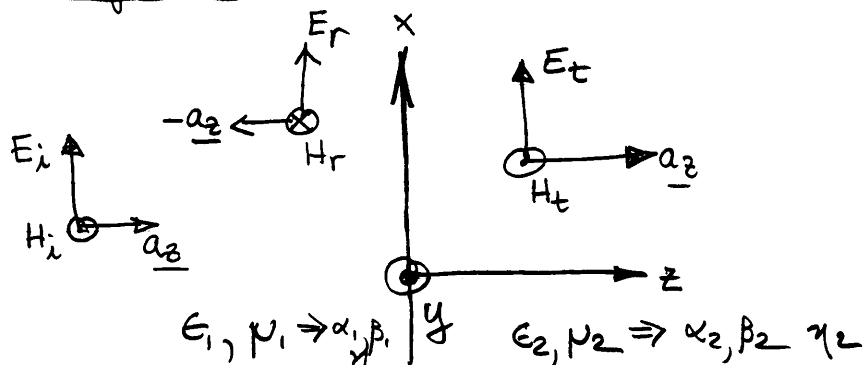
reflection of waves



we talk about polarization $\rightarrow$ direction of E field.
This is E field parallel to interface.

if σ not given assume $\sigma = 0$

$\Rightarrow$ lossless media $\hat{\eta} = \sqrt{\frac{\mu}{\epsilon}}$ in each media

incident	transmitted	reflected.
$\hat{E}_i = E_i e^{-j\beta_1 z}$	$\hat{E}_t = E_t e^{-j\beta_2 z}$	$\hat{E}_r = E_r e^{+j\beta_1 z}$
$\hat{H}_i = \frac{E_i}{\eta_1} e^{-j\beta_1 z}$	$\hat{H}_t = \frac{E_t}{\eta_2} e^{-j\beta_2 z}$	$\hat{H}_r = -\frac{E_r}{\eta_1} e^{+j\beta_1 z}$

what do we know about fields at interface

Tangential E is continuous

tangential H $\qquad H_{1t} - H_{2t} = J_s$

$\therefore$ tan E is continuous.

total field $z<0$

$$E_i e^{-j\beta_1 z} + E_r e^{+j\beta_1 z}$$

$$\frac{E_i}{\eta_1} e^{-j\beta_1 z} - \frac{E_r}{\eta_1} e^{+j\beta_1 z}$$

total field $z>0$

$$E_t e^{-j\beta_2 z}$$

$$\frac{E_t}{\eta_2} e^{-j\beta_2 z}$$

all continuous at $z=0$

$$E_i + E_r = E_t$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

E_i is usually known $\Rightarrow$ find E_r and E_t

$$E_r = \underbrace{\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}_{\hat{\Gamma}} E_i$$

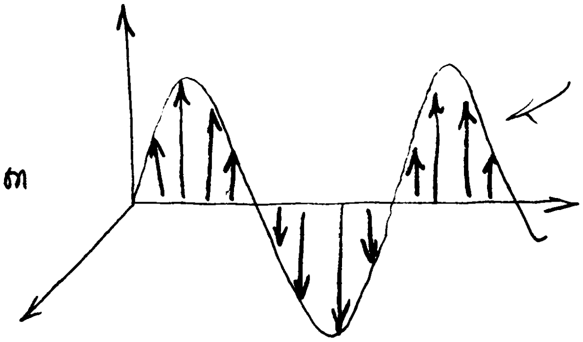
$$E_t = \underbrace{\frac{2\eta_2}{\eta_2 + \eta_1}}_{\hat{T}} E_i$$

$$1 + \hat{\Gamma} = \hat{T}$$

these are general results with complex impedances

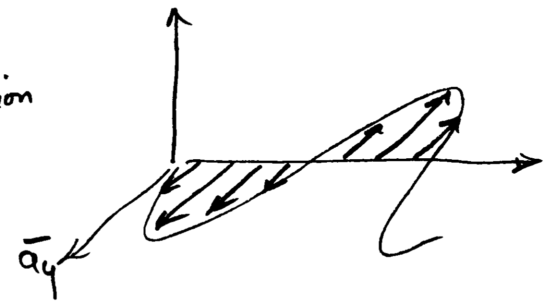
Polarization

linear polarization



$$\underline{\mathcal{E}} = \hat{E}_1^+ e^{-j\beta z}\, \underline{a_x}$$
$$\underline{H} = +\frac{\hat{E}_1^+}{\eta} e^{-j\beta z}\, \underline{a_y}$$

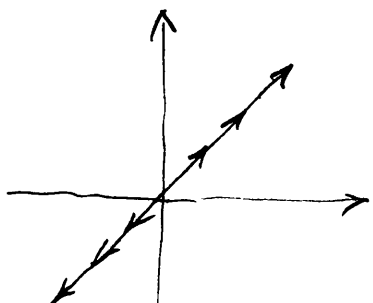
linear polarization



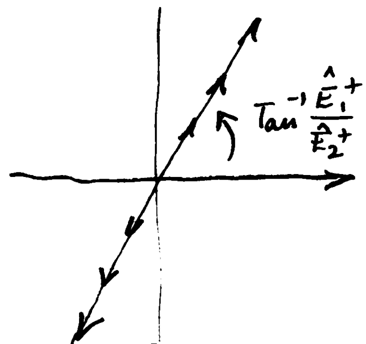
$$\underline{\mathcal{E}} = \hat{E}_2^+ e^{-j\beta z - j\theta}\, \underline{a_y}$$
$$H = -\frac{\hat{E}_2^+}{\eta} e^{-j\beta z - j\theta}\, \underline{a_x}$$

if both of these are added together

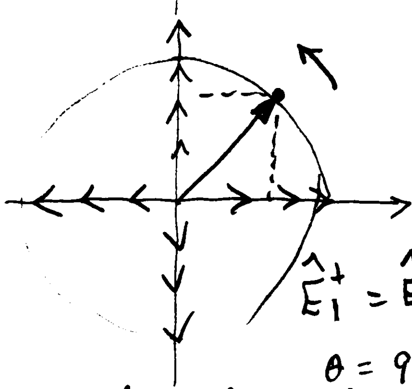
$$\underline{\mathcal{E}}_{tot} = \hat{E}_1^+ e^{-j\beta z}\, \underline{a_x} + \hat{E}_2^+ e^{-j\beta z - j\theta}\, \underline{a_y}$$



$\hat{E}_1^+ = \hat{E}_2^+$
$\theta = 0$

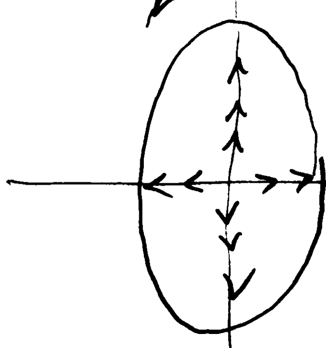


$\hat{E}_1^+ \neq \hat{E}_2^+$
$\theta = 0$



$\hat{E}_1^+ = \hat{E}_2^+$
$\theta = 90^\circ$

Circular polarization



$\hat{E}_1^+ = \hat{E}_2^+$
$\theta \neq 0^\circ$
$\theta \neq 90^\circ$

Elliptical polarization

<u>reflection from a perfect conductor</u>

what is $\hat{\eta}$ for a perfect conductor?

(here is error)

$$\hat{\eta} = \frac{j\omega\mu}{\gamma} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}}$$

as $\sigma \to \infty$ $\quad \hat{\eta} \to 0$

This confirms that there are no fields in perfect conductors for

η_2 in conductor
η_1 in dielectric

$$\hat{\Gamma} = \frac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \longrightarrow \frac{-\hat{\eta}_1}{\hat{\eta}_1} = -1$$

$$\hat{T} = \frac{2\hat{\eta}_2}{\hat{\eta}_2 + \hat{\eta}_1} \longrightarrow 0$$

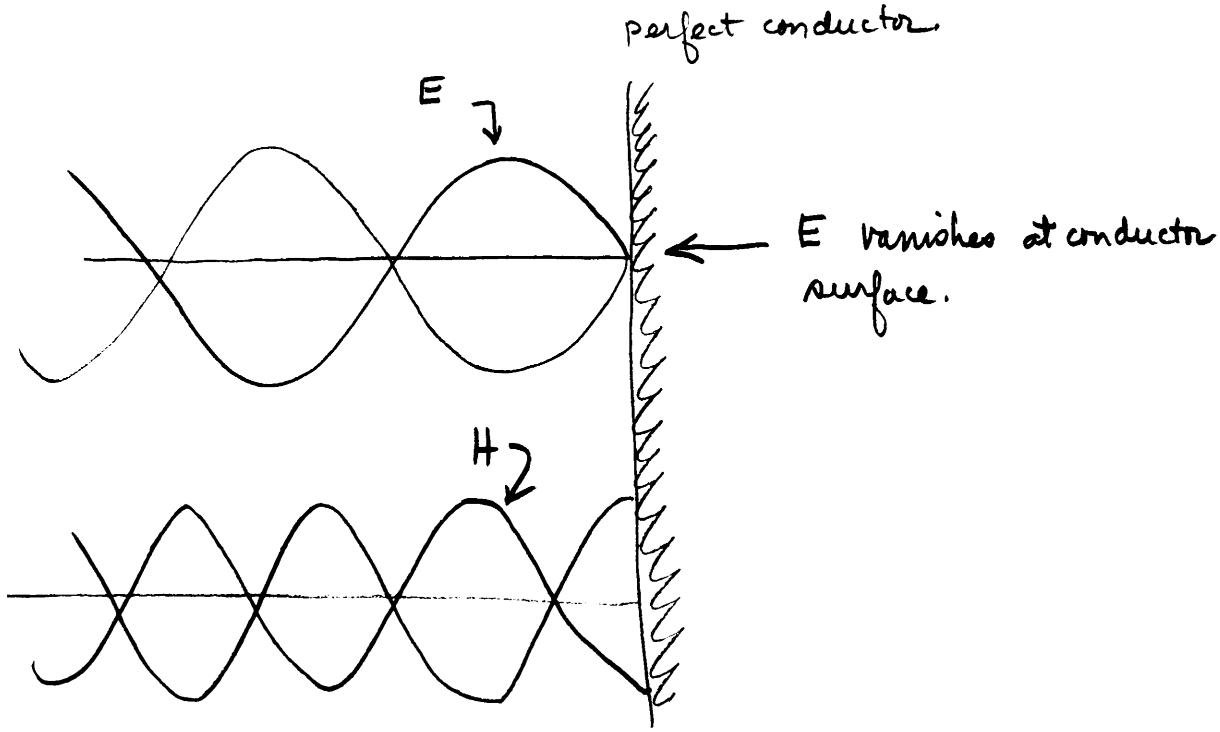
All the wave is reflected producing a standing wave, i.e. a wave which does not move in space.

$$\hat{E}(z,t) = \hat{E}_i \underbrace{\left(e^{-j\beta_1 z} + \frac{E_r}{E_i} e^{j\beta_1 z}\right)}_{\text{total field}} \to \hat{E}_i \left(e^{-j\beta_1 z} - e^{-j\beta_1 z}\right)$$

$$H(z,t) = \frac{\hat{E}_i}{\hat{\eta}_1}\left(e^{-j\beta_1 z} - \frac{\hat{E}_r}{\hat{E}_i} e^{j\beta_1 z}\right) \longrightarrow \hat{H}_i \left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right)$$

$$\hat{E}(z,t) \to -2j\,\hat{E}_i \sin \beta_1 z$$

$$\hat{H}(z,t) \to 2\,\frac{\hat{E}_i}{\eta} \cos \beta_1 z$$


perfect conductor
E
E vanishes at conductor surface.
H

propagation in arbitrary direction

up to now we've assumed waves propagate along coordinate axes so description was simple.

complex Hemholtz wave equations:

$$\nabla^2 \underline{\hat{E}} = \gamma^2 \underline{\hat{E}} \ , \ \nabla^2 \underline{\hat{H}} = \gamma^2 \underline{\hat{H}}$$

with solutions of form for a plane wave propagating in +z direction

$$\underline{\hat{E}} = \underline{\hat{E}}^+ e^{-\gamma z} + \underline{\hat{E}}^- e^{+\gamma z}$$

$$\underline{\hat{H}} = \underline{\hat{H}}^+ e^{-\gamma z} + \underline{\hat{H}}^- e^{+\gamma z}$$

In most general case (restrict attention to E field)

$$\underline{E} = \underline{\hat{E}}^+ e^{-\gamma \underline{n} \cdot \underline{r}} + \underline{\hat{E}}^- e^{+\gamma \underline{n} \cdot \underline{r}}$$

where $r = x\,\underline{a_x} + y\,\underline{a_y} + z\,\underline{a_z}$ general position vector

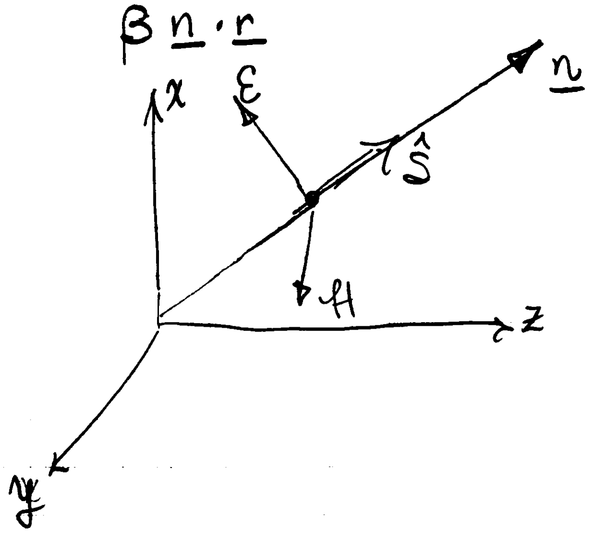
$n = n_x \underline{a_x} + n_y \underline{a_y} + n_z \underline{a_z}$ unit vector in direction of propagation

recall $\gamma = \alpha + j\beta$

For moment, pick $\alpha = 0$

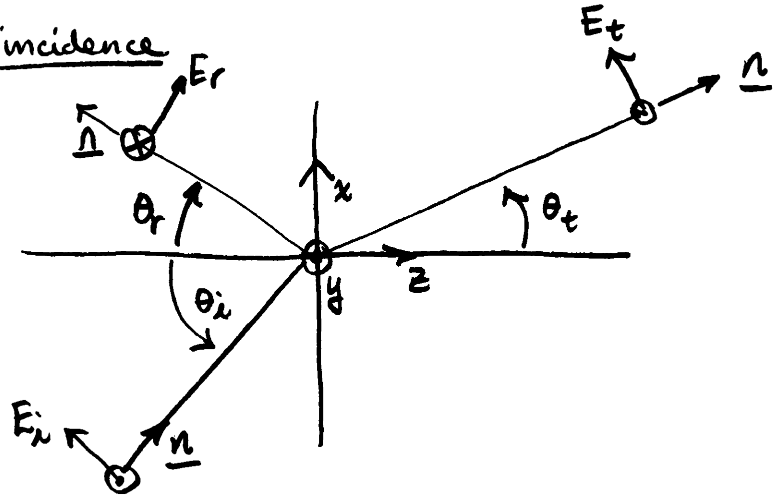
surface of constant phase is described by.

$\beta\, \underline{n} \cdot \underline{r}$



oblique incidence

H parallel to boundary



what is incident wave?

$$\hat{\underline{E}}_i = E_i\,(\underline{a_x}\cos\theta_i - \underline{a_z}\sin\theta_i)\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\beta\,\underline{n}\cdot\underline{r} = \beta_1\,(\underline{a_x}\sin\theta_i + a_z\cos\theta_i)\cdot(x\,\underline{a_x} + z\,\underline{a_z})$$

$$= \beta_1 x\sin\theta_i + \beta_1 z\cos\theta_i$$

$$\hat{\underline{H}}_i = a_y\,\underbrace{\frac{E_i}{\eta_1}}\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

note that $\hat{E_i}$ had to be decomposed, H does not since it lies along a principal axis

what is reflected wave?

$$\hat{\underline{E}}_r = \hat{E}_r\,(\underline{a_x}\cos\theta_r + \underline{a_z}\sin\theta_r)\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\hat{\underline{H}}_r = -\frac{\hat{E}_r}{\eta_1}\,a_y\,e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

what is transmitted wave? looks essentially like incident wave

$$\hat{\underline{E}}_t = \hat{E}_t(\underline{a_x}\cos\theta_t - \underline{a_z}\sin\theta_t)\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\hat{H}_t = \frac{\hat{E}_t}{\hat{\eta}_2}\, \underline{a_y}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

what happens at $z=0$.

E_i, H_i $\quad E_r, H_r \quad E_t, H_t$

(E) $\quad E_i(\underline{a_x}\cos\theta_i - \underline{a_z}\sin\theta_i)e^{-j\beta_1 x\sin\theta_i} \quad E_r(\underline{a_x}\cos\theta_r + \underline{a_z}\sin\theta_r)e^{-j\beta_1 x\sin\theta_r}$

$E_t(\underline{a_x}\cos\theta_t - \underline{a_z}\sin\theta_t)\, e^{-j\beta x_2\sin\theta_t}$

(H) $\quad \frac{E_i}{\eta_1}\underline{a_y}\, e^{-j\beta x\sin\theta_i} \quad -\frac{E_r}{\eta}\underline{a_y}\, e^{-j\beta_1 x\sin\theta_r} \quad \frac{E_t}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$

what do we equate?

at $z=0$ tangential components must be continuous

E_{tan}: $\quad E_i\cos\theta_i\, e^{-j\beta_1 x\sin\theta_i} + E_r\cos\theta_r\, e^{-j\beta_1 x\sin\theta_r} = E_t\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$

H_{tan}: $\quad \frac{E_i}{\eta_1} e^{-j\beta_1 x\sin\theta_i} - \frac{E_r}{\eta_1} e^{-j\beta_1 x\sin\theta_r} = \frac{E_t}{\eta_2} e^{-j\beta_2 x\sin\theta_t}$

Since this must be true for all x.

$$\beta_1 x\sin\theta_i = \beta x\sin\theta_r = \beta_2 x\sin\theta_t$$

or $\quad \sin\theta_i = \sin\theta_r \Rightarrow \theta_i = \theta_r$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_2}{\beta_1}$$

Snell's Laws

If exponents are equal:

$$E_i \cos\theta_i + E_r \cos \overset{\theta_i}{\cancel{\theta_r}} = E_t \cos\theta_t$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

Solutions are

Fresnel Equations

$$\Gamma_{\parallel} = \frac{E_r}{E_i} = \frac{\hat{\eta}_2 \cos\theta_t - \hat{\eta}_1 \cos\theta_i}{\hat{\eta}_2 \cos\theta_t + \hat{\eta}_1 \cos\theta_i}$$

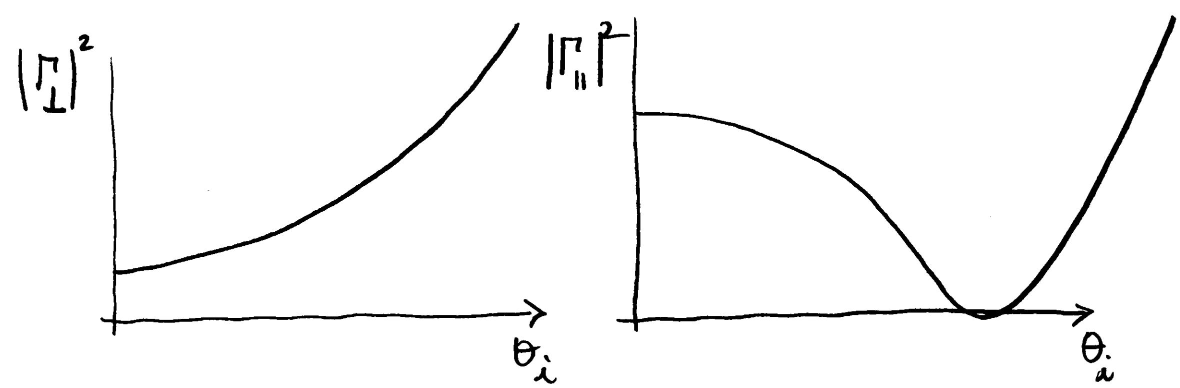
$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2\hat{\eta}_2 \cos\theta_i}{\hat{\eta}_2 \cos\theta_t + \hat{\eta}_1 \cos\theta_i}$$

---

fr $E_{\parallel}$ to interface (i.e. H perpendicular)

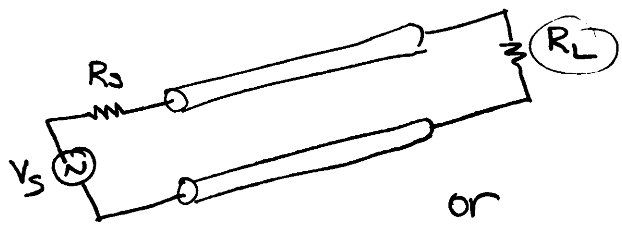
$$\Gamma_{\perp} = \frac{\hat{\eta}_2 \cos\theta_i - \hat{\eta}_1 \cos\theta_t}{\hat{\eta}_2 \cos\theta_i + \hat{\eta}_1 \cos\theta_t}$$

$$T_{\perp} = \frac{2\hat{\eta}_2 \cos\theta_i}{\hat{\eta}_2 \cos\theta_i + \hat{\eta}_1 \cos\theta_t}$$

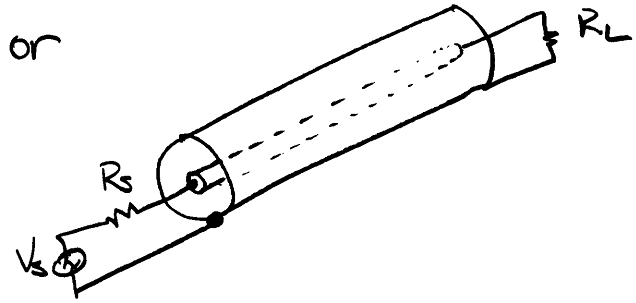


transmission lines

guide propagation of energy from one point to another
typically two conductors (infinitely long at first)



or



These are uniform, i.e. cross-section is uniform
so static E & H field problems we have already solved!

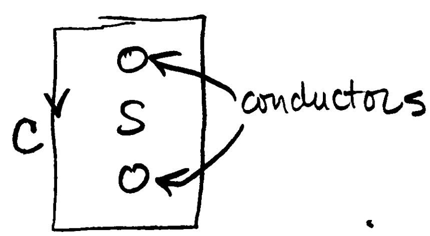
Maxwell's Equations:

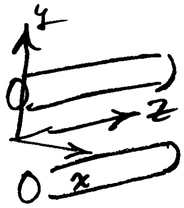
$$\oint_C \underline{\mathcal{E}} \cdot \underline{dl} = -\mu \frac{\partial}{\partial t} \int \underline{H} \cdot \underline{dS}$$

$$\oint_C \underline{H} \cdot \underline{dl} = \int_S \underline{\mathcal{J}} \cdot \underline{dS} + \frac{\partial}{\partial t} \int \underline{D} \cdot \underline{dS}$$

for a transmission line pick contour C in transverse direction

i.e.





call these x, y coordinates

since $\underline{dS} = \underline{a_z}\, dx\, dy$

Then:

$$\oint (\mathcal{E}_x dx + \mathcal{E}_y dy) = -\mu \frac{\partial}{\partial t} \int H_z\, dx\, dy$$

$$\oint (H_x dx + H_y dy) = \int \mathcal{J}_z\, dx\, dy + \epsilon \frac{\partial}{\partial t} \int \mathcal{E}_z\, dx\, dy$$

but $H_z = \mathcal{E}_z = 0$ transverse components only for plane waves.

this means

$$\oint \underline{\mathcal{E}} \cdot \underline{dl} = 0$$

$$\oint \underline{H} \cdot \underline{dl} = \int \underline{J}_z \cdot \underline{ds}$$

<u>These are the static solutions</u> !

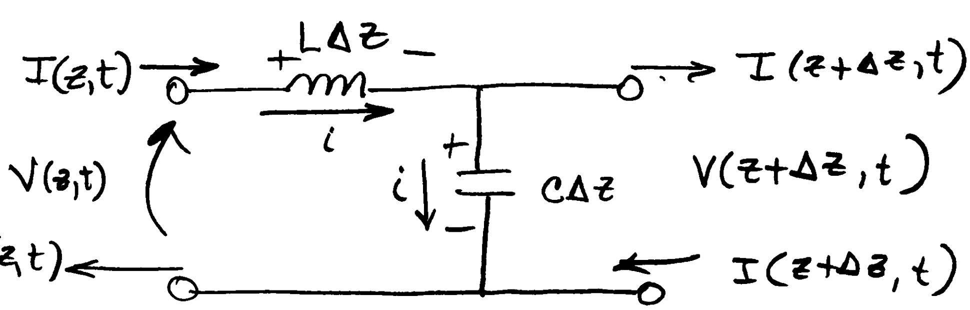
This means that

① electric field is conservative even though field is time varying

② per unit length inductance is constant

③ per unit length capacitance is constant

What we will do now is see if these lumped constant per unit length parameters can lead to plane waves.



I can always stick these together to make a long line.

circuits:

voltage drop across inductor

$$V(z,t) - V(z+\Delta z,t) = L\Delta z \frac{\partial I(z,t)}{\partial t}$$

$$\therefore \quad \frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -L \frac{\partial I(z,t)}{\partial t}$$

voltage across capacitor $\quad i = C\dfrac{dv}{dt}$ or $v = \frac{1}{C}\int i\,dt$.

$$I(z,t) - I(z+\Delta z,t) = C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t}$$

from ①

$$\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = -C\frac{\partial}{\partial t}\left\{ V(z,t) - L\Delta z \frac{\partial I(z,t)}{\partial t} \right\}$$

$$\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = -C\frac{\partial V(z,t)}{\partial t} + LC\Delta z \underbrace{\frac{\partial^2 I(z,t)}{\partial t}}_{\to 0}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}$$

going the other way we get,

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}$$

differential each again to get

$$\frac{\partial^2 I}{\partial z^2} = -C\frac{\partial}{\partial t}\frac{\partial}{\partial z}V(z,t)$$

$$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial}{\partial t}\frac{\partial}{\partial z}I(z,t)$$

interchange order of $\dfrac{\partial}{\partial z}\dfrac{\partial}{\partial t}$

what do we get

$$\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}$$

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}$$

define $v = \frac{1}{\sqrt{LC}}$

Look just like Hemholtz equations

Solutions:

$$V(z,t) = V^+(z-ut) + V^-(z+ut)$$

$$I(z,t) = I^+(z-ut) + I^-(z+ut)$$

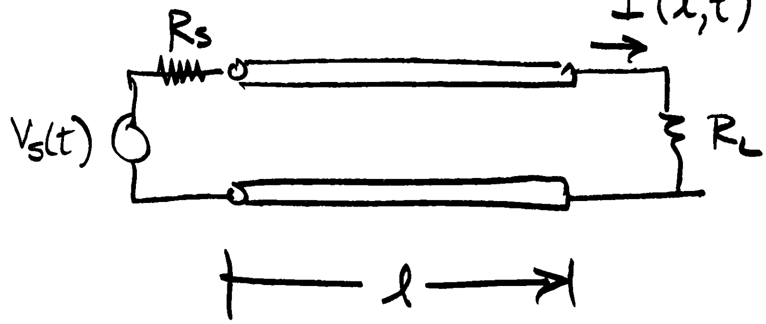
just as for traveling (plane) waves V and I are related by impedance.

$$I^+(z-ut) = \frac{V^+(z-ut)}{R_c}$$

$$I^-(z+ut) = -\frac{V^-(z-ut)}{R_c}$$

$$R_c = \sqrt{\frac{L}{C}}$$

problem



what is $V(l,t)$?

$$V(l,t) = I(l,t)\,R_L \quad \Big\} \text{ at the load}$$

use time instead of z as variable of interest

$$\left.\begin{aligned} V^+(t-\tfrac{l}{u}) &= R_c\, I^+(t-\tfrac{l}{u}) \\ V^-(t+\tfrac{l}{u}) &= -R_c\, I^-(t+\tfrac{l}{u}) \end{aligned}\right\} \text{ on the line}$$

transmission line ....

we must sum these up and see what meets B.C.'s.

$$V(l,t) = V^+(t-\tfrac{l}{u}) + V^-(t+\tfrac{l}{u})$$

$$= V^+(t-\tfrac{l}{u})\left[1+\Gamma_L\right]$$

$$\text{where} \quad \Gamma_L = \frac{V^-(t+\frac{l}{u})}{V^+(t-\frac{l}{u})} \quad \begin{matrix}\text{reflected}\\ \text{incident}\end{matrix}$$

$$I(l,t) = I^+(t-\tfrac{l}{u}) + I^-(t+\tfrac{l}{u})$$

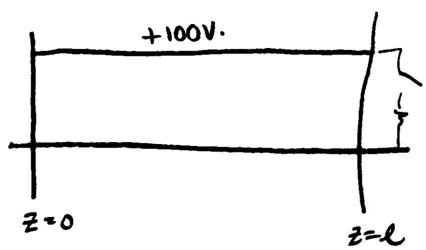
$$= \frac{V^+(t-\frac{l}{u})}{R_c} - \frac{V^-(t+\frac{l}{u})}{R_c}$$

$$= \frac{1}{R_c} V^+(t-\tfrac{l}{u})\left[1-\Gamma_L\right]$$

12B A transmission line is charged to $V_0 = +100V$ and left with both ends open. The line has a distributed capacitance of 88.6 pf/m and a distributed inductance of 2.22×10^{-7} Hy/m. The line is 3 meters long.

(a) if at time $t=0$ a 33Ω resistor is connected across one end of the line, how long is the pulse that is applied to the load and what is $V_L(t)$?

(b) If the resistor is 25Ω, sketch $V_L(t)$



$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{2.22 \times 10^{-7}}{88.6 \times 10^{-12}}}$$

$$= \sqrt{25050} \cong 50\,\Omega$$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2.22 \times 10^{-7})(88.6 \times 10^{-12})}}$$

$$= 2.25 \times 10^{8} \text{ m/sec.}$$

$$V(z,t) = V^{+}(z+ut) + V^{-}(z+ut)$$

$$I(z,t) = \frac{V^{+}(z-ut)}{Z_0} - \frac{V^{-}(z+ut)}{Z_0}$$

$$T = \frac{\ell}{u} = \frac{3}{2.25 \times 10^{8}}$$

$$= 1.33 \times 10^{-8} \text{ sec}$$

for a steady line with no loading

$$V(0,t) = V^{+} + V^{-} = 100$$

$$I(0,t) = \frac{V^{+}}{50} - \frac{V^{-}}{50} = 0$$

$$V^{+} + V^{-} = 100$$

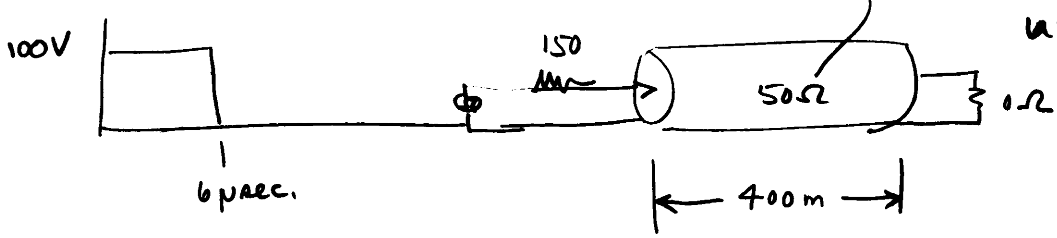
$$V^{+} - V^{-} = 0$$

$$\therefore \; V^{+} = V^{-} = 50$$

Example: short pulse on a line tough

$C = 100\ pf/m$

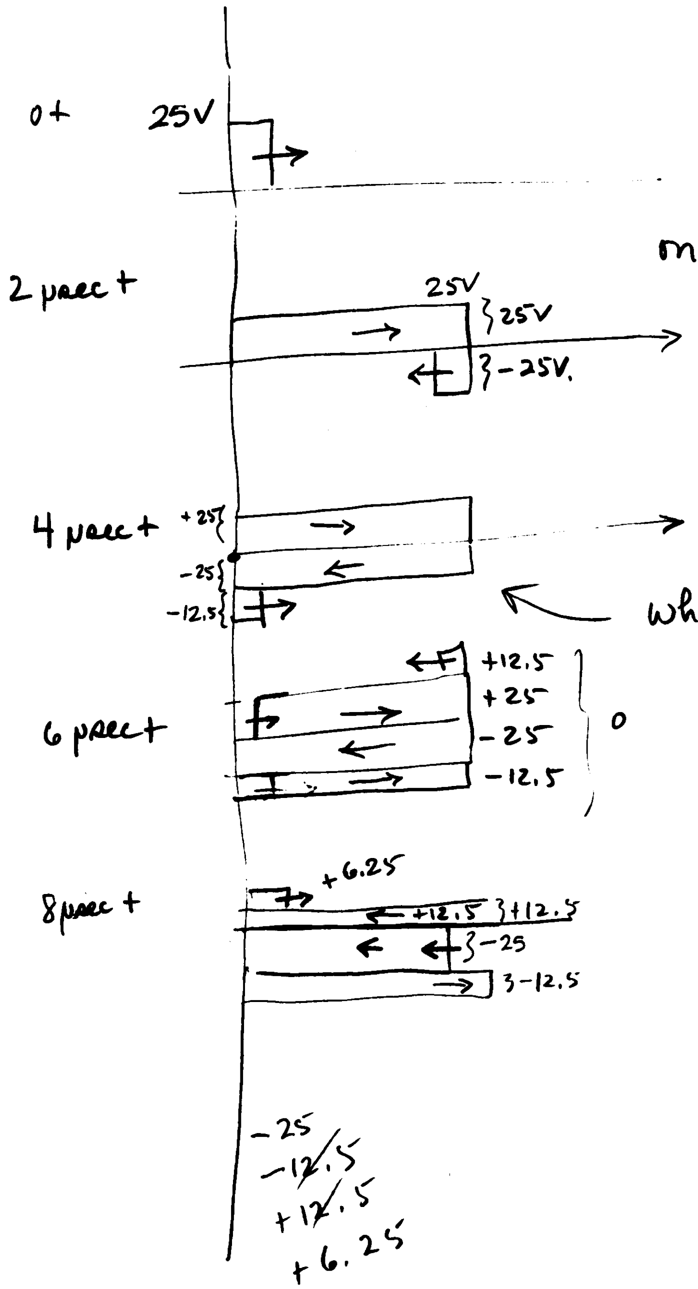
$L = 0.25\ \mu H/m$



$$u = \frac{1}{\sqrt{LC}} = 200 \times 10^{6}\ m/sec$$

$$Z = \sqrt{\frac{L}{C}} = 50\Omega$$

$$T = \frac{400m}{200 \times 10^{+6}\ m/sec} = 2\ \mu sec$$



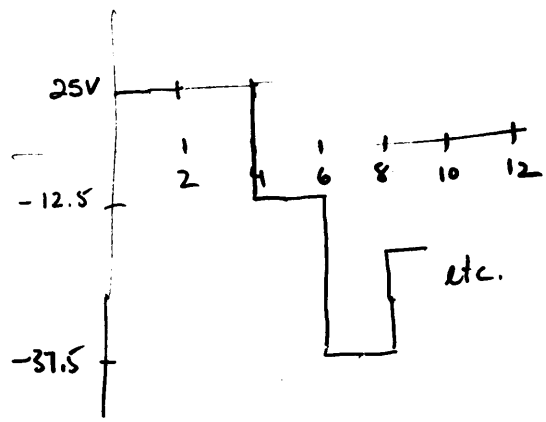
onto line voltage divider

$$\frac{50}{50+150}(100) = 25V$$

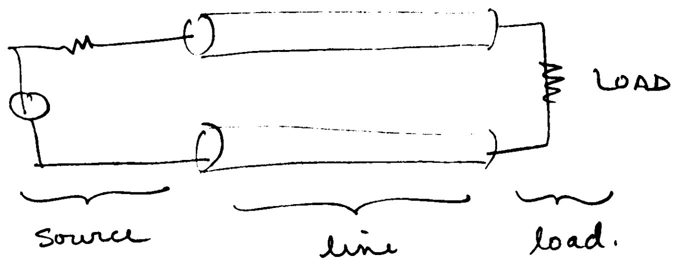
$\Gamma = -1$

what is Γ at source

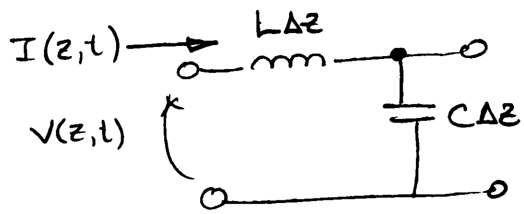
$$\Gamma_{source} = \frac{150-50}{150+50} = \frac{100}{200} = +\frac{1}{2}$$



summary of transmission line results



has inductance & capacitance per unit length,



transmission line equations:

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} \qquad u = \frac{1}{\sqrt{LC}}$$

$$\frac{\partial^2 I}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 I}{\partial t^2}$$

from before solutions look like

$$V(z,t) = V^+(z-ut) + V^-(z+ut)$$

$$\underbrace{I(z,t)}_{\text{total wave.}} = \underbrace{I^+(z-ut)}_{\text{wave in } +z \text{ direction}} + \underbrace{I^-(z+ut)}_{\text{wave in } -z \text{ direction}}$$

reflection coefficient

$$V(z,t) = V^+\left(1 + \frac{V^-}{V^+}\right) = V^+(1+\Gamma_V)$$

$$I(z,t) = I^+\left(1 + \frac{I^-}{I^+}\right) = I^+(1+\Gamma_I)$$

relationship between voltage & current is resistance, or impedance

opposite directions

$$I(z,t) = \frac{V^+}{Z_0}(z-ut) - \frac{V^-(z+ut)}{Z_0} = \frac{V^+}{Z_0}\left(1 - \frac{V^-}{V^+}\right)$$

$$= \frac{V^+}{Z_0}(1-\Gamma_V)$$

at load $\rightarrow$ $V(\ell,t) = Z_L\, I(\ell,t)$

$$V(\ell,t) = V^+ \left(1 + \Gamma_L^{(V)}\right) \qquad \frac{V^+}{Z_0}\left(1 - \Gamma_L^{(V)}\right)$$

$$\cancel{V^+}\left(1 + \Gamma_L^{(V)}\right) = \frac{Z_L}{Z_0}\cancel{V^+}\left(1 - \Gamma_L^{(V)}\right)$$

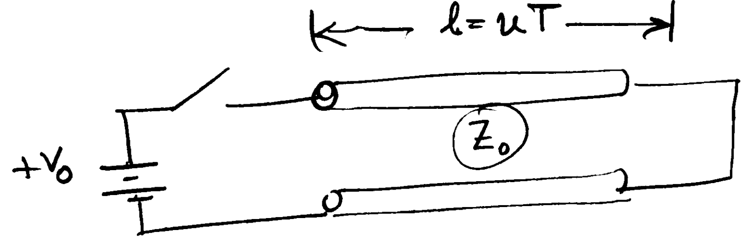
Solving for $\Gamma_L^{(V)}$

$$\Gamma_L(V) = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$\Gamma_L(I) = -\Gamma_L(V)$$

just like $\Gamma_L = \dfrac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1}$

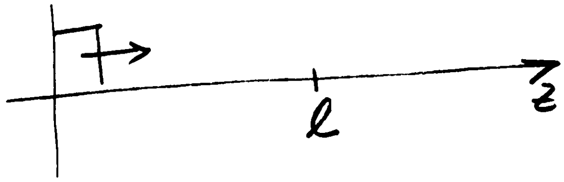
Example: long pulse on shorted line

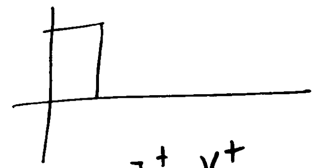


how about I

0^+

pulse moves onto line

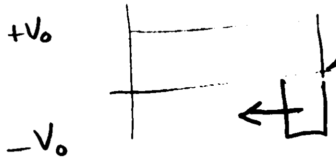




only I^+, V^+

$I^+ = \frac{V^+}{Z_0}$

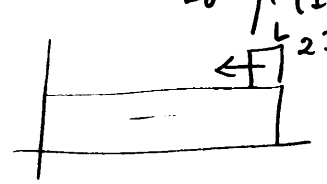
T^+



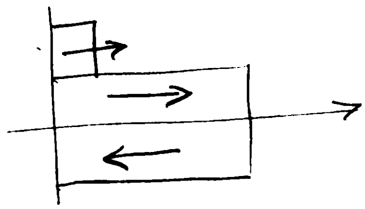
reflection forces B.C. of $v=0$ here.

what is $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \to -1$

$\Gamma_L(I) = +1$

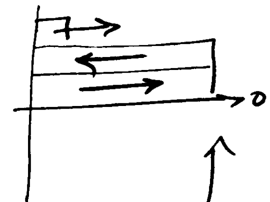


$2T^+$



what is Z of battery? zero

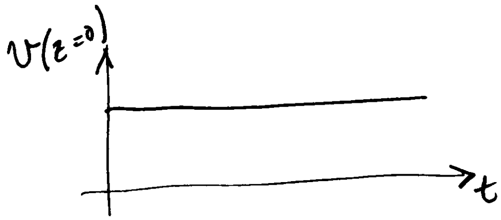
$\Gamma = -1$ at battery

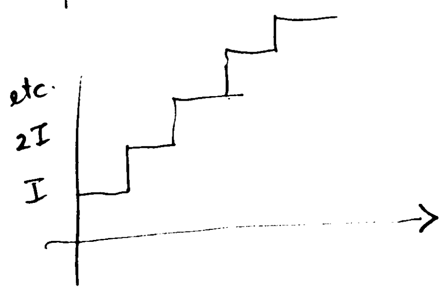


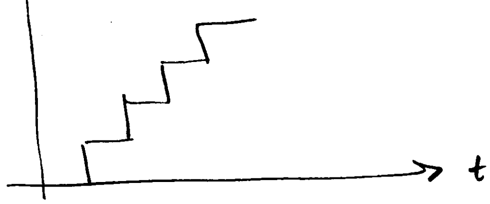
this just goes back and forth

in time

delayed by T

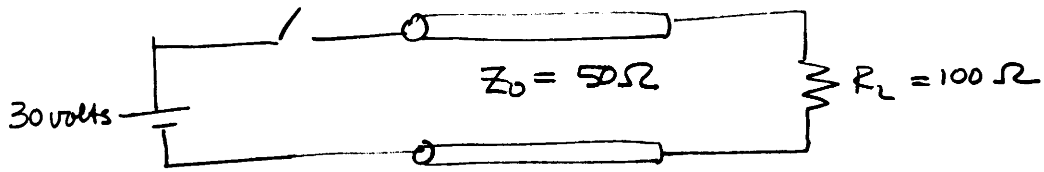






etc.

long pulse w/numbers.



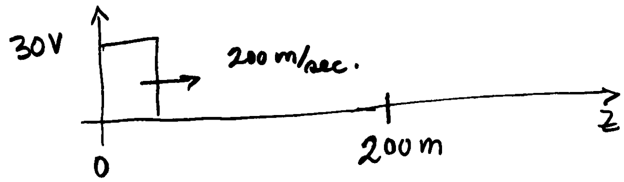
$u = 200\,m/sec.$ $\qquad T = \dfrac{400\,m}{200\,m/sec} = 2$ sec long

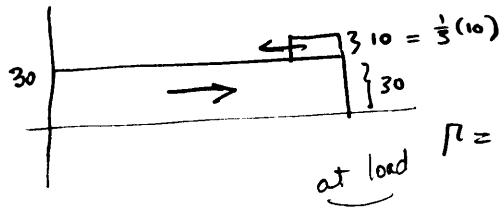
$| \; \mathcal{L} = 400$ meters

what do we expect $\rightarrow$ 30 volts at load

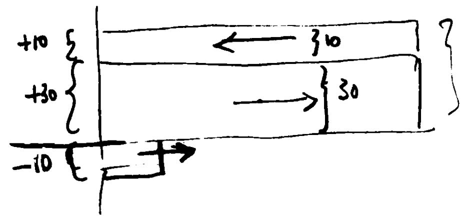
$$i = \frac{\mathcal{E}}{R} = \frac{30}{100} = 0.3 \text{ Amps.}$$

trace pulse. . .





$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = \frac{50}{150} = \frac{1}{3}$$

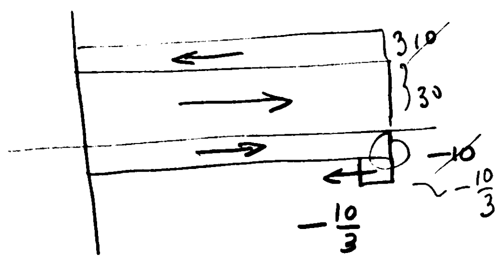


at source $\quad \Gamma = \dfrac{\cancel{Z_L} - Z_0}{\cancel{Z_L} + Z_0} = -1 \quad (Z_L \to 0)$

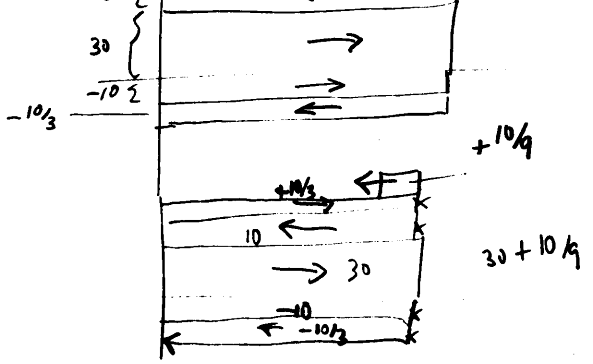
$27\frac{1}{3} + \frac{1}{3}\left(\frac{10}{3}\right)$

$27\frac{1}{3} + 1\frac{1}{9}$

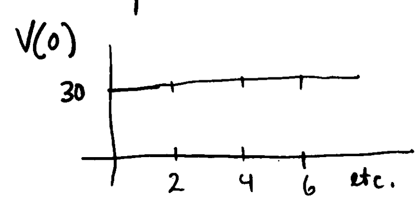
28

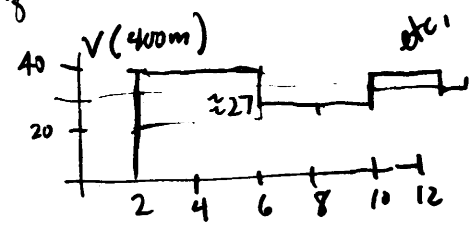


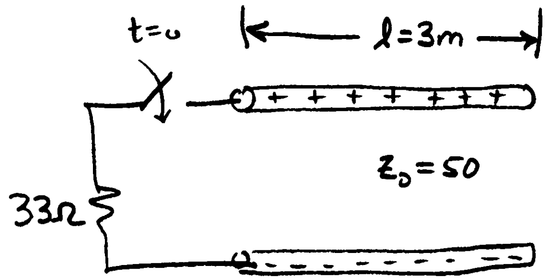
but $\Gamma = \frac{1}{3}$

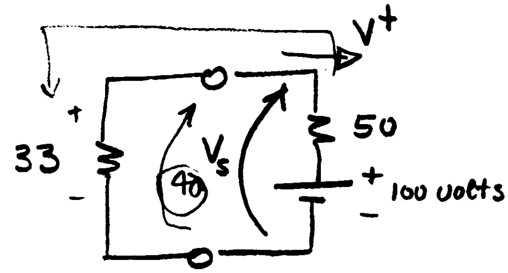


plot as functions of time









Since $\boxed{V_s \neq V_b}$ except when $R = \infty$,
a wave is launched....

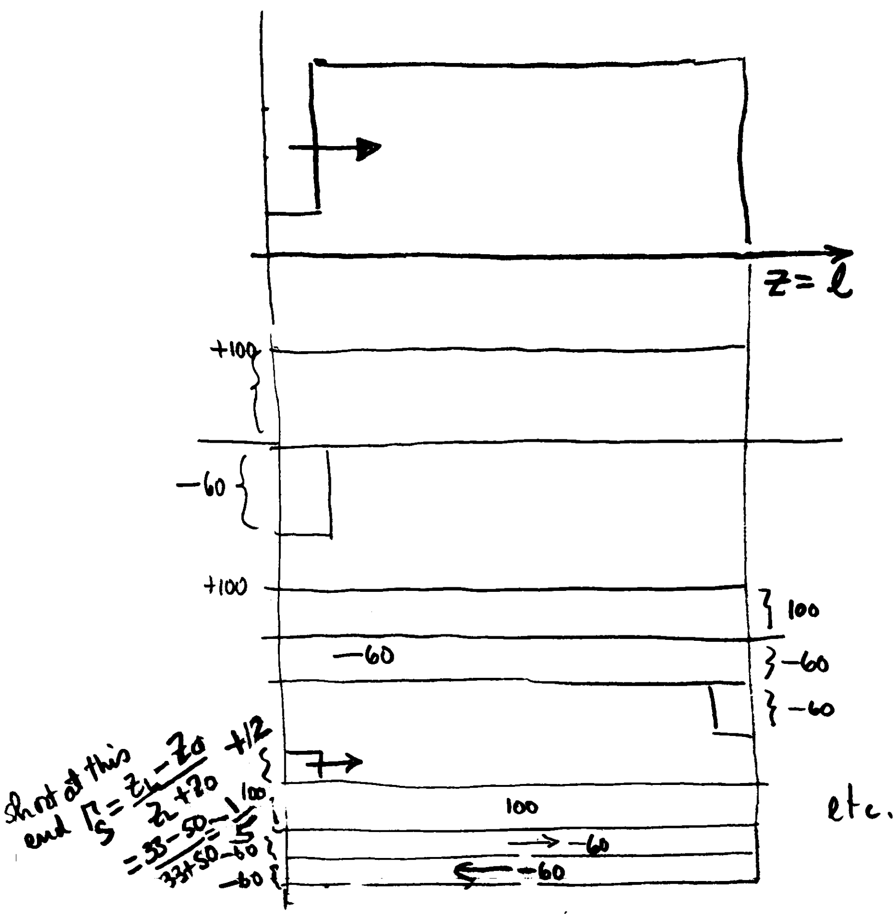
$$V_s = V_b \frac{R_s}{R_s + Z_0}$$

Therefore $V_s = + \frac{100}{50+33} \cdot 33 = +40$ volts.

Voltage across $R = 33\Omega$ resistor is sum of initial voltage V_b and that of launched wave

$$V_s = V_b + V^+$$

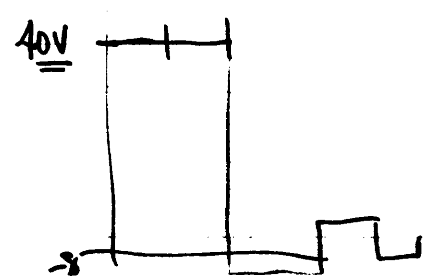
$$40 = 100 + V^+ \qquad \therefore V^+ = -60 \text{ volts.}$$

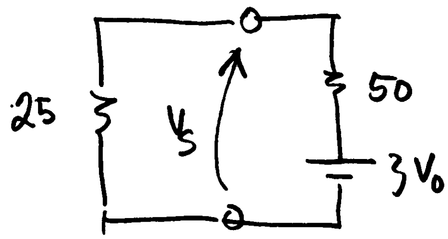


$$\Gamma_L^{(v)} = \frac{Z_L - Z_0}{Z_L + Z_0} = +1$$

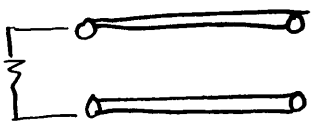
$Z_L = \infty$

etc.





$$V_S = \frac{25}{75} V_0 = \frac{1}{3} V_0$$



$$\Gamma_L^{(v)} = \frac{Z_L - Z_0}{Z_L + Z_0} = +1$$

$$\Gamma_S^{(v)} = \frac{Z_S - Z_0}{Z_S + Z_0} = \frac{25-50}{25+50} = -\frac{25}{75} = \boxed{-\frac{1}{3}}$$

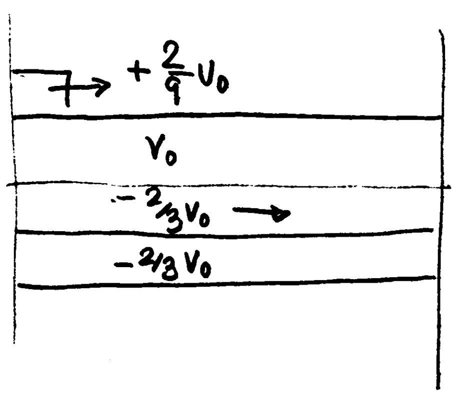
$$V_S = V^+ + V_0 \qquad \text{but } V_S = \frac{1}{3} V_0$$

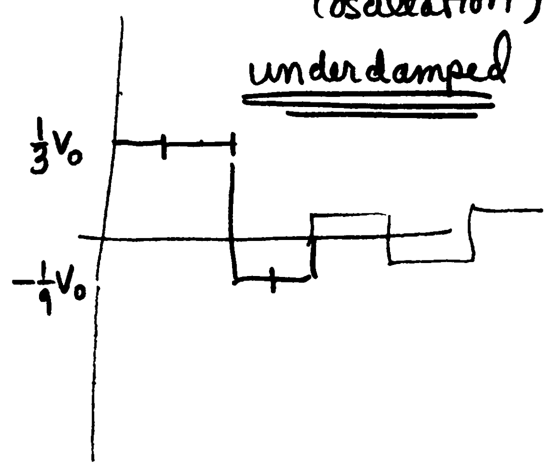
$$\frac{1}{3} V_0 = V^+ + V_0$$

$$\therefore V^+ = -\frac{2}{3} V_0$$

(oscillation)

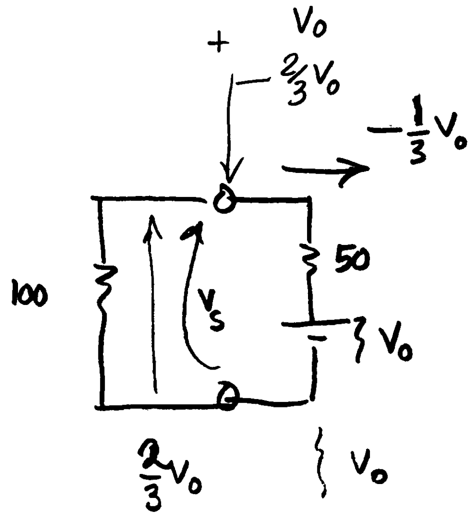
underdamped

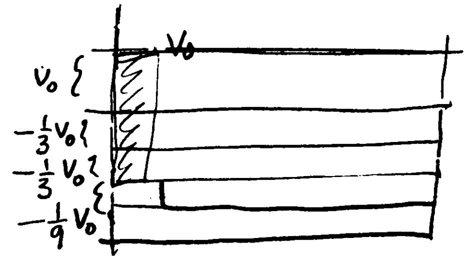




$$V_0 - \frac{4}{3} V_0 + \frac{2}{9} V_0$$

$$\frac{9 - 12 + 2}{9}$$

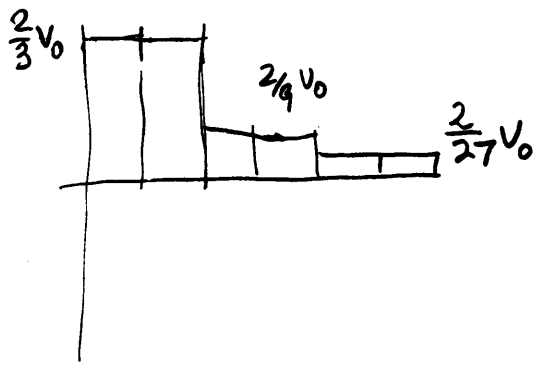




$$V_S = \frac{100}{150} V_0 = \frac{2}{3} V_0$$

NOT -1

$$\Gamma_S(v) = \frac{Z_S - Z_0}{Z_S + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$



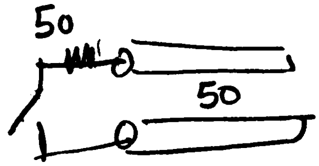
$$+\frac{1}{3}V_0 - \frac{1}{9}V_0$$

$$\frac{3-1}{9} V_0 = \frac{2}{9} V_0$$

$$V_0 - \frac{2}{3}V_0 - \frac{2}{9}V_0 - \frac{1}{27}V_0$$

$$\frac{27 - 18 - 6 - 1}{27} = \frac{2}{27} V_0$$

overdamped (decay)

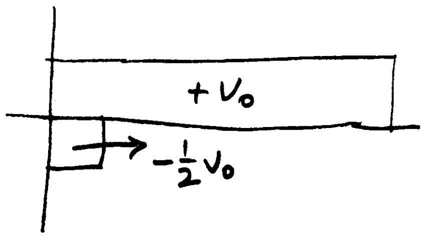


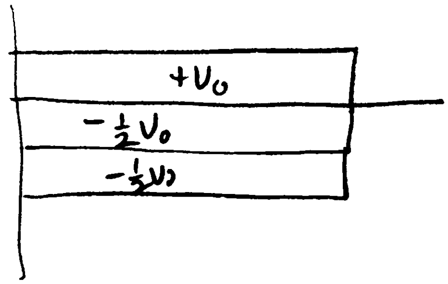
critically damped

$V_S = V^+ + V_B$

$\frac{1}{2}V_0 = V^+ + V_0$

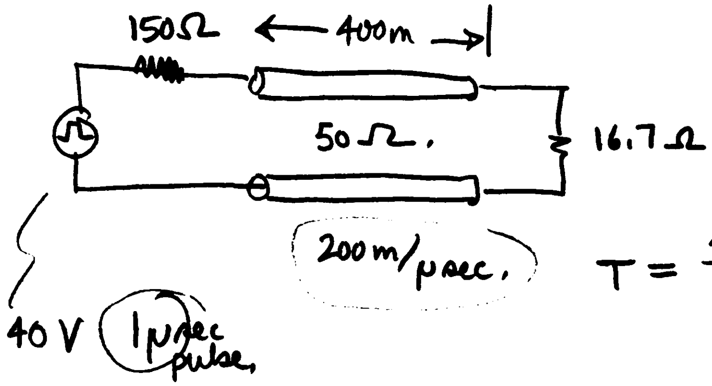
$\therefore V^+ = \frac{1}{2}V_0$





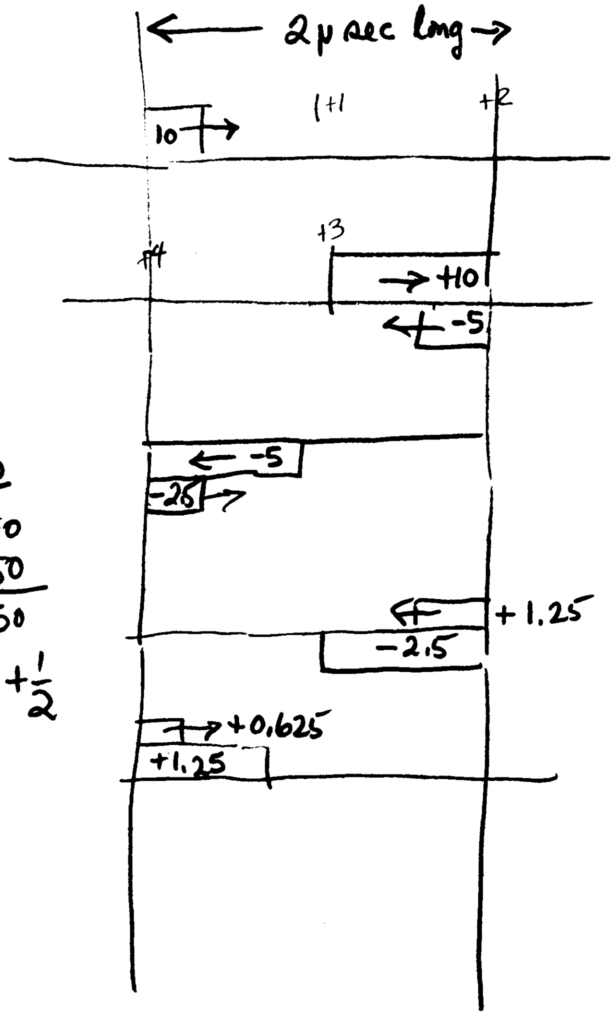
nothing

short pulses



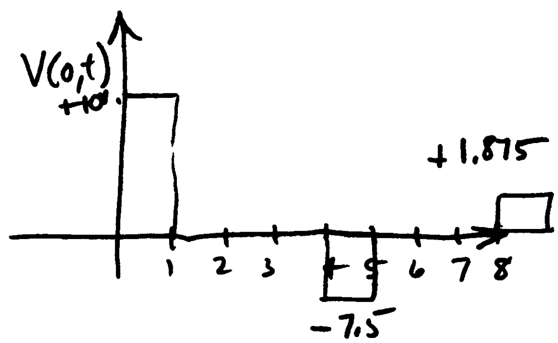
$$T = \frac{400\text{ m}}{200\text{ m}/\mu\text{sec}} = 2\,\mu\text{sec.}$$

into line: $V^+ = \frac{50}{200}(40) = 10\text{V.}$



$$\Gamma_L(V) = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{16.7-50}{16.7+50} \cong -\frac{1}{2}$$

$$\Gamma_S(V) = \frac{Z_S - Z_0}{Z_S + Z_0} = \frac{150-50}{150+50} = \frac{100}{200} = +\frac{1}{2}$$

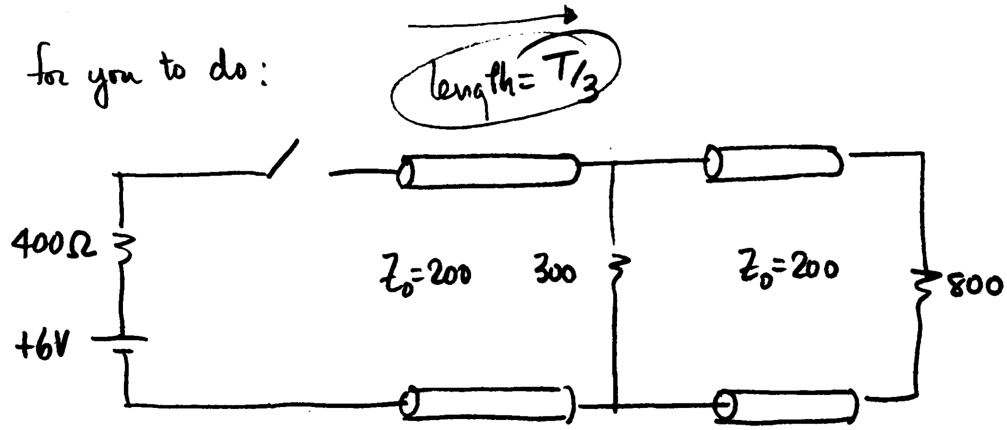


etc.

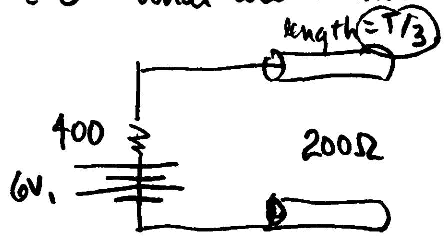
oscillating

TOUGH

Problem for you to do:



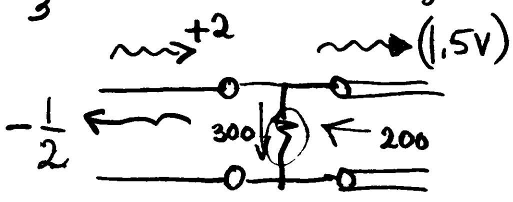
at $t=0$ what are initial voltage & current waves...



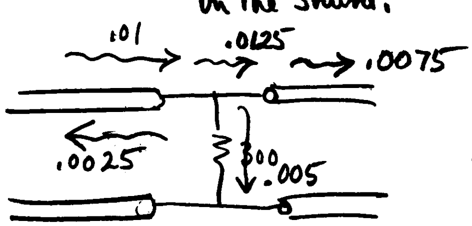
$$V^+ = \left(\frac{200}{600}\right) 6 = 2 \text{ volts.}$$

$$I^+ = \frac{V^+}{Z_0} = \frac{2 \text{ volts}}{200\,\Omega} = 0.01 \text{ Amps.}$$

at $t = T/3$ what is total voltage & current in the shunt



in the shunt.



$$R = \frac{(200)(300)}{200+300} = 120\,\Omega$$

$$\Gamma^{(v)}_{\text{Shunt}} = \frac{Z_{sh} - Z_0}{Z_{sh} + Z_0} = \frac{120-200}{120+200} = -\frac{80}{320}$$

$$= -\frac{1}{4}$$

$$\Gamma_I = -\Gamma_V = +\frac{1}{4}$$

$\therefore$

$$i_{\text{cable}} = \frac{1.5 \text{ volts}}{200\,\Omega} = .0075 \text{ A}$$

$$i_{\text{resistor}} = \frac{1.5 \text{ volts}}{300\,\Omega} = .005 \text{ A.}$$

How about sinusoidal (phasor) solutions of transmission line equations?

motivations:

1. most of the world is analog
2. represent digital pulses by Fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_n t}$$

3. simple solutions

transmission line equations

$$\frac{\partial^2 V}{\partial z^2} = + LC \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = + LC \frac{\partial^2 I}{\partial t^2}$$

$$V(z,t) = \mathrm{Re}\left[\hat{V}(z) e^{j\omega t}\right]$$

$$I(z,t) = \mathrm{Re}\left[\hat{I}(z) e^{j\omega t}\right]$$

$$\frac{d^2\hat{V}}{dz^2} e^{j\omega t} = + LC (j\omega)^2 \hat{V}(z) e^{j\omega t}$$

$$\frac{d^2\hat{V}(z)}{dz^2} = -\omega^2 LC \,\hat{V}(z) \quad \text{cancelling } e^{j\omega t}$$

$$u = \frac{1}{\sqrt{LC}} \qquad \therefore \frac{d^2\hat{V}(z)}{dz^2} = -\frac{\omega^2}{u^2}\hat{V}(z)$$

this is called β

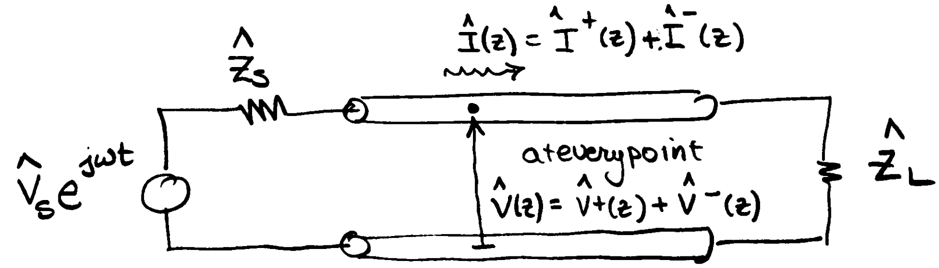
solutions: $e^{\pm j\frac{\omega}{u}z}$

$$\therefore \quad \hat{V}(z) = \hat{V}^+ e^{-j\frac{\omega}{u}z} + \hat{V}^- e^{+j\frac{\omega}{u}z}$$

$$\hat{I}(z) = \hat{I}^+ e^{-j\frac{\omega}{u}z} + \hat{I}^- e^{+j\frac{\omega}{u}z}$$

$$= \frac{V^+}{Z_0} e^{-j\frac{\omega}{u}z} - \frac{V^-}{Z_0} e^{+j\frac{\omega}{u}z}$$

let's look at a circuit



we can define $\hat{Z}(z)$ at every point

$$\hat{Z}(z) = \frac{\hat{V}(z)}{I(z)} = \frac{\hat{V}^+(z) + \hat{V}^-(z)}{\hat{I}^+(z) + \hat{I}^-(z)} \neq Z_0$$

furthermore, since $\hat{Z}(z)$ is distributed we can also write ρ, the reflection coefficient, everywhere.

$$\hat{\rho}(z) = \frac{\hat{V}^-(z)}{\hat{V}^+(z)} = \frac{\hat{V}^- e^{+j\beta z}}{\hat{V}^+ e^{-j\beta z}} = \frac{\hat{V}^-}{\hat{V}^+} e^{j2\beta z}$$

$$\hat{V}(z,t) = \hat{V}^+ e^{-j\beta z} + \hat{V}^- e^{j\beta z}$$

$$= \hat{V}^+ e^{-j\beta z}\left[1 + \frac{\hat{V}^- e^{j\beta z}}{\hat{V}^+ e^{-j\beta z}}\right] = \underbrace{V^+ e^{-j\beta z}\left[1 + \rho(z)\right]}_{\text{everywhere.}}$$

$$\hat{I}(z,t) = \frac{\hat{V}^+}{Z_0} e^{-j\beta z} - \frac{\hat{V}^- e^{+j\beta z}}{Z_0}$$

$$= \frac{\hat{V}^+}{Z_0} e^{-j\beta z}\left[1 - \frac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}}\right] = \frac{V^+ e^{-j\beta z}}{Z_0}\left[1 - \hat{\rho}(z)\right]$$

$$\hat{Z}(z) = \frac{\hat{V}(z)}{\hat{I}(z)} = \frac{V^+ e^{-j\beta z}\left[1 + \hat{\rho}(z)\right]}{\frac{V^+ e^{-j\beta z}}{Z_0}\left[1 - \hat{\rho}(z)\right]} = Z_0 \frac{1 - \hat{\rho}(z)}{1 + \hat{\rho}(z)}$$

Equations needed to solve most problems are

$$\boxed{\begin{aligned} \hat{\rho}(z) &= \frac{\hat{V}^-}{\hat{V}^+}\, e^{j2\beta z} \\ \hat{Z}(z) &= Z_0\, \frac{1+\hat{\rho}(z)}{1-\hat{\rho}(z)} \end{aligned}}$$

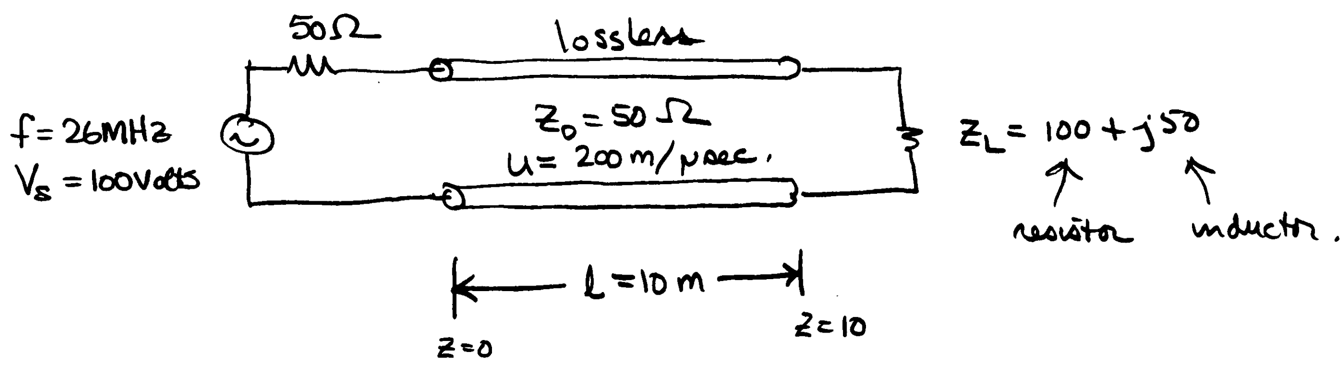
simple proof:

$$\hat{\rho}(z) = \frac{V^-}{V^+}\, e^{j2\beta z}$$

$$\hat{\rho}(z=l) = \hat{\rho}_L = \frac{V^-}{V^+}\, e^{j2\beta l}.$$

dividing

$$\frac{\hat{\rho}(z)}{\hat{\rho}_L} = e^{j2\beta(z-l)}$$



what is Z at z=0

V at z=0 and z=ℓ

$$\beta = \frac{\omega}{u} = \frac{2\pi(26\times10^6)}{200\times10^6} = 0.82 \text{ rad/meter}$$

electrical length

$$2\beta\ell = (2)(0.82\,\text{rad}/\text{m})(10\text{m}) = \boxed{16.34 \text{ radians}}$$

electrical length

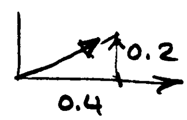
$$\boxed{2\beta\mathcal{L}} = 936^\circ = 936 - 2\cdot360 = \boxed{216^\circ}$$

No matter how long our line really is this is its phase shift, or electrical length....

what is $\hat{\rho}(z=\ell)$?

$$\hat{\rho}_L = \frac{\hat{Z}_L - \hat{Z}_0}{\hat{Z}_L + \hat{Z}_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50}$$

$$= (0.4 + j0.2)$$



transform ρ_L to $\rho(0)$

$$\rho(0) = \rho_L e^{-j2\beta L}$$
$$= (0.4 + j0.2)(e^{-j216^\circ})$$
$$= (0.4 + j0.2)(-0.809 + j0.58)$$
$$= -0.44 + j0.0702.$$

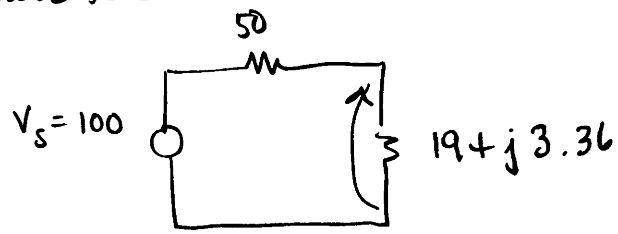
knowing ρ_L we relate it to Z_{IN}

$$\hat{Z}_{IN} = Z(0) = Z(0)\frac{1 + \rho(0)}{1 - \rho(0)}$$

$$= (50)\left(\frac{1 - 0.44 + j0.07}{1 + 0.44 - j0.07}\right) = 50\,\frac{0.56 + j0.07}{1.44 - j0.07}$$

$$= 50\,(0.38 + j0.067) = \boxed{19 + j3.36\ \Omega}$$

generator see's



$$\hat{V}(0) = \frac{\hat{Z}_{IN}}{\hat{Z}_S + \hat{Z}_{IN}} V_s = \frac{19 + j3.36}{50 + 19 + j3.36} \cdot 100 = \frac{19 + j3.36}{69 + j3.36}(100)$$

$$= (0.277 - j0.035)\, 100$$

$$= 27.7 - j3.5 \text{ volts}$$

how do we get $\hat{V}(z = \ell)$. Ans:

$$\hat{V}(z) = V^+ e^{-j\beta z} + V^- e^{-j\beta z}$$

$$= \underbrace{V^+}_{\text{unknown}} e^{-j\beta z}\left[1 + \underbrace{\boxed{\frac{V^-}{V^+} e^{-j2\beta z}}}_{\text{this is } \rho(z)}\right]$$

at $z=0$ $\quad 27.7 - j3.5) = V^+ \, (1) \left[1 + \rho(0)\right]$

what is $\rho(0)$? $\quad \rho(0) = \rho_L e^{-j2\beta \ell}$

$$= \rho_L e^{-j216°}$$

$$= (0.4 + j0.2)(\cos 216 - j\sin 216)$$

$$= (0.4 + j0.2)(-0.809 + j0.5878)$$

$$= -0.44 + j0.074$$

$$(27.7 - j3.5) = V^{+}(1)\left[1 - 0.44 + j\,0.074\right]$$

$$V^{+} = \frac{27.7 - j\,3.5}{.56 + j\,0.074} = (147.8 - j\,12.57) \text{ volts}$$

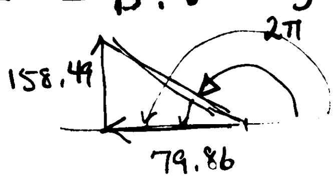
$$\hat{V}(z=\ell) = V^{+} e^{-j\beta z}\left[1 + \rho(z=\ell)\right]$$

$$= (47.8 - j\,12.57)(e^{-j\,108^\circ})(1 + 0.4 + j\,0.2)$$

$$2\beta\ell = 936^\circ$$
$$\beta\ell = 468^\circ$$
$$\;-360$$
$$\beta\ell = 108^\circ$$

$$= (47.8 - j12.57)(-.3090 + j\,0.9511)(1.4 + j0.2)$$

$$= -13.8 + j\,68.52$$



This example was a simple example but note how complex the algebra was

usually we are only interested in impedances, not voltages. There is a quick way of finding impedance Transformations.

Recall $\hat{Z}(z=\ell) = \hat{Z}_L = \hat{Z}_0 \dfrac{1+\hat{\rho}(z=\ell)}{1-\rho(z=\ell)}$

$$\rho(z=\ell) = \frac{\hat{Z}_L - \hat{Z}_0}{\hat{Z}_L + \hat{Z}_0}$$

$$\hat{\rho}(z) = \rho_L e^{+j2\beta(z-\ell)}$$

Combine these equations to get

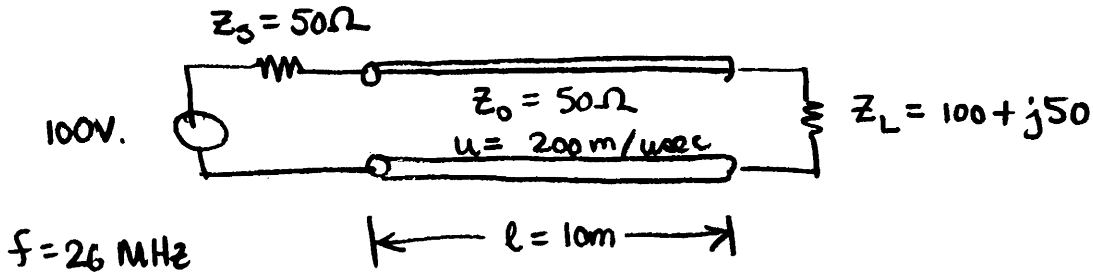
$$\boxed{\hat{Z}_{IN}(z) = \hat{Z}_0 \frac{Z_L + j\hat{Z}_0 \tan\beta(\ell - z)}{Z_0 + jZ_L \tan\beta(\ell - z)}}$$

$$\hat{Z}_{IN}(z=0) = \hat{Z}_0 \frac{Z_L + j\hat{Z}_0 \tan\beta\ell}{Z_0 + jZ_L \tan\beta\ell}$$

This is very easy to use if ℓ is in wavelengths.

say $\ell = \dfrac{\lambda}{4}$

but $\beta = \dfrac{2\pi}{\lambda}$



$$\beta = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{(26\times 10^6)\,2\pi}{200\,m/10^{-6}} = 0.8168 \text{ rad/m.}$$

$$2\beta\ell = 2\,(0.8168 \text{ rad/m})(10m) = 16.34 \text{ rad} = 936.2^\circ$$

$$= 936 - 720 \approx 216.2^\circ$$

* we can compute ρ_L since Z_L and Z_0 are known.

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50}$$

$$= 0.4 + j0.2 = 0.4472\, e^{j26.57^\circ}$$

* since $\rho(z) = \rho_L\, e^{-j2\beta(\ell - z)}$

$$\rho(0) = \left(0.4472 e^{j26.57^\circ}\right)\left(e^{-j216.2^\circ}\right)$$

$$= 0.4472\, e^{-j189.63}$$

$$= -0.441 + j\,0.07$$

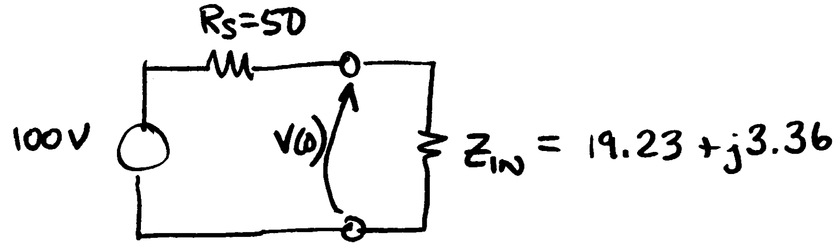
* we can now calculate the input impedance

$$Z_{in}^{(0)} = Z_0 \frac{1 + \rho(0)}{1 - \rho(0)} = 50\,\frac{1 + (-.441 + j.07)}{1 - (-.441 + j.07)}$$

$$= 50\left(\frac{.559 + j\,0.07}{1.441 - j.07}\right) = 50\,(0.385 + j\,.067)$$

$$= 19.23 + j3.36 = 19.52\, e^{j9.9^\circ}$$

★ the input voltage comes from a complex voltage divider



$$\hat{V}(0) = \frac{Z_{in}}{Z_{in} + 50} 100 = \frac{19.23 + j3.36}{19.23 + j3.36 + 50}(100) =$$

$$= (27.95 + j\,3.49)\text{volts} = 28.16\,e^{+j7.13^\circ}$$

★ What is the voltage <u>anywhere</u> on the line

$$\hat{V}(z) = V^+ e^{j2\beta z}\left[1 + \rho(z)\right]$$

idea we know $\hat{V}(0)$, $\hat{\rho}(0)$

$\hat{\rho}(\ell)$

want $\hat{V}(\ell)$. So if we can determine the $V^+ e^{j2\beta z}$ we can determine $\hat{V}(z)$ anywhere.

$$\hat{V}(0) = V^+ \,(1)\left[1 + \rho(0)\right]$$

$$V^+ = \frac{\hat{V}(0)}{1 + \hat{\rho}(0)} = \frac{27.95 + j3.49}{1 + (-.441 + j0.07)}$$

$$= \frac{27.95 + j3.49}{0.559 + j0.07} = 50.0 - j\,0.0176 = 50.0e^{-j.02}$$

now calculate $\hat{V}(\ell = 10\text{m})$

$$\hat{V}(z) = V^+ e^{-j\beta z}\left[1 + \rho(z)\right]$$

$$\hat{V}(10) = \left(50e^{-j.02}\right)\left(e^{-j128^\circ}\right)\left(1 + \rho_L\right)$$

$1 + .441 + j0.07 = 1.441 + j.07$

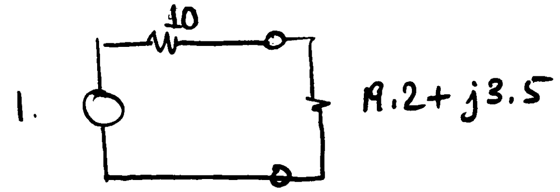
$1.443\,e^{j2.78^\circ}$

$$= 50e^{-j0.02}\,e^{-j128^\circ}\,1.443e^{j2.78^\circ}$$

$$= 72.15e^{-j125.2^\circ} = (-41.6 - j58.9)\text{ volts}$$

$$Z_{in} = 19.2 + j3.5\ \Omega = 19.52\, e^{j10.33}$$

* voltage into line



$$\hat{V}_i(0) = 1\,\frac{19.2+j3.5}{10+19.2+j3.5} = \frac{19.2+j3.5}{29.2+j3.5}$$

$$\hat{V}(0) = 0.66 + j0.040\ (\text{volts}) = 0.66\, e^{j3.46^\circ}$$

* current into line

$$\hat{I}(0) = \frac{\hat{V}(0)}{Z_{in}(0)} = \frac{0.66+j0.04}{19.2+j3.5} = (.0336 - j0.0041)\ \text{Amps}$$

$$= 0.034\, e^{-j6.86^\circ}$$

* input power

$$P_{in} = \tfrac{1}{2}\,\mathrm{Re}\left[\hat{V}(0)\,\hat{I}^*(0)\right]$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left[(0.66+j0.04)(.0336+j0.0041)\right]$$

$$= \tfrac{1}{2}\,\mathrm{Re}\left[0.022 + j0.0041\right]$$

$$= 0.011 \text{ watts into line}$$

* loss less line so $P_{LOAD} = P_{in} = 0.011$ watts.

★ compute load voltage as before...

$$\hat{V}(z) = V^+ e^{-j\beta z}\left[1 + \hat{\rho}(z)\right]$$

known: $\hat{V}(0)$, $\hat{\rho}(0)$, $\hat{\rho}_L$

at $z=0$. $\hat{\rho}(z=0) = \rho_L e^{-j2\beta l} = 0.447\, e^{j26.57}\, e^{-j216^\circ}$

$$= 0.447\, e^{-189.4}$$

$$\hat{\rho}(0) = -0.441 + j0.07$$

$$V^+ = \frac{\hat{V}(0)}{1+\hat{\rho}(0)} = \frac{0.66+j0.04}{1-0.441+j0.07} = \frac{0.66+j0.04}{0.559+j0.07}$$

$$V^+ = 1.17 - j0.075 = 1.17\, e^{-j3.67^\circ}$$

$$\hat{V}(l) = \left(1.17\, e^{-j3.67^\circ}\right) e^{-j\beta l}\left[1+\hat{\rho}_L\right]$$

$$= \left(1.17\, e^{-j3.67^\circ}\right)\left(e^{-j108^\circ}\right)\left(\underbrace{1+0.4+j0.2}_{1.4+j0.2\,=\,1.414 e^{j8.13}}\right)$$

$$\hat{V}(l) = \left(1.17\, e^{-j3.67^\circ}\right)\left(e^{-j108^\circ}\right)\left(1.414\, e^{j8.13}\right)$$

$$= 1.65\, e^{-j103.5}$$

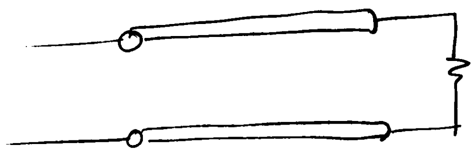
$$\hat{I}(l) = \frac{\hat{V}(l)}{Z_L} = \frac{1.65\, e^{-j103.5}}{\boxed{100+j50} = 111.8\, e^{j26.6}} = 0.0148\, e^{-j130.1}$$

$$\text{Power} = \frac{1}{2}\mathrm{Re}\left[\hat{V}(l)\,\hat{I}^*(l)\right] = \frac{1}{2}\mathrm{Re}\left[1.65\, e^{-j103.5}\, 0.0148\, e^{+j130.1}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[.02442\, e^{j26.6}\right] = \frac{1}{2}\mathrm{Re}\left[0.022 + j0.011\right]$$

$\approx$ 0.011 watts.

Let's look at getting line voltages simply.



Recall $\hat{V}(z) = \hat{V}^{+} e^{-j\beta z}\left[1 + \hat{\rho}(z)\right]$

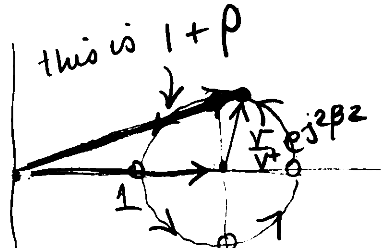
$$|\hat{V}(z)| = \underbrace{|\hat{V}^{+}|}_{\text{independent of } z}\;\underbrace{|1 + \rho(z)|}_{\text{look at } z\text{-dependence of this}}$$

$$\hat{\rho}(z) = |\hat{\rho}(z)|\, e^{j\theta_\rho}$$

but $\hat{\rho}(z) = \dfrac{V^{-}}{V^{+}} e^{j2\beta z}$

so $|\rho(z)| = \dfrac{V^{+}}{V^{-}}$ } assume V^{+}, V^{-} real

$$|1 + \rho(z)| = \left|1 + \frac{V^{-}}{V^{+}} e^{j2\beta z}\right| =$$



this looks like the crank of a locomotive!

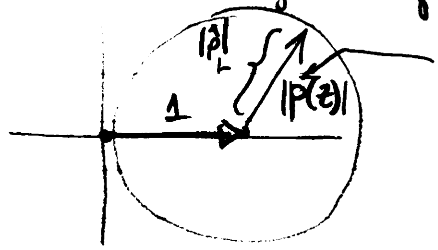
how can we interpret this

$$\hat{V}(z) = V^+ e^{-j\beta z}\left[1 + \hat{\rho}(z)\right]$$

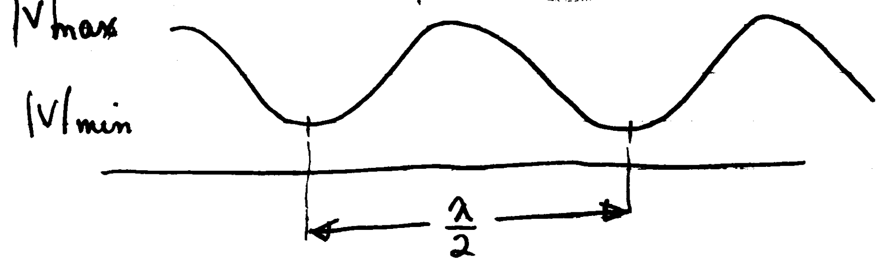
$$|\hat{V}(z)| = |V^+|\,|e^{-j\beta z}|\,|1 + \hat{\rho}(z)|$$

$$= |V^+|\,|1 + \hat{\rho}(z)|$$

magnitude of 1



obviously $|\hat{V}(z)|$ is periodic



$$\hat{\rho}(z) = \hat{\rho}_L e^{j2\beta(z-\ell)}$$

$$2\beta(\lambda) = 2\,\frac{2\pi}{\lambda}\cdot x = 2\pi$$

$$\frac{2x}{\lambda} = 1$$

$$x = \frac{\lambda}{2}$$

$$|\hat{V}|_{max} = |V^+|\left(1 + |\rho_L|\right)$$

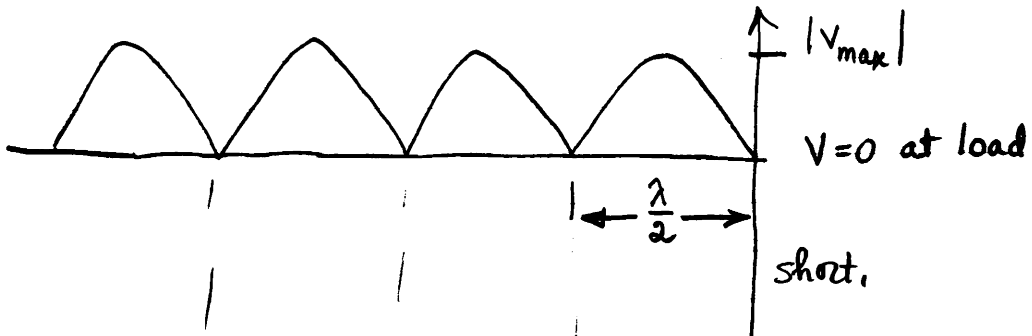
$$|\hat{V}|_{min} = |V^+|\left(1 - |\rho_L|\right)$$

$$VSWR = \frac{1 + |\rho_L|}{1 - |\rho_L|} \qquad 0 \le |\rho_L| \le 1$$

$$1 \le VSWR < \infty$$

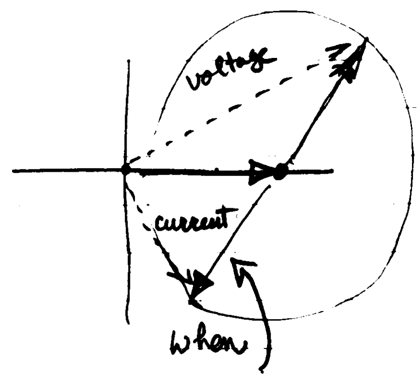
plot $|\hat{V}_{max}|$ $\quad Z_L = 0$ $\quad \rho_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = -1$

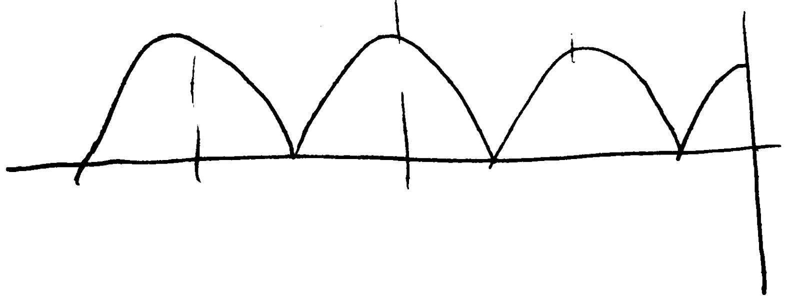
$|\rho_L| = 1$



how about $\hat{I}(z) = \dfrac{V^+}{Z_0} e^{-j\beta z} \left[1 - \hat{\rho}(z)\right]$

$|\hat{I}(z)| = \left|\dfrac{V^+}{Z_0}\right| \left[1 - |\rho(z)|\right]$

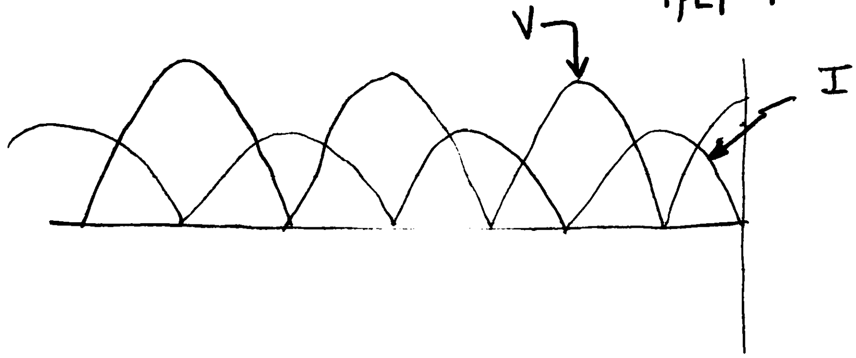




when voltage = max
current = min

at load when $Z_L = \infty$ $\quad \rho_L = \dfrac{Z_L - Z_0}{Z_L + Z_0} = +1$

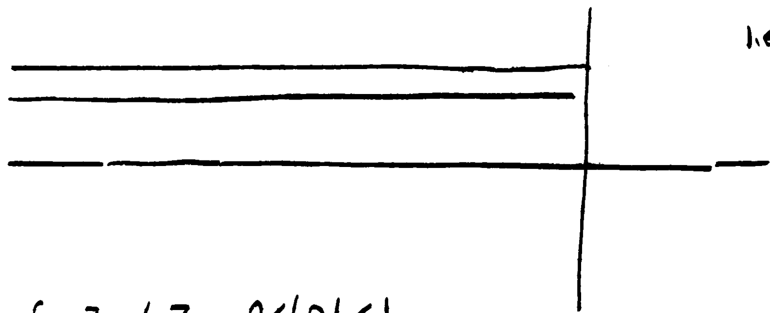
$|\rho_L| = 1$



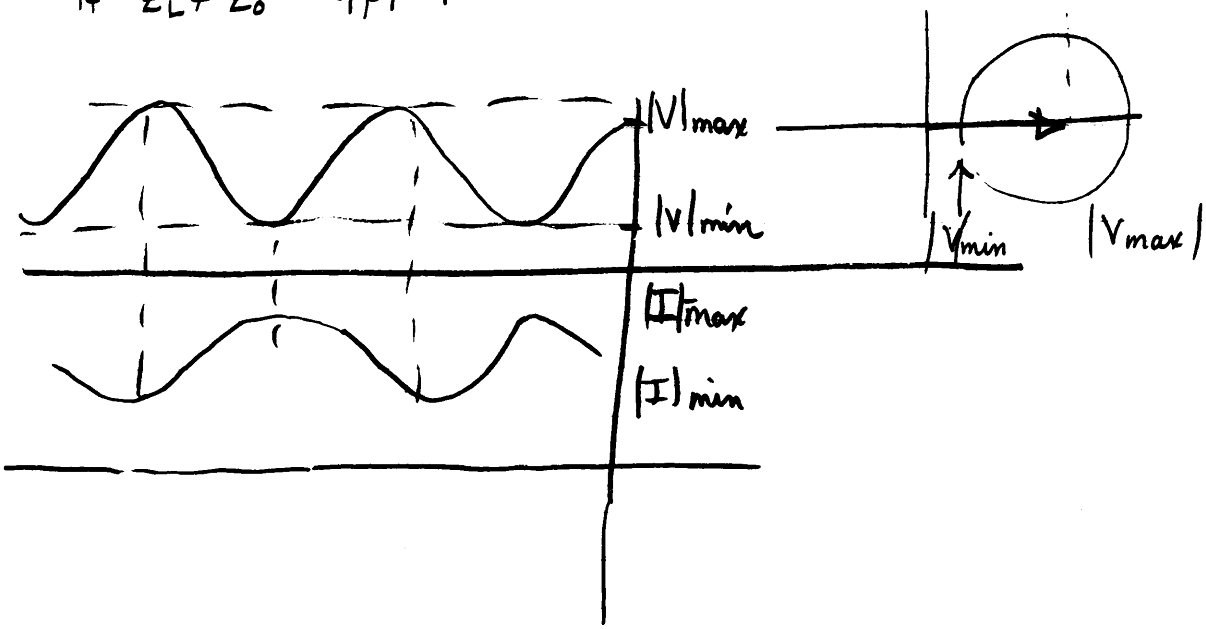
if $Z_L = Z_0$ $\rho \to 0$ and $|V| =$ constant

$|I| =$ constant.

i.e.



if $Z_L \neq Z_0$ $0 < |\rho| < 1$



Notes about things on the line

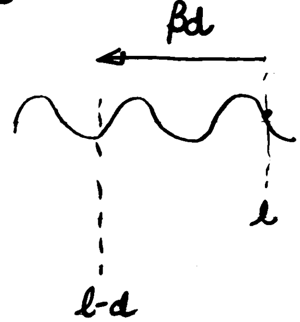
$$\hat{\rho}(z) = \rho_L e^{-j2\beta(\ell - z)}$$

this is periodic in 2π

$$\hat{V}(z) = V^+ e^{-j\beta z}\left[1 + \hat{\rho}(z)\right]$$

$$\hat{V}(\ell) = V^+ e^{-j\beta\ell} + V^- e^{+j\beta\ell}$$

$$\hat{I}(\ell) = \frac{V^+}{Z_0} e^{-j\beta\ell} - \frac{V^-}{Z_0} e^{+j\beta\ell}$$



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with respect to the load....

$$\hat{V}(\ell-d) = \left(V^+ e^{-j\beta\ell}\right) e^{j\beta d} + \left(V^- e^{j\beta\ell}\right) e^{-j\beta d}$$

$$\hat{I}(\ell-d) = \left(\frac{V^+ e^{-j\beta\ell}}{Z_0}\right) e^{j\beta d} - \left(\frac{V^-}{Z_0} e^{j\beta\ell}\right) e^{-j\beta d}$$

---

expand $e^{\pm j\beta d}$

$$\hat{V}(\ell-d) = V^+ e^{-j\beta\ell}\cos\beta d + jV^+ e^{-j\beta\ell}\sin\beta d + V^- e^{j\beta\ell}\cos\beta d - jV^- e^{j\beta\ell}\sin\beta d$$

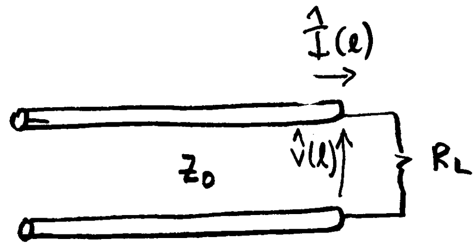
$$\hat{I}(\ell-d) = \frac{V^+ e^{-j\beta\ell}}{Z_0}\cos\beta d + jV^+\frac{e^{-j\beta\ell}}{Z_0}\sin\beta d - \frac{V^- e^{j\beta\ell}}{Z_0}\cos\beta d + \frac{jV^-}{Z_0} e^{j\beta\ell}\sin\beta d$$

$$\hat{V}(\ell-d) = \left(V^+ e^{-j\beta\ell} + V^- e^{j\beta\ell}\right)\cos\beta d + j\left(V^+ e^{-j\beta\ell} - V^- e^{j\beta\ell}\right)\sin\beta d$$

$$\hat{I}(\ell-d) = \left(\frac{V^+ e^{-j\beta\ell}}{Z_0} - \frac{V^- e^{j\beta\ell}}{Z_0}\right)\cos\beta d + j\left(\frac{V^+ e^{-j\beta\ell} + V^- e^{j\beta\ell}}{Z_0}\right)\sin\beta d.$$

$$\hat{V}(\ell-d) = \hat{V}(\ell)\cos\beta d + jZ_0\hat{I}(\ell)\sin\beta d$$

$$\hat{I}(\ell-d) = \hat{I}(\ell)\cos\beta d + j\frac{\hat{V}(\ell)}{Z_0}\sin\beta d.$$

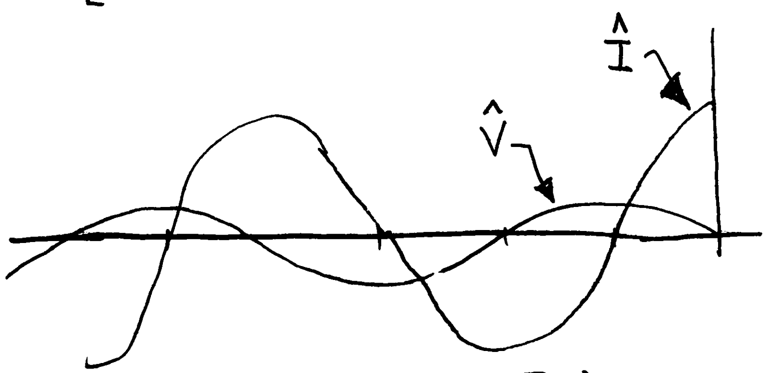


at load $Z_L = R_L$ and $I(\ell) = \dfrac{V(\ell)}{R_L}$

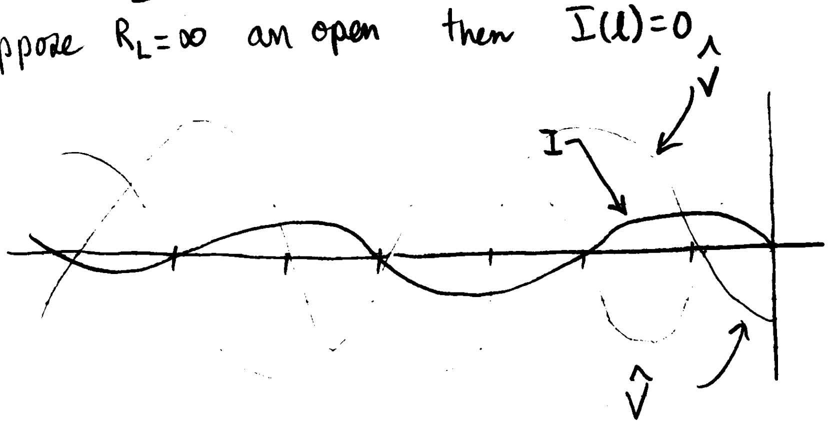
then $$\hat{V}(\ell-d) = \hat{V}(\ell)\cos\beta d + jZ_0\frac{\hat{V}(\ell)}{R_L}\sin\beta d = \hat{V}(\ell)\left[\cos\beta d + j\frac{Z_0}{R_L}\sin\beta d\right]$$

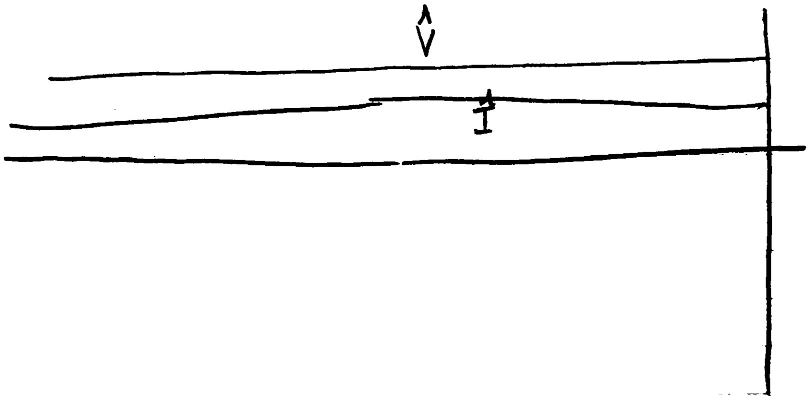
$$\hat{I}(\ell-d) = \frac{\hat{V}(\ell)}{R_L}\cos\beta d + j\frac{\hat{V}(\ell)}{Z_0}\sin\beta d = \underbrace{\frac{\hat{V}(\ell)}{R_L}}_{I(\ell)}\left[\cos\beta d + j\frac{R_L}{Z_0}\sin\beta d\right]$$

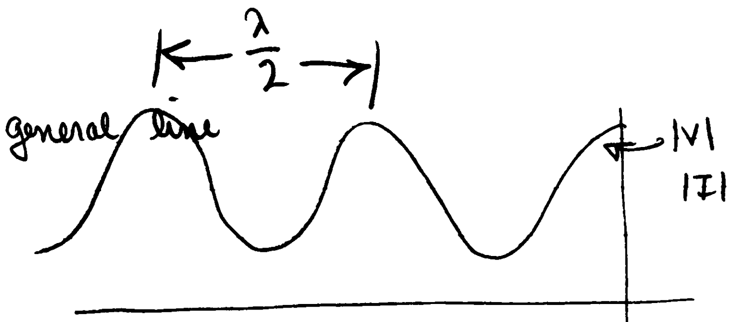
suppose $R_L = 0$ a short then $V(\ell) = 0$



suppose $R_L = \infty$ an open then $I(\ell) = 0$

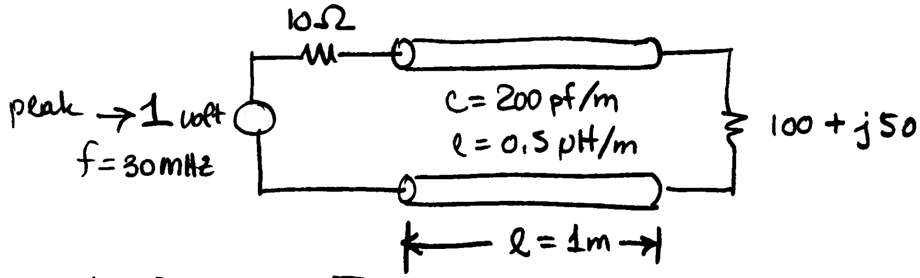






simpler way $\hat{V}(z) = V^+ e^{-j\beta z}$

load voltage, average power to load



★ <u>basic parameters</u>

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.5\times10^{-6}}{200\times10^{-12}}} = 50\Omega$$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(0.5\times10^{-6})(200\times10^{-12})}} = 10^8 \text{ m/sec.}$$

$$\beta = \frac{\omega}{u} = \frac{2\pi\ 30\times10^6}{10^8} = 1.8849 = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1.8849} = 3.333 \text{ meters}$$

line is $\frac{1}{3.3333}\lambda = 0.3\lambda$ long

★ reflection coefficient at load.

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50+j50}{150+j50} = 0.4 + j0.2$$

$$= 0.447 e^{j26.57°}$$

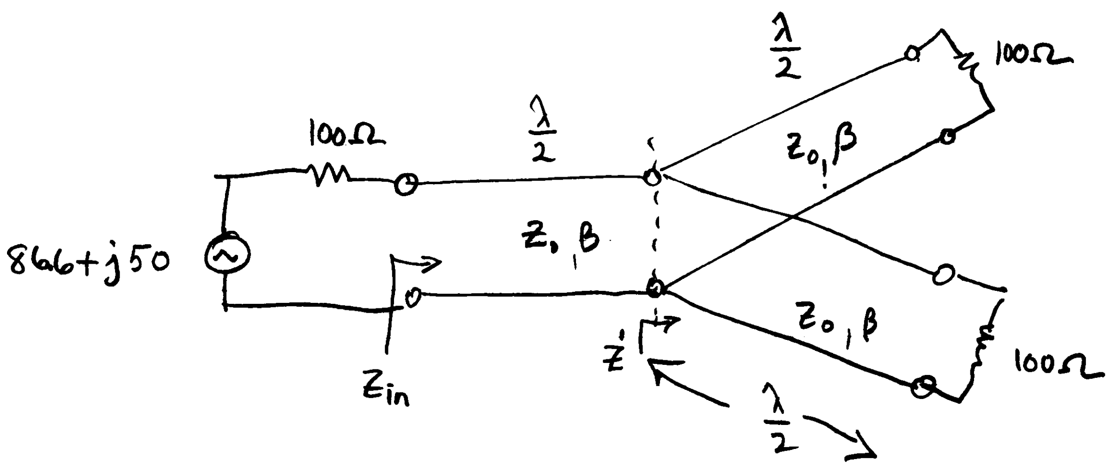
★ what is the input impedance

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + j Z_L \tan\beta l}$$

$$\tan\beta l = \tan \frac{2\pi}{\lambda} .03\lambda = \tan(1.885 \text{ radians}) = \tan(108°)$$

$$= -3.078$$

$$Z_{in} = \frac{(100+j50) + j(50)(-3.078)}{(50) + j(100+j50)(-3.078)} = \frac{100 - j103.88}{203.9 - j307.8} = .384 + j.07$$



a $\frac{\lambda}{2}$ line $\quad Z' = Z_0 \dfrac{Z_L + j Z_0 \tan \beta l}{Z_0 + j Z_L \tan \beta l}$

$$\tan \frac{2\pi}{\lambda} \cdot \frac{\lambda}{2} = \tan \pi \rightarrow 0$$

$$\therefore Z' = Z_0 \frac{Z_L}{Z_0} = Z_L$$

two 100Ω resistors in parallel $\Rightarrow$ 50Ω

$$Z_{in} = Z_0 \frac{50 + jZ_0 \cancelto{0}{\tan \pi}}{Z_0 + j50 \cancelto{0}{\tan \pi}} \rightarrow 50\Omega$$

$$V_{in} = \left(\frac{50}{50+100}\right)(86.6 + j50) = 28.87 + j16.6 = 33.3e^{j30}$$

$$V(z) = \hat{V}^+ e^{-j\beta l}(1 + \hat{\rho}(z))$$

$$V_{in} = V(0) = V^+ (1 + \rho(0)) \qquad \rho(0) = \frac{Z_s - Z_0}{Z_s + Z_0}$$

$\beta l = \pi$

$$V\left(\frac{\lambda}{2}\right) = V^+ e^{-j\beta l}\left(1 + \hat{\rho}\left(\frac{\lambda}{2}\right)\right)$$

but $\rho(z) = \rho_L e^{j2\beta l}$ goes to 2π

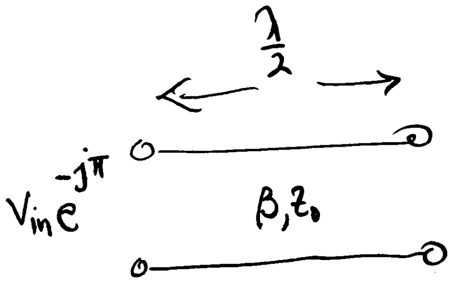
so $\rho(z)$ is periodic in π

and $\rho\left(\frac{\lambda}{2}\right) = \rho(0)$

$$\therefore \frac{V_{in}}{V\left(\frac{\lambda}{2}\right)} = \frac{\cancel{V^+} \cancel{(1+\rho(0))}}{\cancel{V^+} e^{-j\beta l} \cancel{(1+\rho(\frac{\lambda}{2}))}} = e^{j\beta l}$$

$$V\left(\frac{\lambda}{2}\right) = V_{in} e^{-j\beta l} = V_{in} e^{-j\pi}$$

same relationship for rest of line



$$V(z) = V^+ e^{-j\beta l}\left[1 + \hat{\rho}(z)\right]$$

$$V(0) = V^+ \left[1 + \hat{\rho}(0)\right]$$

$$\therefore \; V^+\left[1 + \hat{\rho}(0)\right] = V_{in} e^{-j\pi}$$

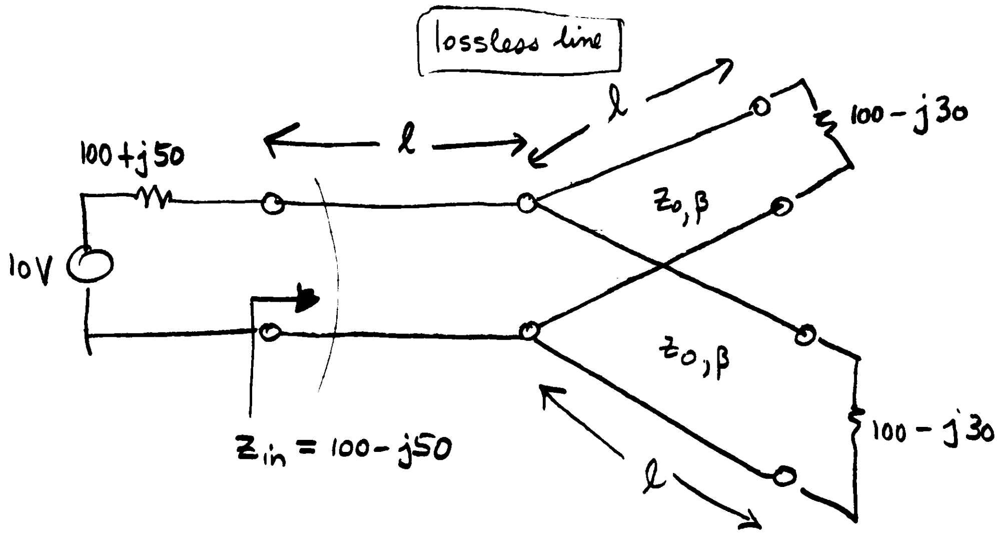
$$\underset{ant}{V\left(z = \tfrac{\lambda}{2}\right)} = V^+ e^{-j\pi}\left[1 + \hat{\rho}\left(\tfrac{\lambda}{2}\right)\right] = V^+ e^{-j\pi}\left[1 + \hat{\rho}(0)\right]$$

[periodic]

$$= V^+\left[1 + \hat{\rho}(0)\right] e^{-j\pi} = V_{in} e^{-j\pi} e^{-j\pi}$$

$$= V_{in} \boxed{e^{-j2\pi}} \rightarrow 1$$

$$\boxed{\therefore \; V_{ANT} = V_{in}}$$



$$V_{in} = \frac{Z_{in}}{100 + j50 + Z_{in}} \, 10V$$

$$= \left(\frac{100 - j50}{100 + \cancel{j50} + 100 - \cancel{j50}}\right) 10 = \left(\frac{100 - j50}{200}\right) 10 = (0.5 - j0.25)\, 10$$

$$= 5 - j2.5 \text{ volts.} = 5.59 e^{-j26.6^\circ}$$

lossless

use power, NOT voltage

$$Z_{in} = 100 - j50 = 111.8 e^{-j26.6}$$

what is P_{IN}?

$$I_{in} = \frac{\hat{V}(0)}{Z_{in}} = \frac{V_{in}}{Z_{in}} = \frac{5.59 e^{-j26.6}}{111.8 e^{-j26.6}} = 0.05$$

$$P_{in} = \frac{1}{2} \mathrm{Re}\left[\hat{V}(0)\hat{I}(0)^*\right] = \frac{1}{2}\mathrm{Re}\left[(5 - j2.5)(.05)\right] = \frac{1}{2}(.25)$$

$$= 0.125 \text{ watts.}$$

Power to one antenna $\quad .125/2 = 62.5\text{mW.}$

$$P_A = \frac{1}{2}\mathrm{Re}\left[\hat{V}(0)\frac{\hat{V}^*}{Z_A^*}\right] = \frac{1}{2}|\hat{V}^+|^2 \,\mathrm{Re}\left(\frac{1}{Z_A^*}\right)$$

$$\therefore \; |\tilde{V}|^2 = \frac{2P_A}{\mathrm{Re}\left(\frac{1}{Z_A^*}\right)} = \frac{2(.0625)}{\mathrm{Re}(.009 - j0.0027)} = \frac{2(.0625)}{.009}$$

$$= 13.88 \text{ volts}^2$$

average voltage $\rightarrow |\hat{V}| = 3.72$ volts.

Simple line transformations.

if $Z_L = 0$ a short

$$Z_{IN}(z=0) = Z_0 \frac{jZ_0 \tan \beta l}{Z_0} = \boxed{jZ_0 \tan \beta l}$$

this looks like an inductor $j\omega L$

$$\beta = \frac{2\pi}{\lambda}$$

for $l = \frac{\lambda}{4}$ $\quad \beta l \to \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$.

Then $Z_{IN}(z=0) = jZ_0 \tan \frac{\pi}{2} \to \infty$ <u>this is an open</u>

suppose $Z_L = \infty$ an open $\quad$ this looks like a capacitor $\frac{1}{j\omega C} = \frac{-j}{\omega C}$

$$Z_{IN}(z=0) = Z_0 \frac{\cancel{\hat{Z}_L}}{j\cancel{\hat{Z}_L} \tan \beta l} = \boxed{-j\frac{Z_0}{\tan \beta l}} \to 0 \quad \text{a } \underline{\text{short}}$$

Note that these are only for a.c. impedances
<u>NOT</u> for d.c,

A quarter wavelength line transforms everything

Consider $\beta l \to \frac{\pi}{2}$ $\quad Z_L = j\omega L$

$$Z_{IN}(z=0) = Z_0 \frac{j\omega L + jZ_0 \tan\frac{\pi}{2}}{Z_0 + j(j\omega L)\tan\frac{\pi}{2}} \to Z_0 \frac{\cancel{j} Z_0}{\cancel{j}(j\omega L)} = -j\frac{Z_0^2}{\omega L},$$

looks like a capacitor.

This also holds true for any multiple of $\frac{\lambda}{2}$ plus $\frac{\lambda}{4}$.

$$Z_{IN}(z=0) = Z_0 \frac{j\omega L + jZ_0 \tan \beta l}{Z_0 + j(j\omega L)\tan \beta l}$$