2.8 Energy Storage in Dispersive Media

For an electric field $\vec{E}(\vec{r},t) = Re.\ \vec{E}(\vec{r})e^{j\omega t}$ the time average stored electric energy is commonly regarded as given by $Re \int_V \frac{\vec{E}\cdot\vec{D}^*}{4} dV = \int_V \epsilon' \frac{\vec{E}\cdot\vec{E}^*}{4} dV$ where ϵ' is the real part of $\epsilon = \epsilon' - j\epsilon'' = \epsilon_0(1+\chi_e' - j\chi_e'')$. This expression is, in fact, only valid if ϵ is independent of ω in the range of ω of interest and ϵ is pure real (no loss). If the medium is very lossy the concept of stored energy does not have a very precise meaning as a thermodynamic state variable since it can no longer be related directly to the work done in establishing the field. That is, because of losses the work done in establishing the field is in part done against dissipative forces. The latter work appears in the form of heat and is not recovered upon reducing the field to zero, i.e. the process is irreversible. (For further discussion see Landau & Lifshitz - "Electrodynamics of Continuous Media" Sec's 10 11 (ol)

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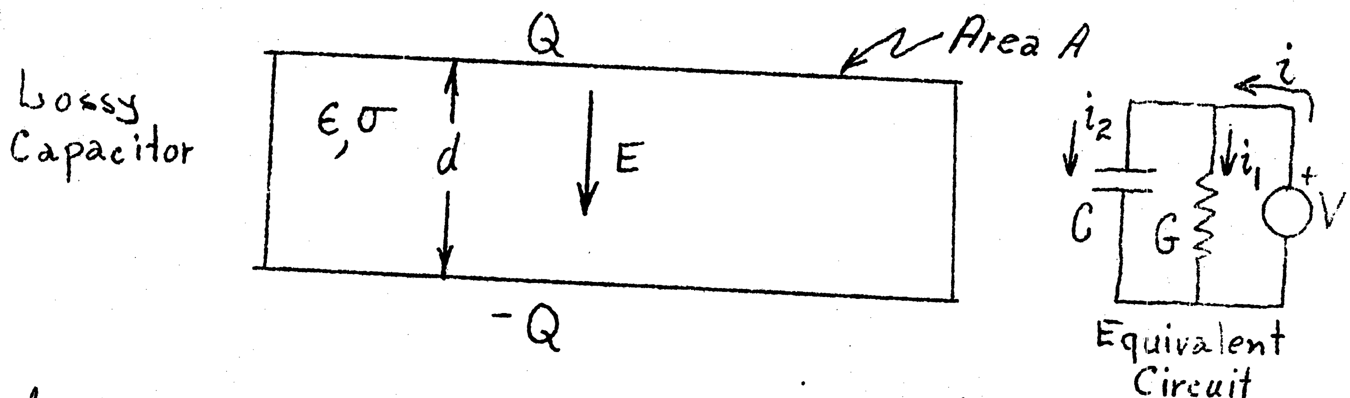
To illustrate the effects of losses consider the following example. The figure shows a parallel plate capacitor containing an isotropic dielectric with permittivity ϵ and conductivity σ. We will assume that ϵ and σ are true constants (not functions of ω). From $\nabla \times \vec{H} = \epsilon \dfrac{\partial \vec{E}}{\partial t} + \sigma \vec{E}$ we obtain $\nabla \times \vec{H}(\vec{r},\omega)$ $= j\omega\epsilon\, \vec{E}(\vec{r},\omega) + \sigma \vec{E}(\vec{r},\omega) = j\omega\left(\epsilon - \dfrac{j\sigma}{\omega\epsilon}\right)\vec{E}(\vec{r},\omega)$ in ω space. Thus $\epsilon - \dfrac{j\sigma}{\omega\epsilon} = \epsilon' - j\epsilon''$ can be viewed as a complex equivalent permittivity in ω space.



Let us apply a voltage $V = V_0 e^{\alpha t}$, $-\infty \le t \le 0$ to build up the field exponential from an initial value of zero to a final value of $E_0 = V_0/d$. From the equivalent circuit an increment of work

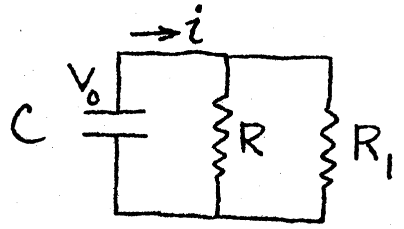
dW done on the system is given by $dW = Vi\,dt$. Also we have $i_1 = VG$, $i_2 = C\frac{dV}{dt}$ where $C = \frac{\epsilon A}{d}$, $G = \frac{\sigma A}{d}$. The total work done is

$$W = \int_{-\infty}^{0} (i_1 + i_2) V\,dt = \int_{-\infty}^{0} \left(V^2 G + CV\frac{dV}{dt}\right) dt$$

$$= \frac{V_0^2 C}{2} + \frac{V_0^2}{2\alpha R} = \frac{V_0^2 C}{2} + \frac{V_0^2 G}{2\alpha} \qquad (2.56)$$

The first term $V_0^2 C/2$ represents the usual term for energy stored in a capacitor but because of losses <u>does</u> <u>not</u> equal the work done. Only if we make $G = 0$ or $\alpha \to \infty$ does the work done become equal to $V_0^2 C/2$ which is the value of $\int_V \frac{\epsilon E_0^2}{2} dV$.

Let us now consider recovering energy from the system by letting the capacitor discharge into an external resistance R_1.



Let $R_e = \frac{RR_1}{R+R_1}$ and then $iR_e = V_0 - \frac{1}{C}\int^t i\,dt$

or $i + CR_e \frac{di}{dt} = 0$ which gives $i = I_0 e^{-t/\tau}$

where $I_0 = V_0/R_e$, $\tau = CR_e$. The current in R_1 is $\frac{R}{R+R_1} i$ and hence the total energy delivered to R_1 will be W_s where

$$W_s = \int_0^\infty \frac{R_1 R^2}{(R+R_1)^2} i^2 dt = \frac{V_0^2 C}{2} \frac{R_e}{R_1} = \frac{V_0^2 C}{2} \frac{R}{R+R_1} \quad (2.57)$$

Thus we do not recover energy of amount $V_0^2C/2$ external to the device because an amount $\frac{R_1}{R+R_1}\frac{V_0^2 C}{2}$ is dissipated internally. Furthermore, the amount of energy which can be recovered depends on the rate (on τ) at which it is taken out just as the work done on the system depended on the rate of building up the field.

For stored energy to be a thermodynamic state function it must depend only on the final state of the system and not on how the system was brought to that state. In a loss free system the work done on a system is stored as internal energy and is a state function. In the example above if we put $G = \frac{1}{R} = 0$ we find that the work done $W = V_0^2 C/2$ and the energy which can be recovered equals W.

We will show that in a dispersive medium (one for which ϵ is a function of ω) the stored energy (average) is given by $\int_V \frac{\vec{E}\cdot\vec{E}^*}{4} \frac{\partial(\omega\epsilon')}{\partial\omega} dV$

when ϵ'' is small (the result is exact when $\epsilon'' = 0$; however, we will show later that this is never possible in a normal passive medium). If ϵ' is essentially constant for those values of ω of interest then $\partial(\omega\epsilon')/\partial\omega \approx \epsilon'$ and we obtain the earlier expression.

To establish the above result we will use a simple model for the polarization $\vec{P}$, i.e. we let $\vec{P}$ be described by

$$m\frac{d^2\vec{P}}{dt^2} + m\nu\frac{d\vec{P}}{dt} + k\vec{P} = q^2\vec{\mathcal{E}}(\vec{r},t) \qquad (2.58)$$

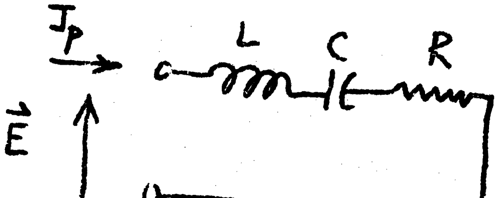
The polarization current $\vec{J}_p$ is given by $\frac{\partial\vec{P}}{\partial t}$. The rate at which $\vec{\mathcal{E}}$ does work to maintain $\vec{J}_p$ is $\vec{\mathcal{E}}\cdot\vec{J}_p = \vec{\mathcal{E}}\cdot\frac{\partial\vec{P}}{\partial t}$ by analogy with the expression

$\vec{\mathcal{E}} \cdot \vec{J}$ for the rate of doing work in supporting a conduction current $\vec{J}$.

It will be more convenient to deal with phasor quantities. Thus let $\vec{\mathcal{E}} = \mathrm{Re}\, \vec{E} e^{j\omega t}$ and let the polarization be $\mathrm{Re}\, \vec{P} e^{j\omega t}$ and the polarization current be $\mathrm{Re}\, \vec{J}_p e^{j\omega t} = \mathrm{Re}\, j\omega \vec{P} e^{j\omega t}$ where $\vec{E}$, $\vec{P}$, and $\vec{J}_p$ are now complex phasor vectors. From our governing equation we have

$$\left(j\omega m + \nu m + \frac{k}{j\omega} \right) \vec{J}_p = q^2 \vec{E} \qquad (2.59)$$

We may model this equation by an equivalent network with $L \equiv m/q^2$, $R \equiv \nu m/q^2$, $C = q^2/k$



as shown. The quantity $\frac{1}{2} \vec{E} \cdot \vec{J}_p^*$ will give us some information about the energy loss and storage in a unit volume of dielectric due to the polarization. The real part $\frac{1}{2} \mathrm{Re}\, \vec{E} \cdot \vec{J}_p^*$

gives the energy loss per second (power loss) due to the damping which depends on ν. This power loss is equal to that in the resistor R in the model or equivalent network. The energy stored in the polarization due to the kinetic energy of the charges depends on the mass m and is equal to that stored in L in the equivalent network. The energy (potential form) due to the elastic restoring forces which depend on k is equal to that stored in C. Thus we have

$$P_l' = \frac{1}{2} R \vec{J}_p \cdot \vec{J}_p^{\,*} = \text{power loss}/m^3$$

$$W_m' = \frac{1}{4} L \vec{J}_p \cdot \vec{J}_p^{\,*} = \text{average stored kinetic energy}/m^3$$

$$W_e' = \frac{1}{4} C \left(\frac{\vec{J}_p}{j\omega C} \right) \left(\frac{\vec{J}_p^{\,*}}{-j\omega C} \right) = \text{average stored potential energy}/m^3$$

where the primes are used to refer to quantities associated with the polarization only.

If we solve for $\vec{J}_p$ we find that

$$\frac{1}{2}\vec{E}\cdot\vec{J}_p^{\,*} = \cancel{\frac{1}{2}\left(\vec{E}\cdot\vec{E}^*\right)\frac{R + j\omega L - j/\omega C}{R^2 + \left(\omega L - 1/\omega C\right)^2}} \quad \text{GARBAGE}$$

$$= \frac{1}{2}\vec{J}_p\cdot\vec{J}_p^{\,*}\left(R + j\omega L - j/\omega C\right) = P_l' + 2j\omega\left(W_m' - W_e'\right) \tag{2.60}$$

This result shows immediately that

$$\langle W\rangle = \frac{\epsilon^*\vec{E}\cdot\vec{E}^*}{4} = \cancel{\frac{\epsilon_0}{4}\vec{E}\cdot\vec{E}^* + \frac{\epsilon_0\chi_e'}{4}\vec{E}\cdot\vec{E}^*} \text{ GARBAGE } \cancel{+ j\epsilon_0\chi_e''\frac{\vec{E}\cdot\vec{E}^*}{4}}$$

$$= \frac{\epsilon_0\vec{E}\cdot\vec{E}^*}{4} + \frac{\vec{E}\cdot\vec{P}^*}{4} = \epsilon_0\frac{\vec{E}\cdot\vec{E}^*}{4} + \frac{1}{2j\omega}\left(\frac{\vec{E}\cdot\vec{J}_p^{\,*}}{2}\right)$$

will not give the total time average stored energy density since $\frac{\vec{E}\cdot\vec{J}_p^{\,*}}{2}$ evaluates only $2j\omega$ times the difference between the average kinetic and potential energy instead of the sum $W_m' + W_e'$, plus of course the power loss. Below we show how the sum $W_m' + W_e'$ can be evaluated in terms of a derivative of $\omega\epsilon'$ with respect to ω.

First note that $\vec{J}_p = Y_{in}\vec{E}$ where Y_{in} is the input admittance of the equivalent network. But $\vec{J}_p = j\omega\vec{P} = j\omega\epsilon_0\chi_e\vec{E}$ also and hence $\underline{j\omega\epsilon_0\chi_e \equiv Y_{in}}$.

Now note that if we take $Y_{in} = G_0 + jB$

$$= \frac{R}{R^2+(\omega L-\frac{1}{\omega C})^2} + \frac{j\left(\frac{1}{\omega C}-\omega L\right)}{R^2+(\omega L-1/\omega C)^2}$$ we have

this does not follow from the above.

$$\frac{\partial B}{\partial\omega} = +\frac{\left(L+\frac{1}{\omega^2 C}\right)}{R^2+(\omega L-1/\omega C)^2}\left(1-2RG_0\right)$$

$$\approx +\frac{\left(L+\frac{1}{\omega^2 C}\right)}{R^2+(\omega L-1/\omega C)^2} \quad \text{for } RG_0 \ll \frac{1}{2} \text{ (small loss)}$$

$x = \omega L - \frac{1}{\omega C}$; $\frac{\partial B}{\partial\omega} = \frac{L+\frac{1}{\omega^2 C}}{R^2+x^2}\left\{-1+\frac{2(\omega L-\frac{1}{\omega C})^2 G}{R}\right\}$

Next note that $\frac{\vec{E}\cdot\vec{E}^*}{4}\frac{\partial B}{\partial\omega} = (W_m' + W_e')$.

Since $j\omega\epsilon_0\chi_e = Y_{in}$ we have $j\omega\epsilon_0\chi_e' = jB$ and hence $W_m' + W_e' = +\frac{\vec{E}\cdot\vec{E}^*}{4}\frac{\partial}{\partial\omega}(\omega\epsilon_0\chi_e')$. Thus the total average stored energy density is

$\epsilon' = \epsilon_0\chi'$

$$\epsilon_0\frac{\vec{E}\cdot\vec{E}^*}{4} + W_m' + W_e' = \frac{\vec{E}\cdot\vec{E}^*}{4}\left(\epsilon_0 + \frac{\partial\omega\epsilon_0\overset{\prime!}{\chi_e}}{\partial\omega}\right) = \frac{\vec{E}\cdot\vec{E}^*}{4}\frac{\partial\omega\epsilon'}{\partial\omega}.$$

If we had considered $Z_{in} = R + jX$ $= R + j(\omega L - 1/\omega C)$ and formed $\partial X/\partial \omega$ we would obtain $\frac{\partial X}{\partial \omega} = L + 1/\omega^2 C$ and hence

$$W_m' + W_e' = \frac{1}{4} \vec{J}_p \cdot \vec{J}_p^* \frac{\partial X}{\partial \omega}$$ without any approximation.

Since $\omega \epsilon_0 \chi_e' = B$ and $\omega \epsilon_0 \chi_e'' = G$ and $X = \frac{-B}{B^2+G^2}$

we would have instead

$$W_m' + W_e' = \frac{1}{4} \vec{J}_p \cdot \vec{J}_p^* \frac{\partial}{\partial \omega}\left(\frac{-\omega \epsilon_0 \chi_e'}{(\omega\epsilon_0)^2(\chi_e'^2 + \chi_e''^2)}\right)$$

$$= \frac{1}{4} \vec{J}_p \cdot \vec{J}_p^* \frac{\partial}{\partial \omega} \frac{-\chi_e'}{\omega \epsilon_0 (\chi_e'^2 + \chi_e''^2)}$$

The use of this more complicated expression is <u>not justified in practise for reasons discussed below.</u>

The model which we used for the polarization is a very simple one. In general we would have $\mathcal{L} \vec{J}_p = \vec{E}$ where $\mathcal{L}$ is a linear differential operator with constant coefficients and of order higher than the second, i.e. it may involve 3rd, 4th, etc. derivatives of $\vec{P}$ and hence of $\vec{J}_p$.

We could still represent the polarization mechanism in terms of an equivalent network but this network would now be a more complicated connection of L's, C's and R's than a simple series connection. But if the losses were zero the following results would still hold exactly,

$$\frac{1}{4}\,\vec{E}\cdot\vec{E}^*\,\frac{\partial B}{\partial \omega} = W_m' + W_e' = \frac{1}{4}\,\vec{J}_p\cdot\vec{J}_p^*\,\frac{\partial X}{\partial \omega} \qquad (2.62)$$

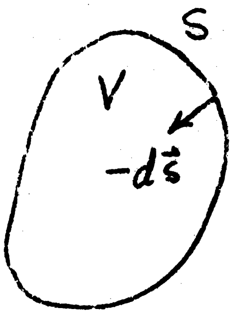
where jB and jX are the input susceptance and reactance for the equivalent network. We will derive these relations later. In network theory these relations are the basis for Foster's reactance theorem. If the losses are finite but sufficiently small the above relations would still be essentially correct.

For similar reasons the energy stored in the magnetic field is given by

$$W_m = \int_V \frac{\vec{H}\cdot\vec{H}^*}{4}\,\frac{\partial\, \omega\mu'}{\partial \omega}\, dV \qquad (2.63)$$

Energy Storage and Power Loss in Anisotropic Media

In anisotropic media the constitutive relations are $\vec{D} = \bar{\epsilon}\cdot\vec{E}$, $\vec{B} = \bar{\mu}\cdot\vec{H}$, where $\bar{\epsilon}$ and $\bar{\mu}$ are dyadics (tensors of rank 2). Maxwell's curl equations are $\nabla\times\vec{E} = -j\omega\bar{\mu}\cdot\vec{H}$, $\nabla\times\vec{H} = j\omega\bar{\epsilon}\cdot\vec{E} + \vec{J}$. By expanding $\nabla\cdot(\vec{E}\times\vec{H}^*)$ and using the curl equations we obtain



$$\frac{1}{2}\oint_S \vec{E}\times\vec{H}^*\cdot(-d\vec{s}) = -\int_V \nabla\cdot(\vec{E}\times\vec{H}^*)\,dV$$

$$= \int_V [(-\nabla\times\vec{E})\cdot\vec{H}^* + (\nabla\times\vec{H}^*)\cdot\vec{E}]\,dV$$

$$= \frac{1}{2}\int_V \vec{E}\cdot\vec{J}^*\,dV + 2j\omega\int_V\left(\frac{\vec{H}^*\cdot\bar{\mu}\cdot\vec{H}}{4} - \frac{\vec{E}\cdot\bar{\epsilon}^*\cdot\vec{E}^*}{4}\right)dV \qquad (2.64)$$

Define $\bar{\epsilon}' = \frac{1}{2}(\bar{\epsilon} + \bar{\epsilon}_t^*)$, $-j\bar{\epsilon}'' = \frac{1}{2}(\bar{\epsilon} - \bar{\epsilon}_t^*)$ so that $\bar{\epsilon} = \bar{\epsilon}' - j\bar{\epsilon}''$ and where $\bar{\epsilon}_t^*$ is the transposed complex conjugate dyadic, i.e. element ϵ_{ij} replaced by ϵ_{ji}^*.

We now note that $(\bar{\epsilon}')_t^* = \frac{1}{2}(\bar{\epsilon}_t^* + \bar{\epsilon}) = \bar{\epsilon}'$ and $\bar{\epsilon}'$ is a hermitian dyadic. Also we have $(-j\bar{\epsilon}'')_t^* = \frac{1}{2}(\bar{\epsilon}_t^* - \bar{\epsilon}) = j\bar{\epsilon}''$ and hence $-j\bar{\epsilon}''$ is an anti-hermitian dyadic. Similarly we will write $\bar{\mu} = \bar{\mu}' - j\bar{\mu}''$ where $\bar{\mu}' = \frac{1}{2}(\bar{\mu} + \bar{\mu}_t^*)$, $-j\bar{\mu}'' = \frac{1}{2}(\bar{\mu} - \bar{\mu}_t^*)$ and $\bar{\mu}'$ is then hermitian while $-j\bar{\mu}''$ is antihermitian

Consider now the term $-2j\omega\, \vec{E}\cdot\epsilon^{*}\cdot\vec{E}^{*}$
$= -2j\omega\, \vec{E}\cdot\overline{\epsilon}'^{*}\cdot\vec{E}^{*} + 2\omega\, \vec{E}\cdot\epsilon''^{*}\cdot\vec{E}^{*}$. Since $-j\overline{\epsilon}''$ is antihermitian $\overline{\epsilon}''$ is hermitian. That is,

$$(\overline{\epsilon}'')^{*}_{t} = \left(\frac{j}{2}\right)^{*}(\overline{\epsilon}^{*}_{t} - \overline{\epsilon}) = -\frac{j}{2}(\overline{\epsilon}^{*}_{t} - \overline{\epsilon}) = \overline{\epsilon}''.$$

The scalar product $\vec{E}\cdot\overline{\epsilon}''^{*}\cdot\vec{E}^{*}$ is pure real when $\overline{\epsilon}''$ is hermitian because the complex conjugate of $\vec{E}\cdot\overline{\epsilon}''^{*}\cdot\vec{E}^{*}$, or $\vec{E}^{*}\cdot\overline{\epsilon}''\cdot\vec{E}$ is equal to $\vec{E}\cdot\overline{\epsilon}''^{*}\cdot\vec{E}^{*}$. To show this note that $\overline{\epsilon}''\cdot\vec{E}$ can be written as $\vec{E}\cdot\overline{\epsilon}''_{t} = \vec{E}\cdot\overline{\epsilon}''^{*}$ and the scalar product between the two vectors $\vec{E}^{*}$ and $\overline{\epsilon}''\cdot\vec{E}$ can be written in any order so that $\vec{E}^{*}\cdot(\overline{\epsilon}''\cdot\vec{E}) = \vec{E}^{*}\cdot(\vec{E}\cdot\overline{\epsilon}''^{*}) = \vec{E}\cdot\overline{\epsilon}''^{*}\cdot\vec{E}^{*}$. Hence $2\omega\,\vec{E}\cdot\overline{\epsilon}''^{*}\cdot\vec{E}^{*}$ is pure real. For the same reason $\vec{E}\cdot\overline{\epsilon}'^{*}\cdot\vec{E}^{*}$ is pure real since $\overline{\epsilon}'$ is also hermitian. Thus the complex Poynting vector theorem can be written as

$$\frac{1}{2}\oint_S \vec{E}\times\vec{H}^{*}\cdot(-d\vec{S}) = \int_V \left\{ \frac{1}{2}\,\vec{E}\cdot\vec{J}^{*} + \frac{\omega}{2}\left(\vec{H}^{*}\cdot\bar{\mu}''\cdot\vec{H} + \vec{E}\cdot\bar{\epsilon}''^{*}\cdot\vec{E}^{*}\right)\right\} dV$$

$$+ 2j\omega \int_V \left\{ \frac{\vec{H}^{*}\cdot\bar{\mu}'\cdot\vec{H}}{4} - \frac{\vec{E}\cdot\bar{\epsilon}'^{*}\cdot\vec{E}^{*}}{4} \right\} dV$$

$$= \int_V \frac{1}{2}\left\{ \vec{E}\cdot\vec{J}^{*} + \frac{\omega}{2}\left(\vec{H}^{*}\cdot\bar{\mu}''\cdot\vec{H} + \vec{E}^{*}\cdot\bar{\epsilon}''\cdot\vec{E} \right)\right\} dV$$

$$+ 2j\omega \int_V \left\{ \frac{\vec{H}^{*}\cdot\bar{\mu}'\cdot\vec{H}}{4} - \frac{\vec{E}^{*}\cdot\bar{\epsilon}'\cdot\vec{E}}{4} \right\} dV \quad (2.65)$$

The first volume integral gives the power loss in the medium. The second integral is related to the stored reactive energy. An anisotropic medium is lossless only if $\bar{\epsilon}$ and $\bar{\mu}$ are hermitian, that is, the loss arises only from the anti-hermitian part.

A Variational Theorem

Below we will derive a variational theorem which is useful in determining group and energy velocities, Foster's reactance theorem, and other similar results.

$$\nabla \times \vec{E}^* = j\omega \bar{\mu}^* \cdot \vec{H}^* \quad , \quad \nabla \times \vec{H}^* = -j\omega \bar{\epsilon}^* \cdot \vec{E}^*$$

$$\nabla \times \frac{\partial \vec{E}^*}{\partial \omega} = j\omega \bar{\mu}^* \cdot \frac{\partial \vec{H}^*}{\partial \omega} + j\frac{\partial \omega \bar{\mu}^*}{\partial \omega} \cdot \vec{H}^* ,$$

$$\nabla \times \frac{\partial \vec{H}^*}{\partial \omega} = -j\omega \bar{\epsilon}^* \cdot \frac{\partial \vec{E}^*}{\partial \omega} - j\frac{\partial \omega \bar{\epsilon}^*}{\partial \omega} \cdot \vec{E}^*$$

Consider
$$\nabla \cdot \left[\vec{E} \times \frac{\partial \vec{H}^*}{\partial \omega} + \frac{\partial \vec{E}^*}{\partial \omega} \times \vec{H} \right] = \frac{\partial \vec{H}^*}{\partial \omega} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \frac{\partial \vec{H}^*}{\partial \omega}$$
$$+ \vec{H} \cdot \nabla \times \frac{\partial \vec{E}^*}{\partial \omega} - \frac{\partial \vec{E}^*}{\partial \omega} \cdot \nabla \times \vec{H} = -j\omega \frac{\partial \vec{H}^*}{\partial \omega} \cdot \bar{\mu} \cdot \vec{H}$$
$$+ j\omega \vec{H} \cdot \bar{\mu}^* \cdot \frac{\partial \vec{H}^*}{\partial \omega} + j\omega \vec{E} \cdot \bar{\epsilon}^* \cdot \frac{\partial \vec{E}^*}{\partial \omega} + j\vec{E} \cdot \frac{\partial \omega \bar{\epsilon}^*}{\partial \omega} \cdot \vec{E}^*$$
$$+ j\vec{H} \cdot \frac{\partial \omega \bar{\mu}^*}{\partial \omega} \cdot \vec{H}^* - j\omega \frac{\partial \vec{E}^*}{\partial \omega} \cdot \bar{\epsilon} \cdot \vec{E} \qquad (2.66)$$

For zero loss $\bar{\mu}^* = \bar{\mu}_t$, $\bar{\epsilon}^* = \bar{\epsilon}_t$ and the above simplifies to (the complex conjugate has been taken also)

$$-\nabla \cdot \left[\vec{E} \times \frac{\partial \vec{H}^*}{\partial \omega} + \frac{\partial \vec{E}^*}{\partial \omega} \times \vec{H} \right]^* = j\vec{H}^* \cdot \frac{\partial \omega \bar{\mu}}{\partial \omega} \cdot \vec{H} + j\vec{E}^* \cdot \frac{\partial \omega \bar{\epsilon}}{\partial \omega} \cdot \vec{E}$$

The right hand side is pure imaginary and hence the left hand side must also be pure imaginary.

Then we have (note that $\vec{H}^* \cdot \frac{\partial \omega \overline{\mu}}{\partial \omega} \cdot \vec{H}$ and $\vec{E}^* \cdot \frac{\partial \omega \overline{\epsilon}}{\partial \omega} \cdot \vec{E}$ are real)

$$\frac{1}{2} \nabla \cdot \left[\vec{E}^* \times \frac{\partial \vec{H}}{\partial \omega} + \frac{\partial \vec{E}}{\partial \omega} \times \vec{H}^* - \text{complex conjugate} \right]$$

$= -4j\,(U_e + U_m)$ where $U_e + U_m$ is the volume density of stored energy, i.e.

$$U_e + U_m = \frac{1}{4}\left[\vec{H}^* \cdot \frac{\partial \omega \overline{\mu}}{\partial \omega} \cdot \vec{H} + \vec{E}^* \cdot \frac{\partial \omega \overline{\epsilon}}{\partial \omega} \cdot \vec{E} \right] = \frac{1}{4}\,\mathrm{Imag}\, \nabla \cdot \left(\vec{E}^* \times \frac{\partial \vec{H}}{\partial \omega} + \frac{\partial \vec{E}}{\partial \omega} \times \vec{H}^* \right) \quad (2.67)$$

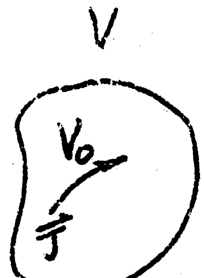
We will show next that this last expression is indeed the time average stored energy in a lossless, anisotropic medium per unit volume. This proof will be independent of the one given earlier which was based on an assumed model for the polarization.

Derivation of Expression for Stored Energy*

If we assume that the medium is free of loss then the stored energy at any time t can be evaluated in terms of the work done to establish the field from an initial value of zero at $t = -\infty$.

* See also A. Tonning, IRE Trans., vol. AP-8, July 1960, Landau and Lifshitz, "Electrodynamics of Continuous Media," sec. 61.

Consider a system of sources (currents) confined to a finite volume V_0. Let the currents have a time dependence of $\cos \omega t$. In the region V outside V_0 a field is produced which is given by



$$\vec{\mathcal{E}}(\vec{r},t) = \mathcal{R}e \; \vec{E}(\vec{r},\omega)\, e^{j\omega t}$$

$$\vec{\mathcal{H}}(\vec{r},t) = \mathcal{R}e \; \vec{H}(\vec{r},\omega)\, e^{j\omega t}$$

or
$$\vec{\mathcal{E}}(\vec{r},t) = \frac{\vec{E}}{2} e^{j\omega t} + \frac{\vec{E}^*}{2} e^{-j\omega t}$$

$$\vec{\mathcal{H}}(\vec{r},t) = \frac{\vec{H}}{2} e^{j\omega t} + \frac{\vec{H}^*}{2} e^{-j\omega t}$$

The fields $\vec{E}(\vec{r},\omega)$ and $\vec{H}(\vec{r},\omega)$ are solutions to the problem where the current has a time dependence $e^{j\omega t}$ while $\vec{E}^*$, $\vec{H}^*$ are the solutions for a time dependence of $e^{-j\omega t}$. If now the current source is assumed to have a dependence $e^{\alpha t} \cos \omega t$ for $-\infty \leq t \leq 0$ and $\alpha \ll \omega$ then this is equivalent to two problems with time dependence $e^{\alpha t + j\omega t}$ and $e^{\alpha t - j\omega t}$. The solutions for $\vec{E}$ and $\vec{H}$ may be obtained by using the first two terms of a Taylor series expansions about ω. Thus

$$\vec{E}(\vec{r}, \omega - j\alpha) = \vec{E}(\vec{r}, \omega) + \frac{\partial \vec{E}(\vec{r}, \omega)}{\partial \omega}(-j\alpha) \quad (2.68a)$$

$$\vec{H}(\vec{r}, \omega - j\alpha) = \vec{H}(\vec{r}, \omega) + \frac{\partial \vec{H}(\vec{r}, \omega)}{\partial \omega}(-j\alpha) \quad (2.68b)$$

are the solutions for the time dependence of $e^{\alpha t + j\omega t}$ while $\vec{E}^*(\vec{r}, \omega) + \frac{\partial \vec{E}^*(\vec{r}, \omega)}{\partial \omega} j\alpha$ and $\vec{H}^*(\vec{r}, \omega) + \frac{\partial \vec{H}^*(\vec{r}, \omega)}{\partial \omega} j\alpha$ are the solutions for the conjugate time dependence. The expansions are valid for $\alpha << \omega$. The physical fields are given by $\frac{\vec{E}(\vec{r}, \omega - j\alpha)}{2} e^{\alpha t + j\omega t} + \frac{\vec{E}^*(\vec{r}, \omega - j\alpha)}{2} e^{\alpha t - j\omega t}$

$$= \frac{\vec{E}(\vec{r}, \omega)}{2} e^{\alpha t + j\omega t} + \frac{\vec{E}^*(\vec{r}, \omega)}{2} e^{\alpha t - j\omega t} - j\alpha \frac{\partial \vec{E}(\vec{r}, \omega)}{\partial \omega} \frac{e^{\alpha t + j\omega t}}{2}$$

$$+ j\alpha \frac{\partial \vec{E}^*(\vec{r}, \omega)}{\partial \omega} \frac{e^{\alpha t - j\omega t}}{2}$$

and a similar expression for $\vec{\mathcal{H}}(\vec{r}, t)$.

The energy supplied per unit volume outside the source region is given by

$\int_{-\infty}^{t} -\nabla\cdot\ \vec{\mathcal{E}}(\vec{r},t)\times\vec{\mathcal{H}}(\vec{r},t)\,dt$ an expression which comes from interpreting the Poynting vector $\vec{\mathcal{E}}\times\vec{\mathcal{H}}$ as giving the instantaneous energy flow. That is, rate of energy flow into a volume V is $\oint_S \vec{\mathcal{E}}\times\vec{\mathcal{H}}\cdot(-d\vec{S})$ $= \int_V -\nabla\cdot\vec{\mathcal{E}}\times\vec{\mathcal{H}}\,dV$ and hence $-\nabla\cdot\vec{\mathcal{E}}\times\vec{\mathcal{H}}$ is the rate of energy flow into a unit volume. Substituting for $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ gives

$$\frac{1}{4}\int_{-\infty}^{t} -\nabla\cdot\Big[\big(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\big)e^{2\alpha t} + j\alpha\Big(-\frac{\partial\vec{E}}{\partial\omega}\times\vec{H}^* + \frac{\partial\vec{E}^*}{\partial\omega}\times\vec{H}$$

$$+ \vec{E}\times\frac{\partial\vec{H}^*}{\partial\omega} - \vec{E}^*\times\frac{\partial\vec{H}}{\partial\omega}\Big)e^{2\alpha t}\Big]dt + \text{terms in } e^{\pm 2j\omega t}$$

+ terms multiplied by α^2.

and where $\vec{E}$, $\vec{H}$, $\frac{\partial\vec{E}}{\partial\omega}$, $\frac{\partial\vec{H}}{\partial\omega}$ are evaluated for $\alpha=0$, i.e. at ω.

In a loss free medium $\nabla\cdot(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H})$ $= 2\,\mathrm{Re}\ \nabla\cdot(\vec{E}\times\vec{H}^*) = 0$ since from the complex Poynting vector theorem $\mathrm{Re}\oint_S \vec{E}\times\vec{H}^*\cdot(-d\vec{S}) = 0$ since there is no power dissipation in the volume bounded by S. We are going to eventually average over one period and let α tend to zero and then the terms involving $e^{\pm 2j\omega t}$ and those multiplied by α^2 will vanish. Hence we will not consider these terms any further (in detail).

Thus up to time t work done to establish field is

$$U(t) = -\frac{j\alpha}{4}\,\nabla\cdot\left(-\frac{\partial \vec{E}}{\partial \omega}\times \vec{H}^* + \frac{\partial \vec{E}^*}{\partial \omega}\times \vec{H} + \vec{E}\times\frac{\partial \vec{H}^*}{\partial \omega} - \vec{E}^*\times\frac{\partial \vec{H}}{\partial \omega}\right)\frac{e^{2\alpha t}}{2\alpha}$$

$$+ \text{ terms in } e^{\pm 2j\omega t} + \text{ terms multiplied by } \alpha^2$$

Averaging over one period and letting α tend toward zero we obtain the average work done and hence the average stored energy, i.e.

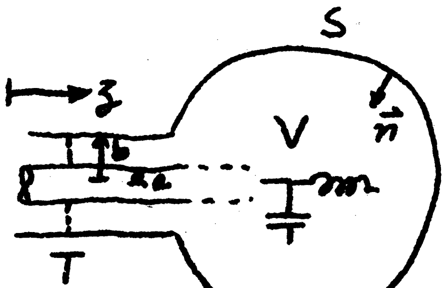
$$U = U_e + U_m = -\frac{j}{8}\nabla\cdot\left(-\frac{\partial \vec{E}}{\partial \omega}\times \vec{H}^* + \frac{\partial \vec{E}^*}{\partial \omega}\times \vec{H} + \vec{E}\times\frac{\partial \vec{H}^*}{\partial \omega} - \vec{E}^*\times\frac{\partial \vec{H}}{\partial \omega}\right)$$

$$= \frac{1}{4}\left[\vec{H}^*\cdot\frac{\partial\, \overline{\omega\mu}}{\partial \omega}\cdot\vec{H} + \vec{E}^*\cdot\frac{\partial\, \omega\overline{\epsilon}}{\partial \omega}\cdot\vec{E}\right]$$ upon comparison with the variational theorem. Thus the latter expression can be interpreted as average stored energy since it is equal to the work done in establishing the field.

Note that (1) a lossless medium has to be assumed so that we can equate the work done to stored energy, (2) α must be taken very small so that the field at time t can be regarded as periodic, (3) the terms in $e^{\pm 2j\omega t}$ represent energy flowing into and out of a unit volume in a periodic manner and does not contribute to the stored energy. The stored energy is only associated with the creation of the field itself.

Foster's Reactance Theorem

Consider a lossless network composed of L's and C's that may incorporate lossless dispersive media. Let the network be fed by a coaxial line and be shielded by an enclosure S as illustrated. Choose a terminal plane T on which the fields consist of only an incident and reflected TEM wave. The volume integral of the variational theorem gives



$$\text{Imag} \int_V \nabla \cdot \left(\vec{E}^* \times \frac{\partial \vec{H}}{\partial \omega} + \frac{\partial \vec{E}}{\partial \omega} \times \vec{H}^* \right) dV$$

$$= \text{Imag} \oint_S -\vec{n} \cdot \left(\vec{E}^* \times \frac{\partial \vec{H}}{\partial \omega} + \frac{\partial \vec{E}}{\partial \omega} \times \vec{H}^* \right) dS = -4j \int_V (U_e + U_m) dV$$

$$= -4j (W_e + W_m) = -4j \,(\text{total average energy stored in } V).$$

On S $\vec{n} \times \vec{E}^*$ and $\vec{n} \times \dfrac{\partial \vec{E}}{\partial \omega} = 0$ so we obtain

$$\text{Imag} \int_T \left(\vec{n} \times \vec{E}^* \cdot \frac{\partial \vec{H}}{\partial \omega} + \vec{n} \times \frac{\partial \vec{E}}{\partial \omega} \cdot \vec{H}^* \right) dS = 4j(W_e + W_m)$$

On T, $\vec{E} = \left\{ \frac{V^+ e^{-jkz}}{r \ln b/a} + \frac{V^- e^{jkz}}{r \ln b/a} \right\} \vec{a}_r = V \frac{\vec{a}_r}{r \ln b/a}$

$$\vec{H} = \left(\frac{I^+ e^{-jkz}}{2\pi r} - \frac{I^- e^{jkz}}{2\pi r} \right) \vec{a}_\phi = \frac{I \, \vec{a}_\phi}{2\pi r}$$

where V and I are the input voltage and current.

We now note that $\int_0^{2\pi} \int_a^b \vec{a}_z \times \vec{a}_r \cdot \vec{a}_\phi \frac{r \, d\phi \, dr}{2\pi r^2 \ln b/a}$

$= 1$ and hence we obtain

$$\text{Imag.} \left(V^* \frac{\partial I}{\partial \omega} + \frac{\partial V}{\partial \omega} I^* \right) = 4j \, (W_e + W_m)$$

But for a lossless network $I = jBV$ so $\frac{\partial I}{\partial \omega} = jB \frac{\partial V}{\partial \omega} + jV \frac{\partial B}{\partial \omega}$. Therefore $V^* \left(B \frac{\partial V}{\partial \omega} + V \frac{\partial B}{\partial \omega} \right) + \frac{\partial V}{\partial \omega} (-BV^*)$

$= VV^* \frac{\partial B}{\partial \omega} = 4 (W_e + W_m)$ and hence

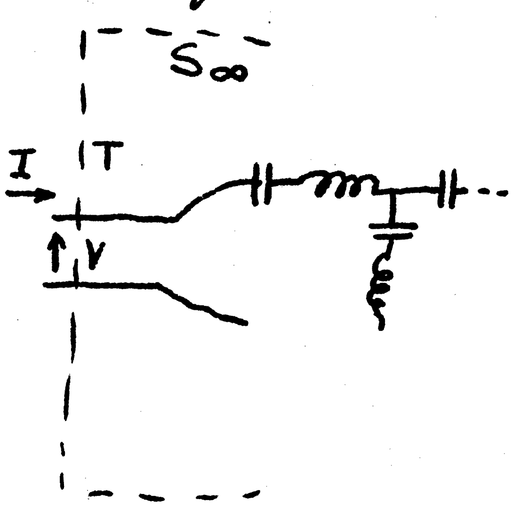
$$\frac{\partial B}{\partial \omega} = \frac{4 (W_e + W_m)}{VV^*} = \text{positive quantity} \qquad (2.69)$$

If we put $V = jXI$ we can also show that

$$\frac{\partial X}{\partial \omega} = \frac{4 (W_e + W_m)}{II^*} \qquad (2.70)$$

Thus the susceptance B and reactance X always have positive slopes. From this property one may show that B and X have only simple poles and zeroes and that these are interlaced or occur alternatingly as B (or X) are plotted as functions of ω. This latter property is called Foster's reactance theorem.

In the case of general waveguide inputs the voltage V, current I, and susceptance jB and reactance jX are defined in an equivalent way only, i.e. $V \propto \vec{E}$, $I \propto \vec{H}$, and $I = jBV$, $V = jXI$, but these quantities may still be used in the above formulas. For a conventional low frequency



network the surface S is chosen as the surface of an infinite hemisphere S_∞ on which the fields vanish (no radiation from the circuit is assumed). On the cross sectional plane T the fields are essentially those of a TEM transmission line mode for which results analogous to those given above hold.

2.9 Analytic Properties of $\bar{\chi}_e$

It is not possible to have a dispersive medium that is loss free, although there may be frequency bands throughout which the loss is very small. Each element of the constitutive parameters $\bar{\epsilon}$ and $\bar{\mu}$ are analytic functions for which the real part determines the imaginary part and vice versa by means of the well-known Hilbert transforms. To see how these properties come about, consider the polarization $\vec{P}(\omega) = \epsilon_0 \bar{\chi}_e(\omega) \cdot \vec{E}(\omega)$. Let

$$\vec{E}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{E}(\omega)\, e^{j\omega t}\, d\omega, \qquad \vec{E}(\omega) = \int_{-\infty}^{\infty} \vec{E}(t)\, e^{-j\omega t}\, dt$$

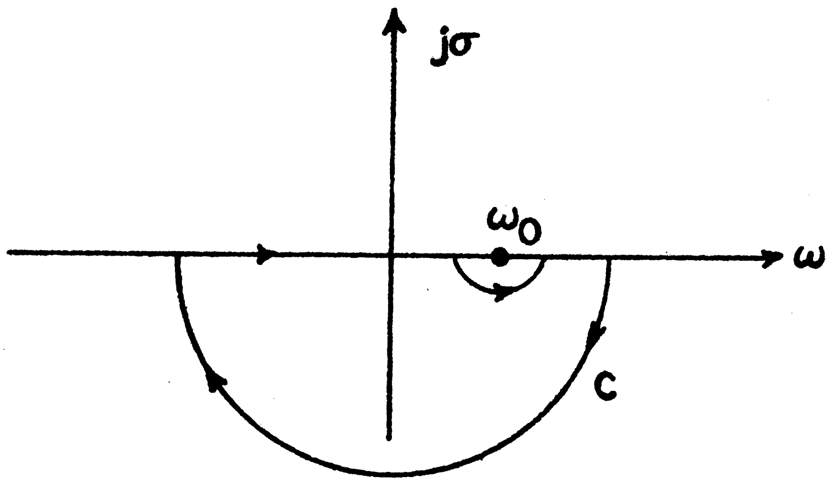
which are the usual Fourier transform relations. If $\vec{E}(t) \equiv 0$ for $t < 0$ then $\vec{E}(\omega)$ is free of all singularities in the lower $\omega + j\sigma$ complex plane since for $t < 0$ the integral for $\vec{E}(t)$ can be evaluated by closing the contour in the lower half plane and must give zero. For $\vec{E}(t)$ to be a real function, $\vec{E}(\omega) = \vec{E}^*(-\omega)$. When $\vec{E}(t)$ is zero for $t < 0$, causality requires the polarization $\vec{P}(t)$ to also be zero for $t < 0$. Hence, $\bar{\chi}_e(\omega) \cdot \vec{E}(\omega)$, which is the Fourier transform of $\vec{P}(t)$, is also analytic in the lower half plane. Thus, $\bar{\chi}_e(\omega)$ is analytic in the lower half plane. For a physical medium, each element χ_{ij} of $\bar{\chi}_e$ will vanish to order ω^{-a}, $a > 0$, as $|\omega|$ tends to infinity. For later use, we also note that $\chi_{ij}(\omega) = \chi^*_{ij}(-\omega)$ since $\vec{P}(t)$ must be real. The above analytic properties of $\bar{\chi}_e$ are a direct result of the physical properties of $\vec{P}(t)$ as given by

$$\vec{P}(t) = \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty} \bar{\chi}_e(\omega) \cdot \vec{E}(\omega)\, e^{j\omega t}\, d\omega$$

For the element $\chi_{ij} = \chi'_{ij} - j\chi''_{ij}$ the contour integral

$$I = \frac{1}{2\pi j}\int_C \frac{\chi_{ij}(\omega)}{\omega - \omega_0}\,d\omega = 0$$

since χ_{ij} is analytic within C and the contour C shown in the figure excludes the point ω_0.



The contribution to the integral over the semi-circle is zero and hence

$$I = \pi j\,\chi_{ij}(\omega_0) + P\int_{-\infty}^{\infty} \frac{\chi_{ij}(\omega)}{\omega - \omega_0}\,d\omega = 0 \qquad (2.75)$$

where P denotes the principal value. From this expression it is easy to show, as required by the exercise below, that

$$\chi'_{ij}(\omega_0) = \frac{2}{\pi}\,P\int_0^{\infty} \frac{\omega\chi''_{ij}(\omega)}{\omega^2 - \omega_0^2}\,d\omega \qquad (2.76a)$$

$$\chi''_{ij}(\omega_0) = -\frac{2}{\pi}\,P\int_0^{\infty} \frac{\omega_0\chi'_{ij}(\omega)}{\omega^2 - \omega_0^2}\,d\omega \qquad (2.76b)$$

These relations, which are Hilbert transforms, are known in the literature as the Kröning-Kremers relations (for a discussion and reference to the original papers see Kerr)* and were extended to a ferrite medium by Gourey.**

Exercise

Split (2.75) into its real and imaginary parts, use the property $\chi_{ij}(\omega) = \chi^*_{ij}(-\omega)$ to obtain $\chi'_{ij}(\omega) = \chi'_{ij}(-\omega)$ and $\chi''_{ij}(\omega) = -\chi''_{ij}(-\omega)$, and thus derive (2.76).

2.10 Principal Axis

For a loss free medium, $\bar{\epsilon}$ is hermitian and for non-gyrotropic media it is real and symmetrical (which is a special case of a hermitian dyadic). In a principal axis reference frame, $\vec{D}$ and $\vec{E}$ are parallel along a principal axis, by definition. Hence, $\vec{D} = \bar{\epsilon} \cdot \vec{E} = \epsilon\vec{E}$. For a solution of $(\bar{\epsilon} - \bar{I}\epsilon) \cdot \vec{E} = 0$ we must have the determinant $|\bar{\epsilon} - \epsilon\bar{I}| = 0$. When $\bar{\epsilon}$ is hermitian there are three real eigenvalues, $\epsilon = \epsilon_1, \epsilon_2, \epsilon_3$, for this equation. For each eigenvalue ϵ_i there exists an eigenvector with a unit vector $\vec{n}_i$ defining its orientation. The three $\vec{n}_i$ specify the principal axis. When the ϵ_i are all unequal (non-degenerate) the $\vec{n}_i$ are mutually orthogonal. For degenerate eigenvalues the corresponding eigenvectors are linearly dependent but may be chosen to form an orthogonal set.

Proof of Orthogonality

We have $\bar{\epsilon} \cdot \vec{n}_i = \epsilon_i\vec{n}_i$ by definition, which can also be written as

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*Kerr, D. E., "Propagation of Short Radio Waves," McGraw-Hill Book Company, Inc., New York, 1951, Chapter 8.

**Gourey, B. S., "Dispersion Relations for Tensor Media and Their Application to Ferrites," J. Appl. Phys., Volume 28, Pages 283- , 1957.

$$\hat{n}_i \cdot \bar{\varepsilon}^*_t = \varepsilon_i \hat{n}_i$$

since $\bar{\varepsilon}$ is hermitian. Also, we have

$$\bar{\varepsilon} \cdot \hat{n}_j = \varepsilon_j \hat{n}_j$$

From these two equations we form

$$\hat{n}_i \cdot \bar{\varepsilon}^*_t \cdot \hat{n}_j - \hat{n}_i \cdot \bar{\varepsilon} \cdot \hat{n}_j = (\varepsilon_i - \varepsilon_j)\hat{n}_i \cdot \hat{n}_j$$

But the left hand side equals $\hat{n}_i \cdot (\bar{\varepsilon}^*_t - \bar{\varepsilon}) \cdot \hat{n}_j = 0$ because $\bar{\varepsilon}$ is hermitian. Hence, if $\varepsilon_i \neq \varepsilon_j$ we must have $\hat{n}_i \cdot \hat{n}_j = 0$.

Real Property of Eigenvalues

As above we can write

$$\bar{\varepsilon} \cdot \hat{n}_i = \varepsilon_i \hat{n}_i$$

$$\bar{\varepsilon}^* \cdot \hat{n}^*_i = \varepsilon^*_i \hat{n}^*_i$$

from which we can form

$$\hat{n}^*_i \cdot \bar{\varepsilon} \cdot \hat{n}_i - \hat{n}_i \cdot \bar{\varepsilon}^* \cdot \hat{n}^*_i = (\varepsilon_i - \varepsilon^*_i)\hat{n}_i \cdot \hat{n}^*_i$$

As before, the left hand side vanishes because

$$\vec{n}^*_i \cdot \bar{\varepsilon} \cdot \vec{n}_i = \vec{n}_i \cdot \bar{\varepsilon}_t \cdot \vec{n}^*_i \quad \text{and} \quad \bar{\varepsilon}_t = \bar{\varepsilon}^*$$

for a hermitian dyadic. Hence, $\varepsilon_i = \varepsilon^*_i$ and is real.

When $\bar{\varepsilon}$ is real and symmetric then $\bar{\varepsilon}^* = \bar{\varepsilon}$ and

$$\bar{\varepsilon} \cdot \vec{n}_i = \varepsilon_i \vec{n}_i, \; \bar{\varepsilon}^* \cdot \vec{n}^*_i = \bar{\varepsilon} \cdot \vec{n}^*_i = \varepsilon_i \vec{n}^*_i$$

so both $\vec{n}_i$ and $\vec{n}^*_i$ are eigenvectors. If $\vec{n}_i$ and $\vec{n}^*_i$ are linearly dependent we choose $\frac{1}{2}(\vec{n}_i + \vec{n}^*_i)$ as a real eigenvector. If $\vec{n}_i$ and $\vec{n}^*_i$ are linearly independent then

$$\frac{1}{2}(\vec{n}_i + \vec{n}^*_i) \quad \text{and} \quad \frac{j}{2}(\vec{n}_i - \vec{n}^*_i)$$

are new linearly independent real eigenvectors.

In the general case when $\bar{\varepsilon}$ is hermitian the eigenvectors are complex but the eigenvalues are real as shown above.

$$\hat{n}^*_i \cdot \bar{\varepsilon} \cdot \hat{n}_i = \hat{n}_i \cdot \bar{\varepsilon}_t \cdot \hat{n}^*_i \quad \text{and} \quad \bar{\varepsilon}_t = \bar{\varepsilon}^*$$

for a hermitian dyadic. Hence, $\varepsilon_i = \varepsilon^*_i$ and is real.

When $\bar{\varepsilon}$ is real and symmetric then $\bar{\varepsilon}^* = \bar{\varepsilon}$ and

$$\bar{\varepsilon} \cdot \hat{n}_i = \varepsilon_i \hat{n}_i, \ \bar{\varepsilon}^* \cdot \hat{n}^*_i = \bar{\varepsilon} \cdot \hat{n}^*_i = \varepsilon_i \hat{n}^*_i$$

so both $\hat{n}_i$ and $\hat{n}^*_i$ are eigenvectors. If $\hat{n}_i$ and $\hat{n}^*_i$ are linearly dependent we choose $\frac{1}{2}(\hat{n}_i + \hat{n}^*_i)$ as a real eigenvector. If $\hat{n}_i$ and $\hat{n}^*_i$ are linearly independent then

$$\frac{1}{2}(\hat{n}_i + \hat{n}^*_i) \quad \text{and} \quad \frac{j}{2}(\hat{n}_i - \hat{n}^*_i)$$

are new linearly independent real eigenvectors.

In the general case when $\bar{\varepsilon}$ is hermitian the eigenvectors are complex but the eigenvalues are real as shown above.