CHAPTER 10

RADIATION AND ANTENNAS

10.1 Introduction. Any system of conductors and material media which is connected to a power source so as to produce a time-varying electromagnetic field in an external region will radiate energy. When the system is arranged so as to optimize or accentuate the radiation of energy from some portion of the system while at the same time minimizing or suppressing radiation from the rest of the system, that portion of the system which radiates energy is called an *antenna*.

Thus antenna theory tacitly assumes that the antenna is connected to a nonradiating power source by means of a nonradiating transmission line. This idealization can usually be achieved in practice, and although, in some practical antenna problems, achieving this idealization may be the most difficult part of the problem, in this chapter we presume that it has been solved, and we concern ourselves only with the antenna.

10.2 The Radiation Problem. In Chap. 2 we showed that we could cast Maxwell's equations into a form involving a scalar wave equation and a vector wave equation plus some subsidiary equations. In particular, we were able to show that the set of equations

$$\nabla^2\phi - \mu\epsilon\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{10-1}$$

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{10-2}$$

where

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla\times\mathbf{A} \tag{10-3}$$

with A and ϕ connected by the Lorentz condition

$$\nabla\cdot\mathbf{A} = -\mu\epsilon\frac{\partial\phi}{\partial t} \tag{10-4}$$

was an alternative statement of Maxwell's equations. This formulation is particularly useful for radiation problems in that it directly relates the scalar

and vector potentials to the sources of the fields. The scalar potential ϕ is not really necessary in antenna problems since **B** can be obtained from **A**, and then Maxwell's equation

$$\nabla\times\mathbf{H} = \epsilon\frac{\partial\mathbf{E}}{\partial t} \tag{10-5}$$

can be integrated with respect to time to give

$$\mathbf{E} = \frac{1}{\epsilon}\int\nabla\times\mathbf{H}\,dt \tag{10-6}$$

Thus it is evident that what we need is the solution to Eq. (10-2).

Although a rigorous solution† of this equation is possible, the details are involved, and hence we present only arguments which make the result seem logical. With this in mind, we note that, in rectangular coordinates, Eq. (10-2) can be expressed as three scalar equations in the three components of **A**, and that each of these scalar equations is of the same mathematical form as Eq. (10-1). In source-free regions, $\rho = 0$, and Eq. (10-1) is the scalar wave equation in ϕ whose general solution is a completely arbitrary, analytic function of the arguments $t - r/v$ and $t + r/v$, where r denotes distance measured along the direction of propagation and $v = 1/\sqrt{\mu\epsilon}$. Also, for time-independent source distributions, Eq. (10-1) is just Poisson's equation, whose solution is

$$\phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho}{r}\,dv \tag{10-7}$$

We should expect the nonhomogeneous time-dependent case to incorporate the features of both types of solutions, since these are just special cases of the general solution. More precisely, it seems intuitively reasonable that the correct solution would be obtained by simply substituting $t - r/v$ for t in the integral of Eq. (10-7).

$$\phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho(t - r/v)}{r}\,dv \tag{10-8}$$

That this expression is indeed a valid solution of the time-dependent nonhomogeneous wave equation can be shown by direct substitution. However, the computations are rather involved, and are not presented at this time.

Notice that the time t in Eq. (10-8) is the time at the point of observation. On the other hand, $t' = t - r/v$ is the time at the source point. Thus the equation says that sources which had the configuration ρ at $t' = t - r/v$

† J. A. Stratton, "Electromagnetic Theory," chap. 8, McGraw-Hill Book Company, New York, 1941.

produce a potential ϕ at a time t which is later† than the time t' by an amount that takes into account the finite velocity of propagation of waves in the medium. Because of this time-delay aspect of the solution, the potential ϕ is known as the *retarded potential*, and the phenomenon itself, *retardation*.

In antenna problems it is convenient to eliminate the scalar potential ϕ and to cast the entire problem in terms of the vector potential **A**. In rectangular coordinates the time-dependent nonhomogeneous vector wave equation in **A** can be written as three simultaneous scalar wave equations, namely,

$$\nabla^2 A_x - \mu\epsilon \frac{\partial^2 A_x}{\partial t^2} = -\mu J_x \tag{10-9}$$

$$\nabla^2 A_y - \mu\epsilon \frac{\partial^2 A_y}{\partial t^2} = -\mu J_y \tag{10-10}$$

$$\nabla^2 A_z - \mu\epsilon \frac{\partial^2 A_z}{\partial t^2} = -\mu J_z \tag{10-11}$$

Since these are mathematically the same equations as the equation in ϕ, we can write down their solutions, by inspection, as

$$A_x = \frac{\mu}{4\pi}\int_V \frac{J_x(t - r/v)}{r}\,dv \tag{10-12}$$

$$A_y = \frac{\mu}{4\pi}\int_V \frac{J_y(t - r/v)}{r}\,dv \tag{10-13}$$

$$A_z = \frac{\mu}{4\pi}\int_V \frac{J_z(t - r/v)}{r}\,dv \tag{10-14}$$

or more compactly, in vector notation,

$$\mathbf{A} = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(t - r/v)}{r}\,dv \tag{10-15}$$

The problem of calculation of the field of an antenna of known current distribution thus reduces essentially to the evaluation of Eq. (10-15).

10.3 The Field of a Current Element (Hertzian Dipole).

A large class of antennas consists of conducting wires arranged so as to produce desired radiation properties. In most cases the cross-sectional size of the wires can be neglected, and the wires can be treated as perfectly conducting filamentary conductors. With this idealization, Eq. (10-15) can be written

$$\mathbf{A} = \frac{\mu}{4\pi}\int_C \frac{I(t - r/v)}{r}\,d\mathbf{l} \tag{10-16}$$

† The alternative argument, $t + r/v$, represents advanced time, implying that the phenomenon represented by the quantity ϕ can be observed before it has been generated by the sources. This is physically inconceivable, and that part of the solution which depends on $t + r/v$ is henceforth discarded.

where $I(t - r/v)$ is the current carried by the wire along the contour C, and $d\mathbf{l}$ is a vector element of length in the direction of the wire. An *isolated* infinitesimal section of the wire is known as a *current element*, or *Hertzian dipole*. Although, obviously, a current element cannot be isolated from the rest of the antenna, it is still very useful to calculate the fields which an isolated current element would produce. The fields of an actual antenna can be calculated from the fields of a current element by integration.

In this section we propose to calculate the field of the current element $I\,d\mathbf{l}$.

From Eq. (10-16) we have that the vector potential **A** is

$$\mathbf{A} = \frac{\mu}{4\pi r} I\left(t - \frac{r}{v}\right) d\mathbf{l} \tag{10-17}$$

or if I is a sinusoidal current,

$$\mathbf{A} = \frac{\mu}{4\pi r} I \cos\omega\left(t - \frac{r}{v}\right) d\mathbf{l} \tag{10-18}$$

From this expression it is apparent that the *phase delay*, corresponding to a time delay of r/v, is $\omega r/v$ rad. It will be convenient to use the spherical geometry of Fig. 10-1, where $d\mathbf{l}$ is in the z direction, and for notational

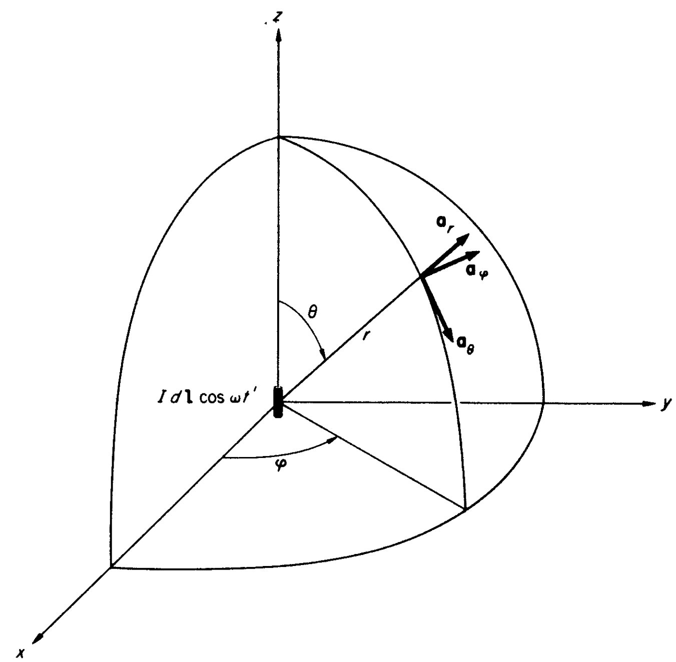


FIGURE 10-1. The geometry of a current element (Hertzian dipole).

convenience we shall write $t - r/v$ as t' in the final results. In this notation, **A** has only a z component, which is given by

$$A_z = \frac{\mu I\,dl \cos \omega t'}{4\pi r} \tag{10-19}$$

However, the **E** and **H** fields are more useful when expressed in spherical coordinates. Using Eqs. (1-13), we can write

$$A_r = A_z \cos\theta \qquad A_\theta = -A_z \sin\theta \qquad A_\varphi = 0 \tag{10-20}$$

and using $\mathbf{B} = \nabla \times \mathbf{A}$, we find

$$B_r = (\nabla \times \mathbf{A})_r = 0 \tag{10-21}$$

$$B_\theta = (\nabla \times \mathbf{A})_\theta = 0 \tag{10-22}$$

$$\begin{aligned} B_\varphi &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] \\ &= \frac{\mu I\,dl}{4\pi r}\left\{\frac{\partial}{\partial r}\left[-\sin\theta \cos\omega\left(t - \frac{r}{v}\right)\right] - \frac{\partial}{\partial \theta}\left[\frac{\cos\theta}{r}\cos\omega\left(t - \frac{r}{v}\right)\right]\right\} \\ &= \frac{\mu I\,dl \sin\theta}{4\pi}\left[\frac{-\omega \sin\omega(t - r/v)}{rv} + \frac{\cos\omega(t - r/v)}{r^2}\right] \end{aligned} \tag{10-23}$$

To find the **E** field, we use Eq. (10-6), and obtain

$$E_\theta = \frac{I\,dl \sin\theta}{4\pi\epsilon}\left(\frac{-\omega \sin\omega t'}{rv^2} + \frac{\cos\omega t'}{r^2 v} + \frac{\sin\omega t'}{\omega r^3}\right) \tag{10-24}$$

$$E_r = \frac{2I\,dl \cos\theta}{4\pi\epsilon}\left(\frac{\cos\omega t'}{r^2 v} + \frac{\sin\omega t'}{\omega r^3}\right) \tag{10-25}$$

Lastly, dividing Eq. (10-23) by μ, we have

$$H_\varphi = \frac{I\,dl \sin\theta}{4\pi}\left(\frac{-\omega \sin\omega t'}{rv} + \frac{\cos\omega t'}{r^2}\right) \tag{10-26}$$

We see from these equations† that, even for a simple current element, the exact total field is complicated. Fortunately, we seldom need to consider the exact total field. There are two reasons for this. First, we notice that the terms involve inverse r, r^2, and r^3 terms. For large distances from the current element we can neglect the higher-order terms. For instance, in E_θ and H_φ the inverse r and inverse r^2 terms are equal in magnitude when

$$\frac{\omega}{v} = \frac{1}{r}$$

† The corresponding forms in the frequency domain are given in Prob. 10-3.

that is, for

$$r = \frac{v}{\omega} = \frac{\lambda}{2\pi} \approx \frac{\lambda}{6} \tag{10-27}$$

The second reason is more fundamental. Since the antenna's primary function is to radiate energy, it will be possible most of the time to ignore those terms which do not contribute to energy radiation. The next section will show that only the inverse r terms contribute to the time-average radiated power.

Accordingly, it is customary in practice to call the field represented by the $1/r$ terms the *radiation field* and, in so doing, to distinguish it from the *induction field*, which is represented by the $1/r^2$ terms and which predominates at small distances r. Note that, aside from a time dependence, the induction field in Eq. (10-26) is predictable from the Biot-Savart law. Note also that the $1/r^3$ term in Eq. (10-24) is just the electric field intensity of an electric dipole if the time dependence were to be suppressed. Accordingly, the $1/r^3$ term is sometimes called the *electrostatic field* term.

10.4 Power Radiated by a Current Element.

In order to calculate the power radiated by a current element, we need to calculate Poynting's vector. The instantaneous Poynting's vector is given by $\mathscr{P} = \mathscr{E} \times \mathscr{H}$. For the fields given by Eqs. (10-24) to (10-26), $\mathscr{P}$ has a θ component and an r component, namely,

$$P_\theta = -E_r H_\varphi \qquad P_r = E_\theta H_\varphi \tag{10-28}$$

It is obvious that the radial component is the only component which contributes to the net outward power flow. Thus

$$\begin{aligned} P_r = \frac{I^2\,dl^2 \sin^2\theta}{16\pi^2\epsilon}\bigg(&\frac{\omega^2 \sin^2\omega t'}{r^2 v^3} - \frac{\omega \sin\omega t' \cos\omega t'}{r^3 v^2} \\ &- \frac{\sin^2\omega t'}{r^4 v} - \frac{\omega \sin\omega t' \cos\omega t'}{r^3 v^2} + \frac{\cos^2\omega t'}{r^4 v} + \frac{\sin\omega t' \cos\omega t'}{\omega r^5}\bigg) \end{aligned} \tag{10-29}$$

or after application of some trigonometric identities,

$$\begin{aligned} P_r = \frac{I^2\,dl^2 \sin^2\theta}{16\pi^2\epsilon}\bigg[&\frac{\sin 2\omega t'}{2\omega r^5} + \frac{\cos 2\omega t'}{r^2 v^2} \\ &- \frac{\omega \sin 2\omega t'}{r^3 v^2} + \frac{\omega^2(1 - \cos 2\omega t')}{2r^2 v^3}\bigg] \end{aligned} \tag{10-30}$$

Noting that the time average of both $\sin 2\omega t'$ and $\cos 2\omega t'$ is zero, we can write the time average of P_r as

$$P_{r(\text{av})} = \frac{\omega^2 I^2\,dl^2 \sin^2\theta}{32\pi^2\epsilon r^2 v^3} \tag{10-31}$$

The important feature of this result, for our present use, is that the time-average value of the radial component of Poynting's vector is one-half times the product of the inverse r terms in **E** and **H**. Hence the *far field* of our isolated current element, which is specified by

$$\mathbf{E} = -\frac{\omega I\,dl \sin\theta}{4\pi\epsilon r v^2}\sin\omega t'\,\mathbf{a}_\theta \qquad \mathbf{H} = -\frac{\omega I\,dl\sin\theta}{4\pi r v}\sin\omega t'\,\mathbf{a}_\varphi \tag{10-32}$$

is all that is needed for calculation of radiated power and is also a valid approximation to the total field for large distances. This is true for the far field of any antenna. For this reason the far field is frequently called the *radiation field*.

Before we calculate the total radiated power from our current element, let us examine the radiation field further. First, we note that the **E** and the **H** fields are in time phase and normal to each other. Second, we note that

$$|\mathbf{E}| = \eta\,|\mathbf{H}| \tag{10-33}$$

where $\eta = \sqrt{\mu/\epsilon}$. Thus, except for a $(\sin\theta)/r$ term in both **E** and in **H**, the radiation field has the properties of a uniform plane wave. For spherical surfaces of large r and for regions on the surface which are small enough so that $\sin\theta$ can be considered constant, the far field appears to be a uniform plane wave.

Returning to the problem of calculation of the total radiated power, we see that

$$\begin{aligned}\text{Power radiated} &= \oint_\Sigma P_{r(\text{av})}\,da = \oint_\Sigma \frac{\eta\,|H_\varphi|^2}{2}\,da \\ &= \frac{\eta\omega^2 I^2\,dl^2}{32\pi^2 v^2}\int_0^{2\pi}\int_0^{\pi}\frac{\sin^2\theta}{r^2}\,r^2\sin\theta\,d\theta\,d\varphi \\ &= \frac{\eta\omega^2 I^2\,dl^2}{12\pi v^2}\end{aligned} \tag{10-34}$$

Generally, it is useful to assume that the antenna is in free space, for which η has the value of 120π, and to note that $\omega/v = \beta = 2\pi/\lambda$ and that $I^2/2 = I_{\text{rms}}^2$. With these substitutions we obtain

$$\text{Power radiated} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 I_{\text{rms}}^2 \tag{10-35}$$

By analogy with circuit theory we like to write power $= I_{\text{rms}}^2 R$, and we define

$$R_{\text{rad}} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2 \tag{10-36}$$

as the *radiation resistance* of a current element.

10.5 The General Nature of the Far Field of an Antenna. In general terms, we can write for the field of a current element

$$E_\theta = \frac{E_0}{r}\sin\omega t' \qquad H_\varphi = \frac{E_0}{r\eta}\sin\omega t' \tag{10-37}$$

where E_0 contains all the amplitude factors in Eqs. (10-32), or if we absorb a 90° phase factor into E_0, we can write

$$E_\theta = \text{Re}\left(\frac{E_0 e^{j(\omega t-\beta r)}}{r}\right) \qquad H_\varphi = \text{Re}\left(\frac{E_0 e^{j(\omega t-\beta r)}}{\eta r}\right) \tag{10-38}$$

which suggests that we can make use of the simplification of manipulation afforded by working with the complex fields†

$$E_\theta = \frac{E_0 e^{-j\beta r}}{r} \qquad H_\varphi = \frac{E_0 e^{-j\beta r}}{\eta r} \tag{10-39}$$

For most antennas, the far field is of a similar form, such that we can, in general, write the far field

$$\mathbf{E} = \mathbf{E}_0\frac{e^{-j\beta r}}{r} \tag{10-40}$$

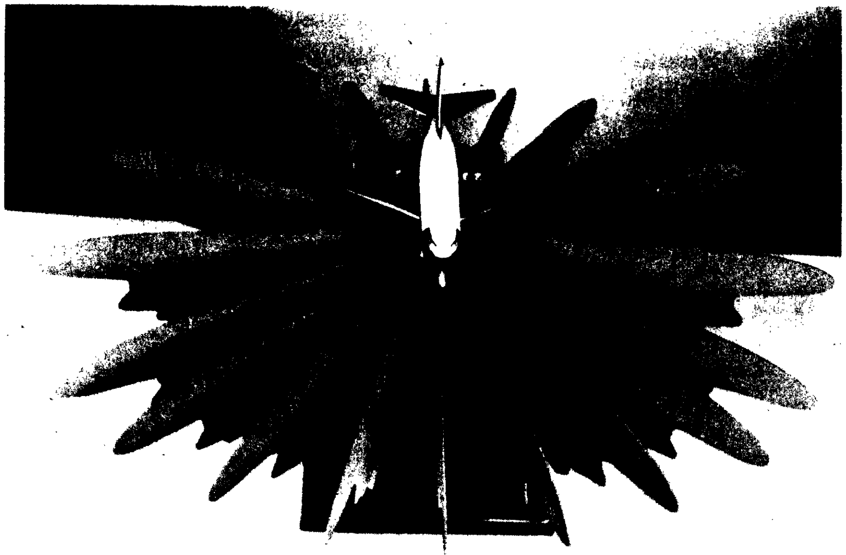
$$\mathbf{H} = \frac{1}{\eta}\,\mathbf{a}_r \times \mathbf{E} \tag{10-41}$$

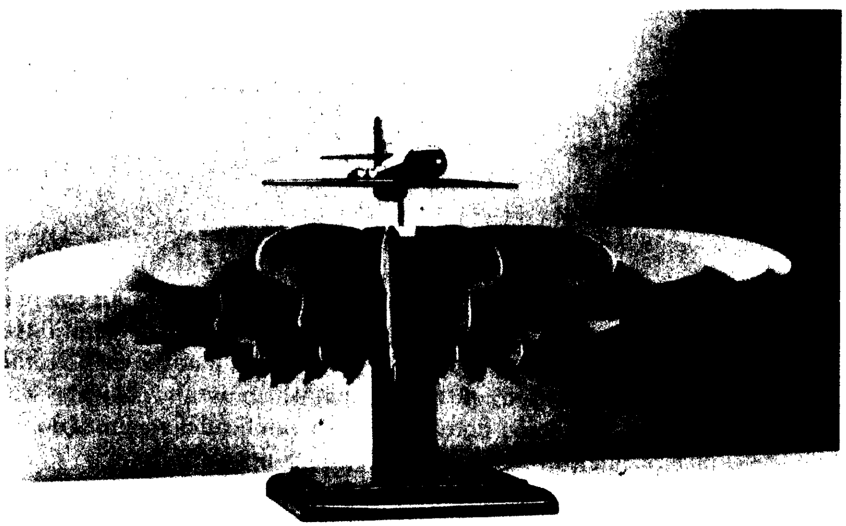
with **E** perpendicular to **H**, and both **E** and **H** perpendicular to $\mathbf{a}_r$, where $\mathbf{a}_r$ is the radius vector from the *phase center* (at a given observation point this is defined‡ as the center of that sphere on which the plane of the field vectors exhibits the least local variation) of the antenna, which usually coincides with its physical center. We should note that some complicated antennas do not have a true phase center, and this simplification is not valid.

In simple cases, the field is linearly polarized. Since the general case of elliptical polarization can usually be treated as superposition of two linear polarizations, in the rest of this chapter we assume linear polarization.

† See Prob. 10-3, previously cited.

‡ For standard definitions see Test Procedures for Antennas, *IEEE Trans. on Antennas and Propagation*, vol. AP-13, pp. 464–466, May, 1965.





Three-dimensional displays of an aircraft antenna pattern.
(*Courtesy of Lockheed-Georgia Company.*)

Usually, of course, E_0 will be a function of angular position, just as it was for a current element. Additionally, the general nature of the far field of an antenna is such that at any point in space it behaves locally as a uniform plane wave.

10.6 Antenna Patterns. An antenna pattern is a three-dimensional plot which shows the antenna's characteristics as a radiator of energy. Three types of antenna patterns are in general use which show the relative angular distribution of (1) field intensity, (2) power density, or (3) radiation intensity.

The* E*-field pattern A plot of $|\mathbf{E}|$ as a function of θ and φ is called the E-*field pattern* (three-dimensional). As a practical matter, it is of course impossible to present a complete three-dimensional plot. In most cases, a plot of $|\mathbf{E}|$ as a function of θ for some particular value of φ plus a plot of $|\mathbf{E}|$ as a function of φ for some particular value of θ give most of the useful information.

Example 10-1 Field Patterns. Given that the E field has only a θ component

$$E_\theta = \frac{E_0 \sin\theta}{r} e^{-j\beta r} \tag{10-42}$$

Plot the E-field pattern for $\theta =$ constant and for $\varphi =$ constant.
These plots are shown in Fig. 10-2*a* and *b*.

Usually, the patterns are normalized so that the maximum magnitude is 1, and are then called *normalized* E-*field patterns*.

The power pattern A plot of the time-average Poynting's vector is called the *power pattern*. The power pattern may be thought of as a plot of

$$\text{Re}\,(S_c) = \tfrac{1}{2}\,\text{Re}\,(\mathbf{E} \times \mathbf{H}^*)$$

Since the complex Poynting's vector for the far field is real, this gives for our example

$$\text{Re}\,(S_c) = S_c = \frac{1}{2}\frac{E_0^2 \sin^2\theta}{\eta r^2} \qquad \text{W/m}^2$$

This pattern is shown plotted in Fig. 10-2*c*. The half-power points in this figure specify the *beamwidth*. This is the angular distance (90° in Fig. 10-2*c*) between the directions at which the power is one-half the maximum power.

The radiation intensity pattern If we multiply $\text{Re}\,(S_c)$ by r^2, we obtain the *radiation intensity U*. Thus

$$U = r^2\,\text{Re}\,(S_c) \qquad \text{W/unit solid angle}$$

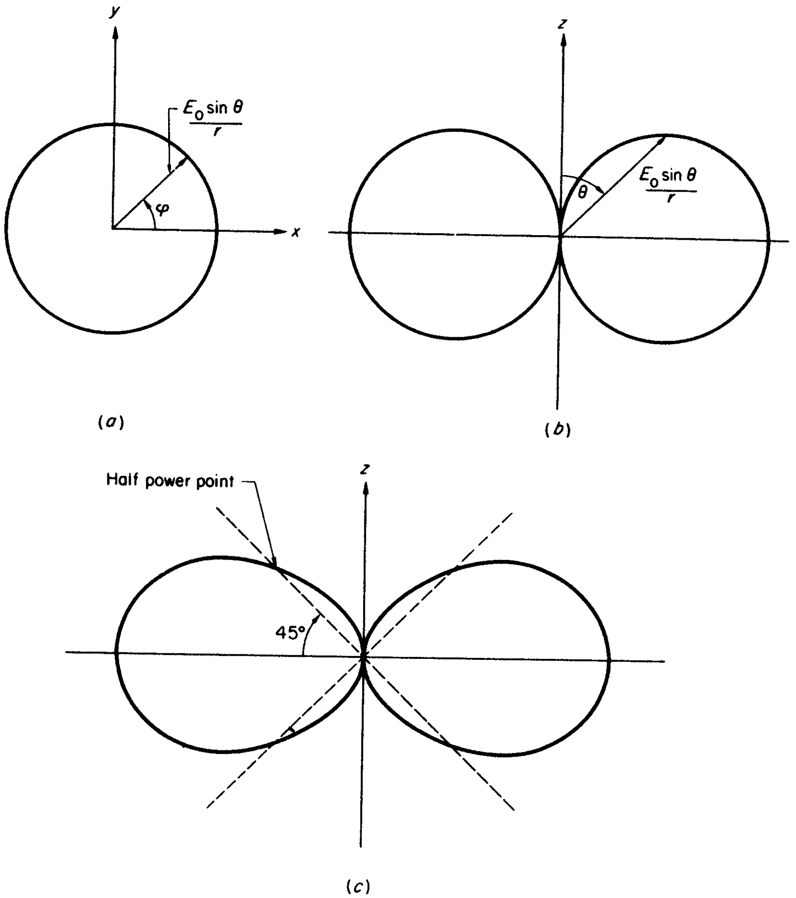


FIGURE 10-2. The field patterns of $E_\theta = (1/r)E_0 \sin\theta\, e^{-j\beta r}$. (a) E_θ for θ = constant; (b) E_θ for φ = constant; (c) power pattern for φ = constant.

A plot of the radiation intensity is called the *radiation intensity pattern.* In our example

$$U = \frac{1}{2}\frac{E_0^2 \sin^2\theta}{\eta} \qquad \text{W/unit solid angle}$$

Both the power patterns and the radiation intensity patterns are usually normalized to unity by dividing by the maximum value at a particular r. Thus

$$\mathrm{Re}\,(S_{c_n}) = \sin^2\theta \tag{10-43}$$

$$U_n = \sin^2\theta \tag{10-44}$$

The normalized patterns are obviously identical, and it is customary in practice to refer to either of them as *the* pattern. Use of this imprecise

terminology is also extended by the methods generally employed to measure patterns experimentally. Most recording techniques actually present the logarithm of the power per unit area relative to an arbitrary reference, and the usual reference is the maximum power per unit area. The quantity

$$10 \log \frac{\mathrm{Re}\,(S_c)}{\mathrm{Re}\,(S_{c_{\max}})} \tag{10-45}$$

expresses the result in decibels (dB).

10.7 Directivity and Gain. Before proceeding further with our discussion, we need precise definitions of two terms which, because of their similarity, are frequently misused.

The *directivity D* of an antenna is defined as the ratio of the maximum radiation intensity to the average radiation intensity. Put mathematically,

$$D = \frac{U_{\max}}{U_{\mathrm{av}}} \tag{10-46}$$

where U_{av} is the average radiation intensity. Notice that, by use of the general method of obtaining an average, we have

$$U_{\mathrm{av}} = \frac{1}{4\pi}\oint_\Sigma U(\theta,\varphi)\, d\Omega \tag{10-47}$$

or

$$4\pi U_{\mathrm{av}} = \oint_\Sigma U(\theta,\varphi)\, d\Omega = P_{\mathrm{rad}} \tag{10-48}$$

Equation (10-48) states that 4π times the average radiation intensity equals the total power radiated by the antenna. Hence we can write Eq. (10-46) as

$$D = \frac{4\pi U_{\max}}{4\pi U_{\mathrm{av}}} = \frac{4\pi U_{\max}}{P_{\mathrm{rad}}} \tag{10-49}$$

which says that we can calculate the directivity by taking the ratio of 4π times the maximum radiation intensity to the total power radiated.

Equation (10-49) is the IEEE standard definition of directivity, and it is frequently the more convenient equation to use for calculation.

A secondary concept which we need is that of a *lossless isotropic radiator* (antenna). For a *lossless antenna*, the power input equals the power radiated. For an *isotropic antenna*, the radiation intensity is the same in all directions, and hence $U(\theta,\varphi)$ = constant = U_{av}. Thus, for a lossless isotropic antenna, we have

$$P_{\mathrm{input}} = P_{\mathrm{radiated}} = 4\pi U_{\mathrm{av}} \tag{10-50}$$

Such an antenna, although conceptually very useful, cannot be achieved in practice.†

Now, the *gain G* of an antenna is defined as the ratio of the maximum radiation intensity from the antenna, U_{max}, to the maximum radiation intensity from a reference antenna, $(U_{max})_r$, with the same power *input*. Thus

$$G = \frac{U_{max}}{(U_{max})_r} \tag{10-51}$$

The gain of an antenna is a relative quantity. It involves the use of a reference antenna and takes into consideration the *efficiency* of both antennas. Efficiency is defined as the ratio of total power radiated by an antenna to the net power accepted by the antenna.

To standardize the gain specification, two reference antennas are commonly used. They are the lossless half-wave dipole (see next section) and the lossless isotropic antenna. For these two cases, we use the terms *gain over a dipole* and *gain over isotropic*. Gain over isotropic is used often enough so that a special symbol G_0 is defined for that case.

$$G_0 = \frac{U_{max}}{U_0} \tag{10-52}$$

where U_0 is the (constant) radiation intensity of a lossless isotropic radiator with the same power input. From these definitions, we have $U_{max} = kU'_{max}$, where k denotes the efficiency of the antenna, and U'_{max} is the maximum value which the radiation intensity would have if the antenna were lossless. Notice that, for a lossless antenna, $k = 1$ and $G_0 = D$; that is, for a lossless antenna, the gain over isotropic is exactly equal to the directivity. Since many antennas have relatively small losses, there is a tendency to be rather lax in distinguishing between gain and directivity.

Frequently, it is convenient to specify gain and directivity in decibels by giving 10 times the logarithm to the base 10 of the actual ratio.

$$\text{Gain in decibels} = 10 \log G_0$$

For example, a gain $G_0 = 20$ would be given as 13 dB.

In concluding our present set of definitions, we should point out that the concepts of gain and directivity are frequently generalized to include the concept of gain and directivity as a function of direction. Specifically,

$$G_0(\theta,\varphi) = \frac{U(\theta,\varphi)}{U_0} \tag{10-53}$$

† It has been shown, however, that one can approach arbitrarily close to this idealization. See W. K. Saunders, On the Unity Gain Antenna, "Symposium on Electromagnetic Theory and Antennas," Copenhagen, June 25–30, 1962 (Pergamon Press).

is the gain over isotropic as a function of direction (θ,φ), and

$$D(\theta,\varphi) = \frac{U(\theta,\varphi)}{U_{av}} \tag{10-54}$$

is the directivity as a function of direction (θ,φ). These expressions are really just normalized radiation intensity patterns. The gain $G_0(\theta,\varphi)$ has been normalized by dividing the radiation intensity $U(\theta,\varphi)$ by the radiation intensity U_0 of a lossless isotropic antenna of the same input power, and $D(\theta,\varphi)$ has been normalized by dividing by the average radiation intensity U_{av}. Note that, for a lossless antenna, $U_{av} = U_0$ and $D(\theta,\varphi) = G_0(\theta,\varphi)$, so that for low-loss antennas the difference between gain and directivity is small, and one tends to be lax about making a distinction.

Example 10-2 Calculation of Directivity. Suppose an antenna has a power input of 40π W and an efficiency of 98 percent. Also, suppose that the radiation intensity has been found to have a maximum value of 200 W/unit solid angle. Find the directivity and gain of the antenna.

We have

$$U_{av} = \frac{P_{rad}}{4\pi} = \frac{(0.98)(40\pi)}{4\pi} = 9.8 \text{ W/sr}$$

Hence

$$D = \frac{200}{9.8} = 20.4, \text{ or } 13.1 \text{ dB}$$

Also,

$$U_0 = \frac{P_{in}}{4\pi} = \frac{40\pi}{4\pi} = 10 \text{ W/sr}$$

and

$$G_0 = 200/10 = 20, \text{ or } 13.0 \text{ dB}$$

10.8 Linear Dipole Antennas. A *linear dipole antenna* is a straight-wire antenna, usually *center-fed* (Fig. 10-3*a*). A *linear monopole antenna* is a straight-wire antenna *fed against* a ground plane (Fig. 10-3*b*). It is fairly obvious that a monopole antenna differs structurally from a dipole antenna. However, the electrical problem of a monopole antenna is basically the same as that of a dipole antenna, and is best handled by the method of images. This method implies that the vector potential and the field intensity of a monopole in the region above the ground plane are exactly the same as those of a center-fed dipole with the same current and an *overall* length which is twice the monopole length. It is clear, then, that analysis of the center-fed dipole includes analysis of the monopole.

Let us fix our attention on the *short dipole*, defined as a center-fed antenna having a length that is very short compared with a wavelength. If the overall length L of a center-fed dipole is short enough so that the contributions to the far field from each infinitesimal element of its length are in time phase with each other, the total fields can be calculated by simple scalar addition of the

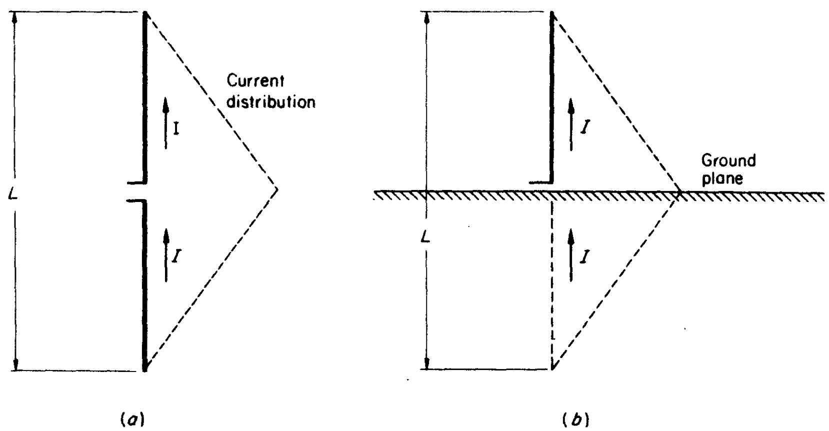


FIGURE 10-3. The current distribution on linear antennas. (*a*) Short dipole; (*b*) short monopole.

infinitesimal fields produced by a chain of Hertzian dipoles. This approximation is valid for $L \leq \lambda/10$, roughly speaking. For such an antenna we have

$$E_\theta = \int_{-L/2}^{L/2} \frac{j\omega I \sin\theta}{4\pi\epsilon v^2 r} e^{-j\beta r}\, dl$$

Our approximations here mean that we can consider the distance r from the point of observation to the source point (position along the antenna) to be constant, allowing us to write

$$E_\theta = \frac{j\omega \sin\theta e^{-j\beta r}}{4\pi\epsilon v^2 r} \int_{-L/2}^{L/2} I\, dl = \frac{j\omega \sin\theta e^{-j\beta r}}{4\pi\epsilon v^2 r} I_{av} L \qquad (10\text{-}55)$$

$$H_\varphi = \frac{E_\theta}{\eta} \qquad (10\text{-}56)$$

This result simply means that the fields of a short dipole are obtained from the expressions for an infinitesimal dipole by simple substitution of L for dl and I_{av} for I, where

$$I_{av} = \frac{1}{L}\int_{-L/2}^{L/2} I\, dz \qquad (10\text{-}57)$$

The current along a short dipole and along a short monopole varies nearly linearly (Sec. 10.9) from I_0 at the center to zero at the end, as shown in Fig. 10-3. From this it is obvious that

$$I_{av} = \frac{I_0}{2} \qquad (10\text{-}58)$$

We can readily calculate the total radiated power from a short antenna by substituting $I_{av} = I_0/2$ and $L = dl$ into Eq. (10-35) and noting that, for a monopole, the radiated power is just ½ that of a dipole (it radiates only into the upper half space). We obtain, for the dipole,

$$P_{rad} = 20\pi^2 \left(\frac{L}{\lambda}\right)^2 I_{rms}^2 \qquad (10\text{-}59)$$

and for the monopole,

$$P_{rad} = 10\pi^2 \left(\frac{L}{\lambda}\right)^2 I_{rms}^2 \qquad (10\text{-}60)$$

where, as in Eq. (10-35), I_{rms}^2 denotes the root-mean-square value of I.

We can also define a radiation resistance for each case by

$$R_{rad}\ (\text{dipole}) = 20\pi^2\left(\frac{L}{\lambda}\right)^2 \quad \Omega \qquad (10\text{-}61)$$

$$R_{rad}\ (\text{monopole}) = 10\pi^2\left(\frac{L}{\lambda}\right)^2 \quad \Omega \qquad (10\text{-}62)$$

Some numerical values are of interest. For $L = \lambda/10$ the two resistances are approximately 2 and 1 Ω, respectively. These values are very small for transmission line loads and cause rather severe problems of transmission line matching. The matching networks required frequently have large losses, with the result that the overall system efficiency is small.

A quantity which is often specified for short dipoles is the *effective length*, defined by the relation

$$L_{eff} = \frac{1}{I_0}\int_{-L/2}^{L/2} I(z)\, dz \qquad (10\text{-}63)$$

where I_0 is the current fed to the antenna which extends from $z = -L/2$ to $z = L/2$. From this definition it is apparent that the effective length of an antenna is that length which, by supporting the feed current I_0 of the actual antenna throughout the entire length, has the same overall effectiveness as the original antenna.

10.9 Current Distribution on a Linear Antenna. We saw that for short antennas we had to know the current distribution on the antenna before we could finish the problem. For longer antennas it is necessary to know it before we start. In principle, we can find the current distribution by solving Maxwell's equations subject to the boundary conditions along the antenna. In practice, it turns out that this is such a formidable problem that it has been

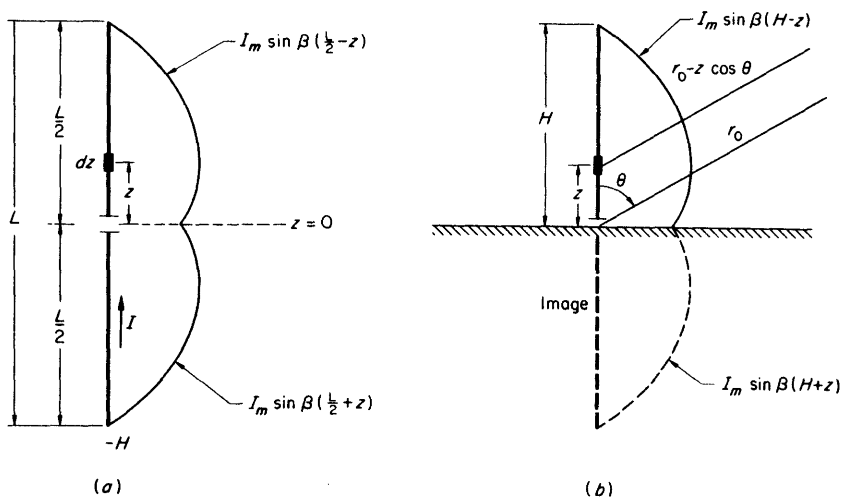


FIGURE 10-4. The geometry and the assumed sinusoidal current distribution for a center-fed dipole and a monopole. (*a*) Dipole; (*b*) monopole ($H = L/2$).

solved rigorously for only one case.† We are therefore faced with the necessity of assuming a distribution, hoping it is correct. Intuitively, we might expect the current to have the standing wave distribution characteristic of an open-circuited transmission line (see previous chapter.) This assumption proves to be correct, at least as a valid engineering approximation, in that antenna calculations based upon this assumption yield quite accurate results.

The standing wave current distribution in the z direction has a magnitude

$$I = \begin{cases} I_m \sin \beta\left(\dfrac{L}{2} - z\right) & z > 0 \\ I_m \sin \beta\left(\dfrac{L}{2} + z\right) & z < 0 \end{cases} \qquad (10\text{-}64)$$

and hence is called a *sinusoidal current distribution* (Fig. 10-4).

10.10 The Longer Linear Dipole and Monopole.

When a center-fed dipole or a monopole, base-fed against a ground plane, has a half length which exceeds approximately $\lambda/10$, the simple analysis of Sec. 10.8 for short linear antennas is no longer a valid approximation, and we must use a more

† R. W. P. King, The Linear Antenna: Eighty Years of Progress, *Proc. IEEE*, vol. 55, pp. 2–16, January, 1967.

exact analysis. The usual procedure is to calculate the vector potential, and the far-field electric and magnetic fields from the vector potential.

If we use the geometry and assume a sinusoidal current distribution, as shown in Fig. 10-4, we shall find that the vector potential has only a z component, given in complex notation by

$$A_z = \int_{-H}^{0} \frac{\mu I_m \sin \beta(H + z)e^{-j\beta r}}{4\pi r}\, dz + \int_{0}^{H} \frac{\mu I_m \sin \beta(H - z)e^{-j\beta r}}{4\pi r}\, dz$$

where $H = L/2$. In these integrands we set $r = r_0 - z \cos \theta$ in the exponential factors and $r = r_0$ in the denominator; we then factor the constant terms, and obtain

$$A_z = \frac{\mu I_m e^{-j\beta r_0}}{4\pi r_0} \int_{-H}^{0} \sin \beta(H + z) e^{j\beta z \cos \theta}\, dz + \frac{\mu I_m e^{-j\beta r_0}}{4\pi r_0} \int_{0}^{H} \sin \beta(H - z) e^{j\beta z \cos \theta}\, dz \qquad (10\text{-}65)$$

Noting that in the first integrand z is negative allows us to change signs on z and change limits so as to obtain

$$A_z = \frac{\mu I_m e^{-j\beta r_0}}{4\pi r_0} \int_{0}^{H} \sin \beta(H - z)(e^{-j\beta z \cos \theta} + e^{j\beta z \cos \theta})\, dz$$

$$= \frac{\mu I_m e^{-j\beta r_0}}{2\pi r_0} \int_{0}^{H} \sin \beta(H - z) \cos (\beta z \cos \theta)\, dz \qquad (10\text{-}66)$$

and after integration,

$$A_z = \frac{\mu I_m e^{-j\beta r_0}}{2\pi \beta r_0} \frac{\cos \beta(H \cos \theta) - \cos \beta H}{\sin^2 \theta} \qquad (10\text{-}67)$$

From A_z we obtain the far field in a manner similar to that employed for the infinitesimal dipole. The details are not displayed here. The results are

$$H_\varphi = \frac{j I_m e^{-j\beta r_0}}{2\pi r_0} \frac{\cos (\beta H \cos \theta) - \cos \beta H}{\sin \theta} \qquad (10\text{-}68)$$

$$E_\theta = \eta H_\varphi \qquad (10\text{-}69)$$

When the half length H is equal to a quarter wavelength, the antenna is known as a *half-wave dipole* (or a *quarter-wave monopole*, if fed against a ground plane). These antennas are of particular practical importance because they have desirable input characteristics and also have desirable

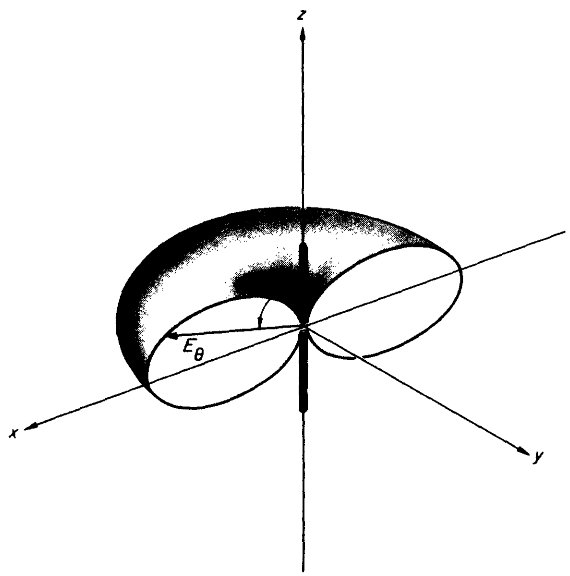


FIGURE 10-5. The normalized E-field pattern for a half-wave dipole. (A quarter-wave monopole has the same pattern for $\theta \leq \pi/2$, but the fields are zero for $\theta > \pi/2$.)

radiation patterns. Substituting $H = \lambda/4$ into the general expressions, Eqs. (10-68) and (10-69), gives the results

$$E_\theta = \frac{j60 I_m e^{-j\beta r}}{r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \tag{10-70}$$

$$H_\varphi = \frac{j I_m e^{-j\beta r}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \tag{10-71}$$

where the subscript on r has been dropped in accordance with usual notation.

The E-field patterns and the power, or radiation intensity, patterns for this special case are relatively easy to plot. A three-dimensional cutaway view of the normalized E-field pattern is shown in Fig. 10-5 for the $\lambda/2$ dipole. The $\lambda/4$ monopole pattern is the same as the $\lambda/2$ dipole in the upper half space, and is zero in the lower half space ($\theta > \pi/2$).

The total time-average power radiated by a half-wave dipole can be calculated by integrating the complex Poynting's vector over a sphere of

radius r. We have

$$\begin{aligned} P_{\text{rad}} &= \oint_\Sigma \tfrac{1}{2} E_\theta H_\varphi^* \, da \\ &= \frac{30}{2\pi} I_m^2 \int_0^{2\pi} d\varphi \int_0^\pi \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta \\ &= 30\, I_m^2 \int_0^\pi \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \, d\theta \end{aligned} \tag{10-72}$$

The remaining integral in Eq. (10-72) is not easy to evaluate. It may be attacked by a change-of-variable technique and eventually cast into a slowly convergent infinite series, or it may be programmed on a digital computer. The result, in any event, is that its value to four significant figures is 1.2186, and we obtain the numerical result that the radiated power is given by

$$P_{\text{rad}} = 73 \frac{I_m^2}{2} \qquad \text{half-wave dipole}$$

Using our previous definition of radiation resistance, we see that

$$R_{\text{rad}} = 73\ \Omega \qquad \text{half-wave dipole} \tag{10-73}$$

Since the fields of a quarter-wave monopole for $\theta < \pi/2$ are exactly the same as those of a half-wave dipole and zero for $\theta > \pi/2$, and since the half-wave dipole fields are symmetrical about $\theta = \pi/2$, we should have just one-half the radiated power of a dipole for the monopole case. That is,

$$P_{\text{rad}} = 36.5 \frac{I_m^2}{2} \qquad \text{quarter-wave monopole}$$

and the radiation resistance would be

$$R_{\text{rad}} = 36.5\ \Omega \qquad \text{quarter-wave monopole} \tag{10-74}$$

A look at Fig. 10-4, specialized to $H = \lambda/4$, shows that the driving-point current is given by

$$I = I_m \sin\beta(H - 0)e^{j\omega t} = I_m e^{j\omega t}$$

In this special case, the resistive part of the driving-point impedance of the antenna is just the radiation resistance. It is beyond the scope of our presentation to derive the driving-point reactance, but it turns out to be approximately zero. It is exactly zero for H slightly less than $\lambda/4$. Most $\lambda/2$ dipoles and $\lambda/4$ monopoles are adjusted to make the driving-point reactance zero. When the length is adjusted, the driving-point resistance is slightly less than 73, or 36.5 Ω. The resulting driving-point impedances of

$$Z_{\text{in}} \approx \begin{cases} 73 + j0 & \text{half-wave dipole} \quad (10\text{-}75) \\ 36.5 + j0 & \text{quarter-wave monopole} \quad (10\text{-}76) \end{cases}$$

are comparatively easy to match to transmission lines, and in a large measure account for the popularity of these antennas.

We complete our discussion of the two special cases by calculating their directivity. From the definition of directivity we obtain, for a $\lambda/2$ dipole,

$$D = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(\frac{1}{2}E_\theta H_\varphi^* r^2)_{\max}}{R_{\text{rad}}(I_m{}^2/2)} = \frac{2\pi(60I_m{}^2/2\pi)}{73(I_m{}^2/2)} = 1.64, \text{ or } 2.15 \text{ dB} \qquad (10\text{-}77)$$

and similarly, for a $\lambda/4$ monopole,

$$D = 3.28, \text{ or } 5.15 \text{ dB} \qquad (10\text{-}78)$$

10.11 Antenna Arrays. When two or more antennas are located in a common region of space and driven either directly or indirectly from a common generator, we have an *antenna array*. In principle, the general antenna-array problem is handled by superposition. That is, the resulting **E** and **H** fields can, in principle at least, be found by writing the vector-phasor sum of the fields produced by the individual antennas.

To obtain specific results, one must consider specific arrays. The simplest array consists of two identical antennas and, more specifically, of two identical short dipoles oriented in space, as shown in Fig. 10-6, and driven with in-phase currents of equal magnitude.

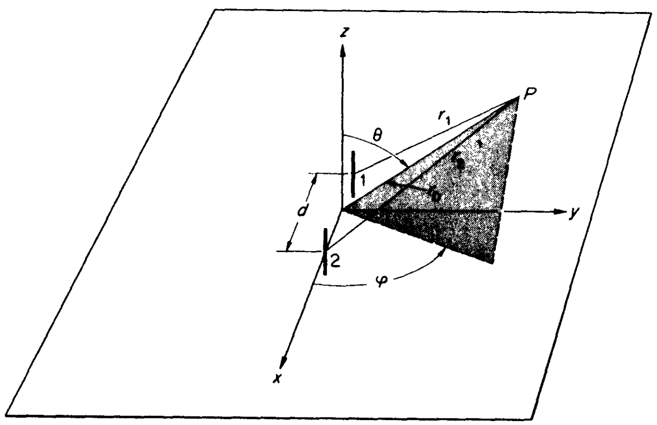


FIGURE 10-6. The geometry of two identical short dipoles.

The far fields of the individual antennas at an ordinary far-field point will be in the θ direction, and will be given by the sum of

$$E_1 = E_m \sin\theta \frac{e^{-j\beta r_1}}{r_1} \qquad (10\text{-}79)$$

and

$$E_2 = E_m \sin\theta \frac{e^{j\beta r_2}}{r_2} \qquad (10\text{-}80)$$

Thus

$$E = E_1 + E_2 = E_m \sin\theta\left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{-j\beta r_2}}{r_2}\right) \qquad (10\text{-}81)$$

In the far-field region, the lines r_1, r_0, r_2 are essentially parallel, and we have

$$r_1 \approx r_0 + \frac{d}{2}\cos\varphi \qquad r_2 \approx r_0 - \frac{d}{2}\cos\varphi \qquad (10\text{-}82)$$

In the far field $r_0 \gg d/2$, and we can use $r_1 \approx r_2 \approx r_0$ in the denominator. However, because of the periodic nature of the exponential, r_1 and r_2 must be expressed as in Eqs. (10-82). This states that the variation of r affects the phase in the integrand but has little effect on the magnitude. Accordingly,

$$E = \frac{E_m \sin\theta}{r_0}\left(e^{-j\beta r_0 - j\beta(d/2)\cos\varphi} + e^{-j\beta r_0 + j\beta(d/2)\cos\varphi}\right) = \frac{E_m \sin\theta e^{-j\beta r_0}}{r_0}(e^{-j\psi/2} + e^{j\psi/2}) \qquad (10\text{-}83)$$

where $\psi = \beta d\cos\varphi$, and finally,

$$E = E_0\left(2\cos\frac{\psi}{2}\right) \qquad (10\text{-}84)$$

where

$$E_0 = \frac{E_m \sin\theta e^{-j\beta r_0}}{r_0} \qquad (10\text{-}85)$$

is the field of the individual short dipole.

To interpret the $2\cos(\psi/2)$ factor, we note that the pattern of two in-phase isotropic radiators of unity magnitude separated by a distance d would be

$$E_i = \left(\frac{e^{-j\beta r_1}}{r_1} + \frac{e^{+j\beta r_2}}{r_2}\right) = \frac{e^{-j\beta r_0}}{r_0}\left(2\cos\frac{\psi}{2}\right)$$

In view of this expression, we see that the pattern of the two identical in-phase dipole antennas can be expressed in normalized form as

$$E_n = (\sin\theta)\left(\cos\frac{\psi}{2}\right)$$

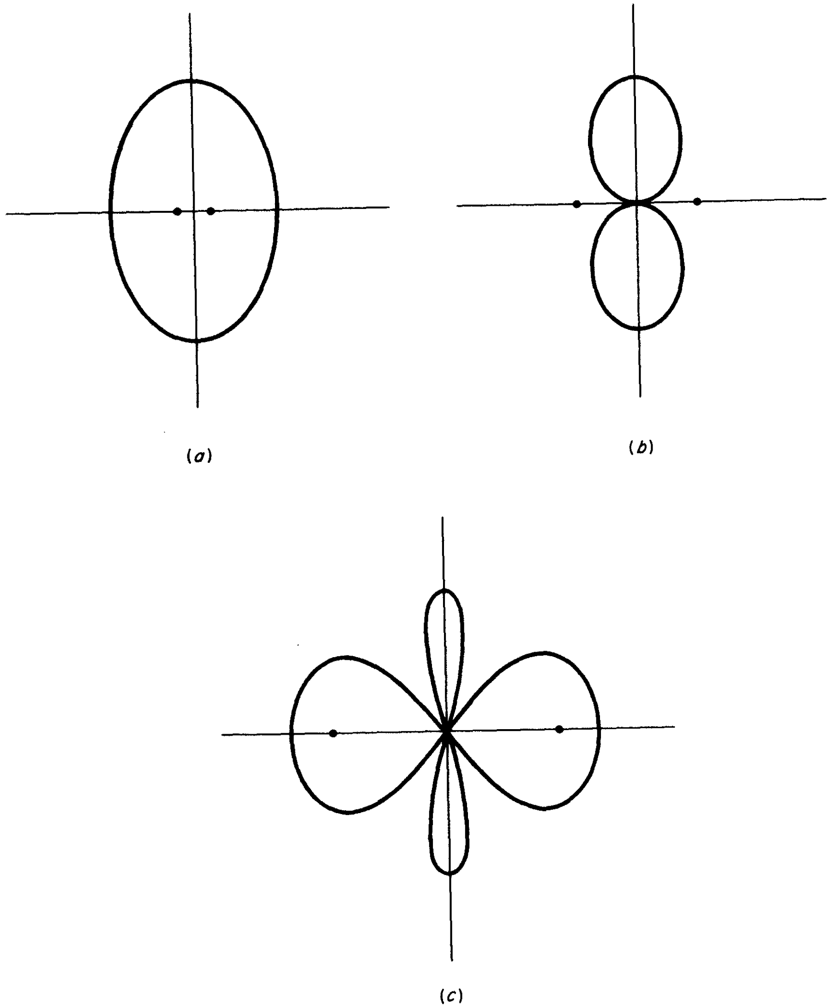


FIGURE 10-7. The array factor for two isotropic in-phase point sources for three different spacings. (a) $d = \lambda/4$; (b) $d = \lambda/2$; (c) $d = \lambda$.

where the first term is the normalized pattern of the individual dipole, and the second term is the normalized pattern of the array of isotropic point sources. This result is general for arrays of identical antennas. It is common practice to analyze arrays of identical antennas by first finding the *array pattern* (frequently called the *array factor*) and multiplying it by the pattern of the individual antenna.

Analysis of arrays of nonidentical antennas is usually a formidable problem, which we do not present in this brief treatment of antennas.

Example 10-3 Arrays of Isotropic In-phase Point Sources. In this example we give the array factor (that is, the normalized E-field pattern) of two isotropic in-phase point sources.

As derived in this section, the array factor is $AF = \cos(\psi/2)$, where $\psi = \beta d \cos\varphi$.

CASE I. $d = \lambda/4$

$$\psi = \beta d \cos\varphi = \frac{2\pi}{\lambda}\frac{\lambda}{4}\cos\varphi = \frac{\pi}{2}\cos\varphi$$

and

$$AF = \cos\left(\frac{\pi}{4}\cos\varphi\right)$$

CASE II. $d = \lambda/2$

$$AF = \cos\left(\frac{\pi}{2}\cos\varphi\right)$$

CASE III. $d = \lambda$

$$AF = \cos(\pi\cos\varphi)$$

These array factors are plotted in Fig. 10-7.

If the two individual antennas of an array are not in phase, the previous results will be modified to include their relative phase. In particular, if we let α be the relative phase by which antenna 2 leads antenna 1, we can let the phase reference of the array be the centerpoint of the array, and let antenna 1 lag this point by $\alpha/2$, and antenna 2 lead this point by $\alpha/2$. Then the result will be

$$E = E_0(e^{-j\beta(d/2)\cos\varphi - j\alpha/2} + e^{j\beta(d/2)\cos\varphi + j\alpha/2}) = E_0\left(2\cos\frac{\psi}{2}\right) \quad (10\text{-}86)$$

where

$$\psi = \beta d\cos\varphi + \alpha \quad (10\text{-}87)$$

which shows that the array factor is still $\cos(\psi/2)$. But in this case ψ includes the relative phase α.

We now present three examples to show typical results.†

Example 10-4 Specific Array Factors. We consider three cases.

CASE I. $d = \lambda/4$, $\alpha = \pi/2$

$$\frac{\psi}{2} = \frac{\beta d}{2}\cos\varphi + \frac{\alpha}{2} = \frac{2\pi}{2\lambda}\frac{\lambda}{4}\cos\varphi + \frac{\pi}{4} = \frac{\pi}{4}(1 + \cos\varphi)$$

$$AF = \cos\left[\frac{\pi}{4}(1 + \cos\varphi)\right]$$

† A rather extensive set of array factors for two isotropic point sources is given in J. D. Kraus, "Antennas," chap. 11, McGraw-Hill Book Company, New York, 1950.

Case II. $d = \lambda/4,\ \alpha = \pi$

$$\frac{\psi}{2} = \frac{\pi}{4}\cos\varphi + \frac{\pi}{2} = \frac{\pi}{4}(2 + \cos\varphi)$$

$$AF = \cos\left[\frac{\pi}{4}(2 + \cos\varphi)\right]$$

Case III. $d = \lambda,\ \alpha = \pi/2$

$$\frac{\psi}{2} = \pi\cos\varphi + \frac{\pi}{4} = \frac{\pi}{4}(1 + 4\cos\varphi)$$

$$AF = \cos\left[\frac{\pi}{4}(1 + 4\cos\varphi)\right]$$

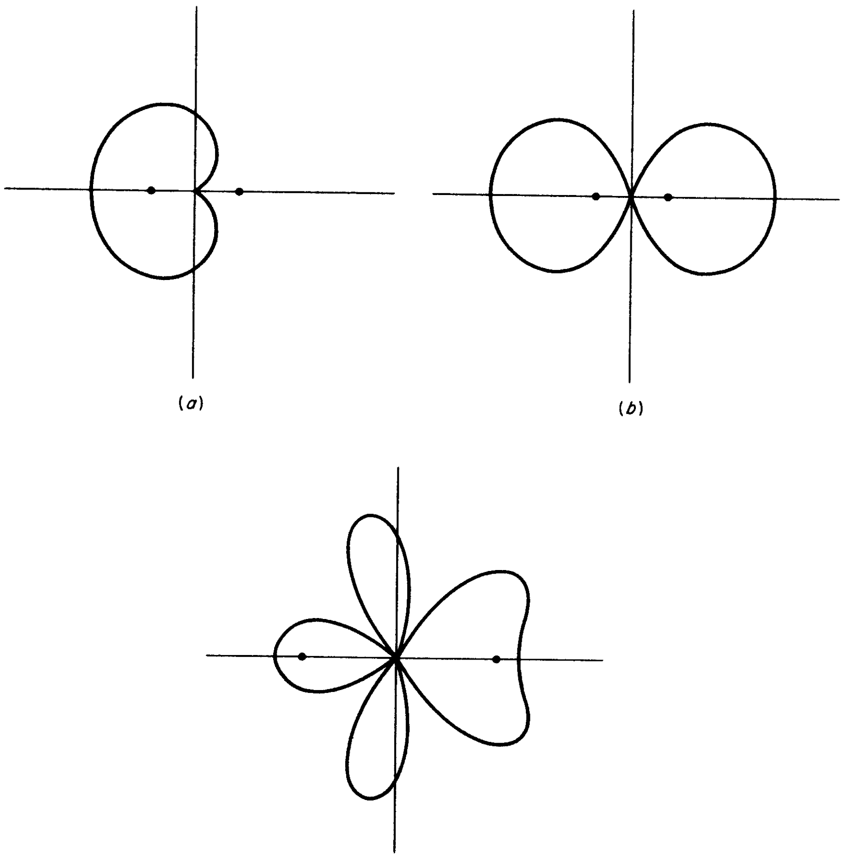


FIGURE 10-8. The array factor for two isotropic point sources for three different conditions of spacing and phase. (*a*) $d = \lambda/4$, $\alpha = \pi/2$; (*b*) $d = \lambda/4$, $\alpha = \pi$; (*c*) $d = \lambda$, $\alpha = \pi/2$.

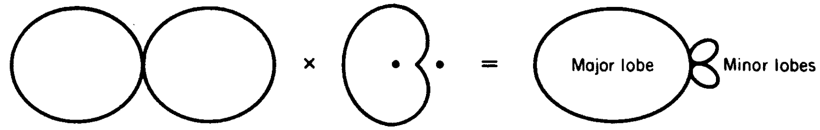


FIGURE 10-9. The individual antenna pattern, the array factor, and the resultant E-field array pattern in the plane of the array for two identical half-wave dipoles spaced $\lambda/4$ apart and fed in phase quadrature. Individual dipole × array factor = resultant pattern.

These array factors are shown plotted in Fig. 10-8. Examination of this figure and Fig. 10-7, which is for $\alpha = 0$, shows that a wide range of array factors can be obtained by adjusting the spacing and phasing of the array. Arrays are commonly used to produce some desired modification of the pattern of the individual antenna. Figure 10-9 shows an example of this for two half-wave dipoles spaced $\lambda/4$ apart and fed with the right-hand antenna leading the left-hand antenna by 90°. Notice that a fairly accurate sketch of the resultant pattern can be made by geometrically multiplying the individual pattern and the array factor.

The example given in Fig. 10-9 shows only the pattern in the plane of the dipoles. The total pattern of the array of course is three-dimensional. A fairly accurate visualization of its three-dimensional behavior can be obtained by applying the same technique in the plane perpendicular to the plane of the dipoles ($\theta = 90°$ plane), and then in the plane perpendicular to the line joining the two dipoles. This is left as an exercise for the student.

10.12 Uniform Linear Arrays. A *uniform linear array* is an array of identical antennas uniformly spaced along a straight line, fed with currents of equal magnitude and having a uniform progressive phase shift. Figure 10-10 shows an n-element uniform linear array. Using the methods already developed, we can write the total radiation **E** field as

$$E = E_0(1 + e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(n-1)\psi}) \qquad (10\text{-}88)$$

where

$$\psi = \beta d\cos\varphi + \alpha \qquad (10\text{-}89)$$

and E_0 is the individual antenna's **E**-field pattern. Since we are primarily interested in the array factor, we suppress E_0 and, after an algebraic manipulation, write

$$|E| = \left|\frac{1 - e^{jn\psi}}{1 - e^{j\psi}}\right| = \left|\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right| \qquad (10\text{-}90)$$

This result has several interesting properties, which we now examine.

1. The angle $\psi/2$ has a maximum value of $(\beta d + \alpha)/2$ at $\varphi = 0$ and a minimum value of $(-\beta d + \alpha)/2$ at $\varphi = \pi$. At $\varphi = 2\pi$, $\psi/2$ returns to

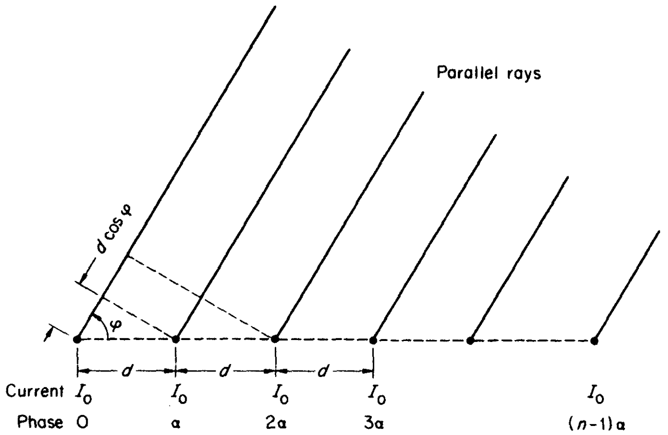


FIGURE 10-10. The spacing, magnitude, and relative phase of an n-element uniform linear array.

its maximum value. Examination of this behavior and a little thought show that the array factor, Eq. (10-90), is symmetrical about the line of the array ($\varphi = 0$, $\varphi = \pi$ line).

2. By differentiation and inspection we find that the *principal maximum* of the array factor occurs at $\psi = 0$ and that the magnitude of this principal maximum is equal to n.
3. The secondary maxima occur at, approximately,

$$\sin \frac{n\psi}{2} = 1$$

which means $\qquad \frac{n\psi}{2} = \pm(2k + 1)\frac{\pi}{2} \qquad k = 1, 2, 3, \ldots$

In particular, the *first secondary maximum* is at

$$\frac{\psi}{2} = \frac{3\pi}{2n}$$

and has a magnitude of $|\sin (3\pi/2n)|^{-1}$. Notice that this has a limiting value, for n large,

$$\lim_{n\to\infty} \frac{1}{|\sin (3\pi/2n)|} = \frac{2n}{3\pi} = 0.212n$$

Since the magnitude of the principal maximum is n, we have that the ratio of the first secondary maximum to the principal maximum is 0.212, or 13.5 dB.

4. The array factor has zero nulls when the numerator is zero; that is, when

$$\sin \frac{n\psi}{2} = 0 \qquad \text{except } \psi = 0$$

which means $\qquad \frac{n\psi}{2} = \pm k\pi \qquad k = 1, 2, 3, \ldots$

Two special cases of uniform linear arrays are of particular interest and practicality.

Case I. Broadside array If all the elements of the array are in phase ($\alpha = 0$), the array is called a *broadside array*, for reasons which are obvious from an examination of the location of the principal maxima. These maxima are at $\psi/2 = 0$ because, for $\alpha = 0$,

$$\psi = \beta d \cos \varphi = 0 \qquad \text{at } \varphi = \pm \frac{\pi}{2}$$

Thus we have the principal maxima perpendicular to the line of the array (or broadside). The angle φ, in Fig. 10-10, is actually an angle of revolution with only positive values from 0 to π. In many practical antenna situations it is convenient to view φ as a planar angle whose range is from 0 to 2π.

Case II. End-fire array If the progressive phase shift α is related to the spacing by $\alpha = -\beta d$, the principal maximum is at $\varphi = 0$, and the array is called an *end-fire array*.

There are several other classes of wire antennas and arrays of linear elements.† However, at frequencies approaching, roughly, 1 GHz, the wavelength is only a fraction of 1 m, and the size and power-handling capability of wire antennas are correspondingly small. For applications in which microwave frequencies are employed, it is necessary to use reflector-type antennas, like those shown in Fig. 10-11, with large physical dimensions. The next section analyzes the radiation from such antennas.

10.13 Huygens' Principle and Aperture Antennas. In our discussion of linear dipole antennas we assumed a well-defined, known current distribution, and we were able to calculate the far field of the antenna by rather straightforward methods. In the case of microwave antennas we do not have a well-defined, known current distribution. However, in most cases of interest, we are able to determine the **E**- and **H**-field distributions over a

† H. Jasik (ed.), "Antenna Engineering Handbook," McGraw-Hill Book Company, New York, 1961.

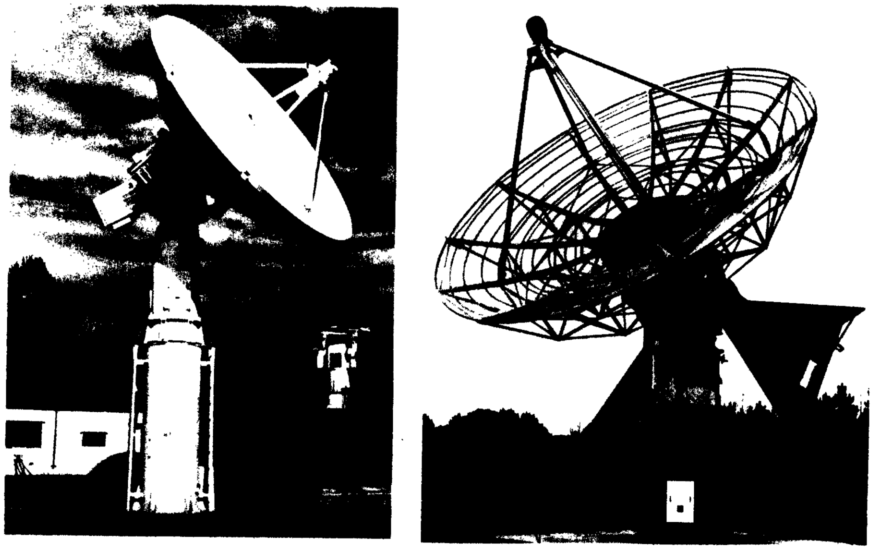


FIGURE 10-11. Reflector-type antennas. (*Courtesy of Scientific-Atlanta, Inc.*)

finite open surface located in front of the antenna. In such cases we call this surface the *aperture*, we call the antenna an *aperture antenna*, and we call the method of calculation of the far-field pattern of the antenna the *aperture field method.*

A basic postulate in the analysis of aperture antennas is the validity of *Huygens' principle.*

Let Σ be a closed surface consisting of a perfect screen S_2, on which the tangential component of the electric field intensity and the normal component of the magnetic field intensity are both zero, and an aperture S_1 which is bounded by a closed contour C (Fig. 10-12). For harmonic ($e^{j\omega t}$) fields, Maxwell's equations predict† that the field at every interior point (x,y,z) of a linear, homogeneous, isotropic, and source-free medium bounded by Σ is given in terms of the *aperture field* $\mathbf{E}_1$, $\mathbf{H}_1$ by the relations

$$\mathbf{E}(x,y,z) = \frac{j}{4\pi\omega\epsilon}\int_{S_1}\{[(\mathbf{n}\times\mathbf{H}_1)\cdot\nabla']\nabla'\phi + \beta^2\phi(\mathbf{n}\times\mathbf{H}_1) + j\omega\epsilon(\mathbf{n}\times\mathbf{E}_1)\times\nabla'\phi\}\,da \quad (10\text{-}91)$$

$$\mathbf{H}(x,y,z) = \frac{-j}{4\pi\omega\mu}\int_{S_1}\{[(\mathbf{n}\times\mathbf{E}_1)\cdot\nabla']\nabla'\phi + \beta^2\phi(\mathbf{n}\times\mathbf{E}_1) - j\omega\mu(\mathbf{n}\times\mathbf{H}_1)\times\nabla'\phi\}\,da \quad (10\text{-}92)$$

† S. Silver (ed.), "Microwave Antenna Theory and Design," chap. 5, McGraw-Hill Book Company, New York, 1949. (Note opposite direction of the vector **n**.)

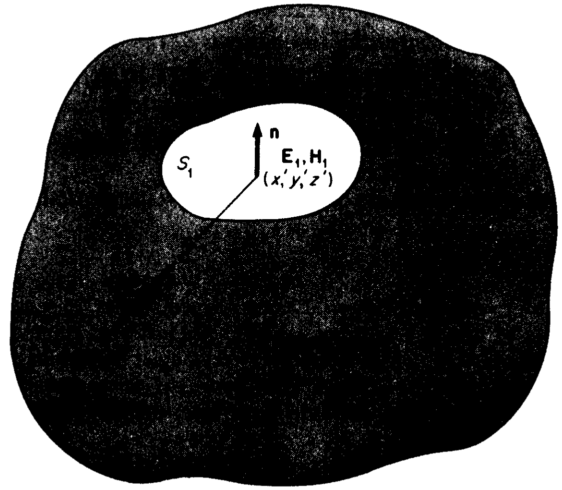


FIGURE 10-12. An aperature S_1 in a perfect screen. The primary sources are outside the closed surface defined by S_1 and S_2.

where

$$[(\mathbf{n}\times\mathbf{H}_1)\cdot\nabla']\nabla'\phi \triangleq \mathbf{a}_x\left[(\mathbf{n}\times\mathbf{H}_1)\cdot\frac{\partial}{\partial x'}(\nabla'\phi)\right] + \mathbf{a}_y\left[(\mathbf{n}\times\mathbf{H}_1)\cdot\frac{\partial}{\partial y'}(\nabla'\phi)\right] + \mathbf{a}_z\left[(\mathbf{n}\times\mathbf{H}_1)\cdot\frac{\partial}{\partial z'}(\nabla'\phi)\right] \quad (10\text{-}93)$$

and where $\phi = e^{-j\beta r}/r$, $\beta = \omega\sqrt{\mu\epsilon}$, and r is the distance from the fixed point of observation (x,y,z) to the variable source point (x',y',z') on the aperture S_1. Furthermore, ∇' denotes differentiation with respect to the primed variables, and $\mathbf{n}$ is the unit outward vector normal to the surface Σ.

Equations (10-91) and (10-92) represent a general mathematical statement of Huygens' principle for harmonic fields and state that, if the field can be described on the boundary, it can be found at any point inside. By means of lengthy calculations requiring the introduction of several simplifying assumptions, this pair of expressions can be reduced to the forms most suitable for analysis and physical interpretation.

Thus, for $r \gg \lambda/2\pi$, the transformed expressions for the field intensities are

$$\mathbf{E}(x,y,z) = \frac{j\beta}{4\pi}\int_{S_1}\mathbf{u}_r\times\left[(\mathbf{n}\times\mathbf{E}_1) - \sqrt{\frac{\mu}{\epsilon}}\,\mathbf{u}_r\times(\mathbf{n}\times\mathbf{H}_1)\right]\frac{e^{-j\beta r}}{r}\,da \quad (10\text{-}94)$$

$$\mathbf{H}(x,y,z) = \frac{j\beta}{4\pi}\int_{S}\mathbf{u}_r\times\left[(\mathbf{n}\times\mathbf{H}_1) - \sqrt{\frac{\epsilon}{\mu}}\,\mathbf{u}_r\times(\mathbf{n}\times\mathbf{E}_1)\right]\frac{e^{-j\beta r}}{r}\,da \quad (10\text{-}95)$$

From these expressions it is clear that the field in the interior can be evaluated from a knowledge of the *tangential components of the field* on the aperture S_1. It is also clear that both integrands are transverse to $\mathbf{u}_r$. Therefore, for all points of observation such that $r \gg \lambda/2\pi$, *the contribution to the field from each infinitesimal Huygens' source on S_1 is perpendicular to $\mathbf{u}_r$, the direction of wave travel.*

An application of the results obtained so far is illustrated in the following example.

Example 10-5 The Pyramidal Horn. Electromagnetic horns, in general, comprise an important class of aperture antennas. Figure 10-13 shows a commonly used type of horn, the *pyramidal horn*, which is fed by a rectangular waveguide in the dominant mode. It is known† that, for all practical purposes, the mouth of the horn is uniformly polarized in one direction (the y direction in this case) and that the *aperture illumination function* is given by

$$\mathbf{E}_1 = E_0 \cos \frac{\pi x}{a} \exp \left[-j\beta \left(\frac{x^2}{2l_H} + \frac{y^2}{2l_E} \right) \right] \mathbf{a}_y \qquad (10\text{-}96)$$

where E_0 is a constant. As indicated in Fig. 10-13, the distances l_H and l_E in Eq. (10-96) define the flare of the horn. The aperture S_1 is the mouth of the horn, and the surface S_2 is the rest of the xy plane. Also, $\mathbf{n} = -\mathbf{a}_z$, and

$$\mathbf{u}_r = u_x\mathbf{a}_x + u_y\mathbf{a}_y + u_z\mathbf{a}_z = \frac{x - x'}{r}\mathbf{a}_x + \frac{y - y'}{r}\mathbf{a}_y + \frac{z}{r}\mathbf{a}_z$$

where u_x, u_y, u_z are the direction cosines of the unit vector $\mathbf{u}_r$.

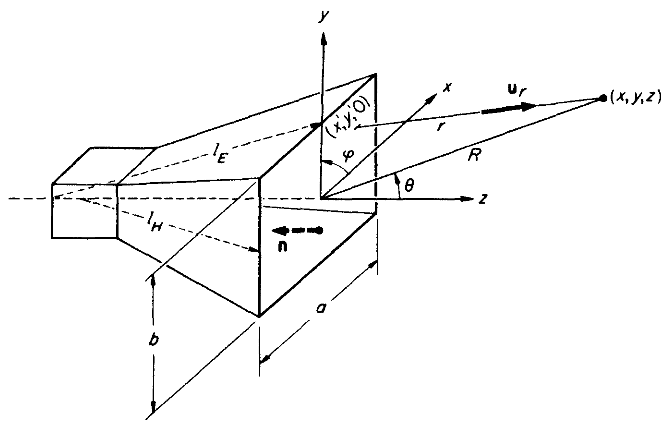


FIGURE 10-13. A pyramidal horn.

† S. A. Schelkunoff and H. T. Friis, "Antennas," chap. 16, John Wiley & Sons, Inc., New York, 1952.

Since the aperture field is nearly a uniform plane wave, the magnetic field intensity is

$$\mathbf{H}_1 = -\frac{E_0}{120\pi} \cos \frac{\pi x}{a} \exp \left[-j\beta \left(\frac{x^2}{2l_H} + \frac{y^2}{2l_E} \right) \right] \mathbf{a}_x \qquad (10\text{-}97)$$

where 120π represents the intrinsic impedance $\sqrt{\mu_0/\epsilon_0}$.

Setting

$$\delta = \beta \left[r + \frac{1}{2} \left(\frac{x'^2}{l_H} + \frac{y'^2}{l_E} \right) \right]$$

then reduces Eqs. (10-94) and (10-95) to

$$\mathbf{E}(x,y,z) = \frac{j}{2\lambda} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_0 \cos \frac{\pi x}{a} \times [-u_x u_y \mathbf{a}_x + (u_z + u_x^2 + u_z^2)\mathbf{a}_y - u_y(1 + u_z)\mathbf{a}_z] \frac{e^{-j\delta}}{r} \, dy \, dx \qquad (10\text{-}98)$$

$$\mathbf{H}(x,y,z) = \frac{j}{240\pi\lambda} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} E_0 \cos \frac{\pi x}{a} \times [-(u_z + u_y^2 + u_z^2)\mathbf{a}_x + u_x u_y \mathbf{a}_y + u_x(1 + u_z)\mathbf{a}_z] \frac{e^{-j\delta}}{r} \, dy \, dx \qquad (10\text{-}99)$$

These integrals are best evaluated on the digital computer. For numerical results, we consider the case of a standard horn, for which $a = 5.984$ in., $b = 4.908$ in., $l_H = 14.333$ in., $l_E = 13.633$ in. For this horn, the power patterns calculated at 16 GHz are shown plotted in Fig. 10-14, and are seen to agree well with the measured patterns throughout the dynamic range of the recording apparatus.

The complexity of the use of the vector equations (10-94) and (10-95) was largely hidden in the statement of the preceding example that calculations are best made on the digital computer. The computational difficulties can be greatly reduced, and the working equations considerably simplified, by introducing several additional approximations. These approximations reduce the analysis of aperture antennas to a scalar problem. The final result is given by Eq. (10-113), below. The algebraic details follow.†

Let us consider a TEM wave impinging upon a *plane* aperture S_1, along a direction $\mathbf{p}$, as indicated in Fig. 10-15. The source field is zero everywhere except over S_1. Then

$$\mathbf{H}_1 = \sqrt{\frac{\epsilon}{\mu}}\,(\mathbf{p} \times \mathbf{E}_1) \qquad \mathbf{n} \times \mathbf{H}_1 = -\mathbf{a}_z \times \left[\sqrt{\frac{\epsilon}{\mu}}\,(\mathbf{p} \times \mathbf{E}_1) \right] \qquad (10\text{-}100)$$

and with $\beta = 2\pi/\lambda$, Eq. (10-94) becomes

$$\mathbf{E}(x,y,z) = \frac{j}{2\lambda} \int_{S_1} \mathbf{u}_r \times \{(-\mathbf{a}_z \times \mathbf{E}_1) + \mathbf{u}_r \times [\mathbf{a}_z \times (\mathbf{p} \times \mathbf{E}_1)]\} \frac{e^{-j\beta r}}{r} \, da \qquad (10\text{-}101)$$

† Use of the final result, Eq. (10-113), does not require a full understanding of these algebraic details. They are essential only for a full realization of the implications of the approximations.

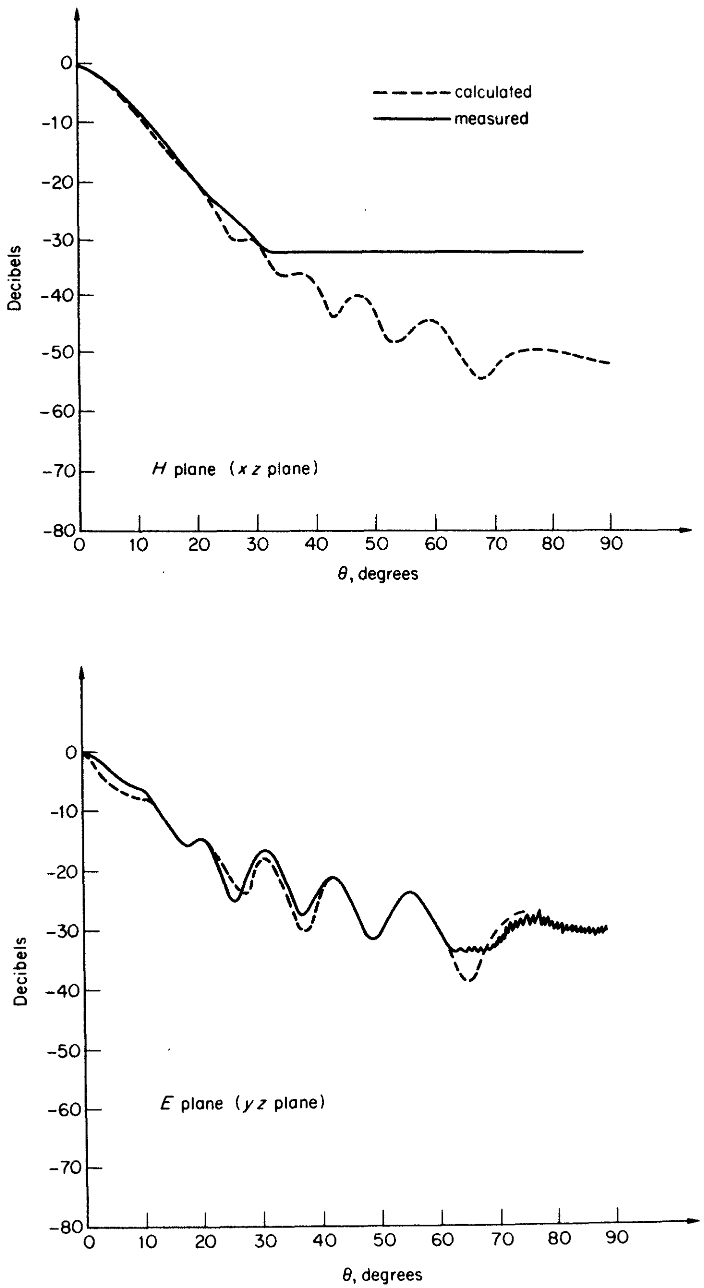


FIGURE 10-14. Power patterns of the pyramidal horn shown in Fig. 10-13.

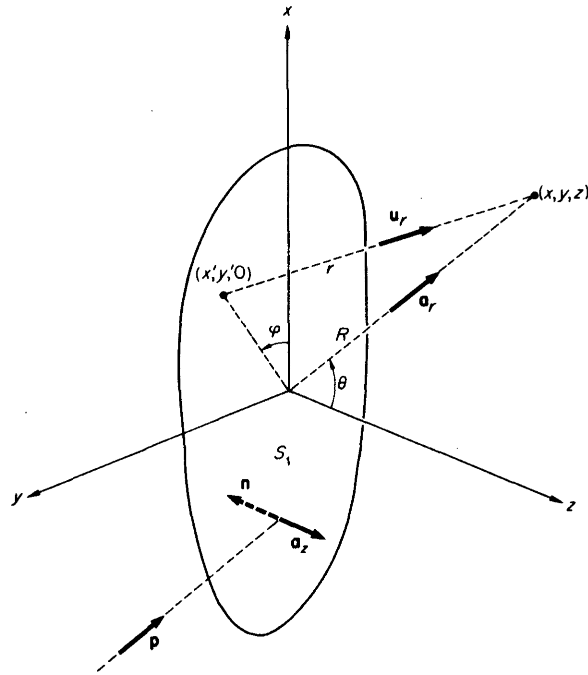


FIGURE 10-15. Geometry of a plane aperture. The wave is incident from $z < 0$, giving rise to a diffraction field in the region $z > 0$.

Over the plane aperture S_1 the direction of the source field vector $\mathbf{E}_1$ is arbitrary. However, no generality will be lost if, for the purposes of this analysis, we consider it to be uniformly polarized in the x direction, namely,

$$\mathbf{E}_1 = F(x',y',0)\mathbf{a}_x = A(x',y')e^{-j\beta L(x',y')}\mathbf{a}_x \qquad (10\text{-}102)$$

Here A and L denote amplitude and phase functions, respectively. Then

$$\mathbf{a}_z \times (\mathbf{p} \times \mathbf{E}_1) = \mathbf{a}_z \times (\mathbf{p} \times \mathbf{a}_x)F = [\mathbf{p}(\mathbf{a}_z \cdot \mathbf{a}_x) - \mathbf{a}_x(\mathbf{p} \cdot \mathbf{a}_z)]F$$

and since $\mathbf{a}_z \cdot \mathbf{a}_x \equiv 0$, the vector integrand $\mathbf{I}$ in Eq. (10-101) transforms to

$$\mathbf{I} = \mathbf{u}_r \times \{-(\mathbf{a}_z \times \mathbf{a}_x) - \mathbf{u}_r \times [(\mathbf{p} \cdot \mathbf{a}_z)\mathbf{a}_x]\}F = -\mathbf{u}_r \times \{[\mathbf{a}_z + (\mathbf{p} \cdot \mathbf{a}_z)\mathbf{u}_r] \times \mathbf{a}_x\}F$$

$$= -[\mathbf{a}_z + (\mathbf{p} \cdot \mathbf{a}_z)\mathbf{u}_r](\mathbf{u}_r \cdot \mathbf{a}_x)F + \mathbf{a}_x[(\mathbf{a}_z \cdot \mathbf{u}_r) + (\mathbf{p} \cdot \mathbf{a}_z)(\mathbf{u}_r \cdot \mathbf{u}_r)]F$$

For points near the z axis, that is, for θ small, $\mathbf{u}_r \cdot \mathbf{a}_x \approx 0$, and the first term on the right vanishes. Then

$$\mathbf{I} \approx \mathbf{a}_x(\mathbf{a}_z \cdot \mathbf{u}_r + \mathbf{a}_z \cdot \mathbf{p})F$$

The amended expression for the **E** field is

$$\mathbf{E}(x,y,z) = \mathbf{a}_x \frac{j}{2\lambda} \int_{S_1} F(\mathbf{a}_z \cdot \mathbf{u}_r + \mathbf{a}_z \cdot \mathbf{p}) \frac{e^{-j\beta r}}{r}\, da \qquad (10\text{-}103)$$

This equation expresses the so-called *diffraction field* (Sec. 8.9). It is customary in practice to divide this diffraction field into three general zones:

1. The near-field zone
2. The Fresnel zone
3. The Fraunhofer, or far-field, zone

The near-field zone is the immediate neighborhood of the aperture. No further simplifying approximation can be made, and Eq. (10-103) applies.

The Fresnel region is far enough from the aperture so that in Eq. (10-103) $\mathbf{a}_z \cdot \mathbf{u}_r \approx \mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$, and $r \approx R$ in every term except in the phase factor $e^{-j\beta r}$. With these approximations, Eq. (10-103) becomes

$$E_x = \frac{j}{2\lambda R} \int_{S_1} F(\mathbf{a}_z \cdot \mathbf{p} + \cos\theta) e^{-j\beta r}\, da \qquad (10\text{-}104)$$

The variation of the phase factor $e^{-j\beta r}$ can be determined from a consideration of the distance

$$r = [(x - x')^2 + (y - y')^2 + z^2]^{1/2}$$

in terms of spherical variables

$$x = R \sin\theta \cos\varphi$$
$$y = R \sin\theta \sin\varphi$$
$$z = R \cos\theta$$

We have

$$r = [R^2 - 2R(x' \sin\theta \cos\varphi + y' \sin\theta \sin\varphi) + x'^2 + y'^2]^{1/2} \qquad (10\text{-}105)$$

Adding and subtracting the term

$$T^2 = (x' \sin\theta \cos\varphi + y' \sin\theta \sin\varphi)^2$$

under the radical and, subsequently, factoring out the completed square, we obtain

$$r = (R - T)\left[1 + \frac{x'^2 + y'^2 - T^2}{(R - T)^2}\right]^{1/2} \qquad (10\text{-}106)$$

Now the earlier hypothesis $r \approx R$ carries with it the tacit implication that $R^2 \gg T^2$. Therefore the radical can be approximated by the first two terms

of a binomial expansion, giving

$$\begin{aligned} r &\approx (R - T)\left[1 + \frac{1}{2}\frac{x'^2 + y'^2 - T^2}{(R - T)^2}\right] \\ &= R - T + \frac{x'^2 + y'^2 - T^2}{2(R - T)} \\ &\approx R - T + \frac{x'^2 + y'^2 - T^2}{2R} \\ &= R + r_1 \end{aligned} \qquad (10\text{-}107)$$

where

$$r_1 = -(x' \sin\theta \cos\varphi + y' \sin\theta \sin\varphi) + \frac{x'^2 + y'^2 - (x' \sin\theta \cos\varphi + y' \sin\theta \sin\varphi)^2}{2R} \qquad (10\text{-}108)$$

The diffraction field, Eq. (10-104), for the Fresnel region then becomes

$$E_x = \frac{je^{-j\beta R}}{2\lambda R} \int_{S_1} Fe^{-j\beta r_1}(\mathbf{a}_z \cdot \mathbf{p} + \cos\theta)\, da \qquad (10\text{-}109)$$

In the Fraunhofer region, the second term on the right of Eq. (10-108) is neglected, and

$$r_1 \approx -(x' \sin\theta \cos\varphi + y' \sin\theta \sin\varphi) \qquad (10\text{-}110)$$

For apertures perpendicular to the direction of propagation, $\mathbf{p} = \mathbf{a}_z$, and

$$E_x = \frac{je^{-j\beta R}}{2\lambda R}(1 + \cos\theta) \int_{S_1} Fe^{j\beta \sin\theta (x' \cos\varphi + y' \sin\varphi)}\, da \qquad (10\text{-}111)$$

This equation is valid for small θ only. Therefore

$$1 + \cos\theta \approx 2 \qquad (10\text{-}112)$$

so that, finally,

$$E_x = \frac{je^{-j\beta R}}{\lambda R} \int_{S_1} Fe^{j\beta \sin\theta (x' \cos\varphi + y' \sin\varphi)}\, da \qquad (10\text{-}113)$$

When the illumination function F can be represented as a product of two functions,

$$F = Ae^{-j\beta L} = F_1(x')F_2(y') \qquad (10\text{-}114)$$

the scalar wave function E_x is the product of two Fourier integrals,

$$E_x = \frac{je^{-j\beta R}}{\lambda R} \int_{x'} F_1(x') e^{j\beta x' \sin\theta \cos\varphi}\, dx' \int_{y'} F_2(y') e^{j\beta y' \sin\theta \sin\varphi}\, dy' \qquad (10\text{-}115)$$

which usually provide the point of departure in the analysis of aperture antennas.

From the preceding derivation it is clear that Eq. (10-115) is subject to several restrictions, namely:

1. Harmonic time variations
2. Linear, homogeneous, isotropic, and source-free medium
3. Zero tangential field intensities over the complement of S_1
4. $r \gg \lambda/2\pi$
5. Plane aperture S_1
6. $\mathbf{u}_r \cdot \mathbf{a}_x \approx 0$
7. $\mathbf{a}_z \cdot \mathbf{u}_r \approx \mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$
8. $\dfrac{e^{-j\beta r}}{r} \approx \dfrac{e^{-j\beta R}}{R} e^{j\beta \sin\theta(x'\cos\varphi + y'\sin\varphi)}$
9. Incidence along the normal to the aperture ($\mathbf{a}_z \cdot \mathbf{p} = 1$)
10. $1 + \cos\theta \approx 2$
11. Separable illumination function

In closing, it is important to remark that the radial distance which marks the boundary between the Fresnel and Fraunhofer regions is taken in practice to be $2D^2/\lambda$, where D is the maximum linear dimension of the aperture. In terms of Eq. (10-108), this corresponds to a maximum phase deviation of $\pi/8$ deg. To prove this statement, we note that the second term in the numerator of the fraction may be neglected in comparison with the first, so that

$$\frac{2\pi}{\lambda}\frac{x'^2 + y'^2}{2R} \le \frac{2\pi}{\lambda}\frac{(D/2)^2}{2R} < \frac{\pi}{8}$$

or

$$\frac{2D^2}{\lambda} < R \tag{10-116}$$

Let us now apply Eq. (10-113) to a specific problem.

Example 10-6 Uniformly Illuminated Rectangular Aperture. Let a rectangular aperture in the xy plane (Fig. 10-16) be centered about the origin, and suppose that the E field is uniform and is polarized in the x direction; that is, in Eq. (10-113), let

$$F(x',y') = \begin{cases} E_0 & -\frac{a}{2} \le x' \le \frac{a}{2}, \quad -\frac{b}{2} \le y' \le \frac{b}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{10-117}$$

We wish to derive expressions for the scalar far field and to determine the 3-dB, or half-power, beamwidths.

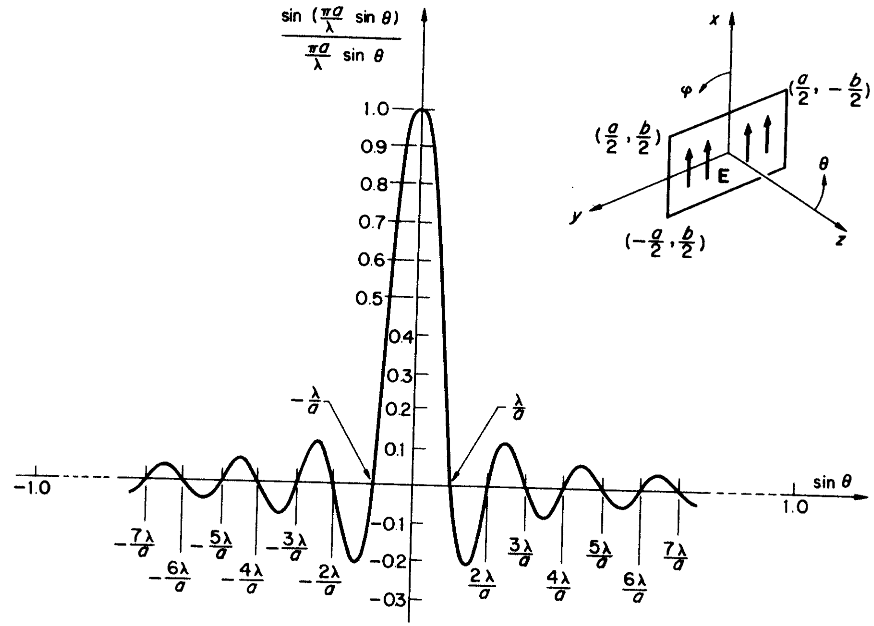


FIGURE 10-16. Radiation from a uniformly illuminated rectangular aperture.

We have

$$\begin{aligned} E_x &= \frac{je^{-j\beta R}}{\lambda R}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} E_0 e^{j\beta\sin\theta(x'\cos\varphi + y'\sin\varphi)}\,dx'\,dy' \\ &= \frac{je^{-j\beta R}}{\lambda R}\int_{-a/2}^{a/2} E_0 e^{jx'\beta\sin\theta\cos\varphi}\,dx' \int_{-b/2}^{b/2} e^{jy'\beta\sin\theta\sin\varphi}\,dy' \\ &= \frac{je^{-j\beta R}}{\lambda R}\left[aE_0\frac{\sin\left(\frac{a}{2}\beta\sin\theta\cos\varphi\right)}{\frac{a}{2}\beta\sin\theta\cos\varphi}\right]\left[b\frac{\sin\left(\frac{b}{2}\beta\sin\theta\sin\varphi\right)}{\frac{b}{2}\beta\sin\theta\sin\varphi}\right] \end{aligned} \tag{10-118}$$

In the $\varphi = 0°$ plane, which in this case is the **E** plane, this expression reduces to

$$E_x(\varphi = 0°) = \frac{je^{-j\beta R}}{\lambda R}(ab)E_0\frac{\sin[(\pi a/\lambda)\sin\theta]}{(\pi a/\lambda)\sin\theta}$$

while in the $\varphi = 90°$ plane, which in this case is the **H** plane, it becomes

$$E_x(\varphi = 90°) = \frac{je^{-j\beta R}}{\lambda R}(ab)E_0\frac{\sin[(\pi b/\lambda)\sin\theta]}{(\pi b/\lambda)\sin\theta}$$

In either plane, the radiation pattern exhibits a dependence of the type displayed in Fig. 10-16.

The 3-dB beamwidth in the $\varphi = 0°$ plane is obtained by setting

$$0.707 = \frac{\sin [(\pi a/\lambda) \sin \theta_0]}{(\pi a/\lambda) \sin \theta_0}$$

and solving for the angle θ_0. (For the $\varphi = 90°$ plane, a is simply replaced by b in this expression.) For small values of the argument $x = (\pi a/\lambda) \sin \theta_0$,

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$$

Therefore $$0.707 = \frac{x - x^3/6 + x^5/120}{x}$$

from which it follows that $x \approx 1.4$, or $\theta_0 \approx \sin^{-1} (1.4\, \lambda/\pi a)$. Thus, in the $\varphi = 0°$ plane,

$$\text{3-dB beamwidth} = 2 \sin^{-1} \left(0.445 \frac{\lambda}{a}\right) \approx 51 \frac{\lambda}{a} \quad \text{deg} \qquad (10\text{-}119)$$

while in the $\varphi = 90°$ plane,

$$\text{3-dB beamwidth} \approx 51 \frac{\lambda}{b} \quad \text{deg} \qquad (10\text{-}120)$$

Clearly, the larger the aperture, the smaller the beamwidth.

The usual type of aperture-antenna problem consists in finding the aperture field which will optimize the far-field pattern to specific requirements. A few general characteristics are:

1. Symmetrical apertures with A and L [Eq. (10-102)] symmetrical about the center of the aperture produce symmetrical far-field patterns.
2. A = constant, $L = 0$, and a symmetrical aperture produces zero nulls in the far-field pattern, and the main beam is the narrowest that can be obtained for the given aperture size. However, the secondary maxima (*side lobes*) are high (see previous example).
3. Larger apertures give narrower beams.
4. $L = 0$ and A symmetrical but monotonically decreasing from the center of the aperture will, in general, decrease the secondary maxima.
5. In general, $L \neq 0$ will produce a moderate increase in main beamwidth compared with the uniform-phase case, and the pattern will have nonzero nulls.

Since narrow beamwidth and low side-lobe level are very common aperture-antenna requirements, a great deal of work has been done on defining the "best" amplitude distribution to optimize the conflicting requirements.

To sum up, this section has given a brief introduction to aperture antennas. Such antennas may be analyzed with the aid of Eqs. (10-94) and (10-95) or, when more assumptions are allowed, with the aid of Eq. (10-113).

10.14 Aperture Synthesis.† A field synthesis problem of great importance is that of designing radiating systems to produce desired radiation characteristics. In most practical situations, the designer of a linear or planar array is required to solve the following problem. Given the specified radiation pattern, how must the array be arranged, and how must the individual elements be excited in order to produce the best (in some prespecified sense) approximation to the given pattern? One method of attacking this problem is known as the *Fourier synthesis method*, based on Schelkunoff's‡ early mathematical treatment of arrays of isotropic sources.

Recall that the array factor for uniform linear arrays is given by Eq. (10-88). If the amplitude and phase progression are not constant, the corresponding expression for the radiation pattern factor will be

$$F(\psi) = B_0 + B_1 e^{j\psi} + B_2 e^{j2\psi} + \cdots + B_{n-1} e^{j(n-1)\psi} \qquad (10\text{-}121)$$

where $B_k = |B_k|\, e^{j\alpha k}$ is the complex amplitude and phase of the kth element, and

$$\psi = \beta d \cos \varphi \qquad (10\text{-}122)$$

as in Sec. 10.12.

In Eq. (10-121) the left-end element is the reference element. If the center element is taken as the reference element, as in Fig. 10-17, the radiation pattern for an array with $2N + 1$ elements will be

$$F(\psi) = A_{-N} e^{-jN\psi} + \cdots + A_{-1} e^{-j\psi} + A_0 + A_1 e^{j\psi} + \cdots + A_N e^{jN\psi} \qquad (10\text{-}123)$$

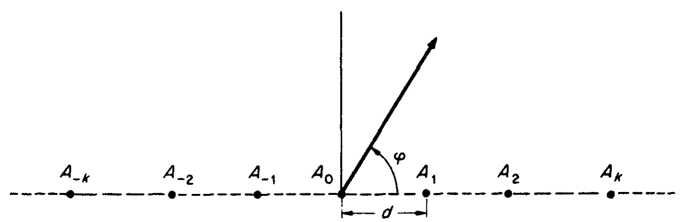


FIGURE 10-17. Array of $2N + 1$ equally spaced isotropic point sources.

† This section was written in collaboration with G. W. Breland. This section may be omitted with no loss in continuity.

‡ S. A. Schelkunoff, A Mathematical Theory of Linear Arrays, *Bell System Tech. J.*, vol. 22, pp. 80–107, 1943.