Energy stored in fields

With the conclusion of the previous section on inductance we are ready to consider the subject of electromagnetic forces and energy stored in fields. We will first examine electrostatics and extend our results to magnetostatics.

Obviously, energy must be stored in fields. Consider the classic case of a charged capacitor. This is not connected to any supply but there is energy stored in the potential difference between the plates, i.e. the field.

Let us consider the work done in assembling a collection of charges. For convenience, let $\Phi(\infty) = 0$. As we bring charge Q_1 in from infinity no work is done because no fields are present. If we place Q_1 at P_1 and bring in another charge Q_2 work is done against the field of Q_1. Recall that voltage = work / unit charge. Let us define W_{21} to be the work done in bringing Q_2 into some point P_2. The work done is then $W_{21} = Q_2 V_{21}$

where V_{21} is the potential between P_2 and P_1 due to the field of Q_1. If we bring in a charge Q_3 with Q_1 and Q_2 already in place. The work now done is the sum of the work done against each field, i.e.

$$W_{31} + W_{32} = Q_3 V_{31} + Q_3 V_{32}$$

For a collection of N charges in space, the work necessary to assemble these charges is

$$W_e = W_{21} + (W_{31} + W_{32}) + \cdots$$

Order cannot be important since it makes no difference which charge we bring in first. So let's bring in Q_2 with Q_3 present, then bring in Q_1 with Q_2 and Q_3 present, etc.

$$W_e = W_{23} + (W_{12} + W_{13}) + \cdots$$

If we add these results together we get

$$2W_e = (W_{12} + W_{21}) + (W_{31} + W_{13} + W_{32} + W_{23}) + \dots$$

$$= Q_1 V_{12} + Q_2 V_{21} + Q_3 V_{31} + Q_1 V_{13} + Q_3 V_{32} + Q_2 V_{23} + \dots$$

$$= Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}) + \dots$$

$$2W_e = \sum_{i=1}^{N} Q_i V_i$$

where V_i is defined to be the potential of Q_i relative to all other charges, i.e.

$$V_i = \sum_{\substack{j=1 \\ i \neq j}}^{N} V_{ij}$$

For a discrete system of charges the total energy (work) expended in assembly the system is then

$$W_e = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i$$

This can be simply extended to the case of continuous charge by replacing Q_i by ρ, V_i by V and replacing the sum by a continuous integral

$$W_e = \frac{1}{2} \int \rho V \, dv.$$

We can write this equation directly in terms of field quantities by noting that $\nabla \cdot D = \rho$ and that V is a scalar.

$$W_e = \frac{1}{2} \int \nabla \cdot D \, V \, dv$$

but $(\nabla \cdot D) V = \nabla \cdot (VD) - D \cdot (\nabla V)$

$$W_e = \frac{1}{2}\int \nabla\cdot(VD)\,dv - \frac{1}{2}\int D\cdot(\nabla V)\,dv.$$

The first term is best examined by converting it to a surface integral

$$\int \nabla\cdot(VD)\,dv = \oint_S V\underline{D}\cdot\underline{ds}$$

and noting that we want to pick S at $r=\infty$ if we want the total energy of a system. Consider a single point charge:

$$E \sim \frac{1}{r^2} \qquad S \sim 4\pi r^2$$

$$D \sim \frac{1}{r^2}$$

$$V \sim \frac{1}{r}$$

Then $\underline{V}\underline{D}\cdot\underline{ds} \sim \frac{1}{r}\cdot\frac{1}{r^2}\cdot 4\pi r^2 = \frac{1}{r}$ which goes to zero as $r\to\infty$. Thus,

$$W_e = -\frac{1}{2}\int D\cdot(\nabla V)\,dv.$$

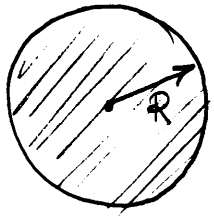
Recalling that $D = \epsilon E$ and $E = -\nabla V$ this can be re-written in its normally used form

$$W_e = \frac{1}{2}\int \epsilon\,|E|^2\,dv$$

A similar result can be obtained for magnetic fields except that we use the vector potential $\underline{A}$ and the work done in assemblying the field is $\underline{A}\cdot\underline{J}$ not QV as for electrostatic fields. The result of such a calculation is

$$W_m = \frac{1}{2}\int \mu\,|H|^2\,dv$$

Example: energy stored in a uniformly charged sphere



$$\rho = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases}$$

The field for this charge distribution is

$$\underline{E} = \begin{cases} \dfrac{r\rho_0}{3\epsilon}\,\underline{a_r} & 0 \leq r \leq R \\[2ex] \dfrac{R^3\rho_0}{3\epsilon r^2}\,\underline{a_r} & r > R \end{cases}$$

From the formula just derived

$$W_e = \frac{1}{2}\int \epsilon |\underline{E}|^2 \, dv$$

$$= \frac{1}{2}\epsilon \left\{ \int_0^R \int_0^\pi \int_0^{2\pi} \frac{r^2\rho_0^2}{9\epsilon^2} r^2 \sin\theta \, d\theta \, d\phi \, dr + \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{R^6\rho_0^2}{9\epsilon^2 r^4} r^2 \sin\theta \, d\theta \, d\phi \, dr \right\}$$

$$= \frac{1}{2}\epsilon \left\{ \int_0^R \frac{r^2\rho_0^2}{9\epsilon^2} 4\pi \, dr \, r^2 + \int_R^\infty \frac{R^6\rho_0^2 \, 4\pi r^2 \, dr}{9\epsilon^2 r^4} \right\}$$

$$= \frac{1}{2}\epsilon \left\{ \frac{4\pi\rho_0^2}{9\epsilon^2} \frac{R^5}{5} + \frac{4\pi R^6 \rho_0^2}{9\epsilon^2} \frac{1}{R} \right\}$$

$$= \frac{1}{2}\epsilon \, \frac{4\pi\rho_0^2}{9\epsilon^2} \left\{ \frac{R^5}{5} + R^5 \right\} = \frac{2\pi\rho^2}{9\epsilon} \left\{ \frac{6R^5}{5} \right\}$$

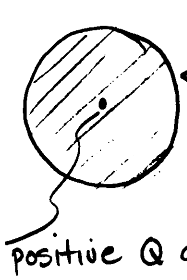
If we note that the total charge Q in the sphere is

$$Q = \frac{4}{3}\pi R^3 \rho_0$$

we can write this result as

$$W_e = \frac{3Q^2}{20\pi\epsilon R}$$

Example: energy in assemblying an atom.



We will do this one in two parts, i.e. we will assemble the electron cloud and then add the positive nucleus. From the previous example the energy expended in assembying the electron cloud is

$$W_e = \frac{3Q^2}{20\pi\epsilon R}.$$

To then bring $+Q$ in from infinity we use our original definition $W_e = QV$ to compute the energy required. The potential for a spherical charge distribution is given by

$$\Phi = \begin{cases} -\frac{3Q}{8\pi\epsilon R^3}\left(R^2 - \frac{r^2}{3}\right) & r<R \\ -\frac{Q}{4\pi\epsilon R} & r>R \end{cases}$$

Then $\quad W_e = Q\Phi(r=0) = -\dfrac{3Q^2}{8\pi\epsilon R}.$

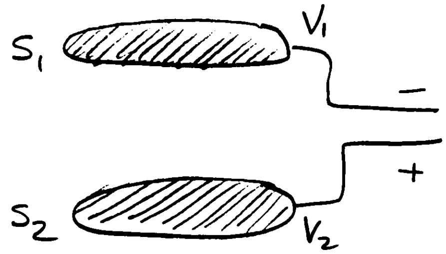
The total energy required is then

$$W_e(\text{total}) = \frac{3Q^2}{20\pi\epsilon R} - \frac{3Q^2}{8\pi\epsilon R}$$

$$= \frac{3Q^2}{\pi\epsilon R}\left[\frac{1}{20} - \frac{1}{8}\right]$$

$$W_e(\text{total}) = -\frac{9Q^2}{40\pi\epsilon R}$$

Example: energy stored in a capacitor



$$W_e = \frac{1}{2}\int \rho V \, dv$$

$$= \frac{1}{2}\int_{S_1} \rho_s V_1 \, dS_1 + \frac{1}{2}\int_{S_2} \rho_s V_2 \, dS_2$$

as the surfaces are conductors and, thus, equipotentials

$$W_e = \frac{1}{2} V_1 \int_{S_1} \rho_s dS_1 + \frac{1}{2} V_2 \int_{S_2} \rho_s dS_2$$

For a capacitor these integrals are simply the total charges on the plates, $-Q$ and $+Q$ respectively as drawn.

$$W_e = \frac{1}{2} V_1 (-Q) + \frac{1}{2} V_2 (+Q)$$

$$W_e = \frac{1}{2}(V_2 - V_1) Q = \frac{1}{2} V Q$$

But $C = \frac{Q}{V}$ so we can write for any capacitor that

$$W_e = \frac{1}{2} C V^2$$

Example: energy in an inductor.

$$W_m = \frac{1}{2}\int \mu |H|^2 \, dv$$

$$W_m = \frac{1}{2} \int \underline{H} \cdot \underline{B} \, dv$$

$$W_m = \frac{1}{2} \int (\nabla \times \underline{A}) \cdot \underline{H} \, dv$$

Use the vector identity $\nabla(\underline{A} \times \underline{H}) = (\nabla \times \underline{A}) \cdot \underline{H} - (\nabla \times \underline{H}) \cdot \underline{A}$

$$W_m = \frac{1}{2} \int \nabla(\underline{A} \times \underline{H}) \quad dv + \frac{1}{2} \int (\nabla \times \underline{H}) \cdot \underline{A} \, dv$$

$$= \frac{1}{2} \oint_S (\underline{A} \times \underline{H}) \cdot \underline{dS} + \frac{1}{2} \int \underline{J} \cdot \underline{A} \, dv$$

where we have converted the first integral into a surface integral via the divergence theorem, and use ampère's law on the second ($\nabla \times \underline{H} = \underline{J}$). As for electrostatics, the fields at $r=\infty$ are zero so the first integral vanishes and we are left with

$$W_m = \frac{1}{2} \int \underline{J} \cdot \underline{A} \, dv$$

Since $\underline{J} = 0$ except along a circuit element $\underline{J} \, dv = J \, dS_0 \, \underline{dl}$

$$W_m = \frac{1}{2} \int_{S_0} \oint_C J \, dS_0 \, \underline{dl} \cdot \underline{A}$$

where we now have integrals both over the cross-section of the current carrying element and along the element.

$$W_m = \frac{1}{2} \int_{S_0} J \, dS_0 \oint_{C_1} \underline{A} \cdot \underline{dl}$$

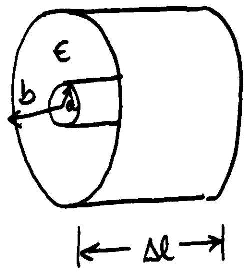
The first integral is just the current, the second integral is the flux since

$$\oint_{C_1} \underline{A} \cdot \underline{dl} = \int_{S_1} (\nabla \times A) \cdot dS = \int_{S_1} \underline{B} \cdot \underline{dS} = \varphi$$

$$\therefore \quad W_m = \frac{1}{2} I \varphi$$

but $L = \dfrac{\varphi}{I}$ for an inductor, so $W_m = \frac{1}{2} L I^2$.

Example: Use energy density to find capacitance of coaxial cable.



From Gauss' Law the fields are

$$\int \underline{D} \cdot \underline{dS} = \frac{q}{\Delta l}$$

$$\epsilon E_r \cdot 2\pi r \cancel{\Delta l} = \frac{q}{\Delta l} \cdot \cancel{\Delta l}$$

$$E_r = \frac{q}{\Delta l} \frac{1}{2\pi \epsilon r}$$

The energy stred in the cable is

$$W_e = \frac{1}{2}\epsilon \int |E|^2 \, dv = \frac{1}{2}\epsilon \int_a^b \int_0^{\Delta l} \int_0^{2\pi} \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi^2 \epsilon^2 r^2} \, r \, dr \, d\phi \, dz$$

$$= \frac{1}{2}\epsilon \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi^2\epsilon^2} \, 2\pi \Delta l \int_a^b \frac{dr}{r}$$

$$W_e = \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi\epsilon} \left[\ln b - \ln a\right]$$

But $W_e = \frac{1}{2} C V^2$

If $E = \frac{q}{\Delta l} \frac{1}{2\pi\epsilon}$

$$\Phi = -\int_a^R \underline{E} \cdot \underline{dl} = -\int_a^b \frac{q}{\Delta l} \frac{1}{2\pi\epsilon} \frac{dr}{r}$$

$$= \frac{q}{\Delta l} \frac{1}{2\pi\epsilon} \ln(b/a)$$

$$\therefore \quad \frac{1}{2} C V^2 = W_e$$

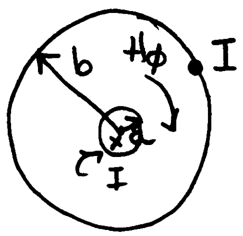
$$\frac{1}{2} C \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi^2\epsilon^2} \ln^2(b/a) = \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi\epsilon} \ln(b/a)$$

Solving for C:

$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

The same result we got before.

Example: inductance/unit length of a coaxial cable



by Ampère's law $H_\phi = \begin{cases} \dfrac{I}{2\pi r} & a<r<b \\ 0 & \text{elsewhere} \end{cases}$

$$W_m = \frac{1}{2}\mu \int |H|^2 \, dv$$

$$W_m = \frac{1}{2}\mu \int_a^b \int_0^{\Delta l} \int_0^{2\pi} \frac{I^2}{4\pi^2 r^2} \, r \, dr \, d\phi \, dz$$

$$W_m = \frac{1}{2}\mu \int_a^b \frac{I^2}{4\pi^2 r^2} \, r \, 2\pi \Delta l \, dr$$

$$= \frac{\mu I^2 \Delta l}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu I^2}{4\pi} \Delta l \, \ln(b/a)$$

But $W_m = \frac{1}{2} L I^2$

$$\frac{1}{2} L I^2 = \frac{\mu I^2}{4\pi} \Delta l \, \ln(b/a)$$

$$\frac{L}{\Delta l} = \frac{2\mu}{4\pi} \ln(b/a)$$

$\therefore$ inductance/unit length $= \dfrac{\mu}{2\pi} \ln(b/a)$

Electromagnetic forces

The general subject of electromagnetic forces will be addressed in later courses. However, all forces are governed by the Lorentz force Law

$$\underline{f} = q\underline{E} + q\underline{v} \times \underline{B}$$

However, we can also relate forces to the inductance and capacitance of a system. Consider an electrostatic system

input electric energy = mechanical work done + energy stored in field

In terms of differentials

$$VI\,dt = F\,dx + dW_e$$

For a magnetic system we have a somewhat different electrical input

$$I\,d\lambda$$

This comes from $V_L = L\dfrac{dI}{dt}$ and $I = \dfrac{\lambda}{L}$

since $L\,dI = V\,dt =$

$$VI\,dt = IL\,dI = I\,d(LI) = I\,d\lambda$$

So that

$$I\,d\lambda = F\,dx + dW_m$$

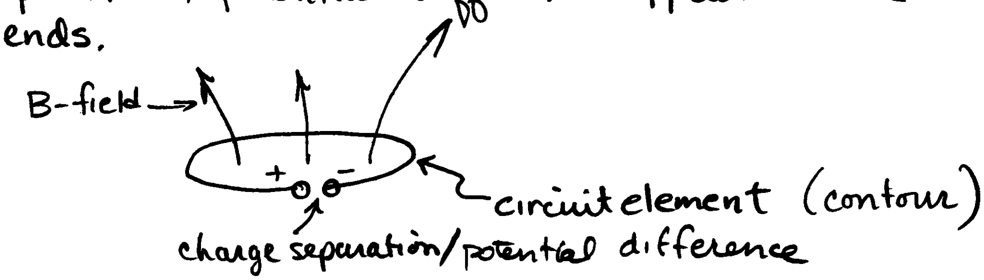
As a general topic prior to looking at time-dependent Maxwell's Equations and waves we will look at adding low frequency time dependence to Maxwell's Equations and the basic validity of Maxwell's Equations in the static formulation.

Let's consider what happens to electromagnetic fields as we introduce a basic time dependence. Two fundamental effects appear: Faraday's Law and displacement current.

Let's look at Faraday's Law first. Faraday's Law states that the voltage induced across an open circuit element is proportional to the time rate of change of magnetic flux through the associated contour. Mathematically,

$$emf = -\frac{d\Phi_m}{dt}$$

Emf is used because $d\Phi/dt$ actually induces a field which causes charge separation in the circuit; However, if we let our circuit element be a perfect conductor there can be no E-field in the circuit and a charge separation / potential difference appear at the element ends.



Note that in the drawing above there is no field in the contour, It is only in the gap. This is why Faraday's Law can only apply to an open contour. A closed, perfectly conducting loop would have no measurable potential anywhere.

NOTE: You may use a right-hand rule to determine the sign of the induced potential for the contour. If your thumb points in the direction of the B field your curled fingers will point to the positive (+) potential end of your contour.

To understand the practical significance of Faraday's law let's return to integral form and re-examine some of the properties we found for static fields.

Faraday's law in integral form:

$$\oint \underline{E} \cdot \underline{dl} = -\frac{d}{dt} \int \underline{B} \cdot \underline{dS}.$$

We found that electric fields (static) were conservative as defined by

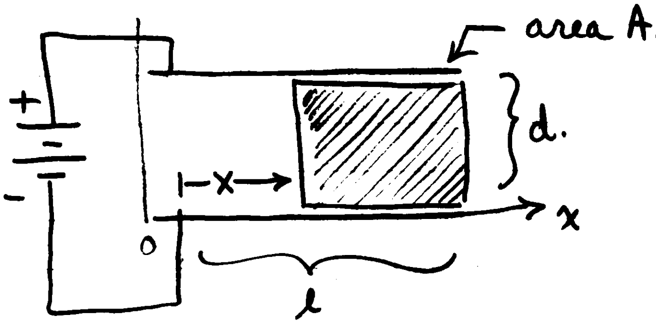
$$\oint \underline{E} \cdot \underline{dl} = 0$$

Obviously, this cannot be true if $\lambda = \int \underline{B} \cdot \underline{dS}$ does not remain constant. In fact, let's break up the contour and look at the problems this realization implies. If we write

$$\oint \underline{E} \cdot \underline{dl} = \int_{P_1}^{P_2} \underline{E} \cdot \underline{dl} + \int_{P_2}^{P_1} \underline{E} \cdot \underline{dl} \neq 0$$

We see that the contour integrals from P_1 to P_2 and P_2 to P_1 are now no longer equal.

Example: force on a capacitor dielectric



For an electric system $VIdt = Fdx + dW_e$

In this case $VIdt = (\Delta q)V$, the energy from the battery. We need an additional equation before we can solve this problem.

The field without the dielectric is the same as that with, i.e. $\frac{V}{d}$
If there were no dielectric in any section dx, the total energy would be

$$W_e = \frac{1}{2}\int \epsilon_0 |E|^2 dv = \frac{1}{2}\epsilon_0\left(\frac{V}{d}\right)^2 Ad$$

If we change the dielectric $W_e = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 Ad$
So that $\Delta W_e = \frac{1}{2}(\epsilon - \epsilon_0)\left(\frac{V}{d}\right)^2 Ad$.
V cannot change due to the insertion of a dielectric since E is constant, However, $\underline{D}$ increases increasing the charge Q on each plate. With no dielectric

$$Q = CV = \frac{\epsilon_0 A}{d} V$$

with a dielectric $\rightarrow$ $Q' = \frac{\epsilon A}{d} V$. Thus,

$$\Delta Q = (\epsilon - \epsilon_0)\left(\frac{V}{d}\right) A$$

The amount of work supplied by the battery is then

$$W_{BAT} = (\epsilon - \epsilon_0)\left(\frac{V}{d}\right)^2 Ad$$

It is a general that $W_e = \frac{1}{2} W_{BAT}$. The rest goes into mechanical forces. Hence,

$$F_x dx + dW_e = 2dW_e$$

$$F_x\, dx = - \; dW_e$$

or $$F_x = -\frac{dW_e}{dx}$$

$$W_e = \frac{1}{2}\int \epsilon |E|^2 dv = \frac{1}{2}\epsilon_0 \left(\frac{V}{d}\right)^2 \frac{x}{\ell} Ad + \frac{1}{2}\epsilon \left(\frac{V}{d}\right)^2 \left(\frac{\ell - x}{\ell}\right) Ad.$$

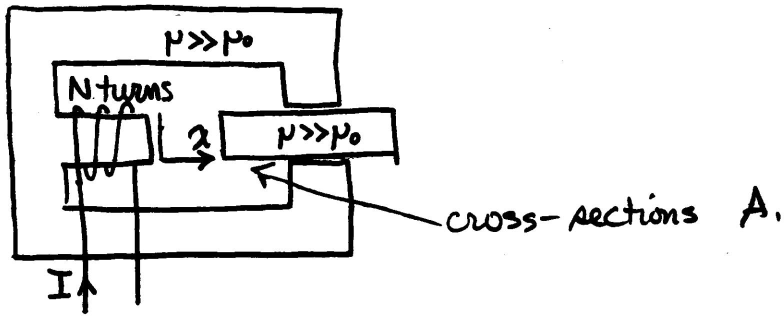
$$W_e = \frac{1}{2}\left(\frac{V}{d}\right)^2 \frac{Ad}{\ell}\left\{ \epsilon_0 x + \epsilon(\ell - x) \right\}$$

$$\frac{dW_e}{dx} = \frac{1}{2}\left(\frac{V}{d}\right)^2 \frac{Ad}{\ell}\left\{\epsilon_0 - \epsilon\right\}$$

$$\therefore \quad F_x = +\frac{1}{2}\left(\frac{V}{d}\right)^2 \frac{Ad}{\ell}(\epsilon - \epsilon_0)$$

which acts to force the dielectric out of the capacitor.

Example: solenoid



As before $\quad f_x = +\dfrac{dW_m}{dx}$

but $W_m = \frac{1}{2} L I^2$

$$f_x = +\frac{d}{dx}\left(\tfrac{1}{2} L I^2\right) = +\frac{1}{2} I^2 \frac{dL}{dx}$$

To find $L(x)$ we will use the magnetic circuit approach.

For the source $\mathcal{F} = NI$

As before, nearly all the reluctance will be concentrated in the gap

$$\mathcal{R}_g = \frac{l}{\mu_0 A} = \frac{x}{\mu_0 A}$$

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}_g} = \frac{NI}{x/\mu_0 A} = \frac{NI\mu_0 A}{x}$$

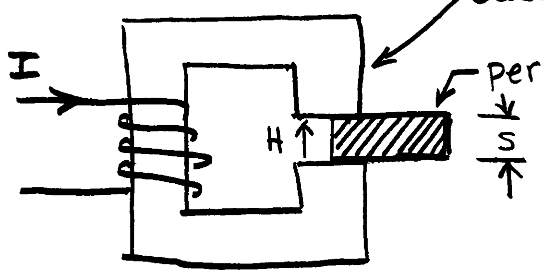
$$L = \frac{N\Phi}{I} = \frac{N^2 \mu_0 A}{x}$$

$$f_x = \frac{dW_m}{dx} = +\frac{1}{2} I^2 \left(N^2 \mu_0 A \; - \frac{1}{x^2} \right)$$

$$f_x = -\frac{1}{2} I^2 \frac{N^2 \mu_0 A}{x^2}$$

and the plunger is pulled in.

Example: Solenoid with a permanent magnet



As before we use $W_m = \frac{1}{2}\int \mu_0 |M|^2 dv$ except we will not use fields concepts but inductance concepts.

For an inductor $W_m = \frac{1}{2} L I^2$ which we can find using magnetic circuits.

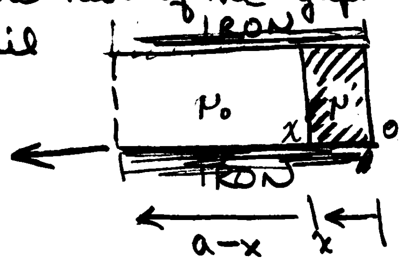
To find the flux in the gap

$$\mathcal{F} = NI \qquad \mathcal{R}_{gap} = \frac{s}{\mu_0 A}$$

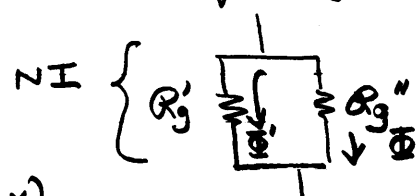
$$\Phi_{gap} = \frac{\mathcal{F}}{\mathcal{R}} = \frac{NI\mu_0 A}{s}$$

Now let's look at how Φ_{gap} splits up between the magnet and the rest of the gap.

gap detail



Assume thickness t and width a of the gap.

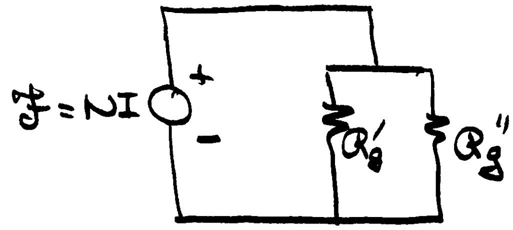


$$\mathcal{R}_g' = \frac{s}{\mu t x} \qquad \mathcal{R}_g'' = \frac{s}{\mu_0 t (a-x)}$$

$$\Phi' = \frac{NI\mu t x}{s} \qquad \Phi'' = \frac{NI\mu_0 t (a-x)}{s}$$

$$\Phi_{total} = \frac{NIt}{s}\left[\mu x + \mu_0 (a-x)\right]$$

Note that we have treated the gap as a parallel resistance circuit. Because the reluctance of iron is so low all reluctance is concentrated in the gap.



We have found Φ_{total} in the gap. As Φ is continuous everywhere (just like current) we can write the inductance in terms of gap parameters

$$L = \frac{N \Phi_{total}}{I} \leftarrow \text{total flux linked by coil}$$

$$L = \frac{N}{I} \frac{N I t}{s} \left[\mu x + \mu_0 (a-x) \right]$$

$$L = \frac{N^2 t}{s} \left[\mu x + \mu_0 (a-x) \right]$$

$$W_m = \frac{1}{2} L I^2 = \frac{1}{2} I^2 \frac{N^2 t}{s} \left[\mu x + \mu_0 (a-x) \right]$$

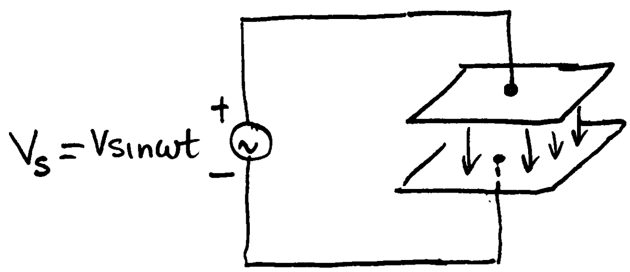
$$f_x = \frac{d}{dx} (W_m) = \frac{1}{2} I^2 \frac{N^2 t}{s} \left[\mu - \mu_0 \right]$$

Since $\mu > \mu_0$ f_x is in $+x$ direction and electromagnet pulls magnet into gap.

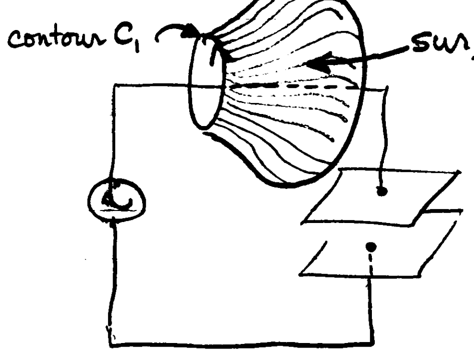
Faraday's Law is basically a time dependent description of electric fields. The similar extension for magnetic fields is the full form of Ampère's Law;

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

$\frac{\partial D}{\partial t}$ must have the same units as J, i.e. current density, and is called the displacement current even though it is not an actual current. To understand how $\frac{\partial D}{\partial t}$ can be interpreted as a current let's re-examine the parallel plate capacitor as shown below.

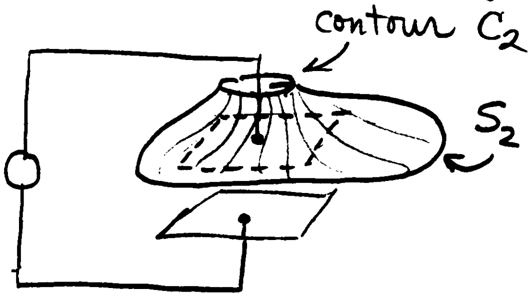


The voltage source is now a function of time. As V_s varies charge flows into and out from the capacitor plates. If we draw a contour and a surface as shown below we see that our conventional Ampère's Law applies since there is no $\underline{E}$ (and $\underline{D}$) field in a perfect conductor.



$$\nabla \times H = \underline{J} + \underbrace{\frac{\partial D}{\partial t}}_{0}$$

Now let us pick the contour and surface as shown below.



Obviously, the same $\underline{H}$ field exists around the conductor at contour C_2; However, there is NO J going through S_2. If we had just $\nabla \times \underline{H} = \underline{J}$ this would imply that $H=0$

However, there is a quasi-current flow through S_2. As charge flows onto (or off) the capacitor plates it changes D (and E) in the capacitor, This induces

an equivalent charge density to the other capacitor plate by pulling it from the source thus creating a continuous current even though there is no physical connection between the capacitor plates.

This apparant current is called the displacement current because it is due entirely to the behavior of the displacement vector and is given by $\frac{\partial D}{\partial t}$.

The ratio of the free current density J to the Displacement

Current density $\frac{\partial D}{\partial t}$ is called the conduction angle and tells us when such time dependent effects as displacement current become important. To get a handle on the magnitude of the numbers we are discussing let's let $\underline{E} = \underline{E_0} e^{j\omega t}$. Then $\underline{J} = \sigma \underline{E_0} e^{j\omega t}$ and $\underline{D} = \epsilon \underline{E_0} e^{j\omega t}$.

Evaluating $\frac{\partial D}{\partial t} = j\omega \epsilon \underline{E_0} e^{j\omega t}$

The conduction angle is then $\dfrac{J}{\frac{\partial D}{\partial t}} = \dfrac{\sigma \underline{E_0} e^{j\omega t}}{j\omega\epsilon \underline{E_0} e^{j\omega t}} = \dfrac{\sigma}{j\omega\epsilon}$

The j gives us no information we want know so we just use the magnitude of the ratio to define the conduction angle θ_c

$$\theta_c = \frac{\sigma}{\omega\epsilon}$$

For copper at $f = 1\,\text{MHz}$

$$\frac{\sigma}{\omega\epsilon} = \frac{5.8\times10^{7}}{2\pi\,(10^{6})\times\frac{1}{36\pi}\times10^{-9}} \approx 10^{12}$$

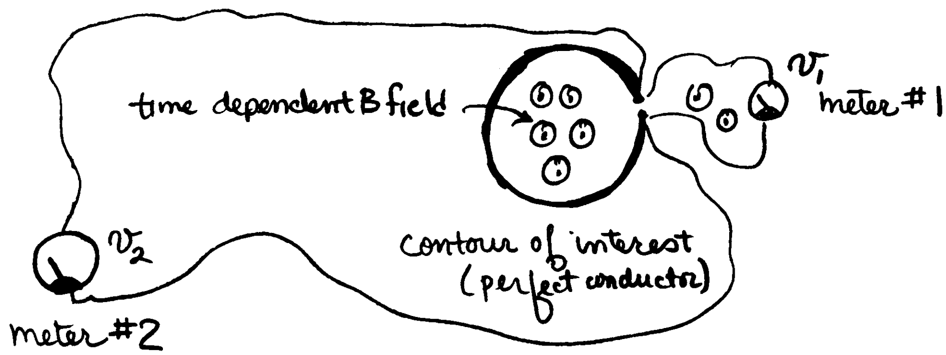
so that we can effectively ignore the displacement current in copper at such low frequencies.

For Teflon at $f = 1\,\text{MHz}$

$$\frac{\sigma}{\omega\epsilon} = \frac{3\times10^{-8}}{2\pi\,(10^{6})\times\frac{1}{36\pi}\times10^{-9}} \approx 3\times10^{-4}$$

we see that the opposite occurs. The conduction current can be neglected and the displacement current dominates. Note that this is a function of frequency, however, and at low frequencies the conduction current and the displacement current may become comparable.

Example: significance of meter loads



for v_1:
$$v_1 + \int_{\text{leads}} \underline{\mathcal{E}} \cdot \underline{d\ell} + \int_{\text{contour}} \underline{\mathcal{E}} \cdot \underline{d\ell} = -\frac{d}{dt}\left[\int \underline{B} \cdot \underline{dS} + \int_S B \cdot dS\right]$$

surface defined by meter leads

surface of contour to be interested

Since there can be no contribution to $\underline{\mathcal{E}} \cdot \underline{d\ell}$ along a perfect conductor the above equation reduces to

$$v_1 = -\frac{d}{dt}\left[\int_{\text{meter}} \underline{B} \cdot \underline{ds} + \int_{\substack{\text{surface}\\ \text{of interest}}} \underline{B} \cdot \underline{dS}\right]$$

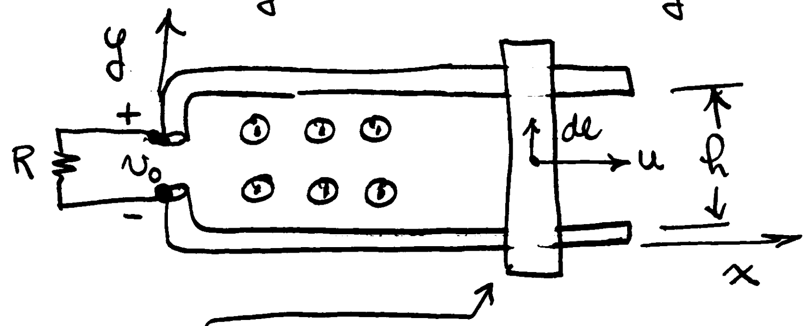
Only if the meter leads do NOT intercept any time dependent $\underline{B}$ field does the first integral vanish and we get the correct answer;

$$v_1 = -\frac{d}{dt}\left[\int_{\text{surface of interest}} \underline{B} \cdot \underline{dS}\right]$$

To show how important the choice of contour is, consider the lead arrangement for meter #2. The contour encloses NO B-field and as a result

$$v_2 = 0.$$

Example: Faraday's Law for moving short



the metal bar slides over a pair of conducting rails in a uniform magnetic field $\underline{B} = B_0 \underline{a_z}$, with a constant velocity $\underline{u}$.

from Faraday's Law

$$v_o = -\frac{d}{dt}\int \underline{B}\cdot\underline{ds}$$

$$= -\frac{d}{dt}\left\{ B h x \right\}$$

Note that $u = \frac{dx}{dt}$ so

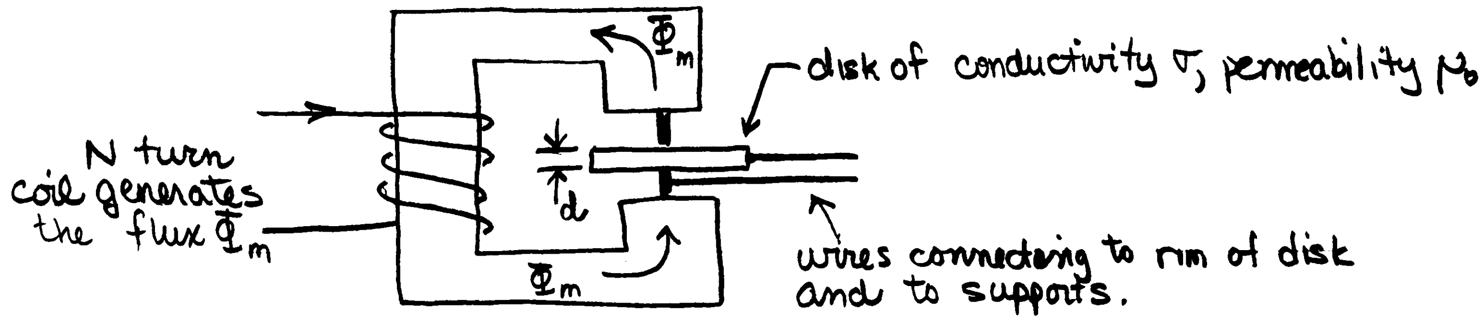
$$v_o = -Bh\frac{dx}{dt} = -Bhu$$

Now that we know the output voltage lets look at how much power is developed.

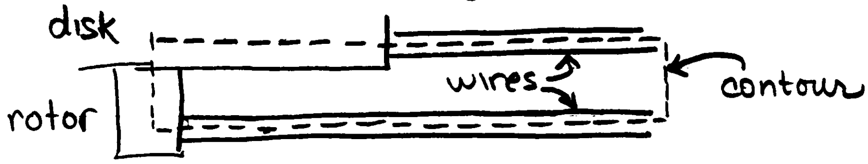
$$I_{OUT} = \frac{v_o}{R} = \frac{Bhu}{R}$$

$$P_{OUT} = I^2R = \frac{(Bhu)^2}{R}$$

Example: Faraday disk homopolar generator



Choose a contour following wires through disk.



To solve problem we need first field in gap. Get by reluctance.

$$\mathcal{F} = NI$$

$$\mathcal{R}_{gap} = \frac{s}{\mu_0 A}$$

$$\Phi_m = \frac{\mathcal{F}}{\mathcal{R}_{gap}} = \frac{NI}{s/\mu_0 A} = \frac{\mu_0 NIA}{s}$$

The flux $\Phi_m = BA$ so

$$B_{gap} = \frac{\mu_0 NI}{s}$$

To find current out of such a device we need to determine E field in gap. However, the disk is in motion and we need to take the disk's radial velocity into account. This is not hard if we return to the Lorentz force Law

$$\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$$

If I travel with velocity $\underline{v}$ I see field $q\underline{E}'$ where

$$q\underline{E}' = q(\underline{E} + \underline{v} \times \underline{B})$$

thus $\underline{E}' = \underline{E} + \underline{v} \times \underline{B}$

Solving for the radial field we have

$$\underline{E}(r) = \underset{\substack{\uparrow \\ \text{field on} \\ \text{moving disk}}}{\underline{E}'} - \underbrace{(\underline{v} \times \underline{B})}_{\text{radial due to angular velocity}}$$

The current is the same in all frames so $E' = \frac{J_r}{\sigma}$

$$E(r) = \frac{J_r}{\sigma} - \omega r B_{gap}$$

$$E(r) = \frac{i_r}{2\pi r d}\frac{1}{\sigma} - \omega r B_{gap}$$

Now we use Faraday's Law on the contour shown earlier.

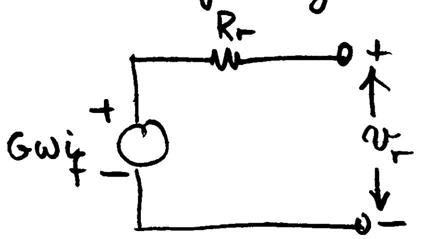
$$\oint E \cdot dl = \underbrace{\int_1^2 E_r\, dr + \int_3^4 \underline{E} \cdot \underline{dl}}_{\text{since no fields along conductors}} = -\underbrace{\frac{dB}{dt}}_{\text{since B is constant}} = 0$$

$$v_r \triangleq \int_3^4 E \cdot dl = \int_1^2 E_r\, dr = \int_{R_1}^{R_2} \left[\frac{i_r}{2\pi\sigma d\, r} - \omega r B_{gap}\right] dr$$

$$= i_r \underbrace{\frac{\ln R_2 - \ln R_1}{2\pi\sigma d}}_{} - \omega B_{gap} \frac{[R_2^2 - R_1^2]}{2}$$

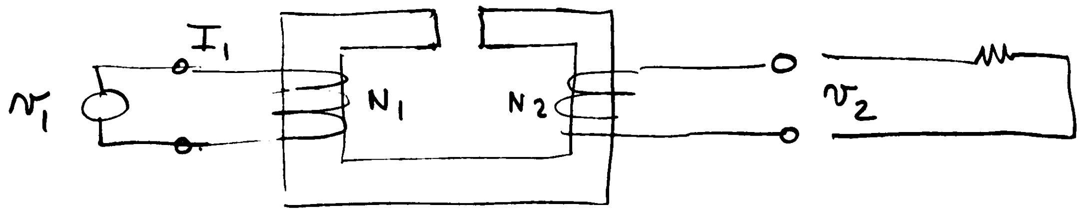
$$= i_r \underset{\substack{\uparrow \\ \text{rotor resistance}}}{R_r} - G\omega \underset{\substack{\uparrow \\ i_f \text{ is the field current.}}}{i_f}$$

This is typically a high current low voltage device



for $\sigma = 6 \times 10^7$ S/m
$d = 1$ mm
$\omega = [3600\text{ rpm}]$ 120π radians/sec.
$R_1 = 1$ cm
$R_2 = 10$ cm

$G\omega i_f = -1.9$ volts
short circuit current $\approx$ 350,000 Amperes.



(a) find flux due to coil #1.

$$\mathcal{F} = N_1 I_1$$

$\mathcal{R}_{gap}$.

$$\lambda = \frac{\mathcal{F}}{\mathcal{R}_{gap}}.$$

goes thru coil #2.

$$I_2 = -\frac{N_2 N_1}{\mathcal{R}_{gap}} \frac{dI_1}{dt}.$$