Waves

Up to now, with the notable exception of Faraday's Law, all our field equations were time independent and electric and magnetic fields did not interact. It was Maxwell who recognized the unity of electrostatics and magnetostatics and formulated the equations which bear his name. In fact it is this unity which tightly couples the electric and magnetic fields that gives rise to electromagnetic waves.

First we will derive the wave equation for homogeneous, source free space starting from Maxwell's equations. Using the script quantities $\mathcal{E}, \mathcal{D}, \mathcal{H}, \mathcal{B}$, etc. to represent time-dependent quantities, Maxwell's equations are

$$\nabla \times \underline{\mathcal{E}} = -\frac{\partial \mathcal{B}}{\partial t}$$

$$\nabla \times \underline{\mathcal{H}} = \mathcal{J} + \frac{\partial \mathcal{D}}{\partial t}$$

$$\nabla \cdot \underline{\mathcal{B}} = 0$$

$$\nabla \cdot \underline{\mathcal{D}} = 0$$

As our first step in the solution we will write all equations in terms of only $\mathcal{E}$ and $\mathcal{H}$ using the constitutive relationships $\underline{\mathcal{D}} = \epsilon \underline{\mathcal{E}}$; $\underline{\mathcal{B}} = \mu \underline{\mathcal{H}}$ and $\mathcal{J} = \sigma \underline{\mathcal{E}}$

$$\nabla \times \underline{\mathcal{E}} = -\mu \frac{\partial \underline{\mathcal{H}}}{\partial t}$$

$$\nabla \times \underline{\mathcal{H}} = \sigma \underline{\mathcal{E}} + \epsilon \frac{\partial \mathcal{E}}{\partial t}$$

$$\nabla \cdot \mathcal{H} = 0$$

$$\nabla \cdot \mathcal{E} = 0$$

We can now mathematically manipulate these equations and arrive at a single set of two equations based upon the $\mathcal{E}, \mathcal{H}$ coupling seen in the above equations. Taking the curl of each of the first two equations we get:

$$\nabla \times \nabla \times \mathcal{E} = -\mu \underline{\nabla} \times \frac{\partial \mathcal{H}}{\partial t} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathcal{H})$$

$$\nabla \times \nabla \times \mathcal{H} = \sigma(\nabla \times \mathcal{E}) + \epsilon \frac{\partial}{\partial t}(\nabla \times \mathcal{E})$$

note that the order of time and space differentiation was reversed on the right-hand side. Now, substituting for $\nabla \times \mathcal{H}$ and $\nabla \times \mathcal{E}$ from our original equations

$$\nabla \times \nabla \times \mathcal{E} = -\mu \frac{\partial}{\partial t}\left(\sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t}\right)$$

$$\nabla \times \nabla \times \mathcal{H} = \sigma\left(-\mu \frac{\partial \mathcal{H}}{\partial t}\right) + \epsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \mathcal{H}}{\partial t}\right)$$

And further re-writing the right-hand sides

$$\nabla \times \nabla \times \mathcal{E} = -\mu\sigma \frac{\partial \mathcal{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \mathcal{H} = -\mu\sigma \frac{\partial \mathcal{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathcal{H}}{\partial t^2}$$

What we have done is written the electromagnetic field unity between electric and magnetic fields as <u>two</u> equations in just $\mathcal{E}$ and $\mathcal{H}$ with second derivatives in time. Before trying to understand what these equations mean we will need to simplify the left-hand sides of the above equations. by the vector identity

$$\nabla \times \nabla \times \underline{A} = \nabla(\nabla \cdot \underline{A}) - \nabla^2 \underline{A}$$

In the case of $\mathcal{E}$ and $\mathcal{H}$ $\nabla \cdot \underline{\mathcal{E}} = \nabla \cdot \underline{\mathcal{H}} = 0$ and the left hand sides reduce to $-\nabla^2 \mathcal{E}$ and $-\nabla^2 \mathcal{H}$ respectively.

$$\nabla^2 \mathcal{E} = \mu\sigma \frac{\partial \mathcal{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$\nabla^2 \mathcal{H} = \mu\sigma \frac{\partial \mathcal{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathcal{H}}{\partial t^2}$$

At this point we have a set of equations but have no idea of what these equations mean. To do that let's look at some solutions to these so called wave equations or vector Hemholtz equations

We will first look at general solutions of these equations in a source free region of space. In such a region there are no currents ($\sigma = 0$) and no charge ($\rho = 0$). The only change this has on our wave equations is that the first derivative terms disappear leaving us with

$$\nabla^2 \mathcal{E} = \mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$\nabla^2 \mathcal{H} = \mu\epsilon \frac{\partial^2 \mathcal{H}}{\partial t^2}$$

We can put these in a more conventional form by noting that the speed of light in the medium $\mu\epsilon$ is given by

$$c = \frac{1}{\sqrt{\mu\epsilon}} \qquad \left(\text{in vacuum } c = \frac{1}{\sqrt{\mu_0\epsilon_0}} \approx 3\times 10^8 \text{ m/sec}\right)$$

so that we may re-write the above equations as

$$\boxed{\begin{aligned} \nabla^2 \underline{\mathcal{E}} - \frac{1}{c^2}\frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} &= 0 \\ \nabla^2 \underline{\mathcal{H}} - \frac{1}{c^2}\frac{\partial^2 \underline{\mathcal{H}}}{\partial t^2} &= 0 \end{aligned}}$$

Now let's look at simplified solutions of the above equations. Note that

$$\begin{aligned} \nabla^2 \underline{\mathcal{E}} = &\left(\frac{\partial^2 \mathcal{E}_x}{\partial x^2} + \frac{\partial^2 \mathcal{E}_x}{\partial y^2} + \frac{\partial^2 \mathcal{E}_x}{\partial z^2}\right) \underline{a_x} \\ &+ \left(\frac{\partial^2 \mathcal{E}_y}{\partial x^2} + \frac{\partial^2 \mathcal{E}_y}{\partial y^2} + \frac{\partial^2 \mathcal{E}_y}{\partial z^2}\right) \underline{a_y} \\ &+ \left(\frac{\partial^2 \mathcal{E}_z}{\partial x^2} + \frac{\partial^2 \mathcal{E}_z}{\partial y^2} + \frac{\partial^2 \mathcal{E}_z}{\partial z^2}\right) \underline{a_z} \end{aligned}$$

This is a complex express. For our simplified solution let's let $\mathcal{E}_y = \mathcal{E}_z = 0$ and assume it is a function of only one coordinate, say z. Then

$$\nabla^2 \underline{\mathcal{E}} = \frac{\partial^2 \mathcal{E}_x}{\partial z^2}$$

We will see later what these simplifying assumptions really meant.

Substituting this result back into our wave equation we get a simplified one-dimensional wave equation

$$\frac{\partial^2 \mathcal{E}_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \mathcal{E}_x}{\partial t^2}$$

It turns out that <u>this</u> equation has a very simple general solution

$$\mathcal{E}_x = f(z-ct)$$

where f can be <u>ANY</u> function. Let's check this. By the chain rule:

$$\frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial f(z-ct)}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial z} = f'(z-ct)$$

$$\frac{\partial^2 \mathcal{E}_x}{\partial z^2} = \frac{\partial f'(z-ct)}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial z} = f''(z-ct)$$

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\partial f(z-ct)}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial t} = -c f'(z-ct)$$

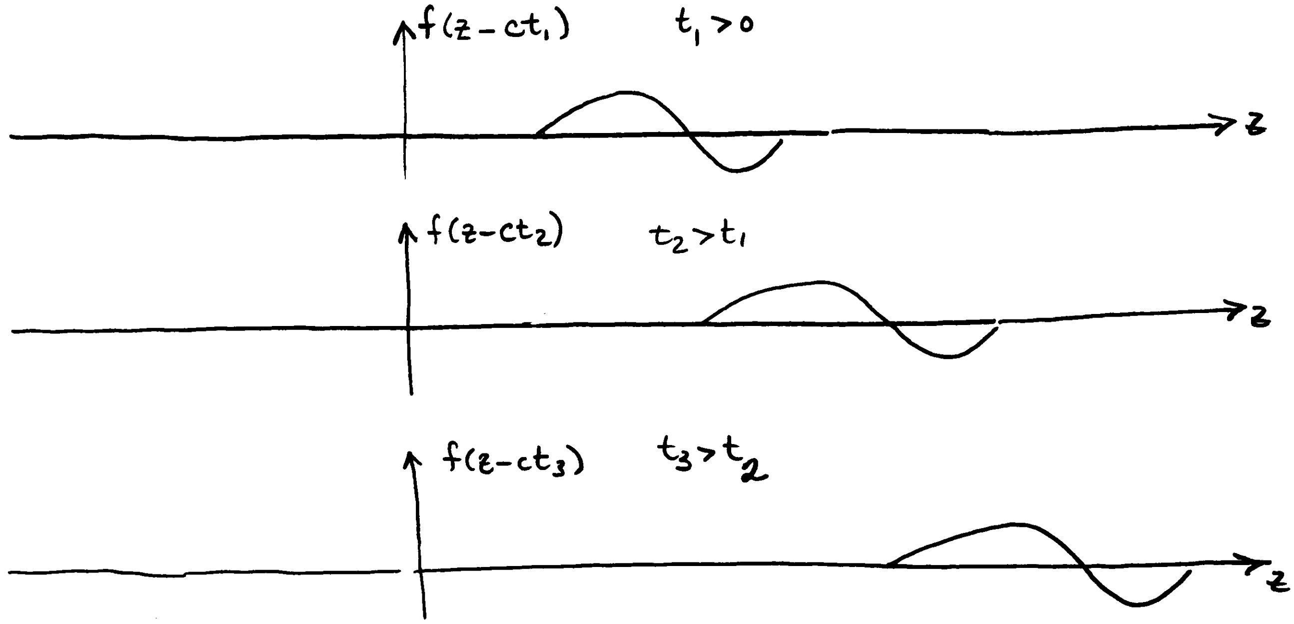
$$\frac{\partial^2 \mathcal{E}}{\partial t^2} = \frac{\partial \left(-cf'(z-ct)\right)}{\partial (z-ct)}\frac{\partial (z-ct)}{\partial t} = +c^2 f''(z-ct)$$

Therefore, our simplified 1-D wave equation becomes

$$f''(z-ct) = \frac{1}{c^2}\left(c^2 f''(z-ct)\right),$$

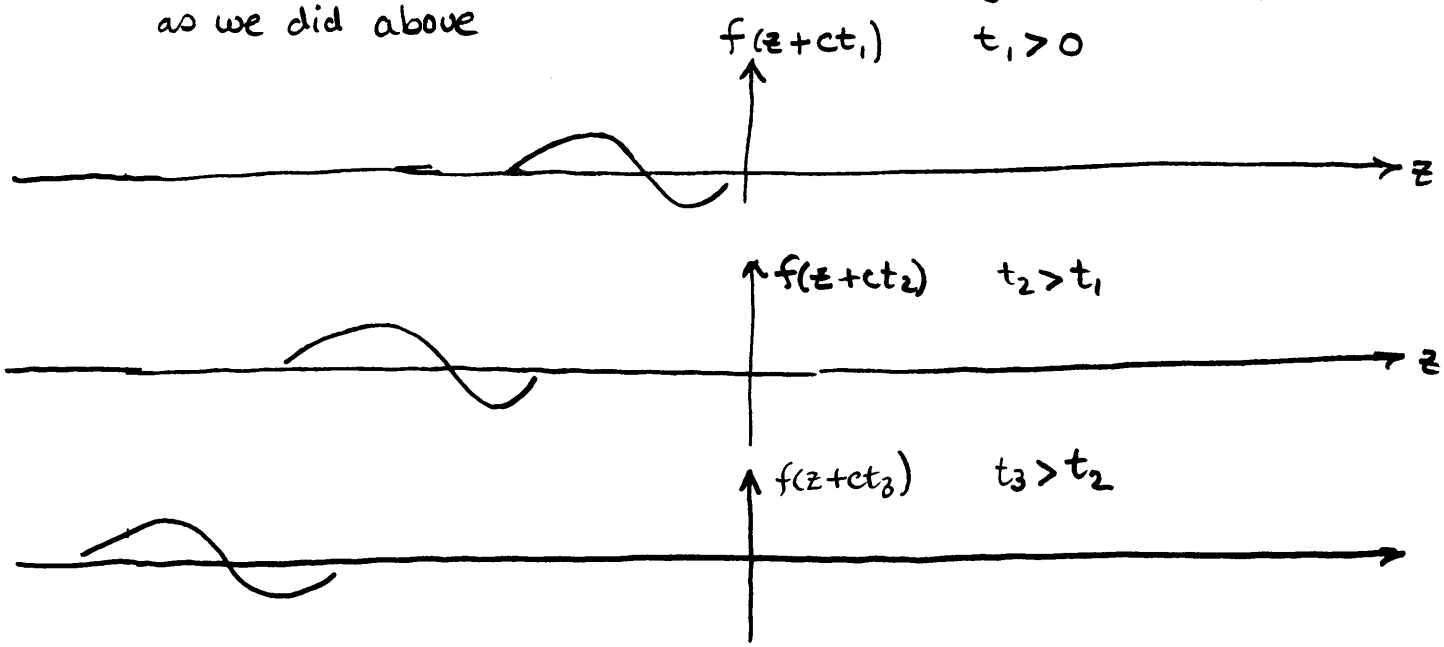
an identity. What this means is best illustrated by a figure. Assume f (z) has the simple form shown below:

If we plot $f(z-ct)$ for various t we get the following results.



As you see $f(z)$ only moves along the z-axis it does not change its form at all. This ability to move without changing form is what we usually mean by wave propagation.

Before we look at some other aspects of our solution, note that $f(z+ct)$ is also a valid solution. Try it! Plotting $f(z+ct)$ as we did above



Note that there is NO attenuation of the wave; however this solution is for $\sigma = 0$. Suppose $\sigma \neq 0$.
The wave equations are then:

$$\nabla^2 \mathcal{E} - \mu\sigma \frac{\partial \mathcal{E}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathcal{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathcal{H} - \mu\sigma \frac{\partial \mathcal{H}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \mathcal{H}}{\partial t^2} = 0$$

Note that $\nabla \cdot \mathcal{E} = 0$ even though there are charges in motion. These charges are a conduction current due to the field $\mathcal{E}$. Any free charge present in a good conductor will decay to zero in a very short time since the relaxation time for a good conductor is so small.

Basically, returning to the above equations, the second term in both equations is a damping term which results in an exponential decay of the wave as it moves away from its source.

Power Flow

It should come as no suprise that we can transport energy through space via electromagnetic waves. Recall that for static fields we determined that one could store energy in the field. For a moving or traveling wave we can say that moving fields represent a moving packet of energy.

It is at this point that we get result akin to classical circuit theory. In circuit theory power is given as the product of volts and amperes. For distributed fields E is in units of volts/meter and H in units of amperes/meter. It thus seems natural to expect that power for distributed fields will be given by the product of E and H. This, however, gives us a power density $P = \text{volts}\cdot\text{amperes}/m^2$

Power flow is a vector quantity and the full vector expression is given by

$$\underline{P} = \underline{E} \times \underline{H}$$

To show that $\underline{P}$ represents the power or energy flow for E and H let us consider the divergence of $\underline{P}$ over a closed surface S. Expanding the above equation, the total power thru S must be given by

$$P_{total} = \oint_S \underline{P} \cdot d\underline{S} = \oint_S \underline{E} \times \underline{H} \cdot d\underline{S}$$

Converting this to a volume integral

$$P_{total} = \int \nabla \cdot (\underline{E} \times \underline{H})\, dv$$

We now expand $\nabla \cdot (\underline{E} \times \underline{H})$ as

$$\nabla \cdot (\underline{E} \times \underline{H}) = \underline{H} \cdot \nabla \times \underline{E} - \underline{E} \cdot \nabla \times \underline{H}$$

From Maxwell's Equations

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{H} = \frac{\partial \underline{D}}{\partial t} + \sigma \underline{E}$$

Substituting into the above equation

$$\nabla \cdot (\underline{E} \times \underline{H}) = -\underline{H} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot \frac{\partial \underline{D}}{\partial t} - \sigma \underline{E} \cdot \underline{E}$$

For an isotropic region of space where μ, σ, ϵ are constants we get

$$\nabla \cdot (\underline{E} \times \underline{H}) = -\mu \underline{H} \cdot \frac{\partial \underline{H}}{\partial t} - \epsilon \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} - \sigma |E|^2$$

But $\quad \underline{H} \cdot \frac{\partial \underline{H}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} |H|^2$ and $E \cdot \frac{\partial E}{\partial t} = \frac{1}{2} \frac{\partial |E|^2}{\partial t}$

$$\nabla \cdot (E \times H) = -\frac{1}{2} \mu \frac{\partial |H|^2}{\partial t} - \frac{1}{2} \epsilon \frac{\partial}{\partial t} |E|^2 - \sigma |E|^2$$

Putting this result into our last integral we get

$$P_{total} = \int \left\{ -\frac{\partial}{\partial t} \left[\frac{1}{2} \mu |H|^2 + \frac{1}{2} \epsilon |E|^2 \right] - \sigma |E|^2 \right\} dv$$

To interpret this result we first note that the minus signs indicate that power is being lost (i.e. is flowing out of the original surface S). The term $\sigma |E|^2$ is nothing more than a ohmic loss due to the conduction current $\sigma \underline{E}$. Finally, $\frac{1}{2} \mu |H|^2 + \frac{1}{2} \epsilon |E|^2$ is the sum of the energies in the electric and magnetic fields just as for static fields.

This result is extremely important. The vector

$$\underline{P} = \underline{E} \times \underline{H}$$

is called the Poynting vector and is the instaneous flow of power (both magnitude and direction) per unit area. Note that energy flow is perpendicular to both the E and H fields.

Plane Waves

Let us assume that the only electric field component present is E_x which varies sinusoidally, i.e.

$$\underline{E_x} = E_0 \sin(z - ct)\, \underline{a_x}$$

If we further assume that the time variation occurs at an angular frequency we get

$$\underline{E} = E_0 \sin \frac{\omega}{c}(z - ct) \underline{a_x} = E_0 \sin\left(\frac{\omega}{c} z - \omega t\right) \underline{a_x}$$

From Faraday's law we get

$$\nabla \times \underline{E} = -\mu_0 \frac{\partial \underline{H}}{\partial t}$$

$$\text{But } \nabla \times \underline{E} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = \underline{a_y} \frac{\partial E_x}{\partial z} = -\underline{a_y}\, E_0 \frac{\omega}{c} \cos\left(\frac{\omega}{c} z - \omega t\right)$$

and, after substitution and integrating with respect to time,

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\left(-E_0 \frac{\omega}{c} \cos\left(\frac{\omega}{c} z - \omega t\right)\right)$$

$$\frac{\partial H_y}{\partial t} = +\frac{E_0 \omega}{\mu_0 c} \cos\left(\frac{\omega}{c} z - \omega t\right)$$

$$H_y = \frac{E_0 \omega}{\mu_0 c}\left(\frac{+\sin\left(\frac{\omega}{c} z - \omega t\right)}{\omega}\right)$$

$$H_y = \frac{E_0}{\mu_0 c} \sin\left(\frac{\omega}{c} z - \omega t\right)$$

$$H_y = \frac{E_0}{\sqrt{\frac{\epsilon_0}{\mu_0}}} \sin\left(\frac{\omega}{c} z - \omega t\right)$$

This quantity $\sqrt{\frac{\epsilon_0}{\mu_0}}$ is called the intrinsic impedance of free space and is denoted by Z_0. The value of Z_0 is

$$Z_0 = 120\pi = 377\,\Omega$$

Our total solution is then

$$\underline{E} = E_0 \sin\left(\frac{\omega}{c}z - \omega t\right) \underline{a_x}$$

$$\underline{H} = \frac{E_0}{Z_0} \sin\left(\frac{\omega}{c}z - \omega t\right) \underline{a_y}$$

This is a uniform plane electromagnetic wave since even though the wave moves in the z-direction, $\underline{E}$ and $\underline{H}$ are in a transverse plane. To show that this is consistent let's examine the poynting vector

$$\underline{P} = \underline{E} \times \underline{H} = \frac{E_0^2}{Z_0} \sin^2\left(\frac{\omega}{c}z - \omega t\right) \underbrace{\underline{a_x} \times \underline{a_y}}_{\underline{a_z}}$$

This shows that the wave is moving or transporting power in the z-direction. This is the instaneous power. To get an expression for the time-average power flow per unit are we integrate over time to get.

$$\overline{P} = \frac{1}{T}\int_0^T \frac{E_0^2}{Z_0} \sin^2\left(\frac{\omega}{c}z - \omega t\right) dt \qquad \text{where } T = \frac{1}{2\omega}$$

$$= \frac{1}{T}\frac{E_0^2}{Z_0}\int_0^T \frac{1}{2}\left\{1 - \cos 2\left(\frac{\omega z}{c} - \omega t\right)\right\} dt$$

$$= \frac{1}{T}\frac{E_0^2}{Z_0}\,\frac{1}{2}\left[t - \frac{1}{2\omega}\sin 2\left(\frac{\omega z}{c} - \omega t\right)\right]_0^T$$

As the integral is over one period the sine term contribution is zero and our result is

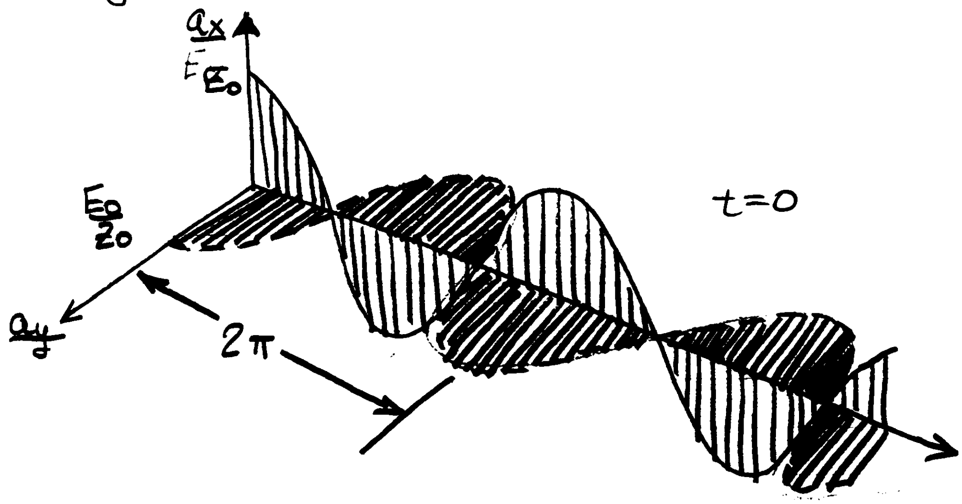
$$P = \frac{1}{2}\frac{E_0^2}{Z_0}.$$

This solution allows us to examine some general properties of wave type solutions to Maxwell's Equations. Starting with our basic solutions

$$\underline{E} = E_0 \sin\left(\frac{\omega}{c} z - \omega t\right) \underline{a_x}$$

$$\underline{H} = \frac{E_0}{Z_0} \sin\left(\frac{\omega}{c} z - \omega t\right) \underline{a_y}$$

Plotting our result.



$\underline{E}$ and $\underline{H}$ vary with a temporal frequency ω. They also vary with a spatial frequency $\beta = \frac{\omega}{c}$. To illustrate, examine the above diagram for $t=0$. The wave repeats itself every 2π. If we call this distance the wavelength

$$\frac{\omega}{c}\lambda = \frac{\beta}{\lambda} = 2\pi$$

Note that the wavelength λ is the distance over which the waveform repeats, β is the frequency of that spatial variation.

Define $\varphi = \beta z - \omega t$ as the wave phase function.

$\varphi =$ constant defines a point of constant phase on the wave. Pick this constant to be zero for simplicity. Then,

$$\varphi = \beta z - \omega t = 0$$

$$\beta z = \omega t$$

$$z = \frac{\omega}{\beta} t$$

This represents the spatial movement of a point of constant phase with time. In general, this is called the phase velocity v_ϕ.

$$v_\phi \triangleq \frac{dz}{dt} = \frac{\omega}{\beta}.$$

If we have two waves of a similar frequency, i.e. ω_1, ω_2 and β_1, β_2 where $|\omega_1 - \omega_2| << |\omega_1|$ and $|\beta_1 - \beta_2| << |\beta_2|$.

Let us examine their sum

$$S(t) = \sin(\beta_1 z - \omega_1 t) + \sin(\beta_2 z - \omega_2 t)$$

$$S(t) = 2 \underbrace{\sin\left(\frac{\beta_1+\beta_2}{2} z - \frac{\omega_1+\omega_2}{2} t\right)}_{\text{this is a very high frequency - the carrier}} \underbrace{\cos\left(\frac{\beta_1-\beta_2}{2} z - \frac{\omega_1-\omega_2}{2} t\right)}_{\text{this is a low frequency modulation and is the frequency at which information propagates}}$$

define $$\varphi_{mod} = \frac{\beta_1 - \beta_2}{2} z - \frac{\omega_1 - \omega_2}{2} t$$

For constant φ_{mod} pick $\varphi_{mod} = 0$ and examine velocity of propagation.

$$\varphi_{mod} = \frac{\beta_1 - \beta_2}{2} z - \frac{\omega_1 - \omega_2}{2} t = 0$$

$$\frac{\beta_1 - \beta_2}{2} z = \frac{\omega_1 - \omega_2}{2} t$$

$$z = \frac{\omega_1 - \omega_2}{\beta_1 - \beta_2} t$$

We see that the modulation propagates at velocity v_g given by

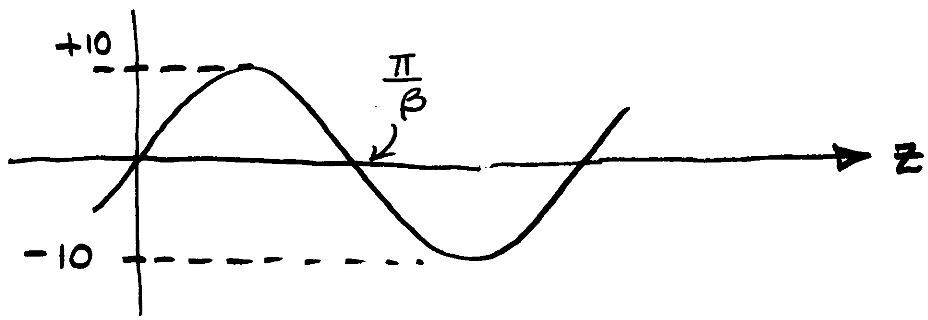
$$v_g = \frac{\omega_1 - \omega_2}{\beta_1 - \beta_2}.$$

In the limiting case of these differences being small

$$v_g = \frac{d\omega}{d\beta}$$

Example:

A traveling wave is described by $y = 10 \sin(\beta z - \omega t)$. Sketch the wave at $t=0$ and at $t=t_1$ when it has advanced $\lambda/8$. The wave velocity $\beta = 3\times10^8$ m/sec, and $\omega = 10^6$/sec.



zero at $\beta z - \omega t = 0$ if $t=0$ $z=0$

zero again at $\beta z = \pi$ $\therefore z = \frac{\pi}{\beta} = 1.047\times10^{-8}$ m.

How fast does this wave travel $\lambda/8$

As before $\beta z - \omega t_1 = 0$

$$\omega t_1 = \frac{\beta z}{\omega} = \frac{(3\times10^8 \text{ m})\left(\frac{1}{235.6\text{ m}}\right)}{10^6/\text{sec}}$$

Example:

The approximate radiation fields of an antenna are

$$H_\phi = \frac{1}{r}\sin\theta\cos(\omega t - \beta r)$$

$$E_\theta = 377 H_\phi$$

Determine the energy flow in watts out of the volume whose boundary is the spherical surface S of radius r centered at the origin.

$$\underline{P} = \underline{E}\times\underline{H} = E_\theta H_\phi\, \underline{a_r}$$

$$= \frac{377}{r^2}\sin^2\theta\cos^2(\omega t - \beta r)\,\underline{a_r}$$

$$P_{total} = \oint_S (\underline{E}\times\underline{H})\cdot \underline{dS} = \oint \frac{377}{\cancel{r^2}}\sin^2\theta\cos^2(\omega t-\beta r)\,\underline{a_r}\cdot\cancel{r^2}\sin\theta\, d\theta\, d\phi\, \underline{a_r}$$

$$= 377\cos^2(\omega t-\beta r)\cdot\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi$$

$$= 377\cos^2(\omega t-\beta r)\; 2\pi\left[-\frac{1}{3}\cos\theta(\sin^2\theta+2)\right]\Big|_0^{\pi}$$

$$= 377\cos^2(\omega t-\beta r)\; 2\pi\left(-\frac{1}{3}\left[\overset{-1}{\cos\pi} - \overset{1}{\cos 0}\right][2]\right)$$

$$= 377\cdot 2\pi\cdot\frac{2}{3}\cdot 2\,\cos^2(\omega t-\beta r)$$

$$P_{total} = 3158.3\cos^2(\omega t-\beta r)$$

Time averaged power is

$$\overline{P_{total}} = 3158.3\cdot\frac{1}{2} = 1579.2 \text{ watts.}$$

Motivation for phasor notation.

We have seen that $\mathcal{E}$ & H are simply related to each other. But the time dependent terms are hard to deal with, particularly if we are interested in say, digital pulses on a transmission

what do we do we Fourier transform everything

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{j\omega t}\, dt$$

and vice versa...

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j i\omega t}\, dt$$

Thus, let's solve Maxwell's equations for each frequency component then transform, convolve, etc. as needed:

Maxwell's equations

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$$
$$\nabla \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t}$$
$$\nabla \cdot \underline{D} = 0$$
$$\nabla \cdot \underline{B} = 0$$

If we let
$$\underline{E} = Re[\hat{\underline{E}} e^{j\omega t}]$$
$$\underline{H} = Re[\hat{\underline{H}} e^{j\omega t}]$$
etc.

$$\nabla \times \hat{\underline{E}} e^{j\omega t} = -(j\omega)\mu \hat{\underline{H}} e^{j\omega t}$$
$$\nabla \times \hat{\underline{H}} e^{j\omega t} = \sigma \hat{\underline{E}} e^{j\omega t} + (j\omega)\epsilon \hat{\underline{E}} e^{j\omega t}$$

Dropping the $e^{j\omega t}$ terms

$$\nabla \times \hat{\underline{E}} = -j\omega\mu \hat{\underline{H}}$$
$$\nabla \times \hat{\underline{H}} = (j\omega\epsilon + \sigma) \hat{\underline{E}}$$

Taking the divergence of each side

$$\nabla \times \nabla \times \hat{\underline{E}} = -j\omega\mu(\nabla \times \hat{\underline{H}})$$
$$\nabla \times \nabla \times \hat{\underline{H}} = (j\omega\epsilon + \sigma)(\nabla \times \hat{\underline{E}})$$

but
$$\nabla \times \nabla \times \hat{\underline{E}} = \nabla(\underbrace{\nabla \cdot \hat{\underline{E}}}_{0}) - \nabla^2 \hat{\underline{E}}$$
$$\nabla \times \nabla \times \hat{\underline{H}} = \nabla(\underbrace{\nabla \cdot \underline{H}}_{0}) - \nabla^2 \underline{H}$$

$$-\nabla^2 \hat{\underline{E}} = -j\omega\mu(j\omega\epsilon + \sigma) \hat{\underline{E}}$$
$$-\nabla^2 \hat{\underline{H}} = (j\omega\epsilon + \sigma)(-j\omega\mu) \hat{\underline{H}}$$

These are the vector Hemholtz equations

define $\quad \gamma^2 = j\omega\mu(j\omega\epsilon + \sigma)$

Then $\quad \nabla^2 \hat{\underline{E}} = \gamma^2 \hat{\underline{E}}$

$$\nabla^2 \hat{\underline{H}} = \gamma^2 \hat{\underline{H}}$$

Let us restrict ourselves to a one dimensional solution

suppose $\hat{\underline{E}} = \hat{E}_x(z)\underline{a}_x$

Then $\quad \nabla^2 \hat{\underline{E}} = \dfrac{\partial^2 \hat{E}_x}{\partial z^2} \underline{a_x}$

and our first equation becomes

$$\frac{\partial^2 \hat{E}_x}{\partial z^2} = \gamma^2 \hat{E}_x$$

Since E_x is only a function of z

$$\frac{d^2 \hat{E}_x}{dz^2} = \gamma^2 \hat{E}_x$$

which has solution $\quad \hat{E}_x = \hat{E}^+ e^{-\gamma z} + \hat{E}^- e^{+\gamma z}$

where $\hat{E}^+, \hat{E}^-$ are complex constants

Now that I have $\hat{\underline{E}}$, how about $\hat{\underline{H}}$

First, it __MUST__ satisify Faraday's Law

$$\nabla \times \hat{\underline{E}} = -j\omega\mu \hat{\underline{H}}$$

$$\nabla \times \hat{\underline{E}} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(z) & 0 & 0 \end{vmatrix} = \underline{a_y} \frac{\partial E_x}{\partial z}$$

This tells us that $\hat{\underline{H}}$ must be in the $\underline{a_y}$ direction

so let $\hat{\underline{H}} = \hat{H}_y(z)\,\underline{a}_y$

This has a similar solution & equation

$$\frac{d^2\hat{H}_y}{dz^2} = \gamma^2 \hat{H}_y$$

with solutions $\hat{H}_y = \hat{H}_y^+ e^{-\gamma z} + \hat{H}_y^- e^{+\gamma z}$

We can get relationships between these various coefficients by using Faraday's & Ampère's Law

Faraday's Law: $\frac{\partial E_x}{\partial z} = -j\omega\mu H_y$

$$\therefore \frac{\partial}{\partial z}\left[\hat{E}^+ e^{-\gamma z} + \hat{E}^- e^{+\gamma z}\right] = -j\omega\mu\left[\hat{H}^+ e^{-\gamma z} + \hat{H}^- e^{+\gamma z}\right]$$

$$-\gamma \cdot \hat{E}^+ e^{-\gamma z} + \gamma \hat{E}^- e^{+\gamma z} = -j\omega\mu \hat{H}^+ e^{-\gamma z} - j\omega\mu \hat{H}^- e^{+\gamma z}$$

Equating coefficients

$$-\gamma\hat{E}^+ = -j\omega\mu\hat{H}^+ \qquad +\gamma\hat{E}^- = -j\omega\mu\hat{H}^-$$

$$\frac{\hat{E}^+}{\hat{H}^+} = \frac{j\omega\mu}{\gamma} \qquad \frac{\hat{E}^-}{\hat{H}^-} = -\frac{j\omega\mu}{\gamma}$$

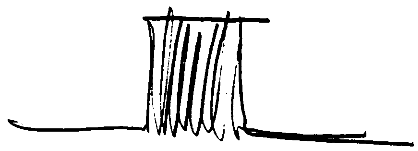
define $\hat{\eta} = \frac{j\omega\mu}{\gamma}$

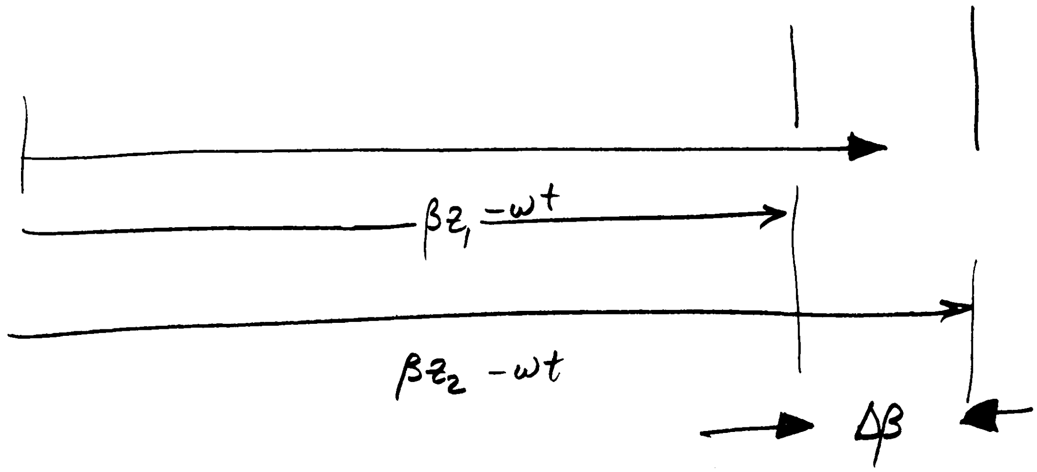
Then, solutions are

$$\underline{\hat{E}} = \left(\hat{E}^+ e^{-\gamma z} + \hat{E}^- e^{+\gamma z}\right) \underline{a_x}$$

$$\underline{\hat{H}} = \left(\hat{H}^+ e^{-\gamma z} + \hat{H}^- e^{-\gamma z}\right) \underline{a_y}$$

$$= \left(\frac{\hat{E}^+}{\eta} e^{-\gamma z} - \frac{\hat{E}^-}{\eta} e^{+\gamma z}\right) \underline{a_y}$$





$$\Delta\beta \leq \frac{2\pi}{10}$$

then $$\frac{\beta}{\omega} c^2 = \frac{\Delta\omega}{\Delta\beta} = \frac{6.28 \times 10^9}{2\pi/10} = 10^{10}$$

$$\boxed{\frac{\beta}{\omega} c^2 = 10^{10}}$$

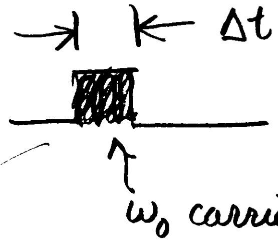
$$\frac{\sqrt{\dfrac{\omega^2 - \omega_p^2}{c^2}}\, c^2}{\omega} = 10^{10}$$

$$\sqrt{\frac{\omega^2 - \omega_p^2}{\omega^2}} = \frac{10^{10}}{c} = \frac{10^{10}}{3 \times 10^8}$$

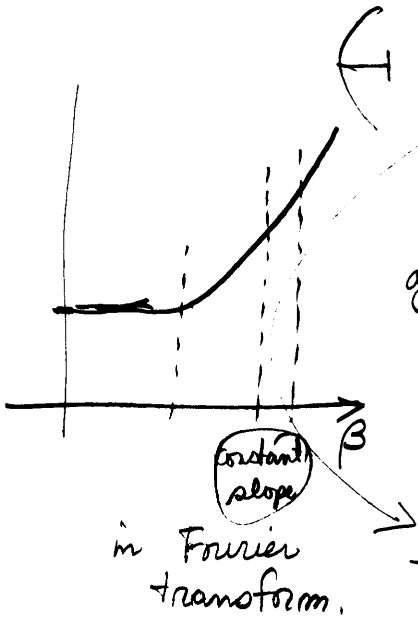
$$\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{3} \times 10^2$$

$$1 - \frac{\omega_p^2}{\omega^2} = \frac{1}{9} \times 10^4$$

Example: Satellite communications.

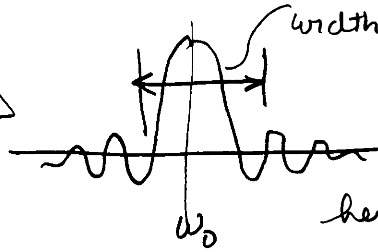


$\Delta t = 1\,\text{nsec}$
(10^{-9} sec)



going through atmosphere.

$$\omega^2 = \omega_p^2 + \beta^2 c^2$$



hence $\Delta\omega \approx \dfrac{2\pi}{\Delta t}$

For communications to occur
$\Delta\omega < \dfrac{2\pi}{\Delta t}$

if $\Delta t = 10^{-9}$ sec $\quad \Delta\omega < \dfrac{2\pi}{\Delta t} = 6.28\times10^{9}$ i.e. the pulse can't smear out.

How do we pick the carrier frequency?

$$v_g = \frac{\Delta\omega}{\Delta\beta}$$ ← picked what $\Delta\omega$ must be....

but $\cancel{2}\omega\, d\omega = \cancel{2}\beta\, d\beta\, c^2$

$$\frac{d\omega}{d\beta} = \frac{\beta}{\omega}c^2$$

$$\therefore \quad \frac{\beta}{\omega}c^2 = \frac{\Delta\omega}{\Delta\beta}$$

Electromagnetic waves in the ionosphere

$$\omega^2 = \omega_p^2 + c^2\beta^2$$

where $\omega_p = 2\pi \nu_p$

where $\nu_p \approx 20$ MHz

this accounts for the atmosphere reflecting frequencies below about 20 MHz

$$v_\phi = \frac{\omega}{\beta} = \frac{\sqrt{\omega_p^2 + c^2\beta^2}}{\beta} = \sqrt{\frac{\omega_p^2 + c^2\beta^2}{\beta^2}}$$

$$= \sqrt{c^2 + \frac{\omega_p^2}{\beta^2}} = c\sqrt{1 + \frac{\omega_p^2}{c^2\beta^2}} \geqslant c$$

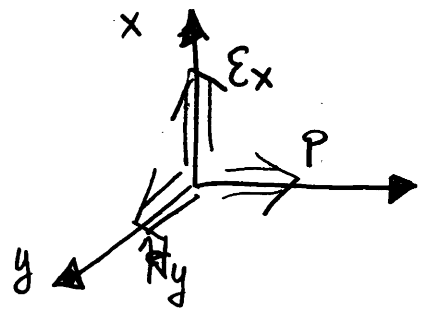
$$v_g = \frac{d\omega}{d\beta} \qquad \therefore \quad \cancel{2}\omega \, d\omega = c^2 \cancel{2}\beta \, d\beta$$

$$\frac{d\omega}{d\beta} = \frac{c^2}{\omega}\beta$$

note that $v_\phi \cdot v_g = \frac{\omega}{\beta} \, c^2 \frac{\beta}{\omega} = c^2$

This is always true ∎

Simple review



Solution of vector Hemholtz equation for $\mathcal{E}$

$$\nabla^2 \underline{\mathcal{E}} = \mu\sigma \frac{\partial \underline{\mathcal{E}}}{\partial t} + \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2}$$

for lossless medium $\sigma = 0$

$$\boxed{\nabla^2 \underline{\mathcal{E}} = \mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} = \boxed{\frac{1}{v^2}\frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2}}}$$

velocity of propagation

plane wave solution

$\underline{\mathcal{E}}(x,y,z,t) = \underline{a_x}\,\mathcal{E}_x(z,t)$ ① no transverse dependence
② one transverse component.

sinusoidal

$$\underline{a_x}\,\underline{\mathcal{E}_x}(z,t) = \mathrm{Re}\left[\underline{a_x}\,\hat{E}_x(z)\,e^{j\omega t}\right]$$

[we will return to solution for $\hat{E}_x(z)$ later]

we get $\mathcal{H}$ exactly the same way but we don't know which direction so we use Faraday's law

$$\nabla \times \underline{\mathcal{E}} = -j\omega\mu\underline{\mathcal{H}}$$

$$\nabla \times \underline{\mathcal{E}} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{E}_x & 0 & 0 \end{vmatrix} = \underline{a_y}\,\frac{\partial \hat{E}_x(z)}{\partial z} - \underline{a_z}\,\cancelto{0}{\frac{\partial \hat{E}_x(z)}{\partial y}}$$

no transverse dependence.

$$\underline{\mathcal{H}} = \underline{a_y}\,\frac{\dfrac{\partial \hat{E}_x(z)}{\partial z}}{-j\omega\mu}$$

let's return to solutions for $\hat{E}_x(z)$

$$\nabla^2 \underline{\mathcal{E}} \rightarrow \frac{d^2\hat{E}_x(z)}{dz^2}\, \underline{a_x}$$

$$\mu\epsilon \frac{\partial^2 \underline{\mathcal{E}}}{\partial t^2} \rightarrow \mu\epsilon(j\omega)^2\, \hat{E}_x(z)\, \underline{a_x}$$

$$\mu\sigma \frac{\partial \underline{\mathcal{E}}}{\partial t} \rightarrow \mu\sigma(j\omega)\, \hat{E}_x(z)\, \underline{a_x}$$

---

vector Hemholtz eqn:

$$\frac{d^2\hat{E}_x(z)}{dz^2} = +j\omega\mu\sigma\, \hat{E}_x(z) - \omega^2\mu\epsilon\, \hat{E}_x(z)$$

$$\frac{d^2\hat{E}_x(z)}{dz^2} = \Big[\underbrace{-\omega^2\mu\epsilon + j\omega\mu\sigma}_{\gamma^2}\Big]\hat{E}_x(z)$$

$$\frac{d^2\hat{E}_x(z)}{dz^2} = \gamma^2\, \hat{E}_x(z)$$

$$\hat{E}_x(z) = \hat{E}_x^{+} e^{-\gamma z} + \hat{E}_x^{-} e^{+\gamma z}$$

for the moment $\gamma = \alpha + j\beta$

$$\hat{E}_x(z) = \hat{E}_x^{+} e^{-\alpha z} e^{-j\beta z} + \hat{E}_x^{-} e^{+\alpha z} e^{+j\beta z}$$

$$\underline{H} = \underline{a_y} \frac{\frac{\partial E_x(z)}{\partial z}}{-j\omega\mu} = \underline{a_y}\left[\frac{\hat{E}_x^{+}(-\gamma)}{-j\omega\mu}\, e^{-\gamma z} + \frac{\hat{E}_x^{-}(+\gamma)}{-j\omega\mu}\, e^{+\gamma z}\right]$$

$$\hat{H}_y = \frac{\hat{E}_x}{\underbrace{\left(\frac{j\omega\mu}{\gamma}\right)}_{\hat{\eta}_+}}\, e^{-\alpha z} e^{-j\beta z} + \frac{\hat{E}_x}{\underbrace{\left(-\frac{j\omega\mu}{\gamma}\right)}_{\hat{\eta}_-}}\, e^{+\alpha z} e^{+j\beta z}$$

$$H_y = \frac{\hat{E}_x}{\hat{\eta}_+} e^{-\alpha z} e^{-j\beta z} + \frac{\hat{E}_x}{\hat{\eta}_-} e^{+\alpha z} e^{+j\beta z}$$

Summary of solutions

$$\hat{E}_x = \hat{E}_x^+ e^{-\alpha z} e^{-j\beta z} + \hat{E}_x^- e^{+\alpha z} e^{+j\beta z}$$

$$\hat{H}_y = \frac{\hat{E}_x^+}{\hat{\eta}_+} e^{-\alpha z} e^{-j\beta z} - \frac{\hat{E}_x}{\hat{\eta}_{\cancel{+}}} e^{+\alpha z} e^{+j\beta z}$$

$$\hat{\eta}_+ = \frac{j\omega\mu}{\gamma}$$

$$\hat{\eta}_- = -\frac{j\omega\mu}{\gamma}$$

For a real problem what are $\hat{E}_x^+, \hat{E}_x^-$ determined by initial conditions & B.C.'s

η_+, η_- determined by materials & γ

$\gamma = \alpha + j\beta$ determined by material

lossless media $\sigma = 0$

$$\gamma^2 = -\omega^2 \mu\epsilon$$

$$\gamma = \pm j\omega\sqrt{\mu\epsilon} \quad \Rightarrow \quad \alpha = 0 \quad \text{no loss}$$

$$\beta = \omega\sqrt{\mu\epsilon}$$

$$\epsilon = \epsilon' - j\epsilon''$$

low loss media:

$$\frac{\sigma}{\omega\epsilon} \ll 1$$

conduction current

displacement current

$$\gamma^2 = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

$$= -\omega^2\mu\epsilon\left[1 + \frac{j\omega\mu\sigma}{-\omega^2\mu\epsilon}\right]$$

$$= -\omega^2\mu\epsilon\left[1 - j\frac{\sigma}{\omega\epsilon}\right]$$

$$\gamma = \pm j\omega\sqrt{\mu\epsilon}\left[1 - j\frac{\sigma}{\omega\epsilon}\right]^{1/2}$$

use Taylor Series

$$\left[1 - j\frac{\sigma}{\omega\epsilon}\right]^{\frac{1}{2}} \approx 1 - \frac{1}{2}j\frac{\sigma}{\omega\epsilon}$$

$$\Rightarrow \gamma = \pm j\omega\sqrt{\mu\epsilon}\left[1 - j\frac{\sigma}{2\omega\epsilon}\right]$$

$$= \pm\left(j\omega\sqrt{\mu\epsilon} + \frac{\cancel{\omega}\sqrt{\mu\epsilon}\,\sigma}{2\cancel{\omega}\epsilon}\right)$$

$$= \pm\left(\frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\,\sigma + j\omega\sqrt{\mu\epsilon}\right)$$

$$\alpha = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\,\sigma$$

$$\beta = \omega\sqrt{\mu\epsilon}$$

If $\epsilon = \epsilon' - j\epsilon''$

$$\gamma^2 = -\omega^2\mu(\epsilon' - j\epsilon'')$$

$$= -\omega^2\mu\epsilon' + j\boxed{\omega^2\mu\epsilon''}$$

looks like σ

so just add

$$\frac{\sigma}{\omega\epsilon'}$$

conducting media:

$$\frac{\sigma}{\omega\epsilon} \gg 1$$

$$\gamma^2 = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

$$= -\omega^2\mu\epsilon\left[1 + \frac{j\omega\mu\sigma}{-\omega^2\mu\epsilon}\right]$$

$$= -\omega^2\mu\epsilon\left[1 - j\frac{\sigma}{\omega\epsilon}\right]$$

$$\gamma = \pm j\omega\sqrt{\mu\epsilon}\left[1 - j\frac{\sigma}{\omega\epsilon}\right]^{\frac{1}{2}}$$

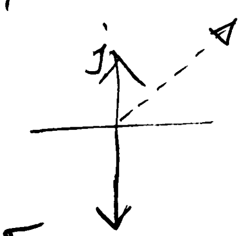
$$\left[1 - j\frac{\sigma}{\omega\epsilon}\right]^{\frac{1}{2}} \approx \sqrt{-j}\sqrt{\frac{\sigma}{\omega\epsilon}}$$

$$\gamma = \pm j\omega\sqrt{\mu\epsilon}\sqrt{-j}\sqrt{\frac{\sigma}{\omega\epsilon}}$$

$$= \pm\sqrt{-j^3}\sqrt{\frac{\omega^2\mu\epsilon\sigma}{\cancel{\omega}\cancel{\epsilon}}}$$

$$= \pm\sqrt{j}\sqrt{\omega\mu\sigma}$$

$$+j = e^{+j\frac{\pi}{2}}$$



$$= \pm\sqrt{\left(e^{+j\frac{\pi}{2}}\right)}\sqrt{\omega\mu\sigma}$$

$$= \pm e^{+j\frac{\pi}{4}}\sqrt{\omega\mu\sigma}$$

$$= \pm\left[\cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right)\right]\sqrt{\omega\mu\sigma}$$

$$= \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\Rightarrow \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} \qquad \beta = \sqrt{\frac{\omega\mu\sigma}{2}}.$$

If you know γ you know η.

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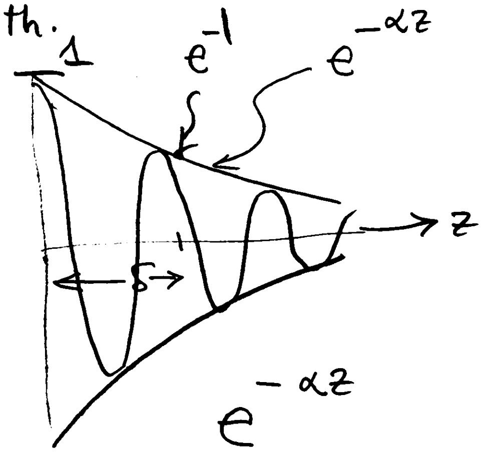
attenuation depth.



e^{-1} so $\alpha\delta = 1$

skindepth.

$\therefore \delta = \frac{1}{\alpha}$

This tells us how good a conductor is

$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\therefore \delta = \sqrt{\frac{2}{\omega\mu\sigma}}$$

as $\sigma \gg 1$

$\delta \ll 1$

so waves do not propagate into conductors.

in general γ, η are complex.

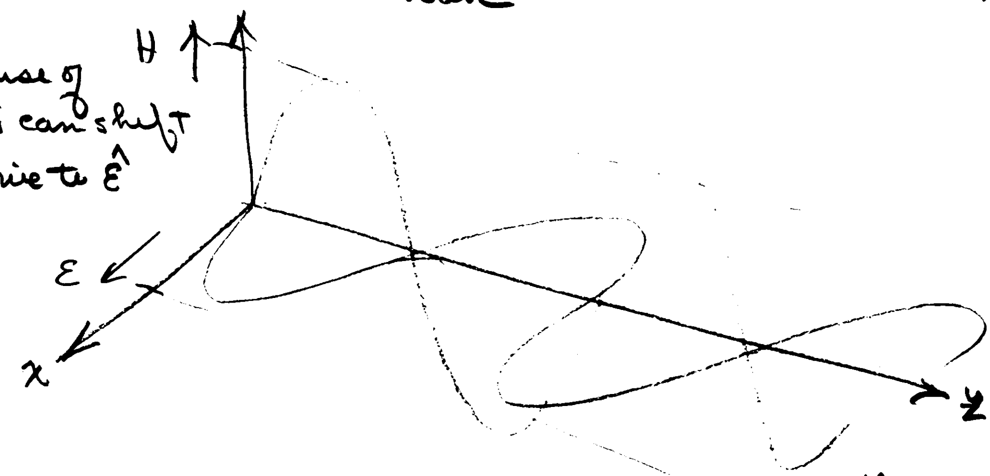
$$\gamma = \alpha + j\beta \leftarrow \text{ for lossless media } \alpha = 0$$
$$\beta = \omega\sqrt{\mu\epsilon} = \omega/c$$
$$\hat{\eta} = \eta e^{j\theta_\eta}$$

$$\hat{E} = \hat{E}^{+} e^{-\alpha z} e^{-j\beta z} + \hat{E}^{-} e^{+\alpha z} e^{+j\beta z}$$

solutions really are

$$\mathcal{E} = \mathcal{E}^{+} e^{-\alpha z} \underbrace{\cos(\omega t - \beta z)}_{\text{+ traveling wave}} + \mathcal{E}^{-} e^{+\alpha z} \underbrace{\cos(\omega t + \beta z)}_{\text{− traveling wave.}}$$

because of $\hat{\eta}$ $\hat{H}$ can shift relative to $\hat{\mathcal{E}}$



for free space $\hat{\eta} = \dfrac{\gamma}{j\omega\mu} = \dfrac{\overset{0}{\alpha} + \overset{\omega\sqrt{\mu\epsilon} = \frac{\omega}{c}}{j\beta}}{j\omega\mu} = \dfrac{j\omega\sqrt{\mu\epsilon}}{j\omega\mu} = \sqrt{\dfrac{\epsilon}{\mu}}$

$\approx 377\Omega$

in a slightly more general case.

$$\hat{E}^{+} = E_m^{+} e^{j\theta_+}$$

$$\hat{E}^{-} = E_m^{-} e^{j\theta_-}$$

$$\underline{E} = \hat{E}_m^{+} e^{-\alpha z} e^{-j\beta z} e^{j\theta_+} + \hat{E}_m^{-} e^{+\alpha z} e^{+j\beta z} e^{j\theta_-}$$

$$\mathcal{E} = E_m^{+} e^{-\alpha z} \cos(\omega t - \beta z + \theta_+) + E_m^{-} e^{\alpha z} \cos(\omega t + \beta z + \theta_-)$$

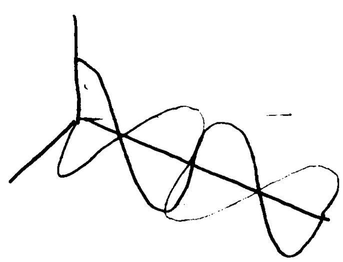
this is not interesting but consider $\underline{H}$ where $\hat{\eta} = \eta e^{j\theta_n}$

$$H = \frac{\hat{E}_m^{+}}{\eta e^{j\theta_\eta}} e^{-\alpha z} e^{-j\beta z} e^{j\theta_+} + \frac{E_m^{-}}{\eta e^{j\theta_\eta}} e^{+\alpha z} e^{+j\beta z} e^{j\theta_-}$$

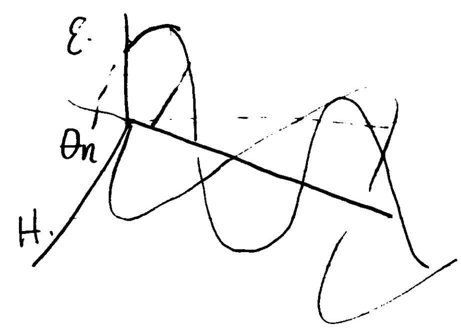
$$= \frac{E_m^{+}}{\eta} e^{-\alpha z} \cos(\omega t - \beta z + \theta_+ - \theta_\eta) + \frac{E_m^{-}}{\eta} e^{+\alpha z} \cos(\omega t + \beta z + \theta_- - \theta_\eta)$$

note phase shift
this delays
the $\hat{H}$ wave relative to the $\underline{E}$

For lossless medium



For lossy



group & phase velocity

phase velocity is the movement of a transverse plane (single frequency) with time

group velocity refers to the difference in velocity between frequency components, it is the velocity at which information propagates.

lets look at the phase function

$$\varphi_+ = \beta z - \omega t$$

pick any point of constant phase, say zero.

$$0 = \beta z - \omega t$$

or $$z = \underbrace{\left(\frac{\omega}{\beta}\right)}_{\text{this is the phase velocity}} t$$

difference in phase between different components.

$$\varphi_2 - \varphi_1 = \Delta\varphi = (\beta_2 z - \omega_2 t) - (\beta_1 z - \omega_1 t)$$

this is actually φ_{mod} why

$$\underbrace{\sin\varphi_1 + \sin\varphi_2}_{\text{fourier sum}} \rightarrow 2\sin\underbrace{\left(\frac{\varphi_2+\varphi_1}{2}\right)}_{\text{sum}}\cos\underbrace{\left(\frac{\varphi_2-\varphi_1}{2}\right)}_{\text{difference}}$$

carrier modulation

from above if we look at $\Delta\varphi = 0$ we get.

$$0 = (\beta_2 - \beta_1)z - (\omega_2 - \omega_1)t$$

or $z = \left(\frac{\omega_2 - \omega_1}{\beta_2 - \beta_1}\right)t$ — group velocity $v_g = \frac{d\omega}{d\beta}$ as $\Delta\beta \to 0$

Lecture period #36 — dielectric media, Normally incident plane waves

continue from before with Taylor Series expansion for good dielectric $\sigma/\omega\epsilon \ll 1$

$$\gamma^2 = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

$$\gamma^2 = -\omega^2\mu\epsilon\left(1 + \frac{j\omega\mu\sigma}{-\omega^2\mu\epsilon}\right)$$

$$= -\omega^2\mu\epsilon\left(1 - j\frac{\sigma}{\omega\epsilon}\right)$$

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{\sigma}{\omega\epsilon}\right)^{1/2}$$

$$= j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{1}{2}\left(\frac{\sigma}{\omega\epsilon}\right)\right)$$

$$= j\omega\sqrt{\mu\epsilon}\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right)$$

$\therefore$ if $\gamma = \alpha + j\beta$

(loss) $$\alpha = -\cancel{j\omega}\sqrt{\mu\epsilon}\ \cancel{j}\frac{1}{2}\frac{\sigma}{\cancel{\omega}\epsilon} = +\frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}}$$

(propagation) $$\beta = \omega\sqrt{\mu\epsilon}$$

(impedance) $$\hat{\eta} = \frac{j\omega\mu}{\gamma} = \frac{\cancel{j\omega}\mu}{\cancel{j\omega}\sqrt{\mu\epsilon}}\frac{1}{\left(1 - j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right)}$$

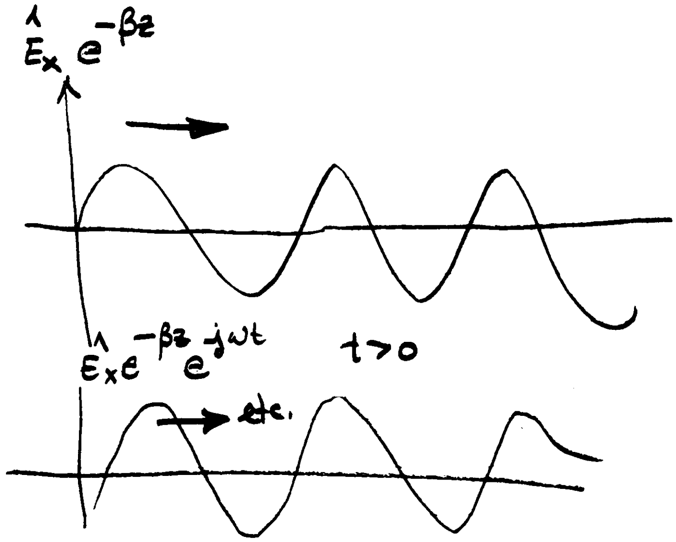
$$= \sqrt{\frac{\mu}{\epsilon}}\left(1 + j\frac{1}{2}\frac{\sigma}{\omega\epsilon}\right)$$

Examples in several media

I. lossless $\quad \alpha = 0$

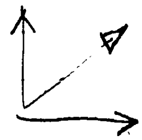
$$\beta = \omega\sqrt{\mu\epsilon}$$

$$\hat{\eta} = \sqrt{\frac{\mu}{\epsilon}} = 377\,\Omega$$



II. dielectric sea water

$$\begin{bmatrix} \mu_0 \\ \epsilon = 79\epsilon_0 \\ \sigma = 3\ \mathrm{S/m.} \end{bmatrix}$$



(a) at 20kHz $\quad \dfrac{\sigma}{\omega\epsilon} = \dfrac{3}{2\pi \times 20\times 10^3 \times 79 \times 4\pi\times 10^{-7}} = 12.4 \gg 1 ..$

this is a conductor. dielectric current can be neglected...

(o) $\omega = 2\pi f$

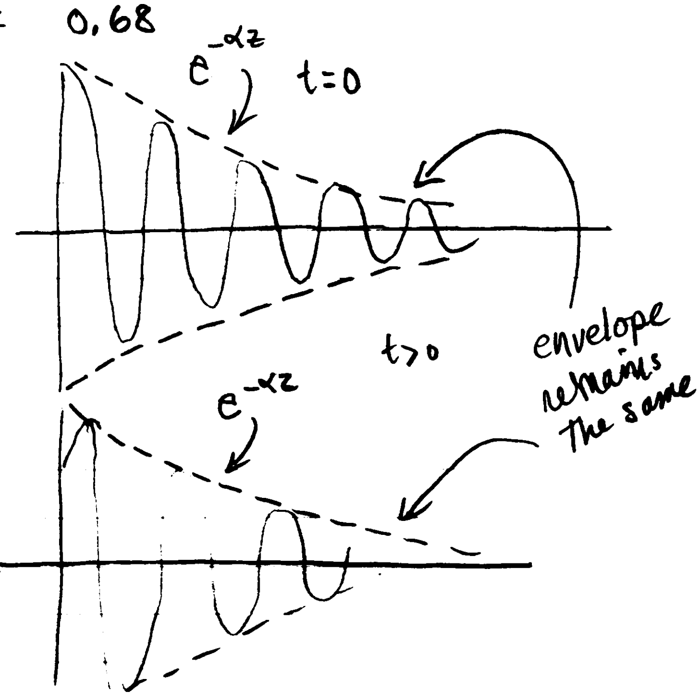
$$\therefore \quad \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\cancel{2\pi}\, 20\times 10^3 \times 3 \times 4\pi\times 10^{-7}}{\cancel{2}}}$$

$$= \sqrt{10.473} = 0.68$$

$$\alpha = \frac{1}{\delta} = 1.45\,\text{meters}$$

e^{-1} distance in 1.45 meters

very little penetration.



very little H in a good conductor.

$$\hat{\eta} \cong \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = (1+j)(7.45\times 10^{-4})\,\Omega$$

very small $\underline{\hat{H}}$

dielectric sea water at 20 GHz

$$\frac{\sigma}{\omega\epsilon} = \frac{3}{2\cancel{\pi} \times 20 \times \cancel{10^9} \times \frac{79}{36\cancel{\pi}} \times \cancel{10^{-9}}} = .03 \ll 1$$

imperfect dielectric

$$\alpha = \frac{1}{2}\sigma\sqrt{\frac{\mu}{\epsilon}} = \frac{1}{2}\cdot 3 \cdot \sqrt{\frac{4\pi\times 10^{-7}}{\frac{79}{36\pi}\times 10^{-9}}} = 63.63/\text{m.} \qquad \frac{1}{\alpha} = .015\text{m}$$

$\delta = \frac{1}{\alpha} = 15\text{mm.}$

$$\beta = \omega\sqrt{\mu\epsilon} = 2\pi \times 20\times 10^9 \sqrt{\cancel{4\pi}\times 10^{-7}\times \frac{1}{\underset{9}{\cancel{36\pi}}}\times 10^{-9}}$$

$$= 1.48 \times 10^3/\text{m.}$$

many cycles/meter . . .

$$\hat{\eta} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{63.63 + j\,1.48\times 10^3}$$

$$\cong \sqrt{\frac{\mu}{\epsilon}}\left(1 + \frac{1}{2}j\frac{\sigma}{\omega\epsilon}\right) = 42.41(1+j.015)\ \Omega$$

II. metallic conductor copper. $(\omega = 10^6)$

$$\sigma = 5.8\times 10^7$$
$$\epsilon = \frac{1}{36\pi}\times 10^{-9}$$
$$\mu = 4\pi\times 10^{-7}$$

$$\frac{\sigma}{\omega\epsilon} = \frac{5.8\times 10^7}{10^6 \times \frac{1}{36\pi}\times 10^{-9}} = 6.6\times 10^{12} \gg 1$$

$$\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{10^6 \times \cancel{4\pi}\times 10^{-7} \times 5.8\times 10^7}$$

$$= 8.5\times 10^3$$

$$\delta = \frac{1}{\alpha} = 0.1 \text{ millimeters}$$

$$\hat{\eta} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\,1041\ \Omega$$

typical of a conductor

$$\sqrt{\frac{\omega\mu}{2\sigma}} = \sqrt{\frac{(10^6)(4\pi\times 10^7)}{2\cdot 5.8\times 10^7}} = 1041$$

What are the fields for this situation

let $E = 100\ \mathrm{V/m}$.

Then $$\underline{\mathcal{E}}(z,t) = \mathcal{R}e\Big[100\ \tfrac{V}{m}\ \overbrace{e^{-\gamma z}}^{\text{just in + z direction}}\ e^{j\omega t}\Big] \quad \text{pick } \underline{a_x}$$

$$= \mathcal{R}e\Big[100\, e^{-\alpha z} e^{-j\beta z} e^{j\omega t}$$

$$= 100\, e^{-\alpha z} \cos(\omega t - \beta z)$$

where α and β are known.

How about $$\underline{\mathcal{H}} = \mathcal{R}e\Big[H e^{-\gamma z} e^{j\omega t}\Big]$$

$$= \mathcal{R}e\Big[\frac{\hat{E}}{\hat{\eta}} e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\Big]$$

$$= \mathcal{R}e\Big[\frac{100}{(1+j)(1041)} e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\Big]$$

$$= \mathcal{R}e\Big[\frac{100}{736\, e^{j\frac{\pi}{4}}} e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\Big]$$

$$= \mathcal{R}e\Big[\frac{100}{736} e^{-j\frac{\pi}{4}} e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\Big]$$

$$= 0.135\, e^{-\alpha z} \cos\Big(\omega t - \beta z - \underbrace{\tfrac{\pi}{4}}_{\text{phase shift}}\Big)$$

How about energy transport for a complex wave?

$\underline{\mathcal{S}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}}$ is time dependent power flow.

write $\underline{\mathcal{E}}(x,y,z,t) = \frac{1}{2}\left[\underline{\hat{E}}\, e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$

$\underline{\mathcal{H}}(x,y,z,t) = \frac{1}{2}\left[\underline{\hat{H}}\, e^{j\omega t} + \underline{H}^* e^{-j\omega t}\right]$

$$\mathcal{S}(x,y,z,t) = \underline{\mathcal{E}} \times \underline{\mathcal{H}}$$

$$= \frac{1}{4}\left[\underline{\hat{E}} \times \underline{\hat{H}}^* + \hat{E}^* \times \hat{H}\right] + \frac{1}{4}\left[\underline{\hat{E}} \times \underline{\hat{H}}\, e^{j2\omega t} + \hat{E}^* \times \hat{H}^* e^{-j2\omega t}\right]$$

$$= \frac{1}{4}\left[\underline{\hat{E}} \times \underline{\hat{H}}^* + (\underline{\hat{E}} \times \underline{\hat{H}}^*)^*\right] + \frac{1}{4}\left[\hat{E} \times \hat{H} e^{j2\omega t} + \left[\hat{E} \times \hat{H} e^{j2\omega t}\right]^*\right]$$

$$= \underbrace{\frac{1}{2} \mathrm{Re}\,(E \times H^*)}_{\text{not time dependent}} + \frac{1}{2} \mathrm{Re}\underbrace{\left\{(\hat{E} \times \hat{H})^* e^{j2\omega t}\right\}}_{\text{time dependent}}$$

if we pick time-average power average of second term is zero

$\therefore$ Complex Poynting vector

$$\hat{S} = \frac{1}{2}(\hat{E} \times \hat{H}^*)$$

$\mathcal{S}(x,y,z,t) = \mathrm{Re}\, \hat{S}$ time average power flow.

How about the Poynting vector for a complex wave?

$$\boxed{\hat{S} = \frac{1}{2}\,\hat{E} \times \hat{H}^*}$$

$\hat{S}$ can be complex. What does this mean?

let $\epsilon = \epsilon' - j\epsilon''$

$\mu = \mu' - j\mu''$

$$\oint \hat{\underline{S}} \cdot d\underline{S} = ?$$

$$= \underbrace{\frac{\omega}{2}\int\left(\epsilon''|\hat{E}|^2 - \mu''|\hat{H}|^2\right)dv}_{\text{polarization damping terms}} - \frac{1}{2}\int \sigma |\hat{E}|^2\, dv \quad \text{just like before}$$

$$\left.\oplus\; j\omega \underbrace{\int\left(\epsilon'|E|^2 - \mu'|H|^2\right)}_{\text{energy stored in fields}} dv\right\} \text{what we had before}$$

This is useful but Too complex

$$\hat{S} = \frac{1}{2}\,\hat{E} \times \hat{H}^*$$

but $\hat{H} = \dfrac{\hat{E}}{\hat{\eta}}$

$$\therefore \quad \hat{S} = \frac{1}{2}\,\hat{\underline{E}} \times \frac{|\hat{E}^*|\, a_{\underline{H}}}{\hat{\eta}^*} = \frac{1}{2}\,\frac{|\hat{E}|^2}{\eta^*}\, a_n$$

$\underline{a_n}$ = direction $\hat{E} \times \hat{H}$