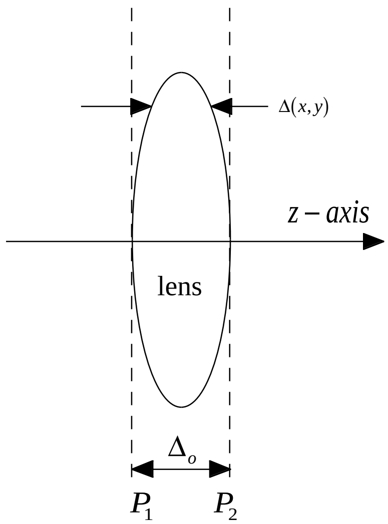
APPENDIX I

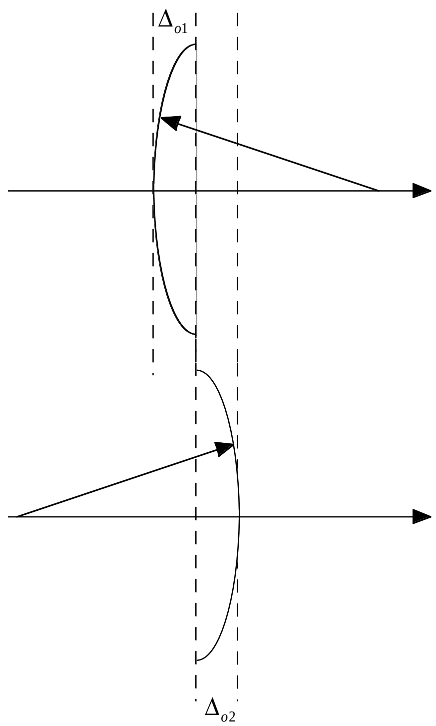
Let the maximum thickness of the lens be Δ_o and let the thickness at coordinates (x, y) be $\Delta(x, y)$. We consider the phase delay between two planes P_1 and P_2 separated by Δ_o along the z-axis and enclosing the lens as shown below. The total phase delay from P_1 to P_2 is given by

$$\phi(x,y) = kn\Delta(x,y) + k\left[\Delta_o - \Delta(x,y)\right] = k\Delta_o + k\Delta(x,y)(n-1) \qquad (1)$$

where n is the index of refraction of the lens material, $kn\Delta(x,y)$ is the phase delay introduced by the lens, and $k\left[\Delta_o - \Delta(x,y)\right]$ is the phase delay due to the remaining free space between P_1 and P_2.



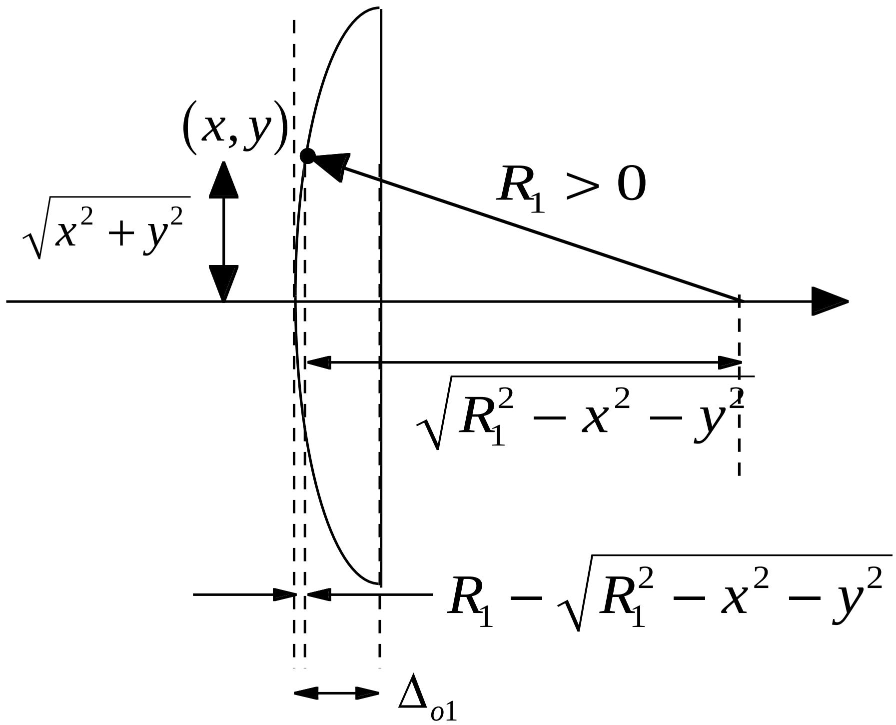
To develop a functional form for $\Delta(x,y)$ we split the lens into two parts as shown below.



And write the total thickness function as the sum of the two individual thickness functions

$$\Delta(x,y) = \Delta_1(x,y) + \Delta_2(x,y) \tag{2}$$

where Δ_1 and Δ_2 are the thickness of each lens section. For the first lens section



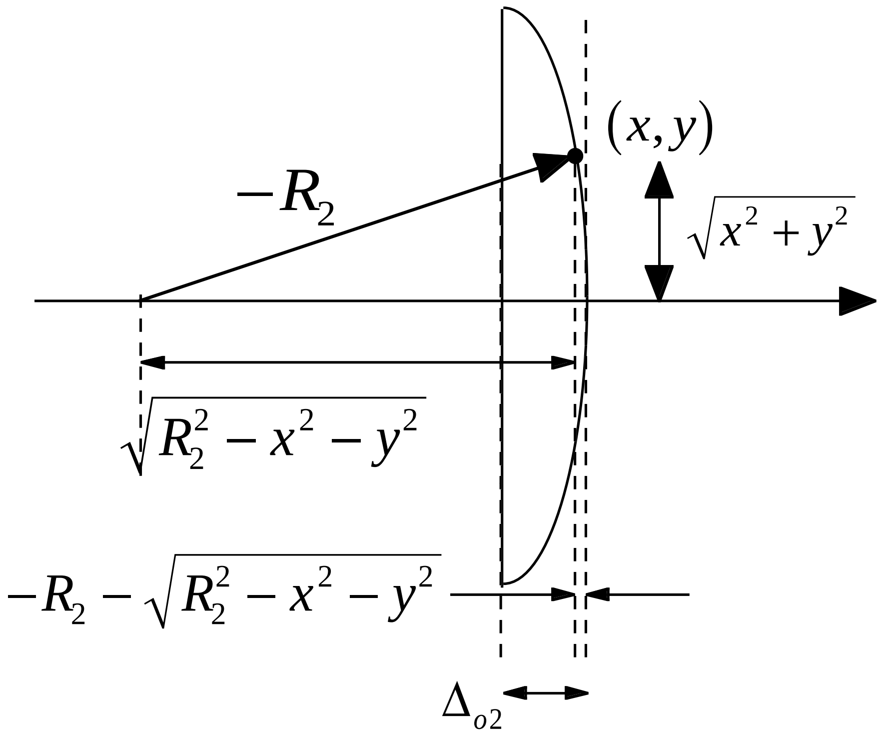
where $R_1 > 0$ by our sign convention. From the drawing

$$\Delta_1(x,y) = \Delta_{01} - \left(R_1 - \sqrt{R_1^2 - x^2 - y^2}\right) = \Delta_{01} - R_1\left(1 - \sqrt{1 - \frac{x^2 + y^2}{R_1^2}}\right)$$

and, using the paraxial ray approximation,

$$\Delta_1(x,y) \approx \Delta_{01} - R_1\left(1 - 1 + \frac{x^2 + y^2}{2R_1^2}\right) = \Delta_{01} - \frac{x^2 + y^2}{2R_1} \qquad (3)$$

For the second lens segment



The calculations proceed in exactly the same manner as for equation (3) but R_2 is negative so for our distances to be positive we use $-R_2$ to get

$$\Delta_2(x,y) = \Delta_{02} - \left(-R_2 - \sqrt{R_2^2 - x^2 - y^2}\right) = \Delta_{02} + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

Note here that we factored $-R_2$ out of the square root to get

$$\Delta_2(x,y) \approx \Delta_{02} + \frac{x^2+y^2}{2R_2} \qquad (5)$$

Substituting (4) and (5) into (3) we get

$$\Delta(x,y) = \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \qquad (6)$$

Substituting (6) into (1) we get

$$\phi(x,y) = k\Delta_o + k\Delta_o(n-1) - k(n-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = kn\Delta_o - k(n-1)\frac{\left(x^2+y^2\right)}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

(7)

This may be simplified by recalling our geometric optics definition of the focal length of a thin lens

$$\frac{1}{f} = P_1 + P_2 = (n_2 - n_1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{8}$$

Using (8) we may re-write (7) as

$$\phi(x,y) = kn\Delta_o - k\frac{(x^2+y^2)}{2f} \tag{9}$$

The thin lens transformation may be written as a pure phase transformation of the incident beam u_1

$$u_2(x,y) = e^{-j\phi(x,y)}u_1(x,y) = e^{-jkn\Delta_o + jk\frac{(x^2+y^2)}{2f}}u_1(x,y) \tag{10}$$

where the reason for the choice of a minus sign will become apparent.

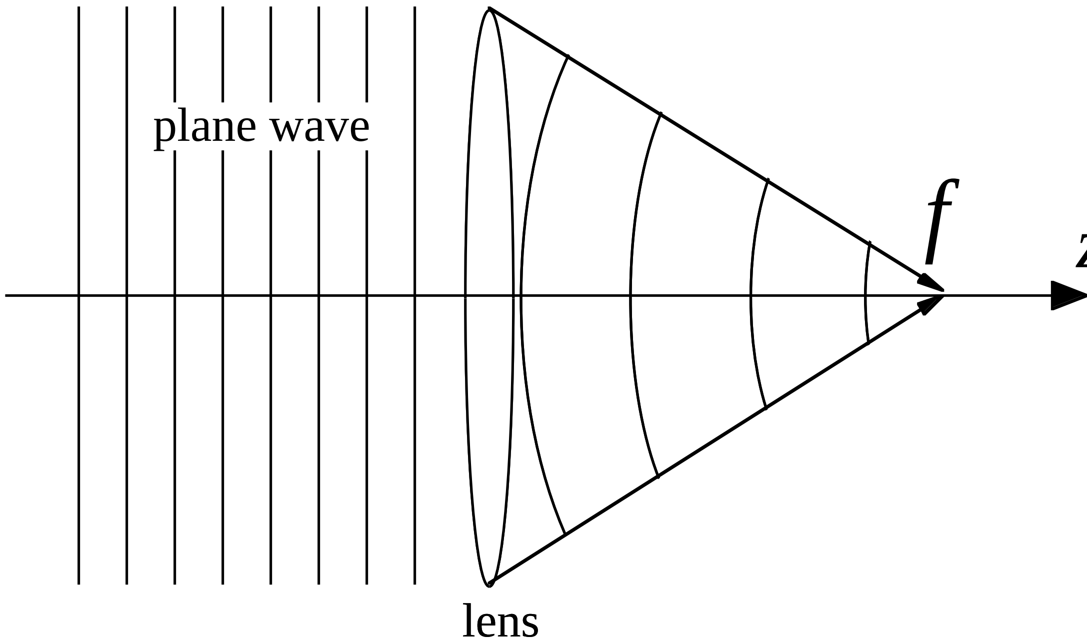
To illustrate that this is a valid model of a lens let

$$u_1(x,y) = E_o e^{-jkz} \tag{11}$$

a plane wave propagating in the $+z$ direction. From (9) and (10) we have

$$u_2(x,y) = e^{-jkn\Delta_o + jk\frac{(x^2+y^2)}{2f}}\left(E_o e^{-jkz}\right) = \left(E_o e^{-jkn\Delta_o}\right)e^{+jk\frac{(x^2+y^2)}{2f} - jkz} \tag{12}$$

where the factor $E_o e^{-jkn\Delta_o}$ is simply a complex constant. It is the exponential phase factor that results in the lens focusing the wave to a spot. From equation (2), page 6 we may identify the exponential phase factor $e^{+jk\frac{(x^2+y^2)}{2R(z)} - jkz}$ as corresponding to a "paraxial" ray approximation of a wave propagating in the $+z$ direction with spherical wavefronts of radius of curvature $-f$ as shown below.



The sign convention for the wavefront curvature is the opposite for that of curved reflecting mirrors so the wavefront curvature shown above is negative, i.e., $R(z) = -f$. This describes a spherical wave with a wavefront that if propagating in the $+z$ direction will converge to a point at a distance f from the lens.

It is interesting to observe that equation (17) can yield the wavefront transformation predicted by geometric optics, equation (6), p.61

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

Let $u_1(x,y) = E_o e^{-jkz - jk\frac{(x^2+y^2)}{2R_1}}$, a spherical wave propagating in the $+z$ direction. From (17) and (16) we have

$$u_2(x,y) = E_o e^{-jkz - jk\frac{(x^2+y^2)}{2R_1}} e^{-jkn\Delta_o + jk\frac{(x^2+y^2)}{2f}} = \left[E_o e^{-jkn\Delta_o}\right] e^{-jkz - jk\frac{(x^2+y^2)}{2}\left(\frac{1}{R_1} - \frac{1}{f}\right)} \qquad (13)$$

Let us write u_2 as a "paraxial" spherical wave of amplitude $E_o e^{-jkn\Delta_o}$. Then,

$$u_2(x,y) = \left[E_o e^{-jkn\Delta_o}\right] e^{-jkz - jk\frac{(x^2+y^2)}{2R_2}} = \hat{E}_o e^{-jkz - jk\frac{(x^2+y^2)}{2R_2}} \tag{14}$$

From (13) and (14) we then have the desired result

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{15}$$

APPENDIX II

$$z_1 = \frac{R_1}{2} \pm \frac{1}{2}\sqrt{R_1^2 - 4z_R^2}$$

$$z_2 = \frac{R_2}{2} \pm \frac{1}{2}\sqrt{R_2^2 - 4z_R^2}$$

$$z_2 - z_1 = \ell$$

$$\frac{R_2}{2} \pm \frac{1}{2}\sqrt{R_2^2 - 4z_R^2} - \frac{R_1}{2} \mp \frac{1}{2}\sqrt{R_1^2 - 4z_R^2} = \ell$$

$$R_2 \pm \sqrt{R_2^2 - 4z_R^2} - R_1 \mp \sqrt{R_1^2 - 4z_R^2} = 2\ell$$

$$\pm\sqrt{R_2^2 - 4z_R^2} \mp \sqrt{R_1^2 - 4z_R^2} = 2\ell + R_1 - R_2$$

$$R_2^2 - 4z_R^2 + R_1^2 - 4z_R^2 \mp \sqrt{\left(R_2^2 - 4z_R^2\right)\left(R_1^2 - 4z_R^2\right)} = \left[2\ell + R_1 - R_2\right]^2 = 4\ell^2 + 4\ell\left(R_1 - R_2\right) + R_1^2 - 2R_1R_2 + R_2^2$$

$$\mp\sqrt{\left(R_2^2 - 4z_R^2\right)\left(R_1^2 - 4z_R^2\right)} = 4\ell^2 + 4\ell\left(R_1 - R_2\right) + 8z_R^2 - 2R_1R_2$$

$$4\left(R_2^2 - 4z_R^2\right)\left(R_1^2 - 4z_R^2\right) = 16\ell^4 + 16\ell^2\left(R_1 - R_2\right)^2 + 64z_R^4 + 32\ell^3\left(R_1 - R_2\right) + 64\ell^2 z_R^2 + 64\ell z_R^2\left(R_1 - R_2\right)$$

$$4\left(R_2^2R_1^2 - 4z_R^2\left(R_1^2 + R_2^2\right) + 16z_R^4\right) = 16\ell^4 + 16\ell^2\left(R_1 - R_2\right)^2 + 64z_R^4 + 32\ell^3\left(R_1 - R_2\right) + 64\ell^2 z_R^2 + 64\ell z_R^2\left(R_1 - R_2\right)$$

$$4R_2^2R_1^2 - 16z_R^2R_1^2 - 16z_R^2R_2^2 + 16z_R^4 = 16\ell^4 + 16\ell^2\left(R_1 - R_2\right)^2 + 64z_R^4 + 32\ell^3\left(R_1 - R_2\right) + 64\ell^2 z_R^2 + 64\ell z_R^2\left(R_1 - R_2\right)$$

$$4R_1^2R_2^2 - 16\ell^4 - 16\ell^2\left(R_1 - R_2\right)^2 + 32\ell^3\left(R_1 - R_2\right) = 16z_R^2\left(R_1^2 + R_2^2\right) + 64\ell^2 z_R^2 + 64\ell z_R^2\left(R_1 - R_2\right)$$

$$z_R^2 = \frac{-16\ell^4 - 16\ell^2\left(R_1 - R_2\right)^2 + 32\ell^3\left(R_1 - R_2\right)}{16\left(R_1^2 + R_2^2\right) + 64\ell^2 + 64\ell\left(R_1 - R_2\right)}$$

$$z_R^2 = \frac{-\ell^4 - \ell^2\left(R_1 - R_2\right)^2 + 2\ell^3\left(R_1 - R_2\right)}{\left(R_1^2 + R_2^2\right) + 4\ell^2 + 4\ell\left(R_1 - R_2\right)}$$

$$z_R^2 = \frac{-\ell^4 + 2\ell^3\left(R_1 - R_2\right) + \ell^2\left(R_1R_2 - \left(R_1 - R_2\right)^2\right) + \ell R_1R_2\left(R_1 - R_2\right)^2}{\left(R_1^2 - 2R_1R_2 + R_2^2\right) + 4\ell\left(R_1 - R_2\right) + 4\ell^2}$$

$$z_R^2 = \frac{\ell\left\{R_1R_2\left(R_1 - R_2\right) + \ell\left[R_1R_2 - R_1^2 + 2R_1R_2 - R_2^2\right] + 2\ell^3\left(R_1 - R_2\right) - \ell^3\right\}}{\left(2\ell + R_1 - R_2\right)^2}$$

$$z_R^2 = \frac{\ell\left\{R_1^2R_2 - R_1R_2^2 + \ell^3R_1R_2 - \ell R_1^2 - \ell R_2^2 + 2\ell^2R_1 - 2\ell^2R_2 - \ell^3\right\}}{\left(2\ell + R_1 - R_2\right)^2}$$

$$z_R^2 = \frac{\ell\left(-R_1 - \ell\right)\left(R_2 - \ell\right)\left(R_2 - R_1 - \ell\right)}{\left(2\ell + R_1 - R_2\right)^2}$$

APPENDIX III

Show that a symmetrical confocal resonator has a minimum spot size.

For a symmetrical resonator $\omega_{1,2} = \sqrt{\frac{\lambda\ell}{2\pi}}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right)^{\frac{1}{4}}$

$$\frac{\partial\omega_{1,2}}{\partial R} = \sqrt{\frac{\lambda\ell}{2\pi}}\,\frac{1}{4}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right)^{-\frac{3}{4}}\frac{\partial}{\partial R}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right)$$

where

$$\frac{\partial}{\partial R}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right) = \frac{2R^2}{\ell}\left(R-\frac{\ell}{2}\right)^{-1} = \frac{2R}{\ell}\left(R-\frac{\ell}{2}\right)^{-1} + \frac{2R^2}{\ell}(-1)\left(R-\frac{\ell}{2}\right)^{-2}$$

$$\frac{\partial}{\partial R}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right) = \frac{4}{\ell}\,\frac{R}{R-\frac{\ell}{2}} + \frac{4}{\ell}\,\frac{\frac{R^2}{2}}{\left(R-\frac{\ell}{2}\right)^2} = \frac{4}{\ell}\left\{\frac{R^2 - \frac{R\ell}{2} + \frac{R^2}{2}}{\left(R-\frac{\ell}{2}\right)^2}\right\}$$

$$\frac{\partial}{\partial R}\left(\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}\right) = \frac{4}{\ell}\left\{\frac{\frac{R^2}{2} - \frac{R\ell}{2}}{\left(R-\frac{\ell}{2}\right)^2}\right\} = \frac{2}{\ell}\,\frac{R(R-\ell)}{\left(R-\frac{\ell}{2}\right)^2}$$

$$\therefore \frac{\partial\omega_{1,2}}{\partial R} = 0 \Rightarrow (R-\ell) = 0$$

For a minimum spot size we require $R = \ell$. This is a symmetrical confocal resonator.

$$\omega_{o,confical} = \sqrt{\frac{\lambda}{\pi}\left(\frac{\ell}{2}\right)^{\frac{1}{4}}\left(\ell-\frac{\ell}{2}\right)^{\frac{1}{4}}} = \sqrt{\frac{\lambda\ell}{2\pi}}$$

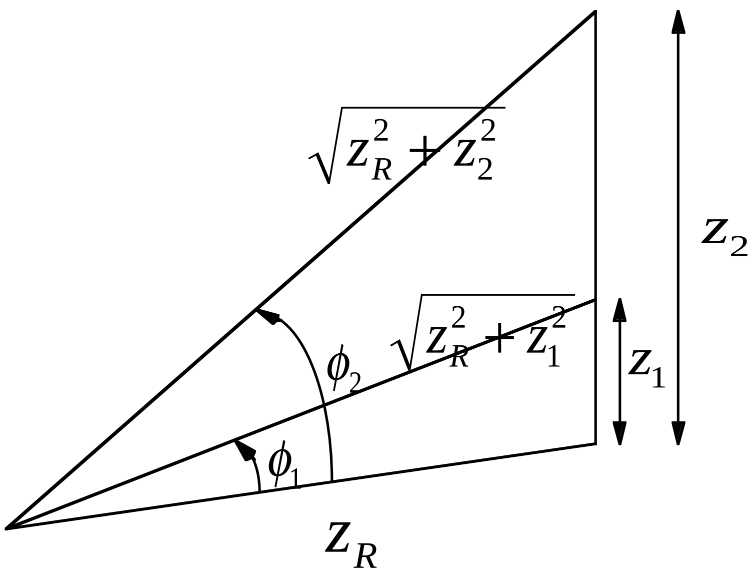
$$\omega_{1,2,confocal} = \sqrt{\frac{\lambda\ell}{2\pi}}\left(\frac{2\ell^2}{\ell\left(\ell - \frac{\ell}{2}\right)}\right)^{\frac{1}{4}} = \sqrt{\frac{\lambda\ell}{2\pi}}\left(\frac{2\ell^2}{\ell(\ell)}\right)^{\frac{1}{4}} = \sqrt{\frac{\lambda\ell}{\pi}} = \sqrt{2}\omega_{o,confical}$$

i.e., the cavity configuration is a Rayleigh distance collimated beam.

APPENDIX IV

$$\phi_2 - \phi_1 = Tan^{-1}\left(\frac{z_2}{z_R}\right) - Tan^{-1}\left(\frac{z_1}{z_R}\right)$$

where $z_R^2 = \dfrac{\ell(-R_1 - \ell)(R_2 - \ell)(R_2 - R_1 - \ell)}{(2\ell + R_1 - R_2)^2}$



Using the law of cosines $c^2 = a^2 + b^2 - 2ab\cos(\phi_2 - \phi_1)$ we can write

$$\cos(\phi_2 - \phi_1) = \frac{a^2 + b^2 - c^2}{2ab}$$

We can then identify

$$a = \sqrt{z_1^2 + z_R^2}$$

$$b = \sqrt{z_2^2 + z_R^2}$$

$$c = z_2 - z_1$$

and solve for $\cos(\phi_2 - \phi_1)$

$$\cos(\phi_2 - \phi_1) = \frac{z_1^2 + z_R^2 + z_2^2 + z_R^2 - z_2^2 + 2z_1z_2 - z_1^2}{2\sqrt{(z_1^2 + z_R^2)(z_2^2 + z_R^2)}} = \frac{z_R^2 + z_1z_2}{\sqrt{(z_1^2 + z_R^2)(z_2^2 + z_R^2)}}$$

The goal is to now write this expression in terms of d and R.

$$z_1 = \frac{R_1 - \sqrt{R_1^2 - 4z_R^2}}{2}$$

$$z_2 = \frac{R_2 + \sqrt{R_2^2 - 4z_R^2}}{2}$$

$$\cos(\phi_2 - \phi_1) = \frac{z_R^2 + \frac{R_1R_2}{4} - \frac{R_2}{4}\sqrt{R_1^2 - 4z_R^2} + \frac{R_1}{4}\sqrt{R_2^2 - 4z_R^2} - R_1^2R_2^2}{\sqrt{(z_1^2 + z_R^2)(z_2^2 + z_R^2)}}$$

$$z_2 = \frac{g_1(1-g_2)}{(g_1 + g_2 - 2g_1g_2)}d$$

$$z_R^2 + z_1z_2 = d^2\left\{\frac{g_1g_2 - g_1^2g_2^2}{(g_1 + g_2 - 2g_1g_2)^2} + \frac{g_1(1-g_2)(-g_2)(1-g_1)}{(g_1 + g_2 - 2g_1g_2)^2}\right\}$$

$$z_R^2 + z_1z_2 = \frac{d^2}{(g_1 + g_2 - 2g_1g_2)^2}\left\{g_1g_2 - g_1^2g_2^2 - g_1g_2 + g_1g_2^2 + g_1^2g_2 - g_1^2g_2^2\right\}$$

$$z_R^2 + z_1z_2 = d^2\,\frac{-2g_1^2g_2^2 + g_1g_2^2 + g_1^2g_2}{(g_1 + g_2 - 2g_1g_2)^2} = d^2\,\frac{g_1g_2}{(g_1 + g_2 - 2g_1g_2)}$$

$$(z_1^2 + z_R^2)(z_2^2 + z_R^2) = d^4\left\{\frac{\left[g_2^2(1-2g_1 + g_1^2) + g_1g_2 - g_1^2g_2^2\right]\left[g_1^2(1-2g_2 + g_2^2) + g_1g_2 - g_1^2g_2^2\right]}{(g_1 + g_2 - 2g_1g_2)^4}\right\}$$

$$(z_1^2 + z_R^2)(z_2^2 + z_R^2) = \frac{d^4}{(g_1 + g_2 - 2g_1g_2)^4}\left\{\left[g_2^2(1-2g_1 + g_1^2) + g_1g_2 - g_1^2g_2^2\right]\left[g_1^2(1-2g_2 + g_2^2) + g_1g_2 - g_1^2g_2^2\right]\right\}$$

$$(z_1^2 + z_R^2)(z_2^2 + z_R^2) = \frac{d^4}{(g_1 + g_2 - 2g_1g_2)^4}\left\{\left[g_2^2 - 2g_1g_2^2 + g_1g_2\right]\left[g_1^2 - 2g_1^2g_2 + g_1g_2\right]\right\}$$

$$(z_1^2 + z_R^2)(z_2^2 + z_R^2) = \frac{d^4}{(g_1 + g_2 - 2g_1g_2)^4}\,g_1g_2\left\{\left[g_2 - 2g_1g_2 + g_1\right]\left[g_1 - 2g_1g_2 + g_2\right]\right\}$$

$$(z_1^2 + z_R^2)(z_2^2 + z_R^2) = \frac{d^4}{(g_1 + g_2 - 2g_1g_2)^4}\,g_1g_2\left\{2g_1g_2 - 4g_1g_2^2 - 4g_1^2g_2 + g_1^2 + g_2^2 + 4g_1^2g_2^2\right\}$$

Using these expressions we can solve

$$\cos(\phi_2 - \phi_1) = \frac{d^2 \dfrac{g_1 g_2}{(g_1 + g_2 - 2g_1 g_2)}}{\dfrac{d^2}{(g_1 + g_2 - 2g_1 g_2)^2}\sqrt{g_1 g_2}\sqrt{2g_1 g_2 - 4g_1 g_2^2 - 4g_1^2 g_2 + g_1^2 + g_2^2 + 4g_1^2 g_2^2}}$$

$$\cos(\phi_2 - \phi_1) = \sqrt{g_1 g_2}\frac{g_1 + g_2 - 2g_1 g_2}{\sqrt{2g_1 g_2 - 4g_1 g_2^2 - 4g_1^2 g_2 + g_1^2 + g_2^2 + 4g_1^2 g_2^2}} = \sqrt{g_1 g_2}\frac{g_1 + g_2 - 2g_1 g_2}{\sqrt{(g_1 + g_2 - 2g_1 g_2)^2}} = \sqrt{g_1 g_2}$$

$$\cos(\phi_2 - \phi_1) = \sqrt{g_1 g_2}$$

APPENDIX V
BASIC FORMULAS SUMMARY

TEM_{00} Gaussian beam formulas

E-field solutions of wave equation under assumptions

1. $k^2(\vec{r}) = k^2 = \left(\frac{2\pi}{\lambda}\right)^2$

2. $\frac{\partial}{\partial \phi} = 0$ (radial symmetry)

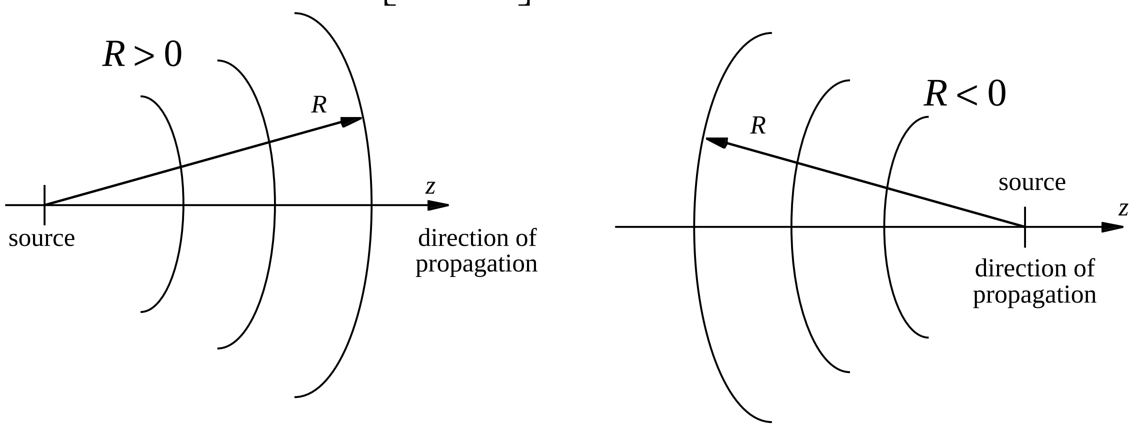
3. $\left|\frac{\partial^2 \varphi}{\partial t^2}\right| << \left|2k\frac{\partial \varphi}{\partial t}\right|$

$$E(x,y,z) = E_0 \frac{\omega_0}{\omega(z)} e^{-ik\frac{r^2}{2\hat{q}} - ikz + i\phi}$$

$$E(x,y,z) = E_0 \frac{\omega_0}{\omega(z)} e^{-ik\frac{r^2}{2R(z)} - \frac{r^2}{\omega^2(z)} - ikz + i\phi}$$

Complex radius of curvature $\hat{q}(z)$: $\frac{1}{\hat{q}(z)} = \frac{1}{\hat{q}_0 + z} = \frac{1}{R(z)} - i\frac{\lambda}{\pi\omega^2(z)}$

Radius of curvature: $R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right] = z + \frac{z_R^2}{z} \approx z$ if $z >> z_R$

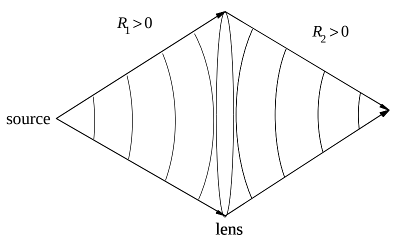
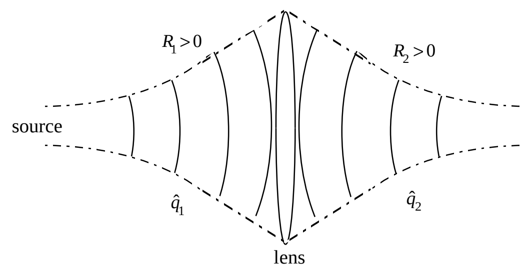


e^{-1} amplitude beam radius: $\omega(z) = \omega_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2}$

Rayleigh distance (collimated beam distance): $z_R = \frac{\pi\omega_0^2}{\lambda}$

Divergence angle: $\theta \approx \frac{\lambda}{\pi\omega_0}$ for $z >> z_R$

Transformation of waves

	spherical	gaussian
Through space	$R_2 = R_1 + (z_2 - z_1)$	
Through a lens	$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$	$\frac{1}{\hat{q}_2} = \frac{1}{\hat{q}_1} - \frac{1}{f}$
		
Through optical systems	$R_2 = \frac{AR_1 + B}{CR_1 + D}$	$\hat{q}_2 = \frac{A\hat{q}_1 + B}{C\hat{q}_1 + D}$
ABCD Law		where $\begin{bmatrix} r_2' \\ r_2 \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} r_1' \\ r_1 \end{bmatrix}$

Power transmission of a Gaussian beam of radius $\omega(z)$ through an aperture of radius a
If $a = \omega(z)$ then 86% of the incident power will be transmitted
If $a = 1.5\omega(z)$ then 99+% of the incident power will be transmitted

Stability of Gaussian beam resonators

For two mirror cavity $0 \le \left(1 - \frac{\ell}{R_1}\right)\left(1 - \frac{\ell}{R_2}\right) \le 1$

For general cavities $0 \le \left(\frac{A + D}{2}\right)^2 \le 1$

where A and D are elements of the ABCD matrix

Measured in plane from which ray matrix makes the transformation

$$R_1 = \frac{2B}{A - D}$$

$$\omega_1^2 = \frac{2B\lambda}{\pi\sqrt{4 - (A + D)^2}}$$

Resonant frequencies of Gaussian beam resonators

$$f_{mnq} = \left[q + \frac{(m+n+1)}{\pi} \cos^{-1}\left(\sqrt{g_1 g_2}\right) \right] \frac{c}{2\ell}$$

transverse mode spacing $\Delta f_{transverse}$

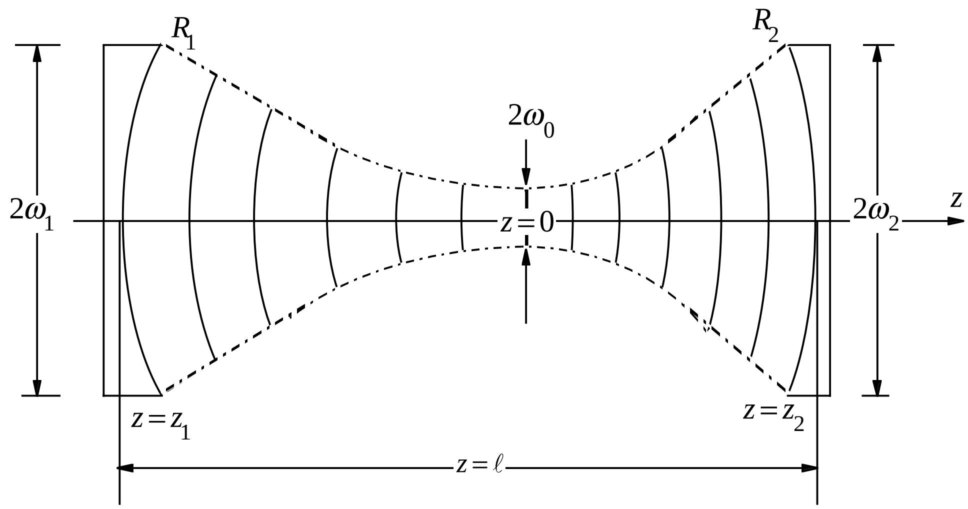
$$\Delta f_{transverse} = \frac{\cos^{-1}\left(\sqrt{g_1 g_2}\right)}{\pi} \Delta f_{long}$$

longitudinal mode spacing Δf_{long}

$$\Delta f_{long} = \frac{c}{2\ell}$$

Optical resonators for Gaussian beams

Note that $z_1 < 0$ and $R_1 < 0$, all other variables are positive.



$$z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4z_R^2}}{2} \qquad z_2 = \frac{R_2 \pm \sqrt{R_2^2 - 4z_R^2}}{2}$$

In general, $z_R^2 = \frac{\ell(-R_1-\ell)(R_2-\ell)(R_2-R_1-\ell)}{(2\ell+R_1-R_2)^2} = \ell^2 \frac{g_1g_2(1-g_1g_2)}{(g_1+g_2-2g_1g_2)^2}$

Where $g_1 = 1-\frac{\ell}{R_1}$, $g_2 = 1-\frac{\ell}{R_2}$

$$z_1 = \frac{-g_2(1-g_1)}{g_1+g_2-2g_1g_2}\ell, \; z_2 = \frac{g_1(1-g_2)}{g_1+g_2-2g_1g_2}\ell = z_1+\ell$$

$$\omega_1^2 = \omega^2(z_1) = \frac{\ell\lambda}{\pi}\left[\frac{g_2}{g_1(1-g_1g_2)}\right]^{\frac{1}{2}}, \; \omega_2^2 = \frac{\ell\lambda}{\pi}\left[\frac{g_1}{g_2(1-g_1g_2)}\right]^{\frac{1}{2}}$$

For symmetrical resonators where R is the unsigned radius of curvature

$$\omega_0 = \sqrt{\frac{\lambda}{4}}\sqrt[4]{\frac{\ell}{2}\left(R-\frac{\ell}{2}\right)}$$

$$-z_1 = z_2 = \frac{\ell}{2}$$

$$\omega_1 = \omega_2 \cong \sqrt{\frac{\lambda}{\pi}}\sqrt[4]{\frac{R\ell}{2}}$$

For confocal symmetric cavity where $R = \ell$

$$(\omega_0)_{conf} = \sqrt{\frac{\lambda\ell}{2\pi}}$$

$$(\omega_1)_{conf} = (\omega_2)_{conf} = (\omega_0)_{conf}\sqrt{2}$$

Note page numbers for original notes.

Appendix I
Derive equation 4, p.79

Appendix II
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Appendix III
Page 81, smallest mirror spot size possible for a symmetrical cavity

Appendix IV
Derivation of equation (7), p.91