1}{j\omega\mu_0\varepsilon_0} \nabla \times \nabla \times \vec{A}_e = \frac{1}{j\omega\varepsilon_0} \nabla \times \vec{H}$$

$$= -\frac{1}{j\omega\varepsilon_0} \int_V \nabla \times (\vec{J}_e \times \nabla \frac{e^{-jk_0R}}{4\pi R}) dV'$$

If we use the vector expansion $\nabla \times (\vec{A} \times \vec{B}) = \vec{A}\nabla \cdot \vec{B} - \vec{B}\nabla \cdot \vec{A} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$ and te that $\vec{J}_e(\vec{r}\,')$ is not a function of $\vec{r}$ we obtain

$$\vec{E}(\vec{r}) = \frac{1}{j\omega\varepsilon_0} \int_V [\vec{J}_e(\vec{r}\,')k_0^2 \frac{e^{-jk_0R}}{4\pi R} + \vec{J}_e(\vec{r}\,') \cdot \nabla\nabla \frac{e^{-jk_0R}}{4\pi R}] dV' \tag{38}$$

where the term $\nabla^2 (\frac{e^{-jk_0R}}{4\pi R})$ has been replaced by $-k_0^2 \frac{e^{-jk_0R}}{4\pi R}$ since this latter function is a solution of the scalar Helmholtz equation. The solution (38) holds only in the region external to the volume containing the current source $\vec{J}_e$.

In the general case the evaluation of the integrals in (37) and (38) is difficult to carry out. In radiation problems three regions of physical space are usually distinguished. The first region is the near-zone field region for which no general approximations may be made in the evaluation of (37) and (38) for the fields. The second region is called the Fresnel region and is the region of physical space between the near-zone region and the far zone or Fraunhofer region. The Fresnel and Fraunhofer regions are characterized by type of approximations that may be made in the integrands in (37) and (38). Since the far zone region is the least stringent we will consider it in detail first.

The far zone region corresponds to the region in which the radiation field predominates and hence is the region of most interest in connection with antennas. The far zone region is characterized by the conditions that $\vec{r}$ is much greater than the maximum value of $\vec{r}'$ and also much greater than the free space wavelength λ_0, i.e., $k_0 r \gg 1$. Using the binomial expansion we have

$$R = |\vec{r} - \vec{r}'| = (r^2 + r'^2 - 2\vec{r} \cdot \vec{r}')^{1/2}$$

$$\approx r - \frac{\vec{r} \cdot \vec{r}'}{r} = r - \vec{a}_r \cdot \vec{r}' = r - r' \cos \psi$$

where ψ is the angle between $\vec{r}$ and $\vec{r}'$ as in Fig. 1.7. We now approximate

$$\frac{1}{R} e^{-jk_0 R} \quad \text{by} \quad \frac{1}{r} e^{-jk_0(r - \vec{a}_r \cdot \vec{r}')}$$

and obtain

$$\nabla \frac{e^{-jk_0(r - \vec{a}_r \cdot \vec{r}')}}{r} = \{\vec{a}_r \left(\frac{-jk_0}{r} - \frac{1}{r^2} \right) - \frac{jk_0 r'}{r^2} \sin \psi \, \vec{a}_\psi \}$$

$$\cdot \{e^{-jk_0(r - \vec{a}_r \cdot \vec{r}')}\} \approx \frac{-jk_0}{r} e^{-jk_0 r} e^{jk_0 \vec{a}_r \cdot \vec{r}'} \vec{a}_r \tag{39}$$

where $\vec{a}_\psi$ is a unit vector in the direction of ψ increasing.

Using this result in (37) gives

$$\vec{H} = \frac{jk_0}{4\pi r} e^{-jk_0 r} \int_V \vec{J}_e(r') \times \vec{a}_r \, e^{jk_0 \vec{a}_r \cdot \vec{r}'} \, dV' \tag{40a}$$

for the far zone magnetic field. To the same order of approximation it may be verified that in the far zone field (38) gives

$$\vec{E} = -Z_0\vec{a}_r \times \vec{H} = \frac{jk_0Z_0e^{-jk_0r}}{4\pi r} \int_V [\vec{a}_r \cdot \vec{J}_e\vec{a}_r - \vec{J}_e]\, e^{jk_0\vec{a}_r \cdot \vec{r}'} dV'$$

$$= -\frac{jk_0Z_0e^{-jk_0r}}{4\pi r} \int_V (J_\theta\vec{a}_\theta + J_\phi\vec{a}_\phi)\, e^{jk_0\vec{a}_r \cdot \vec{r}'} dV' \qquad (40b)$$

Thus it is seen that in the far zone field the relation between $\vec{E}$ and $\vec{H}$ is that which is characteristic of a spherical TEM wave. In the evaluation of (40) it is useful to be able to transform the components in a spherical coordinate frame to a rectangular coordinate frame and vice versa. For this purpose the following relations are useful:

$$\vec{a}_x = \vec{a}_r \sin\theta\cos\phi + \vec{a}_\theta \cos\theta\cos\phi - \vec{a}_\phi \sin\phi$$

$$\vec{a}_y = \vec{a}_r \sin\theta\sin\phi + \vec{a}_\theta \cos\theta\sin\phi + \vec{a}_\phi \cos\phi$$

$$\vec{a}_z = \vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta \qquad (41)$$

$$\vec{a}_r = \vec{a}_x \sin\theta\cos\phi + \vec{a}_y \sin\theta\sin\phi + \vec{a}_z \cos\theta$$

$$\vec{a}_\theta = \vec{a}_x \cos\theta\cos\phi + \vec{a}_y \cos\theta\sin\phi - \vec{a}_z \sin\theta$$

$$\vec{a}_\phi = -\vec{a}_x \sin\phi + \vec{a}_y \cos\phi$$

We now return to a consideration of the Fresnel region. The usual approximations that are considered in defining the Fresnel region are $r >> r'$, $k_0 r >> 1$ but with r, r', and λ_0 such that terms in r'^2 must be retained in the phase term in the exponential. We have $R = r - \vec{a}_r \cdot \vec{r}' + \frac{1}{2r}[r'^2 - (\vec{a}_r \cdot \vec{r}')^2]$ to terms of order $(r'/r)^2$. Thus in the Fresnel region the vector potential is given by

$$\vec{A}_e(\vec{r}) = \frac{\mu_0 e^{-jk_0 r}}{4\pi r} \int_V \vec{J}_e(\vec{r}') e^{jk_0[\vec{a}_r \cdot \vec{r}' + \frac{(\vec{a}_r \cdot \vec{r}')^2}{2r} - \frac{r'^2}{2r}]} dV' \qquad (42)$$

In the evaluation of the field from $\vec{A}_e$ terms with amplitudes decreasing faster than $1/r$ are neglected. Thus the essential difference between the approximations involved in the Fraunhofer and Fresnel regions is only in the phase term in the exponential. There is no clearly marked boundary between the three regions, i.e. the near-zone, Fresnel, and far zone regions, since the range of r in which the approximations outlined above may be made e dependent on the current distribution $\vec{J}_e(\vec{r}')$. In the case of radiation from a plane aperture with a maximum linear dimension D the far zone or Fraunhofer region is commonly considered to begin for r somewhere between D^2/λ_0 and $2D^2/\lambda_0$.* In order to have a significant Fresnel region we require $(k_0 r')\, r'/r$ to assume values as large as several π radians. Thus we have the condition $1 << \frac{r}{r'} << \frac{r'}{\lambda_0}$ in the Fresnel zone.

1.8 Modified Lorentz Reciprocity Theorem

Consider a volume V bounded by a surface S and containing two possible sets of sources $\vec{J}_{e1}$, $\vec{J}_{m1}$ and $\vec{J}_{e2}$, $\vec{J}_{m2}$. The medium in V is assumed characterized by general unsymmetrical permeability and permittivity tensors $\bar{\mu}$ and $\bar{\varepsilon}$. In dyadic form $\bar{\mu}$ is expressed as

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*S. Silver, "Microwave Antenna Theory and Design," McGraw-Hill Book Company, Inc., New York, 1949, RLS Vol. 12, Sec. 6.9.

$$\bar{\mu} = \hat{a}_x\hat{a}_x\mu_{xx} + \hat{a}_x\hat{a}_y\mu_{xy} + \hat{a}_x\hat{a}_z\mu_{xz}$$

$$+ \hat{a}_y\hat{a}_x\mu_{yx} + \hat{a}_y\hat{a}_y\mu_{yy} + \hat{a}_y\hat{a}_z\mu_{yz} \qquad (43)$$

$$+ \hat{a}_z\hat{a}_x\mu_{zx} + \hat{a}_z\hat{a}_y\mu_{zy} + \hat{a}_z\hat{a}_z\mu_{zz}$$

and similarly for $\bar{\epsilon}$. The scalar product of $\bar{\mu}$ with $\hat{H}$ i.e. $\bar{\mu} \cdot \hat{H}$, is a vector obtained by dotting the unit vectors immediately adjacent to the dot sign.

Let the sources $\hat{J}_{e1}$, $\hat{J}_{m1}$ produce a field $\hat{E}_1$, $\hat{H}_1$ in V and on S. This field satisfies the equations

$$\nabla \times \hat{E}_1 = - j\omega\bar{\mu} \cdot \hat{H}_1 - \hat{J}_{m1} \qquad (44a)$$

$$\nabla \times \hat{H}_1 = j\omega\bar{\epsilon} \cdot \hat{E}_1 + \hat{J}_{e1} \qquad (44b)$$

Let $\hat{E}_2$, $\hat{H}_2$ be the field produced in V and on S by $\hat{J}_{e2}$, $\hat{J}_{m2}$ when the medium in V is replaced by a medium characterized by permeability and permittivity tensors which are the transpose of those considered under the first situation. The field $\hat{E}_2$, $\hat{H}_2$ is a solution of

$$\nabla \times \hat{E}_2 = - j\omega\bar{\mu}_t \cdot \hat{H}_2 - \hat{J}_{m2} \qquad (45)$$

$$\nabla \times \hat{H}_2 = j\omega\bar{\epsilon}_t \cdot \hat{E}_2 + \hat{J}_{e2} \qquad (46)$$

where the subscript t denotes the transposed tensor i.e. μ_{xy} replaced by μ_{yx}, etc.

The expansion of $\nabla \cdot (\hat{E}_1 \times \hat{H}_2 - \hat{E}_2 \times \hat{H}_1)$ gives

$$\nabla \cdot (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) = (\nabla \times \vec{E}_1) \cdot \vec{H}_2 - (\nabla \times \vec{H}_2) \cdot \vec{E}_1 - (\nabla \times \vec{E}_2) \cdot \vec{H}_1$$

$$+ (\nabla \times \vec{H}_1) \cdot \vec{E}_2 = \vec{H}_2 \cdot (-j\omega\bar{\mu} \cdot \vec{H}_1 - \vec{J}_{m1}) - \vec{E}_1 \cdot (j\omega\bar{\varepsilon}_t \cdot \vec{E}_2 + \vec{J}_{e2})$$

$$- \vec{H}_1 \cdot (-j\omega\bar{\mu}_t \cdot \vec{H}_2 - \vec{J}_{m2}) + \vec{E}_2 \cdot (j\omega\bar{\varepsilon} \cdot \vec{E}_1 + \vec{J}_{e1})$$

$$= (\vec{E}_2 \cdot \vec{J}_{e1} - \vec{E}_1 \cdot \vec{J}_{e2}) - (\vec{H}_2 \cdot \vec{J}_{m1} - \vec{H}_1 \cdot \vec{J}_{m2}) \qquad (47)$$

since $\vec{H}_2 \cdot \bar{\mu} \cdot \vec{H}_1 = \vec{H}_1 \cdot \bar{\mu}_t \cdot \vec{H}_2$ and $\vec{E}_2 \cdot \bar{\varepsilon} \cdot \vec{E}_1 = \vec{E}_1 \cdot \bar{\varepsilon}_t \cdot \vec{E}_2$.

The integral of (47) over the volume V gives

$$\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{S} = \int_V [\vec{E}_2 \cdot \vec{J}_{e1} - \vec{E}_1 \cdot \vec{J}_{e2}$$

$$- \vec{H}_2 \cdot \vec{J}_{m1} + \vec{H}_1 \cdot \vec{J}_{m2}]\, dV \qquad (48)$$

after converting the volume integral of the divergence term to a surface integral by means of the divergence theorem. Equation (48) is the general form of the reciprocity theorem for the electromagnetic field.*

If the surface S encloses all of the sources the surface integral on the left hand side in (48) vanishes. To show this consider the application of (48) to the volume bounded by S and the surface S_∞ of a sphere of infinite radius. Let the medium have small but finite losses. In this case the fields will decrease in amplitude faster than 1/r and hence on S_∞ the fields vanish. Since there are no sources in the volume bounded by S and S_∞

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*R. F. Harrington, A.J. Villeneuve, RECIPROCAL RELATIONS FOR GYROTROPIC MEDIA, IRE ns., Vol. MTT-6, pp. 308-310, July, 1958.

the surface integral over S must also be zero. Now it seems reasonable to assume that the fields are analytic functions of the loss parameters. Hence the surface integral

$$\oint_S (\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \cdot d\vec{S}$$

will be an analytic function of the loss parameters. Since the integral vanishes for infinitely small but finite loss parameters it must still vanish as the loss parameters are reduced to zero in view of the analytic properties of the integral.

In the case of isotropic media the field on S_∞ satisfies the relation $\vec{E} = -Z\vec{a}_r \times \vec{H}$ and the integrand $(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1) \times \vec{a}_r$ vanishes identically at infinity independent of any loss that may be present. Note that $Z = \sqrt{(\mu/\varepsilon)}$ in a general isotropic medium.

If there are no sources in V we obtain from (48) the generalization of the Lorentz form of the reciprocity theorem

$$\oint_S \vec{E}_1 \times \vec{H}_2 \cdot d\vec{S} = \int_S \vec{E}_2 \times \vec{H}_1 \cdot d\vec{S} \tag{49a}$$

If the volume V contains all of the sources the generalized Rayleigh-Carson form of the reciprocity theorem is obtained

$$\int_V (\vec{E}_1 \cdot \vec{J}_{e2} - \vec{H}_1 \cdot \vec{J}_{m2})dV = \int_V (\vec{E}_2 \cdot \vec{J}_{e1} - \vec{H}_2 \cdot \vec{J}_{m1})dV$$

These relations are also valid in isotropic media, in which case the fields $\vec{E}_2$, $\vec{H}_2$ are the fields produced by $\vec{J}_{e2}$, $\vec{J}_{m2}$, in the actual medium.

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See also References 1 and 3 above. (Ref. 3 gives a very complete discussion).

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See also Reference 1, Chapter 1, Reference 5, Chapter 9.

CHAPTER 2

SCATTERING BY SMALL OBSTACLES

In this chapter we will examine the electromagnetic scattering problem for small obstacles, that is, obstacles whose largest dimensions are of the order of $\lambda_0/10$ or less. The scattering from small obstacles may be determined in terms of radiation from induced electric and magnetic dipoles. Furthermore, the dipole moments may be determined by static field analysis since the incident electromagnetic field is essentially uniform and constant over the extent of the obstacle. We will give a detailed treatment for a small circular disk but similar methods apply to other kinds of small obstacles.

We will also introduce the various "cross sections" which are used to describe the scattering phenomena. In addition, some important cross section theorems are developed.

2.1 Scattering by a Small Circular Disk

Assume an infinitely thin, perfectly conducting, disk of radius a located at the origin as in Fig. 2.1.

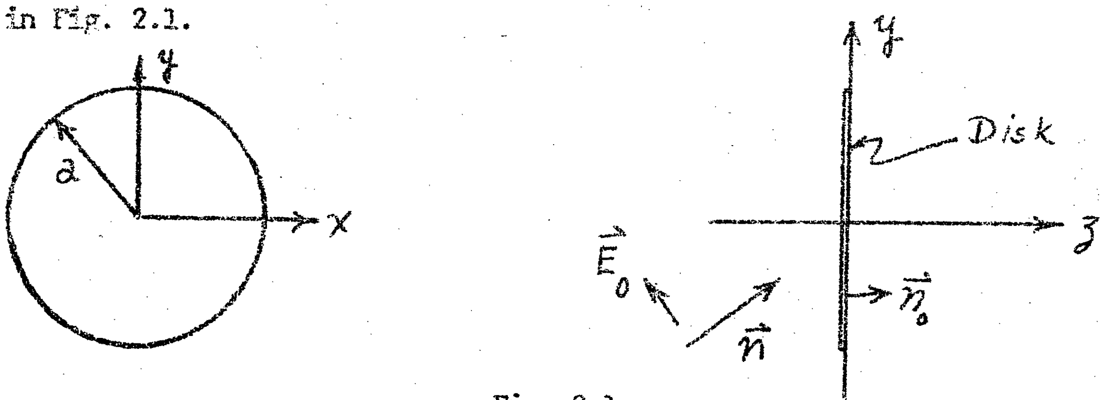

Fig. 2.1

A plane wave is incident along a direction defined by the unit wave normal $\bar{n}$, $\bar{E}_i = \bar{E}_0 e^{-jk_0\bar{n}\cdot\bar{r}}$, $\bar{E}_0$ = constant vector, $\bar{n}\cdot\bar{E}_0 = 0$, $\nabla\times\bar{E}_i = -\bar{E}_0\times\nabla e^{-jk_0\bar{n}\cdot\bar{r}} = jk_0\bar{n}\times\bar{E}_i = -j\omega\mu_0\bar{H}_i$ and hence $\bar{H}_i = Y_0\bar{n}\times\bar{E}_i$ where $Y_0 = \sqrt{\epsilon_0/\mu_0}$. The scattered field is $\bar{E}_s$, $\bar{H}_s$ and the total field is $\bar{E} = \bar{E}_i + \bar{E}_s$, $\bar{H} = \bar{H}_i + \bar{H}_s$. On the disk $\bar{n}_0\times\bar{E} = 0$, (SINCE IT IS PERFECTLY CONDUCTING) $\bar{n}_0\cdot\bar{H} = 0$ where $\bar{n}_0$ is a unit vector normal to the disk surface S. Note that $\bar{n}_0\times\bar{H}_i$ is continuous across the disk but $\bar{n}_0\times\bar{H}_s$ is discontinuous by an amount equal to the current $\bar{J}$ that flows on the disk. The scattered field may be expressed in terms of $\bar{J}$.

The vector potential from a current element at $\vec{r}'$ is

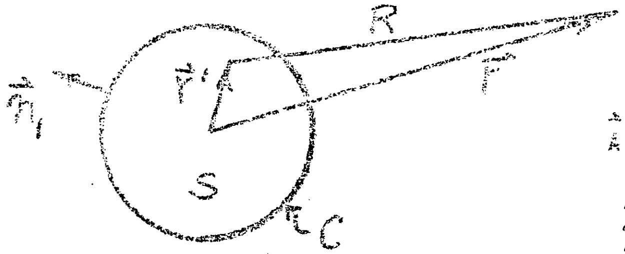

Fig. 2.2

$$\vec{A}(x,y,z) = \frac{\mu_0}{4\pi}\int_S \frac{\vec{J}(x',y')}{R} e^{-jk_0R} dS' \qquad (2.1)$$

$$\vec{R} = \vec{r} - \vec{r}'$$

Let $f(x',y'; x,y,z) = e^{-jk_0R}/R$ and expand f in a taylor series about the origin. Thus

$$f = f\Big|_{x'=y'=0} + (\partial f/\partial x')\Big|_{x'=y'=0} x' + (\partial f/\partial y')\Big|_{x'=y'=0} y' + \ldots$$

Note that $\partial/\partial x' = -\partial/\partial x$ etc. We obtain from the first three terms of the expansion[†]

$$\frac{e^{-jk_0R}}{R} = \frac{e^{-jk_0r}}{r} + (1 + jk_0r)\frac{e^{-jk_0r}}{r^3}(xx' + yy'), \quad r > a$$

The first term gives

$$\vec{A}_0 = \frac{\mu_0}{4\pi r} e^{-jk_0r}\int_S \vec{J}(x',y')dS' \qquad (2.2)$$

while the second term gives

$$\vec{A}_1 = \frac{\mu_0}{4\pi r^3}(1 + jk_0r)\int_S (\hat{r}\cdot\vec{r}')\vec{J}dS' \quad e^{-jk_0r} \qquad (2.3)$$

Lemma: Let ψ = any scalar function of x',y',z', then

$$\int_S \nabla'\cdot(\psi\vec{J})dS' = \int_S (\psi\nabla'\cdot\vec{J} + \vec{J}\cdot\nabla'\psi)dS' = \oint_C \psi\vec{J}\cdot\hat{n}_1 dl'$$

[†]This is the standard multipole expansion, see for example, Stratton, Electromagnetic Theory, McGraw-Hill, 1941, Sec. 8.4.

by using the divergence theorem, where C is the disk contour and $\hat{n}_1$ is a unit normal to C in the plane of the disk. By the edge condition $\vec{H}_{tan} \cdot \hat{n}_1 = \hat{n}_0 \times \vec{H} \cdot \hat{n}_1 = 0$, i.e., the total $\vec{H}$ along contour C is zero, so $\vec{J} \cdot \hat{n}_1 = 0$. Hence, since $\nabla' \cdot \vec{J} = -j\omega\rho$

$$\int_S \vec{J} \cdot \nabla'\psi dS' = j\omega \int_S \rho\psi dS'$$

This gives

$$\int_S J_x dS' = j\omega \int_S x'\rho dS' \qquad \text{for } \psi = x', \text{ similarly for } y';$$

$$\int_S x' J_x dS' = \frac{j\omega}{2} \int_S x'^2 \rho dS' \qquad \text{for } \psi = x'^2, \text{ etc;} \qquad (2.4)$$

$$\int_S (x' J_y + y' J_x) dS' = j\omega \int_S x'y'\rho dS' \qquad \text{for } \psi = x'y'$$

By using the above lemma we find that

$$\vec{A}_0 = \frac{\mu_0}{4\pi r} e^{-jk_0 r} \int_S \vec{J} dS'$$

can be expressed as

$$\vec{A}_0 = \frac{j\omega\mu_0 \vec{P}}{4\pi r} e^{-jk_0 r} \qquad (2.5)$$

since $\int_S \vec{J} dS' = j\omega \int_S \vec{r}'\rho dS' = j\omega\vec{P}$ where $\vec{P}$ is called the electric dipole moment of the current distribution. To evaluate $\vec{A}_1$ note that

$$(\hat{r} \cdot \vec{r}')\vec{J} = \frac{1}{2}[(\vec{r}' \times \vec{J}) \times \hat{r} + (\hat{r} \cdot \vec{r}')\vec{J} + (\hat{r} \cdot \vec{J})\vec{r}']$$

The magnetic dipole moment $\vec{M}$ is defined by

$$\vec{M} = \frac{1}{2} \int_S \vec{r}' \times \vec{J} dS' \qquad (2.6)$$

so one term leads to a magnetic dipole field. The electric quadrupole dyadic moment of the charge on the disk is

$$\overleftrightarrow{Q} = \vec{a}_x\vec{a}_x \int_S x'^2 \rho dS' + (\vec{a}_x\vec{a}_y + \vec{a}_y\vec{a}_x) \int_S x'y'\rho dS'$$

$$+ \vec{a}_y\vec{a}_y \int_S y'^2 \rho dS' \tag{2.7}$$

Consider $(\vec{r}\cdot\vec{r}')\vec{J} + (\vec{r}\cdot\vec{J})\vec{r}' = (xx' + yy')(\vec{a}_x J_x + \vec{a}_y J_y) + (xJ_x + yJ_y)(\vec{a}_x x' + \vec{a}_y y')$
$= 2xx'\vec{a}_x J_x + 2yy'\vec{a}_y J_y + (\vec{a}_x y + \vec{a}_y x)(x'J_y + y'J_x)$. By Lemma

$$\int_S [(\vec{r}\cdot\vec{r}')\vec{J} + (\vec{r}\cdot\vec{J})\vec{r}']dS' = j\omega\,[\vec{a}_x x \int_S x'^2\rho dS' + \vec{a}_y y \int_S y'^2 \rho dS'$$

$$+ (\vec{a}_x y + \vec{a}_y x)\int_S x'y'\rho dS'] = j\omega\overleftrightarrow{Q}\cdot\vec{r} \tag{2.8}$$

Hence

$$\vec{A}_1 = \frac{\mu_0}{4\pi r^3} e^{-jk_0 r} (1 + jk_0 r)\,[\vec{M}\times\vec{r} + \frac{j\omega}{2}\overleftrightarrow{Q}\cdot\vec{r}] \tag{2.9}$$

Evaluation of Scattered Fields

Assume that the disk is so small that $e^{-jk_0\vec{n}\cdot\vec{r}} \approx 1$ on the disk.* Then we have a uniform tangential electric field and a uniform normal magnetic field applied to the disk. This results in an electric dipole and a magnetic dipole being induced such that the electric dipole field cancels $\vec{n}_0 \times \vec{E}_i$ on the disk and the magnetic dipole field cancels

* Note that $\vec{E}_i = \vec{E}_o(1 - jk_o\vec{n}\cdot\vec{r} + \cdots) = \vec{E}_o(1 - jk_o a\,\frac{\vec{n}\cdot\vec{r}}{a} + \cdots)$ so second term is of order $\frac{2\pi a}{\lambda_o}$ times the first term. For a small disk $|\vec{r}/a| \ll 1$ so this term can be dropped to 1st. order.

$\hat{n}_0 \cdot \hat{H}_i$ on the disk. These "static-like" field problems can be solved and yield[†]

$$\hat{P} = \alpha_e \epsilon_0 \hat{E}_t \tag{2.10}$$

where $\hat{E}_t$ is the tangential component of $\hat{E}_i$ on the disk and the electric dipole polarizability α_e is given by

$$\alpha_e = \frac{16a^3}{3} \tag{2.11}$$

Also,

$$\hat{M} = \alpha_m \hat{H}_n \tag{2.12}$$

where $\hat{H}_n$ is the normal component of $\hat{H}_i$ on the disk and the magnetic dipole polarizability is given by

$$\alpha_m = -\frac{8a^3}{3} \tag{2.13}$$

To this order of approximation $\bar{Q}$ equals zero.

We have $\hat{E}_t = \hat{a}_x E_{ox} + \hat{a}_y E_{0y}$ and $\hat{H}_n = \hat{a}_z Y_o (n_x E_{oy} - n_y E_{ox})$. Let us choose a coordinate system uvz with the polar axis u along $\hat{E}_t$. Then $\hat{P} = P\hat{a}_u$ and $\hat{M} = M\hat{a}_z$. In spherical coordinates r, θ, φ, we can write

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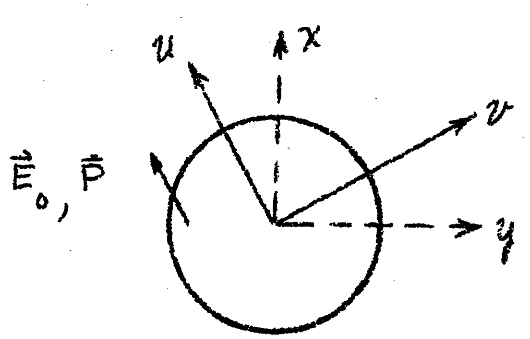
[†]See: R. E. Collin, Field Theory of Guided Waves, McGraw-Hill, 1960, Chapt. 7, Sec. 7.3, and Chapt. 12, Sec. 12.4.



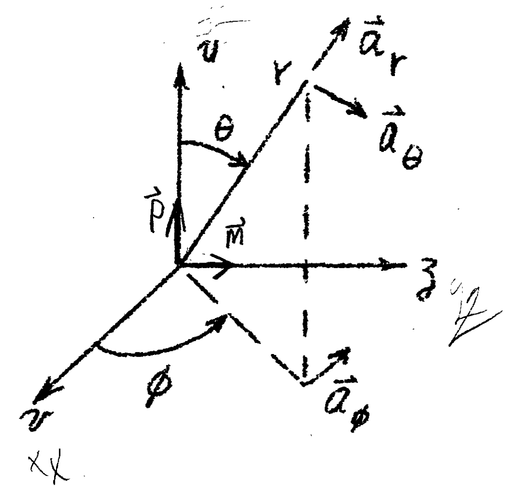


Fig. 2.3

$$\vec{A}_0 = \frac{j\omega\mu_0}{4\pi r} e^{-jk_0 r} P(\vec{a}_r \cos\theta - \vec{a}_\theta \sin\theta) \qquad (2.14)$$

$$\vec{E} = -j\omega\vec{A}_0 + \frac{\nabla\nabla \cdot \vec{A}_0}{j\omega\epsilon_0 \mu_0}$$

NOTE DIRECTIONS $\vec{E}$ FIELD DUE TO OSC. ELECTRIC DIPOLE MOM.

$$= \vec{a}_r \frac{P}{2\pi\epsilon_0}\left(\frac{1}{r^3} + \frac{jk_0}{r^2}\right)\cos\theta e^{-jk_0 r} + \vec{a}_\theta\left(\frac{1}{r^3} + \frac{jk_0}{r^2} - \frac{k_0^2}{r}\right)\frac{P}{4\pi\epsilon_0}\sin\theta e^{-jk_0 r} \qquad (2.15)$$

$$\vec{H} = \frac{1}{\mu_0}\nabla \times \vec{A}_0 = \vec{a}_\phi \frac{j\omega P}{4\pi}\left(\frac{1}{r^2} + \frac{jk_0}{r}\right)\sin\theta e^{-jk_0 r} \qquad (2.16)$$

$\vec{H}$ FIELD DUE TO OSCILLATING ELECTRIC DIPOLE MOMENT (NOTE DIRECTION)

To evaluate the magnetic dipole field from

$$\vec{A}_1 = \frac{\mu_0}{4\pi}\left(\frac{1}{r^3} + \frac{jk_0}{r^2}\right)\vec{M} \times \vec{r} e^{-jk_0 r}$$

$\longrightarrow M\vec{a}_z \times \vec{r} e^{-} = Mr \ldots \times (\vec{a}_r \sin\theta\sin\phi + \vec{a}_\theta \cos\theta \ldots + \vec{a}_\phi \cos\phi) \times \vec{a}_r$

note that $\vec{a}_z = \vec{a}_r \sin\theta \sin\phi + \vec{a}_\theta \cos\theta \sin\phi + \vec{a}_\phi \cos\phi$ so

$$\vec{A}_1 = \frac{\mu_0 M}{4\pi}\left(\frac{1}{r^2} + \frac{jk_0}{r}\right)e^{-jk_0 r}(\vec{a}_\theta \cos\phi - \vec{a}_\phi \cos\theta \sin\phi) \qquad (2.17)$$

Straightforward substitution in the equations giving the field from the vector potential now yields

$$\vec{H} = \frac{1}{\mu_0}\nabla \times \vec{A}_1 = \vec{a}_r \frac{M}{2\pi}\left(\frac{1}{r^3} + \frac{jk_0}{r^2}\right)\sin\theta \sin\phi e^{-jk_0 r}$$

$\vec{H}$ FIELD DUE TO OSCILLATING MAGNETIC DIPOLE MOMENT.

$$+ (\hat{a}_\theta \cos\theta \sin\phi + \hat{a}_\phi \cos\phi) \frac{M}{4\pi} \left(\frac{k_0^2}{r} - \frac{jk_0}{r^2} - \frac{1}{r^3} \right) e^{-jk_0 r} \tag{2.18}$$

$$\hat{\vec{E}} = \frac{MZ_0}{4\pi} \left(\frac{k_0^2}{r} - \frac{jk_0}{r^2} \right) e^{-jk_0 r} (\hat{a}_\theta \cos\phi - \hat{a}_\phi \cos\theta \sin\phi) \tag{2.19}$$

ELECTRIC FIELD DUE TO OSCILLATING MAGNETIC DIPOLE MOMENT.

The total far zone scattered field is thus

$$\hat{\vec{E}}_s = [(MZ_0 k_0^2 \cos\phi - \frac{k_0^2 P}{\epsilon_0} \sin\theta)\hat{a}_\theta - MZ_0 k_0^2 \hat{a}_\phi \cos\theta \sin\phi] \frac{e^{-jk_0 r}}{4\pi r} \tag{2.20a}$$

$$\hat{\vec{H}}_s = [Mk_0^2 \cos\theta \sin\phi \hat{a}_\theta + (Mk_0^2 \cos\phi - \omega P k_0 \sin\theta)\hat{a}_\phi] \frac{e^{-jk_0 r}}{4\pi r} = \frac{\vec{r} \times \hat{\vec{E}}_S}{Z_0 r} \tag{2.20b}$$

The radial component of the complex Poynting vector is

$$(\hat{\vec{E}}_s \times \hat{\vec{H}}_s^*)_r = |\hat{\vec{E}}_s|^2 Y_0$$

The power flow through a solid angle $d\Omega = \sin\theta d\theta d\phi$ is

$$\frac{1}{2} r^2 Y_0 |\hat{\vec{E}}_s|^2 \sin\theta d\theta d\phi = dP_s = \frac{Y_0}{32\pi^2} [(MZ_0 k_0^2)^2 (\cos^2\phi + \cos^2\theta \sin^2\phi)$$

$$+ \left(\frac{k_0^2 P}{\epsilon_0} \right)^2 \sin^2\theta - \frac{2MZ_0 k_0^4 P}{\epsilon_0} \cos\phi \sin\theta] \sin\theta d\theta d\phi \tag{2.21}$$

The differential scattering cross-section in the direction θ, ϕ is defined by

$$\sigma(\theta, \phi) = \frac{dP_s(\theta, \phi)}{d\Omega} \Big/ \text{Power density in incident wave per unit area}$$

or

$$\sigma(\theta, \phi) = \frac{dP_s(\theta, \phi)}{\frac{1}{2} Y_0 |\vec{E}_0|^2 d\Omega} \tag{2.22}$$

The total scattering cross section is given by

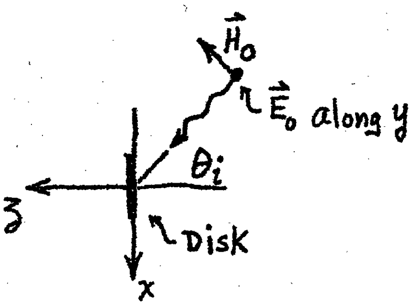
$$\sigma_s = \int_0^{\pi}\int_0^{2\pi} \sigma(\theta,\ \phi)d\Omega$$

$$= \frac{\text{total scattered power}}{\text{incident power density per unit area}} = \frac{32k_0^4a^6}{27E_0^2}\left[|\vec{n}\times\vec{E}_0\cdot\vec{a}_z|^2 + 4|\vec{E}_0-\vec{a}_z\cdot\vec{E}_0\vec{a}_z|^2\right] \qquad (2.23)$$

Note that σ is defined so that

$$dP_s(\theta,\ \phi) = (\text{incident power density per unit area})\sigma(\theta,\ \phi)d\Omega.$$

A Particular Example



Let

$$-\vec{k}_0\cdot\vec{r} = k_0(\sin\theta\,\vec{a}_x + \cos\theta\,\vec{a}_z)\cdot(x\vec{a}_x + z\vec{a}_z)$$

$$\vec{E}_i = \vec{a}_y E_0 e^{-jk_0(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i = Y_0E_0(-\vec{a}_x\cos\theta_i + \vec{a}_z\sin\theta_i)e^{-jk_0(x\sin\theta_i + z\cos\theta_i)}$$

$$= (\sin\theta\,\vec{a}_x + \cos\theta\,\vec{a}_z)\times\vec{a}_y \quad (\vec{n}\times\vec{E}_i)$$

θ_i = angle of incidence relative to z axis

We then find that

$$\vec{P} = \frac{16}{3}a^3\epsilon_0E_0\vec{a}_y, \quad \vec{M} = -\frac{8}{3}a^3Y_0E_0\sin\theta_i\vec{a}_z$$

ONLY NORMAL COMPONENT IS TAKEN

$\vec{M} \sim \vec{H}_{i\,norm} \sim \vec{H}_i\cdot\vec{n}$ (UNIT NORMAL TO SURFACE)

The differential scattering cross-section is found to be given by

$$\sigma(\theta,\ \phi) = \frac{4k_0^4a^6}{9\pi^2}[\sin^2\theta_i(\cos^2\phi + \cos^2\theta\sin^2\phi)$$

$$+ 4 \sin^2\theta + 4 \sin\theta_i \cos\phi \sin\theta] \qquad (2.24)$$

In this case the uvz correspond to the y(-x)z coordinates.

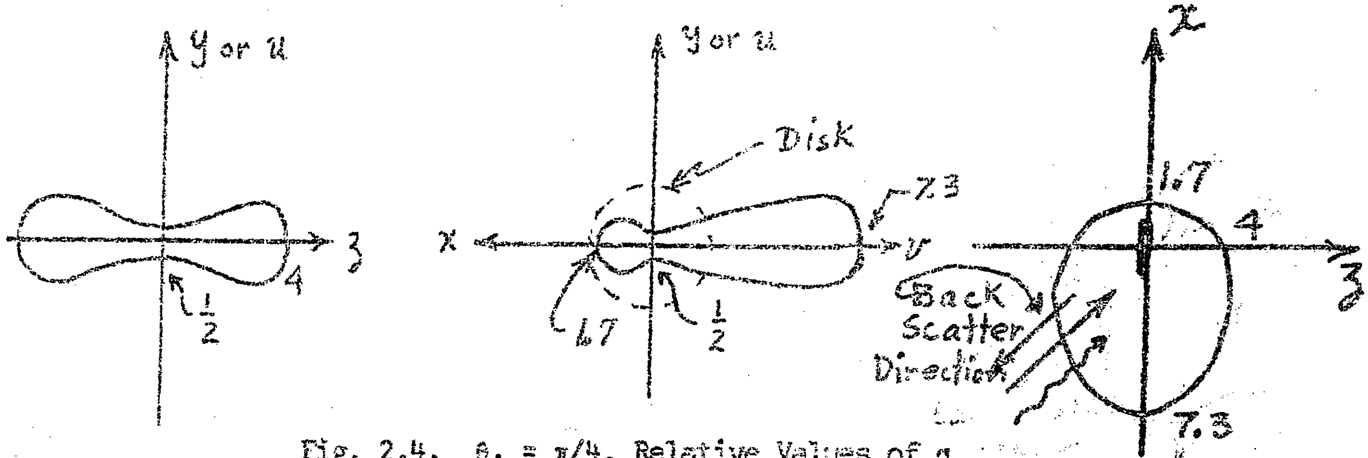


Fig. 2.4. $\theta_i = \pi/4$, Relative Values of σ

Let the plane wave be incident *in* the direction $\theta = \pi/2$, $\phi = \frac{3\pi}{4}$ (in the uvz system). The differential scattering cross-section in the backward direction $\theta = \frac{\pi}{2}$, $\phi = -\frac{\pi}{4}$ is found from (2.24) to be

$$\sigma(\pi/2, -\pi/4) = \frac{25a^6k_0^{\ 4}}{9\pi^2}$$

since $\theta_i = \pi/4$ in this case. The total scattering cross-section is

$$\sigma_s = \int_0^\pi \int_0^{2\pi} \sigma \sin\theta d\theta d\phi = \frac{32}{27\pi} k_0^{\ 4} a^6 (4 + \sin^2\theta_i)$$

The equivalent isotropic scattering cross-section σ_i is defined as follows;

$$\text{Total power scattered} = \sigma_i(\text{incident power density per unit area}) \qquad (2.25)$$

But $P_s = \sigma_s$(incident power density per unit area) and hence $\sigma_i = \sigma_s$. An isotropic scatterer would produce a back scattered power density $P_s/4\pi$ watts per steradian in the direction the incident wave arrived from. The actual power density in this direction, for the particular example discussed above, is $\sigma(\pi/2, -\pi/4)$ times the incident power per unit area.. The radar cross-section or echo area σ_R is defined as the equivalent isotropic scattering cross-section that would produce the same back scattered power density. Thus

$$\frac{\sigma_R}{4\pi} = \sigma\left(\frac{\pi}{2}, -\frac{\pi}{4}\right)$$

or

$$\sigma_R = 4\pi\sigma\left(\frac{\pi}{2}, -\frac{\pi}{4}\right) \tag{2.26}$$

in the present case. Back scattered power density can be expressed by

$$P_{BSc.} = \frac{(\sigma_R)(\text{incident power density/unit area})}{4\pi r^2} \text{ watt per unit area.} \tag{2.27}$$

2.2 Scattering by a Circular Aperture in an Infinite Perfectly Conducting Plane Screen.

The scattering properties of a circular aperture in an infinite plane perfectly conducting screen is closely related to the circular disk problem. In essence the two problems are duals of each other. The solution for one may be found in terms of the other by using Babinet's principle which is based on the duality principle. We will derive the solution from basic principles and give a discussion of Babinet's principle later on.

We will consider the electric field only in the disk [illegible] the magnetic field only in the aperture problem. The other field ($\vec{H}$ or $\vec{E}$) can be found from Maxwell's equations.

Problem A, Disk Problem

Let (t) designate x and y components and (z) the z component. Let the incident field be given by

$$\vec{E}_i = \frac{1}{2}(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)} \quad , \quad \text{from} \quad z < 0 \tag{2.28a}$$

$$\vec{E}_i = \frac{1}{2}(-\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)} \quad , \quad \text{from} \quad z > 0 \tag{2.28b}$$

Note that $\vec{E}_{ot} + \vec{E}_{oz}$ is the constant vector $\vec{E}_o$ introduced earlier and the argument in the exponential is $-jk_o\vec{n}\cdot\vec{r}$. By symmetry the xy plane can be replaced by a conducting plane, or by superimposing the two incident fields it is seen that on the z=0 plane the total tangential electric field vanishes. Hence the scattered field is zero

$$\vec{E}_{sA} = 0 \quad , \quad z \gtrless 0 \tag{2.29}$$

Problem B, Disk Problem

Let the incident field be chosen to have even symmetry, thus

$$\vec{E}_i = \frac{1}{2}(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)} \quad , \quad \text{from} \quad z < 0 \tag{2.30a}$$

$$\vec{E}_i = \frac{1}{2}(\vec{E}_{ot} - \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)} \quad , \quad \text{from} \quad z > 0 \tag{2.30b}$$

Let the scattered electric field be

$$\vec{E}_{sB} = \vec{E}_{sBt}(x,y,z) + \vec{E}_{sBz}(x,y,z) \quad , \quad z < 0 \tag{2.31a}$$

$$\vec{E}_{sB} = \vec{E}_{sBt}(x,y,-z) - \vec{E}_{sBz}(x,y,-z) \quad , \quad z > 0 \tag{2.31b}$$

The total field is the sum of the incident plus scattered fields. The scattered field must be determined so that $\vec{E}_{ot} + \vec{E}_{sBt}(x,y,0)$ vanishes on the disk. Because of the way in which the incident fields have been chosen the total z component of the electric field is an odd function of z. On the disk surface we have

$$\rho_s = \epsilon_o E$$

on the side $z = 0^-$ and a charge density equal to *the same as* this on the side $z = 0^+$. On the $z = 0$ plane outside the disk $\rho_s = 0$, of course, and hence $E_{sBz}(x,y,0) = 0$ on the $z = 0$ plane outside the disk because the total z component of electric field is an odd function of z. It now follows, since $(\nabla \times \vec{H})_z = j\omega\epsilon_o E_z$ that the tangential component of the magnetic field outside the disk vanishes on the $z = 0$ plane. Hence, the plane outside of the disk can be replaced by a magnetic wall on which $\vec{n} \times \vec{H} = 0$.

<u>Problem C, Disk Problem</u>

Superimpose problems A and B, thus

$$\vec{E}_1 = (\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)} \quad , \text{ from } z < 0 \tag{2.32a}$$

$$\vec{E}_1 = 0 \quad , \text{ from } z > 0 \tag{2.32b}$$

For $z < 0$ the scattered electric field is $\vec{E}_s$ (for a small disk $\vec{E}_s$ is the dipole field evaluated earlier) and is given by

$$\vec{E}_s = \vec{E}_{sBt}(x,y,z) + \vec{E}_{sBz}(x,y,z) \quad , \quad z < 0 \tag{2.33a}$$

$$= \vec{E}_{sBt}(x,y,-z) - \vec{E}_{sBz}(x,y,-z), \quad z > 0 \tag{2.33b}$$

For $z > 0$ the total field is $\vec{E}_s$ plus the incident field $(\vec{E}_{ot}+\vec{E}_{oz})\exp -jk_o(xn_x+yn_y+zn_z)$ which propagates into the region $z > 0$ since the total field is $\vec{E}_i+\vec{E}_s$.

Problem a, Aperture Problem

Choose a dual incident field (the prime designates the field for the aperture problem),

$$\vec{H}_i' = \frac{1}{2} Y_o(\vec{E}_{ot}+\vec{E}_{oz})e^{-jk_o(xn_x+yn_y+zn_z)}, \text{ from } z < 0 \tag{2.34a}$$

$$\vec{H}_i' = \frac{1}{2} Y_o(\vec{E}_{ot}-\vec{E}_{oz})e^{-jk_o(xn_x+yn_y-zn_z)}, \text{ from } z > 0 \tag{2.34b}$$

By symmetry the xy plane can be replaced by an electric wall and hence the scattered field is zero.

$$\vec{H}_{sa} = 0, \; z \gtrless 0 \tag{2.35}$$

Problem b, Aperture Problem

We now choose for incident fields the following:

$$\vec{H}_i' = \frac{1}{2} Y_o(\vec{E}_{ot}+\vec{E}_{oz})e^{-jk_o(xn_x+yn_y+zn_z)}, \text{ from } z < 0 \tag{2.36a}$$

$$\vec{H}_i' = \frac{1}{2} Y_o(-\vec{E}_{ot}+\vec{E}_{oz})e^{-jk_o(xn_x+yn_y-zn_z)}, \text{ from } z > 0 \tag{2.36b}$$

By symmetry the aperture can be closed by a magnetic wall. Let the scattered field be

$$\vec{H}_{sb}' = \vec{H}_{sbt}'(x,y,z) + \vec{H}_{sbz}'(x,y,z) \;, \; z < 0 \tag{2.37a}$$

$$\vec{H}_{sb} = -\vec{H}_{sbt}'(x,y,-z)+\vec{H}_{sbz}'(x,y,-z) \;, \; z > 0 \tag{2.37b}$$

For $z < 0$ we can view the scattering problem as that of an incident field on a screen part of which is an electric wall and part of which is a magnetic wall. Thus in the space $z < 0$ problem B is the exact dual of problem b, including a dual incident field and dual boundary conditions. Therefore, the total field for $z < 0$ for problems B and b must be dual. Hence

$$(\vec{H}'_b)_{total} = \frac{1}{2} Y_o(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)}$$

$$+ \frac{1}{2} Y_o(-\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)} + \vec{H}'_{sb} = Y_o(\vec{E}_B)_{total}$$

$$= \frac{1}{2} Y_o(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)} + \frac{1}{2} Y_o(\vec{E}_{ot} - \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)}$$

$$+ Y_o\vec{E}_{sB}$$

from which we find that

$$\vec{H}'_{sb} = Y_o\vec{E}_{sB} + Y_o(\vec{E}_{ot} - \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)} \qquad z < 0 \qquad (2.38a)$$

The plane wave may be viewed as specular reflection from the infinite $z = 0$ plane. For $z > 0$ we have, from (2.37b)

$$\vec{H}'_{sb} = -Y_o\vec{E}_{sBt}(x,y,-z) + Y_o\vec{E}_{sBz}(x,y,-z) - Y_o(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)}, \quad z > 0 \qquad (2.38b)$$

<u>Problem c, Aperture Problem</u>

Superimpose problems a and b, thus

$$\vec{H}'_i = Y_o(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)}, \quad \text{from } z < 0 \qquad (2.39a)$$

$$\vec{H}'_i = 0, \quad \text{from } z > 0 \qquad (2.39b)$$

$$\vec{H}'_s = Y_o\vec{E}_{sB} + Y_o(\vec{E}_{ot} - \vec{E}_{oz})e^{-jk_o(xn_x + yn_y - zn_z)} \qquad z < 0 \qquad (2.40)$$

$$\vec{H}'_s = -Y_o\vec{E}_{sBt}(x,y,-z) + Y_o\vec{E}_{sBz}(x,y,-z) - Y_o(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o(xn_x + yn_y + zn_z)} \qquad (2.41)$$
$$z > 0$$

where $\vec{E}_{sB} = \vec{E}_s$ from (2.33). We thus have a solution for the aperture problem in terms of the solution for the disk problem. Note that the second term in (2.41) cancels the incident field in $z > 0$.

This solution is perfectly general in the sense that if $\vec{E}_s(x,y,z) = \vec{E}_{st}(x,y,z) + \vec{E}_{sz}(x,y,z)$ is the field scattered by an arbitrary shaped disk in the region $z < 0$ for an incident field $(\vec{E}_{ot} + \vec{E}_{oz})e^{-jk_o\vec{n}\cdot\vec{r}}$ then the field scattered through an aperture into the region $z > 0$, with the same shape as the disk is given by (2.41) when the incident field is the dual field specified by (2.39a). This follows from the derivation since no appeal to any particular shaped disk was necessary in order to carry out the derivation.

Consider now the case of a small disk and a small aperture. To understand the nature of the solution note that if $\vec{E}, \vec{H}$ is a solution of the equations

$$\nabla \times \vec{E} = -j\omega\mu_0\vec{H} - j\omega\mu_0\vec{M}$$

$$\nabla \times \vec{H} = j\omega\epsilon_0\vec{E} + j\omega\vec{P}$$

then a dual field $\vec{E}' = \pm Z_0\vec{H}$, $\vec{H}' = \mp Y_0\vec{E}$ is a solution of

$$\nabla \times \vec{E}' = -j\omega\mu_0\vec{H}' - j\omega\mu_0\vec{M}'$$

$$\nabla \times \vec{H}' = j\omega\epsilon_0\vec{E}' + j\omega\vec{P}'$$

where the dual dipole sources are given by

$$\vec{P}' = \pm \mu_0 Y_0 \vec{M},\ \vec{M}' = \mp \frac{Z_0}{\mu_0}\vec{P},\ \vec{M}' \cdot \vec{P}' = -\vec{M} \cdot \vec{P}$$

and similarly $\vec{E}' \cdot \vec{H}' = -\vec{E} \cdot \vec{H}$. For $z < 0$ the field $\vec{H}'_s$ is a reflected plane wave (as though all of xy plane was a perfect conductor) plus a field $Y_0\vec{E}_s$. But $\vec{E}_s$ is a field radiated by dipoles $\vec{P}$, $\vec{M}$ where $\vec{P} = \alpha_e\epsilon_0\vec{E}_{0t}$, $\vec{M} = \alpha_m\vec{H}_n$. The dual field $Y_0\vec{E}_s$ is a field radiated by dual dipole sources

$$\vec{P}' = \mu_0 Y_0 \vec{M} = \mu_0 Y_0 \alpha_m \vec{H}_n = \mu_0 Y_0 \alpha_m(-Y_0\vec{E}'_n) = -\alpha_m\epsilon_0\vec{E}'_n$$

$$\vec{M}' = -\frac{Z_0}{\mu_0}\vec{P}' = -\frac{Z_0}{\mu_0}\alpha_e\epsilon_0\vec{E}_{0t} = -\frac{Z_0}{\mu_0}\alpha_e\epsilon_0 Z_0\vec{H}'_{0t} = -\alpha_e\vec{H}'_{0t}$$

where $\vec{E}'_n$ = normal component of incident electric field for the aperture problem, $\vec{H}'_{0t}$ = tangential component of incident magnetic field for the aperture problem. Thus for $z < 0$ the scattered field is a reflected plane wave plus a field radiated by dipoles $\vec{M}' = -\alpha_e\vec{H}'_{0t}$, $\vec{P}' = -\alpha_m\epsilon_0\vec{E}'_n$ located in aperture. For $z > 0$ the field radiated through the aperture is a dipole field but the transverse part of the magnetic field and z component of electric field has been changed in sign. Such a field would be radiated by aperture dipole sources $-\vec{M}'$, $-\vec{P}'$.

The overall problem may thus be reduced simply to that illustrated in Fig. 2.5. We may state the procedure as follows:

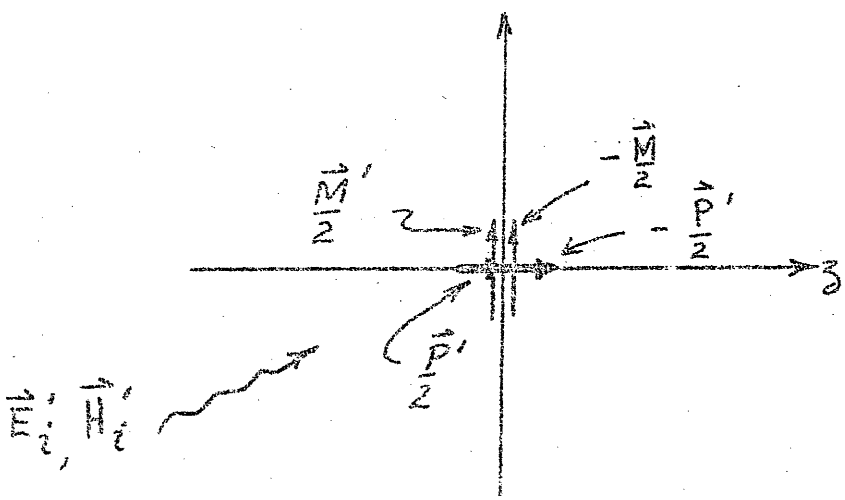


Fig. 2.5. Aperture equivalent dipole sources

For $z < 0$, replace all of the xy plane by a perfectly conducting plane, on this plane place sources $\vec{M}'/2$, $\vec{P}'/2$ at the origin. For $z > 0$, place sources $-\vec{M}'/2$, $-\vec{P}'/2$ at the origin on the perfectly conducting plane. The two regions $z \gtrless 0$ are thereby separated or uncoupled.

Since we closed aperture by a perfectly conducting surface the dipole moments are reduced in half because when we solve for the field radiated by, for example, $\vec{P}'/2$ we do in fact get the same field as would be radiated by a dipole $\vec{P}'$ in an unclosed aperture (by image theory field due to $\vec{P}'/2$ on a conducting plane is the same as field due to $\vec{P}'$ in absence of plane.

We see that there are no net dipole sources in the aperture. This is necessary because $\vec{P}'$, $\vec{M}'$ would be due to magnetic currents flowing on a magnetic wall (disk) and such currents do not have any physical significance.

Sometimes it is advantageous to consider as the exciting field for the aperture dipoles the sum of the normal component of $\vec{E}$ and the tangential components of $\vec{H}$ due to the incident plane wave and the reflected plane wave. In this case the excitation fields are $2\vec{E}'_n$, $2\vec{H}'_{0t}$. In terms of these the effective aperture dipoles are given by

$$\vec{P}'_0 = \frac{\vec{P}'}{2} = -\frac{1}{2}\alpha_m\varepsilon_0\vec{E}'_n = \alpha'_e\varepsilon_0(2\vec{E}'_n)$$

$$\vec{M}'_0 = \frac{\vec{M}'}{2} = -\frac{1}{2}\alpha_e\vec{H}'_{0t} = \alpha'_m(2\vec{H}'_{0t})$$

whence the effective polarizabilities for a circular aperture are

$$\alpha'_e = -\frac{\alpha_m}{4} = 2a^3/3 \tag{2.42a}$$

$$\alpha'_m = -\frac{\alpha_e}{4} = -4a^3/3 \tag{2.42b}$$

This formulation is convenient in connection with waveguide problems since a waveguide mode is the sum of an incident and a reflected wave from a conducting surface.[†]

[†]See: R. E. Collin, Field Theory of Guided Waves, McGraw-Hill, 1960, Chap. 7.

The transmission cross-section σ_T of an aperture is defined as follows:

$$\sigma_T = \frac{\text{Total scattered power through aperture}}{\text{Incident power density per unit area}} \tag{2.43}$$

The transmission cross-section is that equivalent area that intercepts an amount of power from the incident wave equal to the total power transmitted (or scattered) through the aperture. It is related to the total scattering cross-section for the complimentary disk (disk with the same shape as the aperture opening in the infinite screen). A flat disk has currents induced in it that flow only in the plane of the disk. By symmetry it, therefore, scatters identically in both the forward and backward directions. Hence, the total power scattered by the disk in the forward direction ($z > 0$) will be $1/2\ \sigma_S P_i$ where P_i is the incident power per unit area for the incident plane TEM wave. P_i is given by

$$P_i = \frac{1}{2} Y_0 |\hat{\bar{E}}_0|^2$$

For the aperture problem the incident power per unit area for the dual incident field is

$$P'_i = \frac{1}{2} Z_0 |\hat{\bar{H}}'_i|^2 = \frac{1}{2} Y_0 |\hat{\bar{E}}_0|^2 = P_i \tag{2.44}$$

by using (2.39a). The total power transmitted into the region $z > 0$ is obtained from an integral of the Poynting vector over the half sphere $\theta \leq \pi/2$, thus for r very large $\hat{a}_r \cdot \hat{\bar{H}}'_s$ vanishes so

$$P_{Tr} = \frac{1}{2} Z_o \int_0^{\pi/2} \int_0^{2\pi} |\vec{H}_s'|^2 r^2 \sin\theta d\theta d\phi$$

$$= \frac{1}{2} Y_o \int_0^{\pi/2} \int_0^{2\pi} (|\vec{E}_{st}|^2 + |\vec{E}_{sz}|^2) r^2 \sin\theta d\theta d\phi$$

since $\vec{E}_s' \times (H_s')^* \cdot \vec{a}_r = (\vec{a}_r \times \vec{E}_s') \cdot (\vec{H}_s')^* = \vec{H}_s'^{\,2}$ since for r very large $\vec{H}_s' = Y_o \vec{a}_r \times \vec{E}_s'$. Also $|\vec{H}_s'|^2 = Y_o(|\vec{E}_{st}|^2 + |\vec{E}_{sz}|^2)$ from (2.41). But the integral gives the total scattered power by the disk into the half space $z > 0$ so we obtain

$$\sigma_T = \frac{1}{2} \sigma_s \tag{2.45}$$

The transmission cross-section for the aperture equals one half of the total scattering cross-section for the complimentary disk.

2.3 Babinet's Principle

Let the incident field on a disk located in the $z = 0$ plane be $\vec{E}_i(x,y,z)$, $\vec{H}_i(x,y,z)$ from $z < 0$. Let the scattered field in the region $z > 0$ be $\vec{E}_s(x,y,z)$, $\vec{H}_s(x,y,z)$. For the complimentary screen let the incident field be the dual field

$$\vec{E}_i'(x,y,z) = -Z_o \vec{H}_i(x,y,z) \ , \quad \vec{H}_i'(x,y,z) = Y_o \vec{E}_i(x,y,z)$$

and let the scattered field be $\vec{E}_s'(x,y,z)$, $\vec{H}_s'(x,y,z)$ for $z > 0$. Babinet's principle states that

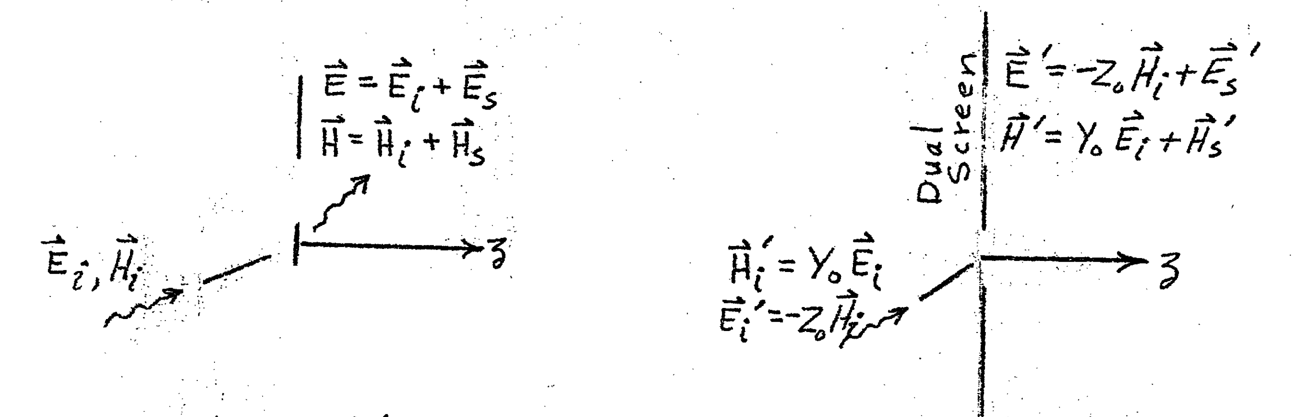
$$\vec{E} + Z_o \vec{H}' = \vec{E}_i \quad , \quad z > 0 \tag{2.46a}$$

$$\vec{H} - Y_o \vec{E}' = \vec{H}_i \quad , \quad z > 0 \tag{2.46b}$$

That is, a suitable superposition of the total fields in $z > 0$ for the two cases gives simply just the incident field. The principle is illustrated in Fig. 2.5 .

Thus if the scattered field for the disk problem is known the scattered field for the aperture problem may be found very rapidly by Babinet's principle. Note that the excitation in the two cases are, however, duals.

The proof of Babinet's principle is essentially that given in the preceeding section. For example, from (2.31b) and (2.41) we find that $\vec{E}_s + Z_o\vec{H}'_s = -\vec{E}_i$ from which (2.46a) follows immediately by adding $2\vec{E}_i$ to both sides.



For $z>0$, $\vec{E}_s + Z_o\vec{H}_s' = -\vec{E}_i$, $\vec{H}_s - Y_o\vec{E}_s' = -\vec{H}_i$, Add $2\vec{E}_i$ and $2\vec{H}_i$ to these to obtain Eq's 2.46

Fig. 2.5, *Illustration of Babinets' Principle*

2.4 Cross Sections

For convenience the definitions of a number of cross sections used in practice is given below. The incident field is a plane TEM wave and P_i denotes the incident power per unit area. Also let $\vec{k}_i$ denote the propagation vector for the incident wave and $\vec{k}_s$ denote the propagation vector for the scattered field.

$$\sigma(\theta, \phi) = \text{differential scattering cross-section} = \frac{1}{P_i}$$

(power scattered per unit solid angle in direction θ, ϕ)

But σ is a function of both $\vec{k}_i$ and $\vec{k}_s$ so it will be denoted by $\sigma(\vec{k}_i, \vec{k}_s)$ in general.

σ_s = scattering cross-section = $\frac{1}{P_i}$ (total power scattered)

σ_a = absorption cross-section = $\frac{1}{P_i}$ (total power absorbed by scatterer)

σ_e = extinction cross-section = $\sigma_s + \sigma_a$

σ_R = radar cross-section = $4\pi\sigma(\hat{k}_i, -\hat{k}_i)$

$\sigma_{R,Bs}$ = radar bi-static cross-section = $4\pi\sigma(\hat{k}_i, \hat{k}_s)$

σ_T = transmission cross-section for an aperture

= $\frac{1}{P_i}$ (total power transmitted through aperture)

2.5 Reciprocity for Scattering

Consider sources $\vec{J}_1$ and $\vec{J}_2$ which radiate in the presence of a scattering obstacle. Let the free space fields radiated by $\vec{J}_1$ and $\vec{J}_2$ be $\vec{E}_{i1}$, $\vec{H}_{i1}$, and $\vec{E}_{i2}$, $\vec{H}_{i2}$ respectively and let the corresponding scattered fields be $\vec{E}_{s1}$, $\vec{H}_{s1}$ and $\vec{E}_{s2}$, $\vec{H}_{s2}$. According to the reciprocity theorem we have

$$\int_V \vec{J}_1 \cdot (\vec{E}_{i2} + \vec{E}_{s2})dV = \int_V \vec{J}_2 \cdot (\vec{E}_{i1} + \vec{E}_{s1})dV \qquad (2.47)$$

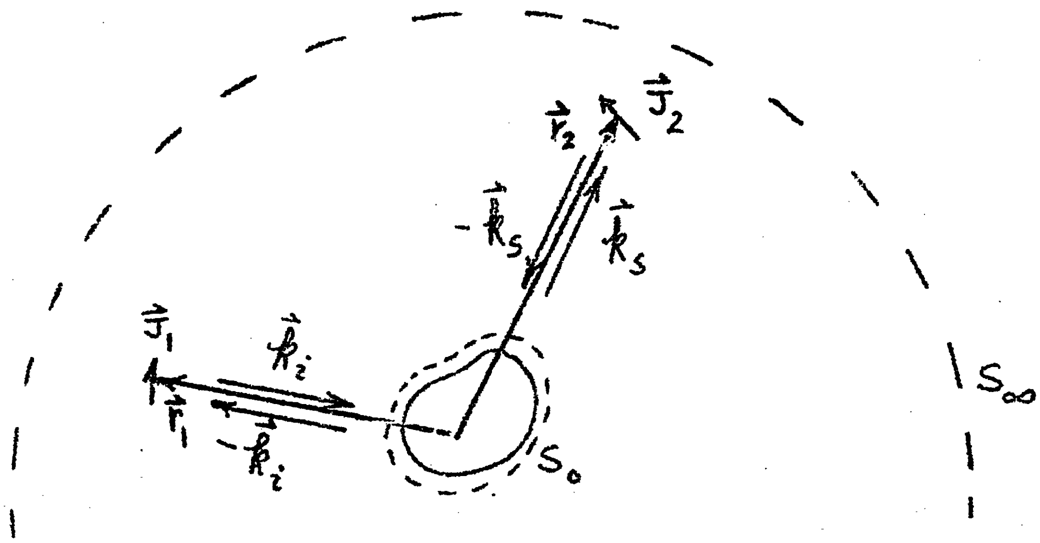


Fig. 2.6. Illustration for Scattering Reciprocity

But we also have

$$\int_V \vec{J}_1 \cdot \vec{E}_{i2} dV = \int_V \vec{J}_2 \cdot \vec{E}_{i1} dV \tag{2.48}$$

for the free space or incident fields so we then obtain

$$\int_V \vec{J}_1 \cdot \vec{E}_{s2} dV = \int_V \vec{J}_2 \cdot \vec{E}_{s1} dV \tag{2.49}$$

Let $\vec{J}_1 = \vec{a}\delta(\vec{r} - \vec{r}_1)$ and $\vec{J}_2 = \vec{a}\delta(\vec{r} - \vec{r}_2)$ where $\vec{a}$ is an arbitrary vector. Thus we get

$$\left[\vec{E}_{s2}(\vec{r}_1) - \vec{E}_{s1}(\vec{r}_2)\right] \cdot \vec{a} = 0$$

or

$$\vec{E}_{s2}(\vec{r}_1) = \vec{E}_{s1}(\vec{r}_2) \tag{2.50}$$

Now $\vec{E}_{s1}(\vec{r}_2)$ is the scattered field in the direction $\vec{k}_s$ due to an incident field in the direction $\vec{k}_i$ while $\vec{E}_{s2}(\vec{r}_1)$ is the scattered field in the direction $-\vec{k}_i$ due to an incident field along the direction $-\vec{k}_s$ (see Fig. 2.5). Hence the relation (2.50) also states that

$$\vec{E}_s(\vec{k}_i, \vec{k}_s) = \vec{F}_s(-\vec{k}_s, \vec{k}_i) \tag{2.51}$$

If the points $\vec{r}_1$ and $\vec{r}_2$ are far away from the scatterer the scattered field at these positions is an outward propagating spherical TEM wave of the form

$$\vec{E}_{s1}(\vec{r}_2) = \vec{F}(\vec{k}_i, \vec{k}_s) \frac{e^{-jk_0 r_2}}{r_2} \left(\frac{e^{-jkr_1}}{r_1}\right) \tag{2.52a}$$

$$\vec{E}_{s2}(\vec{r}_1) = \vec{F}(-\vec{k}_s, -\vec{k}_i) \frac{e^{-jk_0 r_1}}{r_1} \left(\frac{e^{-jk_0 r_2}}{r_2}\right) \tag{2.52b}$$

where $\vec{F}$ gives the directional characteristics of the scattered field and depends on the direction the incident field comes from and the direction in which the scattered field is viewed. The second exponential factor in (2.52) is the propagation factor from the source to the scatterer. Thus reciprocity gives

$$\vec{F}(\vec{k}_i, \vec{k}_s) = \vec{F}(-\vec{k}_s, -\vec{k}_i) \tag{2.53}$$

In other words, the position of the source and observer can be interchanged without affecting the value of the scattered field which would be measured. For a generalization of this result see Problem 2.8.

2.6 The Cross-Section Theorem

The cross-section theorem relates the extinction cross-section to the amplitude of the scattered wave in the forward direction, i.e., in the direction $\vec{k}_s = \vec{k}_i$.

Let the wave incident on the scatterer be a plane TEM wave and consider

$$\frac{1}{2} \operatorname{Re} \oint_S (\vec{E}_i + \vec{E}_s) \times (\vec{H}_i + \vec{H}_s)^* \cdot d\vec{S}$$

where S is any surface enclosing the scatterer and $d\vec{S} = \vec{n}dS$ with $\vec{n}$ pointing outward. The total flux of the real part of the complex Poynting vector through S equals the negative of the power P_a absorbed by the scatterer. Also we have

$$\frac{1}{2} Re \oint_S \vec{E}_i \times \vec{H}_i^* \cdot d\vec{S} = 0$$

$$\frac{1}{2} Re \oint_S \vec{E}_s \times \vec{H}_s^* \cdot d\vec{S} = P_s$$

so we can write

$$P_s + P_a = P_i \sigma_e = -\frac{1}{2} Re \oint_S (\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^*) \cdot \vec{n}\, dS \qquad (2.54)$$

Let us choose S as the surface of a sphere with r very large. Then on S we have

$$\vec{E}_s = \vec{F}(\vec{k}_i, \vec{k}_s) \frac{e^{-j\vec{k}_s \cdot \vec{r}}}{r}, \quad \vec{H}_s = Y_0 \frac{e^{-j\vec{k}_s \cdot \vec{r}}}{r}, \; \vec{a}_r \times \vec{F}$$

Also $\vec{H}_i = \dfrac{\vec{k}_i \times \vec{E}_i}{\omega\mu_0} = \dfrac{\vec{k}_i \times \vec{E}_0}{\omega\mu_0} e^{-j\vec{k}_i \cdot \vec{r}}$

so we obtain

$$P_i \sigma_e = -\frac{Y_0}{2} Re \oint_S \Big[\vec{E}_0 \times (\vec{a}_r \times \vec{F}^*) \cdot \vec{a}_r e^{-j\vec{k}_i \cdot \vec{r} + j\vec{k}_s \cdot \vec{r}}$$

$$+ \vec{F} \times \left(\frac{\vec{k}_i \times \vec{E}_0^*}{k_0} \right) \cdot \vec{a}_r e^{j\vec{k}_i \cdot \vec{r} - j\vec{k}_s \cdot \vec{r}} \Big] \frac{dS}{r} \qquad (2.55)$$

To evaluate the integral we will choose the z axis along $\vec{k}_i$ so $\vec{k}_i \cdot \vec{r} = k_0 r \cos\theta$, then since $\vec{k}_s \cdot \vec{r} = k_0 r$ we get

$$P_i \sigma_e = Re \frac{-Y_0}{2} \int_0^{\pi} \int_0^{2\pi} \Big[\vec{E}_0 \cdot \vec{F}^* e^{jk_0 r(1-\cos\theta)}$$

$$+ (\vec{F} \cdot \vec{E}_0^* \cos\theta - \vec{F} \cdot \vec{a}_z\, \vec{E}_0^* \cdot \vec{a}_r) e^{-jk_0 r(1-\cos\theta)} \Big] r \sin\theta\, d\theta\, d\phi$$

Since r is very large we can evaluate the integral by the method of stationary phase. The exponent $k_0r(1 - \cos\theta)$ is stationary at the points $d(1 - \cos\theta)/d\theta = 0$ or $\sin\theta = 0$ which corresponds to $\theta = 0, \pi$. Thus, the integral becomes equal in the limit as $r \to \infty$ to its value over small regions around $\theta = 0, \pi$. For the region $\theta \le \theta_0$ where θ_0 is so small that the integrand has essentially its value at $\theta = 0$ we get (note that $\vec{F} \cdot \vec{a}_z = 0$ at $\theta = 0, \pi$ since $\vec{F} \cdot \vec{a}_r = 0$)

$$I_1 = -\frac{Y_0}{2} \mathrm{Re} \left[\vec{E}_0 \cdot \vec{F}^*(\vec{k}_i, \vec{k}_i) \int_0^{\theta_0} \int_0^{2\pi} e^{\frac{jk_0r\theta^2}{2}} r\theta d\theta d\phi \right.$$
$$\left. + \vec{E}^*_0 \cdot \vec{F}(\vec{k}_i, \vec{k}_i) \int_0^{\theta_0} \int_0^{2\pi} e^{\frac{-jk_0r\theta^2}{2}} r\theta d\theta d\phi \right]$$

where we have replaced $1 - \cos\theta$ by $\theta^2/2$ and $\sin\theta$ by θ. The value of the integral is essentially unchanged if we make θ_0 infinite, thus since

$$\int_0^\infty e^{\frac{jk_0r\theta^2}{2}} \theta d\theta = \frac{-1}{jk_0r} \tag{2.56}$$

we get

$$I_1 = -\frac{Y_0}{2} \mathrm{Re} \left[\vec{E}_0 \cdot \vec{F}^* \left(\frac{-2\pi}{jk_0} \right) + \vec{E}^*_0 \cdot \vec{F} \frac{2\pi}{jk_0} \right]$$

$$= \frac{2\pi}{k_0} Y_0 \mathrm{Re} \left[j\vec{E}^*_0 \cdot \vec{F}(\vec{k}_i, \vec{k}_i) \right]$$

There is no contribution from the region around $\theta = \pi$ because $\mathrm{Re}\left[\vec{E}_0 \cdot \vec{F}^* e^{jk_0r(1 - \cos\theta)} + \vec{E}^*_0 \cdot \vec{F} \cos\theta e^{-jk_0r(1 - \cos\theta)}\right]$ approaches $\mathrm{Re}\left[\vec{E}_0 \cdot \vec{F}^* - \vec{E}_0 \cdot \vec{F}^*\right] e^{jk_0r(1 - \cos\theta)}$ which vanishes. To obtain the latter expression we took the complex conjugate of the second term and replaced $\cos\theta$ by -1 (note that $\text{Re}(A+B) = \text{Re}(A+B^*) = \text{Re}(A^*+B) = \text{Re}(A^*+B^*)$) .

Our final result thus shows that

$$\sigma_e = \frac{2\pi Y_0}{k_0 P_i} \text{Re}\ [j\vec{E}^*_0 \cdot \vec{F}(\vec{k}_i, \vec{k}_i)]$$
$$= \frac{-2\pi Y_0}{k_0 P_i} \text{imag.}\ \vec{E}^*_0 \cdot \vec{F}(\vec{k}_i, \vec{k}_i) \qquad (2.57)$$

This is a useful formula in practice because it enables the total extinction cross-section to be determined in terms of the amplitude of the scattered wave in the forward direction.

The above result is not an unexpected one for the following reason. The scattered wave can be thought of as a spectrum of plane waves propagating in all directions away from the scatterer. The incident wave is a plane wave propagating in the direction $\vec{k}_i$. These waves with different propagation vectors are orthogonal as regards total energy flow (power). The total scattered and absorbed power must show up as a reduced amplitude for the total field in the direction $\vec{k}_s = \vec{k}_i$ since only in this direction do the incident and scattered waves interact. Thus the extinction cross-section must be related to the amplitude of the scattered wave in the direction $\vec{k}_i$. The stationary phase evaluation of the integral for large r showed that the incident and scattered waves interacted only in the direction $\vec{k}_i$.

Since the transmission cross-section σ_T for an aperture equals $1/2\sigma_s$ for the complementary disk problem and (2.57) is applicable to the latter for $\sigma_a = 0$ (lossless disk) then we must also have

$$\sigma_T = -\frac{\pi Y_0}{k_0 P_i} \text{imag.}\ \vec{E}^*_0 \cdot \vec{F}(\vec{k}_i, \vec{k}_i) \qquad (2.58)$$

2.7 Variational Formulation for Scattering

We will consider a two dimensional problem only because of its scalar nature. Consider an infinite perfectly conducting cylinder (arbitrary cross section). Let $\psi_i = Ce^{-j\vec{k}_i\cdot\vec{r}}$ be the incident z directed electric field, $\vec{k}_i = k_0\vec{n}$, $r = x\vec{a}_x + y\vec{a}_y$ and $\vec{n}$ is a unit normal to the phase front of the incident plane wave. Let $\psi_s(r)$

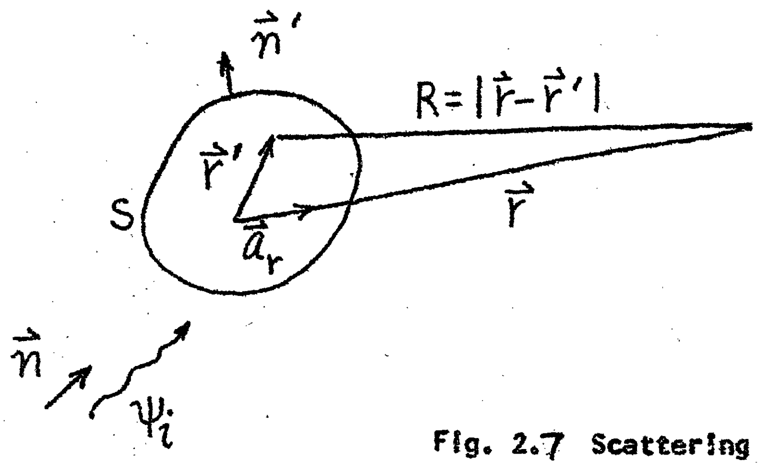


Fig. 2.7 Scattering by a Cylinder

be the scattered electric field. The total field ψ is given by

$$\psi = \psi_i + \psi_s = \psi_i + \oint_S G(\vec{r}|\vec{r}')\ J(\vec{r}')dS' \tag{2.59}$$

where J is the current induced on the cylinder in the z direction and $G = -\frac{\omega\mu_0}{4} H_0^2(k_0|\vec{r}-\vec{r}'|)$ = Green's function = electric field radiated at $\vec{r}$ due to a unit current at $\vec{r}'$.*

*Green's functions will be discussed later, for now we will borrow the result without derivation.

As $r \to \infty$, $|\vec{r}-\vec{r}'| = R \to r - \vec{r}'\cdot\vec{a}_r$ and $G \to -\frac{\omega\mu_0}{4}\sqrt{\frac{2}{\pi k_0 r}}\, e^{-j(k_0 r-\pi/4)}\, e^{j\vec{k}_s\cdot\vec{r}'}$ where $\vec{k}_s = k_0\vec{a}_r$. Also as $r \to \infty$, $\psi_s \to Cf(\vec{k}_i|\vec{k}_s)\frac{e^{-jk_0 r}}{\sqrt{r}}$ where f is a function of the direction of the incident wave and the direction of the scattered wave. The constant C is introduced to make the scattered wave amplitude f in the direction $\vec{a}_r$ due to a wave incident along the direction $\vec{n}$ independent of the amplitude of the incident plane wave. If ψ denotes the total value of E_z then

$$H_{tan} = (j\omega\mu_0)^{-1}\frac{\partial\psi}{\partial n'} = (j\omega\mu_0)^{-1}\psi'(\vec{r}') = J(\vec{r}') \text{ at } \vec{r}' \qquad (2.60)$$

since $-j\omega\mu_0\vec{H} = \nabla \times \vec{a}_z\psi = -\vec{a}_z \times \nabla\psi$.

Hence as $r \to \infty$ we obtain

$$\psi_s \to Cfe^{-jk_0 r}\, r^{-1/2}$$

where

$$f(\vec{k}_i|\vec{k}_s) = -\frac{\omega\mu_0}{4C}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s J(\vec{r}')e^{j\vec{k}_s\cdot\vec{r}'}\, dS'$$

$$= \frac{j}{4C}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s \psi'(\vec{r}')e^{j\vec{k}_s\cdot\vec{r}'}\, dS' \qquad (2.61)$$

We may write

$$\psi(r) = \frac{f\psi_i(\vec{r})}{f} + \psi_s$$

$$= \frac{Ce^{-j\vec{k}_i\cdot\vec{r}}}{f}\left\{\frac{-\omega\mu_0}{4C}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s J(\vec{r}')e^{j\vec{k}_s\cdot\vec{r}'}\right\}dS' + \psi_s \qquad (2.62)$$

On S total $E_z = \psi = 0$ so we have

$$\frac{\omega\mu_0}{4f}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\, e^{-j\vec{k}_i\cdot\vec{r}_1} \oint_s J(\vec{r}\,')e^{j\vec{k}_s\cdot\vec{r}\,'}\, dS'$$

$$= \frac{-\omega\mu_0}{4}\oint_s H_0^2\,(k_0|\vec{r}_1 - \vec{r}\,'|)J(\vec{r}\,')dS' \quad , \quad \vec{r} = \vec{r}_1 \text{ on } S \,. \qquad (2.63)$$

From this expression we can obtain a solution for f that is to first order independent of the current distribution J. Such an expression for f is a stationary (or variational) formulation. If we multiply by some function $g(\vec{r}_1)$ and integrate over S we have

$$\frac{1}{f}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s\oint_s e^{-j\vec{k}_i\cdot\vec{r}_1}\, g(\vec{r}_1)e^{j\vec{k}_s\cdot\vec{r}\,'}\, J(\vec{r}\,')dS'dS_1$$

$$= -\oint_s\oint_s g(\vec{r}_1)J(\vec{r}\,')H_0^2(k_0|\vec{r}_1 - \vec{r}\,'|)dS'dS_1$$

If we evaluated δf for a change δg in g we would find that $\delta f = 0$. If we change J by δJ (δJ is a small amplitude function of $\vec{r}\,'$) then the change δf in f is given by

$$-\frac{\delta f}{f^2}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s\oint_s e^{-j\vec{k}_i\cdot\vec{r}_1 + j\vec{k}_s\cdot\vec{r}\,'}\, gJdS'dS_1$$

$$+\frac{1}{f}\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4}\oint_s\oint_s e^{-j\vec{k}_i\cdot\vec{r}_1 + j\vec{k}_s\cdot\vec{r}\,'}\, g(\vec{r}_1)\delta J(\vec{r}\,')dS'dS_1$$

$$= -\oint_s\oint_s g(\vec{r}_1)\delta J(\vec{r}\,')H_0^2(k_0|\vec{r}_1 - \vec{r}\,'|)dS'dS_1$$

to first order. This may be rewritten as

$$- \frac{\delta f}{f^2} \sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4} \oint_S \oint_S e^{-j\vec{k}_i\cdot\vec{r}_1 + j\vec{k}_s\cdot\vec{r}'}\, g(\vec{r}_1) J(\vec{r}')\, dS' dS_1$$

$$= - \oint_S \delta J(\vec{r}') \{ e^{j\vec{k}_s\cdot\vec{r}'} \frac{1}{f} \sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4} \oint_S g(\vec{r}_1) e^{-j\vec{k}_i\cdot\vec{r}_1}\, dS_1 + \oint_S g(\vec{r}_1) H_0^2(k_0|\vec{r}_1 - \vec{r}'|) dS_1 \} dS' .$$

For δf to vanish we require that the term in braces { } = 0 . If we compare this term with the original definition of $f(\vec{k}_i|\vec{k}_s)$ i.e. (2.61) and the equation $\psi = 0$ on S i.e. (2.63) we see that if $g(\vec{r}_1)$ was the current $J_1(\vec{r}_1)$ on S due to a plane wave incident along $-\vec{a}_r$, i.e. $E_z = \psi_i = Ce^{j\vec{k}_s\cdot\vec{r}}$, then the term in braces { } would vanish and $\delta f = 0$. Note that $f(\vec{k}_i|\vec{k}_s) = f(-\vec{k}_s|-\vec{k}_i)$ which is a statement of the reciprocity principle. Collecting our results we have the following variational expression for the angle distribution function.

$$f(\vec{k}_i|\vec{k}_s) = \frac{\sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4} \oint_S \oint_S J(\vec{r}') J_1(\vec{r}_1) e^{-j\vec{k}_i\cdot\vec{r}_1 + j\vec{k}_s\cdot\vec{r}'}\, dS' dS_1}{- \oint_S \oint_S J(\vec{r}') J_1(\vec{r}_1) H_0^2\, (k_0|\vec{r}_1 - \vec{r}'|) dS' dS_1} \tag{2.64}$$

An approximate solution for J and J_1 when substituted into this expression will lead to an accurate value for f if the approximate solution for J and J_1 is reasonably close to the true value.

The total scattered power $P_s = - \frac{1}{2} \mathrm{Re} \int_0^{2\pi} (E_z H_\theta^*)_s\, r d\theta$ per unit length of cylinder. This Poynting vector can be evaluated over any contour surrounding the cylinder. If we choose a contour just outside the cylinder then $\psi_s = -\psi_i$ and $(H_{tan})_s =$

$$(j\omega\mu_0)^{-1} \frac{\partial \psi_s}{\partial n'} \qquad \text{i.e.} \qquad -j\omega\mu_0 \vec{H}_s = \nabla \times \vec{a}_z \psi_s = -\vec{a}_z \times \nabla\psi_s \quad \text{and}$$

$$-j\omega\mu_0 (H_{tan}^*)_s = \vec{a}_z \times \vec{n}'\, \partial\psi_s^* / \partial n' .$$

and

$$P_s = \frac{1}{2} \mathrm{Re} \oint_s (-j\omega\mu_0)^{-1} C e^{-j\vec{k}_i \cdot \vec{r}'} \frac{\partial \psi_s^*}{\partial n'} dS' \tag{2.65}$$

But $$(j\omega\mu_0)^{-1} \frac{\partial \psi_s}{\partial n'} = (j\omega\mu_0)^{-1} \left(\frac{\partial \psi}{\partial n'} - \frac{\partial \psi_i}{\partial n'}\right) = J(\vec{r}') - (j\omega\mu_0)^{-1} \frac{\partial \psi_i}{\partial n'}$$

Hence

$$P_s = \frac{C}{2} \mathrm{Re} \oint_s J^*(\vec{r}') e^{-j\vec{k}_i \cdot \vec{r}'} dS' + \mathrm{Re} \frac{C}{2j\omega\mu_0} \oint_s e^{-j\vec{k}_i \cdot \vec{r}'} \frac{\partial \psi_i^*}{\partial n'} dS' \tag{2.66}$$

But the flux of the incident field through a closed contour is zero, i.e.

$$\oint_s \psi_i \frac{\partial \psi_i^*}{\partial n'} dS' = \oint_s e^{-j\vec{k}_i \cdot \vec{r}'} \vec{k}_i \cdot \vec{n}' e^{j\vec{k}_i \cdot \vec{r}'} dS' = 0$$

since integral of $\oint_s \vec{k}_i \cdot \vec{n}' dS'$ is clearly zero. Hence,

$$P_s = \frac{C}{2} \mathrm{Re} \oint_s J^*(\vec{r}') e^{-j\vec{k}_i \cdot \vec{r}'} dS' = \frac{C}{2} \mathrm{Re} \oint_s J e^{j\vec{k}_i \cdot \vec{r}'} dS' \tag{2.67}$$

But integral gives (see 2.61))

$$-\frac{2C^2}{\omega\mu_0} \sqrt{\frac{\pi k_0}{2}} \, e^{-j\pi/4} f(\vec{k}_i | \vec{k}_i)$$

so

$$P_s = \mathrm{Re}\left[-\frac{2C^2}{\omega\mu_0} \sqrt{\frac{k_0}{2}} \frac{1-j}{\sqrt{2}} f(\vec{k}_i | \vec{k}_i)\right] \tag{2.68}$$

The incident power density $= \frac{1}{2} C^2 Y_0 = \frac{1}{2} \frac{C^2 k_0}{\omega\mu_0}$ so the cross-section is given by

$$\sigma_s = \text{Re } 2(j - 1)\sqrt{\frac{\pi}{k_0}}\, f(\vec{k}_i|\vec{k}_i) \tag{2.69}$$

If the angle distribution function is defined so that the scattered field at infinity is $\psi_s = CF(\vec{k}_i|\vec{k}_s)H_0{}^2(k_0 r)$ as $r \to \infty$ then

$$f = \sqrt{\frac{2}{\pi k_0}}\, e^{j\pi/4} F \quad \text{and} \quad \sigma_s = -\frac{4}{k_0} \text{Re } F(\vec{k}_i|\vec{k}_i) \tag{2.70}$$

This is the two-dimensional version of the cross-section theorem (2.57). (See also, Morse and Feshback, "Methods of Theoretical Physics," pg. 1544, vol. 2, McGraw-Hill, or Levine and Schwinger, Phys. Rev., vol. 75, p. 1423, 1949, for a similar derivation.) For two dimensional scattering the cross-section is an equivalent scattering width since the scattered power is evaluated on a per unit length basis.

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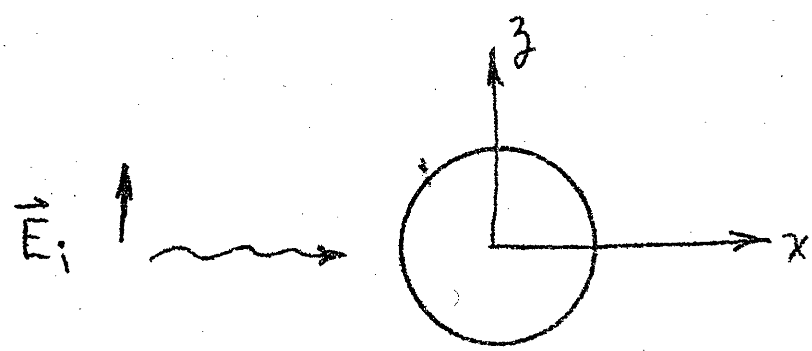
See Also References 1 and 2.

PROBLEMS

1. Solve the following scattering problem: Find the differential scattering cross section and the total scattering cross section for a small perfectly conducting sphere, radius $a \ll \lambda_0$, and located at the origin. The polarizabilities for a small conducting sphere are $\alpha_e = 4\pi a^3$, $\alpha_m = -2\pi a^3$ so that the induced dipole moments are $\vec{P} = \epsilon_0 \alpha_e \vec{E}_t$, $\vec{M} = \alpha_m \vec{H}_n$; $\vec{E}_t$ = tangential component of incident electric field, $\vec{H}_n$ = normal component of incident magnetic field. In your derivation of σ and σ_s retain α_e and α_m as parameters so that you can specialize to the case $\alpha_m = 0$ later. (For the sphere $\vec{E}_t$ and $\vec{H}_n$ become the total incident fields).

Find σ and σ_s for a dielectric sphere for which $\alpha_m = 0$, $\alpha_e = 4\pi a^3 \dfrac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}$. How does σ_s depend on wavelength λ_0? The results you will obtain were obtained by Lord Rayleigh and are referred to as the Rayleigh scattering formulas quite often. The results have been used in calculating the scattering of EM waves by rain drops.

Sketch σ as a function of θ and ϕ when the incident wave on the conducting sphere has $\vec{E}$ in the z direction only and the direction of incidence is along the x axis.



2.2. (a) Consider a sphere with parameters ε, μ. Show that if the sphere is placed in a uniform static electric field that the induced field outside the sphere is a dipole field arising from an electric dipole of moment

$$P = 4\pi a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} (\varepsilon_0 E_0)$$

and placed at the origin. Let ε tend to infinity and show that in the limit the total electric field inside the sphere vanishes. Thus, in the limit you get $\alpha_e = 4\pi a^3$ for the polarizability of a conducting sphere.

(b) When the sphere is placed in a uniform magnetic field H_0 show that the induced magnetic field outside the sphere is a magnetic dipole field arising from a magnetic dipole with moment

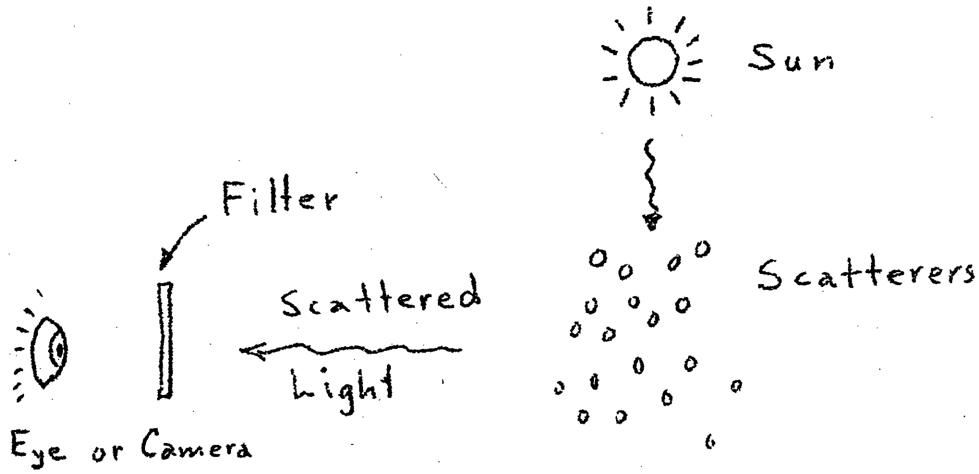
$$M = 4\pi a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0} H_0$$

and located at the origin. Take the limit as μ goes to zero and show that in this case the total normal value of H vanishes at $r = a$. Thus, show that $\alpha_m = -2\pi a^3$ for a perfectly conducting sphere.

2.3. Light is randomly polarized with zero correlation between the two perpendicular polarized components. Consider a randomly polarized wave incident on a dielectric sphere along the x-axis (Problem 2.1). The incident field will have uncorrelated z and y components of electric field with equal magnitude E_0. In what directions is the scattered field completely polarized (linear polarization) and what is the orientation of the electric field in these directions?

The scattering of sun light by minute dust particles and molecules produces linearly polarized light looking in a direction perpendicular to that from which the sun light comes from.

This phenomenon is utilized in photography to get an enhanced 'blue' sky color by using a "polaroid" filter which is a filter which absorbs one component of a linearly polarized wave and not the perpendicular component. Which component is absorbed -- the horizontal or vertical? (see figure below)



Plan view (looking down onto the earth's surface)

2.4. For Problem 2.1, verify that σ_s is given by (2.57). *This requires use of a higher order approximation rather than the dipole approximation. (See sol. for scattered field in Stratton).*

2.5. Consider an aperture cut in an infinite perfectly conducting plane located at $z = 0$. A wave is incident on this aperture from the region $z < 0$. Show that the total tangential magnetic field in the aperture is equal to that of the incident wave only. What is the value of the total z component of electric field in the aperture?

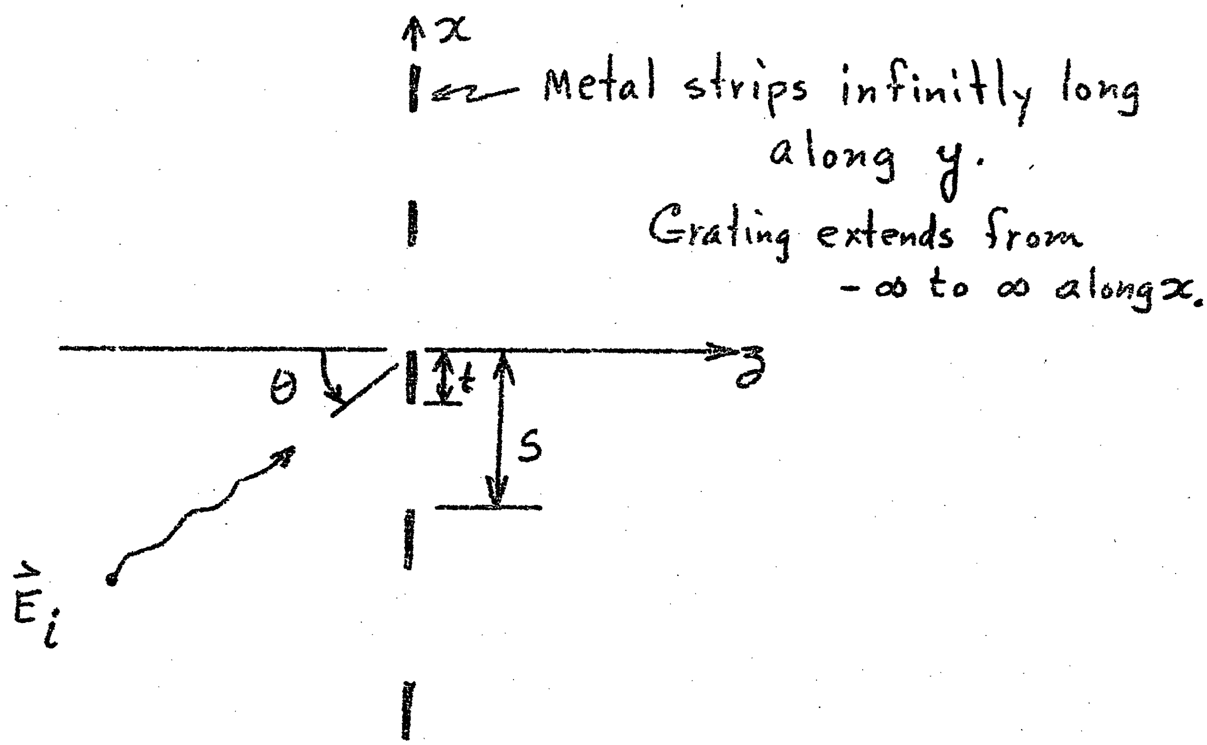
HINT: Superimpose the solutions for even and odd excitations.

2.6. From consideration of the frequency dependence of the scattering from a small dielectric sphere explain why the sky is blue and the sunset and sunrise appears red and orange. Note that sunlight is scattered by air and water vapor molecules as well as dust particles in the atmosphere.

7. For the illustrated grating let the incident field be

$$\vec{E}_i = \vec{a}_y E_0 \, e^{-jk_0 x \sin\theta - jk_0 z \cos\theta}, \quad \text{from } z<0$$

$$\vec{H}_i = Y_0 E_0 \left(-\vec{a}_x \cos\theta + \vec{a}_z \sin\theta\right) e^{-jk_0 x \sin\theta - jk_0 z \cos\theta}, \quad z<0$$



For $z<0$ the scattered field has the form

$$E_y = \Gamma E_0 \, e^{-jk_0 x \sin\theta + jk_0 z \cos\theta} + \sum_{n=-\infty}^{\infty}{}' a_n \, e^{-j\left(k_0 \sin\theta + \frac{2n\pi}{s}\right)x + \gamma_n z}$$

where the prime means ommission of the $n=0$ term and

$\gamma_n^2 = k_0^2 - \left(k_0 \sin\theta + \frac{2n\pi}{s}\right)^2$. For $s < \lambda_0/2$ all modes for $n \neq 0$ are evanescent along z. Γ is the dominant mode reflection coefficient. Also

$$H_x = \Gamma Y_0 E_0 \cos\theta \, e^{-jk_0 x \sin\theta + jk_0 z \cos\theta}$$

$$+ \sum_{n=-\infty}^{\infty}{}' \, Y_0 a_n \frac{\gamma_n}{jk_0} e^{-j\left(k_0 \sin\theta + \frac{2n\pi}{s}\right)x + \gamma_n z}$$

For $z > 0$ the <u>total field</u> has the form

$$E_y = T E_0 \, e^{-jk_0 x \sin\theta - jk_0 z \cos\theta}$$

$$+ \sum_{n=-\infty}^{\infty}{}' \, b_n \, e^{-j\left(k_0 \sin\theta + \frac{2n\pi}{s}\right)x - \gamma_n z}$$

$$H_x = -T Y_0 E_0 \left(e^{-jk_0 x \sin\theta - jk_0 z \cos\theta}\right) \cos\theta$$

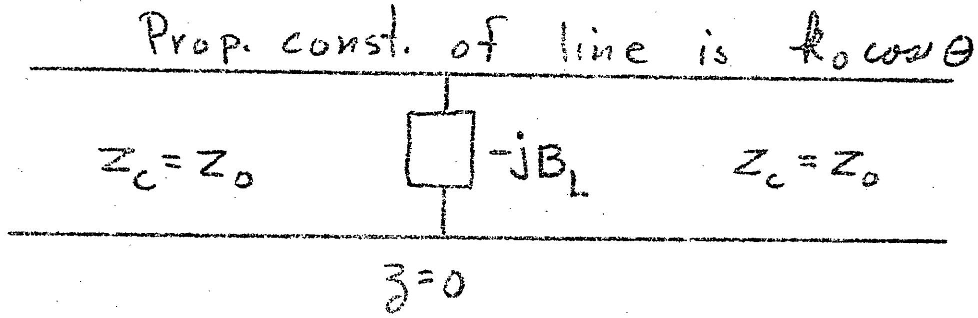
$$- \sum_{n=-\infty}^{\infty}{}' \, Y_0 b_n \frac{\gamma_n}{jk_0} e^{-j\left(k_0 \sin\theta + \frac{2n\pi}{s}\right)x - \gamma_n z}$$

where T is a transmission coefficient. The form of the scattered field is dictated by the periodic nature of the grating.

By Fourier analysis requiring E_y to be contenuous at $z=0$ for each mode we get

$$1+\Gamma = T, \quad a_n = b_n$$

The result $1+\Gamma = T$ shows that the grating acts as a shunt susceptance for the dominant mode. Hence for this mode we can use an equivalent circuit of the form shown.

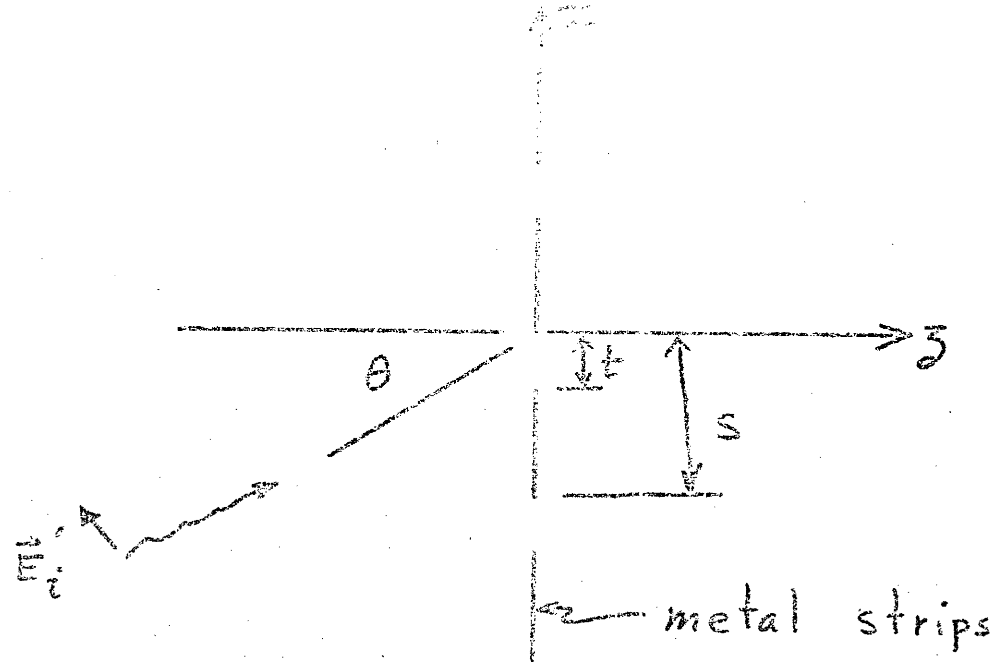


For this circuit $1+\Gamma = T$ and $\Gamma = \dfrac{Y_c - Y_{in}}{Y_c + Y_{in}}$

$$= \frac{Y_c - Y_c + jB_L}{2Y_c - jB_L} = \frac{jB_L}{2Y_c - jB_L}$$

where B_L is the inductive susceptance presented to the incident field by the grating.

Consider now the dual problem with dual incident fields $\vec{E}_i{}' = -Z_0 \vec{H}_i$, $\vec{H}_i{}' = Y_0 \vec{E}_i$



and show that the grating now acts as a capacitive susceptance B_c and that $B_C B_L = 4Y_0^2$. This circuit relation is easily obtained by showing first that $T + T' = -(\Gamma + \Gamma') = 1$ and $TT' = \Gamma\Gamma'$.

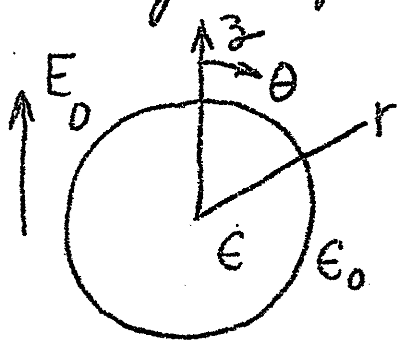
2.8 In Eq. (2.49) let $\vec{J}_1 = \vec{a}_1 \delta(\vec{r} - \vec{r}_1)$, $\vec{J}_2 = \vec{a}_2 \delta(\vec{r} - \vec{r}_2)$ where $\vec{a}_1$ and $\vec{a}_2$ are arbitrary unit vectors. Let $\vec{E}_{s1}(\vec{r}, \vec{a}_1)$, $\vec{E}_{s2}(\vec{r}, \vec{a}_2)$ be the scattered fields from $\vec{J}_1$ and $\vec{J}_2$ respectively. Show that

$$\vec{F}(\vec{k}_i, \vec{k}_s, \vec{a}_1) \cdot \vec{a}_2 = \vec{F}(-\vec{k}_s, -\vec{k}_i, \vec{a}_2) \cdot \vec{a}_1.$$

Give a physical description of this statement. Hint: Consider the special case $\vec{a}_1 = \vec{a}_x$, $\vec{a}_2 = \vec{a}_y$.

Scattering by a Small Dielectric Sphere

Polarization of Sphere - Static Solution



Induced potential is

$$\Phi_i = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} E_0 r\cos\theta, \quad r \le a$$

$$= \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} E_0 \frac{a^3}{r^2}\cos\theta, \quad r \geq a$$

Total electric field in interior is $\frac{3\epsilon_0}{\epsilon+2\epsilon_0} E_0 \vec{a}_z$

Total dipole moment is $\vec{P} = \frac{4}{3}\pi a^3 (\epsilon-\epsilon_0)\vec{E} = 4\pi a^3 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} \epsilon_0 E_0 \vec{a}_z$

Outside sphere $\vec{E}_{ind} = \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} E_0 \frac{a^3}{r^3}\left(2\vec{a}_r\cos\theta + \vec{a}_\theta \sin\theta\right)$

This induced field along with the applied field will satisfy the boundary conditions on the surface $r=a$. $\vec{E}_{ind}$ appears to be due to a point dipole $\vec{P}$ at the center.

If a plane wave $E_0 \vec{a}_z e^{-jk_0 x}$ is incident on a sphere with $a \ll \lambda_0$, then $e^{-jk_0 x} \approx 1$ for $r \le a$. Hence the problem reduces to a static field problem and $\vec{P} = \alpha_e \epsilon_0 \vec{E}_0$, $\alpha_e = 4\pi a^3(\epsilon-\epsilon_0)/(\epsilon+2\epsilon_0)$. We now find the dynamic radiated field from $\vec{P}$ using

$A_z = \frac{\mu_0}{4\pi}\frac{e^{-jk_0 r}}{r} j\omega P$, $\quad \vec{E}_s = -j\omega\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{j\omega\epsilon_0\mu_0}$. This gives

$$\vec{E}_s = \frac{\omega P Z_0}{2\pi k_0}\left(\frac{jk_0}{r^2}+\frac{1}{r^3}\right)\cos\theta\, e^{-jk_0 r}\vec{a}_r + \frac{\omega P Z_0}{4\pi k_0}\left(-\frac{k_0^2}{r}+\frac{jk_0}{r^2}+\frac{1}{r^3}\right)\sin\theta\, e^{-jk_0 r}\vec{a}_\theta$$

For small values of $k_0 r$, $e^{-jk_0 r} = 1 - jk_0 r - \frac{1}{2}k_0^2 r^2 - - - -$

Using this expansion gives, for the scattered dynamic field,

$$\vec{E}_s \simeq \frac{P}{4\pi\epsilon_0}\left[\left(\frac{2\cos\theta}{r^3}\vec{a}_r + \frac{\sin\theta}{r^3}\vec{a}_\theta\right) + \frac{k_0^2}{2r}\left(2\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta\right) - jk_0^3\frac{2}{3}\vec{a}_z + \cdots\right]$$

This is valid for r small but with $r \geq a$ since the multipole expansion holds for field points outside the source region. The first group of terms are the same as for the static solution and thus along with $E_0\vec{a}_z$ satisfy the boundary conditions at $r=a$. The next real term is of order $k_0^2 a^2$ relative to the leading terms. The leading imaginary term is of order $k_0^3 a^3$. Since the dynamic field has these additional terms the boundary conditions at $r=a$ are not satisfied exactly but the error is small when $k_0 a$ is small. The imaginary term, although small, is important since it accounts for the radiated power, i.e.

$$P_s = -\mathrm{Re}\,\frac{1}{2}\vec{E}_s\cdot(j\omega\vec{P})^* = \frac{jk_0^3}{3}\,\frac{-j\omega PP^*}{4\pi\epsilon_0} = \frac{\omega k_0^3}{12\pi\epsilon_0}PP^*$$

where P_s is the total scattered power. The power removed from the incident wave should equal P_s. It is given by $\mathrm{Re}\,\frac{1}{2}E_0(j\omega P)^*$ and is zero when the static solution $P = 4\pi a^3\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\epsilon_0 E_0$ is used for P. For a uniformly polarized sphere the equivalent volume polarization charge is zero. The equivalent surface polarization charge is $\vec{a}_r\cdot\vec{\mathcal{P}}$. The boundary condition $\epsilon_0\,\vec{a}_r\cdot\vec{E}\Big|_{r=a_-}^{r=a_+} = \vec{a}_r\cdot\vec{\mathcal{P}}$ is required at $r=a$, where $\vec{E}$ is the total field.

If we use the static solution $\frac{3\epsilon_0}{\epsilon+2\epsilon_0} E_0 \vec{a}_z$ for the total interior field along with $\vec{p} = (\epsilon-\epsilon_0)\frac{3\epsilon_0}{\epsilon+2\epsilon_0} E_0 \vec{a}_z$ then $\vec{p}\cdot\vec{a}_r$ will support the radial component of the leading term in the scattered field at $r=a$ only. In order to also support the additional leading imaginary term we must increase $\vec{a}_r \cdot \vec{p}$ by the amount $\frac{P}{4\pi}(-jk_0^3 \frac{2}{3}\vec{a}_z\cdot\vec{a}_r) = -jk_0^3 \frac{2}{3}\frac{P}{4\pi}\cos\theta$. In other words we require

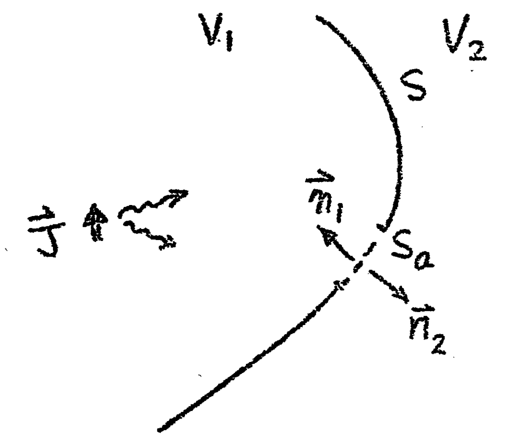
$$\frac{P}{4\pi}\left(\frac{2\cos\theta}{a^3} - jk_0^3\frac{2}{3}\cos\theta\right) + \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}E_0\cos\theta = \frac{P}{\frac{4}{3}\pi a^3}\cos\theta$$

or $P = 4\pi a^3 \frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0} \Big/ \left(1+j\frac{2}{3}k_0^3 a^3\right)$. The correction term $j\frac{2}{3}k_0^3 a^3$ is small but its inclusion will lead to power conservation. The solution for P is equivalent to including $-j\frac{P}{4\pi\epsilon_0}\frac{2}{3}k_0^3\vec{a}_z$, the radiation reaction field, as part of the polarizing field so that

$$\vec{P} = \alpha_e\epsilon_0\left[E_0\vec{a}_z - j\frac{P}{4\pi\epsilon_0}\frac{2}{3}k_0^3\vec{a}_z\right]$$

which then gives the corrected value of $\vec{P}$.

Coupling Through Small Apertures



Prob. 1 Let aperture opening S_a be closed by an electric wall. A source $\vec{J}$ produces a field $\vec{E}_g$, $\vec{H}_g$ with $\vec{n}_1 \times \vec{E}_g = 0$, $\vec{n}_1 \cdot \vec{H}_g = 0$ on S and S_a in V_1. These are the "generator fields" under short-circuit conditions.

Prob. 2 Close S_a by a magnetic wall on which $\vec{n}_1 \times \vec{H} = 0$, $\vec{n}_1 \cdot \vec{E} = 0$. Total field in V_1 is the generator fields plus a scattered field $\vec{E}_{S1}$, $\vec{H}_{S1}$ such that $\vec{n}_1 \times (\vec{H}_{S1} + \vec{H}_g) = 0$ on S_a, $\vec{n}_1 \cdot (\vec{E}_{S1} + \vec{E}_g) = 0$ on S_a. To a first approximation $\vec{E}_{S1}$, $\vec{H}_{S1}$ can be found from induced dipoles $\vec{P}$, $\vec{M}$ located at the center of the aperture. $\vec{P}$ is along $\vec{n}_1$ and $\vec{M}$ is perpendicular to $\vec{M}_1$. $\vec{P}$ and $\vec{M}$ come from equivalent currents $\vec{J}_{ms}$ and charges ρ_{ms} on S_a with $\vec{n}_1 \times \vec{E}_{S1} = -\vec{J}_{ms}$, $\mu_0 \vec{n}_1 \cdot \vec{H}_{S1} = \rho_{ms}$.

Prob. 3 Open aperture surface S_a and place sources $-\vec{J}_{ms}$ on S_a (i.e. place dipoles $-\vec{P}$, $-\vec{M}$ in aperture). These sources radiate into both V_1 and V_2. Let the field produced in V_1 be $\vec{E}_{S1}'$, $\vec{H}_{S1}'$ and that produced in V_2 be $\vec{E}_2$, $\vec{H}_2$. The following boundary conditions hold

$$\vec{n}_2 \times (\vec{E}_2 - \vec{E}_{S1}') = -(-\vec{J}_{ms}) = \vec{J}_{ms}, \quad \vec{n}_2 \times (\vec{H}_2 - \vec{H}_{S1}') = 0 \quad \text{on } S_a.$$

If we superimpose the three solutions we obtain the total correct field produced by $\vec{J}$ in both V_1 and V_2. This field is given by

$$\vec{E}_1 = \vec{E}_g + \vec{E}_{s1} + \vec{E}_{s1}', \quad \vec{H}_1 = \vec{H}_g + \vec{H}_{s1} + \vec{H}_{s1}' \quad \text{in } V_1$$

$$\vec{E}_2, \ \vec{H}_2 \quad \text{in } V_2$$

<u>Proof</u> $\vec{E}_1, \vec{H}_1$ satisfies conditions required by Maxwell's equations in the source region occupied by $\vec{J}$ because $\vec{E}_g, \vec{H}_g$ do and the other fields originated by scattering from S and S_a. At the aperture opening $\vec{n}_2 \times \vec{E}_1 = \vec{n}_2 \times \vec{E}_2$ and $\vec{n}_2 \times \vec{H}_1 = \vec{n}_2 \times \vec{H}_2$. Also $\vec{E}_2, \vec{H}_2$ are found so that all boundary conditions in V_2 are satisfied (e.g. $\vec{n}_2 \times \vec{E}_2 = 0$ on S, radiation condition is obeyed). We have $\vec{n}_2 \times \vec{E}_1 = \vec{n}_2 \times (\vec{E}_g + \vec{E}_{s1} + \vec{E}_{s1}')$ $= \vec{n}_2 \times (\vec{E}_{s1} + \vec{E}_{s1}')$ since $\vec{n}_2 \times \vec{E}_g = 0$ on S and S_a.
But $\vec{n}_2 \times \vec{E}_{s1} = -\vec{n}_1 \times \vec{E}_{s1} = \vec{J}_{ms}$ and $\vec{n}_2 \times \vec{E}_{s1}' = \vec{n}_2 \times \vec{E}_2 - \vec{J}_{ms}$
Hence $\vec{n}_2 \times (\vec{E}_{s1} + \vec{E}_{s1}') = \vec{J}_{ms} + \vec{n}_2 \times \vec{E}_2 - \vec{J}_{ms} = \vec{n}_2 \times \vec{E}_2$ on S_a.
Similarly we can show that $\vec{n}_2 \times \vec{H}_1 = \vec{n}_2 \times \vec{H}_2$ on S_a.
By the uniqueness theorem the postulated field is the correct solution.

CHAPTER III

STURM-LIOUVILLE EQUATION AND GREEN'S FUNCTIONS

In this chapter we review some of the important properties of the Sturm-Liouville differential equation. This is followed by a presentation of several methods of constructing the characteristic Green's function for the Sturm-Liouville system. Finally it is shown how multi-dimensional Green's functions may be found for separable partial differential equations in terms of contour integrals over the spectrum of the associated characteristic one dimensional Green's functions.

3.1 Sturm-Liouville Equation

Many of the boundary value problems encountered in connection with the study of guided electromagnetic waves leads to the Sturm-Liouville equation. In this section we will review the basic properties of the Sturm-Liouville system but without detailed derivation.

The Sturm-Liouville equation is of the form

$$\frac{d}{dx} p(x) \frac{d\psi(x)}{dx} + [q(x) + \lambda\sigma(x)]\psi(x) = 0 \tag{3.1}$$

where λ is a separation constant. In practice both p and σ are usually always positive and continuous functions of x. We will consider the properties of this equation when the range in x is finite, say $0 \leq x \leq a$.

The solutions to (3.1) are fixed only if certain boundary conditions are first specified. The three common boundary conditions are $\psi = 0$, or $\frac{d\psi}{dx} = 0$ or $\psi + K\frac{d\psi}{dx} = 0$ at $x = 0, a$, where K is a suitable constant. Once a set of boundary conditions have been specified (one condition at $x = 0$ and one at $x = a$) then one finds an infinite set of solutions ψ_n to (3.1) corresponding to particular values of λ , say λ_n . The functions ψ_n are called eigenfunctions and the λ_n are called eigenvalues. In all cases the ψ_n form an orthogonal set of functions over the interval $0 \le x \le a$. The orthogonality of the functions may be proved as follows: Multiplying the equation for ψ_n by ψ_m and vice versa, subtracting and integrating gives

$$\int_0^a (\psi_n \frac{d}{dx} p \frac{d\psi_m}{dx} - \psi_m \frac{d}{dx} p \frac{d\psi_n}{dx})dx = \int_0^a (\lambda_n - \lambda_m)\sigma\, \psi_n\psi_m\, dx$$

since the term involving q vanishes. Integrating the left hand side by parts gives

$$(\psi_n \frac{d\psi_m}{dx} - \psi_m \frac{d\psi_n}{dx})p\Big|_0^a - \int_0^a p(\frac{d\psi_m}{dx}\frac{d\psi_n}{dx} - \frac{d\psi_n}{dx}\frac{d\psi_m}{dx})dx$$

$$= p(\psi_n \frac{d\psi_m}{dx} - \psi_m \frac{d\psi_n}{dx})\Big|_0^a = (\lambda_n - \lambda_m)\int_0^a \sigma\,\psi_n\psi_m dx$$

When ψ_n and ψ_m satisfy the same boundary conditions, of the type given earlier, the integrated terms vanish. For example, if $\psi_i + K\frac{d\psi_i}{dx} = 0$ at $x = 0, a$, with $i = n, m$, then we can write

$$\psi_n \frac{d\psi_m}{dx} - \psi_m \frac{d\psi_n}{dx} = (\psi_n + K\frac{d\psi_n}{dx})\frac{d\psi_m}{dx} - (\psi_m + K\frac{d\psi_m}{dx})\frac{d\psi_n}{dx}$$

which clearly vanishes at $x = 0, a$, When $\lambda_n \neq \lambda_m$ we conclude that

$$\int_0^a \sigma\, \psi_n\psi_m = 0 \qquad (3.2)$$

i.e. the ψ_n form an orthogonal set with respect to the weighting function $\sigma(x)$. If $\lambda_n = \lambda_m$ we have a degeneracy but suitable combinations of ψ_n and ψ_m can be chosen to yield an orthogonal set of functions.

For example, if ψ_1 and ψ_2 are eigenfunctions with degenerate eigenvalues $\lambda_1 = \lambda_2$ and $\int_0^a \sigma\, \psi_1 \psi_2 dx \neq 0$ then we can choose new eigenfunctions $\overline{\psi}_1 = \psi_1$ and $\overline{\psi}_2 = \psi_1 + C\psi_2$ and force these to be orthogonal. This means that the constant C is chosen such that $\int_0^a \sigma\,(\psi_1^2 + C\psi_1\psi_2)dx = 0$. The new eigenfunctions are now orthogonal but have the common eigenvalue λ_1 . This procedure may be extended to the case where there are N degenerate eigenvalues with corresponding eigenfunctions.

We will now assume that the ψ_n have been normalized so that they form an ortho-normal set, i.e.

$$\int_0^a \sigma\, \psi_n \psi_m dx = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \qquad (3.3)$$

The eigenfunctions form a complete set and may be used to expand an arbitrary piecewise continuous function f(x) into a Fourier type series. Thus let

$$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x) \qquad (3.4)$$

To find the unknown coefficients multiply both sides by $\sigma\, \psi_m$, integrate over 0 to a and use (3.3), thus

$$\int_0^a \sigma\, f\, \psi_m dx = \sum_{n=1}^{\infty} a_n \int_0^a \sigma\, \psi_n \psi_m dx = a_m \qquad (3.5)$$

Our main interest in the orthogonality property of the eigenfunctions is for the purpose of finding the coefficients in Fourier series type expansions.

3.2 Green's Function G(x, x')

The Green's function is the response of a linear system to a point source of unit strength. A source of unit strength at the position x' can be conveniently represented by the Dirac delta function $\delta(x-x')$ which has the basic operational properties

$$\delta(x-x') = 0 \ , \ x \neq x' \tag{3.6a}$$

$$\int_a^b g(x)\delta(x-x')dx = \begin{cases} g(x') \ , & a < x' < b \\ 0 \ , \ x' & \text{not in} \quad a \to b \end{cases} \tag{3.6b}$$

where $g(x)$ is any function that is continuous at x' . From an heuristic viewpoint we can think of $\delta(x-x')$ as a narrow rectangular pulse of width Δx and height $1/\Delta x$, and thus of unit area, in the limit as $\Delta x \to 0$ as shown in Fig. 3.1 .

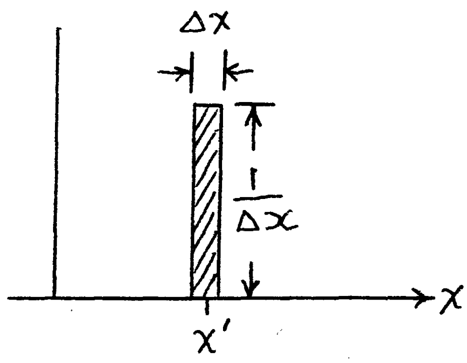


Fig. 3.1 Delta Function

The Green's function $G(x, x')$ is the solution of the Sturm-Liouville equation when the source term or forcing function is a point source of unit strength. The Green's function satisfies the following equation,

$$\frac{d}{dx} p \frac{dG}{dx} + (q + \lambda \ \sigma)G = -\delta(x-x') \tag{3.7}$$

The Green's function must also satisfy appropriate boundary conditions at $x = 0, a$. We will discuss two basic solutions to (3.7) .

Method I

We may expand G in terms of the eigenfunctions ψ_n , with eigenvalues λ_n , that satisfy the same boundary conditions as G . Thus let $G(x, x') = \sum_{n=1}^{\infty} a_n\psi_n(x)$. Substituting in the equation for G gives $\sum_{n=1}^{\infty} a_n(\lambda-\lambda_n)\sigma\psi_n = -\delta(x-x')$ since ψ_n is a solution of (3.1) for $\lambda = \lambda_n$. If we now multiply both sides by ψ_m , integrate over 0 to a , and use (3.3) we obtain

$$\sum_{n=1}^{\infty} a_n(\lambda-\lambda_n)\int_0^a \sigma\, \psi_n\psi_m dx = a_m(\lambda-\lambda_m) = -\int_0^a \psi_m\delta(x-x')dx = -\psi_m(x')$$ upon using

(3.6b) to evaluate the last integral. Using this solution for a_m we obtain

$$G(x, x') = -\sum_{n=1}^{\infty} \frac{\psi_n(x)\ \psi_n(x')}{\lambda - \lambda_n} \tag{3.8}$$

Note that G has poles at the point $\lambda = \lambda_n$. We will make use of this property later on.

Method II

We note that at all points $x \neq x'$ the Green's function is a solution of the homogeneous equation $\frac{d}{dx}\, p\, \frac{dG}{dx} + (q + \lambda\sigma)G = 0$. At x' G must be continuous but $p\,\frac{dG}{dx}$ must be discontinuous by a unit amount (see Fig. 3.2) . The second derivative $\frac{d}{dx}\, p\, \frac{dG}{dx}$ then yields a delta function singularity. If G was not continuous at x' the resulting singularity would be of to high an order.

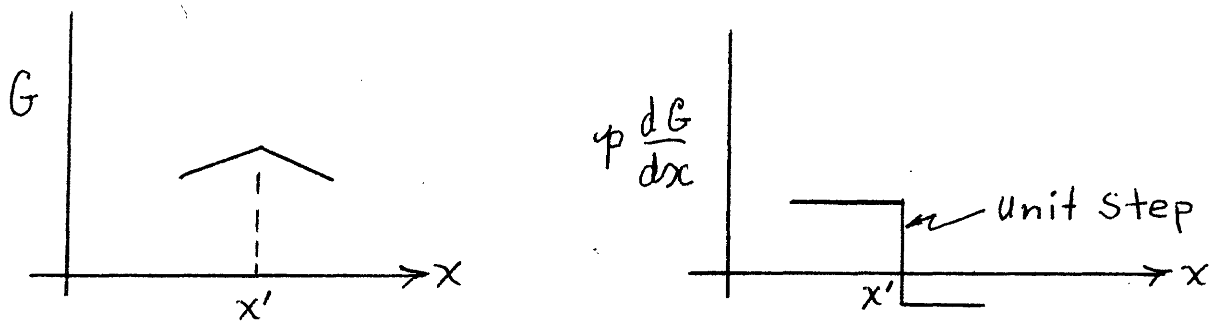


Fig. 3.2 Behavior of G and its Derivative near x'

Let $A\phi_1(x)$ be a solution of the homogeneous equation that satisfies the boundary condition at $x = 0$. Similarly let $B\phi_2(x)$ be a solution, linearly independent of ϕ_1 , that satisfies the boundary condition at $x = a$. Thus let $G = A\phi_1(x)$, $x \leq x'$, and $G = B\phi_2(x)$, $x \geq x'$. Continuity of G at $x = x'$ requires that $A\phi_1(x') = B\phi_2(x')$. If we integrate the equation for

G from $x'-\tau$ to $x'+\tau$ and let $\tau \to 0$ and note that $\lim_{\tau\to 0} \int_{x'-\tau}^{x'+\tau} (q+\lambda\sigma) G dx = 0$ since q, σ, and G are all continuous at x' then we obtain

$$\lim_{\tau \to 0} \int_{x'-\tau}^{x'+\tau} \frac{d}{dx} p \frac{dG}{dx} dx = p \left.\frac{dG}{dx}\right|_{x'_-}^{x'_+} = - \int_{x'-\tau}^{x'+\tau} \delta(x-x')dx = -1$$

In other words pdG/dx evaluated between adjacent sides of the point x' must undergo a step change as shown in Fig. 3.2 . Hence we have

$$\left.\frac{dG}{dx}\right|_{x'_-}^{x'_+} = - \frac{1}{p(x')} = B\phi_2'(x') - A\phi_1'(x')$$

where ϕ' stands for $d\phi/dx$. The solution for A and B may be found from the above two equations and is $A = -\phi_2(x')/p(x')W(x')$, $B = -\phi_1(x')/p(x')W(x')$ where the Wronskian determinant $W(x') = \phi_1(x')\phi_2'(x') - \phi_1'(x')\phi_2(x')$ does not equal zero since ϕ_1 and ϕ_2 are linearly independent solutions.

Our solution for the Green's function is thus

$$G(x, x') = \begin{cases} -\dfrac{\phi_1(x)\phi_2(x')}{p(x')W(x')} & 0 \le x \le x' \\ -\dfrac{\phi_1(x')\phi_2(x)}{p(x')W(x')} & x' \le x \le a \end{cases} \qquad (3.9)$$

The Wronskian determinant W will be a function of the parameter λ and will vanish whenever $\lambda = \lambda_n$ so G will have the poles that are explicitly exhibited by the first solution (3.8). Although (3.8) and (3.9) are different in form they are equal.

It is convenient to define the following:

$x_> \equiv$ the greater of x or x'

$x_< \equiv$ the lesser of x or x'

We can then express the solution for $G(x, x')$ in the condensed form

$$G(x, x') = -\frac{\phi_1(x_<)\phi_2(x_>)}{p(x')W(x')} \tag{3.10}$$

which includes both forms given by (3.9) .

Example

We wish to find the eigenfunctions and eigenvalues for the equation $\frac{d^2\psi}{dx^2} + \lambda\psi = 0$ with boundary conditions $\psi = 0$ at $x = 0, a$. This is a special case with $p = \sigma = 1$ and $q = 0$. A general solution for ψ is $A \sin \sqrt{\lambda}x + B \cos \sqrt{\lambda}x$. To make $\psi = 0$ at $x = 0$ we must choose $B = 0$. To make ψ vanish at $x = a$ we must have $\sin\sqrt{\lambda}\, a = 0$. Thus $\sqrt{\lambda}\, a = n\pi$, $n = 1,2,3,\ldots$ or $\lambda_n = (n\pi/a)^2$. Hence the eigenfunctions are $\sin n\pi x/a$. Since $\int_0^a \sin^2 \frac{n\pi x}{a}\, dx = \frac{a}{2}$ the normalized eigenfunctions are $\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} = \psi_n$.

We now wish to solve $\frac{d^2G}{dx^2} + \lambda G = -\delta(x-x')$. According to Method I we let $G = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$. By substituting in the equation for G we find $\sum_{n=1}^{\infty} a_n(\lambda - \frac{n^2\pi^2}{a^2}) \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} = -\delta(x-x')$. We now multiply both sides by $\sqrt{\frac{2}{a}}$ $\sin \frac{m\pi x}{a}$ and integrate from 0 to a and obtain

$$a_m(\lambda - \frac{m^2\pi^2}{a^2}) = -\sqrt{\frac{2}{a}} \int_0^a \delta(x-x')\sin \frac{m\pi x}{a}\, dx = -\sqrt{\frac{2}{a}} \sin \frac{m\pi x'}{a}$$

Thus

$$G(x, x') = -\sum_{n=1}^{\infty} \frac{\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \sqrt{\frac{2}{a}} \sin \frac{n\pi x'}{a}}{\lambda - n^2\pi^2/a^2} \tag{3.11a}$$

We may also use Method II. We let $\phi_1 = \sin \sqrt{\lambda}\, x$ and $\phi_2 = \sin \sqrt{\lambda}(a-x)$.

Note that ϕ_1 and ϕ_2 are linearly independent and satisfy the boundary conditions at $x = 0, a$, respectively. The Wronskian determinant $W(x') = \phi_1(x')\phi_2'(x') - \phi_1'(x')\phi_2(x') = (\sin\sqrt{\lambda}\, x')[-\sqrt{\lambda}\cos\sqrt{\lambda}\,(a-x')] - \sqrt{\lambda}\cos\sqrt{\lambda}\, x'$ $\sin\sqrt{\lambda}(a-x') = -\sqrt{\lambda}\sin\sqrt{\lambda}\, a$. From (3.10) we get

$$G(x, x') = \frac{\sin\sqrt{\lambda}\, x_< \sin\sqrt{\lambda}(a-x_>)}{\sqrt{\lambda}\sin\sqrt{\lambda}\, a} \tag{3.11b}$$

Note that $\sin\sqrt{\lambda}\, a = 0$ whenever $\lambda = (n\pi/a)^2$ so that this second solution for G does have the poles at $\lambda = n^2\pi^2/a^2$ that are shown explicitly in the first solution. A Fourier series expansion of the second solution would yield the first solution.

3.3 Solution of Boundary Value Problems

Let it be required to find a solution to the equation

$$\frac{d}{dx} p \frac{d\psi}{dx} + (q + \lambda\sigma)\psi = -f(x) \tag{3.12}$$

with either $\psi, \frac{d\psi}{dx}$, or $\psi + K\frac{d\psi}{dx}$ specified at $x = 0, a$. In (3.12) $f(x)$ is a distributed forcing function. Let $G(x, x')$ be a Green's function that is a solution of

$$\frac{d}{dx} p \frac{dG}{dx} + (q + \lambda\sigma)G = -\delta(x-x') \tag{3.13}$$

We now replace x by x' for convenience, multiply the equation for ψ by G , the equation for G by ψ, subtract and integrate to obtain

$$\int_0^a [G(x', x)\frac{d}{dx'} p(x')\frac{d\psi(x')}{dx'} - \psi(x')\frac{d}{dx'} p(x')\frac{dG}{dx'}(x', x)]dx'$$

$$= \int_0^a - f(x')G(x', x)dx' + \int_0^a \psi(x')\delta(x'- x)dx'$$

since the terms involving the factor $q + \lambda\sigma$ cancel.

After an integration by parts we obtain

$$\psi(x) = \int_0^a G(x', x)f(x')dx' + [G(x', x)p(x')\frac{d\psi(x')}{dx'}$$

$$-\psi(x')p(x')\frac{dG(x', x)}{dx'}]\Big|_0^a \qquad (3.14)$$

If Ψ satisfies one of the following boundary conditions $\Psi = 0$, $\frac{d\Psi}{dx'} = 0$, $\Psi + K\frac{d\Psi}{dx'} = 0$, at each end $x = 0$ or a , and we choose G to satisfy the same boundary conditions then the boundary terms vanish and the solution for ψ is simply

$$\psi(x) = \int_0^a G(x', x)f(x')dx' \qquad (3.15)$$

This solution is a superposition of the response from each differential element of force $f\,dx$ with G being the response of the system due to a point source of unit strength. It is the ability to represent the solution to a linear system driven by an arbitrary forcing function $f(x)$ by a superposition integral such as (3.15) that makes the theory of Green's functions of great importance in practice.

Sometimes the boundary conditions on ψ may be of the form $K_1\psi + K_2\frac{d\psi}{dx'} = K_3$ (K_1 or K_2 might be zero and the K_i may be different at each end). In this case we choose the boundary conditions on G in such a way that we can evaluate the boundary terms in (3.14) in terms of known quantities. If ψ is given on the boundary ($K_2 = 0$) we choose $G = 0$ on the boundary, if $\frac{d\psi}{dx'}$ is given ($K_1 = 0$) we choose $\frac{dG}{dx'} = 0$, and finally if $K_1\psi + K_2\frac{d\psi}{dx'}$ is given we choose $K_1G + K_2\frac{dG}{dx'} = 0$ on the boundary. We will illustrate the last case explicitly.

I.E., REPLACE ψ WITH G IN THE B.C.S AND SET EQUAL TO ZERO

The boundary terms in (1.14) (3.14) may be written as

$$p(x')[\frac{G}{K_2}(K_1\psi + K_2\frac{d\psi}{dx'}) - \frac{\psi}{K_2}(K_1G + K_2\frac{dG}{dx'})]\Big|_0^a$$

which by virtue of the boundary conditions on ψ and G becomes the known quantity $\frac{K_3}{K_2} p(x')G(x', x)\Big|_0^a$.

3.4 Multi-dimensional Green's Functions and Alternative Representations

We have noted that there is more than one representation for a one dimensional Green's function. Also we have noted that the Green's function has poles for certain values of the separation constant λ . These poles are called the spectrum of the Green's function. For finite range problems the spectrum is discrete but for infinite range problems the spectrum is continuous and the Green's function will have branch point singularities instead of poles.

In two and three dimensions there are many different possible representations for the Green's functions. For this reason we wish to present a unified approach to multi-dimensional Green's functions.

First we will present a theorem that is useful in constructing two and three dimensional Green's functions from Green's functions for one dimensional problems.

Theorem 1 $$\frac{1}{2\pi j} \oint_C G(x, x', \lambda) d\lambda = - \frac{\delta(x-x')}{\sigma(x')} \tag{3.16}$$

where C is a closed contour in the complex λ plane that encloses all the singularities of $G(x, x', \lambda)$.

Proof. Use the form $G(x, x', \lambda) = - \sum_{n=1}^{\infty} \frac{\psi_n(x)\psi_n(x')}{\lambda - \lambda_n}$

and Cauchy's theorem to give

$$- \frac{1}{2\pi j} \sum_{n=1}^{\infty} \oint_C \frac{\psi_n(x)\psi_n(x')}{\lambda - \lambda_n} d\lambda = - \sum_{n=1}^{\infty} \psi_n(x)\psi_n(x')$$

To complete the proof we show that this equals the right-hand side of (3.16) by expanding $\delta(x-x')$ in terms of a series in the ψ_n . Let $\delta(x-x') = \sum_{n=1}^{\infty} C_n \psi_n(x)$. Multiply both sides by $\sigma(x)\psi_m(x)$ and use (3.3) to obtain $C_m = \int_0^a \delta(x-x')\sigma(x)\psi_m(x)dx = \sigma(x')\psi_m(x')$.

Hence

$$\frac{\delta(x-x')}{\sigma(x')} = \sum_{n=1}^{\infty} \psi_n(x)\psi_n(x') \qquad (3.17)$$

which completes the proof.

Example. Consider $\frac{d^2G}{dx^2} + \lambda G = -\delta(x-x')$, $G = 0$ at $x = 0, a$.

By using Method II we readily find that

$$G(x, x', \lambda) = \frac{\sin\sqrt{\lambda}\, x_< \sin\sqrt{\lambda}(a-x_>)}{\sqrt{\lambda}\sin\sqrt{\lambda}\, a}$$

Since G is an even function of $\sqrt{\lambda}$ it has no branch points, its only singularities are poles at $\sqrt{\lambda}a = n\pi$ or $\lambda = (n\pi/a)^2$, $n = 1,2,3,\ldots$. Theorem I gives

$$\frac{1}{2\pi j}\oint_C G\, d\lambda = -\delta(x-x')$$

where C is the contour shown in Fig. 3.3 . Near $\lambda_n = (\frac{n\pi}{a})^2$ we have $\sin\sqrt{\lambda}\, a$

$$= \sin a(\sqrt{\lambda} - \sqrt{\lambda_n} + \sqrt{\lambda_n}) =$$

$$\sin[(\sqrt{\lambda} - \sqrt{\lambda_n})a + n\pi] =$$

$$\cos n\pi \sin(\sqrt{\lambda} - \sqrt{\lambda_n})a$$

$$\to (\cos n\pi)(\sqrt{\lambda} - \sqrt{\lambda_n})a$$

$$= \cos n\pi \frac{\lambda - \lambda_n}{\sqrt{\lambda} + \sqrt{\lambda_n}} a .$$

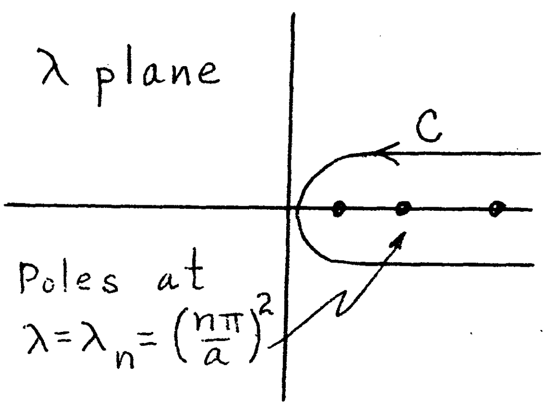


Fig. 3.3

Thus the residue from the term $\frac{1}{\sqrt{\lambda}\sin\sqrt{\lambda}\, a}$ at the n'th pole is $\frac{2\sqrt{\lambda_n}}{a\sqrt{\lambda_n}\cos n\pi} =$

$= (-1)^n \frac{2}{a}$. The residue expansion of the integral may now readily be found and is

$$\frac{1}{2\pi j} \oint_C G d\lambda = \sum_{n=1}^{\infty} (-1)^n \frac{2}{a} \sin \frac{n\pi x_<}{a} \sin \frac{n\pi}{a} (a-x_>)$$

$$= - \sum_{n=1}^{\infty} \frac{2}{a} \sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \text{ since } \sin \frac{n\pi}{a} (a-x_>)$$

$= -\cos n\pi \sin \frac{n\pi x_>}{a}$. The latter series is easily verified to be the expansion of $-\delta(x-x')$ in conformity with Theorem I.

We will use the above Theorem to show that multi-dimensional Green's functions can be constructed in the form of contour integrals taken over the spectrum of a product of associated one dimensional Green's functions. The theory will be developed by means of suitable examples.

Green's Function for Two Dimensional Laplace's Equation

Consider

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x-x')\delta(y-y') \tag{3.18}$$

$$G = 0 \quad \text{at} \quad x = 0, a \; ; \; y = 0, b$$

Apart from a factor $1/\varepsilon_o$ the function G represents the electric potential from a z directed line charge inside a conducting rectangular tube as shown in Fig. 3.4 .

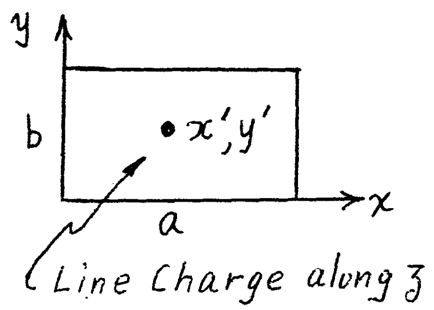


Fig. 3.4

Let $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $L_x = \frac{\partial^2}{\partial x^2}$, $L_y = \frac{\partial^2}{\partial y^2}$.

The equation $L\psi = 0$ separates into two one-dimensional equations of the form $L_x \psi_x(x) + \lambda_x \psi_x = 0$

and $L_y\psi_y(y) + \lambda_y\psi_y = 0$ with $\lambda_x + \lambda_y = 0$ provided we assume a solution in product form i.e. $\psi(x, y) = \psi_x(x)\psi_y(y)$. Corresponding to each of the separated equations there are associated one dimensional Green's function problems of the form

$$\frac{d^2G_x}{dx^2} + \lambda_x G_x = -\delta(x-x') \; , \; G_x = 0 \quad \text{at} \quad x = 0, a$$
$$\frac{d^2G_y}{dy^2} + \lambda_y G_y = -\delta(y-y') \; , \; G_y = 0 \quad \text{at} \quad y = 0, b \tag{3.19}$$

We will now show that the solution to the original problem (3.18) can be expressed in terms of the solutions to the simpler one dimensional problems (3.19). Specifically we will show that

$$G(x, x', y, y',) = \frac{-1}{2\pi j} \oint_{C_x} G_x(x, x', \lambda_x)G_y(y, y', -\lambda_x)d\lambda_x$$
$$= \frac{-1}{2\pi j} \oint_{C_y} G_x(x, x', -\lambda_y)G_y(y, y', \lambda_y)d\lambda_y \tag{3.20}$$

where C_x encloses ONLY the singularities of G_x and excludes those of G_y and similarly C_y encloses only the singularities of G_y . The formula (3.20) synthesises an appropriate two dimensional Green's function from the associated one dimensional Green's functions.

To prove (3.20) we only need to show that it is a solution of (3.18). Clearly G satisfies the correct boundary conditions in view of the conditions imposed on G_x and G_y at the boundaries. We also have $LG = (L_x + \lambda_x + L_y + \lambda_y)G =$

$$= -\frac{1}{2\pi j} \oint_{C_x} (L_x + \lambda_x + L_y + \lambda_y)G_x(x, x', \lambda_x)G_y(y, y', \lambda_y)d\lambda_x$$

$$= -\frac{1}{2\pi j} \oint_{C_x} [-\delta(x-x')G_y(y, y', -\lambda_x) - G_x(x, x', \lambda_x)\delta(y-y')]d\lambda_x$$

upon using (3.19) to replace $(L_x + \lambda_x)G_x$ by $-\delta(x-x')$ and $(L_y + \lambda_y)G_y$ by $-\delta(y-y')$. Since C_x excludes the singularities of G_y we obtain $LG =$ $\frac{1}{2\pi j} \oint_{C_x} G_x(x, x', \lambda_x)d\lambda_x \, \delta(y-y') = -\delta(x-x')\delta(y-y')$ upon using Theorem I, (note that $\sigma = 1$ in this problem). In other words the contour integral of G_y is zero because no singularities are enclosed by virtue of the manner in which C_x was chosen. A similar proof holds if we use the form involving integration around the contour C_y in (3.20) .

We will now illustrate the above procedure by constructing G directly and then show that the same solution is obtained from using (3.20) and solutions to (3.19).

The normalized eigenfunctions of the equation $\frac{d^2G_x}{dx^2} + \lambda_x G_x = 0$ which vanish at $x = 0, a$, are $\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$, $n = 1,2,\ldots$, $\lambda_x = (\frac{n\pi}{a})^2$. These form a complete set so we may assume that

$$G(x, x', y, y') = \sum_{n=1}^{\infty} a_n(y)\Phi_n(x) \ , \ \Phi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

When we substitute into the equation for G , i.e. into (3.18), we obtain

$$\sum_{n=1}^{\infty} [\frac{d^2a_n(y)}{dy^2} - (\frac{n\pi}{a})^2 a_n(y)]\Phi_n(x) = -\delta(x-x')\delta(y-y')$$

We now multiply both sides by $\Phi_m(x)$ and integrate to get

$$\frac{d^2a_m}{dy^2} - (\frac{m\pi}{a})^2 a_m = -\Phi_m(x')\delta(y-y')$$

since the functions Φ_n are orthogonal. We can readily solve this equation for $a_m(y)$ according to the Method II given earlier. We find that

$$a_m(y) = \frac{\Phi_m(x')\sinh \frac{m\pi}{a} y_< \sinh \frac{m\pi}{a} (b-y_>)}{\frac{m\pi}{a} \sinh \frac{m\pi b}{a}}$$

upon choosing $\Phi_1 = \sinh \frac{m\pi y}{b}$, $\Phi_2 = \sinh \frac{m\pi}{a} (b-y)$. Our solution for G is thus

$$G = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \sinh \frac{n\pi}{a} y_< \sinh \frac{n\pi}{a} (b-y_>)}{\sinh \frac{n\pi b}{a}} \tag{3.21}$$

We could equally well have made the first expansion with respect to y and we would then have found that

$$G = \sum_{n=1}^{\infty} \frac{2}{n\pi} \frac{\sin \frac{n\pi y}{b} \sin \frac{n\pi y'}{b} \sinh \frac{n\pi}{b} x_< \sinh \frac{n\pi}{b} (a-x_>)}{\sinh \frac{n\pi a}{b}} \tag{3.22}$$

We can find yet another form for G by assuming that it can be expanded as a double Fourier series. Thus let

$$G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

If we substitute this solution into (3.18) we obtain

$$- \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} C_{nm}[(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2]\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} = -\delta(x-x')\delta(y-y') .$$

The coefficient C_{nm} may be found by multiplying both sides by $\sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$ and integrating over x and y . The final result is

$$G = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{ab} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \sin \frac{m\pi y}{b} \sin \frac{m\pi y'}{b}}{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2} \tag{3.23}$$

We will now show how the different solutions (3.21) - (3.23) are all contained in the general formulation (3.20). We first need the solutions to the one dimensional Green's function problems given in (3.19). By using Method II these are readily found in the following forms:

$$G_x = \frac{\sin \sqrt{\lambda_x}\, x_< \sin \sqrt{\lambda_x}\, (a-x_>)}{\sqrt{\lambda_x} \sin \sqrt{\lambda_x}\, a} \tag{3.24a}$$

$$G_y = \frac{\sin \sqrt{\lambda_y}\, y_< \sin \sqrt{\lambda_y}\, (b-y_>)}{\sqrt{\lambda_y} \sin \sqrt{\lambda_y}\, b} \tag{3.24b}$$

We note that $G_x(x, x', \lambda_x)$ has poles at $\lambda_x = (\frac{n\pi}{a})^2$, $n = 1,2,\ldots$ and that $G_y(y, y', -\lambda_x)$ has poles when $\sin\sqrt{-\lambda_x}\, b = 0$ or at $\lambda_x = -(n\pi/b)^2$. Thus we choose the contour C_x as shown in Fig. 3.5 . According to (3.20) we now have

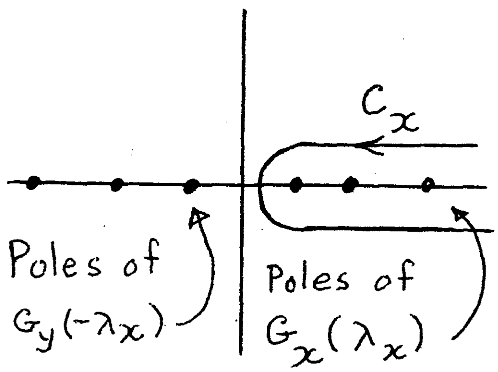


Fig.3.5 The Proper Contour C_x

$$G = -\frac{1}{2\pi j} \oint_{C_x} G_x(\lambda_x) G_y(-\lambda_x) d\lambda_x$$

The residue expansion† of this integral in terms of the residues at the poles of $G_x(\lambda_x)$ gives the following solution for G :

$$G = -\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{a} x_< \sin \frac{n\pi}{a} (a-x_>) \sin j \frac{n\pi}{a} y_< \sin j \frac{n\pi}{a} (b-y_>)}{j \frac{n\pi}{a} \frac{a}{2} \cos n\pi \sin j \frac{n\pi b}{a}}$$

$$= \sum_{n=1}^{\infty} \frac{2\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \sinh \frac{n\pi}{a} y_< \sinh \frac{n\pi}{a} (b-y_>)}{n\pi \sinh \frac{n\pi b}{a}}$$

which is the same as (3.21) .

---

†The residues may be found by evaluating the derivative of the denominator with respect to λ_x at the poles.

The contour C_x may be deformed into the contour C_y as shown in Fig. 3.6 . If this is done the integral may be evaluated in terms of the residues at the poles of $G_y(-\lambda_x)$ which occur at $\lambda_x = -(n\pi/b)^2$. This evaluation would give the same solution as (3.22) .

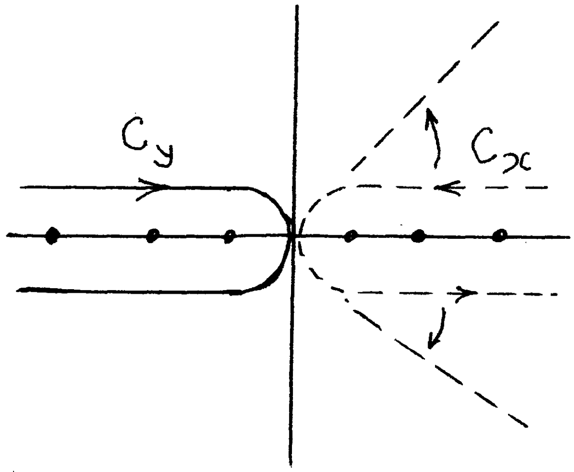


Fig. 3.6 Deformation of Contour C_x into C_y

If we construct the solution for G_y according to Method I we would obtain (see Eq. 3.11a)

$$G_y(y, y', \lambda_y) = -\sum_{m=1}^{\infty} \frac{\frac{2}{b}\sin\frac{m\pi y}{b}\sin\frac{m\pi y'}{b}}{\lambda_y - (m\pi/b)^2} \tag{3.25}$$

If we use (3.24a) for G_x and (3.25) for G_y in the formula (3.20) we would obtain the Green's function solution given by (3.23) provided that we integrate around a contour C_x enclosing the singularities of $G_x(x, x', \lambda_x)$. We leave the details as exercise 3.4 to be carried out by the reader.

3.5 Green's Function for Line Source in a Rectangular Waveguide

Figure 3.7 shows a uniform current filament directed along y , at x', z', in a rectangular waveguide. The current has a time dependence $e^{j\omega t}$ and is not a function of y . This current source will excite TE_{no} modes only. The vector potential is a solution of

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)\psi = -\mu_o I_g \delta(x-x')\delta(z-z') \tag{3.26}$$

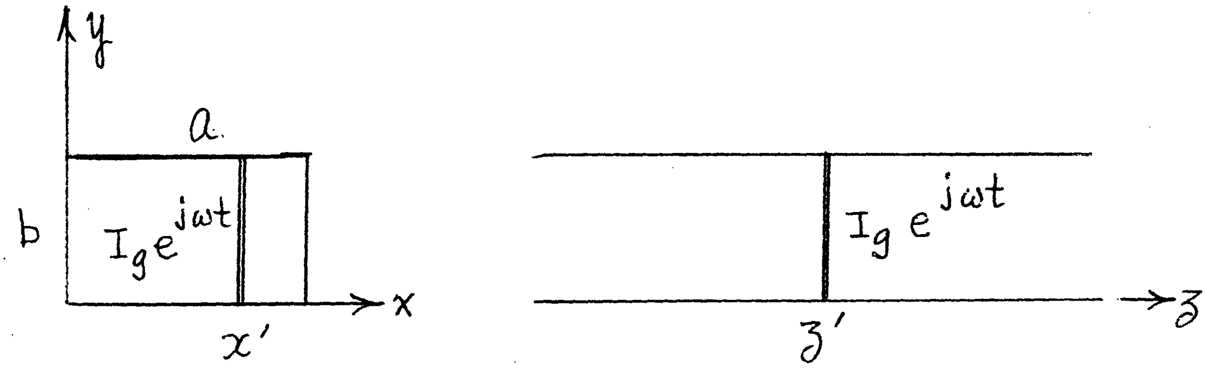


Fig. 3.7 A Line Source in a Rectangular Waveguide

with $\psi = 0$ at $x = 0, a$ and ψ representing outward propagating waves at $|z|$ approaching infinity. If we solve the following Green's function problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)G = -\delta(x-x')\delta(z-z') \tag{3.27}$$

$$G = 0 \ , \ x = 0, a$$

then $\psi = \mu_o I_g G$ and $E_y = -j\omega\mu_o I_g G$. In (3.26) and (3.27) $k_o^2 = \omega^2\mu_o\varepsilon_o = \omega^2/c^2$.

Consider now the homogeneous equation $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k_o^2\right)G = 0$. If we assume $G(x, x', z, z') = G_x(x, x')G_z(z, z')$ we obtain

$$\frac{1}{G_x}\frac{\partial^2 G_x}{\partial x^2} + \frac{1}{G_z}\frac{\partial^2 G_z}{\partial z^2} + k_o^2 = 0 \quad \text{so we must have}$$

$$\frac{\partial^2 G_x}{\partial x^2} + \lambda_x G_x = 0 \ , \quad \frac{\partial^2 G_z}{\partial z^2} + \lambda_z G_z = 0$$

and

$$\lambda_x + \lambda_z = k_o^2 \tag{3.28}$$

From these we can identify the associated one dimensional Green's function problems to be

$$\frac{\partial^2 G_x}{\partial x^2} + \lambda_x G_x = -\delta(x-x') \tag{3.29a}$$

$$G_x = a \ , \ x = 0, a$$

$$\frac{\partial^2 G_z}{\partial z^2} + \lambda_z G_z = -\delta(z-z') \tag{3.29b}$$

G_z is outward propagating wave as $|z| \to \infty$.

We may readily solve (3.29a) by Method I or II. If we use Method I we obtain (see 3.11a)

$$G_x = - \sum_{n=1}^{\infty} \frac{2}{a} \frac{\sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a}}{\lambda_x - n^2\pi^2/a^2}$$

Since the range in z is infinite we use Fourier transforms to solve (3.29b). We will define $\hat{G}_z(\beta)$ to be the transform of $G_z(z)$, i.e.

STRICTLY, ONE SHOULD BREAK THIS INTEGRAL INTO ONE IN THE UHP + ONE IN THE LHP.

HOWEVER, IT HAS BEEN IMPLICITLY ASSUMED HERE THAT λ HAS A SMALL IMAGINARY PART.

$$\hat{G}_z(\beta) = \int_{-\infty}^{\infty} G_z(z) e^{j\beta z}\, dz \tag{3.30a}$$

Then

$$G_z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}_z(\beta) e^{-j\beta z}\, d\beta \tag{3.30b}$$

If we take the Fourier transform of (3.29b) and note that the transform of $\frac{\partial^2 G_z}{\partial z^2}$ is $-\beta^2 \hat{G}_z$ (obtained through two integrations by parts) we find that

$$\hat{G}_z = - \frac{e^{j\beta z'}}{\lambda_z - \beta^2} = \frac{e^{j\beta z'}}{(\beta - \sqrt{\lambda_z})(\beta + \sqrt{\lambda_z})}$$

We wish to consider λ_z as a complex variable and so we will arbitrarily choose that branch of the two valued function $\sqrt{\lambda_z}$ which has $\text{Imag.}\sqrt{\lambda_z} < 0$. In the complex β plane $\hat{G}_z$ then has two poles located as shown in Fig. 3.8 . The solution for G_z is given by

$$G_z = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\beta(z-z')}\, d\beta}{(\beta - \sqrt{\lambda_z})(\beta + \sqrt{\lambda_z})}$$

When $z > z'$ the factor $e^{-j\beta(z-z')}$ becomes exponentially small (we write β as $\beta'+j\beta''$) in the lower half plane (proportional to $e^{\beta''(z-z')}$ with $\beta'' < 0$). Hence we can close the contour by a semi-circle of infinite radius in the lower half plane and evaluate the integral in terms of the residue at the pole at $\beta = \sqrt{\lambda_z}$. For $z < z'$ we can close the contour in the upper half plane and evaluate the integral in terms of the residue at $-\sqrt{\lambda_z}$. Note that there is no contribution to the integral from the semi-circles and that a negative sign enters when the contour is traversed in a clockwise sense. We are thus led to the following solution for G_z :

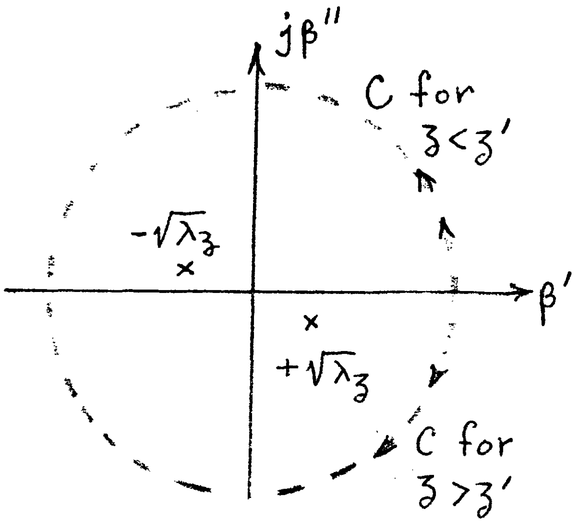


Fig. 3.8 Pole Locations in the Complex β Plane.

$$G_z = \begin{cases} -\dfrac{j}{2\sqrt{\lambda_z}} e^{-j\sqrt{\lambda_z}(z-z')} , & z > z' \\ -\dfrac{j}{2\sqrt{\lambda_z}} e^{j\sqrt{\lambda_z}(z-z')} , & z < z' \end{cases}$$

which may be also expressed as

$$G_z = -\frac{j}{2\sqrt{\lambda_z}} e^{-j\sqrt{\lambda_z}|z-z'|} , \text{ all } z \qquad (3.31)$$

Note that with $\text{Imag.}\sqrt{\lambda_z} < 0$ chosen as the branch of the function $\sqrt{\lambda_z}$ that G_z will tend toward zero as $|z| \to \infty$ for complex λ_z. Also note that G_z does not have any poles but instead has a branch point at $\lambda_z = 0$. If we wish to always remain on the branch for which $\sqrt{\lambda_z}$ has a negative imaginary part then the angle of λ_z in the complex plane must be restricted to -2π to 0. This may be accomplished by placing a branch line (or cut) along the positive real axis but an infinitesimal distance above it as shown in Fig. 3.9 . This forces us to measure the angle of λ_z from the positive real axis in a clockwise or negative sense and hence restricts the angle to the range 0 to -2π . The spectrum of G_z is the continuous range of values of λ_z along the branch cut (singular line, which if crossed changes the sign of $\sqrt{\lambda_z}$) . This is in accord with our earlier statement that for infinite range problems the Green's function has a continuous spectrum.

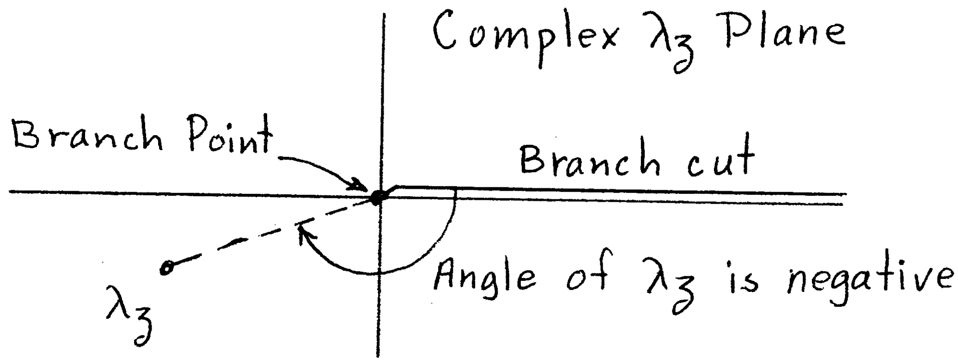


Fig. 3.9 Branch Cut for $G_z(z, z', \lambda_z)$

We now apply (3.20) to obtain G in the form

$$G(x, x', z, z') = -\frac{1}{2\pi j}\oint_{C_x} G_x(x, x', \lambda_x)G_z(z, z', k_o^2 - \lambda_x)d\lambda_x$$
$$= -\frac{1}{2\pi j}\oint_{C_z} G_x(x, x', k_o^2 - \lambda_z)G_z(z, z', \lambda_z)d\lambda_z \qquad (3.32)$$

Note that when we integrate over λ_x that we use the condition (3.28) to express λ_z in terms of λ_x and vice versa. In the λ_x plane the poles of G_x are at $\lambda_x = (n\pi/a)^2$ and the branch line for G_z becomes the line along which $\text{Imag.}\sqrt{k_o^2 - \lambda_x} = 0$. This is the line extending from $\lambda_x = k_o^2$ (branch point) along the negative axis. The contour C_x is chosen to enclose the poles but not the branch point at k_o^2 as shown in Fig. 3.10a . On the other hand if we choose to integrate over λ_z then the contour C_z is chosen to enclose the branch cut for G_z but such that the poles of G_x, which now occur when $k_o^2 - \lambda_z = (n\pi/a)^2$ or $\lambda_z = k_o^2 - (n\pi/a)^2$, are excluded (Fig. 3.10b). If only the TE_{10} mode propagates then $k_o > \pi/a$ but $k_o < n\pi/a$ for $n > 1$.

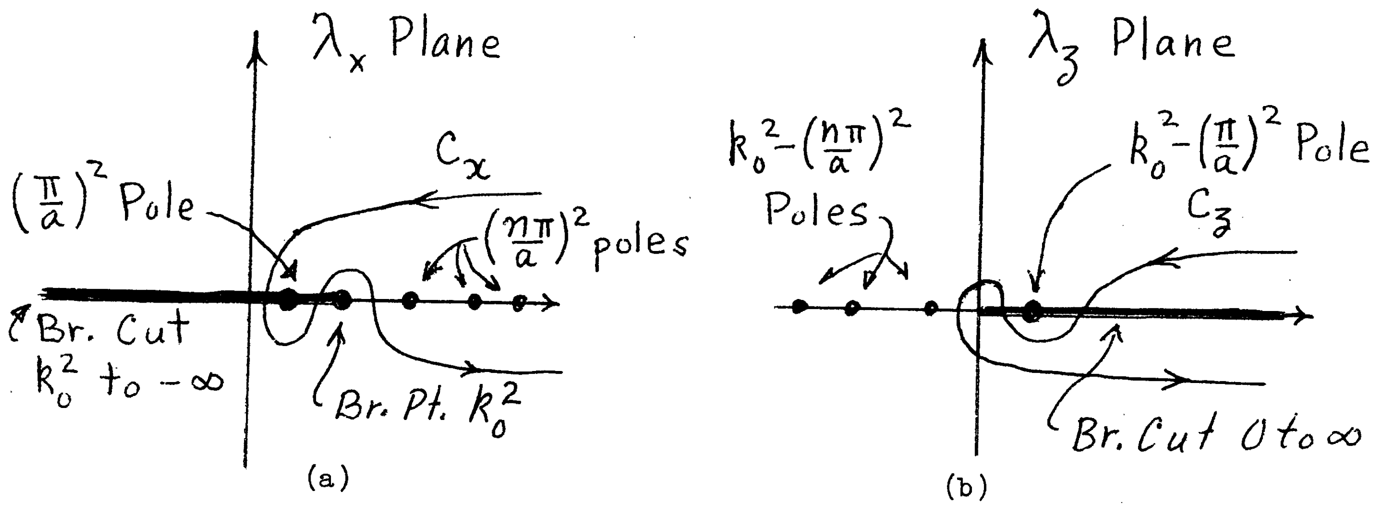


Fig. 3.10 Proper Choice of the Contours C_x and C_z

Either of the two prescriptions given by (3.32) may be used for determining G. If we use the first one then the integral may be evaluated in terms of the residues at the poles. We have

$$G = -\frac{1}{2\pi j}\oint_{C_x} \frac{j\, e^{-j\sqrt{k_o^2-\lambda_x}\,|z-z'|}}{a\sqrt{k_o^2-\lambda_x}} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{\lambda_x - (n\pi/a)^2}\, d\lambda_x$$

$$= \frac{1}{a}\sum_{n=1}^{\infty}\frac{1}{\Gamma_n}\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}\, e^{-\Gamma_n|z-z'|} \qquad (3.33)$$

where $\Gamma_n = \sqrt{(\frac{n\pi}{a})^2 - k_o^2}$ and Imag. $\Gamma_n > 0$.

If $k_o < \frac{n\pi}{a}$ for $n > 1$ then the solution consists of the propagating dominant mode for which $\Gamma_1 = j\sqrt{k_o^2 - \pi^2/a^2}$ plus an infinite number of evanescent or non-propagating TE_{no} modes. These modes exist on both sides of the source point at z' . If $x' = a/2$ then only the modes for $n = 1, 3, 5, \ldots$ etc. are excited.

If the prescription involving the contour C_z in (3.32) is used then the solution is expressed simply as a branch cut integral which may be interpreted as a continuous sum (integral) over the continuous spectrum of G_z in place of the sum over the discrete spectrum of G_x as exhibited in (3.33) .

We could have equally well chosen the branch $\text{Imag.}\sqrt{\lambda_z} > 0$ and would have arrived at the same solution. The reader may find it of interest to make this choice and to show that (3.33) is still obtained for the final result.

The branch point singularity and the considerations that enter into the choice of branch cut is discussed more completely in Appendix A of this chapter.

3.6 Three Dimensional Green's Functions

The theory presented in the previous sections may be applied to three dimensional problems also. We will illustrate the case of the scalar Helmholtz equation in rectangular coordinates. We have

$$(\nabla^2 + k_o^2)G = -\delta(x-x')\delta(y-y')\delta(z-z') \tag{3.34}$$

The equation $(\nabla^2 + k_o^2)\psi = 0$ separates into three one dimensional equations of the form $\frac{\partial^2\psi_x}{\partial x^2} + \lambda_x\psi_x = 0$, $\frac{\partial^2\psi_y}{\partial y^2} + \lambda_y\psi_y = 0$, $\frac{\partial^2\psi_z}{\partial z^2} + \lambda_z\psi_z = 0$ with

$$\lambda_x + \lambda_y + \lambda_z = k_o^2 \tag{3.35}$$

Consequently the associated one dimensional Green's function problems are

$$\frac{d^2G_x}{dx^2} + \lambda_x G_x = -\delta(x-x') \tag{3.36a}$$

$$\frac{d^2G_y}{dy^2} + \lambda_y G_y = -\delta(y-y') \tag{3.36b}$$

$$\frac{d^2G_z}{dz^2} + \lambda_z G_z = -\delta(z-z') \tag{3.36c}$$

along with appropriate boundary conditions.

When (3.36a) - (3.36c) have been solved then the solution to (3.34) is given by

$$G(x, y, z, x', y', z') = \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_x} \oint_{C_z} G_x(\lambda_x)G_y(\lambda_y)$$

$$G_z(k_o^2 - \lambda_x - \lambda_y)d\lambda_x d\lambda_y = \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_x} \oint_{C_z} G_x(\lambda_x)$$

$$G_y(k_o^2 - \lambda_x - \lambda_z)G_z(\lambda_z)d\lambda_x d\lambda_z$$

$$= \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_y} \oint_{C_z} G_x(k_o^2 - \lambda_y - \lambda_z)G_y(\lambda_y)G_z(\lambda_z)d\lambda_y d\lambda_z \tag{3.37}$$

Note that there is an integration over two of the λ_i and that the third λ is expressed in terms of the two being integrated over by means of the condition (3.35) on the separation constants. The contours C_i enclose only the singularities of the G_i with $i = x, y, z$. The proof is readily developed by noting that

$$(\nabla^2 + k_o^2)G = \left[\left(\frac{\partial^2}{\partial x^2} + \lambda_x\right) + \left(\frac{\partial^2}{\partial y^2} + \lambda_y\right) + \left(\frac{\partial^2}{\partial z^2} + k_o^2 - \lambda_x - \lambda_y\right)\right]G$$

$$= \left(\frac{-1}{2\pi j}\right)^2 \oint_{C_x} \oint_{C_y} [-G_y(\lambda_y)G_z(k_o^2 - \lambda_x - \lambda_y)\delta(x-x')$$

$$-G_x(\lambda_x)G_z(k_o^2 - \lambda_x - \lambda_y)\delta(y-y') - G_x(\lambda_x)G_y(\lambda_y)\delta(z-z')]d\lambda_x d\lambda_y \ .$$

The integral around C_x of the first term vanishes while the integral around C_y of the second term vanishes because no singularities of G_z are included. The integral of the third term gives $-\delta(x-x')\delta(y-y')\delta(z-z')$ by application of Theorem 1 twice.

The above technique is applicable in other coordinate systems also. Some applications are given in the papers by Marcuvitz and Felsen cited at the end of this chapter.

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APPENDIX A

In this section we will discuss the nature of the branch point singularity that occurs in Sec. 3.5 in greater deal. The Green's function of interest is given by (3.33) as

$$G = -\frac{1}{2\pi j}\oint_{C_x} \frac{j\, e^{-j\sqrt{k_o^2-\lambda_x}\,|z-z'|}}{a\sqrt{k_o^2-\lambda_x}} \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{a}\,\sin\frac{n\pi x'}{a}}{\lambda_x-(n\pi/a)^2}\, d\lambda_x$$

where C_x is to be chosen as a closed contour that encloses the poles of G_x at $\lambda_x = (n\pi/a)^2$, $n = 1,2,\ldots$ and excludes the singularities of G_z . The function G_z has branch point at $\lambda_x = k_o^2$ because of the factors $\sqrt{k_o^2 - \lambda_x}$ which are present.

To test if a point is a branch point we consider how $\sqrt{k_o^2 - \lambda_x}$ varies on a circle centered on k_o^2 . If when we make one revolution around k_o^2 the function $\sqrt{k_o^2 - \lambda_x}$ does not return to its original value then k_o^2 is a branch point. With reference to Fig. 1 we note that $k_o^2 - \lambda_x$ is the directed line segment shown.

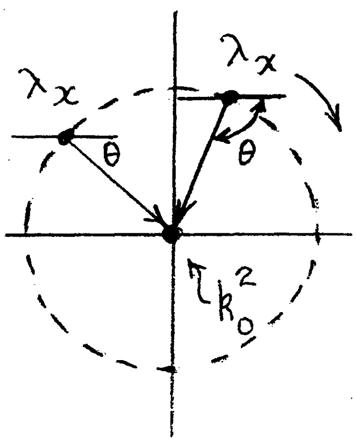


Fig. 1

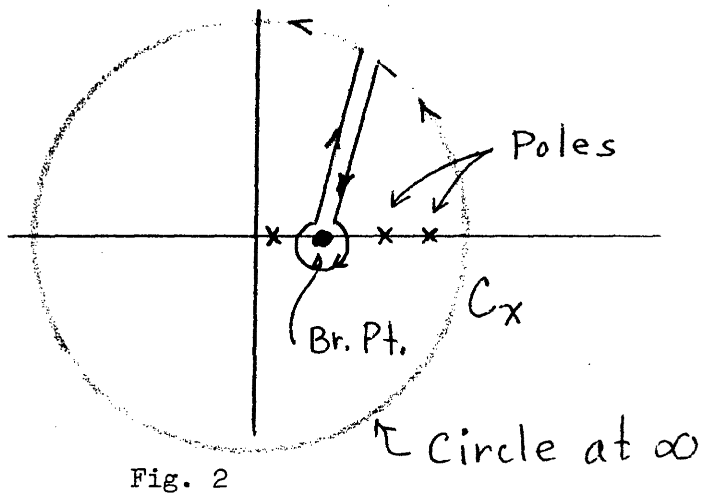


Fig. 2

When λ_x is on the real axis and to the left of k_o^2 the phase angle of $k_o^2 - \lambda_x$ is zero. As we move λ_x clockwise on a circle about k_o^2 the phase angle of $k_o^2 - \lambda_x$ becomes negative. After one revolution $k_o^2 - \lambda_x = \rho e^{j\theta}$ becomes $\rho e^{-2\pi j}$ and $\sqrt{k_o^2 - \lambda_x}$ becomes $\sqrt{\rho}\, e^{-\pi j} = -\sqrt{\rho}$. Thus $\sqrt{k_o^2 - \lambda_x}$ does not return to its original value $\sqrt{\rho}$ and hence k_o^2 is a branch point. The contour C_x can be any closed contour that encloses the poles of G_x and excludes the branch point k_o^2 . It may be chosen as in Fig. 2 if so desired. The contour C_x may be distorted in any arbitrary way, as long as it is not moved across a singular point or pole, without changing the value of the integral. Thus the contours shown in Fig. 3 are fully equivalent to that in Fig. 2 .

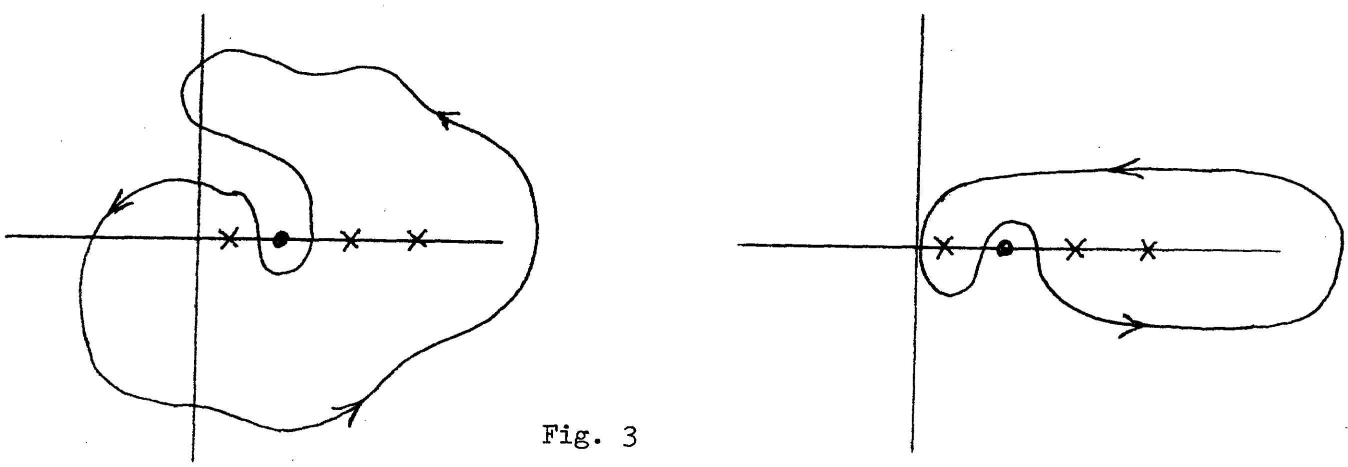


Fig. 3

If we are not going to perform the integration directly but will evaluate the integral in terms of the residues at the enclosed poles the choice of contour at this stage in the analysis is immaterial.

We now consider the possibility of evaluating the integral directly. The contour shown in Fig. 2 would then be a useful one if it should turn out that the contribution to the integral over the circle at infinity was zero. We would then only need to consider the integral along the two radial lines from the branch point to infinity and along the small circle about the branch point. The integral over the circle at infinity will vanish provided the factor $e^{-j\sqrt{k_o^2-\lambda_x}|z-z'|}$ becomes exponentially small at infinity. This will be the case if Imag $\sqrt{k_o^2-\lambda_x}$ is negative, say $\sqrt{k_o^2-\lambda_x} = -j\beta + \alpha$, $\beta > 0$, for then $e^{-j\sqrt{k_o^2-\lambda_x}|z-z'|}$ will decay like $e^{-\beta|z-z'|}$. To ensure that Imag. $\sqrt{k_o^2-\lambda_x} < 0$ we must restrict the phase angle of $\sqrt{k_o^2-\lambda_x}$ to the range -2π to 0 for all points on the circle at infinity. This requires that we choose the contour shown in Fig. 4 as reference to Fig. 1 and the earlier discussion shows. Thus we imagine that we have placed a barrier or branch cut running from the branch point at k_o^2 out to infinity along the negative real axis. The Greens function may then be evaluated as an

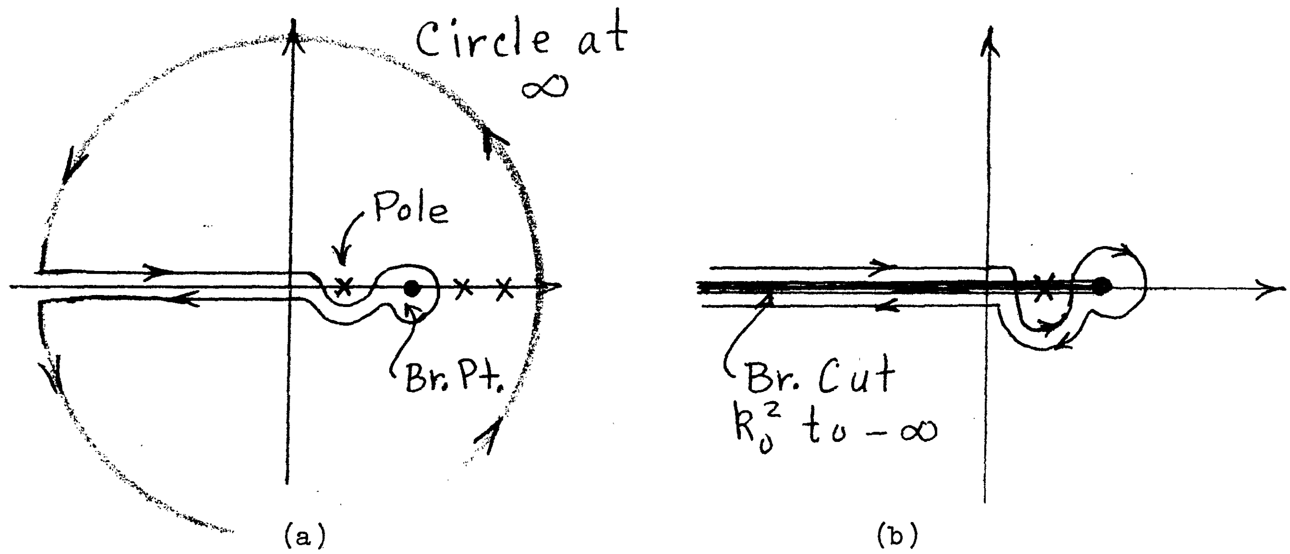


Fig. 4

integral along the contour shown in Fig. 4b . If we change to the variable $\lambda_z = k_o^2-\lambda_x$ then this becomes the contour shown in Fig. 3.10b . If the branch cut was chosen differently the integral over the circle at infinity would not vanish and since C_z in (3.32) must be a closed contour, and it can not be closed across the branch cut, the resultant contour of integration, C_z , would include the circle at infinity. Thus it is only by a proper choice of branch cut that C_z can be chosen as in Fig. 3.10b .

APPENDIX B

NON-SELF ADJOINT SYSTEM

Let L denote a differential operator and consider the equation

$$L\psi + \lambda\psi = f(x) \tag{1}$$

The variable is x where $0 \leq x \leq a$. The domain $\mathcal{D}$ of the operator L is the domain of functions ψ that satisfy certain continuity conditions and specified boundary conditions. We can define an inner product in more than one way - for example

$$< \phi , \psi > = \int_0^a \phi(x)\psi(x)dx \tag{2a}$$

or

$$< \phi , \psi > = \int_0^a \phi(x)\psi(x)\sigma(x)dx \tag{2b}$$

or

$$< \phi , \psi > = \int_0^a \phi(x)\psi^*(x)dx \tag{2c}$$

etc. The definition (2c) is used in quantum mechanics (Hilbert space) while (2b) is used if Eq. (1) has the function σ as a factor with λ .

Once the inner product has been defined the adjoint operator L_a is defined to be that operator which satisfies

$$< \phi, L\psi > = < L_a\phi , \psi > \tag{3}$$

If $L = L_a$ and the domain $\mathcal{D}_a$ of L_a coincides with the domain $\mathcal{D}$ of L then L is called a self-adjoint operator. Sometimes $\mathcal{D}$ and $\mathcal{D}_a$ differ even though $L = L_a$ and in this case L is only formally self-adjoint. In practice L_a is determined by integration by parts to transfer differentiations on ψ to differentiations on ϕ .

Example

$$\frac{d^2\psi}{dx^2} + \lambda\psi = f \ , \quad \psi + j\,\frac{d\psi}{dx} = 0, \ x = 0, \ a \ ,$$

ψ may be expressed in terms of the eigenfunctions of $\frac{d^2\psi}{dx^2} + \lambda\psi = 0$. It is easily verified that $\psi_n = \sin\frac{n\pi x}{a} - j\,\frac{n\pi x}{a}\cos\frac{n\pi x}{a}$. If we choose (2a) for the inner product then

$$\int_0^a \psi_n\psi_m dx = 0, \quad n \neq m \ .$$

The domain of L is the domain of functions which satisfy $\psi + j\,\frac{d\psi}{dx} = 0$ at $x = 0, a$.

With (2a) as the inner product we have

$$< \phi \ , L\,\psi > = \int_0^a \phi\,\frac{d^2\psi}{dx^2}\,dx = \phi\,\frac{d\psi}{dx}\Big|_0^a - \int_0^a \frac{d\phi}{dx}\,\frac{d\psi}{dx}\,dx$$

$$= (\phi\,\frac{d\psi}{dx} - \frac{d\phi}{dx}\,\psi)\Big|_0^a + \int_0^a \psi\,\frac{d^2\phi}{dx^2}\,dx = < L_a\phi, \psi >$$

The integrated terms vanish if $\phi + j\,\frac{d\phi}{dx} = 0$ at $x = 0, a$. Hence $L_a = L$, since $\mathcal{D} = \mathcal{D}_a$ also, the operator is self-adjoint.

However, if we choose (2c) as the inner product then

$$< \phi, L\,\psi > = (\phi\,\frac{d\psi^*}{dx} - \frac{d\phi}{dx}\,\psi^*)\Big|_0^a + \int_0^a \psi^*\,\frac{d^2\phi}{dx^2}\,dx$$

The boundary terms can be written as

$$[(\phi - j\,\frac{d\phi}{dx})\,\frac{d\psi^*}{dx} - (\psi^* - j\,\frac{d\psi^*}{dx})\,\frac{d\phi}{dx}]_0^a$$

and vanish if $\phi - j\,\frac{d\phi}{dx} = 0$ at $x = 0, a$. In this case $L_a = L$ but $\mathcal{D}_a \neq \mathcal{D}$ so the operator is not self-adjoint. Note that the adjointness properties of an operator is dependent on the choice of inner product. When an operator is not self-adjoint the eigenfunctions are not orthogonal with respect to the inner product used.

With the scalar product (2a) L was self-adjoint and $\int_0^a \psi_n \psi_m dx = 0$, $n \neq m$. With the scalar product (2c) L is not self-adjoint and this implies $\int_0^a \psi_n \psi_m^* dx \neq 0$ for $n \neq m$. In a non-self adjoint system the appropriate orthogonality principle is instead

$$\int_0^a \phi_n \psi_m^* \, dx = 0 \text{ , } n \neq m \text{ .} \tag{3}$$

We may show this as follows: Let ϕ_n, μ_n be the eigenfunctions and eigenvalues of L_a , then $L_a \phi_n + \mu_n \phi_n = 0$. Since $L\psi_m + \lambda_m \psi_m = 0$ also we have $< \phi_n, L\psi_m > = -\lambda_m^* < \phi_n, \psi_m > = < L_a \phi_n, \psi_m > = -\mu_n < \phi_n, \psi_m >$ if (2c) is used for the scalar product (if (2a) is used replace λ_m^* by λ_m) . If $\lambda_m^* \neq \mu_n$ then $< \phi_n, \psi_m > = 0$.

The eigenvalues of the adjoint operator are related to those of the original operator. We have $< (L_a + \mu_n)\phi_n, \psi > = 0$ for all ψ in $\mathcal{D}$. But this also equals $< L_a \phi_n, \psi > + \mu_n < \phi_n, \psi > = <\phi_n, L\psi > + < \phi_n, \mu_n^* \psi > = <\phi_n, (L+\mu_n^*)\psi > = 0$. Since ϕ_n is not zero $(L+\mu_n^*)\psi = 0$ so μ_n^* is an eigenvalue of L when (2c) is used for the scalar product. When (2a) is used then μ_n is an eigenvalue of both L_a and L . With proper indexing $\mu_n = \lambda_n^*$ (or $\mu_n = \lambda_n$) .

In our example if we use (2c) for the inner product then $\phi_n = \psi_n^*$ and the orthogonality relation is $< \psi_n, \phi_m > = 0$, $n \neq m$. But this equals $\int_0^a \psi_n \phi_m^* dx = \int_0^a \psi_n \psi_m dx = 0$ in agreement with the results obtained if (2a) is used for the inner product.

For the Sturm-Liouville equation we choose $L = \frac{1}{\sigma} \frac{d}{dx} p \frac{d}{dx} + \frac{q}{\sigma}$ so the equation becomes $L\psi + \lambda\psi = \frac{f}{\sigma}$. We now use (2b) for the inner product. For $< \phi, L\,\psi >$ we get

$$\int_0^a \sigma\phi[\frac{1}{\sigma} \frac{d}{dx} p \frac{d\psi}{dx} + \frac{q}{\sigma} \psi]dx = [p\phi \frac{d\psi}{dx} - p\psi \frac{d\phi}{dx}]_0^a + \int_0^a \sigma\psi[\frac{1}{\sigma} \frac{d}{dx} p \frac{d\phi}{dx} + \frac{q}{\sigma} \phi]dx$$

upon integrating by parts twice. If $K_1\psi + K_2 d\psi/dx = 0$ on the boundary then the integrated terms vanish if $K_1\phi + K_2 d\phi/dx = 0$ on the boundary. Hence $L = L_a$ and $\mathcal{D} = \mathcal{D}_a$ so the Sturm-Liouville equation is self-adjoint.

Most people, outside the field of quantum mechanics, use (2a) for the inner product. We can use this for the Sturm-Liouville equation also if we choose $\frac{d}{dx} p \frac{d}{dx} + q$ as the operator L . The orthogonality condition is then expressed as $< \phi_n, \sigma\phi_m > = 0$, $n \neq m$. In the next section on Green's function we will follow this convention.

Green's Function

Consider the Green's function for a self-adjoint system with

$$LG(x, x') + \lambda G = -\delta(x-x') \tag{4}$$

We then have $< G(x, x_1'), LG(x, x_2') > - < G(x, x_2'), LG(x, x_1') >$

$$+ \lambda < G(x, x_1'), G(x, x_2') > - \lambda < G(x, x_2'), G(x, x_1') >$$

$$= - < G(x, x_1'), \delta(x-x_2') > + < G(x, x_2'), \delta(x-x_1') >$$

But $< G(x,x_1'), LG(x, x_2') > = < LG(x, x_1'), G(x, x_2') >$ for a self-adjoint system so the left hand side vanishes. Thus $< G(x, x_1'), \delta(x-x_2') > = < G(x, x_2'), \delta(x-x_1') >$

$$\text{or} \quad G(x_2', x_1') = G(x_1', x_2') \tag{5}$$

i.e. the Greens's function is symmetrical in x_1', x_2' .

For a non-self adjoint system we consider (4) along with

$$L_a G_a(x, x') + \lambda G_a = -\delta(x-x') \tag{6}$$

We can now write $< G(x, x_1'), L_a G_a(x, x_2') > + \lambda < G, G_a > - < G_a(x, x_2'), LG(x, x_1') >$

$$- \lambda < G_a, G > = - < G(x, x_1'), \delta(x-x_2') > + < G_a(x, x_2'), \delta(x-x_1') > = 0$$

since $< G, L_a G_a > = < LG, G_a > = < G_a, LG >$.

Hence

$$G(x_2', x_1') = G_a(x_1', x_2') \tag{7}$$

For a non-self adjoint system the Green's function is not symmetric.

Consider now a linear system

$$L\,\psi + \lambda\,\psi = f \tag{8}$$

If we solve (4) and make G satisfy the same boundary conditions as ψ does then by superposition the solution for ψ is

$$\psi(x) = -\int_0^a G(x, x')f(x')dx' \tag{9}$$

since $G(x, x')$ is the field at x due to a unit source at x'. From (7) we see that this solution can also be written as

$$\psi(x) = -\int_0^a G_a(x', x)f(x')dx' \tag{10}$$

If ψ_n, λ_n are the eigenfunctions and eigenvalues of L, and ϕ_n, λ_n, those of L_a, then

$$G(x, x') = -\sum_n \frac{\psi_n(x)\phi_n(x')}{\lambda - \lambda_n} \tag{11a}$$

and

$$G_a(x, x') = -\sum_n \frac{\psi_n(x')\phi_n(x)}{\lambda - \lambda_n} \tag{11b}$$

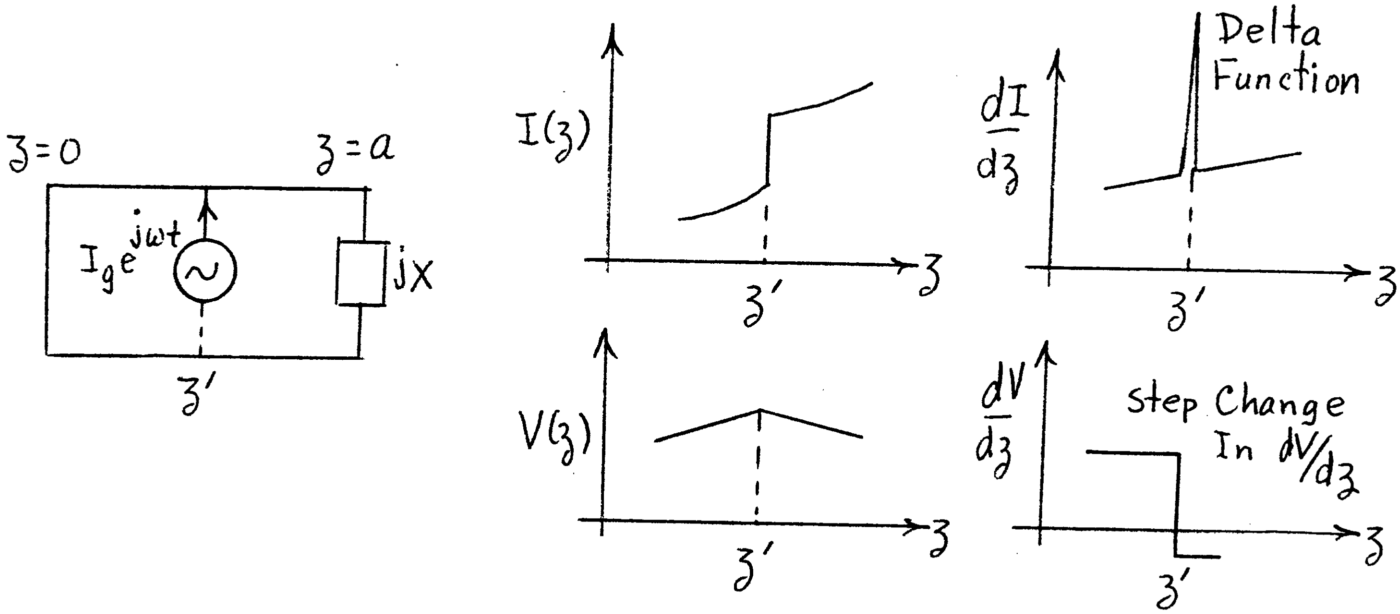
PROBLEMS

3.1 Make a Fourier series expansion of (3.11b) in terms of the eigenfunctions $\sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$ and show that the result (3.11a) is obtained.

3.2 Consider the equation $\frac{d^2\psi}{dx^2} + \lambda\psi = 0$ with boundary conditions $\psi + \frac{2d\psi}{dx} = 0$ at $x = 0$ and $\psi = 0$ at $x = a$. Show that the eigenvalues are determined by the transcendental equation $\tan \sqrt{\lambda}\, a = 2\sqrt{\lambda}$ and that the eigenfunctions (unnormalized) are $\sin\sqrt{\lambda_n}\,x - 2\sqrt{\lambda_n}\cos\sqrt{\lambda_n}\,x$, with λ_n the n'th eigenvalue.

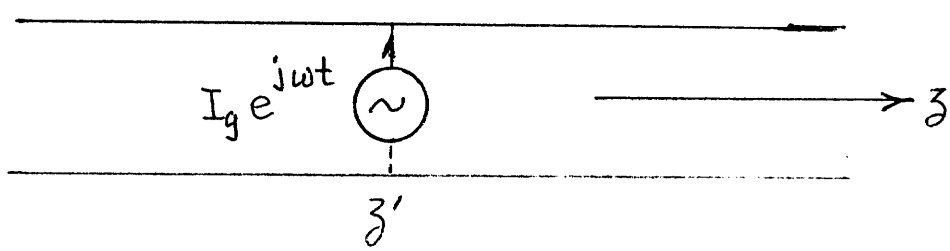
3.3 Consider the lossless transmission line circuit illustrated in the Figure. The line has inductance L and capacitance C per meter. At the end $z = a$ the line is terminated in a reactance jX while at $z = 0$ it is short-circuited. At $z = z'$ the line is excited by a time harmonic current generator with output $I_g e^{j\omega t}$. Because of the current source at z' the current on the line undergoes a step change of amount I_g at z' as shown in the Figure. Consequently $\frac{\partial I}{\partial z}$ has a delta function change $I_g \delta(z-z')$ at z' . The voltage is continuous at z' but its slope, proportional to I , has a step change as shown. The equations describing the line are $\frac{\partial V}{\partial z} = -j\omega LI$, $\frac{\partial I}{\partial z} = -j\omega CV + I_g\delta(z-z')$. Show that $\frac{\partial^2 V}{\partial z^2} + \omega^2 LCV = -j\omega L\, I_g \delta(z-z')$. Find two solutions for V (Methods I and II) . Note that at $z = 0$, $V = 0$ and at $z = a$, $V = j\,X\,I = -\frac{X}{\omega L}\frac{\partial V}{\partial z}$ or $V + \frac{X}{\omega L}\frac{\partial V}{\partial z} = 0$.

Note that the poles determine the resonant frequencies.

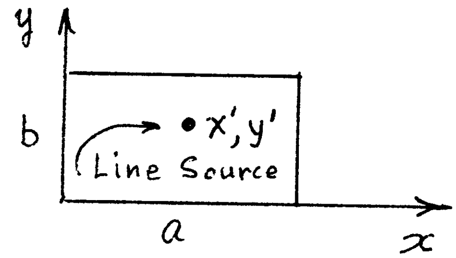


3.4 Use (3.24a) and (3.25) in (3.20) and integrate around a contour C_x enclosing the poles of G_x and show that the solution (3.23) is obtained for G.

3.5 Consider an infinitely long transmission line excited by a current source $I_g\, e^{j\omega t}$ at z'. Solve $\frac{d^2V}{dz^2} + k_o^2 V = -j\omega L\, I_g \delta(z-z')$ by means of a Fourier transform. Hint: assume that there is a small shunt conductance so that $k_o^2 = \omega^2 LC(1 - jG/\omega C)$ and $k_o = k_o' - jk_o''$. This will displace the poles away from the real axis. Note that for $z > z'$ the inversion contour can be closed in the lower half plane and for $z < z'$ it can be closed in the upper half plane. Thus the inverse transform can be evaluated in terms of residues.



3.6 Consider an axial line source inside a rectangular pipe as shown. Find the Green's function which satisfies:



$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -\delta(x-x')\delta(y-y')$$

$$G = 0 \quad \text{on boundary,}$$

in terms of a double Fourier series of eigenfunctions for the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda_{nm}\right)\psi_{nm} = 0 \ , \quad \psi_{nm} = 0 \quad \text{on boundary} \ .$$

3.7 Find G for Problem 3.6 in terms of an eigenfunction expansion along x but as a closed form in the y variable, i.e., in the form

$$G = \sum_{n=1}^{\infty} f_n(y) \sin \frac{n\pi x}{a}$$

3.8 Find the Green's function for Problem3.6 in terms of a contour integral of $G_x(\lambda_x)G_y(\lambda_y)$ where

$$\frac{\partial^2 G_x}{\partial x^2} + \lambda_x G_x = -\delta(x-x')$$

$$\frac{\partial^2 G_y}{\partial y^2} + \lambda_y G_y = -\delta(y-y')$$

Verify your result with that obtained in Problems 3.6 and 3.7 .

3.9 Determine the Green's function of the first kind $(G = 0$ at $r = a)$ for Poisson's equation for a line source parallel and outside a conducting cylinder of radius a . Source is at $r' > a$. Use (a) method of images, (b) expansion in a suitable set of eigenfunctions.

Chapter 4

Two Dimensional Scattering and Diffraction

Electromagnetic scattering and diffraction involving fields and obstacle cross sections that are independent of one coordinate, say z, can be formulated as scalar problems. For three dimensional problems the vector nature of the field must be taken into account and this complicates the analysis.

In this chapter we will consider a representative number of two dimensional scattering and diffraction problems such as scattering by a conducting cylinder, diffraction of a plane wave by a half plane, radiation from a pair of parallel plates excited by a TEM wave, and some problems related to surface waves. In the course of the analysis a number of mathematical techniques such as Fourier transforms in the complex plane, Wiener-Hopf method, and the method of Steepest Descent (Saddle Point method) for the asymptotic evaluation of integrals, will be introduced.

4.1 Radiation From A Line Source

Consider an infinite line source directed in the z direction and given by $\delta(x)\delta(y) = \delta(r)/2\pi r$. The Green's function satisfies

$$(\nabla^2 + k_0^2) G = -\delta(x)\delta(y) = -\frac{\delta(r)}{2\pi r} \tag{4.1}$$

The normalization factor $2\pi r$ is needed to make the source strength equal to unity, i.e.

$$\int_0^{2\pi}\int_0^{r} \frac{\delta(r)}{2\pi r}\, r\, d\theta\, dr = 1$$

If the source is a unit current source then the vector potential is given by $A_z = \mu_0 G$ and the electric field by $E_z = -j\omega A_z = -j\omega\mu_0 G$, $E_r = E_\theta = 0$. In cylindrical coordinates we have $\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial G}{\partial r} + k_0^2 G = -\frac{\delta(r)}{2\pi r}$. For $r \neq 0$, G satisfies Bessel's equation and the two independent solutions are $J_0(k_0 r)$ and $Y_0(k_0 r)$. As $r \to \infty$ we have $J_0 \sim \sqrt{\frac{2}{\pi k_0 r}} \cos(k_0 r - \frac{\pi}{4})$, $Y_0 \sim \sqrt{\frac{2}{\pi k_0 r}} \sin(k_0 r - \pi/4)$.

If the solution is to be an outgoing wave at infinity we must choose $G = C(J_0 - jY_0) = CH_0^2$ where $H_0^2(k_0 r)$ is the Hankel function of the second kind and order zero. Thus as $r \to \infty$ $G \sim C\sqrt{\frac{2}{\pi k_0 r}}\, e^{-jk_0 r + j\frac{\pi}{4}}$. The boundary condition on G at infinity may be stated as

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial G}{\partial r} + jk_0 G\right) = 0 \tag{4.2}$$

which is the Sommerfeld radiation condition and is equivalent to stating that G must be an outward propagating wave at infinity. The constant C must be found so that G has the proper singularity at $r = 0$. For $r \to 0$, $J_0(k_0 r) \to 1$ and $Y_0(k_0 r) \to -\frac{2}{\pi}\ln\frac{2}{\gamma k_0 r} \to \frac{2}{\pi}\ln r$. Hence $G \to -jC\frac{2}{\pi}\ln r$ as $r \to 0$. We have $\nabla^2 G = \nabla\cdot\nabla G$ and consequently

$$\int_0^{2\pi}\int_0^a \nabla\cdot\nabla G\, r\,d\theta\,dr + k_0^2\int_0^{2\pi}\int_0^a G\,r\,d\theta\,dr = -\int_0^{2\pi}\int_0^a \frac{\delta(r)}{2\pi r}\,r\,d\theta\,dr = -1$$

in the limit as $a \to 0$, $\int_0^{2\pi}\int_0^a G\,r\,d\theta\,dr = 0$ so $\int_0^{2\pi}\int_0^a \nabla\cdot\nabla G\,r\,d\theta\,dr = -1$

By using the two dimensional form of the divergence theorem we get

$$\iint_S \nabla\cdot\nabla G\,dS = \oint_C \frac{\partial G}{\partial n}\,dl = \int_0^{2\pi}\frac{\partial G}{\partial r}\,r\,d\theta$$

where C is the circle bounding the area S. But on C, $r\frac{\partial G}{\partial r}$ is constant and hence $2\pi r\frac{\partial G}{\partial r} = -1$. Integrating gives $G = -\frac{1}{2\pi}\ln r$. By comparing this result with our solution shows that $C = -j/4$, thus

$$G = -\frac{j}{4}\,H_0^2(k_0 r) \tag{4.3}$$

The argument of H_0^2 is $k_0 r$ where r is the radial distance away from the line source. Thus if the line source is located at $\vec{r}'$ and the field point at $\vec{r}$ the Green's function will be given by

$$G = -\frac{j}{4}\,H_0^2(k_0|\vec{r}-\vec{r}'|) = -\frac{j}{4}\,H_0^2(k_0 R)$$

since $R = |\vec{r}-\vec{r}'|$ is the distance from the source point to the field point.

It will be instructive to solve the problem in rectangular coordinates also since this will lead to an integral representation for $H_0^2(k_0 r)$ and also permit the asymptotic form of H_0^2 for large $k_0 r$ to be found. The analysis is facilitated if we assume a small loss in the medium, e.g. $\epsilon_0 = \epsilon_0' - j\epsilon_0''$, so that $k_0 = k_0' - jk_0''$ with $k_0'' \ll k_0'$. At the end of the analysis we can set $k_0'' = 0$. Our basic equation is

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} + k_0^2 G = -\delta(x)\delta(y)$$

Let

$$\mathcal{G}(w,y) = \int_{-\infty}^{\infty} e^{jwx}\,G(x,y)\,dx \tag{4.4}$$

which is the Fourier transform of G with respect to x. Since $G \to e^{-(jk_0'+k_0'')|x|}$ as $|x| \to \infty$, y finite, the integral converges uniformly in the strip $-k_0'' <$ Imag. $w < k_0''$. The Fourier transform of the differential equatiob yields $\frac{\partial^2 \mathcal{G}}{\partial y^2} + (k_0^2 - w^2)\mathcal{G} = -\delta(y)$

since the terms integrated by parts vanish at infinity. For $\mathcal{G}$ assume $\mathcal{G} = A_1 e^{-\sqrt{w^2-k_0^2}\, y}$, $y \geq 0$; $\mathcal{G} = A_2 e^{\sqrt{w^2-k_0^2}\, y}$, $y \leq 0$. Continuity at $y = 0$ gives $A_1 = A_2 = A$ while the discontinuity condition on the first derivative gives $\left.\frac{\partial \mathcal{G}}{\partial y}\right|_{0_-}^{0_+} = -1 = -2A\sqrt{w^2-k_0^2}$.

Thus

$$\mathcal{G} = \frac{e^{-\sqrt{w^2-k_0^2}\,|y|}}{2\sqrt{w^2-k_0^2}} \qquad (4.5)$$

and

$$G = \frac{1}{2\pi}\int_C \frac{e^{-jwx-\sqrt{w^2-k_0^2}\,|y|}}{2\sqrt{w^2-k_0^2}}\, dw \qquad (4.6)$$

where C is a contour parallel to the u axis in the strip in which $\mathcal{G}$ is analytic as shown in Fig. 4-2. The integrand for G is a two-valued function of w because of the factor $\sqrt{w^2-k_0^2}$ which occurs.

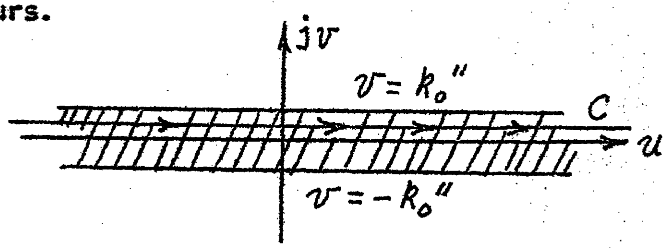


Fig. 4-2. Inversion contour for the Green's function

We require the solution to represent outgoing waves at infinity and this dictates the correct branch of the function to use. We must choose that branch for which $Re\sqrt{w^2-k_0^2} \geq 0$. This is the branch $+\sqrt{w^2-k_0^2}$. The two branches of $\sqrt{w^2-k_0^2}$, i.e. $+\sqrt{w^2-k_0^2}$ and $-\sqrt{w^2-k_0^2}$ may be conveniently considered as being defined on a two-sheeted Riemann surface. In the w plane we can construct cuts by which we can pass from one sheet of the Riemann surface to the other. These cuts must lie along the curves $Re\sqrt{w^2-k_0^2} = 0$ since on one sheet $Re\sqrt{w^2-k_0^2} \geq 0$ while on the other sheet $Re\sqrt{w^2-k_0^2} \leq 0$. Now $w^2 - k_0^2 = u^2 - v^2 - k_0'^2 + k_0''^2 + 2juv + 2jk_0' k_0''$. Thus $Re\sqrt{w^2-k_0^2} = 0$ is given by $uv = -k_0' k_0''$ for when this holds $u^2 - k_0'^2 \leq 0$, $-v^2 + k_0''^2 \leq 0$ so $(u^2 - v^2 - k_0'^2 + k_0''^2)^{1/2}$ is pure imaginary. These hyperbolas are plotted in Fig. 4-3. The curves run from the branch points at $\pm k_0$ and asymptotic to the v axis. As $k_0'' \to 0$ the curves move in towards the axis as shown by the dashed curves in the figure. If we are on the sheet of the Riemann surface for which $Re\sqrt{w^2-k_0^2} \geq 0$ then we may permit the point w to move about at will and as long as we do not cross a branch cut we will always remain on the same sheet. As soon as a branch cut is crossed we obtain the other branch of the function, i.e. we pass through the cut unto the other sheet of the Riemann surface.

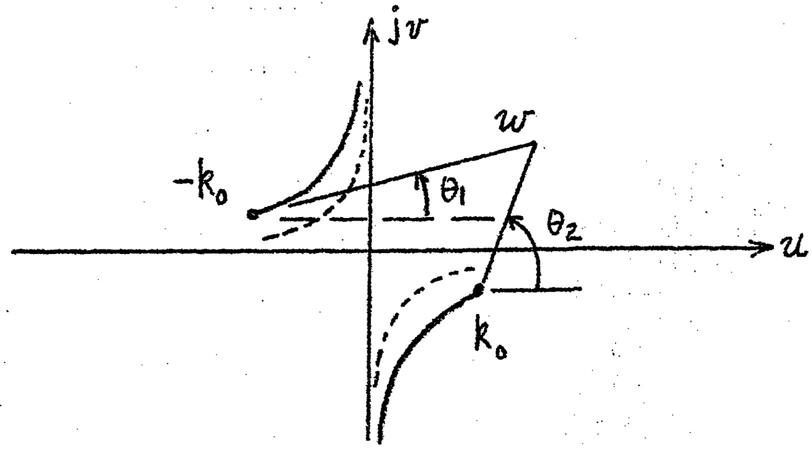


Fig. 4-3 Branch cuts in the w plane

If we cross a cut twice we will return to our original branch (sheet). On the top sheet (which we choose to represent $+\sqrt{w^2-k_o^2}$) we have for any w, phase $w+k_o = \theta_1$, phase $w-k_o = \theta_2$, where $-\frac{\pi}{2} < \frac{\theta_1+\theta_2}{2}$ = phase $\sqrt{w^2-k_o^2} < \frac{\pi}{2}$ and hence Re $\sqrt{w^2-k_o^2} \geq 0$. Along the continuation of the hyperbolas $uv = -k_o'k_o''$ from the branch points and asymptotic to the u axis we have Imag. $\sqrt{w^2-k_o^2} = 0$. Everywhere else in the cross hatched region shown in Fig. 4-4 we have Imag. $\sqrt{w^2-k_o^2} \geq 0$, Re $\sqrt{w^2-k_o^2} \geq 0$.

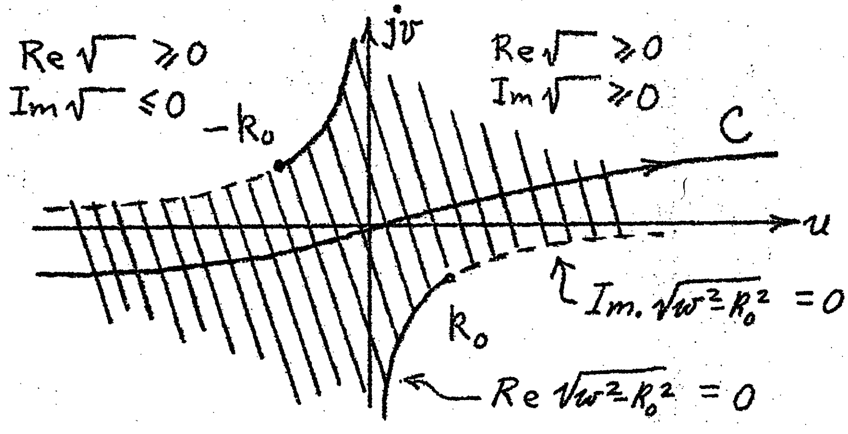


Fig. 4-4

The contour C in the integral for G must lie in this region in order to give rise to outward propagating waves, or attenuated waves.

On the top sheet of the Riemann surface we find that for $x>0$ the integral over the semi-circle at infinity in the lower half plane vanishes while for $x<0$ the integral over a semi-circle in the upper half plane vanishes since $\mathrm{Re}\sqrt{w^2-k_0^2}\geq 0$ always. The contour C can thus be closed as illustrated in Fig. 4-5. Note that since we must remain on the top sheet the branch cuts cannot be crossed so the contour must be deformed around the cuts. The original integral can now be expressed as a branch cut integral since it vanishes on the semi-circle and no poles are enclosed.

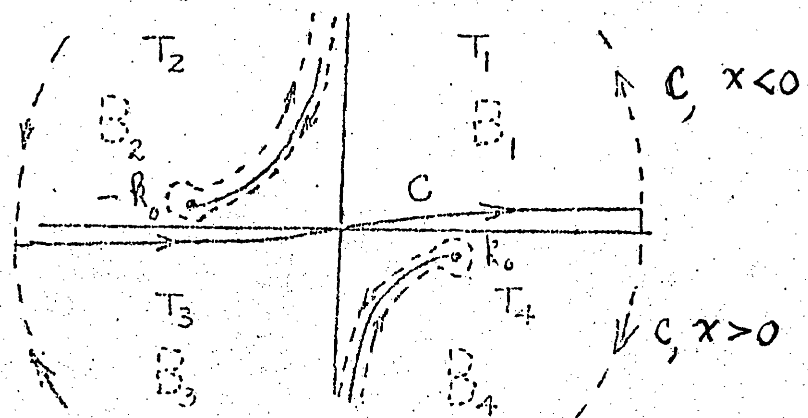


Fig. 4-5

The quadrants of the w plane have been labelled T_1, T_2, T_3, T_4 for the top sheet of the Riemann surface and B_1, B_2, B_3, B_4 for the lower sheet.

In order to arrive at an integral representation for the Hankel function it is convenient to express x and y in polar form and to map the two sheets of the Riemann surface (2 branches of $\sqrt{w^2-k_0^2}$) into a strip of width 2π in the complex $\phi=\sigma+j\eta$ plane by means of the transformations $x=r\cos\theta$, $|y|=r\sin\theta$, $0\leq\theta\leq\pi$; $w=k_0\cos\phi$, $\sqrt{w^2-k_0^2}=jk_0\sin\phi$

We now have $w=k_0(\cos\sigma\cosh\eta-j\sin\sigma\sinh\eta)$ and

$$\sqrt{w^2-k_0^2}=-k_0\cos\sigma\sinh\eta+jk_0\sin\sigma\cosh\eta.$$

The top sheet maps into the region $\mathrm{Re}\sqrt{w^2-k_0^2}\geq 0$ or $\cos\sigma\sinh\eta<0$ while the bottom sheet goes into the region $\cos\sigma\sinh\eta>0$.

The quadrants are located from the expression for w, e.g. for T_1 we require Re $w > 0$ and Imag. $w > 0$ wo $\cos\sigma > 0$ and $\sin\sigma \sinh\eta < 0$. The various quadrants map into the regions illustrated in Fig.4-6.

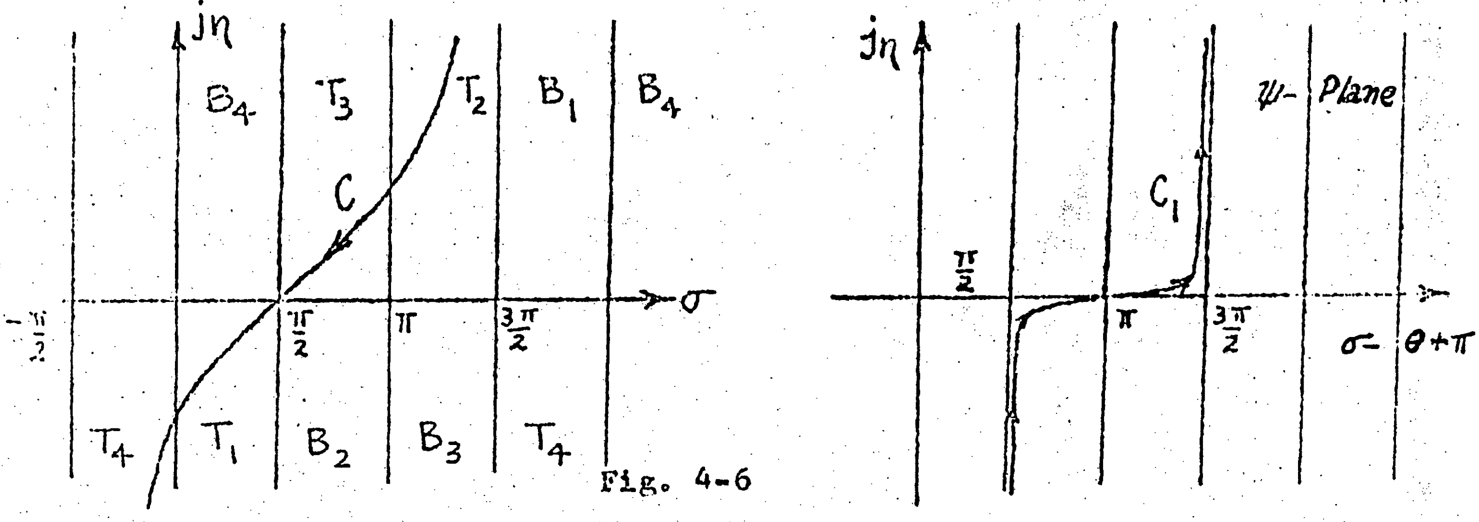


Fig. 4-6

In the ϕ plane the contour C can be any contour begining at ∞ in T_2 or T_3 and ending at ∞ in T_1 or T_4. Since $-jwx - \sqrt{w^2 - k_0^2}\,|y|$ becomes $-jk_0 r \cos\phi \cos\theta - jk_0 r \sin\phi \sin\theta = -jk_0 r \cos(\phi - \theta)$ the integral becomes

$$G = -\frac{1}{4\pi j}\int_C e^{-jk_0 r \cos(\phi-\theta)}\,d\phi$$

But $G = -\frac{j}{4} H_0^2(k_0 r)$

and hence we have $$H_0^2(k_0 r) = -\frac{1}{\pi}\int_C e^{-jk_0 r \cos(\phi-\theta)}\,d\phi .$$

If we reverse the direction of integration and put $\phi - \theta = \psi - \pi$ then $\cos(\phi - \theta) = \cos(\psi - \pi) = -\cos\psi$ and

$$H_0^2(k_0 r) = \frac{1}{\pi}\int_{C_1} e^{jk_0 r \cos\psi}\,d\psi \qquad (4.7)$$

where a particular contour C_1 has been chosen in the ψ plane as shown in Fig.4-6. Clearly for $-\pi/2 \le \sigma \le \pi/2$ we have $-\pi/2 \le Re\,\psi - \pi + \theta \le \pi/2$ or $\pi/2 - \theta \le Re\,\psi \le \frac{3\pi}{2} - \theta$ so for $0 \le \theta \le \pi$ the particular value Re $\psi = \pi/2$ is in the specified range so C_1 may begin at $\pi/2 - j\infty$. Likewise C_1 may end at $3\pi/2 + j\infty$. The above form for H_0^2 was originally due to Sommerfeld.

The integral may be used to obtain the asymptotic form of

$H_0^2(k_0r)$ for large k_0r by evaluating it by the method of steepest descents as described in the appendix.

4.2 Diffraction by a Perfectly Conducting Cylinder

Consider a cylinder of radius a, infinitely long, and a line source at $r = r'$, $\theta = 0$ as in Fig. 4-7.

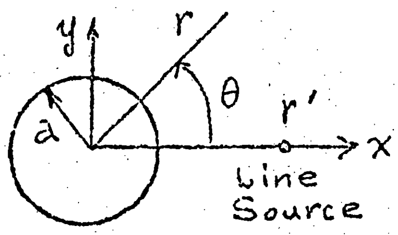


Fig. 4-7.

The Green's function is a solution of

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial \theta^2} + k_0^2 G = -\frac{\delta(r - r')\delta(\theta)}{r'} \tag{4.8}$$

with the boundary conditions $G = 0$ at $r = a$ and

$$\lim_{r\to\infty} \sqrt{r}\left[\frac{\partial G}{\partial r} + jk_0 G\right] = 0.$$

The only component of electric field present is $E_z = -j\omega\mu_0 G$. As $r' \to \infty$ the field from the line source that is incident on the cylinder can be considered as a plane wave over the region of the cylinder. Thus, the problem of diffraction by a plane wave can be obtained as a limiting case. A direct approach to the problem of diffraction by a plane wave incident on the cylinder, say e^{jk_0x}, requires expansion of e^{jk_0x} in terms of Bessel functions and the cos $n\theta$ functions in order to facilitate matching the boundary conditions at $r = a$. This Bessel-Fourier series expansion will be obtained from the analysis of the diffraction problem with a line source, as the source point $r' \to \infty$.

To find G assume

$$G = \sum_{n=0}^{\infty} F_n(r) \cos n\theta \ .$$

Thus

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{r} \frac{d}{dr} r \frac{dF_n}{dr} - \frac{n^2}{r^2} F_n + k_0^2 F_n \right\} \cos n\theta = - \frac{\delta(r - r')\delta(\theta)}{r'} \tag{4.9}$$

Multiplying by cos nθ and integrating leads to the equation

$$\frac{1}{r} \frac{d}{dr} r \frac{dF_n}{dr} - \left(\frac{n^2}{r^2} - k_0^2 \right) F_n = - \frac{\delta(r - r')}{\varepsilon_{0n} \pi r'} \ . \quad \varepsilon_{0n} = \begin{cases} 2, & n = 0 \\ 1, & n > 1 \end{cases} \tag{4.10}$$

ε_{0n} is called the Neumann factor. For $r > r'$ a solution for F_n which represents an outward propagating wave is $AH_n^2(k_0 r)$. For $a \le r < r'$ a solution which vanishes at $r = a$ is readily found in the form $F_n = B[J_n(k_0 r)Y_n(k_0 a) - J_n(k_0 a)Y_n(k_0 r)]$. Continuity at $r = r'$ gives $AH_n^2(k_0 r') + B[J_n(k_0 a)Y_n(k_0 r') - J_n(k_0 r')Y_n(k_0 a)] = 0$. Integrating the equation for F_n from r'_- to r'_+ gives

$$\int_{r'_-}^{r'_+} \frac{1}{r} \frac{d}{dr} r \frac{dF_n}{dr} r dr = r \frac{dF_n}{dr} \Bigg|_{r'_-}^{r'_+} = r' \left[\frac{dF_n}{dr} \Bigg|_{r'_+} - \frac{dF_n}{dr} \Bigg|_{r'_-} \right] = - \int_{r'_-}^{r'_+} \frac{\delta(r - r')}{\varepsilon_{0n} \pi r'} r dr = \frac{-1}{\varepsilon_{0n} \pi}$$

since

$$\int_{r'_-}^{r'_+} F_n \, r dr = 0$$

since F_n is continuous at $r = r'$. We now have

$$r'A \left.\frac{dH_n^{2}(k_0r)}{dr}\right|_{r'} - r'B \left[Y_n(k_0a) \left.\frac{dJ_n(k_0r)}{dr}\right|_{r'} - J_n(k_0a) \left.\frac{dY_n(k_0r)}{dr}\right|_{r'} \right.$$

$$= \frac{-1}{\epsilon_{0n}}$$

To evaluate this latter result we prove a useful property of the Wronskian for Bessel's equation. Let $\psi_1(k_0r)$ and $\psi_2(k_0r)$ be any two linearly independent solutions of Bessel's equation of order n. Multiply the equation for ψ_2 by $r\psi_1$, the equation for ψ_1 by $r\psi_2$, subtract and integrate to obtain

$$\int_{r_1}^{r_2} \left(r\psi_1 \frac{1}{r} \frac{d}{dr} r \frac{d\psi_2}{dr} - r\psi_2 \frac{1}{r} \frac{d}{dr} r \frac{d\psi_1}{dr} \right) dr = 0$$

since the other terms cancel. Integrating by parts gives

$$r \left(\psi_1 \frac{d\psi_2}{dr} - \psi_2 \frac{d\psi_1}{dr} \right)\Bigg|_{r_1}^{r_2} = \int_{r_1}^{r_2} \left(r \frac{d\psi_1}{dr} \frac{d\psi_2}{dr} - r \frac{d\psi_2}{dr} \frac{d\psi_1}{dr} \right) dr \equiv 0.$$

Since r_1 and r_2 are arbitrary we conclude that

$$W \equiv \psi_1(k_0r) \frac{d\psi_2(k_0r)}{dr} - \psi_2(k_0r) \frac{d\psi_1(k_0r)}{dr} = \frac{C}{r} \qquad (4.11)$$

where C is a constant. The beauty of this result is that C can be evaluated for $r \to \infty$ where the asymptotic forms of the Bessel functions are valid. In our case we have

$$\lim_{r\to\infty} \frac{d\psi_n^{2}}{dr} = \frac{d}{dr}\sqrt{\frac{2}{\pi k_0 r}}\; e^{-j(k_0 r - \frac{2n+1}{4}\pi)}$$

$$= -jk_0\sqrt{\frac{2}{\pi k_0 r}}\; e^{-j(k_0 r - \frac{2n+1}{4}\pi)}$$

to order $r^{-1/2}$. Similarly

$$\frac{d}{dr}J_n(k_0 r) \to \frac{d}{dr}\sqrt{\frac{2}{\pi k_0 r}}\cos\left(k_0 r - \frac{2n+1}{4}\pi\right)$$

$$= -k_0\sqrt{\frac{2}{\pi k_0 r}}\sin\left(k_0 r - \frac{2n+1}{4}\pi\right) \text{ as } r \to \infty \quad \text{and}$$

$$\frac{dY_n(k_0 r)}{dr} \to k_0\sqrt{\frac{2}{\pi k_0 r}}\cos\left(k_0 r - \frac{2n+1}{4r}\pi\right).$$

If we call our solution for F_n, $B\psi_1$ for $r \le r'$ and $A\psi_2$ for $r \ge r'$ then the equations to be solved are

$$B\psi_1 - A\psi_2 = 0, \quad B\frac{d\psi_1}{dr}\bigg|_{r'} - A\frac{d\psi_2}{dr}\bigg|_{r'} = \frac{1}{\varepsilon_{0n}\pi r'}$$

The solution is

$$B = \frac{-\psi_2(k_0 r')}{\varepsilon_{0n}\pi W r'}, \quad A = \frac{-\psi_1(k_0 r')}{\varepsilon_{0n}\pi W r'}$$

where the Wronskian

$$W = \left(\psi_1\frac{d\psi_2}{dr} - \psi_2\frac{d\psi_1}{dr}\right)\bigg|_{r'}.$$

But since $W = \frac{C}{r}$ for all r we can evaluate W at $r \to \infty$ and then $r' W(r') = \lim_{r\to\infty} r W(r) = C$. Using the asymptotic forms for $\psi_1 = J_n(k_0r)Y_n(k_0a) - J_n(k_0a)Y_n(k_0r)$, $\psi_2 = H_n^2(k_0r)$ it is readily found that

$$\lim_{r\to\infty} r W(r) = \frac{2}{\pi k_0} \{[Y_n(k_0a) \cos (k_0r - \frac{2n+1}{4}\pi)$$

$$- J_n(k_0a) \sin (k_0r - \frac{2n+1}{4}\pi)] [-jk_0e^{-j(k_0r - \frac{2n+1}{4}\pi)}]$$

$$- e^{-j(k_0r - \frac{2n+1}{4}\pi)} [-k_0Y_n(k_0a) \sin (k_0r - \frac{2n+1}{4}\pi)$$

$$- k_0J_n(k_0a) \cos (k_0r - \frac{2n+1}{4}\pi)]\}$$

$$= \frac{2}{\pi} [J_n(k_0a) - jY_n(k_0a)] = \frac{2}{\pi} H_n^2(k_0a) = C.$$

Hence, $W(r') = \frac{C}{r'} = \frac{2}{\pi r'} H_n^2(k_0a)$. The required solution for G is thus

$$G = - \sum_{n=0}^{\infty} \frac{\cos n\theta}{\epsilon_{0n}\pi r'} \frac{\pi r'}{2H_n^2(k_0a)} \{H_n^2(k_0r_>) [J_n(k_0r_<)Y_n(k_0a)$$

$$- J_n(k_0a)Y_n(k_0r_<)]\}, \qquad \begin{array}{l} r_> \equiv \text{larger of } r, r' \\ r_< \equiv \text{lesser of } r, r' \end{array} \qquad (4.12)$$

For $a < r < r'$ the field consists of an incident field from the line source plus a field scattered from the cylinder. The scattered field is an outward propagating field. In the solution for $r < r'$ the terms involving the $H_n^2(k_0r)$ function would represent outward propagating or scattered waves. These terms are not apparent in the above solution. However, the term in square brackets can be written as

$$J_n(k_0r_<)Y_n(k_0a) - J_n(k_0a)Y_n(k_0r_<)$$

$$- j\,[J_n(k_0r_<)J_n(k_0a) - J_n(k_0r_<)J_n(k_0a)]$$

$$= - j\,J_n(k_0a)\,[J_n(k_0r_<) - j\,Y_n(k_0r_<)]$$

$$+ j\,J_n(k_0r_<)\,[J_n(k_0a) - j\,Y_n(k_0a)]$$

$$= - j\,J_n(k_0a)\,H_n^{\,2}(k_0r_<) + j\,J_n(k_0r_<)H_n^{\,2}(k_0a).$$

Hence, another solution for G is

$$G = - \sum_{n=0}^{\infty} \frac{\cos n\theta}{2j\,\epsilon_{0n}} \frac{H_n^{\,2}(k_0r_>)}{H_n^{\,2}(k_0a)}\,[J_n(k_0a)H_n^{\,2}(k_0r_<) - J_n(k_0r_<)H_n^{\,2}(k_0a)]$$

This form could have been obtained directly by initially choosing for F_n the form in square brackets for $a \le r \le r'$. With this form it is clear that for $r < r'$ we have

$$\text{Outgoing waves} = - \sum_{n=0}^{\infty} \frac{\cos n\theta}{2j\,\epsilon_{0n}} \frac{H_n^{\,2}(k_0r')}{H_n^{\,2}(k_0a)}\,J_n(k_0a)H_n^{\,2}(k_0r) \qquad (4.14)$$

$$\text{Incident Waves} = \sum_{n=0}^{\infty} \frac{\cos n\theta}{2j\,\epsilon_{0n}}\,J_n(k_0r)H_n^{\,2}(k_0r')$$

But the incident field from a line source is simply $-\frac{j}{4}H_0^{\,2}(k_0\rho)$. As $\rho \to \infty$ we know the incident wave is of the form

$$-\frac{j}{4}\sqrt{\frac{2}{\pi k_0\rho}}\;e^{-j(k_0\rho - \frac{\pi}{4})}$$

where ρ is the distance from the line source as in Fig. 4-8. In our case we have $\rho = (r'^2 + r^2 - 2rr' \cos\theta)^{1/2} \approx r' - r\cos\theta = r' - x$, $r \ll r'$. The field at P due to the line source is

$$-\frac{j}{4}\sqrt{\frac{2}{\pi k_0 r'}}\; e^{-j(k_0 r' - \frac{\pi}{4})}\; e^{jk_0 x}$$

But this incident field is also given by

$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2j\epsilon_{0n}}\; J_n(k_0 r) \cdots$$

$$j^n \sqrt{\frac{2}{\pi k_0 r'}}\; e^{-j(k_0 r' - \frac{\pi}{4})} \quad \text{for } r' \gg r.$$

We thus find the following expansion of the plane wave $e^{jk_0 x}$,

$$e^{jk_0 x} = \sum_{n=0}^{\infty} \frac{2}{\epsilon_{0n}}\, j^n \cos n\theta \;\; J_n(k_0 r) = e^{jk_0 r\cos\theta} \tag{4.15}$$

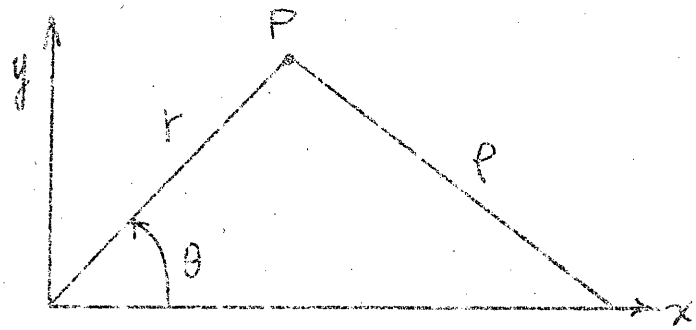


Fig. 4-8.

for arbitrary values of ρ but with $r' > r$ we have the Bessel addition formula, i.e.

$$G_s^2(r,\phi) = \sum_{n=0}^{\infty} \frac{2 \cos n\phi}{\epsilon_{0n}} J_n(k_0 r) H_n^2(k_0 r') \qquad (4.16)$$

When $k_0 a \ll 1$ the series for the scattered field may be summed term by term since it converges rapidly. The following results may be used

$$J_n(k_0 a) \approx \frac{1}{n!} \left(\frac{k_0 a}{2} \right)^n, \quad Y_0(k_0 a) \approx -\frac{2}{\pi} \ln \frac{2}{\gamma k_0 a}, \quad \gamma = 1.781$$

$$Y_n(k_0 a) \approx -\frac{(n-1)!}{\pi} \left(\frac{2}{k_0 a} \right)^n, \quad H_n^2 \approx -j Y_n \quad \text{as} \quad k_0 a \to 0.$$

For large values of $k_0 a$ the series converges very slowly. A large amount of work has been done in obtaining alternate forms for G that will permit numerical evaluation for the case of $k_0 a$ large. The "Watson transformation" is one technique used to convert the above form for G into a more rapidly convergent form for large $k_0 a$. This transformation is introduced in a rather "ad hoc" fashion. We can obtain the same representation by using the general contour representation for G given earlier in Chapt. 3

Associated with the original problem are the two following one dimensional Green's function problems,

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial G_r}{\partial r} - \frac{\lambda}{r^2} G_r + k_0^2 G_r = -\frac{\delta(r - r')}{r'} \qquad (4.17a)$$

$$\frac{\partial^2 G_\phi}{\partial \phi^2} + \lambda G_\phi = -\delta(\phi) \qquad (4.17b)$$

The solution for G_r is similar to that found earlier for $F_n(r)$ with n replaced by $\sqrt{\lambda}$. We readily find

$$G_r = \frac{j\pi}{2} \frac{H_\nu^2(k_0 r_>)}{H_\nu^2(k_0 a)} [J_\nu(k_0 a) H_\nu^2(k_0 r_<) - J_\nu(k_0 r_<) H_\nu^2(k_0 a)] \qquad (4.18)$$

One possible solution for G_θ is $G_\theta = \sum_{n=0}^{\infty} a_n \cos n\theta$. By substituting this series into the equation and evaluating the a_n by the usual procedure it is found that

$$G_\theta = - \sum_{n=0}^{\infty} \frac{\cos n\theta}{\epsilon_{0n} \pi(\lambda - n^2)} \qquad (4.19)$$

An alternate form for G_θ that is more convenient to use in practice can also be constructed. The boundary condition on G_θ requires that it be a periodic function of θ of period 2π and that it have a singularity at $\theta = 0$. The periodicity condition is satisfied if we imagine that θ is a rectangular coordinate and we place line sources at $\theta = 0$, $2n\pi$, $n = \pm 1, \pm 2, \ldots$ From the source at $\theta = 0$ we can write $G_\theta = A e^{j\nu\theta}$, $\theta > 0$ and $G_\theta = B e^{-j\nu\theta}$, $\theta < 0$ where $\nu = \sqrt{\lambda}$. Continuity at $\theta = 0$ gives $A = B$. Also we require $\left.\frac{dG_\theta}{d\theta}\right|_{0_-}^{0_+} = -1$ so $j\nu A + j\nu B = -1$ or $A = B = -\frac{1}{2j\nu}$. Thus, from one line source $G_\theta = -\frac{e^{j\nu|\theta|}}{2j\nu}$. From all sources we obtain

$$G_\theta = - \sum_{n=-\infty}^{\infty} \frac{e^{j\nu|\theta - 2n\pi|}}{2j\nu} \qquad (4.20)$$

If this series is summed it gives the original form for G_θ provided that series is also summed. That is, we find that both series can be summed to yield $G_\theta = -\frac{\cos\sqrt{\lambda}(\pi - |\theta|)}{2\sqrt{\lambda}\sin\sqrt{\lambda}\pi}$ with $-\pi \le \theta \le \pi$. Thus, the two forms for G_θ given are equivalent and equal to the above closed form.

For the Green's function we now have $G = -\frac{1}{2\pi j}\oint_{C_\theta} G_\theta(\lambda) G_r(\lambda)\, d\lambda$ where C_θ encloses the poles of G_θ only as in Fig. 4.8a. These occur where $\sin\sqrt{\lambda}\pi = 0$ or $\lambda = 0, 1, 4, \ldots n^2 \ldots$ A residue expansion of the integral yields

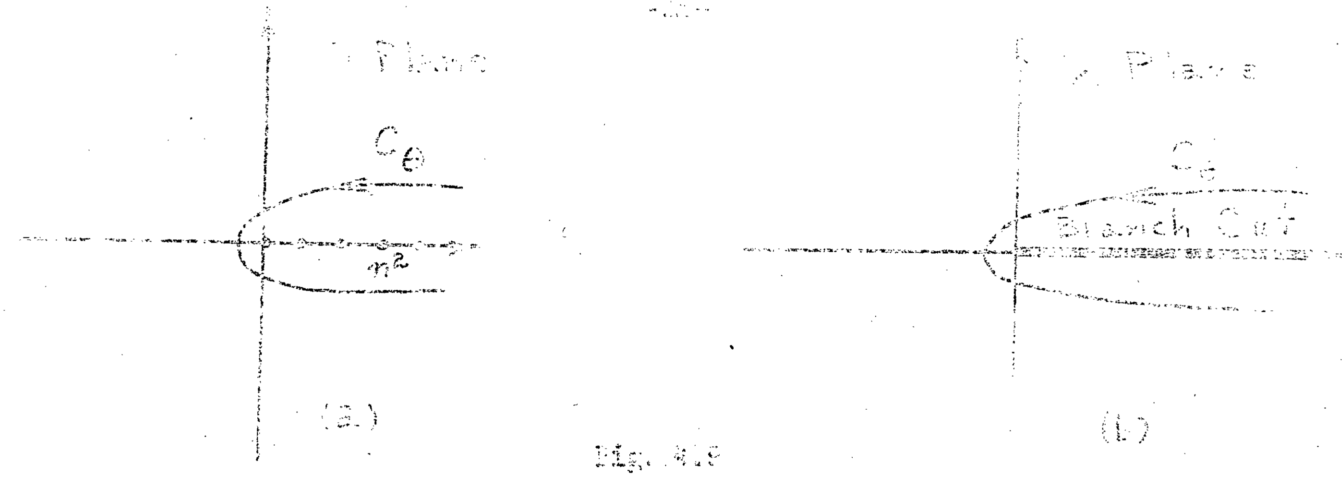


Fig. 4.9

the original solution for G given by (4.13). In order to obtain an alternative form for G it is necessary to deform the contour C_θ such that it encircles the singularities of $G_n(\lambda)$ in the λ plane.

If we use the form $G_\theta = -\sum_{n=-\infty}^{\infty} \frac{e^{j\sqrt{\lambda}|\theta - 2n\pi|}}{2j\sqrt{\lambda}}$

we require the branch Imag. $\sqrt{\lambda} \leq 0$ in order that $e^{j\sqrt{\lambda}|\theta - 2n\pi|}$ shall be exponentially decaying for all λ as $\lambda \to \infty$. If we place a branch cut along the positive real axis in the λ plane then $0 \leq \text{arg}\ \lambda < 2\pi$ and $0 \leq \text{arg}\ \sqrt{\lambda} < \pi$ and Imag. $\sqrt{\lambda} \geq 0$. The contour C_θ now encircles this branch cut as illustrated in Fig. 4.9b. In the complex $\nu = \sqrt{\lambda}$ plane the Bessel functions do not have any branch points. However, the function $H_\nu^{(2)}(k_0 a)$ has an infinity of zeros for complex values of ν. These are located in the upper and lower half of the complex plane as illustrated in Fig. 4.10. Since $\nu = \sqrt{\lambda}$ and the branch Imag. $\sqrt{\lambda}$ = Imag. $\nu \geq 0$ has been chosen only the zeros in the upper half of the ν plane are of

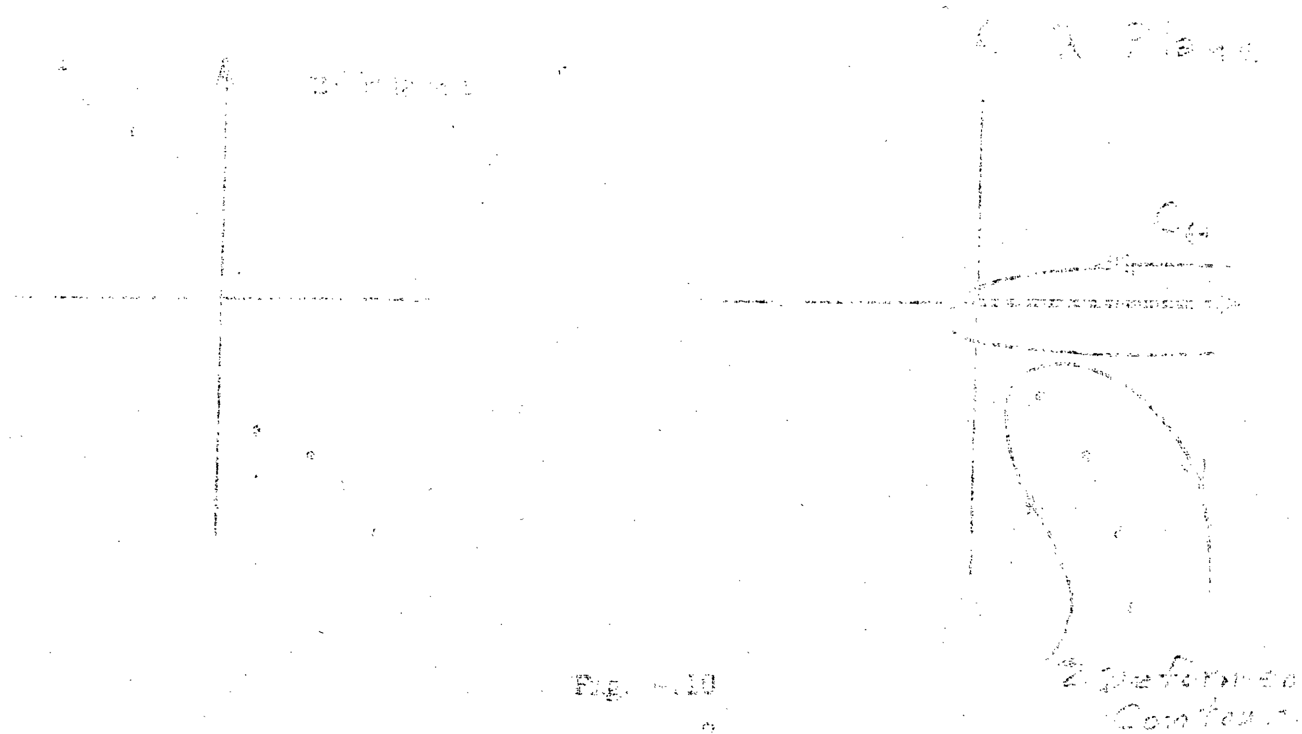


Fig. [illegible].10

Zeroes of $H_\nu^{2}(k_0 a)$

interest. These occur in the lower half of the ν plane. The contour C_6 may be deformed to enclose the poles of $[H_\nu^{2}(k_0 a)]^{-1}$ and G evaluated in terms of the corresponding residues. The evaluation of the zeroes and the residues is a complex task. The great advantage of this formulation stems from the following considerations. The roots of [illegible] are of the form $\nu_n = \alpha_n + j\beta_n$ where $\beta_1 < \beta_2 < \beta_3 \ldots$ and [illegible] $\rightarrow \infty$. Hence, in the series for G_0 only the n = 0 term is important for if β_n is large $e^{-\beta_m|\theta - 2n\pi|}$ is very small and the term n = 1 gives the major contribution. Furthermore, the most important term in the residue expansion is that for the root ν_1 since successive terms are much smaller because $e^{-\beta_m|\theta|}$ [illegible] $e^{-\beta_1|\theta|}$. Thus, the solution for G is well approximated by a single term in a large k a. For the evaluation of [illegible]

[illegible]
[illegible]

[illegible], IRE Trans., Vol. [illegible], Jan. 1957, "Cylindrical Radio Waves," [illegible] has many references to earlier work.

It is found that $\nu_n \approx -k_0 a \cos \alpha_n$

where $\alpha_n = \delta_n \left[1 + \frac{\delta_n^3}{30} + \frac{11}{12,600} \delta_n^6 + \cdots \right]$

with $\delta_n = \left[\frac{3\pi}{4 k_0 a} (4n-1) e^{j\pi} \right]^{2/3}$

For $k_0 a = 25$, $|\delta_1| = 0.655$, $\frac{|\delta_1|^3}{30} = 0.0173$,

$\nu_1 = -\sigma_1 + j\eta_1$, $\eta_1 = 4.65$, while $\eta_2 \approx 1.53 \times 4.65$.

Thus the second term in the series is smaller by a factor of $e^{-.53 \times 4.65} = e^{-2.46} = 0.085$.

For $k_0 a = 100$, $\eta_1 \approx 7.35$, $\eta_2 \approx 11.35$ which makes the second term a factor $e^{-4} = 0.018$ smaller than the first.

Wiener-Hopf Techniques

A number of special types of scattering and diffraction problems lead to integral equations of the Wiener-Hopf type. These can be solved by Fourier transforms together with function theoretic arguements.

The basic Wiener-Hopf integral equation is of the following type:

$$\phi(z) = v(z) + \int_0^\infty G(z-z')\phi(z')\,dz' \qquad (4.21)$$

where v and G are known. We now introduce the following functions

$$\phi_+(z) = \begin{cases} \phi(z), & z>0 \\ 0, & z<0 \end{cases} \qquad \phi_-(z) = \begin{cases} \phi(z), & z<0 \\ 0, & z>0 \end{cases}$$

so that (4.21) can be written as

$$\phi_+(z) + \phi_-(z) = v(z) + \int_{-\infty}^{\infty} G(z-z')\phi(z')\,dz' \quad (4.22)$$

The integral is now of the convolution type so we can take a Fourier transform of both sides to obtain

$$\hat{\phi}_+ + \hat{\phi}_-(k) = \hat{v}(k) + \hat{G}(k)\,\hat{\phi}_+(k) \qquad (4.23)$$

For each problem one must establish that the Fourier transforms $\left[\hat{\phi}_+(k) = \int_{-\infty}^{\infty} \phi_+(z) e^{jkz} dz\right]$ of all functions exist and have a common strip of analyticity in the complex k plane. In that strip (4.23) is valid. We now assume that we can factor $\hat{v}(k)$ into the form $\hat{v}_+(k) + \hat{v}_-(k)$ where $\hat{v}_+$ is analytic in an upper half plane (u.hp) and $\hat{v}_-$ is analytic in the lower half plane (lhp). Note that $\hat{\phi}_+$ is analytic in the upper half plane since

$$\phi_+(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-jkz}\, \hat{\phi}_+(k)\, dk$$

and for $z<0$ we can close the contour in the uhp and must get zero, which is possible only if $\hat{\phi}_+$ is free of all singularities in the uhp (analytic in uhp). For the present problem we have

$$(1-\hat{G})\hat{\Phi}_+ - \hat{v}_+ = \hat{v}_- - \hat{\phi}_- \qquad (4.24)$$

We now factor $1-\hat{G}$ into the form

$$1-\hat{G} = \frac{\hat{H}_+}{\hat{H}_-} \qquad (4.25)$$

so that

$$\hat{H}_+ \hat{\Phi}_+ - \hat{v}_+ \hat{H}_- = \hat{H}_- (\hat{v}_- - \hat{\phi}_-) \qquad (4.26)$$

As a next step we factor $\hat{H}_- \hat{v}_+$ into $\hat{S}_+ + \hat{S}_-$ so that we obtain

$$\hat{H}_+ \hat{\Phi}_+ - \hat{S}_+ = \hat{S}_- + \hat{H}_-(\hat{v}_- - \hat{\phi}_-) \qquad (4.27)$$

The right hand side of (4.27) is analytic in the lhp while the left hand side is analytic in the upper half plane. The two functions are equal in a common strip and hence together they define a function $h(k)$ which is analytic throughout the whole complex k plane. Thus

$$\hat{H}_+ \hat{\Phi}_+ - \hat{S}_+ = h$$

or

$$\hat{\Phi}_+ = \frac{h + \hat{S}_+}{\hat{H}_+} \qquad (4.28)$$

The function h is determined in practise from the known behavior of $\Phi_+(z)$ near $z=0$ (usually an edge condition). Thus provided the appropiate factorization of the various functions of k can be carried out a solution for $\hat{\Phi}_+(k)$ and hence $\Phi_+(z)$ can be found (similarly for $\Phi_-(z)$

In practise the most difficult step in the Wiener-Hopf method is the factorization. A general procedure for carrying this out is given below.

Wiener-Hopf Factorization

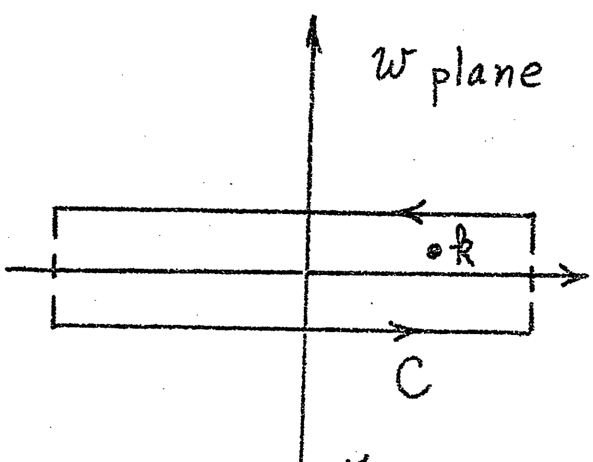


Fig. 4.11

Let $F(w)$ be analytic within and on the contour C shown in Fig. 4.11. By Cauchy's integral formula

$$\frac{1}{2\pi j}\oint_C \frac{F(w)\,dw}{w-k} = F(k)$$

We choose for C two lines parallel to the real w axis. If $F(w)$ is of order w^{α}, $\alpha < 1$, then as we extend the contour C to $\pm\infty$ the contributions from the two ends of the contour vanish so that we obtain

$$F(k) = \frac{1}{2\pi j}\int_{-\infty-j\eta}^{\infty-j\eta} \frac{F(w)\,dw}{w-k} - \frac{1}{2\pi j}\int_{-\infty+j\eta}^{\infty+j\eta} \frac{F(w)\,dw}{w-k}$$

where η is a suitable positive number. The first integral defines a function which is analytic in the u h p $\operatorname{Im} k > -\eta$ since for these values of k the pole at $w = k$ will not cross the contour. Similarly the second integral defines a function of k which is analytic in the l h p $\operatorname{Im} k < \eta$. Hence we have the following factorization:

$$F_+(k) = \frac{1}{2\pi j}\int_{-\infty-j\eta}^{\infty-j\eta} \frac{F(w)}{w-k}\,dw \qquad (4.29a)$$

$$F_-(k) = -\frac{1}{2\pi j}\int_{-\infty+j\eta}^{\infty+j\eta} \frac{F(w)}{w-k}\,dw \qquad (4.29b)$$

To achieve a product type of factorization we use $\ln F$ in Cauchy's formula to obtain

$$M_+(k) = (\ln F)_+ = \frac{1}{2\pi j}\int_{-\infty-j\eta}^{\infty-j\eta} \frac{\ln F(w)}{w-k}\,dw \quad (4.30a)$$

$$M_-(k) = (\ln F)_- = -\frac{1}{2\pi j}\int_{-\infty+j\eta}^{\infty+j\eta} \frac{\ln F(w)}{w-k}\,dw \quad (4.30b)$$

Hence we obtain $\ln F_+ + \ln F_- = M_+ + M_-$

$= \ln F_+ F_-$ and so

$$F_+ F_- = e^{M_+ + M_-} = e^{M_+} e^{M_-}$$

or

$$F_+(k) = e^{M_+(k)} \quad (4.31a)$$

$$F_-(k) = e^{M_-(k)} \quad (4.31b)$$

$$F(k) = F_+(k)\,F_-(k) \quad (4.31c)$$

Infinite Product Expansions

Very often in practise it is necessary to know the infinite product expansion of an integral function in order to carry out the Wiener-Hopf factorization. A procedure for carrying out this type of expansion is given below.

Let $f(z)$ have simple zeroes at $z = z_n$, $n = 1, 2, \cdots$

Then $\frac{d \ln f(z)}{dz} = \frac{f'(z)}{f(z)}$ has simple poles at z_n

and can be written in partial fraction form as

$$\frac{f'(z)}{f(z)} = \sum_n \left(\frac{a_n}{z - z_n} + \frac{a_n}{z_n} \right) + \frac{f'(0)}{f(0)} \qquad (4.32)$$

If there are an infinite number of poles the series $\sum_n \frac{a_n}{z - z_n}$ usually does not converge uniformly so we add on the terms a_n/z_n so the series becomes

$$\sum_n \frac{a_n z}{z_n (z - z_n)} = \sum_n \left(\frac{a_n}{z - z_n} + \frac{a_n}{z_n} \right)$$

which has terms proportional to a_n/z_n^2 for n large. When $z = 0$ the series vanishes so the constant left over is $f'(0)/f(0)$. When we integrate (4.32) from 0 to z we obtain

$$\ln \frac{f(z)}{f(0)} = \sum_n a_n\left[\ln(z-z_n) - \ln(-z_n) + z/z_n\right] + z\frac{f'(0)}{f(0)}$$

$$= \sum_n a_n \ln\left[\frac{z-z_n}{-z_n} e^{z/z_n}\right] + \ln e^{z f'(0)/f(0)} \qquad (4.33)$$

Hence

$$f(z) = f(0)\, e^{z f'(0)/f(0)} \prod_n \left(1-\frac{z}{z_n}\right)^{a_n} e^{a_n z/z_n} \qquad (4.34)$$

If $f(z)$ has simple zeroes then all $a_n = 1$. If $f(z)$ is an even function then $f'(0) = 0$.

Example: The function $\frac{\sin z}{z}$ has simple zeroes at $z = n\pi$, $n = \pm 1, \pm 2, \ldots$. It is also an even function so

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)e^{-z/n\pi} \prod_{n=1}^{\infty}\left(1-\frac{z}{n\pi}\right)e^{z/n\pi}$$

$$= \prod_{n=1}^{\infty}\left(1 - z^2/n^2\pi^2\right) \qquad (4.35)$$

Diffraction by a half plane

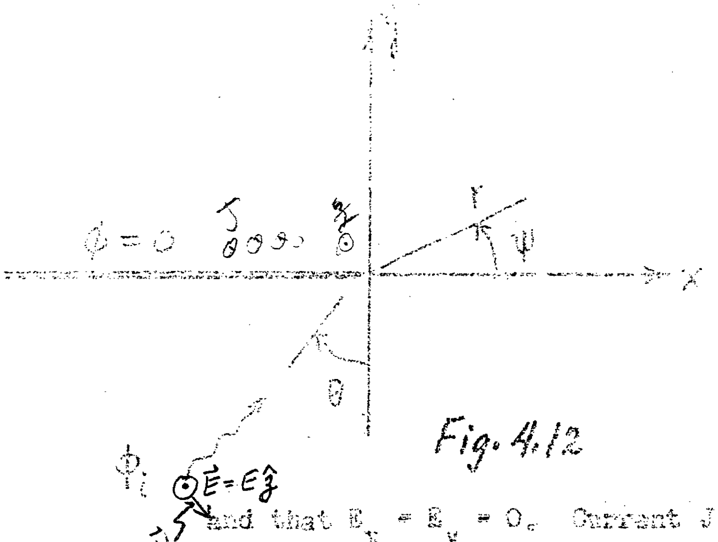


Fig. 4.12

Consider plane wave incident on a perfectly conducting half plane along $-\infty < x < 0$, $y = 0$. Let incident electric field be $\phi_i = Ae^{-jk_0(\vec{n}\cdot\vec{r})} = Ae^{-jk_0(x \sin\theta - y\cos\theta)}$.

Let scattered electric field be $\phi_s(x,y)$. Total field is $\phi = \phi_i + \phi_s$. Note that ϕ_i, ϕ_s equals electric field in z direction and that $E_x = E_y = 0$. Current $J(x)$ induced on screen equals discontinuity in H_x across screen. This is proportional to $\left.\frac{\partial\phi}{\partial y}\right|_{y=0^-}^{y=0^+}$. ϕ_s can be expressed

$$\text{as } \phi_s = \frac{-\omega\mu_0}{4}\int_{-\infty}^{0} H_0^2\left[k_0\left\{(x-x_0)^2+y^2\right\}^{1/2}\right]J(x_0)dx_0 \quad \text{since } \frac{-\omega\mu_0}{4}H_0^2 = \qquad (4.36)$$

field radiated by a unit line current. Integral equation for J is obtained from condition $\phi = 0$ on half plane, i.e.

$$Ae^{-jk_0x\sin\theta} = \frac{\omega\mu_0}{4}\int_{-\infty}^{0} H_0^2\left[k_0|x-x_0|\right]J(x_0)\,dx_0, \quad -\infty < x < 0 \qquad (4.37)$$

In order to take a transform of this equation and to be able to use the convolution theorem we must extend the integral over $-\infty < x_0 < \infty$ and make the equation hold for all x. To do this define

$$J_-(x) = J(x), \quad -\infty < x < 0$$
$$= 0 \quad , x > 0$$

and

$$f_-(x) = \begin{cases} Ae^{-jk_0x\sin\theta}, & -\infty < x < 0 \\ 0 & , x > 0 \end{cases}$$

Thus

$$f_-(x) + f_+(x) = \frac{\omega\mu_0}{4}\int_{-\infty}^{\infty} H_0^2(k_0|x-x_0|)\,J_-(x_0)\,dx_0 \qquad (4.38)$$

all x

$f_+(x)$ is defined by this equation. We do not need to know $f_+(x)$, however, it should be clear that

$$f_+(x) = -\phi_s(x, y=0), \; x>0$$
$$= 0, \; x<0 \qquad (4.39)$$

Transform Equation

$$F_-(w) = \mathcal{L} f_- = \int_{-\infty}^{0} e^{-wx} f_-(x)\,dx = A\int_{-\infty}^{0} e^{-x(w+jk_o \sin\theta)}\,dx$$

$$= \frac{-A}{w+jk_o\sin\theta}\, e^{-x(w+jk_o\sin\theta)}\Big|_{-\infty}^{0}.$$ We assume small losses so that $k_o = k_o' - jk_o''$. Hence F_- exists and is analytic in half plane

$\operatorname{Re} w \equiv u < -k_o'' \sin\theta$, thus , $w = u + jv$

$F_-(w) = -A/(w+jk_o\sin\theta)$.

Let $\hat{J}_- = \mathcal{L} J_- = \int_{-\infty}^{0} e^{-wx} J_-(x)\,dx.$

As $x \to -\infty$, $\phi_s \to -\phi_i = -Ae^{-jk_ox\sin\theta}$. Since $J_- \propto \frac{\partial(\phi_i+\phi_s)}{\partial y}$,

$J_- \to Ce^{-jk_ox\sin\theta} = Ce^{-jk_o'x\sin\theta - k_o''\sin\theta\, x}$ where C is a suitable constant.

Thus $\hat{J}_-$ will exist and be analytic in the half plane $u < -k_o''\sin\theta$.

For $y = 0$, $\phi_s \to C_1 e^{-jk_ox}$ as $x \to \infty$. This decays as $e^{-k_o''x}$ and hence $F_+ = \mathcal{L} f_+$ exists and is analytic in the half plane $u > -k_o''$. Similarily since $H_o^2(k_o|x|) \to C_2 e^{-jk_o'|x| - k_o''|x|}$ for $y = 0$ we have

$$G(w) = \mathcal{L}\; \frac{+\omega\mu_o}{4} H_o^2(k_o|x|)$$ exists and is analytic in the

strip $-k_o'' < u < k_o''$.

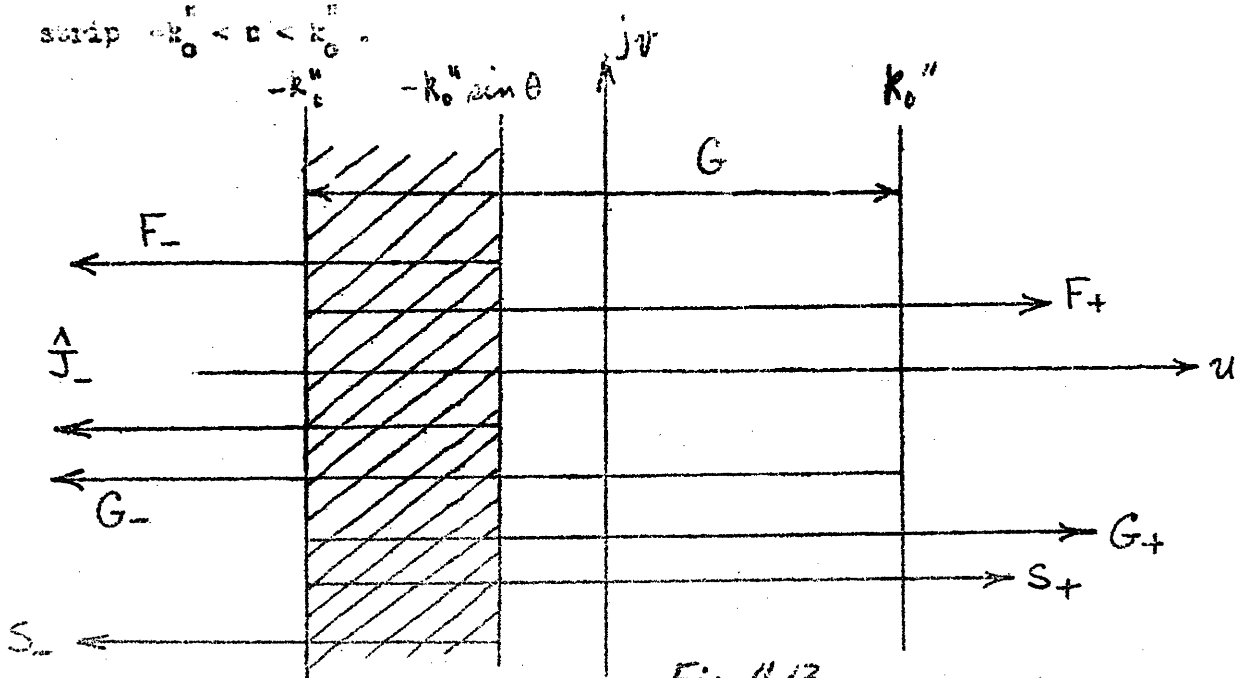


Fig. 4.13

All required transforms exist and are analytic in a common strip $-k_o'' < u < -k_o'' \sin\theta$. In this strip transform of original integral equation is

$F_- + F_+ = G\hat{J}_-$ upon using convolution theorem. To determine

$G = \frac{+\omega\mu_o}{4} \mathcal{L}\, H_o^2$ use the form

$$H_o^2\left[k_o\sqrt{x^2+y^2}\right] = \frac{j}{\pi}\int_C \frac{e^{-jw_o x - \sqrt{w_o^2-k_o^2}\,|y|}}{\sqrt{w_o^2-k_o^2}}\, dw_o$$

as given in previous notes. Thus $\mathcal{L}\ \frac{+\omega\mu_o}{4} H_o^2(k_o\sqrt{\ })$

$$= +\frac{\omega\mu_o j}{4\pi}\int_C \left\{\int_{-\infty}^{\infty} \frac{e^{-\sqrt{w_o^2-k_o^2}\,|y|}}{\sqrt{w_o^2-k_o^2}}\ e^{-jw_o x - wx}\ dx\right\} dw_o$$

$$= +\frac{\omega\mu_o j}{4\pi}\int_C \frac{e^{-\sqrt{w_o^2-k_o^2}\,|y|}}{\sqrt{w_o^2-k_o^2}}\quad 2\pi j\ \delta(w+jw_o)\ \frac{1}{j}\, d(jw_o)$$

$$= \frac{+\omega\mu_0 j}{2\pi j} 2\pi j \frac{e^{-\sqrt{-w^2-k_0^2}\,|y|}}{\sqrt{-w^2-k_0^2}} + \frac{j\omega\mu_0}{4\pi} \frac{e^{-j\sqrt{w^2+k_0^2}\,|y|}}{\sqrt{w^2+k_0^2}} 2\pi j$$

since $\int_{-\infty}^{\infty} e^{-wx-\lambda x}\, dx = 2\pi j\, \delta(w+\lambda)$ and $\frac{1}{2\pi j}\int_C 2\pi j\, \delta(w+\lambda)\, e^{wx}\, dw = e^{-\lambda x}$.

Our transform equation becomes

$$F_- + F_+ = G\hat{J}_- = \frac{-A}{w+jk_o \sin\theta} + F_+ = \frac{-j\omega\mu_o \hat{J}_-(w)}{4\pi\sqrt{w^2+k_o^2}} 2\pi j$$

$$= \frac{+\omega\mu_o}{2} \hat{J}_-(w) \frac{1}{\sqrt{w-jk_o'-k_o''}\sqrt{w+jk_o'+k_o''}} \qquad (4.40)$$

We want to factor G into the form G_-/G_+. Then $F_+G_+ = -F_-G_+ + G_-\hat{J}_-$. Next we factor F_-G_+ into the form $F_-G_+ = S_+ + S_-$. Hence if $F_+G_+ + S_+$ and $-S_- + G_-\hat{J}_-$ have a common strip of analyticity then $F_+G_+ + S_+$ is the analytic continuation of $-S_- + G_-\hat{J}_-$ into the right half plane. Together they define an integral function h(w) having no singularities in complex w plane, i.e.

$$h(w) = \begin{cases} F_+G_+ + S_+, & \text{rhp} \\ -S_- + G_-\hat{J}_-, & \text{lhp} \end{cases} \qquad (4.41)$$

is an integral function.

If we examine G we see that there are two factors $\sqrt{w-jk_o}$ and $\sqrt{w+jk_o}$ that have branch points at $w = jk_o' + k_o''$ and $w = -jk_o' - k_o''$. Clearly $\sqrt{w+jk_o}$ is analytic in rhp $u > -k_o''$. Hence we can choose $G_+ = \sqrt{w+jk_o}$,

$G_- = \frac{\omega\mu_o}{2\sqrt{w-jk_o}}$. It does not matter how we choose the constant factors as long as $G = G_-/G_+$. We now have $F_-G_+ = \dfrac{-A\sqrt{w+jk_o}}{w+jk_o\sin\theta}$ a function with a branch point at $w = -jk_o$ and a pole at $w = -jk_o \sin\theta$. We wish to express F_-G_+ as $S_+ + S_-$. The factor $\sqrt{w+jk_o}$ would be suitable for S_+ while the factor $1/(w+jk_o \sin\theta)$ is suitable for S_-. We can write

$$F_-G_+ = \frac{-A\sqrt{w+jk_o}}{w+jk_o\sin\theta} = \frac{-A(\sqrt{w+jk_o} - \sqrt{-jk_o\sin\theta+jk_o})}{w+jk_o\sin\theta} +$$

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$$+ \frac{-A\sqrt{jk_o-jk_o\sin\theta}}{w+jk_o\sin\theta}.$$ The first term now has no pole at $w = -jk_o \sin\theta$ since the numerator vanishes at this point. This term is analytic for $u > -k_o''$ and hence can be chosen to be S_+. The other term is analytic for $u < -k_o'' \sin\theta$ and hence is S_-.

thus

$$S_+ = -A\,\frac{\sqrt{w+jk_o} - \sqrt{-jk_o\sin\theta+jk_o}}{w+jk_o\sin\theta}$$

$$S_- = -A\,\frac{\sqrt{jk_o-jk_o\sin\theta}}{w+jk_o\sin\theta}$$

We now have

$$F_+\sqrt{w+jk_o} \qquad -A\,\frac{\sqrt{w+jk_o} - \sqrt{jk_o-jk_o\sin\theta}}{w+jk_o\sin\theta} = F_+G_+ + S_+$$

$$= G_-\hat{J}_- - S_- = \frac{\omega\mu_o}{2\sqrt{w-jk_o}}\hat{J}_- + A\,\frac{\sqrt{jk_o-jk_o\sin\theta}}{w+jk_o\sin\theta} = h(w)$$

Thus

$$\hat{J}_- = \frac{-2}{\omega\mu_o}\left\{-\sqrt{w-jk_o}\;h(w) + A\,\frac{\sqrt{jk_o-jk_o\sin\theta}\,\sqrt{w-jk_o}}{w+jk_o\sin\theta}\right\} \qquad (4.42)$$

As $x \to 0$, $J_- \to C \quad x^{-1/2}$, hence by final value theorem $\hat{J}_-(w) \sim w^{-(-\frac{1}{2}+1)} = w^{-1/2}$ as $w \to \infty$. This will be the case only if $h(w) = 0$. Hence

$$\hat{J}_- = \frac{-2A}{\omega\mu_o} \frac{\sqrt{jk_o - jk_o \sin\theta}\,\sqrt{w - jk_o}}{w + jk_o \sin\theta} \qquad (4.43)$$

and $J_- = \frac{1}{2\pi j}\int_C e^{wx}\,\hat{J}_-(w)\,dw$, C in strip $-k_o'' < u < -k_o'' \sin\theta$

or actually anywhere in half plane $u < -k_o'' \sin\theta$. Note that for $x > 0$, C can be closed in lhp and, since no singularities are enclosed, $J_- = 0$ for $x > 0$ as it should.

Let $\psi(w,y) = \mathcal{L}\,\phi_s(x,y) = \mathcal{L}\,\frac{-\omega\mu_o}{4}\int_{-\infty}^{\infty} J_- H_o^2\, dx_o$

$$= \frac{-2A}{\omega\mu_o} \frac{\sqrt{jk_o - jk_o \sin\theta}\;\sqrt{w - jk_o}}{w + jk_o \sin\theta}\left[-\frac{\omega\mu_o}{2}\,\frac{e^{-j\sqrt{w^2+k_o^2}\,|y|}}{\sqrt{w^2+k_o^2}}\right]$$

Hence $\phi_s(x,y) = \frac{1}{2\pi j}\int_C \left[+A\sqrt{jk_o - jk_o \sin\theta}\right] \frac{e^{wx - j\sqrt{w^2+k_o^2}\,|y|}}{(w + jk_o \sin\theta)\sqrt{w + jk_o}}\, dw$

where C lies in strip $-k_o'' < u < -k_o'' \sin\theta$.

If $x < 0$, $y = 0$, C can be closed in rhp. Only singularity of ψ in rhp is pole at $w = -jk_o \sin\theta$. Hence integral gives (a $-$ sign arises because integration is in a clockwise sense)

$$-2\pi j\left\{\frac{1}{2\pi j} A\sqrt{jk_o - jk_o \sin\theta}\quad e^{-jk_o x \sin\theta}\quad \frac{1}{\sqrt{jk_o - jk_o \sin\theta}}\right\}$$

$= -2\pi j(\text{residue at pole}) = -\phi_i(x,0)$. For $y \neq 0$ the branch point occuring at $w = jk_o$ in the factor $e^{-j\sqrt{w^2+k_o^2}\,|y|}$ must be taken into account. The contour can be closed as illustrated.

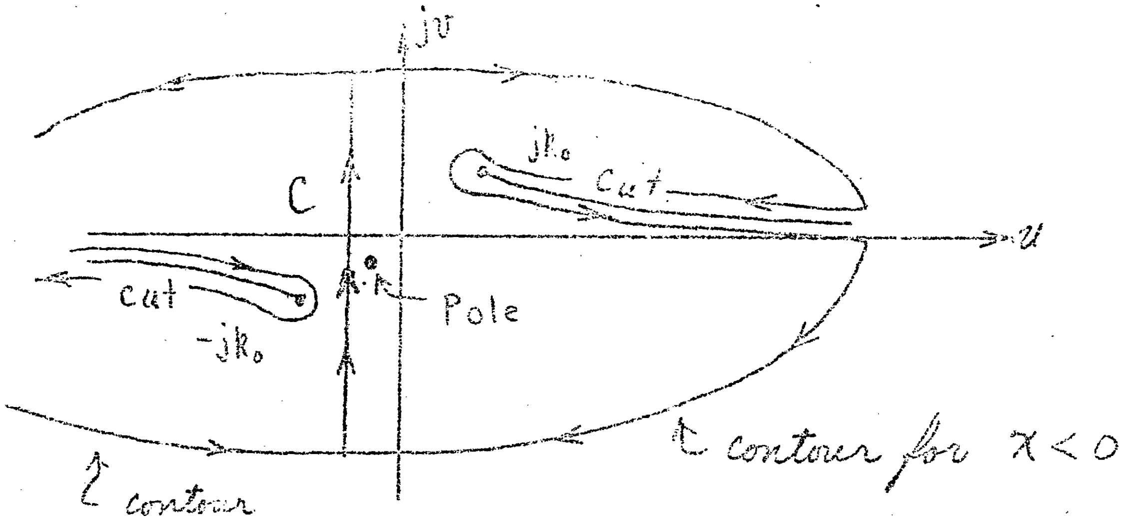

Fig. 4.14

If we solve for F_+ from page ~~(58)~~ 31 we find

$$f_+ = \frac{1}{2\pi j} \int_C e^{wx} \left\{ \frac{A}{w+jk_o \sin\theta} - \frac{A\sqrt{jk_o - jk_o \sin\theta}}{(w+jk_o \sin\theta)\sqrt{w+jk_o}} \right\} dw \qquad (4.44)$$

where C is in rhp $u > -k_o''$. For $x < 0$, C can be closed in rhp. Only singularity is pole at $w = -jk_o \sin\theta$ for both terms. However, residues cancel so $f_+=0$, $x<0$ as it should be. For $x>0$, C can be closed in lhp.

Now $\int_C \frac{e^{wx} A}{w+jk_o \sin\theta} dw = 0$ so it is easy to see that $f_+ = -\phi_s(x, y=0)$ for $x>0$ by comparison with result for ϕ_s on page ~~(57)~~ 32.

Solution for ϕ_s can be expressed in terms of Fresnel integrals which are tabulated. For details see "The Wiener-Hopf Technique" by B. Noble, Pergamon Press, or "Huygen's Principle" by Baker and Copson, Oxford Univ. Press.