In our previous discussion of Maxwell's equation we only looked at Gauss Law and Faraday's Law. Let's quickly review what we did in justifying That. Starting with Maxwell's equations (in differential form)

$$\nabla \cdot \underline{D} = \rho \qquad \nabla \cdot \underline{B} = 0$$
$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}$$

Note that both Faraday's Law and Ampere's Law couple the electric and magnetic fields. In the static limit where all time derivatives are zero this coupling vanishes as shown below

$$\nabla \cdot \underline{D} = \rho \qquad \nabla \cdot \underline{B} = 0$$
$$\nabla \times \underline{E} = 0 \qquad \nabla \times \underline{H} = \underline{J}$$

If we recall our constitutive relationships $\underline{D} = \epsilon \underline{E}$ and $\underline{B} = \mu \underline{H}$ we can rewrite the above equations in terms of just $\underline{E}$ and $\underline{H}$.

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon} \qquad \nabla \cdot \underline{H} = 0$$
$$\nabla \times \underline{E} = 0 \qquad \nabla \times \underline{H} = \underline{J}$$

These two equations for $\underline{E}$ describe a pure irrotational field; those for $\underline{H}$ describe a pure rotational field. As a result we can describe each set of fields independently. However, the mathematics describing the $\underline{H}$ field will be somewhat different from those for the $\underline{E}$ field which we have just finished studying. Note that the integral form of Ampère's Law is not a volume integral as is Gauss Law (its electrostatic counterpart.) Clearly both describe the relationship of the field to the source except that Ampère's Law relates a line integral to a surface integral as shown below

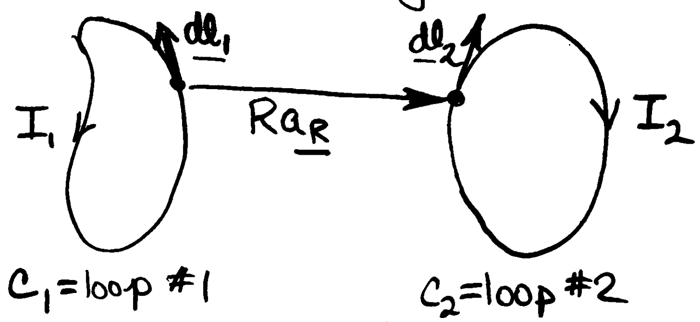
$$\oint \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{dS}$$

(After Plonsey & Collin)

Before continuing with our discussion of magnetic fields and Maxwell's equations we will look at the concept of a magnetic field. When we studied the electric field $\underline{E}$ we described $\underline{E}$ as the force/unit charge exerted by a source of electric charge where the force was given by Coulomb's Law:

$$\underline{F} = \frac{q_1 q_2}{4\pi\epsilon_0 \, |r_{12}|}$$

The corresponding magnetic force is given by Ampère's Force Law which describes the force between two current loops. Consider the geometry shown below.



By experimentation Ampère discovered that the vector force $\underline{F}_{21}$ exerted on C_2 by C_1 as caused by currents I_1 and I_2 is given by

$$\underline{F}_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{I_2 \, d\underline{l}_2 \times [I_1 \, d\underline{l}_1 \times \underline{a}_R]}{R^2} \qquad (1)$$

where I_1, I_2 are in amperes; dl_1, dl_2 and R are in meters; and $\mu_0 = 4\times 10^{-7}$ henrys/meter. We will discuss henrys when we study inductance.

Equation (1) is not very nice nor useful because it is not obviously symmetric, i.e. is $\underline{F}_{12} = -\underline{F}_{21}$. We can expand the vector product in (1) to prove it is. Looking at just the vectors in (1) we can expand (1) as

$$\underline{dl}_2 \times \underline{dl}_1 \times \underline{a}_R$$

$$= (\underline{dl}_2 \cdot \underline{a}_R)\, \underline{dl}_1 - (\underline{dl}_2 \cdot \underline{dl}_1)\, \underline{a}_R$$

As long as the contours are closed the integral of

$$\underline{dl}_2 \cdot \underline{a}_R$$

can be shown to be zero if we recall that we have a R^2 factor in the denominator.

$$\frac{\underline{dl}_2 \cdot \underline{a}_R}{R^2} = \underline{dl}_2 \cdot -\nabla\left(\frac{1}{R}\right)$$

where we have noted that $\frac{a_R}{R^2} = -\nabla(\frac{1}{R})$. If this term is integrated around a closed contour, i.e.

$$\oint \underline{dl}_2 \cdot -\nabla\left(\frac{1}{R}\right) = -\oint \nabla\left(\frac{1}{R}\right) \cdot \underline{dl}_2$$

we note that $\nabla(\frac{1}{R})$ is the directional derivative of $\frac{1}{R}$ and $\underline{dl}_2$ denotes the direction of the derivative. In simplest form

$$\nabla\left(\frac{1}{R}\right) = \frac{d(1/R)}{dl_2}$$

denoting the variation of $\frac{1}{R}$ around C_2. However, if we put this in our integral

$$-\oint \frac{d(\frac{1}{R})}{dl_2}\, dl_2$$

is nothing more than the integral of a single-valued function around a loop and that we know from our study of the electric potential is zero. So, we are left with the symmetric result

$$F_{12} = - \oint_{C_1}\oint_{C_2} \frac{\mu_0}{4\pi} I_1 I_2 \frac{\underline{dl}_1 \cdot \underline{dl}_2}{R^2}$$

this result says that we have a contribution to $\underline{F}$ only when $\underline{dl}_1$ and $\underline{dl}_2$ are parallel to each other; zero when they are perpendicular.

The symmetric in magnetic force is not immediately apparent as it was with coulomb's Law. To continue with our development of magnetic Fields we return to Ampère's Law.

$$\underline{F}_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{I_2 \underline{dl}_2 \times (I dl_1 \times \underline{a}_R)}{R^2}$$

We can separate this equation into two equations as shown below.

$$\underline{F}_{12} = \oint_{C_2} I_2 \underline{dl}_2 \times \underline{B}_{21} \qquad (2a)$$

$$\underline{B}_{21} = \frac{\mu_0}{4\pi} \int \frac{\underline{I}\, dl_1 \times \underline{a}_R}{R^2} \qquad (2b)$$

This method of separation is motivated by a "action at a distance" philosophy. B_{21} is defined in terms of a "source" current element. B_{21} propagates through space where it interacts with another current element to produce a force. Note that the first equation requires no knowledge of the source elements — only their resultant field. And the field is due only to sources not the element being acted upon.

Equation (2) will be our fundamental "working" equation to describe magnetic fields. Historically, eqn (2) was developed by Biot and Savart and is known as the Biot — Savart Law.

Finding magnetic Fields

Basically, we have defined $\underline{B}$ in eqn (2), the force due to $\underline{B}$ in (2a), and the behavior of $\underline{B}$ in maxwell's equations (1). Now, let's look at methods of relating $\underline{B}$ to its sources. If the source terms possess sufficient symmetry, we can use Ampère's Law directly to find B.

$$\oint_C \underline{H}\cdot \underline{dl} = \int \underline{J}\cdot \underline{ds}$$

Just as with Gauss' Law a solution using Ampère's Law requires $\underline{H}$ to be symmetric about the source term $\underline{J}$. In situations where the source does not lead to symmetric fields we can calculate B by use of the Biot – Savart Law. A third and perhaps the most powerful method is by the vector potential. This is very similar in motivation to the scalar potential. However, a scalar potential cannot produce a rotational field. For a rotational field

$$\nabla \times \underline{B} \neq 0$$

If $\underline{B} = -\nabla\Phi$ we get $\nabla \times (-\nabla\Phi)$ which is identically zero. If we instead try a potential function of the form

$$B = \nabla \times \underline{A}$$

we will get a pure rotational field. To prove this let's first examine the divergence.

$$\nabla \cdot \nabla \times \underline{A} \equiv 0$$

which satisfies $\nabla \cdot \underline{B} = 0$.

Now, lets examine the curl of B

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) \qquad (3)$$

We know that $\nabla \times \underline{B} = \mu \underline{J}$. By vector identities

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \nabla \, \nabla \cdot \underline{A} - \nabla^2 \underline{A} \qquad (4)$$

Note that all we require of $\underline{A}$ is that $\underline{B} = \underline{\nabla} \times \underline{A}$. We can put additional constrainsts as needed upon A. Without proving it, it can be shown that substituting $\underline{B} = \nabla \times \underline{A}$ into Maxwell's equations results in the following requirement for a simple solution for $\underline{A}$

$$\nabla \cdot A = -\mu \epsilon \frac{\partial \Phi}{\partial t}$$

[See Seely and Poularikas, <u>Electromagnetics</u>]

For static fields this means $\underline{\nabla} \cdot \underline{A} = 0$ and is known as the Lorentz condition. Combining (3) and (4) we have

$$\nabla \times \underline{B} = -\nabla^2 \underline{A} = \mu \underline{J}.$$

This is known as Poisson's equation for the vector potential because of its similarity to the electrostatic Poisson equation. Note that now both $\underline{A}$ and $\underline{J}$ are vectors so that we have a separate equation in each component, i.e.

$$\nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

These equations are identical in form to Poisson's equation

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon}$$

and have solutions of the same form — the math cannot change even though we are now discussing magnetic fields. The general solution for each component of A is then by analogy

$$A_x = \int \frac{\mu J\, dV}{4\pi r}$$

$$A_y = \int \frac{\mu J\, dV}{4\pi r}$$

etc.

Recall the solution for Φ was

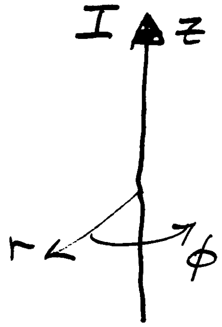
$$\Phi = \int \frac{\rho/\epsilon}{4\pi r}\, dv$$

The scalar forms can be combined to give the general solution

$$\underline{A} = \int \frac{\mu \underline{J}\, dv}{4\pi r}$$

also known as the Green's function solution for $\underline{A}$. This completes our discussion of methods of determining magnetic fields and forces we will do several examples of these methods in the next section.

Example: simple infinitely long wire w/current I

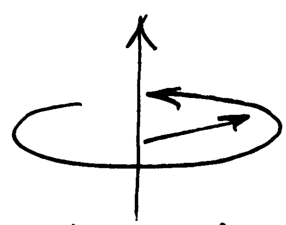


This is a highly symmetric problem so we can use Ampère's Law

$$\oint_C \underline{H} \cdot \underline{dl} = \int \underline{J} \cdot \underline{dS}$$

As the wire is infinitely long we expect no z-dependence. The symmetry in ϕ precludes any ϕ dependence so H must only be dependent upon r. We pick a surface perpendicular to J as shown below to simplify the right hand integral,

$$\int \underline{J} \cdot \underline{dS} = I$$



circular contour of radius r

The field must be in the direction of the contour so H only has a ϕ component (with radial dependence)

$$\oint \underline{H} \cdot \underline{dl} = H_\phi \, 2\pi r$$

Combining these results gives

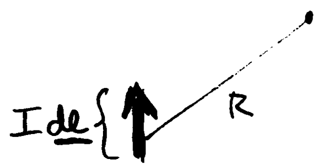
$$H_\phi = \frac{I}{2\pi r}$$

$$B_\phi = \frac{\mu_0 I}{2\pi r}$$

Example: field of a short current element

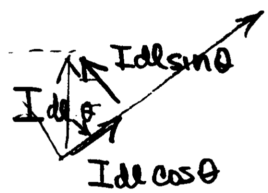
This problem no longer has the symmetry required by Ampère's law. We could do it, however, by the Biot - Savart Law or by the vector potential method.

By the Biot - Savart Law



$$\underline{H} = \frac{I\,\underline{dl} \times \underline{a_R}}{4\pi R^2}$$

$$= \frac{(I\,dl\,\underline{a_\theta}\sin\theta + I\,dl\,\underline{a_r}\cos\theta) \times \underline{a_r}}{4\pi r^2}$$



$$\underline{H} = \frac{I\,dl\,\sin\theta}{4\pi r^2}\,\underline{a_\phi}$$

$$\underline{B} = \frac{\mu_0 I\,dl\,\sin\theta}{4\pi r^2}\,\underline{a_\phi}$$

By the vector potential

$$\underline{A} = \int \frac{\mu \underline{J}\,dv}{4\pi r}$$

units of ampere - meter

$$\underline{A} = \frac{\mu}{4\pi r}\underbrace{\int \underline{J}\,dv}_{} = \frac{\mu}{4\pi r}\underbrace{I\,\underline{dl}}_{}\,\underline{a_z}$$

just as above break into components

$$\underline{A} = A_z\cos\theta\,\underline{a_r} + A_z\sin\theta\,\underline{a_\theta}$$

$$B = \nabla \times \underline{A} = \underline{a_r}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]$$

$$+\ \underline{a_\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]$$

$$+\ \underline{a_\phi}\ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

since $\underline{A}$ has no ϕ components and no ϕ dependence we can drastically reduce this expression to

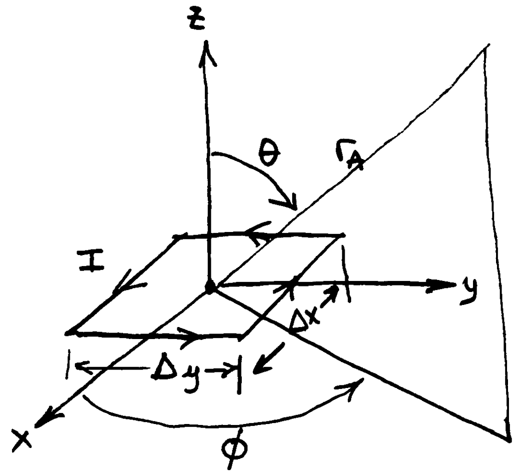
$$\underline{B} = \underline{a_r}\left[0\right] + \underline{a_\theta}\left[0\right] + \underline{a_\phi}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right]$$

$$= \underline{a_\phi}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\mu I}{4\pi r}\,I\,dl\,\sin\theta\right) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\mu}{4\pi r}\,I\,dl\,\cos\theta\right)\right]$$

$$B = \underline{a_\phi}\left[\frac{\mu}{4\pi r^2}\,I\,dl\,\sin\theta\right]$$

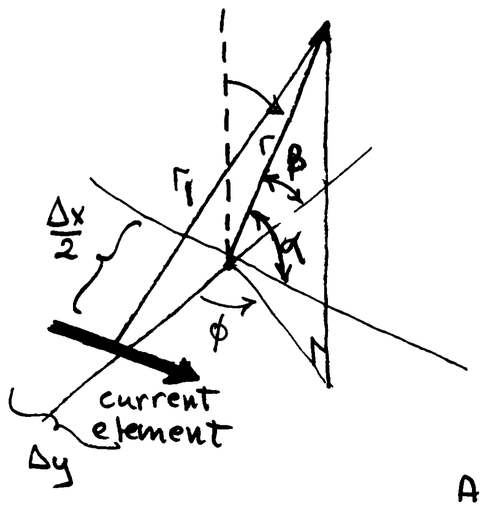
and the two results agree ∎.

Example : square magnetic dipole



We will use only vector potential for this problem. To solve do each side separately, then sum up the results.

For reference $\underline{A} = \int \frac{\mu J}{4\pi r}\, dv$



By law of cosines

$$r_1^2 = r^2 + \left(\frac{\Delta x}{2}\right)^2 - 2(r)\left(\frac{\Delta x}{2}\right)\cos(\pi - \beta)$$

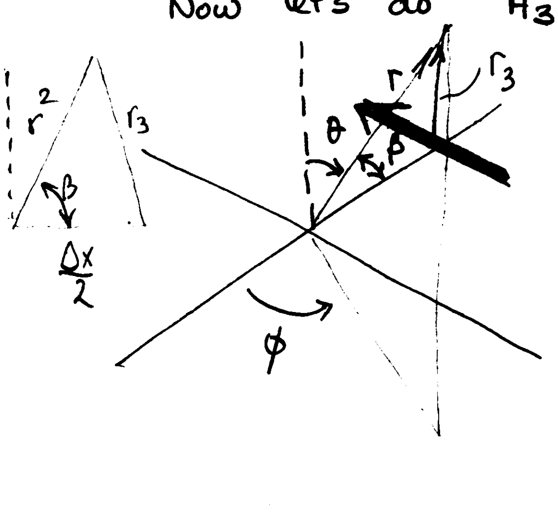
$$= r^2 + \frac{\Delta x^2}{4} + r\Delta x \cos\beta$$

The vector potential from this current element is then

$$\underline{A}_1 = \frac{\mu_0 (I\Delta y)\underline{a}_y}{4\pi r_1}$$

This assumes the current element is very short and concentrated at a point which is good enough if $\Delta y \to 0$.

Now let's do A_3 from the opposite side.



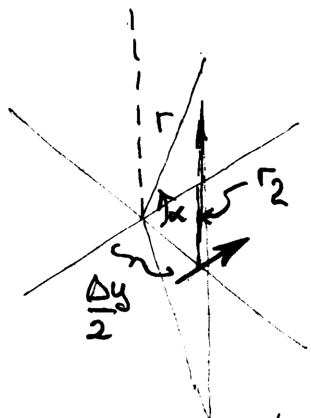
As before

$$r_3^2 = r^2 + \left(\frac{\Delta x}{2}\right)^2 - 2(r)\left(\frac{\Delta x}{2}\right)\cos(\beta)$$

$$r_3^2 = r^2 + \frac{\Delta x^2}{4} - r\Delta x \cos\beta$$

$$\underline{A}_3 = -\frac{\mu_0 (I\Delta y)\underline{a}_y}{4\pi r_3}$$

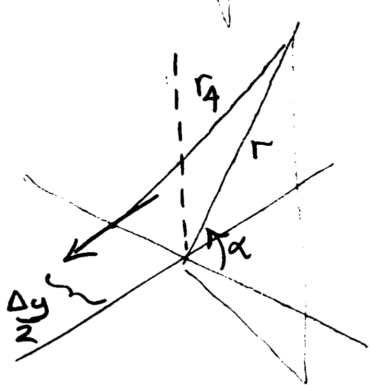
Note that we defined β as we did so as to be able to use the law of cosines. Now we do the same thing for sides 2 and 4. except using angle α now.



$$r_2^2 = r^2 + \left(\frac{\Delta y}{2}\right)^2 - 2(r)\left(\frac{\Delta y}{2}\right)\cos(\alpha)$$

$$r_2^2 = r^2 + \frac{\Delta y^2}{4} - r\Delta y \cos\alpha$$

$$\underline{A}_2 = \frac{-\mu_0 (I\Delta x)\, \underline{a_x}}{4\pi r_2}$$



$$r_4^2 = r^2 + \left(\frac{\Delta y}{2}\right)^2 - 2r\left(\frac{\Delta y}{2}\right)\cos(\pi - \alpha)$$

$$r_4^2 = r^2 + \frac{\Delta y^2}{4} + r\Delta y \cos\alpha$$

$$\underline{A}_4 = \frac{\mu_0 (I\Delta x)\, \underline{a_x}}{4\pi r_4}$$

The obvious thing to do is now to add up the vector potentials in the limit as $\frac{\Delta x}{r}, \frac{\Delta y}{r} \to 0$

Let's look at A_1 first, after substituting in our expression for r_1

$$\underline{A_1} = \frac{\mu_0 (I\Delta y)\, \underline{a_y}}{4\pi\sqrt{r^2 + \frac{\Delta x^2}{4} + r\Delta x\cos\beta}}$$

$$= \frac{\mu_0 (I\Delta y)\, \underline{a_y}}{4\pi r\sqrt{1 + \frac{1}{4}\left(\frac{\Delta x}{r}\right)^2 + \left(\frac{\Delta x}{r}\right)\cos\beta}}$$

$$\approx \frac{\mu_0 (I\Delta y)\, \underline{a_y}}{4\pi r\sqrt{1 + \left(\frac{\Delta x}{r}\right)\cos\beta}}$$ where we keep only first order terms.

$$\underline{A}_1 \approx \frac{\mu_0 I \Delta y\, \underline{a_y}}{4\pi r}\left[1+\left(\frac{\Delta x}{r}\right)\cos\beta\right]^{-\frac{1}{2}}$$

$$\approx \frac{\mu_0 I}{4\pi}\left(\frac{\Delta y}{r}\right)\underline{a_y}\left[1-\frac{1}{2}\frac{\Delta x}{r}\cos\beta\right]$$ after a Taylor Series expansion to first order.

And after some re-arrangement

$$\underline{A}_1 \approx \frac{\mu_0 I}{4\pi}\,\underline{a_y}\left[\left(\frac{\Delta y}{r}\right)-\frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right]$$

We can do the same thing for each term to get

$$\underline{A}_2 \approx -\frac{\mu_0 I}{4\pi}\,\underline{a_x}\left[\left(\frac{\Delta x}{r}\right)+\frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right]$$

$$\underline{A}_3 \approx -\frac{\mu_0 I}{4\pi}\,\underline{a_y}\left[\left(\frac{\Delta y}{r}\right)+\frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\beta\right]$$

$$\underline{A}_4 \approx \frac{\mu_0 I}{4\pi}\,\underline{a_x}\left[\left(\frac{\Delta x}{r}\right)-\frac{1}{2}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha\right]$$

Summing up,

$$\underline{A} = \underline{A}_1+\underline{A}_2+\underline{A}_3+\underline{A}_4$$

$$= \frac{\mu_0 I}{4\pi}\left[\underline{a_y}\left\{\frac{\Delta y}{r}-\frac{1}{2}\frac{\Delta x}{r}\frac{\Delta y}{r}\cos\beta\right\}-\underline{a_x}\left\{\frac{\Delta x}{r}+\frac{1}{2}\frac{\Delta x}{r}\frac{\Delta y}{r}\cos\alpha\right\}\right.$$
$$\left.-\underline{a_y}\left\{\frac{\Delta y}{r}+\frac{1}{2}\frac{\Delta x}{r}\frac{\Delta y}{r}\cos\beta\right\}+\underline{a_x}\left\{\frac{\Delta x}{r}-\frac{1}{2}\frac{\Delta x}{r}\frac{\Delta y}{r}\cos\alpha\right\}\right]$$

After cancellation of like terms

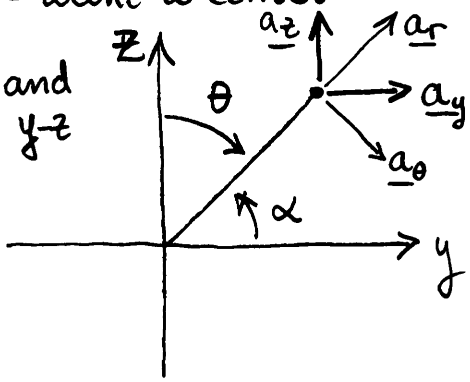
$$\underline{A} = \frac{\mu_0 I}{4\pi}\left[-\underline{a_x}\left(\frac{\Delta x}{r}\right)\left(\frac{\Delta y}{r}\right)\cos\alpha - \underline{a_y}\,\frac{\Delta x}{r}\frac{\Delta y}{r}\cos\beta\right]$$

$$\underline{A} = -\frac{\mu_0 I}{4\pi r^2}\Delta x \Delta y \left[\underline{a_x} \cos\alpha + \underline{a_y} \cos\beta \right]$$

We really want this in spherical coordinates because it will be easier to plot and manipulate later.

We want to convert α and β into θ, ϕ coordinates.

If $\phi = \frac{\pi}{2}$ and we are in y-z plane.



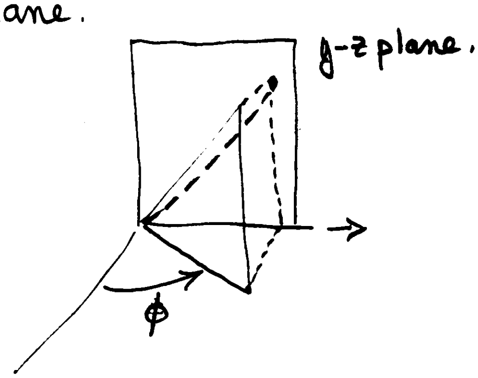
$$\underline{a_r} \cdot \underline{a_y} = |\underline{a_r}||\underline{a_y}|\cos\alpha = \cos\alpha$$

So we can interpret α as the angle between the r-vector and the y-axis (y-vector).

But in terms of spherical coordinates $\cos\alpha = \cos\left(\frac{\pi}{2} - \theta\right)$

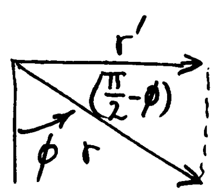
$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = +\sin\theta$$

Now what happens if $\phi \neq \frac{\pi}{2}$ The above relationship holds true for r in the plane so we project r into the y-z plane.



What is projection of r onto y-z plane.

To clarify look at top view.



$\therefore$ r' which is what we want is $r\cos\left(\frac{\pi}{2} - \phi\right)$

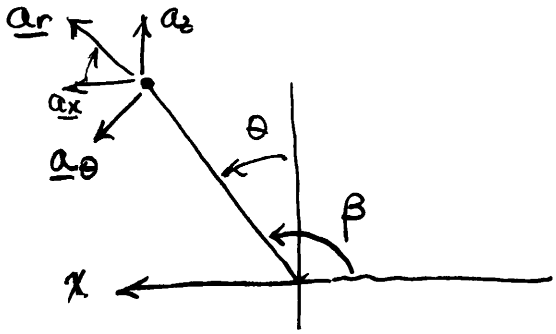
$$= r\left\{\cancel{\cos\frac{\pi}{2}}^{0}\cos\phi + \cancel{\sin\frac{\pi}{2}}^{1}\sin\phi\right\}$$

$$= r\sin\phi$$

Therefore $\cos\alpha = \sin\theta\sin\phi$

because $r'' = r'\sin\theta = r\sin\phi\,\sin\theta = r\cos\alpha$

We can do the same thing in the x-z plane for β.

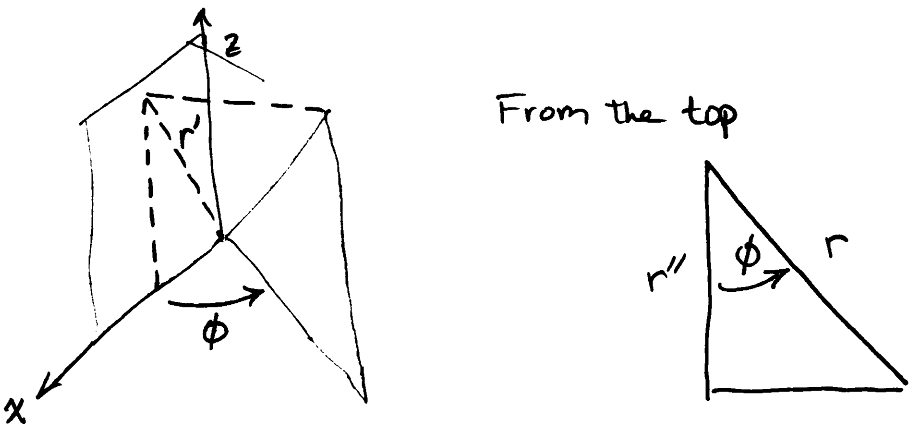


$$\underline{a_r} \cdot \underline{a_x} = |a_r||a_x| \cos(\pi - \beta)$$
$$= \cos\pi \cos\beta + \sin\pi \sin\beta$$
$$= -\cos\beta$$

If $\phi = 0$ $\quad \underline{a_r} \cdot \underline{a_x} = -\cos\beta = -\cos\left(\frac{\pi}{2} + \theta\right)$

$$= -\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$
$$= +\sin\theta$$

If $\phi \neq 0$ we must project $\underline{a_r}$ into the x-z plane then onto the x-axis.



$r'' = r\cos\phi$

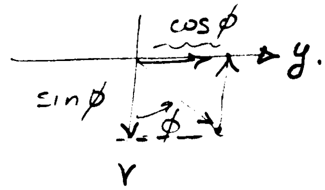
Note The − sign here since $\underline{a_r} \cdot \underline{a_x} = -\cos\beta$.

$$\underline{a_r} \cdot \underline{a_x} = \sin\theta\cos\phi = -\cos\beta$$

Rewriting $\underline{A}$ in spherical coordinates

$$\underline{A} = -\frac{\mu_0 I}{4\pi r^2} \underbrace{(\Delta x \Delta y)}_{\text{call this}} \left\{ \underline{a_x} \sin\phi \sin\theta - \underline{a_y} \sin\theta \cos\phi \right\}$$

ΔS, the area of the loop



$$\underline{A} = \frac{\mu_0 I}{4\pi r^2} \Delta S' \sin\theta \underbrace{\left\{ -\underline{a_x} \sin\phi + \underline{a_y} \cos\phi \right\}}_{\text{this is just } \underline{a_\phi} \text{ in rectangular coordinates}}$$

$$\underline{A} = \frac{\mu_0 I}{4\pi r^2} \Delta S' \sin\theta \, \underline{a_\phi}$$

Now, that we have $\underline{A}$ entirely in spherical coordinates we can compute the resultant $\underline{B}$

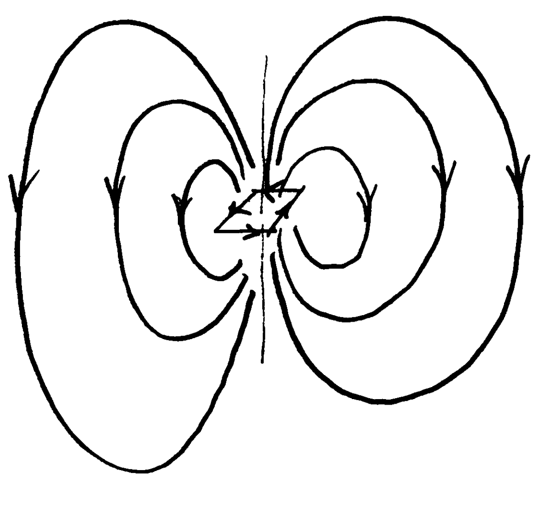
$$B = \underline{\nabla} \times \underline{A} = \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (A_\phi \sin\theta) \, \underline{a_r} - \frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \, \underline{a_\theta}$$

$$= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\mu_0 I}{4\pi r^2} \Delta S' \sin^2\theta \right) \underline{a_r} - \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{\mu_0 I}{4\pi} \Delta S' \sin\theta \, \frac{1}{r} \right) \underline{a_\theta}$$

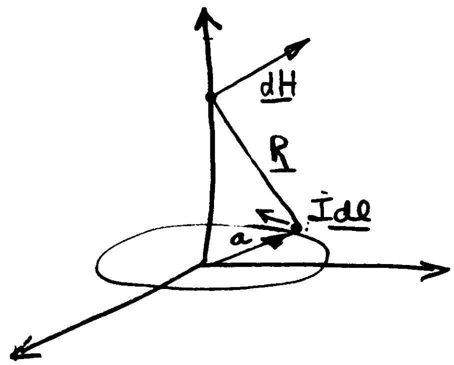
$$= \frac{1}{r\cancel{\sin\theta}} \frac{\mu_0 I}{4\pi r^2} \Delta S' \, 2 \cancel{\sin\theta} \cos\theta \, \underline{a_r} - \frac{1}{r} \frac{\mu_0 I}{4\pi} \Delta S' \sin\theta \left(-\frac{1}{r^2} \right) \underline{a_\theta}$$

$$= \frac{\mu_0 I}{4\pi r^3} \Delta S' \left[2\cos\theta \, \underline{a_r} + \sin\theta \, \underline{a_\theta} \right]$$

This is exactly the field (as) we got for the electric dipole moment.



Example: Field from a single current loop of radius a

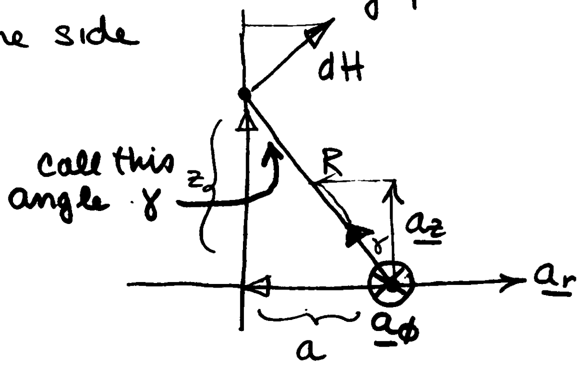


This is a unique problem best solved by the Biot - Savart Law.

$$d\underline{H} = \frac{I\, d\underline{l} \times \underline{a}_R}{4\pi R^2}$$

By inspection $d\underline{H}$ is in the direction shown if $d\underline{l}$ is in the a_ϕ direction in the x-y plane

From the side



To evaluate $d\underline{H}$ we need expressions for $d\underline{l}$ and $\underline{a}_R$.
From the above diagrams

$$d\underline{l} = a\, d\phi\, \underline{a}_\phi$$

$\underline{a}_R$, however, has both $\underline{a}_r$ and $\underline{a}_z$ components.

If $\quad \underline{a}_R = c_1 \underline{a}_r + c_2 \underline{a}_z$

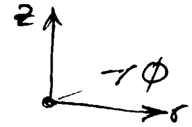
$$\underline{a}_R \cdot \underline{a}_z = c_1 \underbrace{\underline{a}_r \cdot \underline{a}_z}_{0} + c_2 \underbrace{\underline{a}_z \cdot \underline{a}_z}_{1}$$

$$c_2 = \underline{a}_R \cdot \underline{a}_z = |\underline{a}_R||\underline{a}_z| \cos\gamma$$

$$c_1 = \underline{a}_R \cdot \underline{a}_r = |\underline{a}_R||\underline{a}_r| \cos(\pi/2 + \gamma) = -\sin\gamma$$

$$\therefore \quad \underline{a}_R = \cos\gamma\, \underline{a}_z - \sin\gamma\, \underline{a}_r$$

$$d\underline{H} = \frac{I\, a\, d\phi\, \underline{a}_\phi \times (\cos\gamma\, \underline{a}_z - \sin\gamma\, \underline{a}_r)}{4\pi (z^2 + a^2)}$$



$$\underline{dH} = \frac{I a \, d\phi}{4\pi (a^2 + z^2)} \left[\cos\gamma \, \underline{a_r} + \sin\gamma \, \underline{a_z} \right]$$

$$= \frac{I a \, d\phi}{4\pi} \left[\frac{\cos\gamma}{a^2 + z^2} \underline{a_r} + \frac{\sin\gamma}{a^2 + z^2} \underline{a_z} \right]$$

Before integrating we note that

$$\cos\gamma = \frac{z}{(a^2 + z^2)^{1/2}}$$

$$\sin\gamma = \frac{a}{(a^2 + z^2)^{1/2}}$$

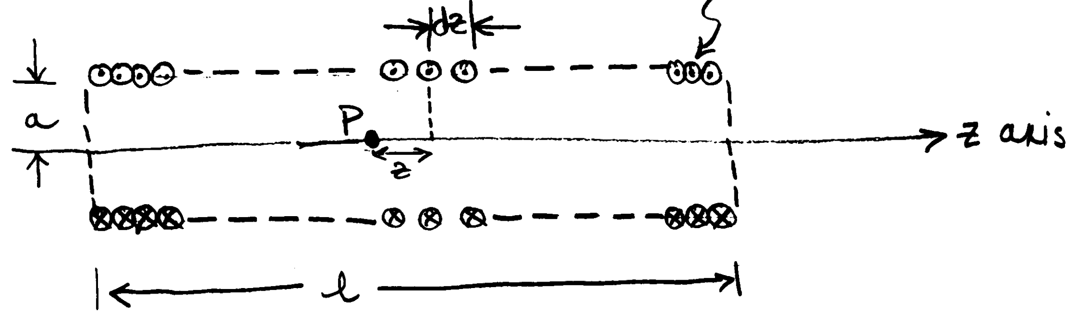
Integrating to get each component

$$(H_r)_{total} = \frac{Ia}{4\pi} \int_0^{2\pi} \frac{z \, \underline{a_r}}{(a^2 + z^2)^{3/2}} \, d\phi \qquad = 0$$

but as we rotate the $\underline{ar}$'s cancel giving

$$(H_z)_{total} = \frac{Ia}{4\pi} \int_0^{2\pi} \frac{a}{(a^2 + z^2)^{3/2}} \, d\phi = \frac{Ia^2}{2 \, (a^2 + z^2)^{3/2}}$$

Example : solenoid



N turns total

Problem: find the field along the z-axis at P (P is at the center)

Solution: sum the fields from a single loop.

from previous example the field from the loop is

$$H_z = \frac{I a^2}{2(a^2+z^2)^{3/2}}$$

call this field dH_z and integrate over length of loop.

$$dH_z = \frac{(dI)\, a^2}{2(a^2+z^2)^{3/2}} = \frac{a^2}{2(a^2+z^2)^{3/2}} \Big(\frac{N}{l} I\, dz\Big)$$

current = # of turns/length × current/turn × length

$$H_z(P) = \int \frac{a^2}{2(a^2+z^2)^{3/2}} \Big(\frac{N}{l} I\, dz\Big) = \frac{a^2 N I}{2l} \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{\cancel{a^2} N I}{2l} \left[\frac{z}{\cancel{a^2}(z^2+a^2)^{1/2}} \Bigg|_{-l/2}^{+l/2} \right]$$

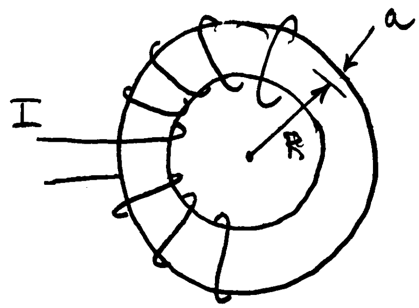
$$H_z(P) = \frac{NI}{2l} \frac{l/2 - (-l/2)}{\left[(\frac{l}{2})^2 + a^2\right]^{1/2}} = \frac{NI\cancel{l}}{2\cancel{l}(\frac{l^2}{4}+a^2)^{1/2}} = \frac{NI}{2(\frac{l^2}{4}+a^2)^{1/2}}$$

Note that if $l \gg a$

$$H_z \approx \frac{NI}{2\,\frac{l}{2}} = \frac{NI}{l}$$

As long as $\frac{l}{a} > 4$ this is a pretty good approx. which we will use later.

Example: toroid



A toroid is simply a solenoid in which the ends are brought together.

The length of this solenoid is

$$l = 2\pi R$$

For the linear solenoid $\quad H_z = \dfrac{NI}{2\left(\frac{l^2}{4} + a^2\right)^{1/2}}$

For the toroid $\quad H_\phi = \dfrac{NI}{2\left(\frac{4\pi^2 R^2}{4} + a^2\right)^{1/2}}$

$$= \frac{NI}{2(\pi^2 R^2 + a^2)^{1/2}}$$

Torque on a magnetic dipole:

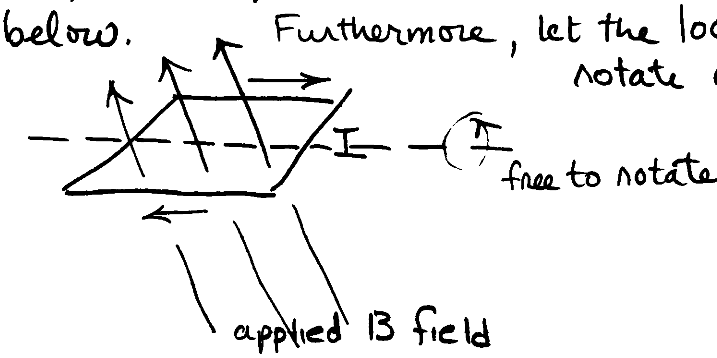
Before we begin our study of magnetic materials we must first examine how magnetic dipoles interact with applied fields. For as we shall see later many materials act like they are composed of many atomic dipoles.

The basic law governing electromagnetic forces is the Lorentz force Law which will discussed at great length when we study magnetic forces, For now it suffices to recall from P2 that the force on a moving charge is given by

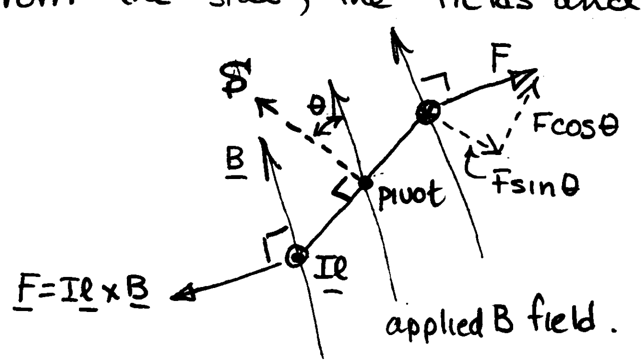
$$F = q\underline{v} \times B$$

where $\underline{v}$ is the velocity of the moving charge. Note That $q\underline{v}$ is nothing more than coulomb - meters/sec, i.e. our old friend $I\underline{l}$, or $\underline{F} = I\underline{l} \times \underline{B}$

Consider, now, a loop of current in an applied $\underline{B}$ field as shown below. Furthermore, let the loop be free to rotate as shown.



From the side, the fields and forces are as shown below.



$\underline{S}$ is the vector surface area of the loop. where the vector direction is determined by the right-hand rule.

The applied $\underline{B}$ field will produce a Force $\underline{F}$ which can be decomposed into a component $F\cos\theta$ which attempts to pull the loop apart and a second component $F\sin\theta$ which acts to rotate the loop. This latter force is what we are interested in.

For the electric dipole we defined the dipole moment

$$\underline{p} = q\underline{d}.$$

In a like manner, for the magnetic dipole we define

$$\underline{m} = I\underline{A}$$

The torque acting on the loop can now be written in terms of $\underline{m}$. First, write the torque on the loop as

$$T = \underbrace{2}_{\text{sides produce torque in same direction}} (\underbrace{F}_{\text{force perpendicular to loop}} \times \text{moment arm})$$

$$T = 2\,(F\sin\theta)\left(\frac{d}{2}\right)$$

$$T = \boxed{2}\,(I\ell B)\sin\theta\,\frac{d}{2}$$

these form area.

$$T = I A B \sin\theta$$

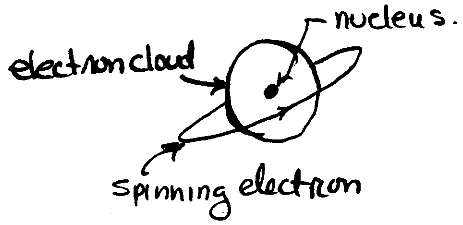
In vector form this becomes

$$\underline{T} = I\underline{A} \times \underline{B}$$

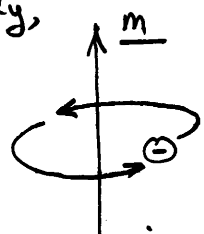
or

$$\underline{T} = m \times \underline{B}$$

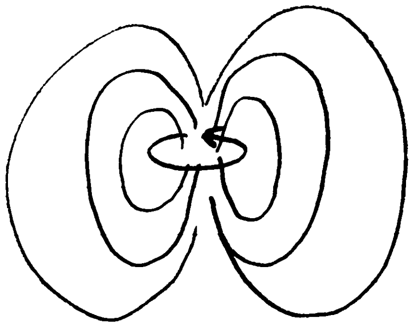
Matter has several basic natural dipole moments. Consider the electron spinning about its nucleus. This is a circular current flow and forms a microscopic magnetic dipole.



Schematically,



The field looks like



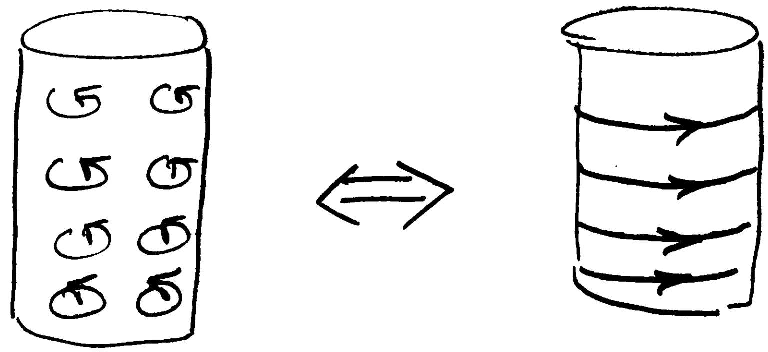
Just as we defined the macroscopic polarization $\underline{P}$ to be the average dipole moment

$$\underline{P} = \lim_{\Delta v \to 0} \frac{\sum_i \underline{p_i}}{\Delta v}$$

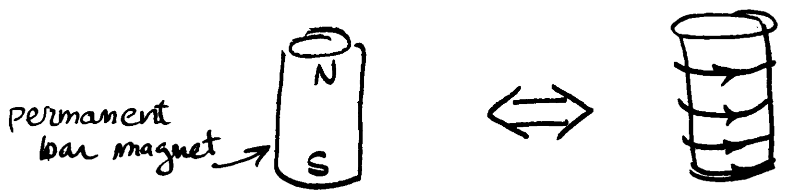
we define the macroscopic magnetization $\underline{M}$ to be the average microscopic dipole moment

$$\underline{M} = \lim_{\Delta v \to 0} \frac{\sum_i \underline{m_i}}{\Delta v}.$$

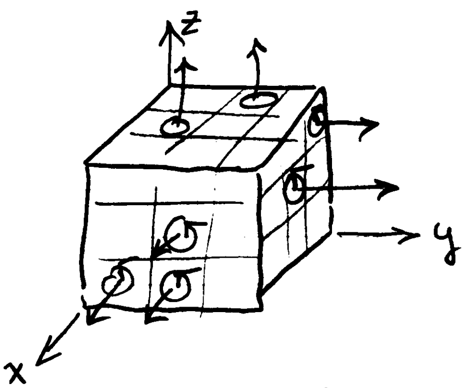
It becomes very complex when we look at the resultant macroscopic magnetizations as the small currents can combine to yield larger equivalent currents as shown below.



Furthermore as we shall show later permanent magnets act like currents

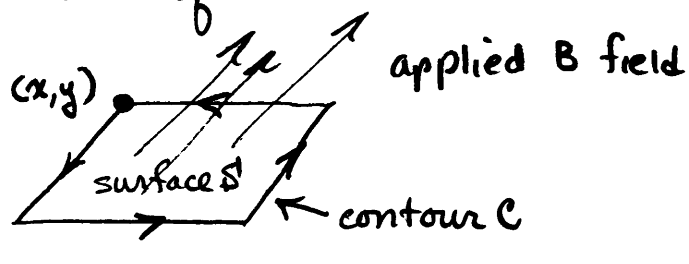


At this point we are ready to look at the macroscopic magnetization of matter. Let us examine the cube of matter shown below



The magnetic dipoles shown are not random.

To develop $\underline{P}$ we considered the divergence of $\underline{D}$ over a similar cube of electric dipoles, i.e. we wrote an expression for the 'flux' source using Gauss' Law. Any corresponding expression for $\underline{M}$ must use Ampere's Law and the vector curl in an analogous manner. This means we must examine a <u>slice</u> of the above cube and not just a cubic volume. Why? Returning to Ampère's Law we see that ampère's law relates the current through a surface (i.e. a slice of the cube) to the field around that surface (i.e. along the contour). This motivates us to consider the "slice" of matter shown below.



The current loop is assumed as is an applied $\underline{B}$ field; we will study their interaction.

Now, as we know any dipoles enclosed by the loop will tend to align with the applied field because of the torque generated by the dipole field interacting with the $\underline{B}$ field.

Recalling that the microscopic dipole moment is

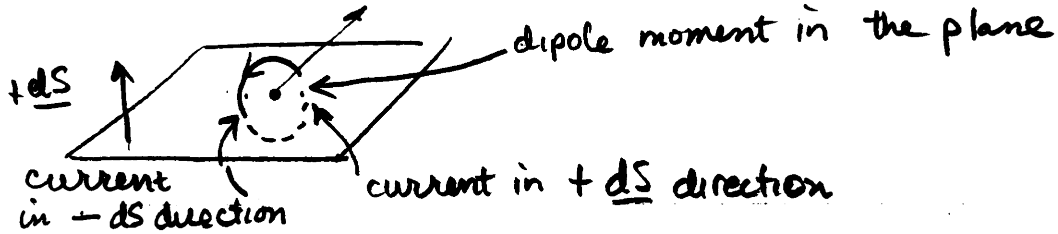
$$\underline{m} = I d\underline{s}$$

where this $\underline{m}$ is due to "probably" individual atoms. The resulting overall magnetization must be given by

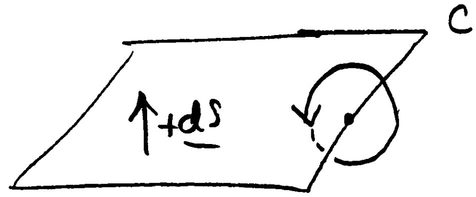
$$\underline{M} = N \underline{m} = N I d\underline{S}$$

where N is the density of magnetic dipoles/unit volume.

Now let's apply Ampère's Law to the above loop. What is the normal component of $\underline{J}$ relative to the contour C? Consider a single dipole moment. If it is in the interior of C there is a net contribution of zero.

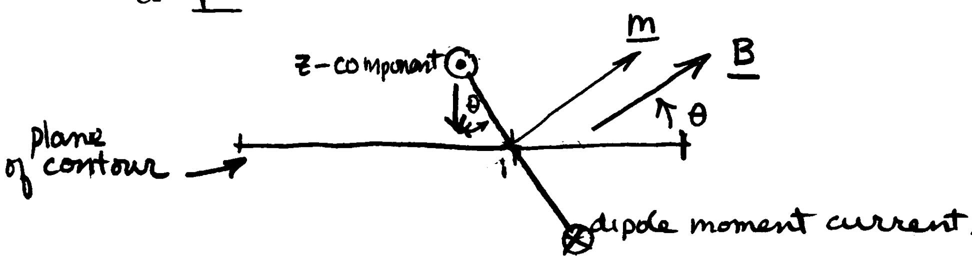


At the surface there is just as much current in the $+d\underline{S}$ direction as in the $-d\underline{S}$ direction. The only place there may be a non-zero contribution is at the edges of the loop. as shown below



Note that the dipole shown has a net current in the $-d\underline{S}$ direction. The dipole only cuts through the surface S once allowing a non-zero net current.

Before proceeding any further we need to develop an expression for this net current. Consider a magnetic dipole where $\underline{B}$ and $\underline{m}$ are at an angle θ relative to $\underline{S}$.



The z-combonent of I is then given by

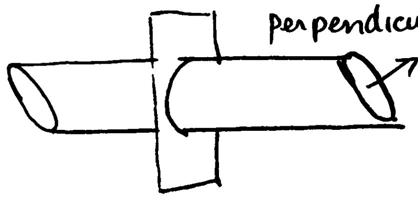
$$-I \quad \cos\theta$$

The rest of the dipole will contribute an equal but opposite current

$$+I \quad \cos\theta$$

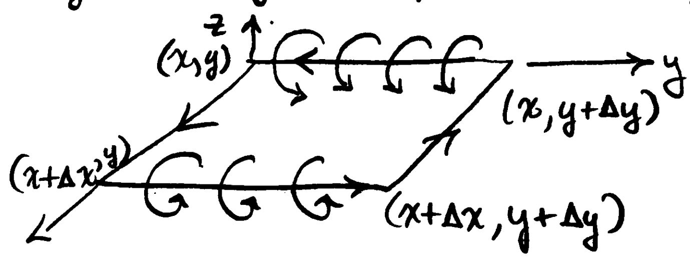
At the edge of the contour only one direction contributes. For the right hand side of the contour shown above that is

$$-I \quad \cos\theta.$$



perpendicular area is $\underline{dS\cos\theta}$
length is Δy
so total volume is $dS\cos\theta\,\Delta y$.

Returning to our original loop. and drawing the dipoles along the contour.



For the section of the contour at x we get a net current of

$$I_z = N\,(-I\ \cos\theta)\big|_x\, dS\,\Delta y$$

where N = # of dipoles/unit volume.
$-I\ \cos\theta$ is the current due to one dipole
Δy = length of contour. and $dS\,\Delta y$ = volume of cyclinder.

The subscript x indicates that all expressions are evaluated at x since N, θ, etc. may be functions of x.

But $\underline{m} = N I\, d\underline{S}$ and we can interpret $N I\, dS\cos\theta$ as the y-component of $\underline{m}$ at x. So, rewriting I_z

$$I_z = -\,m_y\big|_x\,\Delta y.$$

We can do exactly the same thing at the $x+\Delta x$ edge of the contour to get

$$I_z = +\,m_y\big|_{x+\Delta x}\,\Delta y$$

where we noted that the current at $x+\Delta x$ will be in the $-z$ direction.

If the field is aligned along the x-axis we get exactly the same type of result except for the directions

$$+\,m_x\big|_y\ \Delta x = I_z$$

$$-\,m_x\big|_{y+\Delta y}\ \Delta x = I_z$$

The overall current through $d\underline{S}$ is then given by.

$$(I_z)_{total} = (m_y|_{x+\Delta x} - m_y|_x)\Delta y - (m_x|_{y+\Delta y} - m_x|_y)\Delta x$$

$$\frac{(I_z)\ total}{\Delta x \Delta y} = \frac{m_y|_{x+\Delta x} - m_y|_x}{\Delta x} - \frac{m_x|_{y+\Delta y} - m_x|_y}{\Delta y}$$

And after taking the limit of both sides as $\Delta x, \Delta y \longrightarrow 0$

$$J_z\ (total) = \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}$$

But what is this?

$$(\underline{\nabla} \times \underline{M})_z = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m_x & m_y & m_z \end{vmatrix} \cdot \underline{a_z} = \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}$$

Thus, we see that the surface current density in the z direction is given by the z-component of the curl of the magnetization, i.e.

$$J_z = (\underline{\nabla} \times \underline{m})_z$$

If we re-did our analysis for a loop oriented in the $\underline{a_x}$ and $\underline{a}_y$ directions we would get

$$J_x = (\underline{\nabla} \times \underline{m})_x$$

$$J_y = (\underline{\nabla} \times \underline{m})_y$$

Combining these results in vector form we have

$$\underline{J} = \underline{\nabla} \times \underline{m}$$

We will complete our discussion of magnetic materials by relating B, H and M. To do this we return to the historical form of Ampère's law,

$$\oint_{} \underline{H} \cdot d\underline{l} = \int_S \underline{J} \cdot d\underline{S}$$

Converting to differential form

$$\nabla \times \underline{H} = \underline{J}$$

We know that for free space $\underline{B} = \mu_0 \underline{H}$ so let us re-write the above equation as

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

We know now, however, that a magnetic material can contain a current $\underline{J}_m = \underline{\nabla} \times \underline{M}$ of magnetic origin. $\underline{M}$ can be due to either an applied magnetic field or to permanently magnetic material. This means that $\underline{J}$ in the above equation may have free and/or magnetic components, i.e.

$$\underline{J} = \underline{J}_f + \underline{J}_m$$

where J_f = free currents which are sources for magnetic materials

J_m = magnetic currents due to material-field interactor or, sometimes, to permanent magnets

Then,
$$\nabla \times \underline{H} = \underline{J}_f + \underline{J}_m$$

$$\nabla \times \frac{B}{\mu_0} = \underline{J}_f + \underline{\nabla} \times \underline{M}$$

$$\nabla \times \left(\frac{\underline{B}}{\mu_0} - \underline{M}\right) = \underline{J}_f$$

$$\nabla \times \left(\frac{\underline{B}}{\mu_0} - \underline{M}\right) = \underline{J}_f$$

where we recognized $\underline{B}$ as the total magnetic flux density due to both free and magnetic currents.

write
$$\frac{\underline{B}}{\mu_0} - \underline{M} = \underline{H}$$

or
$$\underline{B} = \mu_0(\underline{H} + \underline{M})$$

where $\underline{B}$ is in units of webers/m^2

$$\mu_0 = 4\pi \times 10^{-7} \text{ henrys/meter}$$

$\underline{H}$, $\underline{M}$ are in units of amperes/meter.

The expression $\underline{B} = \mu_0(H+M)$ is deceptively simple since M is often a complex function of H. In general we write

$$\underline{B} = \mu(H)\,\underline{H}$$

where $\mu(\underline{H})$ is not necessarily a linear function. Furthermore, $\underline{M}$ may not be even in the same direction as H.

There are four broad classifications of magnetic materials based upon the relationship of $\mu(\underline{H})$ to μ_0, as shown below

$\mu \lesssim \mu_0$ diamagnetic, due to the orbital motion of electrons

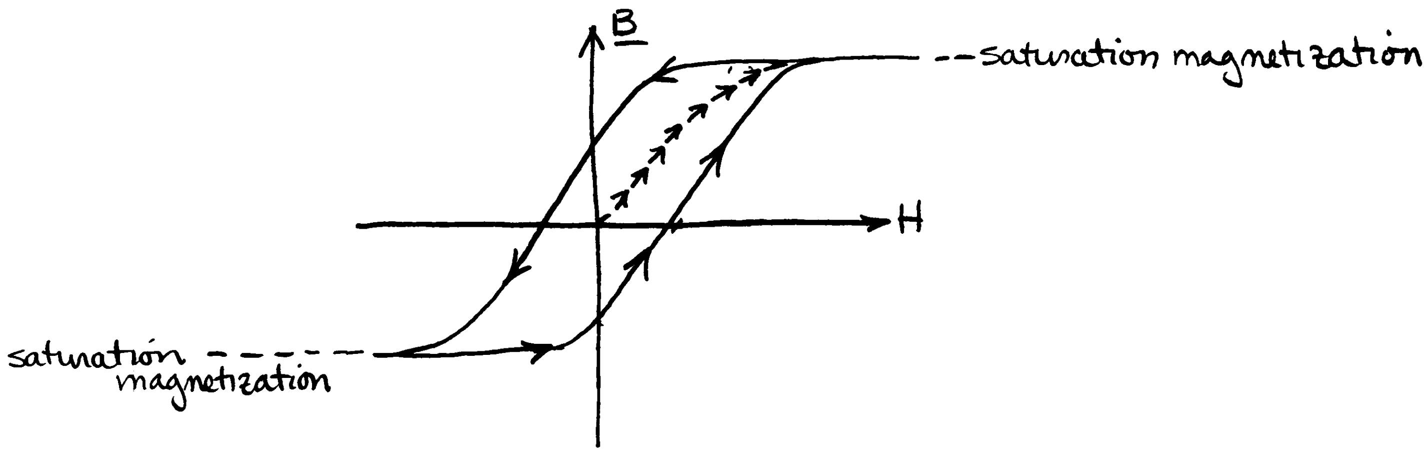
$\mu \gtrsim \mu_0$ paramagnetic, due to electron spin

$\mu \gg \mu_0$ ferromagnetic and ferrimagnetic, both due to electron spin

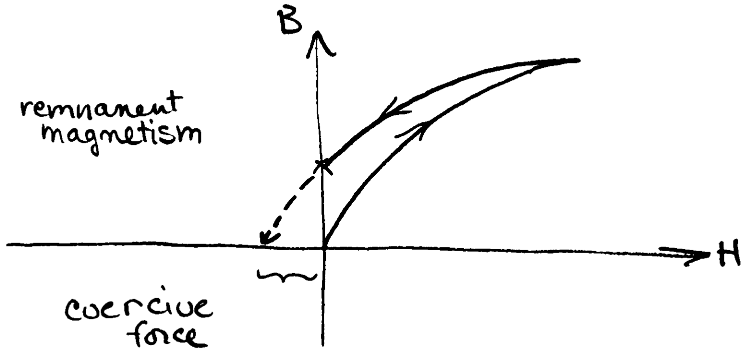
ferromagnetism is due to the quantum mechanical alignment of many electron spin moments. This causes regions called domains in which the magnetization is strong and uniform in the presence of an applied field. Ferromagnets are further classified as to whether they are hard or soft. Iron is a soft ferromagnet. On the other hand cobalt found in speaker magnets is a hard ferromagnetic material.

Ferrimagnetic materials, or ferrites, have large magnetizations but not as large as ferromagnetic materials. In ferrimagnetic materials the electron spins do not have a strong tendency to interact and, as a result, the spin moments often cancel resulting in a lower overall magnetization than ferromagnetic materials.

Ferromagnetic materials possess an unusual property called hysterisis in which $\underline{B}$ is a function not just of $\underline{H}$ but of what $\underline{H}$ has been. This is due to energy being necessary to create and/or destroy magnetic domains. A typical B-H curve for a ferromagnetic material is shown below.



The solid line shows the magnetization due to domain formation. However, at some point in the material's history its magnetization was zero. A magnetic field was applied to this ferromagnetic material forming the original domains. This is shown by the dashed line in the figure above. Note that once magnetized in this fashion the material remains permanently magnetized even when $\underline{H} = 0$. The curve of original magnetization is called the magnetization curve. Once the domains are formed the $\underline{B}$ field follows the solid curve known as the hysterisis curve. Note that it is multi-valued for each value of $\underline{H}$ with $\underline{B}$ depending upon whether $\underline{H}$ is increasing or decreasing. Finally, note that the $\underline{B}$ field saturates to some value even if $\underline{H}$ increases further.

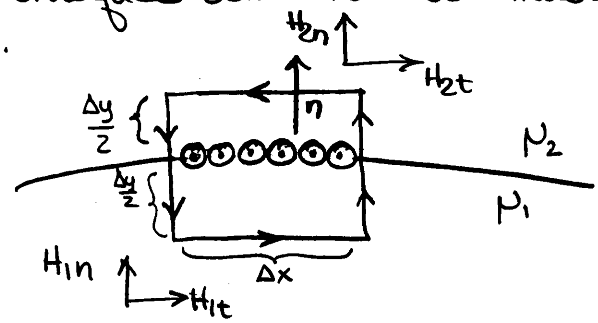


The value of B remaining after H goes to zero is known as the remanent magnetism. The field required to force B to zero is called the coercive force.

Magnetic boundary conditions

Now that we understand the relationships between J, μ, B and M we can look at what happens to the tangential & normal components of the magnetic field at an interface between different types of materials.

Consider an interface between two materials of different permeability.



Apply Ampère's to the contour shown above. where J_f is a "free" surface current.

$$\oint \underline{H} \cdot d\underline{l} = \int \underline{J} \cdot d\underline{S}$$

$$\oint \underline{H} \cdot d\underline{l} = H_{1t} \Delta x + H_{1n} \frac{\Delta y}{2} + H_{2n} \frac{\Delta y}{2} - H_{2t} \Delta x - H_{2n} \frac{\Delta y}{2} - H_{1n} \frac{\Delta y}{2}$$

$$= (H_{1t} - H_{2t}) \Delta x$$

$$\oint \underline{J} \cdot d\underline{S} = J_n \Delta x \Delta y = I, \text{ total current through the loop}$$

Equating our expressions

$$(H_{1t} - H_{2t}) \Delta x = J_n \Delta x \Delta y$$

$$(H_{1t} - H_{2t}) = J_n \Delta y.$$

What is the significance of $J_n \Delta y$ as $\Delta y \to 0$? $J_n \Delta y \to K_s$ the surface current, for as $\Delta y \to 0$ only the surface remains. In vector notation

$$\underline{n} \times (\underline{H_2} - \underline{H_1}) = \underline{K}$$

If we had a permanently magnetized material we would simply replace J_f by J_m and $\underline{H}$ by $\underline{M}$, i.e.

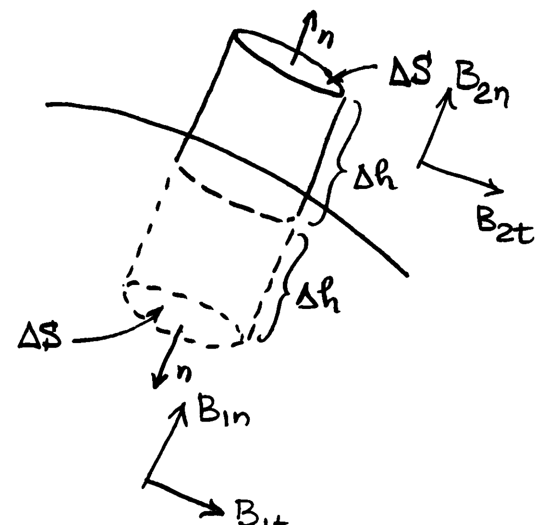
$$n \times (\underline{M_2} - \underline{M_1}) = K_m$$

Note that K_m is the "equivalent" surface current of the magnetization if $\underline{M}_2 = 0$. Recall that magnetization can be represented by equivalent currents.

To examine the normal component we will use Gauss' law which states

$$\oint \underline{B} \cdot \underline{n}\, ds = 0.$$

for the surface shown below.



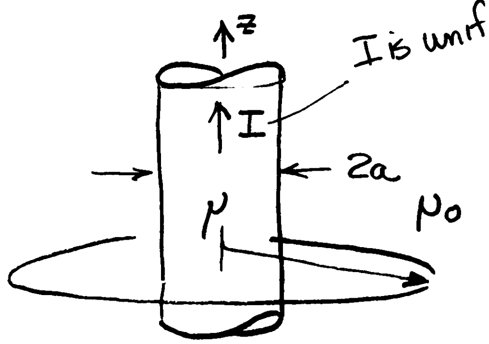
There will be no contribution to the integral from the sides of our surface as long as we pick a cylinder small enough that $\underline{B}$ is uniform across it. Looking at the contributions from the ends

$$\oint \underline{B} \cdot \underline{dS} = -(B_{1n})\,\Delta S + (B_{2n})\,\Delta S = 0$$

Thus, $B_{2n} = B_{1n}$ and we conclude that the normal component of $\underline{B}$ is continuous between differing materials.

Example: Plot $\underline{B}$, $\underline{H}$, m and J_m as functions of r for the conductor shown below

I is uniformly distributed



The conductor has permeability μ and is in free space where $\mu = \mu_o$. Using Ampère's Law for the contour shown,

for $r > a$ $\quad \oint \underline{H} \cdot \underline{dl} = H_\phi 2\pi r = I \quad \Rightarrow H_\phi = \dfrac{I}{2\pi r}$

for $r < a$ $\quad \int \underline{H} \cdot \underline{dl} = H_\phi 2\pi r = \int \underline{J} \cdot \underline{ds} = \dfrac{I}{\pi a^2} \pi r^2 = I \dfrac{r^2}{a^2}$

$$\Rightarrow H_\phi = \frac{I r}{2\pi a^2}$$

For $r > a$ $\quad B_\phi = \dfrac{\mu_o I}{2\pi r}$; for $r < a$ $\quad B_\phi = \dfrac{\mu I r}{2\pi a^2}$

How about m? If $B = \mu_o (H + m)$ then

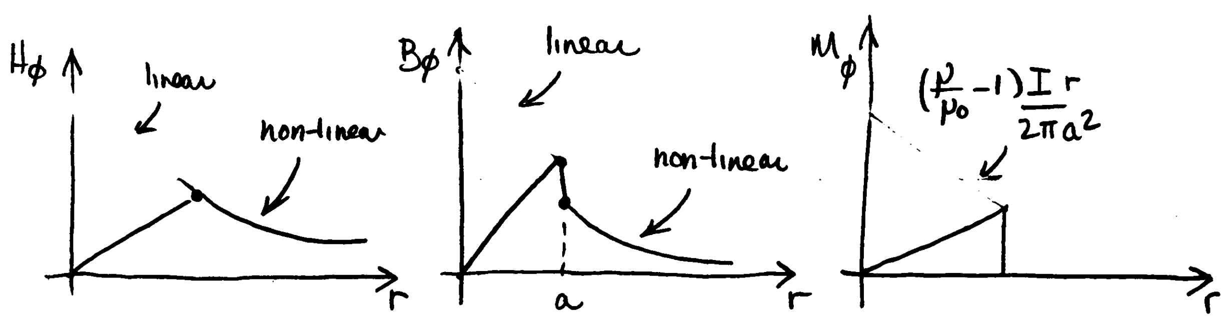
$$\frac{B}{\mu_o} = H + m$$

$$m = \left(\frac{B}{\mu_o} - H\right)$$

For $r > a$ $\quad m = 0$; for $r < a$ $\quad m = \left(\dfrac{\mu}{\mu_o} - 1\right) H$

as $B = \mu H$ there.

How about J_m? Let's plot B, H and m before we look at J_m.



Let's first find $\underline{J_m}$ mathematically, then look at a simpler intuitive approach.

Mathematically, $\underline{J_m} = \underline{\nabla} \times \underline{m}$

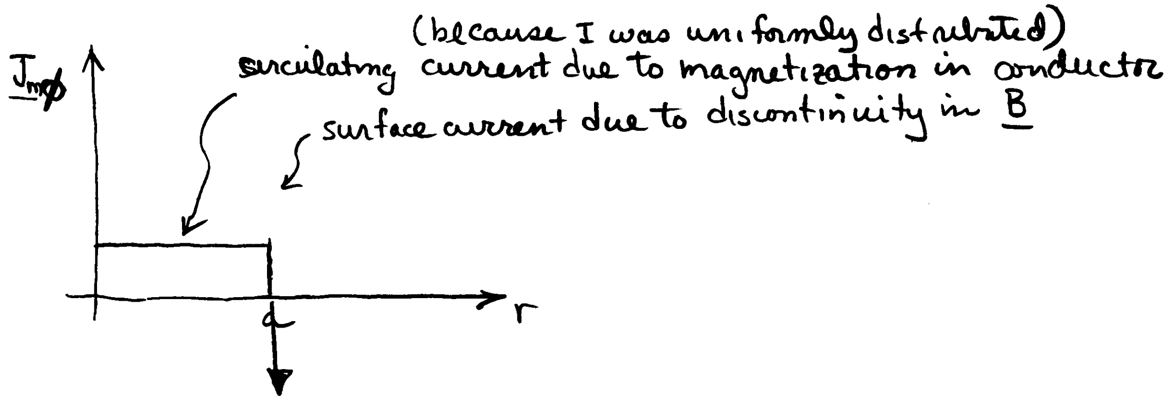
where $\underline{m} = (\mu/\mu_0 - 1)\underline{H} = \begin{cases} (\frac{\mu}{\mu_0}-1)\frac{Ir}{2\pi a^2}\,\underline{a_\phi} & \text{for } r<a \\ 0 & r>a \end{cases}$

$$\underline{J_m} = -\underbrace{\frac{\partial m_\phi}{\partial z}}_{0}\,\underline{a_r} + \frac{1}{r}\frac{\partial}{\partial r}(r\, m_\phi)\,\underline{a_z}$$

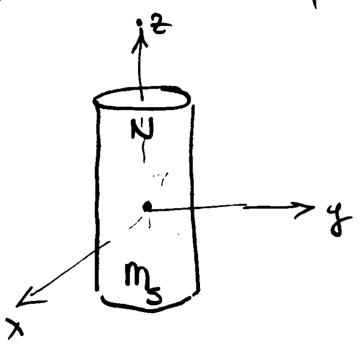
$$= \frac{1}{r}\frac{\partial}{\partial r}\left((\frac{\mu}{\mu_0}-1)\frac{Ir^2}{2\pi a^2}\,\underline{a_\phi}\right)$$

$$= \frac{1}{r}\left[(\frac{\mu}{\mu_0}-1)\frac{2Ir}{2\pi a^2}\,\underline{a_\phi}\right]$$

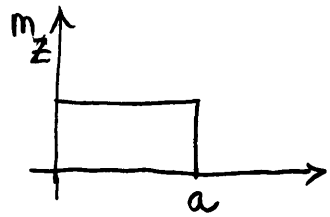
$$= (\frac{\mu}{\mu_0}-1)\frac{I}{\pi a^2}\,\underline{a_z}$$



Example: Field of a permanent magnet: Find the H field on the z-axis a distance z from the center of the rod of length L with magnetization m.



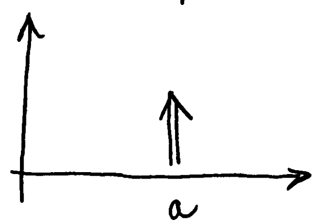
Plot $m(r)$.



$$J_m = (\nabla \times m) = \underline{a_r}\, \frac{1}{r} \underbrace{\frac{\partial m_z}{\partial \phi}}_{0} - \underline{a_\phi}\, \frac{\partial A_z}{\partial r} = -\underline{a_\phi}\, \frac{\partial A_z}{\partial r}$$

surface $\rightarrow$ $\underline{K_m} = m\, \delta(r-a)\, \underline{a_\phi}$

Plot J_m



This now looks like a solenoid with field given (on the z-axis) by

$$d\underline{H_z} = \underline{a_z}\, \frac{\overbrace{dI(z)}^{I\,dl\,=}\, a^2}{2\,(a^2+z^2)^{3/2}} = \underline{a_z}\, \frac{a^2}{2(a^2+z^2)^{3/2}}\, K_m\, dz$$

and the field at z exterior to the magnet is given by

$$H_z(z) = \int_{-\frac{l}{2}}^{+l/2} \frac{a^2\, m}{2(a^2+\underbrace{(z-z')^2}_{\text{represents many individual loops, each at } z'})^{3/2}}\, dz'$$

let $u = z - z'$
$du = -dz'$

$$H_z(z) = \int_{z+l/2}^{z-l/2} \frac{a^2\, m\, (-du)}{2\,(a^2+u^2)^{3/2}}$$

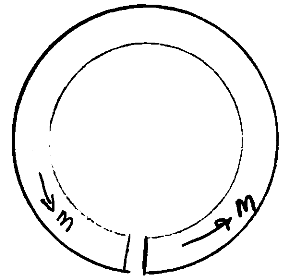
$$= \frac{a^2 M}{2} \int_{z-l/2}^{z+l/2} \frac{du}{(a^2+u^2)^{3/2}} = \frac{a^2 M}{2}\left[\frac{-1}{\sqrt{a^2+u^2}}\right]_{z-l/2}^{z+l/2}$$

$$H_z(z) = \frac{a^2 M}{2}\left[\frac{-1}{\sqrt{a^2+(z+l/2)^2}} - \frac{-1}{\sqrt{a^2+(z-l/2)^2}}\right]$$

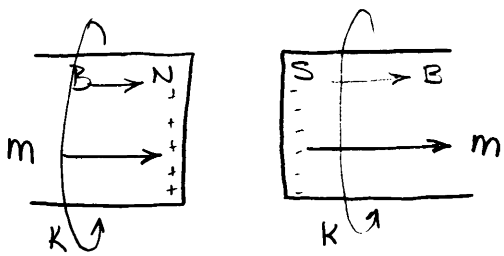
$$= \frac{a^2 M}{2}\left[\frac{1}{\sqrt{a^2+(z-l/2)^2}} - \frac{1}{\sqrt{a^2+(z+l/2)^2}}\right]$$

Example: permanent magnet toroid

permanent magnetization



detail



We know that this is equivalent to a surface current $K = \nabla \times m$.

abrupt.

M

$B = \mu_0 M$

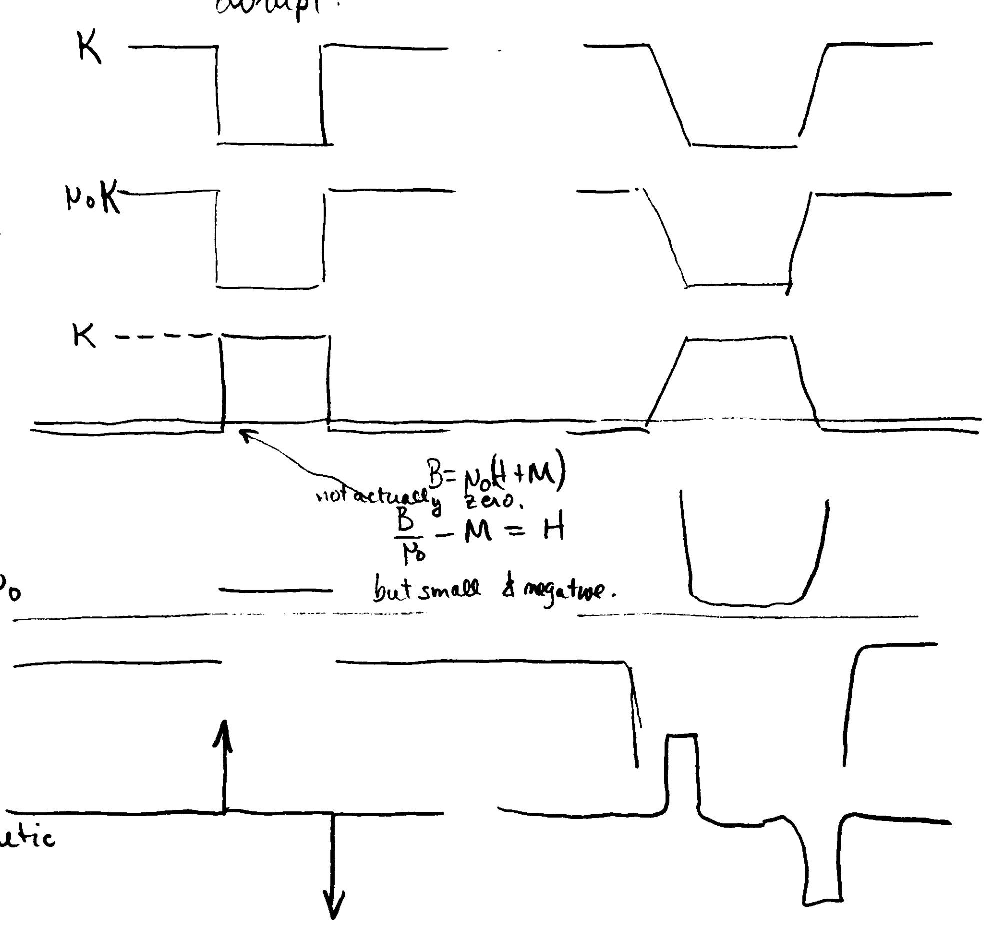
This is **new physics**

H

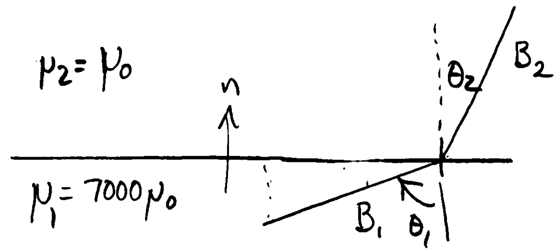
μ

$\rho_m = \nabla \cdot H$

looks like magnetic charge.



Example refractive type boundary problem



$\theta_1 = 85^\circ$

B.C.'s are $B_{1n} = B_{2n}$

$H_{1t} - H_{2t} = K = 0$

$$B_{1n} = B_{2n} \qquad \Rightarrow \qquad B_1 \cos\theta_1 = B_2 \cos\theta_2$$

$$H_{1t} = H_{2t} \qquad\qquad H_1 \sin\theta_1 = H_2 \sin\theta_2$$

$$\frac{H_1}{B_1} \tan\theta_1 = \frac{H_2}{B_2} \tan\theta_2$$

$$\frac{1}{\mu_1} \tan\theta_1 = \frac{1}{\mu_2} \tan\theta_2$$

$$\therefore \quad \frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} = \frac{7000\,\mu_0}{\mu_0} = 1000$$

$\tan 85^\circ = 11.43$

$$\tan\theta_2 = \frac{\tan 85^\circ}{7000} = .0016$$

Magnetic scalar potential:

Consider a source-free region of space in which $J = 0$.

Ampère's law then becomes

$$\nabla \times H = 0.$$

But it was Ampère's Law that prevented our using a scalar potential for H before. [Recall that $\nabla \times (\nabla \phi) = 0$ for any scalar ϕ]. So for the special case of a source free region of space we can define a magnetic scalar potential $\mathcal{F}$ for which

$$\nabla \times (\nabla \mathcal{F}) = 0$$

$$\text{and } H = -\nabla \mathcal{F}$$

Note that Faraday's law now becomes.

$$\oint \underline{H} \cdot \underline{dl} = \oint_{P_1}^{P_2} -\nabla \mathcal{F} = \mathcal{F}(P_2) - \mathcal{F}(P_1) = 0$$

just like for our conservative electric field. We can generalize our result here to a non-closed contour

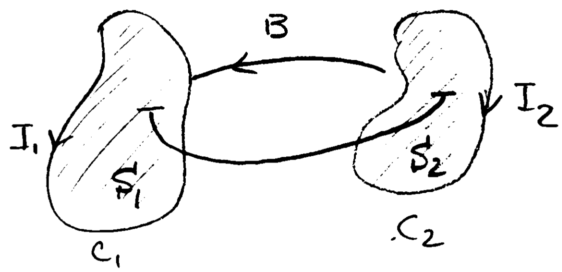
$$\mathcal{F} = -\int_{P_1}^{P_2} \underline{H} \cdot \underline{dl}$$

where this general magnetic scalar potential is called the magnetomotive force. Since $\mathcal{F}$ is caused by a magnetic field (or rather, is the source of a field) we can relate it to the magnetic field. For static fields we use the flux since $\mathcal{F}$ is defined along a contour, it makes sense to define the flux φ through a closed contour.

$$\varphi = \int \underline{B} \cdot \underline{dS}$$

Inductance

Earlier we defined capacitance as a fundamental relationship between the charge and the electrostatic potential for a system of charged bodies. Let us consider a typical magnetic system consisting of two current loops as shown below.



The system consists of two circuits C_1 and C_2, with currents I_1 and I_2. The current I_1 produces a field $\underline{B}_1$ which links the second circuit with a flux $\varphi_{12} = \int_{S_2} \underline{B}_1 \cdot d\underline{s}$.

The field also links itself with a flux $\varphi_{11} = \int_{S_1} \underline{B}_1 \cdot d\underline{s}$.

Recall that the time dependent version of Faraday's Law gives us:

$$\oint_C \underline{E} \cdot d\underline{l} = -\frac{d}{dt} \int \underline{B} \cdot d\underline{s}$$

or in terms of flux

$$\oint \underline{E} \cdot d\underline{l} = -\frac{d\varphi}{dt}$$

In terms of circuit 1 we note the following property know as Lenz's Law: If I_1 increases the flux φ_{11} through C_1 will increase creating an electric field and a, consequent, induced current in the <u>OPPOSITE</u> direction to the original <u>change</u> in current.

In terms of circuit 2: If I_1 increases the flux φ_{12} increases creating an induced current in the direction specified by Ampère's Law.

This ability of a changing current to produce an induced voltage (and current) is called inductance. If we are describing the change in voltage (or current) due to a current change in the same circuit we call it self-inductance. However, if we are describing the voltage (or current) induced in another circuit due to a change in current in one circuit we call it mutual inductance. Self-inductance is always defined with the letter L and is given by

$$L_{11} = \frac{\text{flux linking } C_1 \text{ due to current in } C_1}{\text{current in } C_1}$$

$$L_{11} = \frac{\phi_{11}}{I_1}$$

Note that the subscript is often dropped for self-inductance. Furthermore, whenever anyone describes the inductance of a circuit they are usually referring to its self-inductance.

Mutual inductance can be defined in a similar manner:

$$L_{ij} = \frac{\text{flux linking } C_j \text{ due to current in } C_i}{\text{current in } C_i}$$

$$= \frac{\phi_{ij}}{I_i}$$

Examples of self-inductance:

single loop:

$$H_z(\text{total})\Big|_{z=0} = \frac{Ia^2}{2(a^2)^{3/2}} = \frac{Ia^2}{2a^3} = \frac{Ia}{2}.$$

$$B_z(\text{total}) = \mu_0 \frac{Ia}{2}$$

$$\phi_{11} \cong \mu_0 \frac{Ia}{2} \cdot \pi a^2$$

$$L_{11} \approx \frac{\mu_0 I \pi a^3}{2I} = \frac{\mu_0 \pi a^3}{2}$$

solenoid (long):

$$H_z \approx \frac{NI}{l}$$

$$B_z \approx \frac{\mu NI}{l}$$

$$\phi_{11} \approx \frac{\mu NI}{l} \cdot A$$

$$L_{11} \approx \frac{\mu NIA}{lI} = \frac{\mu NA}{l}$$

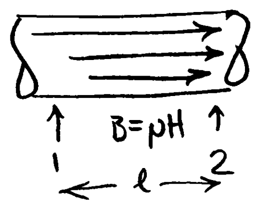
toroid:

$$H_\phi \approx \frac{NI}{2(\pi^2R^2+a^2)^{1/2}}$$

$$B_\phi \approx \frac{\mu NI}{2(\pi^2R^2+a^2)^{1/2}}$$

$$\phi_{11} \approx \frac{\mu NI}{2(\pi^2R^2+a^2)^{1/2}\, I} = \frac{\mu N}{2(\pi^2R^2+a^2)^{1/2}}$$

Let us now consider a simple magnetic circuit — a magnetic material containing a uniform B field.



The mmf from point 1 to point 2 is given by

$$\mathcal{F} = \int_{P_1}^{P_2} H \cdot d\ell = H\ell$$

The flux through any plane between P_1 and P_2 is given by

$$\varphi = \int B \cdot dS = \mu H A$$

In the manner of a circuit we can define an Ohm's Law between $\mathcal{F}$ and φ, i.e.

$$\mathcal{R} = \frac{\mathcal{F}}{\varphi} = \frac{H\ell}{\mu H A} = \frac{\ell}{\mu A}.$$

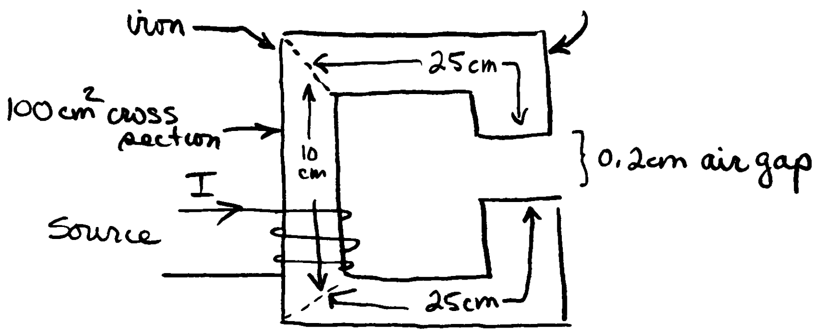
To see how closely this resembles Ohm's Law for a cyclindrical resistor recall that

$$R = \frac{\Phi}{I} \quad \text{where} \quad R = \frac{\ell}{\sigma A}$$

The analogy is very close if we identify the mmf as a voltage-like term and the flux as being like a current.

These arguements only hold true for magnetic materials in which the fluxes are confined to the materials and little field escapes. This would be equivalent to saying than an electric circuit cannot lose current as it travels a circuit. Such an assumption is only true when $\mu >> \mu_0$. Additional assumptions not repeated here are that the circuit is static. Additional considerations may have to be made for hysterisis and magnetic saturation.

Example: Find the number of turns required to give a field of $1 Wb/m^2$ in the gap.

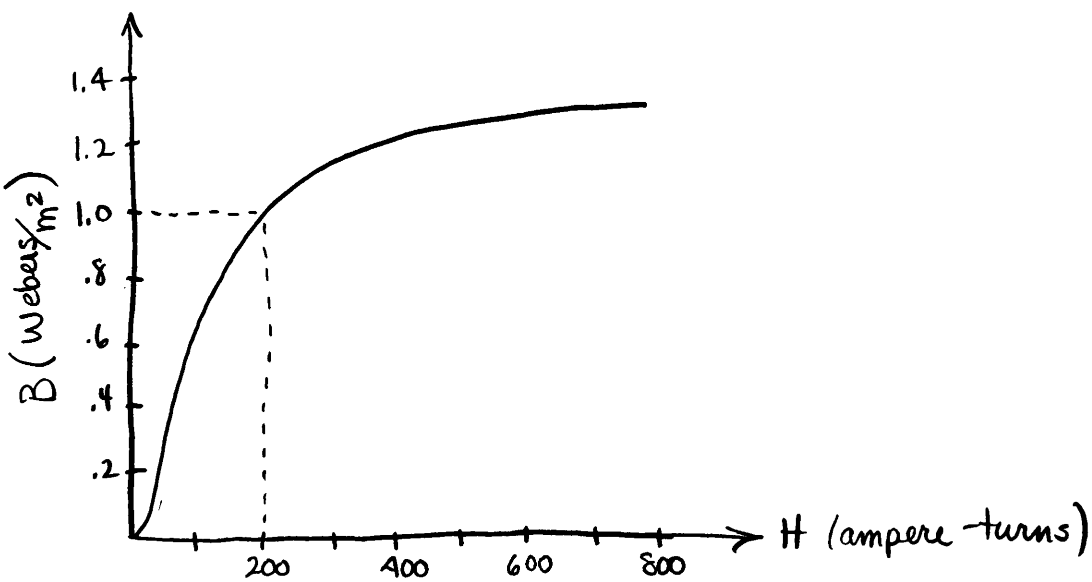


using our result that $\mathcal{R} = \dfrac{l}{\mu A}$

for the gap: $$\mathcal{R}_{gap} = \frac{2\times10^{-3}\ m}{4\pi\times10^{-7}\ H/m \times 25\times10^{-4}\ m^2}$$

$$= 6.36\times10^{5}\ \frac{\text{Ampere-turns}}{\text{weber.}} \quad \text{actually } \frac{1}{H}$$

for the arms we have a problem as iron is a material with hysterisis. Examine B-H curve below.



This is the saturation curve corresponding to an initial turn-on.

The problem is what are B and H? Well, we know B.

Since the fields are continuous (confined to magnetic materials) the field in the arms equals the field in the gap: 1 Weber/m^2

From the curve on the preceeding page this corresponds to an H of 200 Ampere-turns. Then

$$\mu = \frac{B}{H} = \frac{1}{200} = 5 \times 10^{-2}$$

Compare this with $\mu_0 = 4\pi \times 10^{-7}$!

$$R_{arm} = \frac{25 \times 10^{-2} + 25 \times 10^{-2} \text{ meters}}{(5 \times 10^{-2})\,(25 \times 10^{-4})\,m^2} = \frac{2 \times 10^{-2}}{5 \times 10^{-6}}$$

$$= 0.4 \times 10^{4} = .04 \times 10^{5} \ \frac{\text{A-turns}}{\text{weber}}.$$

The reluctance of the arm is negligible compared to the gap!

Continuing into the arm with the coil we have a change in cross-sectional area. ϕ is constant (continuous) so B must decrease

$$\phi = BA$$

The flux in the gap and arms was

$$\phi = \left(1 \ \frac{\text{Weber}}{m^2} \times 25 \times 10^{-4} m^2\right) = 25 \times 10^{-4} \text{ weber.}$$

In the coil region

$$\phi = B \cdot 100 \times 10^{-4} m^2$$

Thus, $$B = \frac{25 \times 10^{-4} \text{ webers}}{100 \times 10^{-4} \ m^2} = .25 \text{ webers}/m^2$$

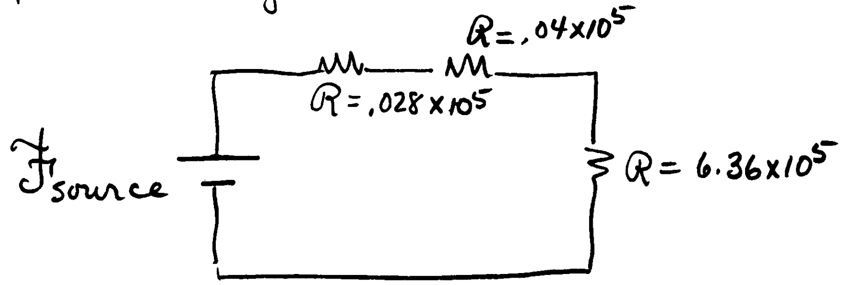
From our B-H curve for iron this is about 70 Ampere-turns for H.

$$\mu = \frac{B}{H} = \frac{.25 \text{ webers}/m^2}{70 \ \text{A-t}} = 36 \times 10^{-4}$$

We can now compute the reluctance of the coil region to be

$$\mathcal{R}_{coil} = \frac{l}{\mu A} = \frac{10 \times 10^{-2}}{35.7 \times 10^{-4} \times 100 \times 10^{-4}} \cong 2.8 \times 10^{3}$$

which is again small compared to the reluctance of the gap. Our magnetic circuit now looks like



The required flux in the gap was

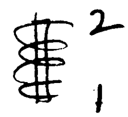
$$\phi = \left(1 \text{ webers}/\text{m}^2 \times 25 \times 10^{-4} \text{ m}^2\right) = 2.5 \times 10^{-3} \text{ Webers.}$$

Since the flux corresponds loosely to current we see that our source must deliver a $\phi = 2.5 \times 10^{-3}$ webers into a reluctance of $\mathcal{R} \cong 6.43 \times 10^{5}$. i.e.

$$\mathcal{F}_{source} = \mathcal{R}\phi = (6.43 \times 10^{5})(2.5 \times 10^{-3})$$
$$= 1.61 \times 10^{3} \text{ Ampere-turns.}$$

Now, we can determine the required # of coil turns. It is basically a solenoid for which we can use the simplified formula

$$H = \frac{NI}{l}$$



$$\therefore \quad \mathcal{F}_{source} = \int_1^2 H \cdot dl = \frac{NI}{l} \cdot l = NI.$$

This is a general result which can be used in most magnetic circuits.

If we assume $I = 10$ Amperes we have.

$$\mathcal{F}_{source} = N\,(10 \text{ Amperes}) = 1.61 \times 10^{3} \text{ Ampere-turns}$$

$$\therefore \quad N \approx 161 \text{ turns.}$$

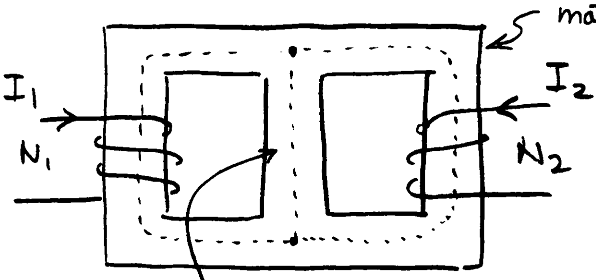
Note that we could have simply neglected the reluctance of the arms and the coil region and almost immediately get the result that

$$\mathcal{F}_{soure} \approx \mathcal{F}_{gap} = (6.36 \times 10^{5} \text{ A-t})(2.5 \times 10^{-4} \text{m})$$
$$= 1.59 \times 10^{3} \text{ A-t}$$

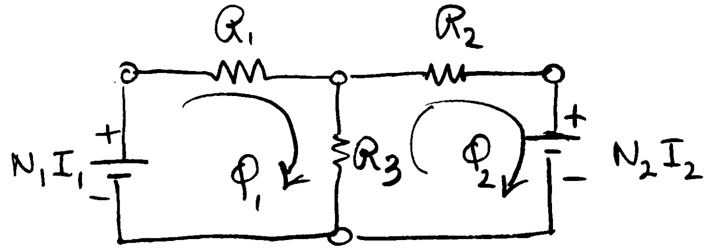
$$\therefore \quad N \approx \frac{1.59 \times 10^{3} \text{ A-t}}{10 \text{ A}} = 159 \text{ turns.}$$

This is a good result compared to the work we did to get a only slightly more accurate result.

Example: transformer type circuit (i.e. parallel)
Find the flux in the center section.



Solution: draw equivalent "electrical" circuit



$$\mathcal{R}_1 = \frac{l_1}{\mu A} \quad ; \quad \mathcal{R}_2 = \frac{l_2}{\mu A} \quad ; \quad \mathcal{R}_3 = \frac{l_3}{\mu A}$$

write loop equations

$$N_1 I_1 = \mathcal{R}_1 \varphi_1 + \mathcal{R}_3(\varphi_1 - \varphi_2)$$

$$-N_2 I_2 = \mathcal{R}_2 \varphi_2 + \mathcal{R}_3(\varphi_2 - \varphi_1)$$

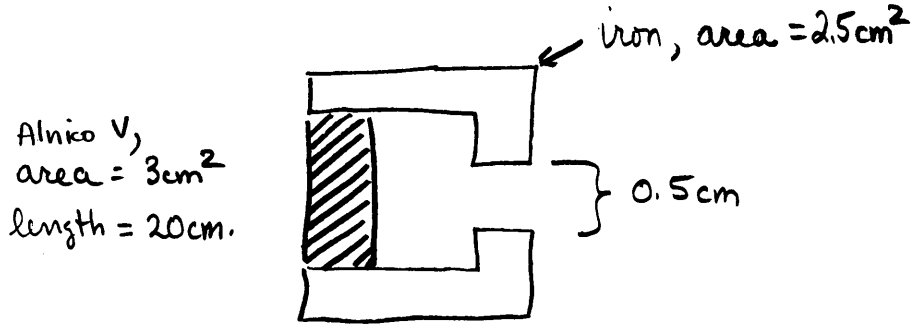
$$(\mathcal{R}_1 + \mathcal{R}_3)\varphi_1 - \mathcal{R}_3 \varphi_2 = N_1 I_1$$

$$-\mathcal{R}_3 \varphi_1 + (\mathcal{R}_2 + \mathcal{R}_3)\varphi_2 = -N_2 I_2$$

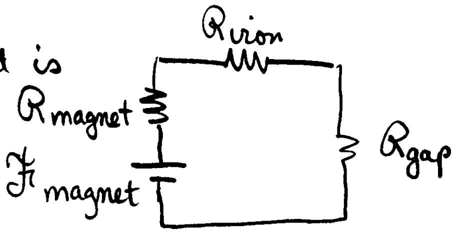
$$\varphi_1 = \frac{\begin{bmatrix} N_1 I_1 & -\mathcal{R}_3 \\ -N_2 I_2 & \mathcal{R}_2 + \mathcal{R}_3 \end{bmatrix}}{\begin{bmatrix} (\mathcal{R}_1 + \mathcal{R}_3) & -\mathcal{R}_3 \\ -\mathcal{R}_3 & (\mathcal{R}_2 + \mathcal{R}_3) \end{bmatrix}}$$

$$\varphi_2 = \frac{\begin{bmatrix} \mathcal{R}_1 + \mathcal{R}_3 & N_1 I_1 \\ -\mathcal{R}_3 & -N_2 I_2 \end{bmatrix}}{\begin{bmatrix} \mathcal{R}_1 + \mathcal{R}_3 & -\mathcal{R}_3 \\ -\mathcal{R}_3 & \mathcal{R}_2 + \mathcal{R}_3 \end{bmatrix}}$$

Example: permanent magnet.
Determine the air-gap field for an alnico V circuit shown below.

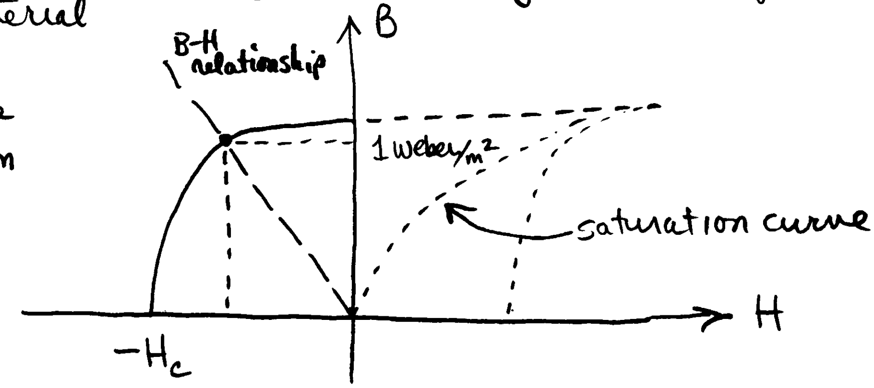


our circuit is



As we have seen $\mathcal{R}_{iron} \ll \mathcal{R}_{gap}$ so it can be neglected.
An Alnico V magnet is simply a second-quadrant magnetic material

This is often called the demagnetization curve



For the permanent magnet we need to determine the BH operating point. Neglecting $\mathcal{R}_{iron}$. Since μ is also high for magnets we can also neglect $\mathcal{R}_{magnet}$. Then, summing around our circuit

$$\mathcal{F}_{magnet} + \mathcal{R}_{gap}\,\varphi_{gap} = 0$$

$$H l_m + \frac{l_g}{\mu_0 A_g} B_g A_g = 0.$$

Our second equation is that

$$\varphi_{magnet} = B A_m = B_g A_g.$$

Combining we get a B-H relationship the magnet <u>must</u> satisify

$$H l_m + \frac{l_g}{\mu_o} B \frac{A_m}{A_g} = 0.$$

$$B = -\mu_o \frac{l_m}{l_g} \frac{A_g}{A_m} H$$

From our data

$$\mu_o \frac{l_m}{l_g} \frac{A_g}{A_m} = (4\pi \times 10^{-7}) \left(\frac{20 \text{ cm}}{.5 \text{ cm}} \right) \left(\frac{2.5 \text{cm}^2}{3 \text{ cm}^2} \right)$$

$$\approx 23{,}900 \ H/m.$$

$$\therefore \quad B = -23{,}900 \ H$$

Plotting this on the B-H curve we get

$$\left. \begin{array}{l} B \approx .975 \ wb/m^2 \\ H \approx 23{,}300 \ A\text{-}t \end{array} \right\} \text{in the magnet}$$

The air gap flux is then

$$\varphi_{gap} = \varphi_{magnet} = \left(.975 \frac{Wb}{m^2} \right) (3 \times 10^{-4} m^2) \cong .029 \, Wb$$