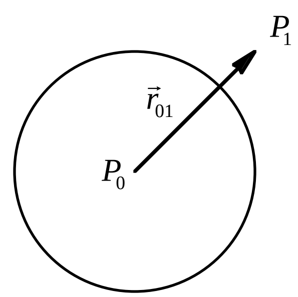
At this point we turn to the problem of reconciling everything we have developed using ray matrices with a wave description of optics. We do this because although laser beams may appear to be pencil-like beams of light they are not and only a wave description of laser beams (and optical resonators) will allow us to predict the properties of laser beams.

A spherical wave is the complex wave radiated by an isotropic point source, i.e., a non-directional point source. We will restrict ourselves to a scalar wave description where only the scalar amplitude of one transverse component of either the electric or magnetic field is considered. The other components may be found by using Maxwell's equations. In this formalism we may write a spherical wave emitted by an isotropic point source at P_0 as

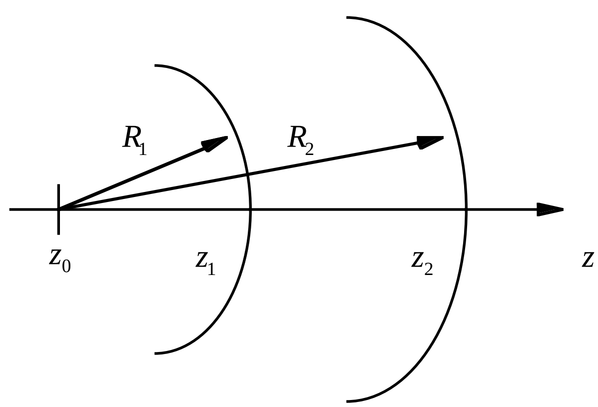
$$G(P_1) = \frac{e^{-ikr_{01}}}{4\pi r_{01}} \qquad (1)$$

where $r_{01} = |\vec{r}_{01}|$ and P_1 is the observation point as shown below



The wavefront of a wave such as (1), the surfaces of equal power perpendicular to the direction of power flow, are spheres; hence, the use of the term spherical waves. [For a more complete description of scalar waves see Goodman, Introduction to Fourier Optics, Chapter 3 "Foundations of Scalar Diffraction Theory."]

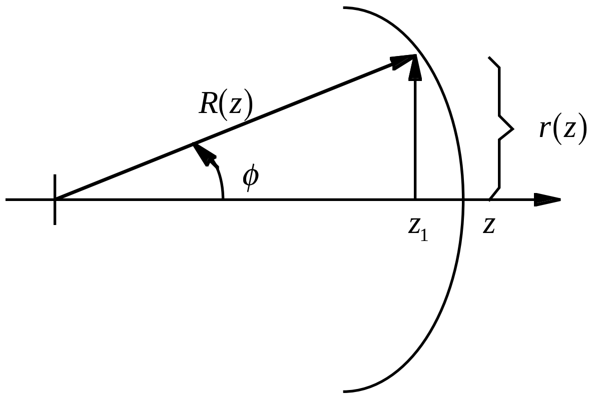
For those of us that are interested in such things it may be noted that G is the Green's function solution of the scalar wave equation in three dimensions, i,e., G satisfies the scalar wave equation $(\nabla^2 + k^2)G = -\delta(P_0)$. [See Collin, "Scattering and Diffraction Theory," EEAP 635 Class Notes]. A wave such as (1) may be visualized as a series of expanding spheres radiating power away from a point source as shown below.



The radius of curvature of a spherical wavefront originating from z_0 will, at z_1, be $R_1 = z_1 - z_0$. As the wavefront propagates from z_1 to z_2 the radius of curvature increases from R_1 to R_2 where R_2 is given by

$$R_2 - R_1 = z_2 - z_1 \qquad (2)$$

Let us consider the propagation of a ray normal to a spherical wavefront.



Because we will eventually concerned with laser beams which are very narrow we use the paraxial ray approximation that

$$r(z) << z_1 \approx z = R(z) \qquad (3)$$

This indicates that φ is small so that $\tan\varphi = r'(z) \approx \varphi$. From this we have

$$r'(z) \approx \frac{r(z)}{R(z)}$$

or

$$R(z) = \frac{r(z)}{r'(z)} \qquad (4)$$

At this point we can relate the ray matrices governing ray propoagation to the propagation of spherical wavefronts.

Recall that

$$\begin{bmatrix} r_2' \\ r_2 \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} r_1' \\ r_1 \end{bmatrix} \qquad (4a)$$

or

$$\begin{aligned} r_2 &= Ar_1 + Br_1' \\ r_2' &= Cr_1 + Dr_1' \end{aligned} \qquad (4b)$$

Using (3) and (4) we can write

$$R(z_2) = \frac{Ar_1 + Br_1'}{Cr_1 + Dr_1'} = \frac{A\left(\frac{r_1}{r_1'}\right) + B}{C\left(\frac{r_1}{r_1'}\right) + D} = \frac{AR(z_1) + B}{CR(z_1) + D}$$

Defining $R_2 = R(z_2)$ and $R_1 = R(z_1)$ we can re-write this result in a form known as the ABCD law

$$R_2 = \frac{AR_1 + B}{CR_1 + D} \qquad (5)$$

which will be important in our understanding of the behavior of laser beams in optical systems.

To illustrate the important of this ABCD law we will derive the wavefront transformation associated with a thin lens. The ABCD matrix for a thin lens is

$$\begin{bmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{bmatrix}$$

so that from (5)

$$R_2 = \frac{(1)R_1 + 0}{\left(-\frac{1}{f}\right)R_1 + 1}$$

or

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{6}$$

which is very similar to the lens law from Gaussian optics

$$\frac{1}{s_2} = \frac{1}{s_1} - \frac{1}{f}$$

In our development of the ABCD law we invoked the paraxial ray approximation. Let us see how this alters our picture of a spherical wave. From (1)

$$G(\vec{r}) = \frac{e^{-ikR}}{4\pi R} \tag{7}$$

where we have placed the point source at the origin ($\vec{r} = 0$) and $R = |\vec{r}|$. In a cartesian coordinate system

$$R = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \frac{x^2 + y^2}{z^2}} \tag{8}$$

If we restrict ourselves to the region where $x^2 + y^2 << z^2$, i.e., a paraxial ray approximation, we have the result that

$$R \approx z\left(1 + \frac{x^2 + y^2}{z^2}\right) = z + \frac{x^2 + y^2}{2z} \tag{9}$$

Substituting (9) into the exponent of (7) and using $R \approx z$ in the denominator we have

$$G(\vec{r}) \approx \frac{1}{4\pi z} e^{ik\left(z + \frac{x^2 + y^2}{2z}\right)}$$

$$G(\vec{r}) \approx \frac{1}{4\pi z} e^{-ikz} e^{-ik\left(\frac{x^2 + y^2}{2z}\right)} \tag{10}$$

The result (10) is basically a plane wave of the form e^{-ikz} propagating in the +z direction with a small transverse distortion given by the $e^{-ik\left(\frac{x^2 + y^2}{2z}\right)}$ term.

<u>Gaussian Beam Solution of the Wave Equation</u>

Starting with Maxwell's equations for homogeneous charge free ledia

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \tag{1}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \tag{2}$$

Taking the curl of (2) and using (1)

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial \left(\nabla \times \vec{H}\right)}{\partial t} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} - \mu\sigma \frac{\partial \vec{E}}{\partial t}$$

Using the identity $\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \vec{E}\right) - \nabla^2 \vec{E}$ where $\nabla \vec{E} = 0$ since this is a charge free media

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \tag{3}$$

Assuming a solution of the form $\vec{E}(x,y,z,t) = \mathrm{Re}\left\{\tilde{E}(x,y,z)e^{i\omega t}\right\}$, (3) becomes

$$\nabla^2 \tilde{E} - \mu\varepsilon(i\omega)^2 \tilde{E} - \mu\sigma(i\omega)\tilde{E} = 0$$

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon\tilde{E} - i\omega\mu\sigma\tilde{E} = 0$$

$$\nabla^2 \tilde{E} + \omega^2 \mu\varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)\tilde{E} = 0 \tag{4}$$

Define $k^2(\vec{r}) = \omega^2 \mu\varepsilon\left(1 - \frac{i\sigma}{\omega\varepsilon}\right)$ to allow for gains and losses in the media. For lasers the only $k(\vec{r})$ we will be interested in is

$$k^2(\vec{r}) = k^2 - kk_2 r^2 \tag{5}$$

Substituting (5) into (4) we have the wave equation

$$\nabla^2 \tilde{E} + k^2(\vec{r})\tilde{E} = 0 \tag{6}$$

The Laplacian ∇^2 may be separated into transverse and longitudinal parts, i.e.,

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{7}$$

In (7) we are assuming that the solution will be cylindrically symmetric since $k(\vec{r})$ is symmetric in r. Assume a solution of the form

$$\tilde{E} = \varphi(x,y,z)e^{-ikz} \tag{8}$$

This is a plane wave propagating in the +z direction modified by the factor φ. Substitute (8) into (7). Note that this k is not $k(\vec{r})$ but corresponds to the right side of equation (5)

$$\nabla^2 \tilde{E} + k^2 \tilde{E} = \nabla_t^2 \tilde{E} + \frac{\partial^2 \tilde{E}}{\partial z^2} + k^2(\vec{r})\tilde{E}$$

$$\nabla^2 \tilde{E} + k^2 \tilde{E} = \left(\nabla_t^2 \varphi\right)e^{-ikz} + \frac{\partial^2\left(\varphi e^{-ikz}\right)}{\partial z^2} + k^2(\vec{r})\left(\varphi e^{-ikz}\right)$$

$$\nabla^2 \tilde{E} + k^2 \tilde{E} = \nabla_t^2 \varphi e^{-ikz} + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z} e^{-ikz} - ik\varphi e^{-ikz}\right) + k^2(\vec{r})\varphi e^{-ikz}$$

$$\nabla^2 \tilde{E} + k^2 \tilde{E} = \nabla_t^2 \varphi e^{-ikz} + \frac{\partial^2 \varphi}{\partial z^2} e^{-ikz} - ik\frac{\partial \varphi}{\partial z} e^{-ikz} - (ik)^2 \varphi e^{-ikz} - ik\frac{\partial \varphi}{\partial z} e^{-ikz} + k^2(\vec{r})\varphi e^{-ikz}$$

$$\nabla_t^2 \varphi + \frac{\partial^2 \varphi}{\partial z^2} - 2ik\frac{\partial \varphi}{\partial z} - k^2 \varphi + k^2(\vec{r})\varphi = 0$$

Using (5)

$$\nabla_t^2\varphi + \frac{\partial^2\varphi}{\partial z^2} - 2ik\frac{\partial\varphi}{\partial z} - k^2\varphi + k^2\varphi - kk_2r^2\varphi = 0$$
$$\nabla_t^2\varphi + \frac{\partial^2\varphi}{\partial z^2} - 2ik\frac{\partial\varphi}{\partial z} - kk_2r^2\varphi = 0$$

We assume that φ is a slowly varying function so that $\frac{\partial^2\varphi}{\partial z^2}$ may be neglected. Then,

$$\nabla_t^2\varphi + -2ik\frac{\partial\varphi}{\partial z} - kk_2r^2\varphi = 0 \qquad (9)$$

The differential equation (9) will have a solution of the form

$$\varphi = e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}} \qquad (10)$$

Substituting (10) into (9) and first explicitly evaluating the terms

$$\frac{\partial\varphi}{\partial r} = -\frac{i}{2}Q(z)2re^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}} = -iQ(z)re^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}$$

$$\nabla_t^2\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{1}{r}e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}\left(-\frac{i}{2}\right)Q(z)2r + \frac{\partial}{\partial r}\left\{e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}\left(-irQ(z)\right)\right\}$$

$$\nabla_t^2\varphi = -iQ(z)e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}} + e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}\left\{\left(-\frac{i}{2}\right)Q(z)2r(-i)rQ(z)\right\} + e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}(-i)rQ(z)$$

$$-2ik\frac{\partial\varphi}{\partial z} = -2ik\left\{-iP'(z) - i\frac{r^2}{2}Q'(z)\right\}e^{-i\left\{P(z)+\frac{Q(z)}{2}r^2\right\}}$$

so that (9) becomes

$$\nabla_t^2\varphi - 2ik\frac{\partial\varphi}{\partial z} - kk_2r^2\varphi = -iQ(z) - r^2Q^2(z) - iQ(z) - 2kP'(z) - kr^2Q'(z) - kk_2r^2 = 0$$
(11)

For this equation (10) to be true for all r we may set terms corresponding to like powers of r equal to 0, i.e.,

$$\left(-Q^2(z) - kQ'(z) - kk_2\right)r^2 = 0 \qquad (12)$$

so that $Q^2(z) + kQ'(z) + kk_2 = 0$. As a corollary of (12) it follows that $-2iQ(z) = +2kP'(z) = 0$, or

$$P'(z) = -\frac{iQ(z)}{k} \qquad (13)$$

Let us now simplify the problem by considering only homogeneous media. This means that $k_2 \to 0$ in (5) and reduces (12) to the form

$$Q^2(z) + kQ'(z) = 0 \qquad (14)$$

This equation may be easily solved by defining $Q(z) = k\frac{is'(z)}{s(z)}$. Then,

$Q'(z) = k\frac{s''(z)}{s(z)} - k\frac{(s'(z))^2}{s^2(z)} = \frac{k}{s^2}\left(s(z)s''(z) - (s'(z))^2\right)$. Substituting into (14)

$$k^2\frac{(s'(z))^2}{s^2(z)}+\frac{k^2}{s^2}\left(s(z)s''(z)-(s'(z))^2\right)=0 \qquad (15)$$

From (15) $k^2\frac{s''(z)}{s(z)}=0$ which implies that $s''(z)=0$, or

$$s'(z)=a \text{ (a constant)}$$
$$s(z)=az+b$$

where a and b are constants determined by the initial conditions. From the definition of s

$$Q(z)=k\frac{s'(z)}{s(z)}=\frac{ka}{az+b}$$

Define

$$q(z)=\frac{k}{Q(z)}=\frac{az+b}{a}=z+\frac{b}{a} \qquad (16)$$

Thus, we can write $q(z)=z+q_0$ where q_0 is $q(0)$. Knowing $Q(z)$ we can use (13) to find $P(z)$

$$P'(z)=-\frac{iQ(z)}{k}=-\frac{i}{q(z)}=-\frac{i}{z+q_0}$$

Integrating, we get

$$P(z)=-i\ln(z+q_0)+c_1$$

where c_1 is a constant of integration. Let $P(0)=0$ so that $c_1=i\ln(q_0)$

$$P(z)=-i\ln(z+q_0)+i\ln(q_0)=-i\ln\left(\frac{z+q_0}{q_0}\right)=-i\ln\left(1+\frac{z}{q_0}\right) \qquad (17)$$

Note that as $\varphi=e^{-iP(z)}e^{\frac{1}{2}Q(z)r^2}$, $P(0)=0$ means that we are setting the phase of our solution $\tilde{E}=\varphi e^{-ikz}$ to zero at $z=0$. Using (16) and (17) in (10) we may write

$$\varphi(x,y,z)=e^{-i\left\{-i\ln\left(\frac{z+q_0}{q_0}\right)+\frac{kr^2}{2(q_0+z)}\right\}} \qquad (18)$$

To further simplify this result assume that q_0 is purely imaginary and may be written in the form

$$q_0\equiv i\frac{\pi\omega_0^2}{\lambda} \qquad (19)$$

where ω_0^2 and λ are real.

$$\varphi(x,y,z)=e^{-\ln\left(1-i\frac{\lambda z}{\pi\omega_0^2}\right)}e^{-i\frac{kr^2}{2(q_0+z)}} \qquad (20)$$

$$e^{-\ln\left(1-i\frac{\lambda z}{\pi\omega_0^2}\right)}=\frac{e^{i\tan^{-1}\left(\frac{\lambda z}{\pi\omega_0^2}\right)}}{\sqrt{1+\frac{\lambda^2z^2}{\pi^2\omega_0^4}}}=\frac{\omega^2(z)}{\omega_0^2}e^{i\phi} \qquad (21)$$

where we have defined

$\phi=\tan^{-1}\left(\frac{\lambda z}{\pi\omega_0^2}\right)$ and

$$\omega^2(z) = \omega_0^2\left[1 + \left(\frac{\lambda z}{\pi\omega_0^2}\right)^2\right]$$

and used the identity $\ln(a+ib) = \ln\sqrt{a^2+b^2} + i\tan^{-1}\left(\frac{b}{a}\right)$, i.e.,

$$e^{-\ln(a+ib)} = e^{-\ln\sqrt{a^2+b^2}+i\tan^{-1}\left(\frac{b}{a}\right)} = e^{\ln\left(\frac{1}{\sqrt{a^2+b^2}}\right)} e^{i\tan^{-1}\left(\frac{b}{a}\right)} = \frac{1}{\sqrt{a^2+b^2}} e^{t\tan^{-1}\left(\frac{b}{a}\right)}$$

$$e^{-\frac{ikr^2}{2(q_0+z)}} = e^{-\frac{ikr^2}{2(q_0+z)}\left(\frac{q_0^*+z}{q_0^*+z}\right)} = e^{-\frac{ikr^2}{2\left(\frac{i\pi\omega_0^2}{\lambda}+z\right)}\left(\frac{z-\frac{i\pi\omega_0^2}{\lambda}}{z-\frac{i\pi\omega_0^2}{\lambda}}\right)} = e^{\frac{-ikr^2 z-\frac{kr^2\pi\omega_0^2}{\lambda}}{2\left(z^2+\left(\frac{\pi\omega_0^2}{\lambda}\right)^2\right)}}$$

$$e^{-\frac{ikr^2}{2(q_0+z)}} = e^{\frac{-ikr^2 z-\frac{kr^2\pi\omega_0^2}{\lambda}}{2\left(z^2+\left(\frac{\pi\omega_0^2}{\lambda}\right)^2\right)}} = e^{\frac{-ikr^2 z}{2z^2\left(1+\left(\frac{\pi\omega_0^2}{\lambda z}\right)^2\right)}-\frac{-kr^2\pi\omega_0^2}{2\lambda\left(\frac{\pi\omega_0^2}{\lambda}\right)^2\left(1+\frac{z^2}{\left(\frac{\pi\omega_0^2}{\lambda}\right)^2}\right)}} = e^{\frac{-ikr^2}{2R(z)}-\frac{-kr^2\pi\omega_0^2}{2\frac{\lambda\pi^2\omega_0^4}{\lambda^2}\left(1+\left(\frac{\lambda z}{\pi\omega_0^2}\right)^2\right)}}$$

$$e^{-\frac{ikr^2}{2(q_0+z)}} = e^{\frac{-ikr^2}{2R(z)}-\frac{-kr^2\lambda}{2\pi^2\omega_0^2\left(1+\left(\frac{\lambda z}{\pi\omega_0^2}\right)^2\right)}} = e^{\frac{-ikr^2}{2R(z)}-\frac{-r^2}{\omega_0^2\left(1+\left(\frac{\lambda z}{\pi\omega_0^2}\right)^2\right)}} = e^{\frac{-ikr^2}{2R(z)}-\frac{-r^2}{\omega^2(z)}}$$

In summary,

$$e^{-\frac{ikr^2}{2(q_0+z)}} = e^{\frac{-ikr^2}{2R(z)}-\frac{-r^2}{\omega^2(z)}} \tag{22}$$

where we have used the fact that $k = \frac{2\pi}{\lambda}$ and the definition

$$R(z) = z\left[1 + \left(\frac{\pi\omega_0^2}{\lambda z}\right)^2\right]$$

Substituting (21) and (22) into (20) we get

$$\varphi(x,y,z) = \frac{\omega^2(z)}{\omega_0^2} e^{-i\phi - r^2\left[\frac{1}{\omega^2(z)}+\frac{ik}{2R(z)}\right]} \tag{23}$$

and from (8)

$$E(x,y,z) = \frac{\omega^2(z)}{\omega_0^2} e^{-i(kz-\phi) - r^2\left[\frac{1}{\omega^2(z)}+\frac{ik}{2R(z)}\right]} \tag{24}$$

Properties of Gaussian Beams

The solution we have obtained to the wave equation

$$E(x,y,z) = E_0 \frac{\omega(z)}{\omega_0} e^{-i(kz-\phi) - r^2\left[\frac{1}{\omega^2(z)} + \frac{ik}{2R(z)}\right]} \qquad (1)$$

is called the fundamental Gaussian beam solution since we assumed a transverse dependence based only on r^2, i.e., $\frac{\partial}{\partial\phi} = 0$. [More on this later.] To understand the result (1) we examine each of the exponential factors in (1).

The parameter $\omega(z)$ is the distance r at which the field amplitude is down by $1/e$ since $\frac{-r^2}{\omega^2(z)}$ is the only real term in the exponential and $e^{-\frac{r^2}{\omega^2(z)}}$ can be regarded as a transverse amplitude modulation of the beam.

The term going as $e^{-ik\frac{r^2}{2R(z)}}$ is most easily understood by combining it with the $e^{-i(kz-\phi)}$ term. Recalling the "paraxial ray" approximation of a spherical wave e^{-ikr} (neglecting the $\frac{1}{4\pi r}$ amplitude factor) from a point source [see (10), p.62]

$$e^{-ikr} \approx e^{-ikz - ik\frac{x^2+y^2}{2z}} \qquad (2)$$

and comparing it with our expression

$$e^{-ik\left(z-\frac{\phi}{k}\right) - ik\frac{x^2+y^2}{2R(z)}}$$

we see that $\frac{\phi}{k}$ is simply a phase shift which we may ignore, e^{-ikz} is a basic plane wave propagating in the +z direction, and $e^{-ik\left(z+\frac{x^2+y^2}{2R(z)}\right)}$ is somewhat like the "paraxial ray" approximation of a spherical wave propagating in the +z direction (2) except for the term $R(z)$. Recall that $R(z)$ is given by

$$R(z) = z\left[1 + \left(\frac{\pi\omega_0^2}{\lambda z}\right)^2\right]$$

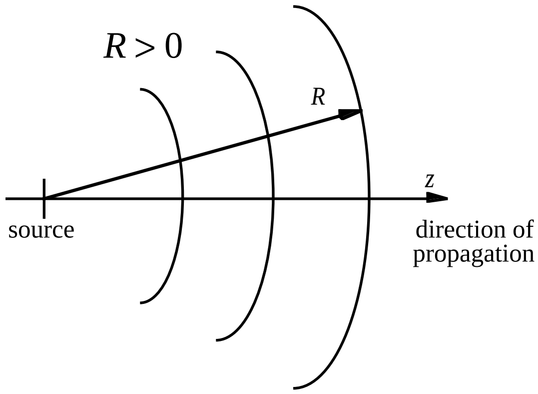
We define $z_R = \frac{\pi\omega_0^2}{\lambda}$ so that

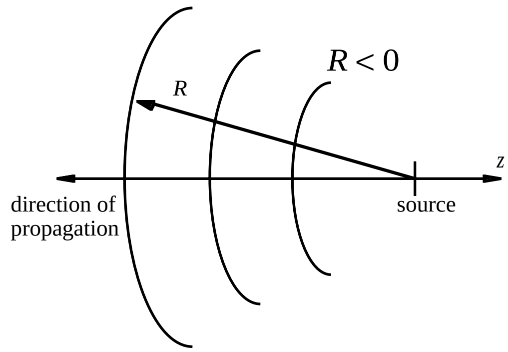
$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right] = z + \frac{z_R^2}{z} \qquad (3)$$

If $z >> z_R$ (3) becomes

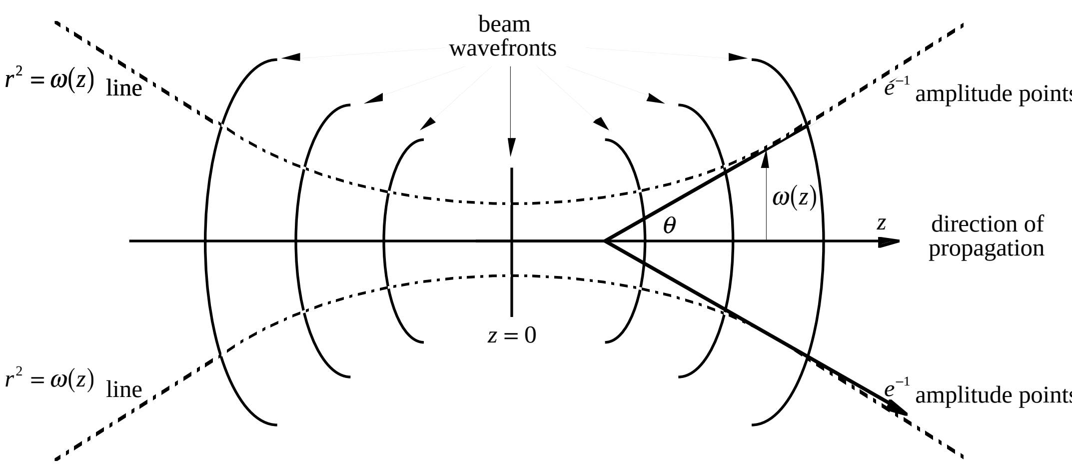
$$R(z) \approx z \qquad (4)$$

In (2) we may interpret the z in the denominator of the second term of the exponential as being the radius of curvature of the wavefront. For a spherical wave from a source at the origin, the radius of curvature is given by $R(z) = z$. From (4) our expression (3) behaves like a spherical wave with radius of curvature $R(z) = z$ for $z >> z_R$. This suggests that we regard $R(z)$ as the curvature of the wavefronts of the wave described by (1). The sign convention for wavefronts is the opposite of that for mirrors and lenses as may be seen below.





Continuing with our examination of (3) as $z \to \infty$ we see that $R(z) \to \infty$ so that the beam wavefronts are initially planar ($R = \infty$) and gradually becomes spherical as z becomes larger than z_R, A suitable picture of what is happening is seen in the drawing below where the transverse Gaussian amplitude dependence is indicated by the dashed lines representing the $r = \omega(z)$, i.e., the e^{-1} amplitude points.



The e^{-1} amplitude lines become straight for $z >> z_R$ and the half angle θ for the cone formed by these lines in three dimensions is illustrated above and is given by

$$\tan\theta = \frac{\omega(z)}{z} \cong \frac{\frac{\lambda z}{\pi\omega_0}}{z} = \frac{\lambda}{\pi\omega_0}$$

since $\omega(z)=\omega_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}\rightarrow\frac{\lambda z}{\pi\omega_0}$ as $z>>z_R$. This angle θ is called the divergence angle. Since θ is small for laser beams (paraxial ray approximation if you prefer), $\tan\theta\approx\theta$ and

$$\theta\approx\frac{\lambda}{\pi\omega_0} \tag{4}$$

The result is that, for $z>>z_R$, a Gaussian beam has a constant divergence angle given by (4).

We return to (22), p. 67 in the handout on Gaussian beams

$$e^{-\frac{ikr^2}{2(\hat{q}_0+z)}}=e^{-\frac{ikr^2}{2R(z)}-\frac{r^2}{\omega^2(z)}} \tag{5}$$

We have already identified $e^{-ikz-\frac{ikr^2}{2R(z)}}$ as a "paraxial" approximation of an outward propagating "spherical" wave with radius of curvature $R(z)$. In an extension of the reasoning that led to interpreting $R(z)$ as a radius of curvature we define the complex radius of curvature $\hat{q}(z)=\hat{q}_0+z$ or, from (5),

$$\frac{1}{\hat{q}(z)}=\frac{1}{\hat{q}_0+z}=\frac{1}{R(z)}+\frac{2}{\omega^2(z)ik}$$

Recalling $k=\frac{2\pi}{\lambda}$ we may write this in slightly different form as

$$\frac{1}{\hat{q}(z)}=\frac{1}{\hat{q}_0+z}=\frac{1}{R(z)}-i\frac{\lambda}{\pi\omega^2(z)} \tag{6}$$

Let us examine how $\hat{q}(z)$ changes as we move along the z-axis from a point z_1 to another point z_2. Using $\hat{q}(z)=\hat{q}_0+z$ we have that $\hat{q}(z_1)=\hat{q}_0+z_1$ at $z=z_1$, and $\hat{q}(z_2)=\hat{q}_0+z_2$. Subtracting, we get $\hat{q}(z_2)-\hat{q}(z_1)=z_2-z_1$, or

$$\hat{q}(z_2)=\hat{q}(z_1)+(z_2-z_1) \tag{7}$$

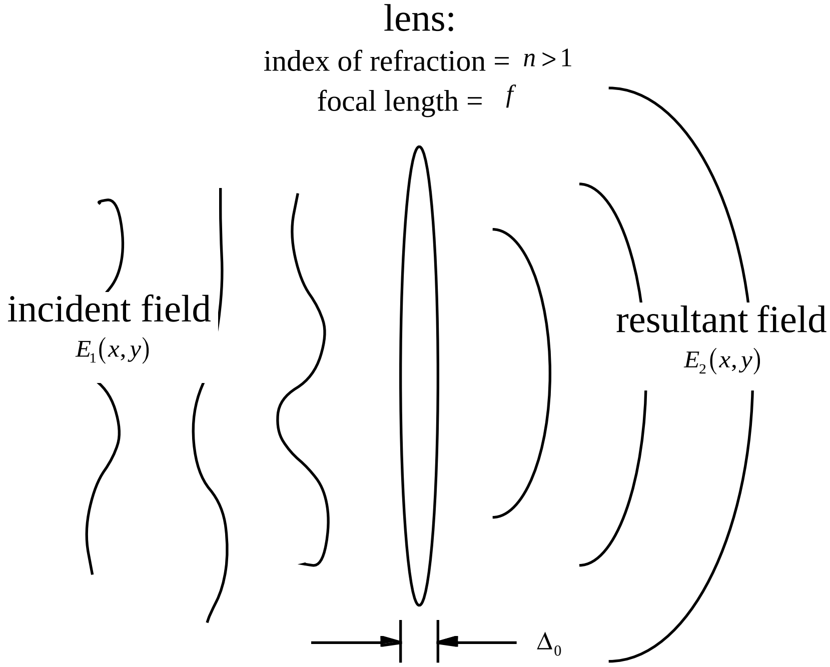
This is exactly the form of the transformation for spherical waves along the z-axis [Eqn. (2), p.60] which is repeated here for comparison.

$$R_2=R_1+(z_2-z_1) \tag{8}$$

We now examine the effect of a lens upon a Gaussian beam with complex radius of curvature $\hat{q}_1$ just to the left of the lens. The lens may be viewed as causing a phase distortion of the incident wave according to equation (10) of Appendix I

$$E_2(x,y)=e^{ikn\Delta_0+ik\frac{x^2+y^2}{2f}}E_1(x,y) \tag{9}$$

where $E_1(x,y)$ is the scalar field incident upon the lens from the left, Δ_0 is the thickness of the lens, n is its index of refraction, f is its focal length, and $E_2(x,y)$ is the beam (field) exiting the lens at the right.



If E_1 and E_2 are spherical waves we get the result that the lens transforms the wavefront curvatures R_1 and R_2 respectively. According to Equation (15) of Appendix I and Eqn. (6), p.61, i.e.,

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \tag{10}$$

If E_1 and E_2 were Gaussian beams we would have exactly the same result except for replacing R_2 by the complex radius of curvature $\hat{q}_2$ and R_1 by $\hat{q}_1$. To show this, let E_1 and E_2 be Gaussian, i.e.,

$$E_1 = \hat{A} e^{-ikz - ik\frac{r^2}{2\hat{q}_1}} \tag{11a}$$

$$E_2 = \hat{B} e^{-ikz - ik\frac{r^2}{2\hat{q}_2}} \tag{11b}$$

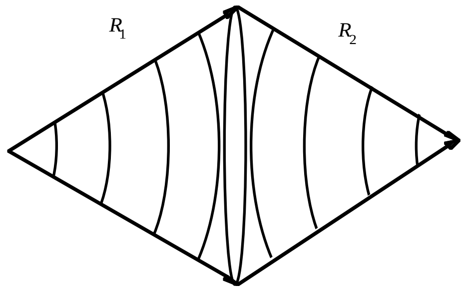
Then, from (9),

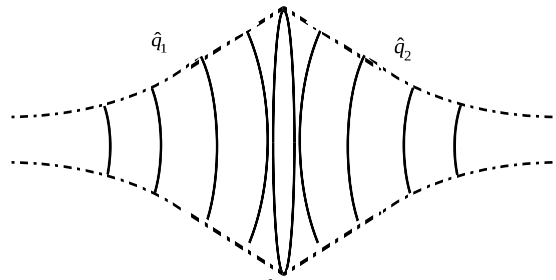
$$E_2 = \hat{B} e^{-ikz - ik\frac{r^2}{2\hat{q}_2}} = e^{-ikn\Delta_0 + ik\frac{r^2}{2f}} \hat{A} e^{-ikz - ik\frac{r^2}{2\hat{q}_1}}$$

For the phase of the waves to be continuous at the lens surface we have

$$\frac{1}{\hat{q}_2} = \frac{1}{\hat{q}_1} - \frac{1}{f} \tag{12}$$

which is the Gaussian beam analog of (10). Inserting (6) into (12) and equating real and imaginary parts we get $\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$ and $\omega_1(z) = \omega_2(z)$ at the lens surfaces which agree with our geometric optic picture of transforming slopes (R) and <u>not</u> changing a ray's displacement from the optic axis (ω). A sketch of beam transformation by a lens is shown below for spherical waves and Gaussian beams.





The solid curved lines represent the wavefront curvature.

It can be shown that for lenses and mirrors equations (7) and (12) will lead to the ABCD law for Gaussian beams

$$\hat{q}_2 = \frac{A\hat{q}_1 + B}{C\hat{q}_1 + D} \tag{13}$$

For more complex structures we argue that since $R = \hat{q}$ yields the Gaussian beam forms of (8) and (10) it will also yield the ABCD law (13) from Equation (5), p.61.

Returning to the transformation of a Gaussian beam by a lens we note that although such a Gaussian beam is largely confined near the z-axis it has an infinite transverse extent and some of the beam power will be lost each time a Gaussian beam passes through a finite-sized aperture such as a lens or mirror. This power loss is called diffraction loss and may be estimated in the following manner. The transverse beam amplitude goes as $e^{-\frac{r^2}{\omega^2(z)}} = e^{-\frac{r^2}{\omega^2}}$ where we have dropped the functional notation for ω for brevity. We define the normalized transverse amplitude distribution ϕ as

$$\phi = \sqrt{\frac{2}{\pi}}\frac{1}{\omega}e^{-\frac{r^2}{\omega^2}}$$

where the normalization is of the transverse beam power, that is,

$$\int_0^\infty \phi^2(r)2\pi r dr = 1$$

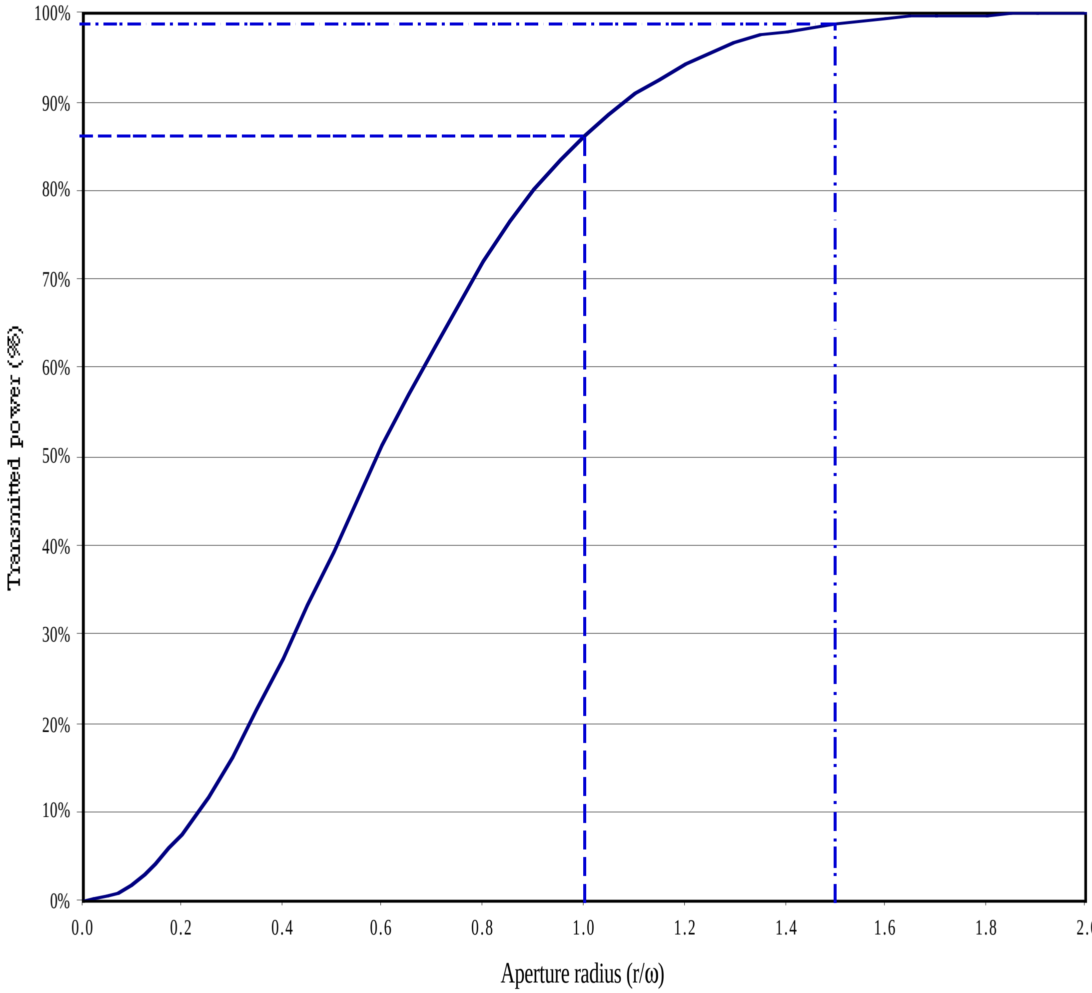
where the square of the electric field amplitude ϕ is proportional to the beam power Φ. Then, in a circular region of radius a about the z-axis we contain that fraction of the total beam power

$$\frac{\Phi(a)}{\Phi(\infty)} = \frac{\int_0^a \phi^2(r)2\pi r dr}{\int_0^\infty \phi^2(r)2\pi r dr} = \int_0^a 4\frac{r}{\omega^2}e^{-2\frac{r^2}{\omega^2}}dr = -e^{-2\frac{r^2}{\omega^2}}\Bigg|_0^a = 1 - e^{-2\frac{a^2}{\omega^2}}$$

or

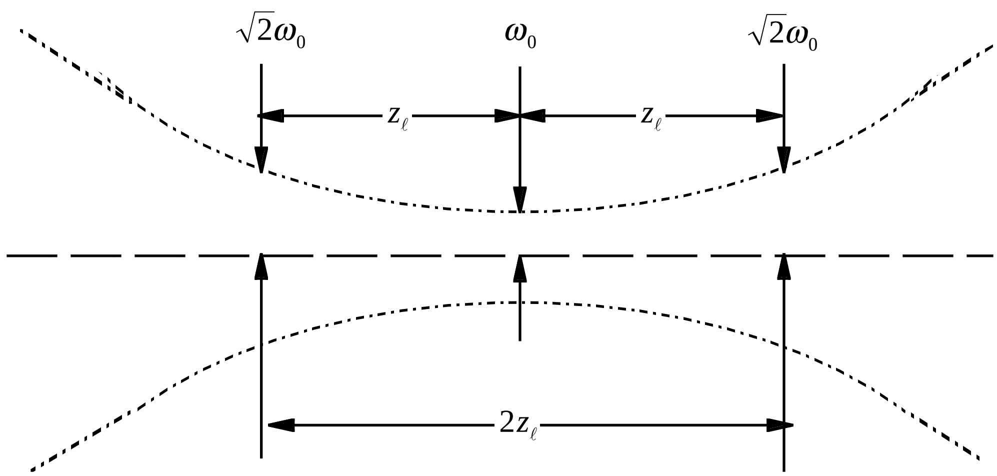
$$\frac{\Phi(a)}{\Phi_{total}} = 1 - e^{-2\frac{a^2}{\omega^2}} \qquad (14)$$

Plotting (14) as a function of a



This result says that if $a = \omega$ approximately 86% of the incident power will be transmitted through the aperture, i.e., 14% power loss. A general rule of thumb is to pick $a \geq 1.5\omega$ at which point 99+% of the incident power will be transmitted through the aperture.

Gaussian Beam Collimation



We rather arbitrarily define a collimated Gaussian beam where the spot size has increased by $\sqrt{2}$ over the beam waist ω_0, or the beam area has doubled. Beyond these limits the beam continues spreading nearly linearly with distance and, hence, is no longer a parallel or collimated beam. The beam spread is given by

$$\omega^2(z) = \omega_0^2\left[1 + \left(\frac{z}{z_R}\right)^2\right]$$

where

$$z_R = \frac{\pi\omega_0^2}{\lambda}$$

Up to now we have not associated any physical significance to z_R; it is simply defined as $\frac{\pi\omega_0^2}{\lambda}$ and nothing more.

$$\left(\sqrt{2}\omega_0\right)^2 = \omega_0^2\left[1 + \left(\frac{z}{z_R}\right)^2\right]$$

$$2 = 1 + \left(\frac{z}{z_R}\right)^2$$

$$z = \pm z_R$$

Hence, z_R is called the Rayleigh distance or range and defines the collimated beam region.

<u>Focusing to a spot</u>
For a collimated beam of spot size ω incident before a lens, the lens will be located a distance z equal to its focal length f behind the focus.

Proof:
If the beam is collimated $z \le z_R$ and

$$R(z) = z\left[1 + \frac{z_R^2}{z^2}\right] = z + \frac{z_R^2}{z} \ge 2z_R$$

The beam wave fronts are planar at $z = 0$ and have $R(z_R) = 2z_R$, hence, $R(z_R) \ge 2z_R$ for $|z| \le z_R$. For some typical numbers like $\lambda = 0.5\mu$, $\omega_0^2 = 17mm$

$$z_R = \frac{\pi\omega_0^2}{\lambda} = \frac{(3.14)(17mm)^2}{5\times10^{-4}mm} \approx 1800m$$

The beam transformation by the lens was given in class as

$$\frac{1}{\hat{q}_2} = \frac{1}{\hat{q}_1} - \frac{1}{f} \qquad (1)$$

By its definition

$$\frac{1}{\hat{q}(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi\omega^2(z)} \qquad (2)$$

Substituting this result into (1) we get

$$\frac{1}{R_2} - \frac{i\lambda}{\pi\omega_2^2} = \frac{1}{R_1} - \frac{i\lambda}{\pi\omega_1^2} - \frac{1}{f}$$

or, equating real and imaginary parts,

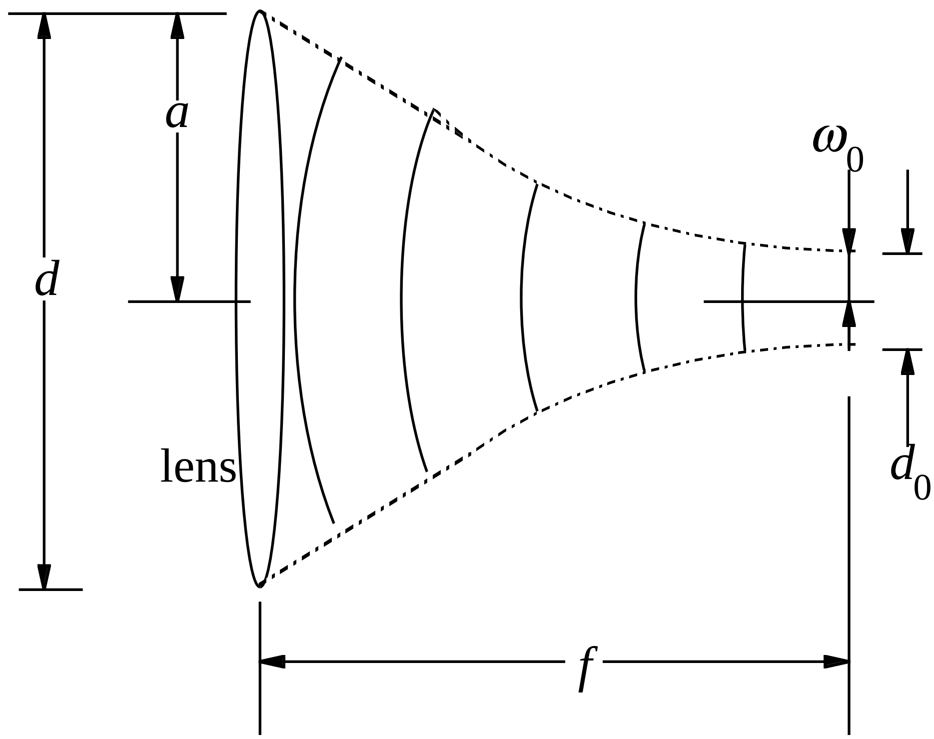
$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f} \text{ and } \omega_1 = \omega_2 \qquad (3)$$

i.e., the beam radius does not change in passing through the lens and the wavefronts transform as spherical waves.

Returning to the problem, if R_1, the incident beam curvature is very large then $\frac{1}{R_1} << \frac{1}{f}$ since f is typically on the order of 1 meter or less and (3) becomes

$$\frac{1}{R_2} \approx -\frac{1}{f}$$

This is a spherical wavefront converging to a point a distance f in front of the lens.



The wavefront leaving the lens is highly curved since f is generally short. This means that $f >> z_R$ since in the collimated beam waist the curvatures are very planar ($R(z)$ very large).

For 99% power transmission through the lens $d = 3\omega$ or $\omega = \frac{d}{3}$ where d is the lens diameter. The relationship between ω and ω_0 is given by

$$\omega(f) = \omega_0\left(1 + \frac{f^2}{z_R^2}\right)^{\frac{1}{2}} \approx \frac{\omega_0 f}{z_R}$$

since $f >> z_R$. Solving for ω_0

$$\omega_0 = \frac{z_R\omega}{f} = \frac{\pi\omega_0^2\omega}{\lambda f}$$

or

$$\omega_0 = \frac{\lambda f}{\pi\omega}$$

The beam waist is ω_0 and the beam spot size d_0 (86% power point) is defined as $d_0 = 2\omega_0$.

$$\frac{d_0}{2} = \frac{\lambda f}{\pi \omega} = \frac{\lambda f}{\pi\left(\frac{d}{3}\right)} \approx \frac{\lambda f}{d}$$

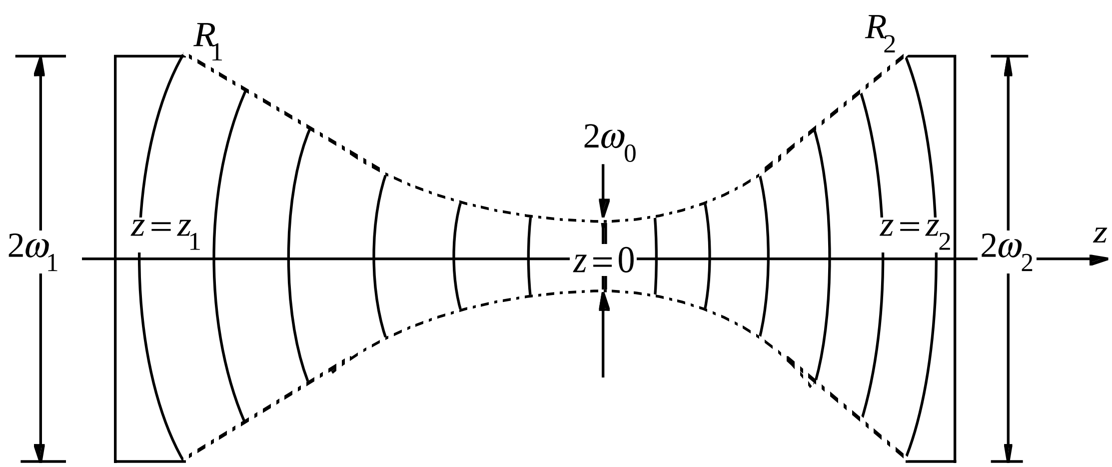
Then, $d_0 \approx \frac{2\lambda f}{d}$. The ratio $\frac{f}{d}$ is known as the f-number of the lens and is defined as $f\# = \frac{f}{d}$. For a typical lens $f\#$ can be as low as 0.5. Picking $f\# = 0.5$ and $\lambda = 10.6\mu$ (a CO_2 laser) we have

$$d_0 \approx 2\lambda f\# = 2\left(10.6 \times 10^{-6}\right)(0.5) = 10.6\mu$$

This shows that we can focus a laser beam as small as one wavelength across.

Resonator Mode Properties

We can now apply Gaussian beam theory to develop the basic mode properties of optical resonators. Let us approach the problem somewhat in reverse by assuming that we have a spherical Gaussian beam, i.e. assume a waist spot size ω_0 at $z = 0$ and spot sizes ω_1 and ω_2 and radii of curvature R_1 and R_2 at $z = z_1$ and $z = z_2$with radii of curvature R_1 and R_2 and diameter much larger than ω_1 and ω_2 at z_1 and z_2, we will have trapped the beam inside the resonator. The beam will be reflected exactly back on itself at each mirror and will form a standing wave with time-independent spot sizes and radii of curvature.



Note that as long as the mirrors are significantly larger than the spot size, the shape of the mode will depend only on R_1, R_2 and $\ell = z_2 - z_1$ (since $z_1 < 0$).

The usual problem in resonator design is to assume two mirrors of radii $\Re_1$ and $\Re_2$ and separation ℓ and find the appropriate beam parameters which result, i.e., find ω_1, ω_2, ω_0, location of the beam waist, etc. We can do this by making use of the relations

$$R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right] \tag{1}$$

and

$$\omega(z) = \omega_0\left[1+\left(\frac{z_R}{z}\right)^2\right]^{\frac{1}{2}} \tag{2}$$

As z_1 and z_2 are the distances of the two mirrors from the beam waist at $z = 0$ we may write the beam radii as being

$$R_1 = R(z_1) = z_1 + \frac{z_R^2}{z_1} \tag{3a}$$

$$R_2 = R(z_2) = z_2 + \frac{z_R^2}{z_2} \tag{3b}$$

Each of these equations may be solved by the quadratic formula for z_1 and z_2 giving

$$z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4z_R^2}}{2} \tag{4a}$$

$$z_2 = \frac{R_2 \pm \sqrt{R_2^2 - 4z_R^2}}{2} \tag{4b}$$

This set of equations (4) is two equations in three unknowns (z_1, z_2, z_R). To allow a unique solution of the problem we use the mirror separation as the third equation

$$z_2 - z_1 = \ell \tag{4c}$$

The algebra of solving (4) is very tedious and, for those interested, the details of the solution are found in Appendix I. The solution of (4) is found to be

$$z_R^2 = \frac{\ell(-R_1 - \ell)(R_2 - \ell)(R_2 - R_1 - \ell)}{(2\ell + R_1 - R_2)^2} \tag{5}$$

A word about signs is in order at this point. R_1 and R_2 are the beam curvatures whereas $\Re_1$ and $\Re_2$ are the mirror curvatures and have the opposite signs from R_1 and R_2. The equation (5) may also be written in terms of our previously defined stability factors

$$g_1 = 1 - \frac{\ell}{R_1}$$

$$g_2 = 1 - \frac{\ell}{R_2}$$

where R_1 and R_2 are signed quantities and

$$z_R^2 = \ell^2 \frac{g_1 g_2 (1 - g_1 g_2)}{(g_1 + g_2 - 2g_1 g_2)^2} \tag{6}$$

From (6) and (11) we may solve for z_1 and z_2 as

$$z_1 = \frac{-g_2(1 - g_1)}{g_1 + g_2 - 2g_1 g_2}\ell \tag{7a}$$

$$z_2 = \frac{g_1(1 - g_2)}{g_1 + g_2 - 2g_1 g_2}\ell = z_1 + \ell \tag{7b}$$

The mirror spot sizes will then be given by (2) and (7) as

$$\omega_1^2 = \omega^2(z_1) = \frac{\ell\lambda}{\pi}\left[\frac{g_2}{g_1(1-g_1g_2)}\right]^{\frac{1}{2}} \tag{8a}$$

$$\omega_2^2 = \frac{\ell\lambda}{\pi}\left[\frac{g_1}{g_2(1-g_1g_2)}\right]^{\frac{1}{2}} \tag{8b}$$

<u>The symmetrical resonator</u>

Let us examine the behavior of a cavity formed by two mirrors of equal radii of curvature. To eliminate signs we define the <u>unsigned</u> beam radius of curvature R as $R_2 = -R_1 = R$. Since R is unsigned it will also correspond to the magnitudes of the mirror radii of curvature. For our purposes (5) is easier to use than (6) so we get

$$z_R^2 = \frac{\ell(R-\ell)(R-\ell)(2R-\ell)}{(2\ell+2R)^2} = \frac{\ell}{4}(2R-\ell) \tag{9}$$

We can determine the beam waist ω_0 by using the definition of the Rayleigh distance $z_R = \frac{\pi\omega_0^2}{\lambda}$ to get

$$\omega_0^4 = \frac{\lambda^2 z_R^2}{\pi^2} \tag{10}$$

and, from (9), for a symmetrical cavity

$$\omega_0 = \sqrt{\frac{\lambda}{4}}\sqrt[4]{\frac{\ell}{2}\left(R-\frac{\ell}{2}\right)} \tag{11}$$

By symmetry we argue that

$$-z_1 = z_2 = \frac{\ell}{2} \tag{12}$$

which, for this case, the spot sizes at the mirrors may be found from (11), (12) and (2) to be

$$\omega_1 = \omega(z_1) = \omega_2 = \omega(z_2) = \sqrt{\frac{\lambda\ell}{2\pi}}\sqrt[4]{\frac{2R^2}{\ell\left(R-\frac{\ell}{2}\right)}} \tag{13}$$

If $R >> \ell$ as in many practical lasers, equations (11) and (13) become

$$\omega_0 \cong \sqrt{\frac{\lambda}{\pi}}\sqrt[4]{\frac{\ell R}{2}} \tag{14}$$

$$\omega_1 = \omega_2 \cong \sqrt{\frac{\lambda}{\pi}}\sqrt[4]{\frac{R\ell}{2}} \tag{15}$$

Equations (14) and (15) show that the beam spread is small since $\omega_1 = \omega_2 \cong \omega_0$. It may also be noted that for a symmetrical confocal cavity where $R = \ell$ we have the smallest mirror spot sizes possible in a symmetrical cavity, that is, $\omega_1 = \omega_2 = \sqrt{2}\omega_0$. This result is developed in Appendix III.

<u>Example:</u>
Design a symmetrical resonator for $\lambda = 10^{-4}$ cm with $\ell = 2$ meters.

If we choose a confocal geometry, i.e., $R = \ell = 2$ meters, equation (14) gives the beam waist as

$$\omega_0 \cong \sqrt{\frac{\lambda}{\pi}} \sqrt[4]{\frac{\ell}{2}\left(\ell - \frac{\ell}{2}\right)} = \sqrt{\frac{\lambda\ell}{2\pi}} = \sqrt{\frac{(10^{-4}\,cm)(2\times 10^{-2}\,cm)}{2\pi}} \approx 0.06cm$$

and, from (13), we have the mirror spot size as

$$\omega_1 = \omega_2 = \sqrt{\frac{\lambda\ell}{2\pi}} \sqrt[4]{\frac{2\ell^2}{\ell\left(\ell - \frac{\ell}{2}\right)}} = \omega_0 \sqrt[4]{4} = \omega_0\sqrt{2} = 0.084cm$$

As shown in Appendix III, this is the smallest mirror spot size possible for a symmetrical cavity. Suppose we wanted a larger mirror spot size for some reason. Let us say that we want $\omega_1 = \omega_2 = 0.3cm$ and calculate what R must be. First, let us assume $R >> \ell$ which will turn out to be a reasonable assumption. We may then use (15) to get

$$0.3 = 0.06\left[\frac{2R}{2}\right]^{\frac{1}{4}}$$

or

$$R = \left[\frac{0.3}{0.06}\right]^4 \approx 600m$$

which justifies our assumption that $R >> \ell = 2m$. Typical gas lasers have mirrors with radii of curvature of a few meters (2 to 10 meters typically) for $\ell = 1m$ so that beam waists and mirror spot sizes tend to be small giving rise to "narrow" laser beams.

<u>Stable resonators</u>

The ability of an optical resonator to lase depends upon its ability to confine radiation within the cavity. As an example, consider the symmetrical resonator where $R_2 = -R_1 = R$. The mirror spot size is given by (13) as

$$\omega_{1,2} = \sqrt{\frac{\lambda\ell}{2\pi}} \sqrt[4]{\frac{2R^2}{\ell\left(R - \frac{\ell}{2}\right)}} \tag{16}$$

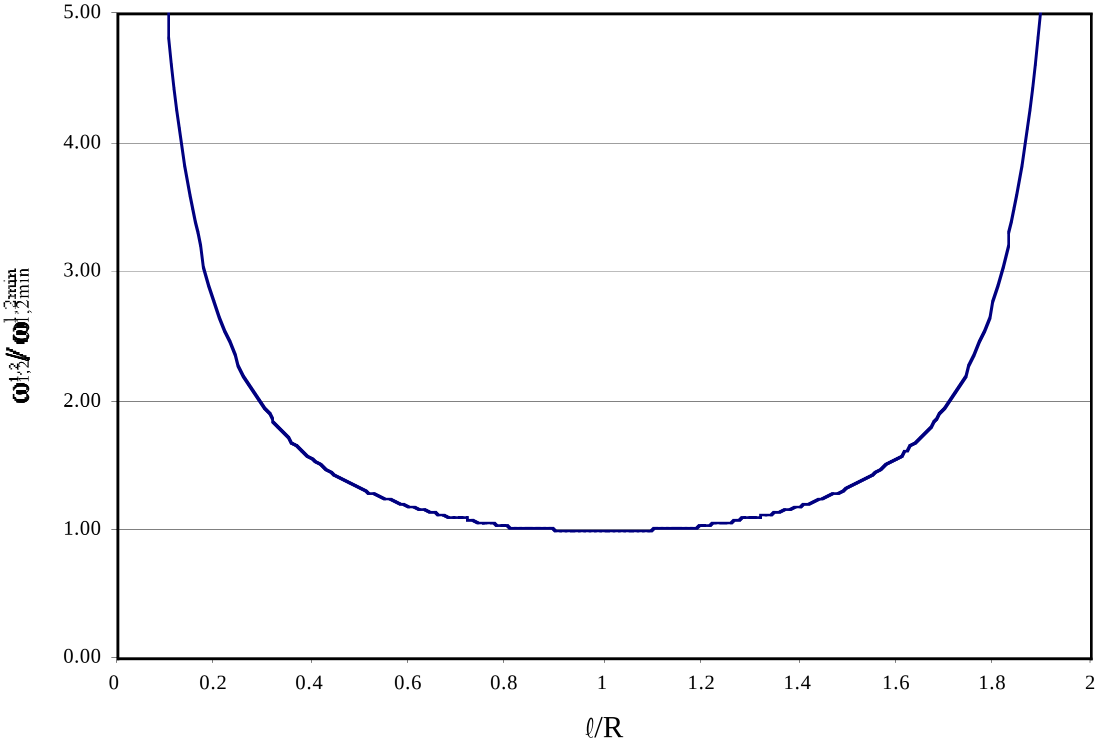
The minimum mirror spot size is found in the confocal symmetrical cavity where $R = \ell$. The minimum spot size in this case is given by

$$\omega_{1,2_{\min}} = \sqrt{\frac{\lambda\ell}{2\pi}}\sqrt{2} \tag{17}$$

(See the example on the previous page for where this formula came from.). The ratio of (16) to (17) is

$$\frac{\omega_{1,2}}{\omega_{1,2_{\min}}} = \frac{1}{\sqrt[4]{\frac{\ell}{R}\left[2 - \frac{\ell}{R}\right]}} \tag{18}$$

Plotted as a function of $\frac{\ell}{R}$



We see that the mirror spot size becomes infinite as $R \to \infty$ ($\frac{\ell}{R} = 0$, plane parallel mirrors) or $R \to \frac{\ell}{2}$ ($\frac{\ell}{R} = 2$, two concentric mirrors), As the mirror spot size goes to infinity, the size of the mirrors required to reflect most of the light back into the cavity also goes to infinity (See p. 74). Obviously we cannot use mirrors of infinite diameter so we say that a resonator is unstable if the mirror spot size becomes infinite. By unstable we mean that it cannot confine light within the resonator.

To determine the conditions under which light will be confined within the optical resonator we have to consider the propagation of Gaussian beams within an optical resonator. In an optical resonator a Gaussian beam starting from some point at which the beam has a complex radius of curvature $\hat{q}_1$ must, after one round trip of the resonator, come back to the starting point, with the same radius of curvature $\hat{q}_2 = \hat{q}_1$. If we know the ABCD matrix for ray propagation through the system it follows that $\hat{q}_1$ and $\hat{q}_2$ are related by the ABCD law, i.e.,

$$\hat{q}_2 = \frac{A\hat{q}_1 + B}{C\hat{q}_1 + D} \tag{19}$$

But, for a standing wave to be created within the resonator, $\hat{q}_2 = \hat{q}_1$ or

$$\hat{q}_1 = \frac{A\hat{q}_1 + B}{C\hat{q}_1 + D} \tag{20}$$

This gives rise to a quadratic in $\hat{q}_1$

$$C\hat{q}_1^2 + (D - A)\hat{q}_1 - B = 0$$

which may be solved by the quadratic formula to give

$$\hat{q}_1 = \frac{(A-D) \pm \sqrt{(A-D)^2 + 4BC}}{2C}$$

We then expand the quantity $(A-D)^2$ under the square root and use the identity $BC - AD = 1$ (since the determinant of a ray matrix is 1) to get

$$\hat{q}_1 = \frac{(A-D) \pm \sqrt{(A+D)^2 - 4}}{2C}$$

which we re-write as

$$\hat{q}_1 = \frac{(A-D) \pm i\sqrt{4-(A+D)^2}}{2C}$$

The reason for this will become apparent when we identify the radius of curvature and beam radius from this expression. To do this we select the solution for $\hat{q}_1$ with the positive root and invert to get

$$\frac{1}{\hat{q}_1} = \frac{2C}{(A-D) \pm i\sqrt{4-(A+D)^2}}$$

Rationalizing the denominator we get

$$\frac{1}{\hat{q}_1} = \frac{(A-D)}{2B} - i\sqrt{\frac{4-(A+D)^2}{4B^2}} \tag{21}$$

But the complex radius of curvature was defined [Equation (6), p.70] as

$$\frac{1}{\hat{q}_1} = \frac{1}{R_1} - i\frac{\lambda}{\pi\omega_1^2} \tag{22}$$

from which we can identify

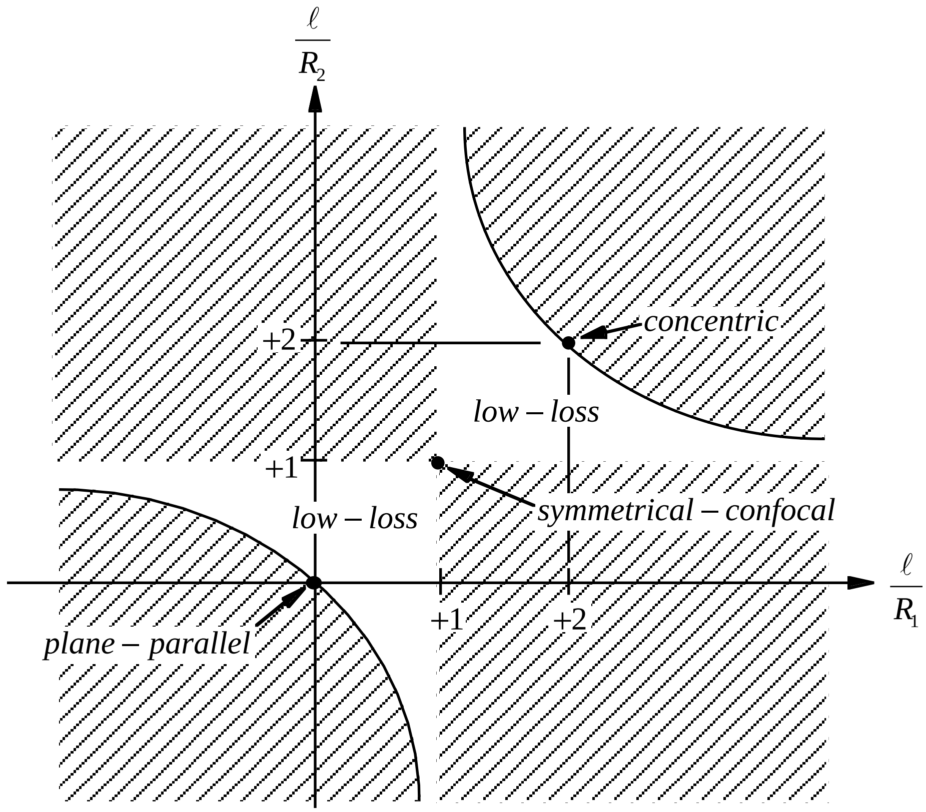
$$R_1 = \frac{2B}{A-D} \tag{23a}$$

$$\omega_1^2 = \frac{2B\lambda}{\pi\sqrt{4-(A+D)^2}} \tag{23b}$$

For the resonator to be stable ω_1 must be real and finite; hence, the denominator of (23b) must be non-zero and $(A+D)^2 < 4$ giving us the stability condition

$$\left(\frac{A+D}{2}\right)^2 < 1 \tag{24}$$

This is basically the same stability condition as we got for the biperiodic lens sequence [Equation 14, p.58] where we now recognize R_1 and R_2 as the radii of curvature of the Gaussian beam trapped in the optical resonator.



Stability diagram of an optical resonator

NOTE: The shading indicates high loss (unstable) regions in which the stability condition (24) is violated.

<u>Higher order modes</u>

The Gaussian spherical wave which we have discussed to this point is only the lowest order solution of the wave equation. It is also the solution which will have the lowest losses for a stable curved mirror cavity; higher order modes are also possible although, as we shall se, their diffraction losses will be progressively higher. The reason for this is that, while the mirrors are of finite size, the amplitude distribution moves further away from the center of the mirrors as the order increases.

If we no longer assume that $\frac{\partial}{\partial\phi} = 0$ [See Equation (7), p.73 and (8), p.74], i.e., that there may be a transverse dependence of the beam that is not symmetric about the optic axis we find that the wave equation is satisfied not only by the spherical Gaussian that we have examined in great detail, but by all members of the doubly infinite set

$$E_{mn}(x,y,z) = \sqrt{\frac{1}{2^{m+n} m!n!}} E_0 \frac{\omega_0}{\omega(z)} H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right) H_n\left(\frac{\sqrt{2}y}{\omega(z)}\right) e^{-i\frac{k}{2}\frac{r^2}{\hat{q}(z)} - ikz} e^{i(m+n+1)\phi(z)} \quad (1)$$

where $\omega(z)$, $\hat{q}(z)$ and $\phi(z)$ are as before, and $H_n(x)$ are the Hermite polynomials given by

$$H_0(x) = 1 \quad (2)$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

If the cavity has cylindrical symmetry the modes may be described in terms of associated Laguerre polynomials, i.e.,

$$E_{p\ell}(r,\theta,z) = E_0 \frac{\omega_0}{\omega(z)}\left(\frac{\sqrt{2}r}{\omega(z)}\right)^{\ell} L_p^{\ell}\left(\frac{2r^2}{\omega^2(z)}\right)\begin{pmatrix}\cos\ell\theta \\ \sin\ell\theta\end{pmatrix} e^{-i\frac{k}{2}\frac{r^2}{\hat{q}(z)}-ikz} e^{i(2p+\ell+1)\phi(z)} \qquad (3)$$

Most laser cavities will have slight asymmetries in them (e.g., mirror misalignments, Brewster angle windows, etc.) which cause mode patterns of rectangular rather than cylindrical symmetry.

We may re-write (1) using $\frac{1}{\hat{q}} = \frac{1}{R} - i\frac{\lambda}{\pi\omega^2(z)}$ to get

$$E_{mn}(x,y,z) = E_0 \frac{\omega_0}{\omega(z)} H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right) H_n\left(\frac{\sqrt{2}y}{\omega(z)}\right) e^{-\frac{r^2}{\omega^2(z)} - i\frac{k}{2}\frac{r^2}{R(z)} - ikz + i(m+n+1)\phi(z)} \qquad (4)$$

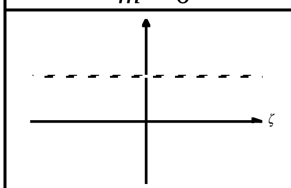
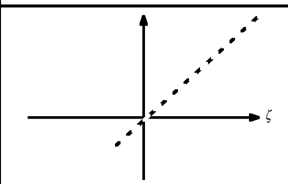
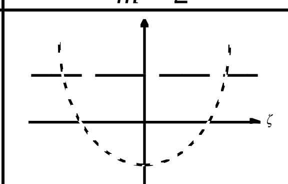
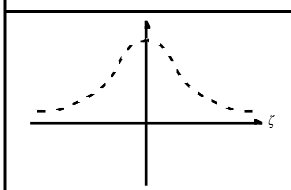
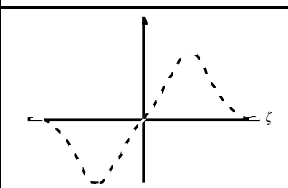
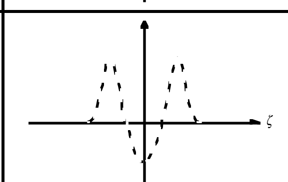
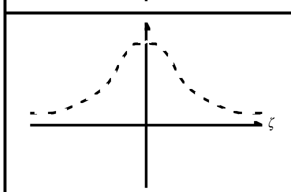
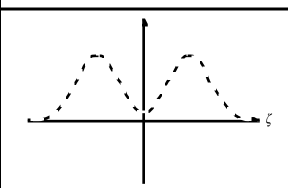
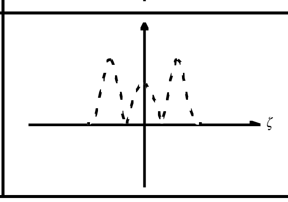
The transverse variation of the electric field is then of the form

$$E_{mn} \propto H_m\left(\frac{\sqrt{2}x}{\omega(z)}\right) H_n\left(\frac{\sqrt{2}y}{\omega(z)}\right) e^{-\frac{r^2}{\omega^2(z)}} \qquad (5)$$

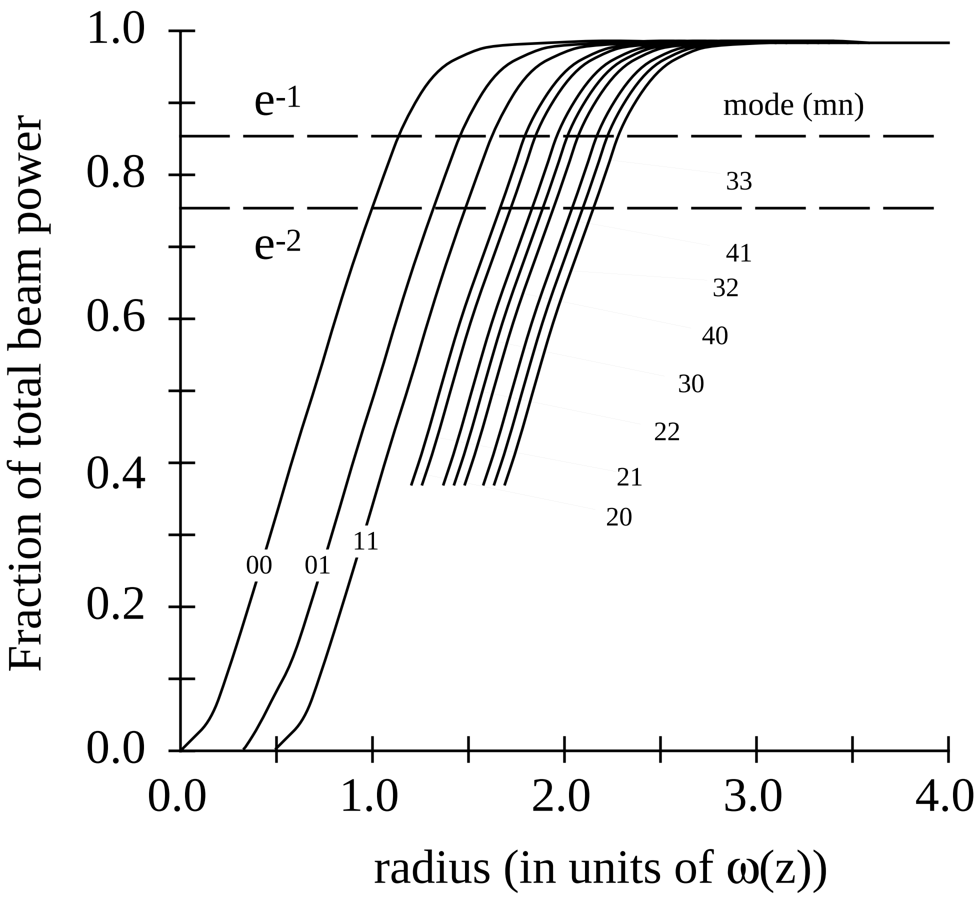
A quick graphical consideration of (5) is relevant here. For ease of drawing we will assume $n = 0$ so that (5) becomes

$$E_{m0} \propto H_m(\zeta) e^{-\frac{\zeta^2}{2}} \qquad (6)$$

where we have defined $\zeta = \frac{\sqrt{2}x}{\omega(z)}$ and are looking at the field distribution only along the x-axis, i.e., y=0, then

$m = 0$	$m = 1$	$m = 2$	mode number
			Polynomial, $H_m(\zeta)$
			field amplitude, $H_m(\zeta)e^{-\frac{\zeta^2}{2}}$
			intensity or power, $\left\lvert H_m(\zeta)e^{-\frac{\zeta^2}{2}}\right\rvert^2$

The presence of the Hermite polynomials in (5) [and (1) and (2)] is seen to shift the intensity distribution further away from the optical axis ($\zeta = 0$) in the figure above. The function $\omega(z)$ no longer gives the $\frac{1}{e^2}$ power points ($\frac{1}{e}$ amplitude points). We cannot talk about a $\frac{1}{e}$ amplitude point because as seen in the above figure the transverse amplitude can become negative for $m > 0$ and we have no basis for defining the beam size. What is done is to talk about the intensity distribution (third row in the above figure) which is everywhere positive. We can now integrate the transverse intensity distribution on a computer to determine the power distribution of the beam.



The above figure plots the fraction of the total beam power for a particular rectangular mode (TEM_{mn}) within a circular cross-section of the beam. Note that TEM_{mn} and TEM_{nm} are represented by the same curve because of their symmetry.

The e^{-1} and e^{-2} power points are shown to show how the beam "size" is increasing as the mode numbers increase. It is possible to define an effective beam size for a particular mode, say $\omega_{eff} = C_{mn}\omega(z)$, where C_{mn} is the $\frac{1}{e^2}$ power point as determined from the graph on p.88. Note that C_{mn} will be different for different m and n.

Stable resonators constructed with mirrors having radii of curvature R_1 and R_2 will reflect higher-order modes as well as the fundamental mode since neither the radius of curvature nor the spot size depends upon the indices. Lasers can, and frequently, do oscillate in a combination of higher order modes. We note, however, that as the mode indices increase the intensity distribution moves farther out on the mirrors and away from the cavity optical axis. We can, therefore, control the mode structure by making the mirrors small and, thus, increase the diffraction losses which the higher order modes experience. In practice, this is done by introducing some aperture inside the optical cavity rather than reducing the mirror size. The effect of the aperture is to "slice off" a large part of the intensity from the higher order mode than from the low-order pattern.

The beam divergence θ was defined as the ratio of $\omega(z)$ to z. This was the half-angle of the cone formed by the $\frac{1}{e^2}$ power points. Because the intensity distribution shifts further away from the optical axis with higher mode numbers we define the divergence using the previously defined ω_{eff} as

$$\theta \approx \frac{\omega_{eff}(z)}{z} = C_{mn}\frac{\omega(z)}{z} \cong C_{mn}\frac{\lambda}{\pi\omega_0}$$

showing that the beam divergence increases with increasing mode number (C_{mn} increases with increasing m and n, see p.88).

Mode frequencies

To this point we have not discussed the resonance frequencies of Gaussian beam resonators. To a first approximation, the validity of which is related to the degree to which the Gaussian spherical waves can be approximated by plane waves, $e^{-ikz} = e^{-i\varphi}$, the resonance frequencies are determined by

$$\varphi(z_2) - \varphi(z_1) = k(z_2 - z_1) = k2\ell = q2\pi \tag{1}$$

where k is the wavenumber ($k = \frac{2\pi}{\lambda}$), ℓ is the cavity length, and q is an integer. This equation (1) says that the phase shift per round trip is an integer multiple of 2π. From (1) the resonance frequencies are

$$f_q = q\frac{c}{2\ell} \tag{2}$$

To develop a more accurate expression for the mode frequencies we must use the requirement that the round trip phase shift experienced by the Gaussian beam must be an integral multiple of 2π, or that the one-way phase shift must be an integer multiple of π, i.e.,

$$\varphi(z_2) - \varphi(z_1) = q\pi \tag{3}$$

The phase shift $\varphi(z)$ for the TEM_{mn} mode from (1), p. 86 is

$$\varphi(z) = kz - (n+m+1)\phi(z) \tag{4}$$

where $\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$. Thus, the resonance condition (3) is

$$\varphi(z_2) - \varphi(z_1) = k(z_2 - z_1) - (n+m+1)\left[\phi(z_2) - \phi(z_1)\right] = q\pi$$

But, $z_2 - z_1 = \ell$ and $k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}$ so that

$$f_{mnq} = \left[q + \frac{(m+n+1)}{\pi}\left[\phi(z_2) - \phi(z_1)\right]\right]\frac{c}{2\ell} \tag{5}$$

Note that (5) specifies the resonance frequencies in terms of the Rayleigh distance, z_R, and the distances of the mirrors from the beam waist, z_1 and z_2. It would be more convenient if we had an expression in terms of the easily measured cavity parameters ℓ, R_1 and R_2. Let us examine the term $\phi(z_2) - \phi(z_1)$ in (5).

$$\cos[\phi(z_2)-\phi(z_1)] = \cos(\phi_2-\phi_1) = \cos\phi_2\cos\phi_1 + \sin\phi_2\sin\phi_1$$

Since $\phi_1 = \tan^{-1}\left(\frac{z_1}{z_R}\right)$ we have

$$\sin\phi_1 = \frac{z_1}{\sqrt{z_1^2+z_R^2}}$$

$$\cos\phi_1 = \frac{z_R}{\sqrt{z_1^2+z_R^2}}$$

and, in a similar manner,

$$\sin\phi_2 = \frac{z_2}{\sqrt{z_2^2+z_R^2}}$$

$$\cos\phi_2 = \frac{z_R}{\sqrt{z_2^2+z_R^2}}$$

Then, we can write

$$\cos(\phi_2-\phi_1) = \frac{z_R^2+z_1z_2}{\sqrt{(z_2^2+z_R^2)(z_1^2+z_R^2)}} \qquad (6)$$

using the expressions (6) and (7), p.79, for z_1, z_2 and z_R^2

$$z_1 = -\frac{g_2(1-g_1)\ell}{(g_1+g_2-2g_1g_2)}$$

$$z_2 = -\frac{g_2(1-g_1)\ell}{(g_1+g_2-2g_1g_2)}$$

$$z_R^2 = \frac{g_1g_2(1-g_1g_2)\ell^2}{(g_1+g_2-2g_1g_2)^2}$$

We have from (6), after considerable algebra,

$$\cos(\phi_2-\phi_1) = \sqrt{g_1g_2}$$

or

$$\phi_2-\phi_1 = \cos^{-1}\left(\sqrt{g_1g_2}\right) \qquad (7)$$

The detailed derivation of this result can be found in Appendix IV. Substituting (7) into (5) we have the result

$$f_{mnq} = \left[q + \frac{(m+n+1)}{\pi}\cos^{-1}\left(\sqrt{g_1g_2}\right)\right]\frac{c}{2\ell} \qquad (8)$$

which gives the resonance frequencies in terms of easily determined parameters.

There are several observations we can make about the mode frequencies in a general curved mirror cavity:

1. The lowest order mode, i.e., the TEM_{00q} modes, will not in general have frequencies corresponding to the simple plane wave analysis which resulted in (1) and (2). Instead,

   $$f_{00q} = \left[q + \frac{1}{\pi}\cos^{-1}\left(\sqrt{g_1g_2}\right)\right]\frac{c}{2\ell}$$

   and only if $\sqrt{g_1g_2} = 1$ do we have

$$f_{00q} = q\frac{c}{2\ell}$$

2. There is considerable degeneracy in the mode frequencies. Note that the frequencies of all modes for which $q = k'$ and $(m+n) = k''$, where k' and k'' are constants, are equal.

A word about terminology used to describe the mode structure in a laser cavity is in order. A laser oscillating at frequencies f_{mnq}, $f_{mnq'}$, $f_{mnq''}$, etc. at the same time is said to be operating on many <u>longitudinal modes</u>, the longitudinal modes being denoted by the various q values, which are usually widely separated in frequency. The indices m and n are used to designate <u>transverse modes</u>. Thus, a laser having frequencies f_{mnq}, $f_{m'n'q}$, etc. is said to be oscillating on several transverse modes and a single longitudinal mode. A single frequency laser will usually oscillate on the lowest order <u>transverse</u> mode, its frequency will be f_{00q}.

Let us examine the frequency spectrum predicted by (8) in more detail. The spacing between adjacent longitudinal mode frequencies (q and $q+1$) will be given by

$$f_{mn(q+1)} - f_{mnq} = \frac{c}{2\ell}(q+1-q) = \frac{c}{2\ell} = \Delta f_{long}$$

showing that the frequency spacing between adjacent longitudinal modes is a constant given by

$$\Delta f_{long} = \frac{c}{2\ell} \tag{9}$$

The spacing between transverse modes will be given by

$$f_{m'n'q} - f_{mnq} = \Delta f_{long}\left[(m'+n') - (m+n)\right]\frac{\cos^{-1}\left(\sqrt{g_1 g_2}\right)}{\pi}$$

For adjacent transverse modes, i.e., $(m'+n') - (m+n) = 1$, we have the uniform transverse mode spacing $\Delta f_{transverse}$ given by

$$\Delta f_{transverse} = \frac{\cos^{-1}\left(\sqrt{g_1 g_2}\right)}{\pi}\Delta f_{long} \tag{10}$$

To illustrate the meaning of (9) and (10) we do an example. Suppose we are considering a near planar symmetrical cavity where $R >> \ell$. Then, $g_1 = g_2 = g = 1 - \frac{\ell}{R}$. For $\frac{\ell}{R} << 1$

$$\cos^{-1}\left(\sqrt{g_1 g_2}\right) = \cos^{-1}(g) = \cos^{-1}\left(1 - \frac{\ell}{R}\right) \approx \frac{2\ell}{R}$$

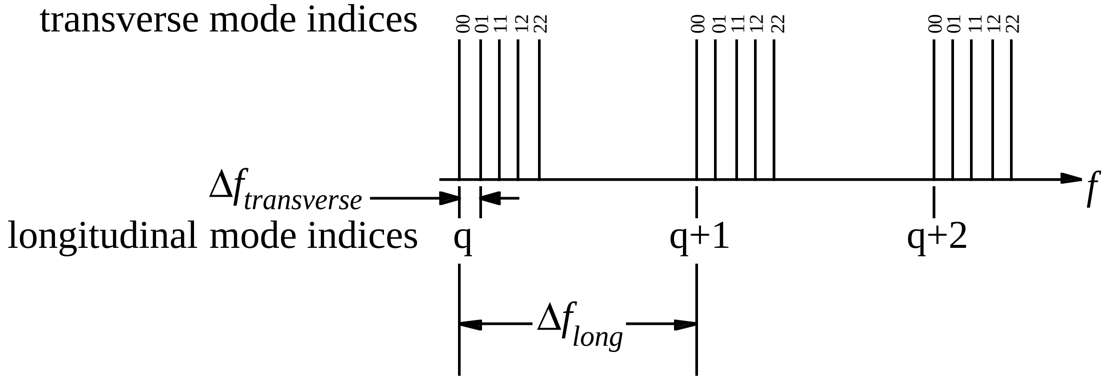
Then, from (10),

$$\Delta f_{transverse} = \frac{2\ell}{R\pi}\Delta f_{long} \tag{11}$$

This shows that the transverse mode frequencies are located very near the longitudinal mode frequencies, i.e.,

$$\Delta f_{transverse} << \Delta f_{long}$$

from (11) and the resulting frequency spectrum is as shown below.

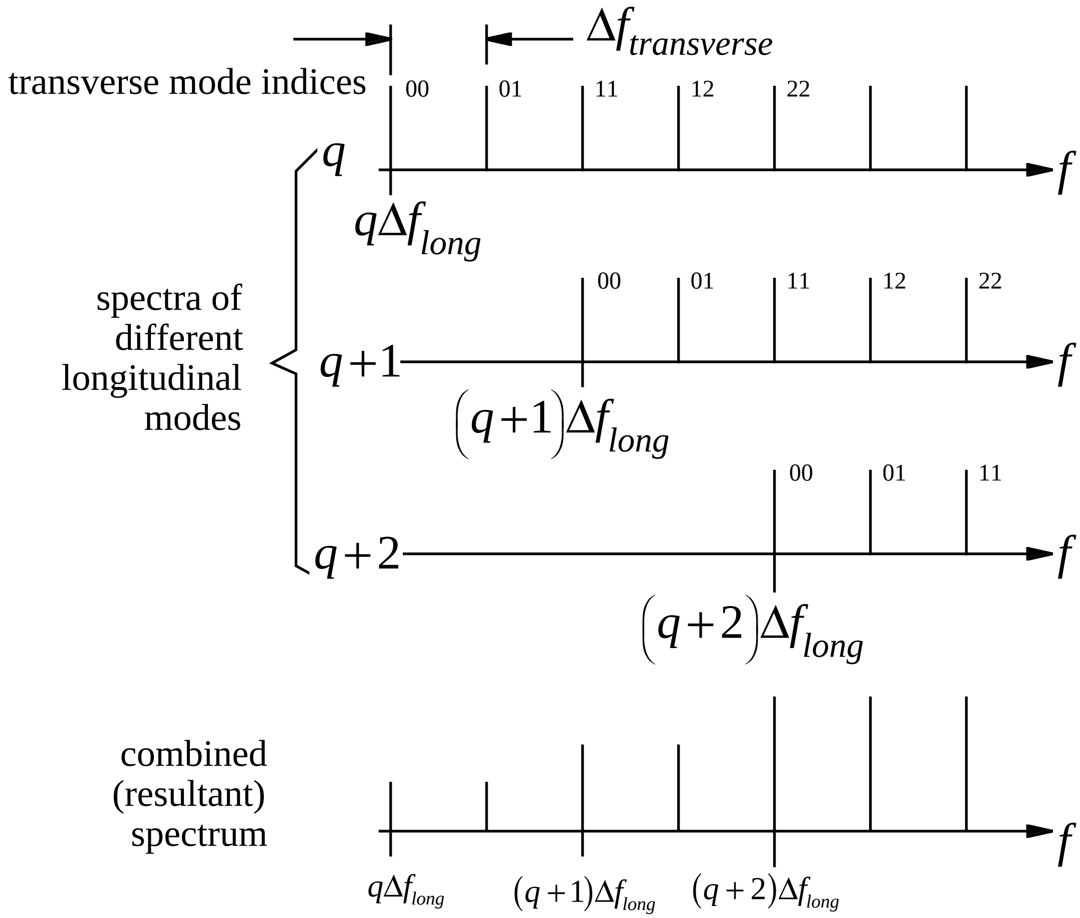


Note that the transverse mode frequencies do not continue indefinitely. Even if the laser is operating in many higher order modes simultaneously there will be a maximum transverse mode TEM_{mnq} beyond which the diffraction losses become too great to allow lasing in these higher order modes.

Consider now the spectrum of a symmetrical confocal cavity. Since $R = \ell$ we have $g = 0$ and $\cos^{-1}(g) = \cos^{-1}(0) = \frac{\pi}{2}$. From (10)

$$\Delta f_{transverse} = \frac{\frac{\pi}{2}}{\pi} \Delta f_{long} = \frac{\Delta f_{long}}{2}$$

The resulting spectrum is then



$\Delta f_{transverse}$
transverse mode indices
00
01
11
12
22
q
f
$q\Delta f_{long}$
spectra of different longitudinal modes
$q+1$
$(q+1)\Delta f_{long}$
$q+2$
$(q+2)\Delta f_{long}$
combined (resultant) spectrum

Note, page numbers for original notes.

Appendix I
Derive equation 4, p. 79

Appendix II
????

Appendix III
Page 81, smallest mirror spot size possible for a symmetrical cavity

Appendix IV
Derivation of equation (7), p.91

Appendix V
Summary of Basic optical formula

<u>TEM_{00} Gaussian beam formulas</u>

E-field solutions of wave equation under assumptions

1. $k^2(\vec{r}) = k^2 = \left(\frac{2\pi}{\lambda}\right)^2$

2. $\frac{\partial}{\partial\phi} = 0$ (radial symmetry)

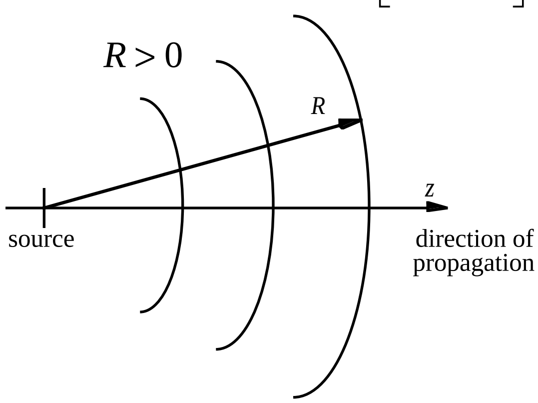
3. $\left|\frac{\partial^2\varphi}{\partial t^2}\right| << \left|2k\frac{\partial\varphi}{\partial t}\right|$

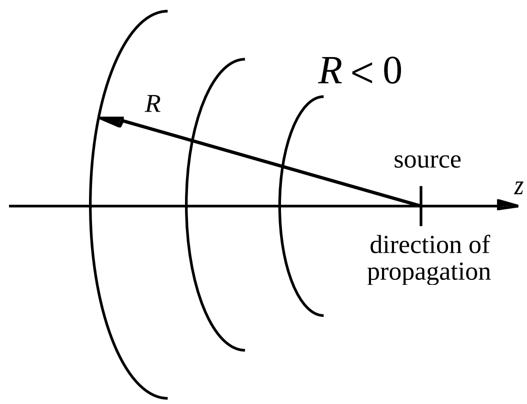
$$E(x,y,z) = E_0\,\frac{\omega_0}{\omega(z)}\,e^{-ik\frac{r^2}{2\hat{q}}-ikz+i\phi}$$

$$E(x,y,z) = E_0\,\frac{\omega_0}{\omega(z)}\,e^{-ik\frac{r^2}{2R(z)}-\frac{r^2}{\omega^2(z)}-ikz+i\phi}$$

Complex radius of curvature $\hat{q}(z)$: $\frac{1}{\hat{q}(z)} = \frac{1}{\hat{q}_0 + z} = \frac{1}{R(z)} - i\frac{\lambda}{\pi\omega^2(z)}$

Radius of curvature: $R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right] = z + \frac{z_R^2}{z} \approx z$ if $z >> z_R$



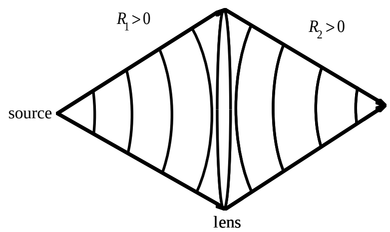
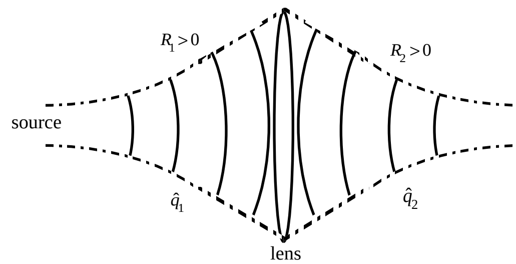


e^{-1} amplitude beam radius: $\omega(z) = \omega_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$

Rayleigh distance (collimated beam distance): $z_R = \frac{\pi\omega_0^2}{\lambda}$

Divergence angle: $\theta \approx \frac{\lambda}{\pi\omega_0}$ for $z >> z_R$

Transformation of waves

	spherical	gaussian
Through space	$R_2 = R_1 + (z_2 - z_1)$	
Through a lens	$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$	$\frac{1}{\hat{q}_2} = \frac{1}{\hat{q}_1} - \frac{1}{f}$
		
Through optical systems	$R_2 = \frac{AR_1 + B}{CR_1 + D}$	$\hat{q}_2 = \frac{A\hat{q}_1 + B}{C\hat{q}_1 + D}$
ABCD Law		where $\begin{bmatrix} r_2' \\ r_2 \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix} \begin{bmatrix} r_1' \\ r_1 \end{bmatrix}$

Power transmission of a Gaussian beam of radius $\omega(z)$ through an aperture of radius a
If $a = \omega(z)$ then 86% of the incident power will be transmitted
If $a = 1.5\omega(z)$ then 99+% of the incident power will be transmitted

Stability of Gaussian beam resonators

For two mirror cavity $0 \le \left(1 - \frac{\ell}{R_1}\right)\left(1 - \frac{\ell}{R_2}\right) \le 1$

For general cavities $0 \le \left(\frac{A + D}{2}\right)^2 \le 1$

where A and D are elements of the ABCD matrix

Measured in plane from which ray matrix makes the transformation

$$R_1 = \frac{2B}{A - D}$$

$$\omega_1^2 = \frac{2B\lambda}{\pi\sqrt{4 - (A + D)^2}}$$

Resonant frequencies of Gaussian beam resonators

$$f_{mnq} = \left[q + \frac{(m+n+1)}{\pi} \cos^{-1}\left(\sqrt{g_1 g_2}\right) \right] \frac{c}{2\ell}$$

transverse mode spacing $\Delta f_{transverse}$

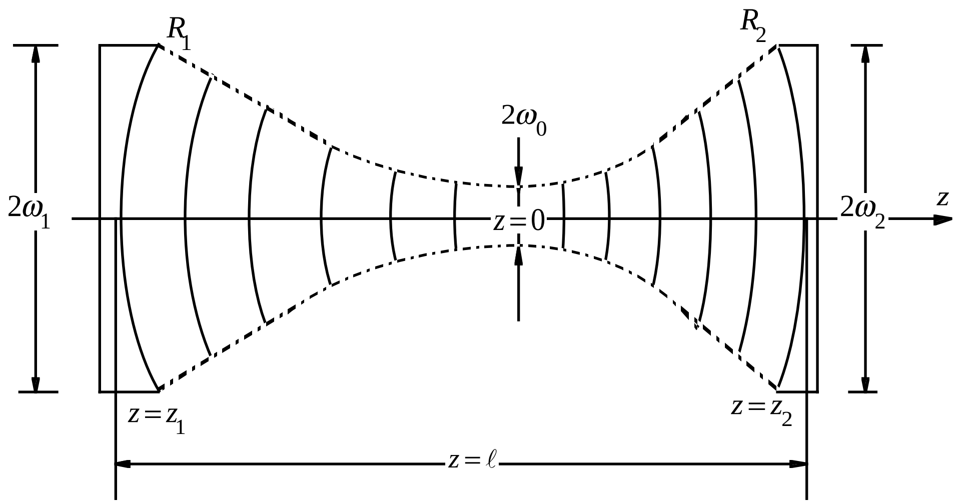
$$\Delta f_{transverse} = \frac{\cos^{-1}\left(\sqrt{g_1 g_2}\right)}{\pi} \Delta f_{long}$$

longitudinal mode spacing Δf_{long}

$$\Delta f_{long} = \frac{c}{2\ell}$$

Optical resonators for Gaussian beams

Note that $z_1 < 0$ and $R_1 < 0$, all other variables are positive.



$$z_1 = \frac{R_1 \pm \sqrt{R_1^2 - 4z_R^2}}{2} \qquad z_2 = \frac{R_2 \pm \sqrt{R_2^2 - 4z_R^2}}{2}$$

In general, $z_R^2 = \dfrac{\ell(-R_1-\ell)(R_2-\ell)(R_2-R_1-\ell)}{(2\ell+R_1-R_2)^2} = \ell^2 \dfrac{g_1 g_2(1-g_1 g_2)}{(g_1+g_2-2g_1 g_2)^2}$

Where $g_1 = 1 - \dfrac{\ell}{R_1}$, $g_2 = 1 - \dfrac{\ell}{R_2}$

$$z_1 = \frac{-g_2(1-g_1)}{g_1+g_2-2g_1 g_2}\ell, \quad z_2 = \frac{g_1(1-g_2)}{g_1+g_2-2g_1 g_2}\ell = z_1 + \ell$$

$$\omega_1^2 = \omega^2(z_1) = \frac{\ell\lambda}{\pi}\left[\frac{g_2}{g_1(1-g_1 g_2)}\right]^{\frac{1}{2}}, \quad \omega_2^2 = \frac{\ell\lambda}{\pi}\left[\frac{g_1}{g_2(1-g_1 g_2)}\right]^{\frac{1}{2}}$$

For symmetrical resonators where R is the <u>unsigned</u> radius of curvature

$$\omega_0 = \sqrt{\frac{\lambda}{4}}\sqrt[4]{\frac{\ell}{2}\left(R-\frac{\ell}{2}\right)}$$

$$-z_1 = z_2 = \frac{\ell}{2}$$

$$\omega_1 = \omega_2 \cong \sqrt{\frac{\lambda}{\pi}}\sqrt[4]{\frac{R\ell}{2}}$$

For confocal symmetric cavity where $R = \ell$

$$(\omega_0)_{conf} = \sqrt{\frac{\lambda\ell}{2\pi}}$$

$$(\omega_1)_{conf} = (\omega_2)_{conf} = (\omega_0)_{conf}\sqrt{2}$$