## Conductors and dielectrics

Up to now we have only considered fields in free space, i.e. vacuum. We are faced with two rather fundamental restrictions due to our development so far. These are:

1. When is $\epsilon \neq \epsilon_0$?
2. What happens at the boundary between media with different $\epsilon$?

The answer to the first question will depend upon an examination of the behavior of atoms in response to an applied electric field. The answer to the second question will come directly from Maxwell's equations after we answer the first question.

Let's begin by considering the simplest (conceptually) material – a conductor. By a conductor we mean an electrically neutral material which possesses a large number of free (mobile) charges. The presence of free electrons is actually due to some rather special quantum mechanical properties of metal atoms and will not be discussed here. In a conductor we can have a movement of these free charges resulting in an electric current, i.e. coulombs/sec or amperes.

Mathematically we usually want to talk about a current density vector $\underline{J}$ which is given in amperes/$m^2$.

The free electrons do not move unless an electric field is present. Hence, $\underline{J}$ is related to $\underline{E}$. This relationship can be written as

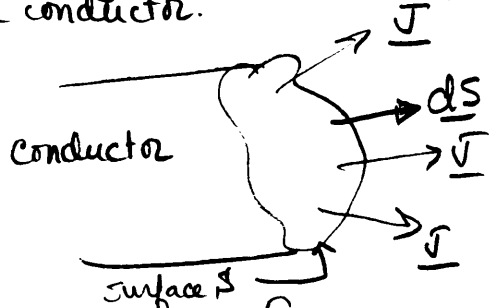
$$\underline{J} = \sigma \underline{E}$$

This is the well known Ohm's Law in a fields context

where $\sigma$ (the conductivity) is a property of the material as determined by its atomic orbitals.

The units of $\sigma$ are siemens/meter. This sounds strange until we inspect Ohm's law. The units of $\underline{J}$ are amperes/$m^2$; those of $\underline{E}$ are volts/meter. Hence, the units of $\sigma$ are $\frac{\text{amperes}}{m^2} \cdot \frac{m}{\text{volt}} = \frac{\text{amperes}}{\text{meter-volt}}$. From circuits I we know that volts/amperes = ohms, so the units of $\sigma$ in more understandable terms are $\frac{1}{\text{ohm-meters}}$. The siemen is a measure of conductivity and is the reciprocal of resistance, i.e. 1 siemen = $\frac{1}{\text{ohm}}$.

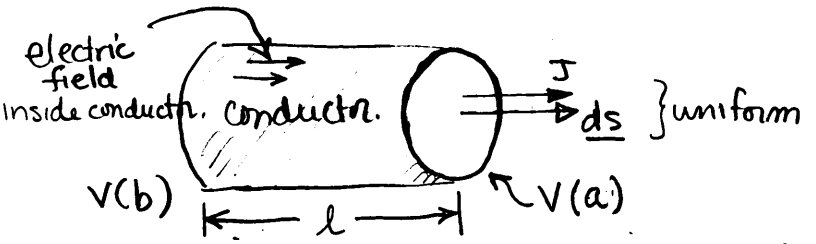
Our more common form of Ohm's Law is obtained by integrating $\underline{J}$ over a surface, usually the cross-section of the conductor.




$$I = \int_S \underline{J} \cdot d\underline{S}$$

$$= \int_S \sigma \underline{E} \cdot d\underline{S} = \int_S \underline{E} \cdot \sigma d\underline{S}$$

this is the voltage; this is the conductivity or 1/resistance.




If E is uniform and perpendicular to $\underline{ds}$ as shown above.

$$I = \sigma E A$$

where A is the cross-sectional area of the conductor.

This is a definition of I; recall that we defined the voltage (potential) in fields form as

$$V_{ab} = -\int_b^a \underline{E} \cdot \underline{dl}$$

which keeping the same practice of assuming uniformity as above gives

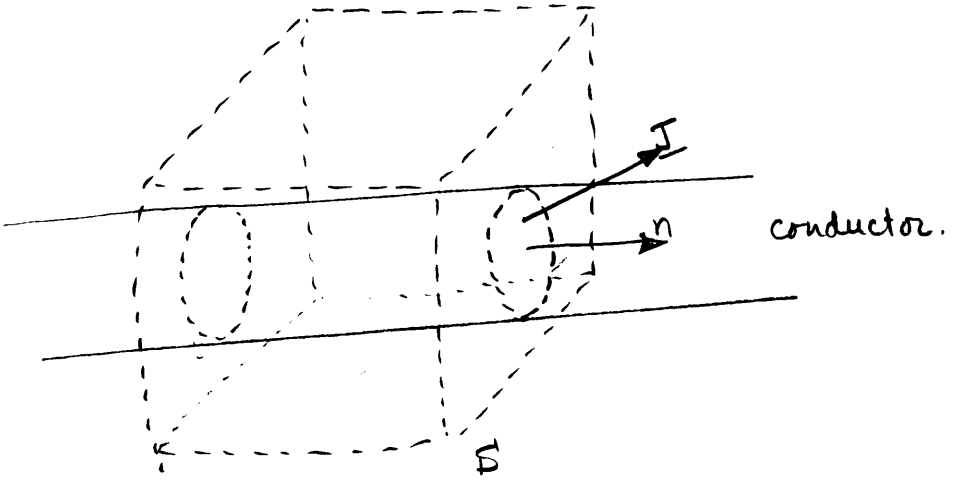
$$V_{ab} = -\underline{E} \cdot \int_b^a \underline{dl} = El.$$

$$\therefore \quad R \triangleq \frac{V_{ab}}{I} = \frac{El}{\sigma E A} = \frac{l}{\sigma A}.$$

this is the resistance of the material and the above relationship is the normal circuits version of Ohm's Law.

## Conservation of charge:

To complete our discussion of conductors we need to examine the $\underline{E}$ fields inside a conductor. Consider a portion of conductor enclosed by a surface $S$




By conservation of charge the net current through a closed surface $S$ gives the rate of change of free charge within the volume; i.e.

$$\oint_S \underline{J} \cdot d\underline{S} = -\frac{dQ_{tot}}{dt} = -\frac{d}{dt}\int \rho \, dv = -\int \frac{\partial \rho}{\partial t} dv$$

The minus sign is due to the fact that charge is leaving $S$, using the divergence theorem the left-hand side of this equation becomes

$$\oint_S \underline{J} \cdot d\underline{S} = \int_V \underline{\nabla} \cdot \underline{J} \, dv$$

Then, by substitution

$$\int \underline{\nabla} \cdot \underline{J} \, dv = -\int \frac{\partial \rho}{\partial t} dv.$$

or

$$\underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

For a conductor $\underline{J} = \sigma \underline{E}$ and

$$\underline{\nabla} \cdot \underline{J} = \underline{\nabla} \cdot \sigma \underline{E} = -\frac{\partial \rho}{\partial t}$$

But from Gauss' Law

$$\underline{\nabla} \cdot \epsilon \underline{E} = \rho$$

If $\sigma$ and $\epsilon$ are linear and isotropic

$$\sigma \left( \frac{\rho}{\epsilon} \right) = -\frac{\partial \rho}{\partial t}$$

Rewritten, this equation represents the behavior of free charge in a conductor.

$$\frac{\partial \rho}{\partial t} + \left( \frac{\sigma}{\epsilon} \right) \rho = 0$$

The solution of this equation is $\rho = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right) t}$
To interpret this solution suppose I put charge density $\rho_0$ into $S$ at time $t=0$. For a perfect conductor $\sigma = \infty$ and hence, instantly, $\rho \equiv 0$. What this means since charge cannot disappear is that it all goes to the surface. However, this does mean that there cannot be any electric field inside a perfect conductor since $\rho \equiv 0$ and $\underline{\nabla} \cdot \underline{E} = \rho/\epsilon$ is also zero.

Even for a good conducta there is essentially no E field in the conductor. Consider copper for which

$$\epsilon \cong \epsilon_0 \cong 10^{-12}$$

$$\sigma = 5.8 \times 10^{7}$$

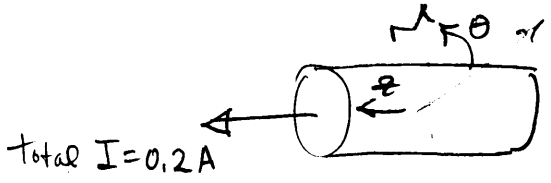
$$\therefore \left( \frac{\epsilon}{\sigma} \right) \approx 1.5 \times 10^{-19} \text{ seconds.}$$

No field exists inside copper after such a such interval that it may be considered zero.

Example: A simple copper wire

If I use a 12 gauge wire, what is the resistance and current density if I pass a 0.2 ampere current through the wire.

The measured conductivity of copper is $\sigma = 5.8 \times 10^7$ ohm-meters.




We have no reason to assume that $\underline{J}$ is not uniform over the end of the wire.

$$I = \int_S \underline{J} \cdot d\underline{S}$$

$$I = J \cdot \pi r^2$$

$$\therefore J = \frac{I}{\pi r^2} = \frac{0.2 \text{ amperes}}{\pi (1.025 \times 10^{-3})^2} = 6.06 \times 10^4 \text{ A/m}^2$$

where $r = 1.025$mm corresponds to 12 gauge wire.

The electric field is given by ohm's Law (fields version): $J = \sigma E$

$$E = \frac{J}{\sigma} = \frac{6.06 \times 10^4 \text{ A/m}^2}{5.8 \times 10^7 \text{ }\Omega\text{-m}} = 1.05 \times 10^{-3} \text{ volts/m}.$$

We get the conventional resistance using ohm's Law (circuits version)

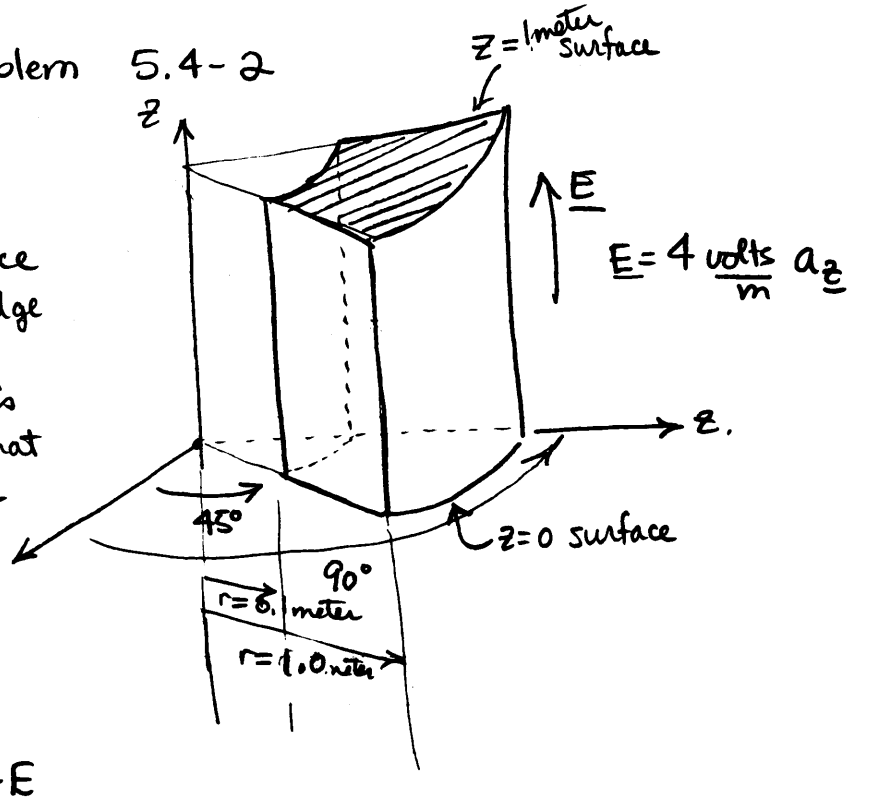
$$R = \frac{E}{I} = \frac{1.05 \times 10^{-3} \text{ volts/m}}{0.2 \text{ amperes}} = 5.3 \times 10^{-3} \text{ }\Omega\text{/m}.$$

Note that the fields version of Ohm's Law relates distributed parameters, whereas the circuits version relates lumped parameters.

The resistance of 100 feet of wire is then

$$R = 5.3 \times 10^{-3} \text{ }\Omega\text{/m} \times \frac{2.54 \times 10^{-2}}{\text{inch}} \times \frac{12 \text{ inches}}{\text{foot}} \times 100 \text{ ft.} = 1.6 \text{ }\Omega$$

Example: Problem 5.4-2




Find the resistance of this copper wedge when the radius of the inner surface is 0.1 meter and that of the outer surface is 1-meter.

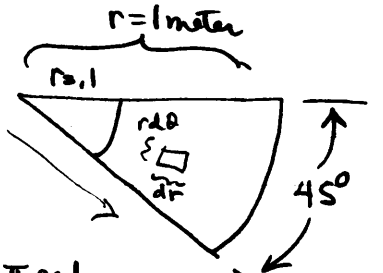
$E = 4\,a_z$

$\underline{J} = \sigma \underline{E}$

for copper $\sigma = 4 \times 10^{+7}$ $\mho/m$ $\quad \therefore J = (4\times10^{+7})\,\mho/m\,(4\ V/m) = 16\times10^{+7}\ A/m^2$.

This is: UNIFORM.

The current is given by




$$I = \int \underline{J} \cdot d\underline{S} = \int_{0.1}^{1}\int_{0}^{\pi/4} 16\times10^{7}\, r\, dr\, d\theta$$

$$= \frac{\pi}{4}\,16\times10^{7}\int_{0.1}^{1} r\,dr \qquad 45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

$$= 16\frac{\pi}{4}\times10^{7}\,\frac{r^2}{2}\Big]_{0.1}^{1} = 4\pi\times10^{7}\left[\frac{1}{2} - \frac{.01}{2}\right] = \frac{.99}{2}\,4\pi\times10^{7}$$

how do we pick $d\underline{S}$ ? $\quad d\underline{S} = dS\,a_z$

$\therefore I = 6.22\times10^{7}$ Amperes.

$$V = -\int_0^1 4\,a_z \cdot dz\,a_z = -4\Big|_0^1 = -4 \text{ volts}$$

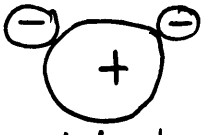
$$\therefore R = \frac{4 \text{ volts}}{6.22\times10^{+7} \text{ amperes}} = 6.4\times10^{-8}\,\Omega$$

Now that we understand what conductors are (to a limited extent) let's consider dielectrics.

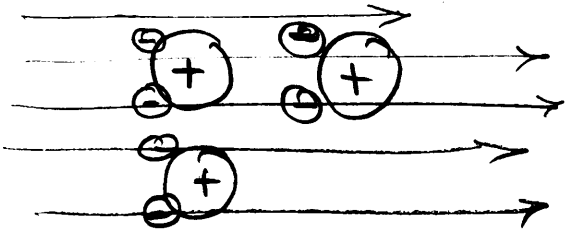
A dielectric is: an electrically insulating material in which electrons are tightly bound to the nucleus but atomic-level charge separation occurs at the macroscopic level due to an applied electric field.

This definition will make more sense after we examine some forms of charge separation.

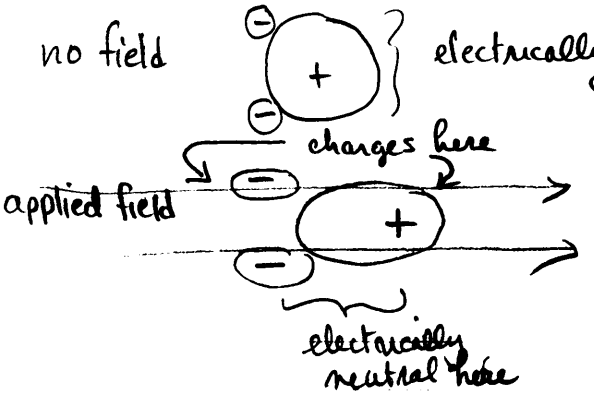
The easiest to grasp (and visualize) is <u>orientational.</u> Consider a water molecule as shown below.



This molecule will rotate to align itself with an applied electric field. If a large number of molecules so rotate one end of the material will be negatively charged, the other positive.

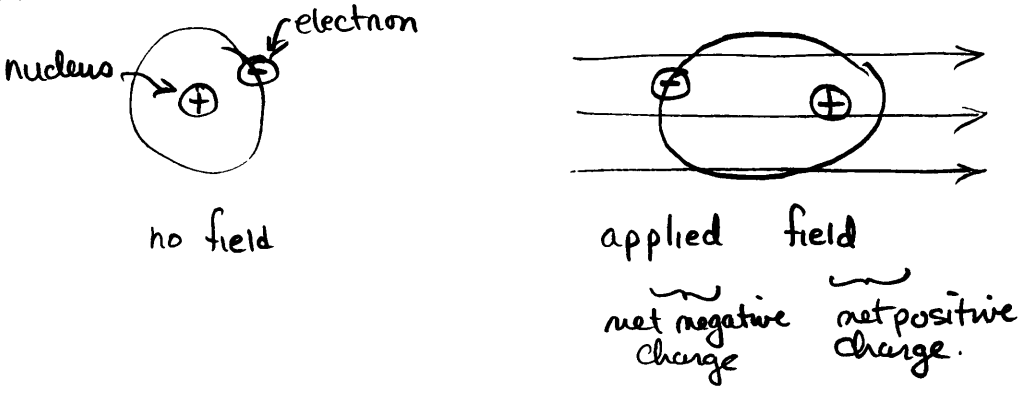



In the same molecule shown above, an applied E-field can also stretch the atom



What this really means is that as the charges (on the ions) move farther away from each other they behave more and more like discrete charges separated by a distance.

The final (and perhaps most important) form of charge separation is electronic in which the electrons and nucleus of an atom separate in response to an applied $\underline{E}$ field as shown below



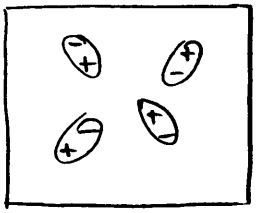

The conclusion of all types of polarization; electronic, ionic, and orientational, is to develop regions of net positive and negative charge at the microscopic level. separated (by a neutral region) by a distance $\ell$, i.e. an electric dipole. Because this occurs at the microscopic level large numbers of these charge separations occur and give a macroscopic average, i.e.

$$\underline{P} \triangleq \lim_{\Delta v \to 0} \frac{\sum_i \underline{p_i}}{\Delta v}$$

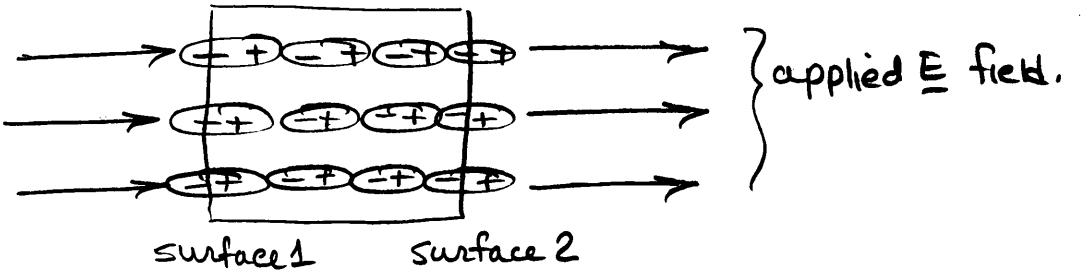
where $\Delta v$ is the volume containing the microscopic dipoles (charge separations) $\underline{p_i}$

We can write this definition as $\underline{P} = N\underline{p}$ in most cases. $\underline{p}$ is a microscopic dipole element and $N$ is the number density of such dipole elements. This relationship is valid as long as the material is uniform,

Let's now consider what happens in the bulk material shown below.



With no applied field the dipoles are in random directions (or may not exist) and the <u>net</u> macroscopic polarization is zero. However, when an electric field is applied electric dipoles (may be created and) align themselves with the electric field as shown below.




Note that all charges cancel in the interior of the volume but <u>net</u> <u>surface</u> charges appear in response to the applied field. Let us examine this phenomena in more detail and try to come up with a model.

Let us begin by assuming that all dipoles are of uniform size and charge, i.e. $p = qd$ at the microscopic level, and that there are $N$ such dipoles/unit volume.

out of order.

This can be related to the macroscopic polarization $\underline{P}$ by noting that P is nothing more than the average polarization per unit volume, i.e.

$$P_x(x) = N(qd)\Big|_{x_0}$$

where the subscript x denotes fields and directions along the x-axis. evaluated at $x = x_0$

Similarily, at surface 2 the net bound charge and surface charge are given by

$$dq_2 = + N(qd)\Big|_{x_0+\Delta x} \Delta y\, \Delta z$$

$$dq_2' = - N(qd)\Big|_{x_0+\Delta x} \Delta y\, \Delta z.$$

Note that $\underline{P}$ is defined only within the volume, NOT on the surface. So that at $x = x_0 + \Delta x$ we note that

$$P_x(x_0+\Delta x) = N(qd)\Big|_{x_0+\Delta x}$$

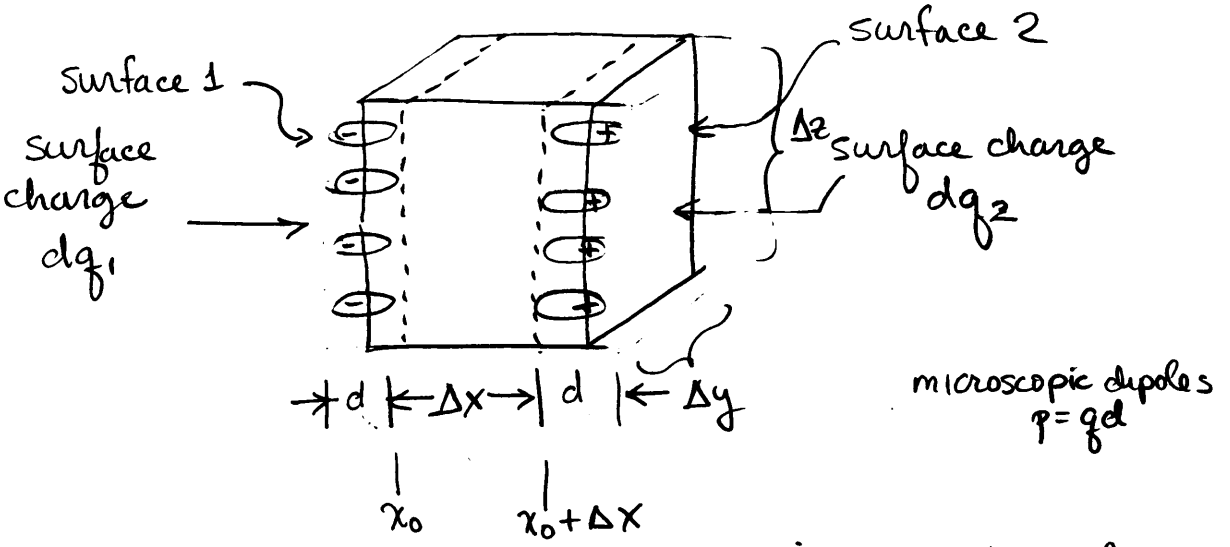
The net bound charges written in terms of P are then

$$\begin{cases} dq_1' = + P_x(x_0)\, \Delta y\, \Delta z \\ dq_2' = - P_x(x_0+\Delta x)\, \Delta y\, \Delta z \end{cases}$$

As all other bound charges cancel the total charge enclosed with the volume $\Delta x \Delta y \Delta z$ is then

$$dq_{TOTAL} = dq_1' + dq_2' = P_x(x_0)\, \Delta y\, \Delta z - P_x(x_0+\Delta x)\, \Delta y \Delta z$$

$$= - \frac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x} \underbrace{\Delta x\, \Delta y\, \Delta z}_{\Delta v}$$



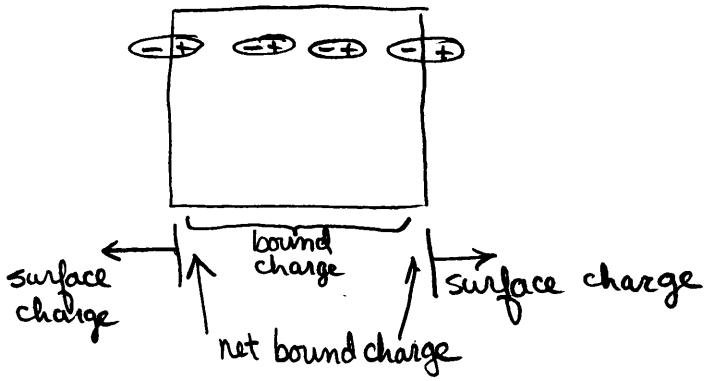

This results in the above semi-microscopic picture of our polarizable material.

By inspection, the net charge on surface 1 is given by.

$$dq_1 = -(N q d)\big|_{x_0} \Delta y \Delta z$$

where $d_x \Delta y \Delta z$ represents the "volume" of the surface layer and $Nq$ represents the charge per unit volume at $x$-coordinate $x$, i.e. on surface 1.

Only those dipoles within a distance $d$ of the surface will contribute a net charge. The charge remaining within the volume has a charge of zero, and is known as the bound charge as it does not move, but can rotate. The net bound charge is that just inside the surface layer.




The net bound charge $dq_1'$ corresponding to $dq_1$ must be

$$dq_1' = +(Nqd)\big|_x \Delta y \Delta z$$

The macroscopic polarization charge density within the volume is then given by

$$\rho_b = \lim_{\Delta v \to 0} - \frac{\dfrac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x}\,\Delta v}{\Delta v}$$

$$\rho_b = -\left.\frac{dP_x}{dx}\right|_{x_0}$$

This result can be generalized to three-dimensions by applying a field sequentially in the $a_y$ and $a_z$ directions in the same manner as for the $a_x$ direction shown above. The result is that in three dimensions

$$\rho_b = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z}$$

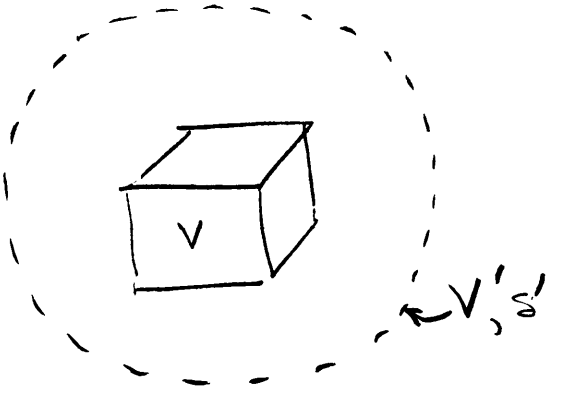
or

$$\rho_b = -\nabla \cdot \underline{P}$$

Note that $\rho_b$ is the net bound charge within the volume V.

This formulation leads to some problems at the surface of our volume. Clearly $\underline{P}$ is discontinuous at the surface of $V$ because polarization only exists within $V$. $P$ is not defined on the surface.

To help us understand what happens in $V$ we will use Gauss' Law on a volume $V'$ which <u>includes</u> $V$ as shown below.




Note that $\epsilon = \epsilon_0$ at $S'$ which is free space.

Then, by Gauss' Law

$$\oint_{S'} \epsilon_0 \underline{E} \cdot \underline{dS'} = \int_{V'} \rho_{tot}\, dV'$$

But what is $Q_{total}$? The only charge within $V'$ is in $V$. However, there is a contribution to $\rho_{tot}$ from free charges (if present) <u>and</u> from bound charges contributing to the polarization of $V$. Thus,

$$\rho_{total} = \rho_f + \rho_b$$

Note that $\rho_f$ is a charge density due to external sources, i.e. it was somehow put there. $\rho_b$ is a bound charge which exists only because of the polarization of $V$ due to the applied field $E$.

$$\oint_{S'} \epsilon_0 \underline{E} \cdot \underline{dS'} = \int_{V'} (\rho_f + \rho_b)\, dV.$$

$$= \int_{V} \rho_f\, dV + \int_{V'} \rho_b\, dV$$

By the divergence theorem

$$\oint_{S'} \epsilon_0 \underline{E} \cdot \underline{dS'} = \int_{V'} \underline{\nabla} \cdot \epsilon_0 \underline{E} \, dv$$

Equating the volume integrals we have.

$$\underline{\nabla} \cdot \epsilon_0 \underline{E} = \rho_f + \rho_b$$

Recalling that $\rho_b = -\underline{\nabla} \cdot \underline{P}$ we can re-write this equation as

$$\underline{\nabla} \cdot \epsilon_0 \underline{E} + \underline{\nabla} \cdot \underline{P} = \rho_f$$

$$\underline{\nabla} \cdot (\epsilon_0 \underline{E} + \underline{P}) = \rho_f$$

We define the displacement vector $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$. Note that it is $\underline{D}$ that is associated with the free charge density $\rho_f$ and not $\underline{E}$. $D$ is also known as the electric flux density vector because its units are coulombs/$m^2$, i.e. a density.

In free space there are no bound charges; hence, $\underline{P} = 0$ and $\underline{D} = \epsilon_0 \underline{E}$.

However, in a material with bound charges $\underline{P} \neq 0$ and $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$. In general, $\underline{P} = f(\underline{E})$. More accurately, $\underline{P} = f(\epsilon_0 \underline{E})$ since $\epsilon_0 \underline{E}$ is a charge density. For <u>most materials</u> this is a simple linear relationship

$$\underline{P} = \chi_e \epsilon_0 \underline{E}$$

where $\chi_e$ is the electric susceptability of the material.

This allows us to write the expression for D in several ways.

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$$= \epsilon_0 \underline{E} + \chi_e \epsilon_0 \underline{E}$$

If we define the permittivity $\epsilon$ to be $\epsilon_0 + \chi_e \epsilon_0$ we can write

$$\underline{D} = \epsilon \underline{E}$$

which we saw in our earlier fields courses. We can also define the dielectric constant $\kappa$ by

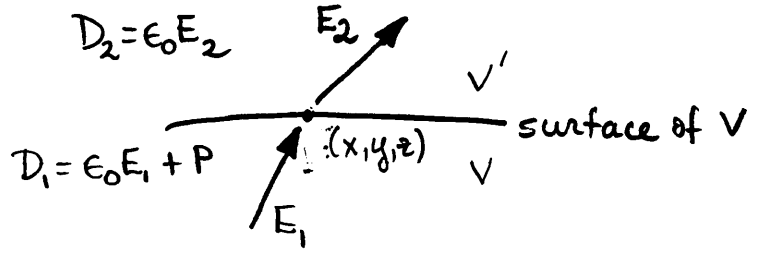
$$\kappa = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

To complete our study of dielectric materials we must examine what happens on the surface of a polarizable (i.e. $\underline{P} \neq 0$) material. Let us look at a small section of the surface S of V. And let us continue to assume that V exists within V', a vacuum (free space) except for V. The relationship between $\underline{D}$ and $\underline{E}$ within V is given by

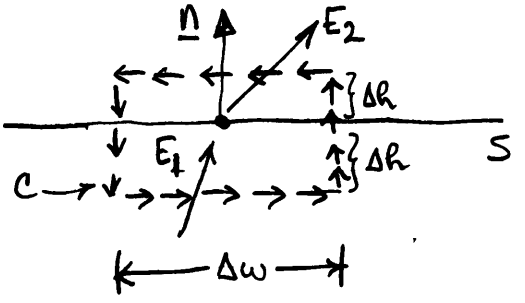
$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

Just outside this volume V, $\underline{D} = \epsilon_0 \underline{E}$ since free space is not polarizable -- There are no bound charges in it. What is the relationship between $\underline{D}$ and $\underline{E}$ in V and V' (outside V)?

let $E_1$ be the field within V and let $E_2$ be the field in V' (outside V) just across the surface.




To determine what the relationship between $\underline{E}_1$ and $\underline{E}_2$ is we will use a contour about (x,y,z) as shown below




Since $\underline{E}$ is conservative $\oint_C \underline{E} \cdot d\underline{\ell} = 0$. Breaking $E_1$ and $E_2$ into components normal and parallel to S we get $E_{1n}, E_{1t}$ and $E_{2n}, E_{2t}$ respectively. Picking $\underline{n}$ to point from V to V' as shown above we can write the contour integral as

$$(\Delta h) E_{1n} + (\Delta h) E_{2n} - E_{2t}(\Delta w) - (\Delta h) E_{2n} - (\Delta h) E_{1n} + (E_{1t}) \Delta w = 0$$

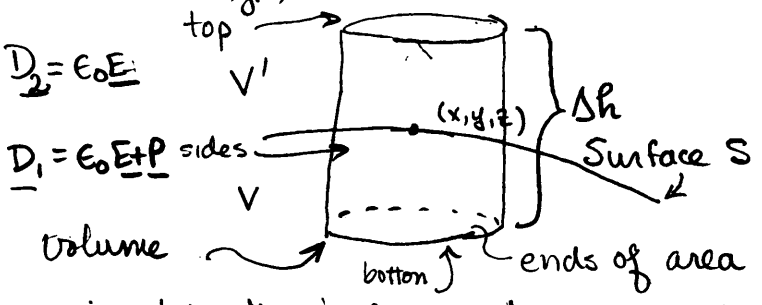
Note that $\Delta h$ and $\Delta w$ are chosen small enough that $\underline{E}_1$ and $\underline{E}_2$ may be treated as constants on C. This allows many terms in the above contour integral expression to cancel leaving

$$E_{1t} \Delta w - E_{2t} \Delta w = 0$$

or $$E_{1t} = E_{2t}.$$

Thus, the tangential component of $\underline{E}$ is continuous. In a more elegant mathematical way this can be expressed as $\underline{n} \times (\underline{E}_2 - \underline{E}_1) = 0$

To discover the relationship between normal components we must use Gauss' law on a small volume centered on $(x,y,z)$ as shown below

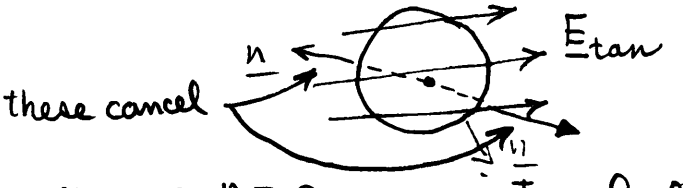



We will let this volume be a small cyclinder of height $\Delta h/2$ on each side of S' and an end surface area of $\Delta S$. Using Gauss' Law

$$\oint_{\text{surface}} \underline{D} \cdot d\underline{S} = \rho_f \Delta h \Delta S \text{ (within volume)}$$

$$\int_{\text{top}} \underline{D} \cdot d\underline{S} + \int_{\text{sides}} \underline{D} \cdot d\underline{S} + \int_{\text{bottom}} \underline{D} \cdot d\underline{S} = \rho_f \Delta h \, \Delta S.$$

Note that as $\Delta S \to 0$ $\underline{D} \cdot d\underline{S} = \epsilon_0 \underline{E}_{tan} \cdot \underline{a}_r$ is everywhere equal in magnitude in the surface. In a top view this looks like




However, as $\underline{n} = \underline{a}_r$ an integral of $\underline{E}_{tan} \cdot \underline{a}_r$ around the circumference of S must be zero due to the cancellation of $\underline{E}_{tan} \cdot \underline{a}_r$ as shown in the drawing above. Thus,

$$\int_{\text{top}} \underline{D} \cdot d\underline{S} + \int_{\text{bottom}} \underline{D} \cdot d\underline{S} = \rho_f \Delta h \, \Delta S$$

Treating D as constant on each surface we get

$$D_{2n} \Delta S + D_{1n} \Delta S = \rho_f \Delta h \, \Delta S$$

where the subscript $\underline{n}$ indicates the normal component of $\underline{D}$.

$$D_{2n} - D_{in} = \rho_f \Delta h.$$

As $\Delta h \to 0$ $\rho_f \Delta h \longrightarrow \rho_s$ the <u>surface free</u> charge density. Thus,

$$D_{2n} - D_{1n} = \rho_s$$

or in terms of the electric field

$$\epsilon_0 E_{2n} - \epsilon E_{1n} = \rho_s.$$

This relationship for $\underline{D}$ can be more elegantly written as

$$\underline{n} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_s$$

The direction of $\underline{n}$ is very important as it determines the <u>sign</u> of $\rho_s$.

We may now conclude that the tangential component of $\underline{E}$ is always continuous at a dielectric interface, but that the normal component of $\underline{E}$ may be discontinuous if there is a surface charge at the interface.

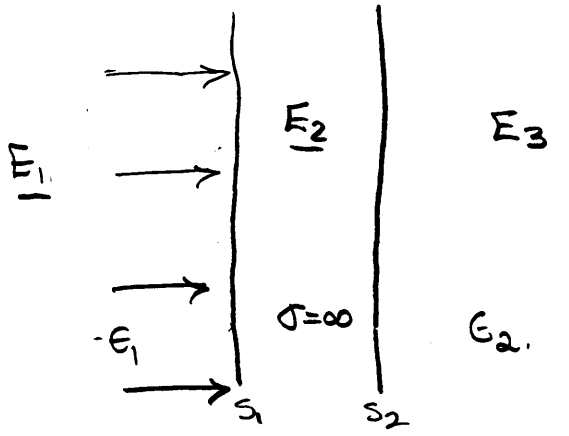
These boundary conditions also apply to perfect conductor provided we recall that there can be no $\underline{E}$ field within a perfect conductor. That means $E_{1n} = E_{1t} = 0$. Then the boundary conditions for a perfect conductor–dielectric interface are:

$$E_{2t} = 0$$

$$E_{2n} = \frac{\rho_s}{\epsilon}$$

This says that there can only be a normal component of $\underline{E}$ at the surface of a conductor and that charges can only exist on the surface of conductors. Recall that $\underline{E} = 0$ almost instaneously in any type of good conductor.

Example: perfectly conducting plates




What are $E_2$ and $E_3$? First $E_1$ can only have normal components. Second, we know $\underline{E} \equiv 0$ in a perfect conductor That means that $E_2 \equiv 0$. As $E_2 \equiv 0$, and tangential $\underline{E}$ is continuous tangential $E_3 \equiv 0$. There is no charge put on $S_2$ so $D_{3n} = D_{2n}$. And as $E_2 \equiv 0$, $D_{2n} \equiv 0$ and, Thus, $D_{3n} \equiv 0$. This means that $\underline{E_3} \equiv 0$ everywhere. Finally,

$$D_{2n} - D_{1n} = \rho_s$$

But $D_{2n} \equiv 0$ so

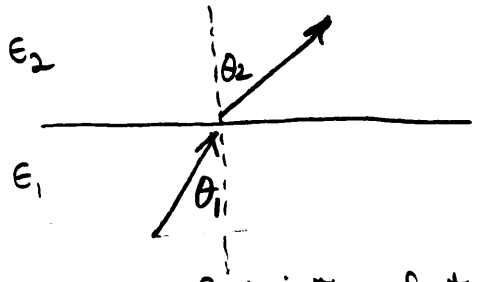
$$-\epsilon_1 E_{1n} = \rho_s$$

and our boundary conditions require a surface charge density

$$\rho_s = -\epsilon_1 E_{1n}$$

at the interface. Incidentally, this is the shielding property of a conducting sheet of metal and why electronics is put in metal boxes.

Example: field components




If there is no $\rho_s$ what is the relationship between $E_1$ and $E_2$?

As tangential $\underline{E}$ is continuous at all points

$$E_1 \sin\theta_1 = E_2 \sin\theta_2.$$

Normal $\underline{D}$ is discontinuous by the surface charge density $\rho_s$.

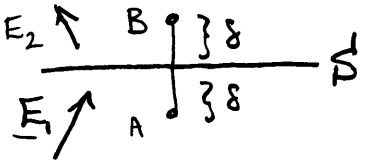
$$D_{2n} - D_{1n} = \rho_s$$
$$\epsilon_2 E_2 \cos\theta_2 - \epsilon_1 E_1 \cos\theta_1 = \rho_s$$

Now, there is never a $\rho_s$ between two perfect dielectrics unless someone puts it there so.

$$\epsilon_2 E_2 \cos\theta_2 = \epsilon_1 E_1 \cos\theta_1$$

Boundary conditions for the electric potential

Since we will use it shortly we will examine boundary conditions for the electric potential. Consider two points A and B on each side of a dielectric interface as shown below separated by a distance $2\delta$.




The electric potential between B and A is then given by

$$\Phi(B) - \Phi(A) = -\int_A^B \underline{E} \cdot d\underline{\ell}$$

We have picked $\overrightarrow{AB}$ to be perpendicular to the boundary to simplify the problem. In terms of the drawing above the line integral may be written as

$$\int_A^B \underline{E} \cdot d\underline{\ell} = E_{2n}\delta - E_{1n}\delta$$

sign is backwards

where the subscript "n" indicates the normal component. Then, by substitution

$$\Phi(B) - \Phi(A) = [E_{1n} - E_{2n}]\,\delta$$

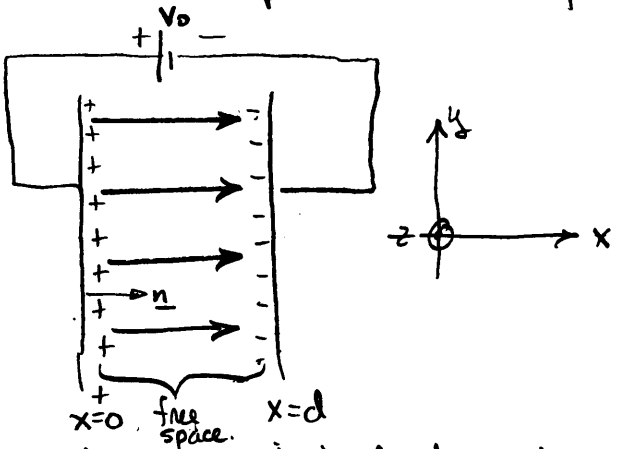
We know that $E_{1n} - E_{2n} = -\rho_s$, the surface charge on S and is finite for any real problem. Therefore, in the limit as $\delta \to 0$ we have

$$\Phi(B) - \Phi(A) = 0$$

and we have the simple, yet very useful result, that $\Phi$ is continuous across a boundary.

Even at the surface of a conductor $\Phi$ is continuous, for the entire conductor must be at the same potential. As $\underline{E} = 0$ inside the conductor, $\Phi$ cannot change anywhere within the conductor; hence, a conductor is an equipotential surface.

Example: basic capacitor with perfect conductor plates




If the plates are infinitely large there can be no $y$ component or $z$ component of the electric field. Furthermore $\underline{E}=0$ everywhere except between the plates. Why? Because the plates are perfect conductors, and confine the field to the region between the plates. Between the plates

$$\Phi(d)-\Phi(0) = -\int_0^d \underline{E}\cdot d\underline{\ell}$$

$$= -\int_0^d E_x \underline{a_x}\cdot \underline{a_x}\,dx = -\int_0^d E_x\,dx$$

If we assume $E_x$ is uniform and we let $\Phi(d)=0$, i.e. our ground reference we get

$$-V_0 = -E_x d \quad \Rightarrow E_x = \frac{V_0}{d}$$

$$\underline{D} = \epsilon_0 \underline{E} \text{ so } \quad D_x = \epsilon_0 E_x = \epsilon_0 \frac{V_0}{d}$$

The $\underline{P}$ vector is zero in free space so there are no bound charges.

$$\Phi(x) - V_0 = -\int_0^x \underline{E}\cdot d\underline{\ell} = -E_x(x-0) = -\frac{V_0}{d}(x)$$

$$\Phi(x) = V_0 - \frac{V_0 x}{d} = V_0\left(1-\frac{x}{d}\right)$$

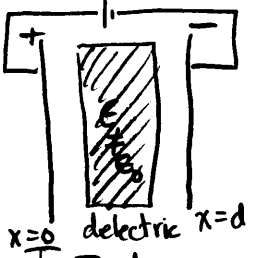
Finally, there must be a surface charge density on the plates to compensate for the field being zero for $x<0$ and $x>d$. At $x=0$

$$n\cdot(D_2 - D_1) = \rho_s$$

$$\underline{a_x}\cdot(\epsilon_0 E_x - 0) = \rho_s$$

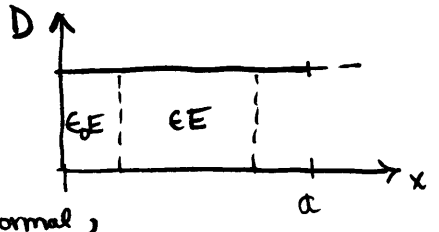
$$\therefore \rho_s = \epsilon_0 \frac{V_0}{d} \text{ at } x=0$$

and $\rho_s = -\epsilon_0 \frac{V_0}{d}$ at $x=d$.

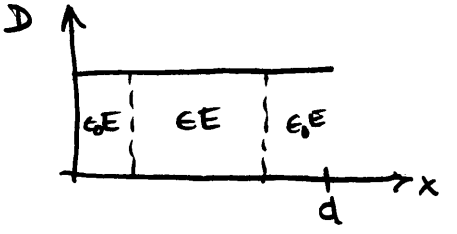



Now what happens to $\Phi, E, d, \rho$ and $P$ if we put a dielectric, polarizable block between the plates.

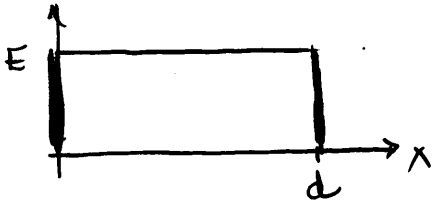
In the original capacitor

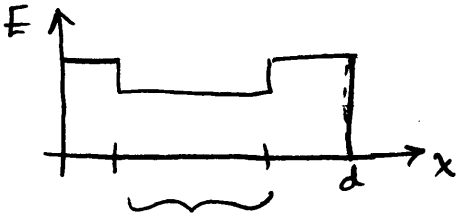



With the dielectric block



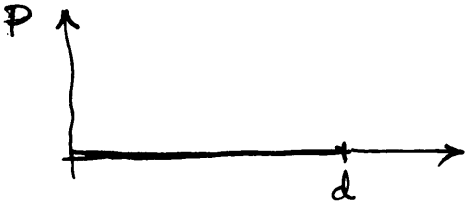

since D is continuous (normal) regardless of dielectric. However, because $D = \epsilon E$ and $\epsilon$ is not uniform, neither is E as shown below

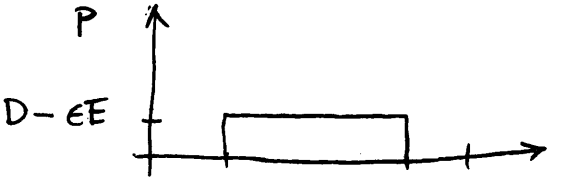


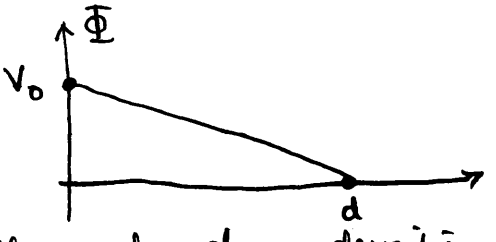

goes down in this region as $\epsilon > \epsilon_0$ usually.

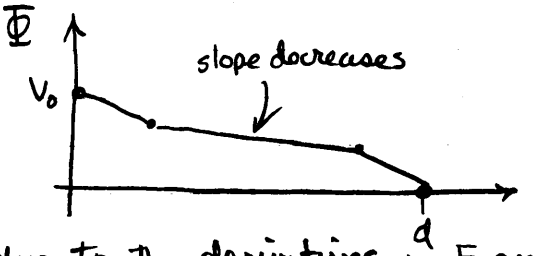
The difference between E and D is P so.



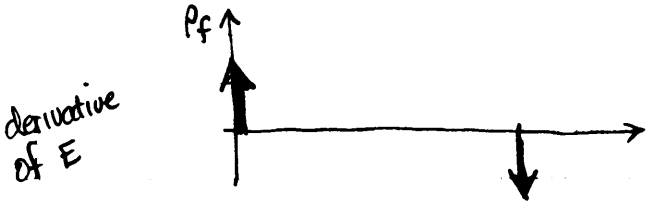



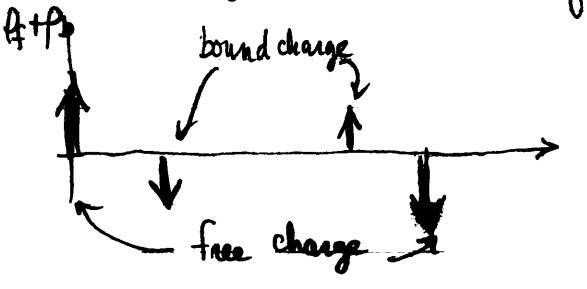

$\Phi$ is nothing more than the integral of E, where $\Phi(d) = 0$ as our reference determines the slope.




Finally, surface charge densities are due to the derivatives of $\underline{E}$ and $\underline{P}$. Free charges are due to derivates of $\underline{E}$ and bound charges due to derivatives of $\underline{P}$.



## Laplace' and Poisson's Equations

From our study of electric potential we know that we can write $\underline{E} = -\nabla\Phi$ for the electric field. If we take the divergence of each side we get.

$$\underline{E} = -\underline{\nabla}\Phi$$

$$\underline{\nabla}\cdot\underline{E} = -\underline{\nabla}\cdot\underline{\nabla}\Phi = -\nabla^2\Phi$$

$$\text{but } \nabla\cdot E = \nabla\cdot\left(\frac{D}{\epsilon}\right) = \frac{\nabla\cdot D}{\epsilon} = \frac{\rho}{\epsilon}$$

$$\therefore \quad \nabla^2\Phi = -\frac{\rho}{\epsilon}$$

This is known as Poisson's equation. If the region contains no free charge we have Laplace's equation

$$\nabla^2\Phi = 0$$

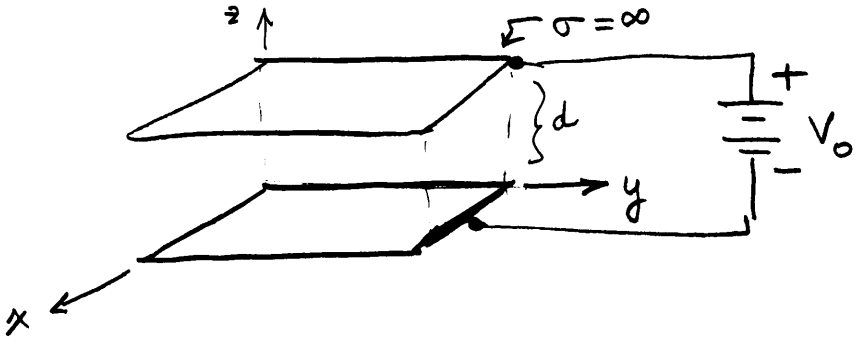
This is probably the most powerful technique for solving electrostatic problems. Boundary conditions for $\Phi$ are usually given; and, if not, we have the simple fact that $\Phi$ is everywhere continuous. There are several techniques that we have already studied that are equivalent to solving Laplace's equation; i.e. Gauss' Law, and methods we have not studied such as the method of images. But solving Laplace's equation directly gives the electric potential $\Phi$ and the resulting electric field in a very straight forward manner.

If the E-field is known everywhere Then by Gauss Law The charges on all surfaces are known. Conversely, given the charge densities $\underline{E}$ can be calculated. In practice neither is likely to be known.

Data that is usually available is the potential upon metallic (conducting) surfaces and the detailed geometry of the problem. The fundamental solution method is to find a potential which satisfies Laplace's equation and matches the boundary conditions. Because the general method is to solve a differential equation and then match the solution to the boundary conditions this is often called a boundary value problem.

We will first example problems where Laplace's equation can be simply integrated.

Example: parallel plates




Consider parallel conducting plates of infinite extent connected to an external battery of potential $V_0$.

From Laplace's eqn $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$

But there can be no x or y dependence since the plates are infinite in extent. Thus, $\Phi = \Phi(x)$ and Laplace's equation becomes

$$\frac{\partial^2\Phi}{\partial z^2} = 0$$

or $$\frac{d^2\Phi}{dz^2} = 0$$

This is a simple equation to integrate to give

$$\frac{d\Phi}{dz} = C_1$$

$$\Phi = C_1 z + C_2$$

where $C_1$ and $C_2$ are constants to be found from the boundary conditions. But what are the boundary conditions: that $\Phi(0) = 0$ and $\Phi(d) = V_0$. Thus

$$\Phi(0) = C_2 = 0$$

$$\Phi(d) = C_1 d + C_2 = C_1 d = V_0$$

This means that

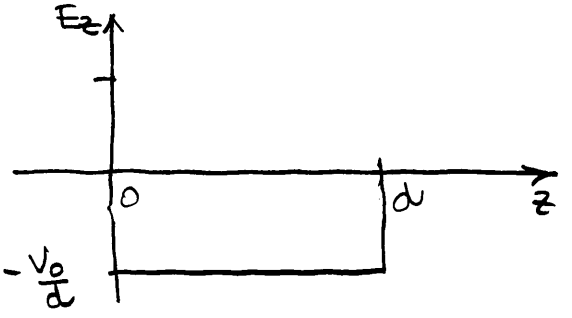
$$\Phi(z) = \frac{V_0}{d} z \qquad \text{for } 0 \le z \le d.$$

and $$\underline{E} = -\nabla\Phi = -a_z \frac{d\Phi}{dz} = -\frac{V_0}{d}\underline{a_z}$$

How about $\underline{\Phi}, \underline{E}$ for $z<0$ and $z>d$. Obviously, $\underline{E} \equiv 0$ in these regions because of the conducting plates. But since $\underline{\Phi}$ is continuous

$$\underline{\Phi} = \begin{cases} 0 & z < 0 \\ V_0 & z > d. \end{cases}$$

How about if the plates are not infinite in extent? As long as we are not near the edges of the plates the solution will be the same. Note what happens to $\underline{E}$ at the plates




As long as it is just air between the plates $\underline{D} = \epsilon_0 \underline{E}$ and we can use the result that

$$\underline{n} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_s$$

at each plate. At $z=0$, for $\underline{n} = a_z$, $D_1 = 0$ and

$$-\epsilon_0 E_1 = -\epsilon_0 \frac{V_0}{d} = \rho_s$$

At $z=d$, for $\underline{n} = a_z$, $D_2 = 0$ and

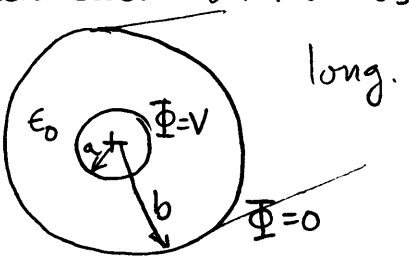
$$+\epsilon_0 E_2 = \rho_s.$$

Thus, there is a surface charge density

$$\rho_s = -\frac{\epsilon_0 V_0}{d}$$

on the plate at $z=0$ and the same charge density (of the opposite sign) on the plate at $z=d$.

Example: coaxial cable with outershield




The problem is that $\nabla^2\Phi=0$ for $b<r<a$ and $\Phi(b)=0, \Phi(a)=V$

Because the cable is cyclindrical, use cyclindrical coordinates

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

This looks formidable to integrate but consider that there can be no $z$-dependence except near the ends if the cable is long. Furthermore, because of cyclindrical symmetry $\Phi$ cannot be a function of $\phi$. Then

$$\nabla^2\Phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right)=0$$

which can be readily integrated to give

$$r\frac{d\Phi}{dr} = C_1$$

$$d\Phi = C_1\frac{dr}{r}$$

$$\Phi = C_1 \ln r + C_2$$

We now match the boundary conditions to give

$$\Phi(a) = C_1 \ln a + C_2 = V$$

$$\Phi(b) = C_1 \ln b + C_2 = 0$$

$$C_1(\ln a - \ln b) = V$$

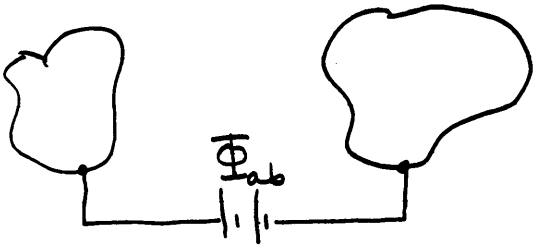
$$C_1 = \frac{V}{\ln a - \ln b} = \frac{V}{\ln(a/b)}$$

$$C_2 = -C_1 \ln b = -\frac{V \ln b}{\ln(a/b)}$$

$$\Phi = \frac{V \ln r}{\ln(a/b)} - \frac{V \ln b}{\ln(a/b)} = V\frac{\ln(r/b)}{\ln(a/b)}$$

# Capacitance

We have deferred capacitance until after studying an introduction to Laplace's equation for a very specific reason: the uniqueness theorem for the Dirichlet problem, i.e. that $\Phi$ is unique if it satisfies $\nabla^2\Phi=0$ and the known surface potentials. Consider two arbitrary-shaped (perfectly) conducting surfaces as shown below




with a potential established by an external source of electrons. By the uniqueness theorem since $\nabla^2\Phi=0$ and $\Phi$ is known on the conducting surfaces there must be a unique $\Phi$. Furthermore, as $\underline{E}=-\nabla\Phi$ $\underline{E}$ (and $\underline{D}$) must also be unique. This means that as soon as the geometry of the problem is specified, the total charge on each surface is UNIQUELY specified through the boundary conditions on $\underline{D}$. We can define this relationship between the total charge on each surface and the potential as the capacitance

$$C = \frac{Q}{\Phi_{ab}}$$

Note that $C$ depends only upon the geometry and is unique for a given geometry. The units of $C$ are coulombs/volt or farads.

For a parallel plate capacitor we have shown that

$$|\rho_s| = \frac{\epsilon_0 V_0}{d}$$

on each plate. Then the total charge on each plate is

$$Q = \frac{\epsilon_0 V_0}{d} A$$

where A is the plate area and we are neglecting fring effects (i.e. edge effects). The capacitance of this parallel plate geometry is then

$$C_{\text{parallel plate}} = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

Note that if there were a dielectric of permeability $\epsilon$ between the plates the capacitance would become

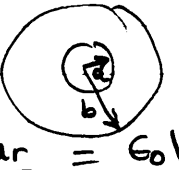
$$C = \frac{\epsilon A}{d}.$$

For the coaxial cable $\Phi(r) = V \dfrac{\ln(\frac{r}{b})}{\ln(\frac{a}{b})}$

$$\underline{E} = -\nabla\Phi = -\underline{a}_r \frac{\partial \Phi}{\partial r} = -\underline{a}_r \frac{V}{\ln(\frac{a}{b})} \frac{1}{(\frac{r}{b})} \frac{1}{b}$$

$$\underline{D} = \epsilon_0 \underline{E} = -\frac{\epsilon_0 V}{\ln(\frac{a}{b})} \frac{1}{r} \underline{a}_r$$

the surface charge density $\rho_s$ is




$$\rho_s = \underline{n} \cdot (\underline{D}_2 - \underline{D}_1) = \underline{a}_r \cdot \left( \frac{\epsilon_0 V}{\ln(\frac{a}{b})} \frac{1}{b} \right) \underline{a}_r = \frac{\epsilon_0 V}{b \ln(\frac{a}{b})}$$

The total surface charge/unit length is then

$$\frac{Q}{\Delta z} = \frac{\epsilon_0 V}{b \ln(\frac{a}{b})} 2\pi b = \frac{2\pi\epsilon_0 V}{\ln(\frac{a}{b})}$$

This gives us a capacitance/unit length of

$$\frac{C}{\Delta z} = \frac{Q}{V}\frac{1}{\Delta z} = \frac{2\pi\epsilon_0}{\ln\left(\frac{a}{b}\right)}$$

## Laplace's equation in two or more dimensions

The most common method of solving Laplace's equation is that of separation of variables. This method works provided that the boundary conditions are specified parallel to the coordinate axes and that the solution can be written in product form in the coordinate system to be used.

As an example consider $\Phi = f(u)\,g(v)$ in $u$-$v$ coordinates. If

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2} = 0$$

we have a separable equation since

$$\nabla^2\Phi = g\frac{\partial^2 f}{\partial u^2} + f\frac{\partial^2 g}{\partial v^2} = 0$$

and dividing by $fg$

$$\frac{1}{f}\frac{d^2 f}{du^2} + \frac{1}{g}\frac{d^2 g}{dv^2} = 0$$

Note that the first term must equal the second term independent of $u$, i.e. $\frac{1}{f}\frac{d^2 f}{du^2} = -\frac{1}{g}\frac{d^2 g}{dv^2}$ for all $u$.

Similarly, $\frac{1}{g}\frac{d^2 g}{dv^2} = -\frac{1}{f}\frac{d^2 f}{du^2}$ for all $v$.

The only way this can hold true is for each term to be equal to a constant independent of $u$ or $v$.

i.e. $$\frac{1}{f}\frac{d^2 f}{du^2} = k^2$$

where $k^2$ is a constant, and

$$\frac{1}{g}\frac{d^2 g}{dv^2} = -k^2.$$

Note that $u, v$ are arbitrary coordinates chosen to show the technique. We will study specific problems in all coordinate systems next.

Boundary conditions have yet to be specified, however. Boundary value problems for $\Phi$ can specify either $\Phi$ or $\frac{\partial \Phi}{\partial n}$ on the boundary. We can easily understand specifying the potential – this might correspond to a battery connected to a metal plate. However, how about $\frac{\partial \Phi}{\partial n}$, the normal derivative of $\Phi$ at a boundary.

To understand this, recall that $E = -\nabla \Phi$. This means that the normal component of $E$ is given by $-\frac{\partial \Phi}{\partial n}$.

Furthermore, since $D_{normal} = \epsilon E_{normal} = -\epsilon \frac{\partial \Phi}{\partial n}$, this boundary condition is equivalent to specifying the charge density on the boundary. Thus, we see that the two types of boundary conditions are necessary to allow us to solve Laplace's equation with either dielectric or metal (conductor) boundaries.

Mathematically, if $\Phi$ is specified everywhere on the boundary we call this a Dirichlet boundary condition. If we specify $\frac{\partial \Phi}{\partial n}$ everywhere on the boundary the boundary condition is called a Neumann boundary condition. Finally, combinations of $\Phi$ and $\frac{\partial \Phi}{\partial n}$ may be used to specify the boundary conditions leading to what is known as a mixed boundary-value problem.

## Laplace's equation in rectangular coordinates:

Laplace's equation in rectangular coordinates is

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

To determine if the solution $\Phi$ is separable we write $\Phi$ in product form as

$$\Phi(x,y,z) = f(x)\, g(y)\, h(z)$$

where $f$, $g$ and $h$ are ONLY functions of $x$, $y$ and $z$ respectively. Substituting this assumed solution into Laplace's equation we get

$$\frac{\partial^2}{\partial x^2}(fgh) + \frac{\partial^2}{\partial y^2}(fgh) + \frac{\partial^2}{\partial z^2}(fgh) = 0$$

$$gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} = 0$$

Since the functions $f, g, h$ are only functions of $x, y$ and $z$ we can convert the partial derivates to complete, ordinary derivates.

$$gh\frac{d^2 f}{dx^2} + fh\frac{d^2 g}{dy^2} + fg\frac{d^2 h}{dz^2} = 0$$

Dividing through by $\frac{1}{fgh}$ (this also means that we cannot let $\Phi = 0$).

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = 0$$

It is at this step that we really solve the original equation. Note that each term is a function of only one variable. If we hold $y$ and $z$ constant and vary $x$, $\frac{1}{f(x)}\frac{d^2 f(x)}{dx^2}$ could change, but

$$\frac{1}{g(y)}\frac{d^2g(y)}{dy^2} + \frac{1}{h(z)}\frac{d^2h(z)}{dz^2}$$

is constant. For these three terms to add to zero always $\frac{1}{f(x)}\frac{d^2f(x)}{dx^2}$ must be a constant independent of $x$. This arguement can be repeated for the terms

$$\frac{1}{g(y)}\frac{d^2g(y)}{dy^2} \text{ and } \frac{1}{h(z)}\frac{d^2h(z)}{dz^2}$$

Let $$\frac{1}{f}\frac{d^2f}{dx^2} = k_x^2$$

where $k_x^2$ is a constant. We can similarly let

$$\frac{1}{g}\frac{d^2g}{dy^2} = k_y^2$$

$$\frac{1}{h}\frac{d^2h}{dz^2} = k_z^2$$

$k_x^2, k_y^2$ and $k_z^2$ are constants (arbitrary as yet) called separation constants. which will be fixed by the given boundary conditions. Note that we have reduced the three dimensional Laplace equation to three equations — this is "separation of variables."
Before examing specific problems, let's look at solving these equations since they are identical and pick $f$ for illustrative purposes.

$$\frac{1}{f}\frac{d^2f}{dx^2} = k_x^2$$

can be re-written as

$$\frac{d^2f}{dx^2} - k_x^2 f = 0$$

If $k_x^2 > 0$ this is the differential equation which defines the exponential function $e^{\pm k_x x}$. Hence, our solutions to this equation are

$$f(x) = c_1 e^{+k_x x} + c_2 e^{-k_x x}$$

and $c_1, c_2, k_x$ must be determined from the boundary conditions.

If $k_x^2 < 0$ this is the differential equation for the trigonometric functions $\sin k_x x$ and $\cos k_x x$. Hence, our solutions now are

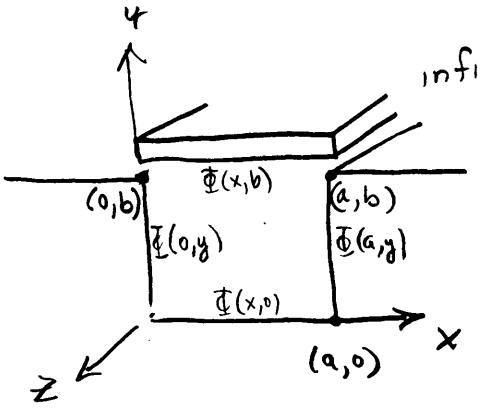
$$f(x) = c_1 \sin k_x x + c_2 \cos k_x x$$

Finally, if $k_x^2 = 0$ we have the simple solution

$$f(x) = c_1 + c_2 x.$$

In all cases $k_x$ must be determined <u>first</u> for it determines both the type of solution, <u>and</u> the interaction between solutions as $k_x^2 + k_y^2 + k_z^2 = 0$.

Example:




with boundary conditions $\Phi(0,y) = 0$

$\Phi(x,0) = 0$

$\Phi(a,y) = 0$

$\Phi(x,b) = V \sin \frac{\pi x}{a}$

let $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$ since the problem is infinitely long in the z-direction and we can expect no z-dependence. Letting $\Phi = f(x)\, g(y)$ we get

$$g \frac{d^2 f}{dx^2} + f \frac{d^2 g}{dy^2} = 0$$

$$\frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2}$$

And, separating

$$\frac{1}{f} \frac{d^2 f}{dx^2} = k_x^2 \qquad \frac{1}{g} \frac{d^2 f}{dy^2} = k_y^2$$

where $k_x^2 + k_y^2 = 0$. The problem is whether $k_x^2$ or $k_y^2$ is greater than zero. — The other must be less than zero. The selection is based upon boundary conditions, do we want exponential, sinusoidal or linear solution for f or g. Obviously, since $\Phi(x,b) = V \sin \frac{\pi x}{a}$ we want f to be sinusoidal so we can match it at the boundary

So pick $k_x^2 > 0$, Then $k_y^2 < 0$ and our solutions become

$$f(x) = C_1 \sin k_x x + C_2 \cos k_x x$$

$$g(y) = C_3 e^{+k_y y} + C_4 e^{-k_y y}$$

and $$\Phi(x,y) = (C_1 \sin k_x x + C_2 \cos k_x x)(C_3 e^{+k_y y} + C_4 e^{-k_y y})$$

As $\Phi(x,b) = V \sin \frac{\pi x}{a}$ we can pick $C_2 = 0$

$$\Phi(x,y) = (\sin k_x x)(C_3 e^{+k_y y} + C_4 e^{-k_y y})$$

where $C_1$ has been absorbed into $C_3$ and $C_4$. Picking $k_x = \frac{\pi}{a}$ we have a match for $\Phi(x,b)$. However, let us consider what must be done to match at $\Phi(0,y)$ and $\Phi(a,y)$. Our first thought is to use $C_3$ and $C_4$ to match at those points but we cannot use exponentials to achieve $\Phi = 0$ for all values of $y$ along $x=0$ and $x=a$. It is instead the sin function that we must look at. For $x=0$ $\Phi(0,y) = 0$ since $\sin(0) = 0$. This means that if $\sin k_x a = 0$ we always will meet our boundary condition. $\sin k_x a = 0$ only if $k_x a = n\pi$ where $n = 1, 2, 3, \ldots$. Now we have all possible values of $n$ allowed and must apply the boundary conditions to try to reduce them, or to determine $n$.

$$\Phi(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)\left(C_3 e^{-\frac{n\pi y}{a}} + C_4 e^{+\frac{n\pi}{a} y}\right)$$

where it was noted that $k_y^2 = -\frac{n^2\pi^2}{a^2}$ since $k_x^2 + k_y^2 = 0$.

Let us not examine what happens at $y = 0$.

$$\Phi(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)(C_3 + C_4)$$

This must equal zero independent of $x$ and can occur only if $C_3 = -C_4$. Therefore,

$$\Phi(x,y) = \sum_{n=1}^{\infty} C \sin\left(\frac{n\pi}{a} x\right)\left(e^{-\frac{n\pi}{a} y} - e^{+\frac{n\pi}{a} y}\right)$$

where we have combined all constants into $C$.

Now, what happens at $y = b$.

$$\Phi(x,b) = \sum_{n=1}^{\infty} C \sin\left(\frac{n\pi}{a}x\right)\left(e^{-\frac{n\pi}{a}b} - e^{+\frac{n\pi}{a}b}\right) = V \sin\frac{\pi x}{a}$$

This can only be true if $n=1$ and

$$C\left(e^{-\frac{\pi}{a}b} - e^{+\frac{\pi}{a}b}\right) = V$$

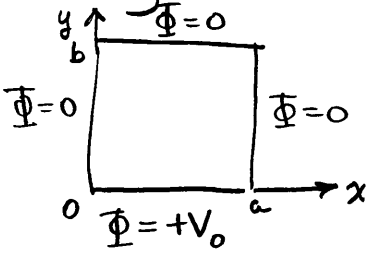
$$C = \frac{V}{e^{-\frac{\pi}{a}b} - e^{+\frac{\pi}{a}b}} = \frac{V}{2\sinh\left(\frac{\pi b}{a}\right)}$$

Our final solution, meeting all boundary conditions, is then

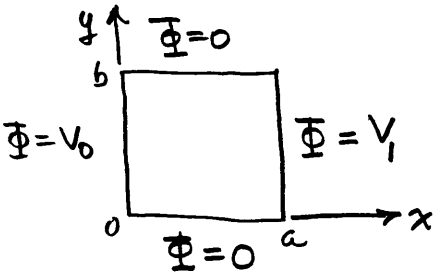
$$\Phi(x,y) = \frac{V}{e^{-\pi b/a} - e^{+\pi b/a}} \sin\left(\frac{\pi x}{a}\right)\left(e^{-\frac{\pi b}{a}} - e^{+\frac{\pi b}{a}}\right)$$

## Complex boundary conditions

The problem with a two-dimensional solution is to determine which side to start from. The choice is based upon which boundaries are zero. In the two dimensional Cartesian problems we have seen the boundaries have been primarily zero, with only one non-zero boundary potential. The general rule is That if any two opposing sides are zero then one should choose the $k^2 < 0$ solution, i.e. sines and cosines, along that coordinate since sinusoidal functions can have a zero at both boundaries. The $k^2 > 0$ exponential solutions cannot meet such a boundary condition.

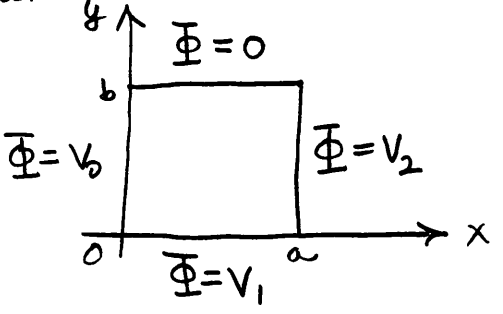


(a)

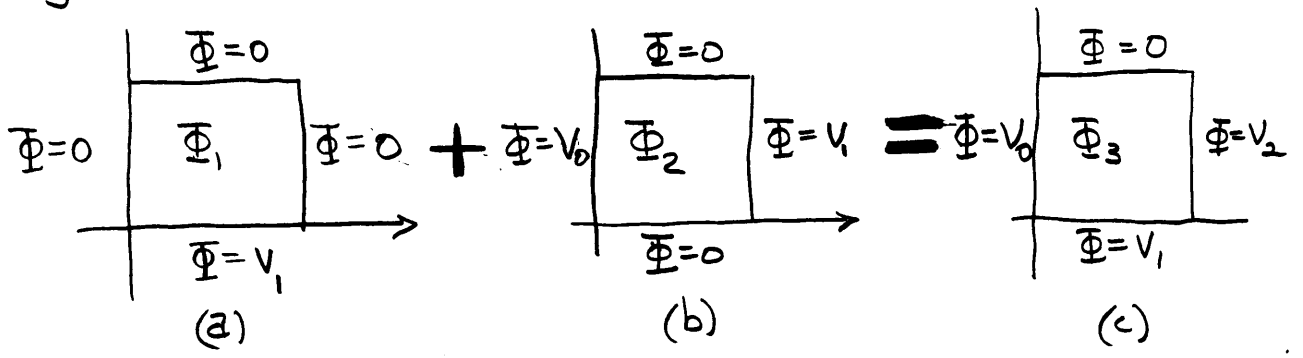


(b)

In Figure (a) we could not use exponential solutions in the x-direction and must use functions of the form $\sin\left(\frac{n\pi x}{a}\right)$ to have a zero at $x=0$, $x=a$. In b the same situation occurs in the y-direction which requires functions of the form $\sin\left(\frac{n\pi x}{b}\right)$. Note n can be 1, 2, 3,... (more on this later). This discussion leads naturally to boundary conditions of the form shown below.



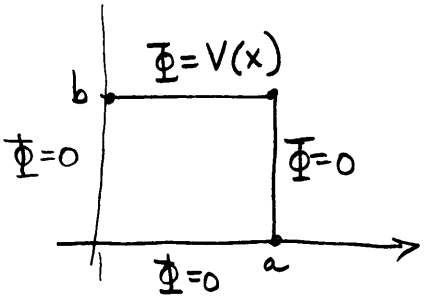
This problem can be solved by using superposition and de-composing the problem into two more easily solved problems.




If $\Phi_1$ solves Laplace's equation for the situation in (a) and $\Phi_2$ solves Laplace's equation for the B.C.'s in (b). Then $\Phi_3 = \Phi_1 + \Phi_2$ solves Laplace's equation for the B.C.'s in (c).

This principle can be applied to solutions of Laplace's equation in all coordinate systems. As long as the boundary conditions sum up to the desired B.C.'s and Laplace's equation is solved for the same region (i.e. the regions must be exactly the same except for the boundary conditions.

Finally, we are led into the realm of complex B.C.'s. Up to now we have dealt only with boundary conditions that are constant. Suppose we examine the problem shown below.




We have let $\Phi = 0$ on the other three sides to avoid situations requiring superposition. As we know this problem can be decomposed by separation

of variables into the two ordinary differential equations

$$\frac{d^2f}{dx^2} - k_x^2 f = 0 \qquad \frac{d^2g}{dy^2} - k_y^2 g = 0$$

where $\Phi(x,y) = f(x)\,g(y)$ and $k_x^2 + k_y^2 = 0$.

The boundary conditions on $f(x)$ require $k_x^2 < 0$ so that

$$f(x) = C_1 \sin k_x x + C_2 \cos k_x x.$$

At $x=0$, $f(x)=0$ which requires $C_2 = 0$. At $x=a$ $f(x)$ must also equal zero which requires

$$k_x a = n\pi \qquad \left(k_x = \frac{n\pi}{a}\right)$$

where $n = 1, 2, 3, \ldots$, etc. The <u>complete</u> formal solution is then

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} x\right)$$

The solution for $g(y)$ is simply

$$g(y) = C_3 e^{+k_y y} + C_4 e^{-k_y y}$$

At $y=0$, $\Phi = 0$ so $C_3 + C_4 = 0$. The complete solution is then

$$\Phi(x,y) = \sum_{n=1}^{\infty} C_n' \sin\left(\frac{n\pi x}{a}\right)\left[e^{k_y y} - e^{-k_y y}\right]$$

Note that $k_y^2 = -k_x^2 = -\left(\frac{n\pi}{a}\right)^2$. and that we have satisfied all B.C.'s except $\Phi(x,b) = V(x)$. Let us examine this B.C. in greater detail. Let $y=b$

$$\Phi(x,b) = \sum_{n=1}^{\infty} C_n' \sin\left(\frac{n\pi x}{a}\right)\left[e^{k_y b} - e^{-k_y b}\right].$$

$$= V(x).$$

At first this seems like a very formidable problem with no simple solution. However, we astutely observe that

$$C_n' e^{kyb} - e^{-kyb} = C_n$$

where $C_n$ is a constant. If we re-write this B.C. using $C_n$ we have

$$V(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right). \qquad (1)$$

Those of you who have been exposed to Fourier analysis will immediately recognize the right-hand side of the above equation as a Fourier sine series expansion of $V(x)$. The coefficients $C_n$ are determined just as one would for a Fourier sine series. First, we recall that $\sin\left(\frac{n\pi x}{a}\right)$ for $n = 1, 2, 3, \ldots$ are orthogonal functions, i.e.

$$\int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2}\delta_{mn} \qquad (2)$$

where $\delta_{mn} = 1$ if $m = n$ and zero otherwise.
To complete our solution for $\Phi(x,y)$ we then multiply both sides of Eqn (1) by $\sin\frac{m\pi x}{a}$ and integrate over $[0,a]$.

$$V(x)\sin\frac{m\pi x}{a} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}$$

$$\int_0^a V(x)\sin\frac{m\pi x}{a}\,dx = \int_0^a \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}\,dx$$

$$= \sum_{n=1}^{\infty} C_n \int_0^a \underbrace{\sin\frac{n\pi x}{a}\sin\frac{m\pi x}{a}}_{\pi\delta_{m,n}}\,dx$$

$$= C_m\,\pi$$

where we used

The results of (b) to eliminate all but one integral from the sum on the right-hand side of the above equation.

Our final result is then

$$\Phi(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\left[e^{k_n y} - e^{-k_n y}\right]$$

where $$k_n = \sqrt{1 - \frac{n^2\pi^2}{a^2}}$$

and $$c_n = \frac{\frac{1}{\pi}\int_0^a V(x) \sin\frac{m\pi x}{a}\, dx}{\left[e^{k_n b} - e^{-k_n b}\right]}$$

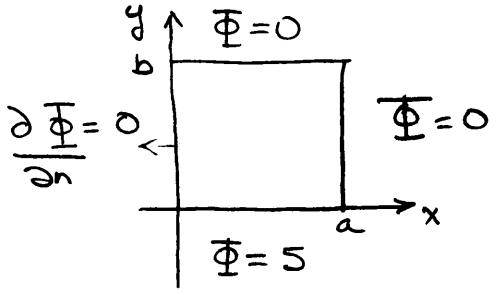
■

## Neumann Boundary conditions:

Up to now we have discussed problems where the boundary conditions have only specified $\Phi$ on the boundary, i.e. Dirichlet boundary conditions. This boundary condition is easy to conceptualize - its simply a voltage. On the other hand, a Neumann type boundary condition requires the specification of $\frac{\partial \Phi}{\partial n}$ on the boundary. First, does $\frac{\partial \Phi}{\partial n}$ make any sense. It certainly does! $E_{normal} = -\frac{\partial \Phi}{\partial n}$ and $D_{normal} = \epsilon E_{normal}$. Thus, specifying $\frac{\partial \Phi}{\partial n}$ is equivalent to specifying $D_{normal}$ at the boundary. As most boundaries are at conducting surfaces, $D_{normal}$ is zero beyond the boundary and a surface charge density $\rho_s = -D_{normal}$ is present at the boundary. Combining our equations $\rho_s = \epsilon \frac{\partial \Phi}{\partial n}$ and we see that when we specify $\frac{\partial \Phi}{\partial n}$ we are really specifying the surface charge density at the boundary. The only mathematical consequence of specifing $\frac{\partial \Phi}{\partial n}$ is that where the coefficient of the sine term in the solution was always zero; it will no longer be zero.

It may be noted that Neumann type boundary conditions often occur at the surfaces of dielectric boundaries where there is a surface charge density on the interface.

## Example of Neumann boundary conditions




let $\Phi = f(x)\, g(y)$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{k_x^2} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{k_y^2} = 0$$

$k_x^2 + k_y^2 = 0$

$$\frac{d^2f}{dx^2} - k_x^2 f = 0$$

$$\frac{d^2g}{dy^2} - k_y^2 g = 0$$

we must use sinusoids to satisfy the Neumann boundary conditions; so, $k_x^2 < 0$.

$$f(x) = c_1 \sin k_x x + c_2 \cos k_x x$$

$$\frac{\partial f}{\partial n} = \frac{df}{dx} = c_1 k_x \cos k_x x - c_2 k_x x \sin k_x x$$

$$\left.\frac{df}{dx}\right|_{x=0} = c_1 k_x \quad \therefore \text{ this B.C. requires } c_1 = 0.$$

$$\therefore f(x) = c_2 \cos k_x x$$

$$f(a) = c_2 \cos k_x a = 0$$

This B.C. requires $k_x a = n\frac{\pi}{2}$ (Note: $\frac{\pi}{2}$ not $\pi$ as we usually have had)

$$\therefore f(x) = \sum_{n=1,3,5,\ldots}^{\infty} c_n \cos\left(\frac{n\pi x}{2a}\right)$$

Since, $k_x^2 < 0$, $k_y^2 > 0$ and

$$g(y) = c_3 e^{+k_y y} + c_4 e^{-k_y y}$$

As a general rule of thumb do boundary conditions with zero's first; other values usually require a Fourier series approach for that B.C..

If we do the B.C. $\Phi(x,b)=0$ it is sufficient to require $g(b)=0$, i.e.

$$g(b)=0= C_3 e^{k_y b} + C_4 e^{-k_y b}.$$

$$C_3 = -C_4 e^{-2k_y b}.$$

At this point we have the following expression for $\Phi(x,y)$

$$\Phi(x,y) = \sum_{n=1,3,5\ldots}^{\infty} C_n \cos\left(\frac{n\pi x}{2a}\right) C_4 \left[e^{-k_y y} - e^{+k_y y} e^{-2k_y b}\right]$$

If we combine $C_n$ and $C_4$ into $C_n'$

$$\Phi(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} C_n' \cos\left(\frac{n\pi x}{2a}\right)\left[e^{-k_y y} - e^{+k_y y} e^{-2k_y b}\right]$$

At this point we let $\Phi(x,0)=5$ and do a Fourier series of the B.C.

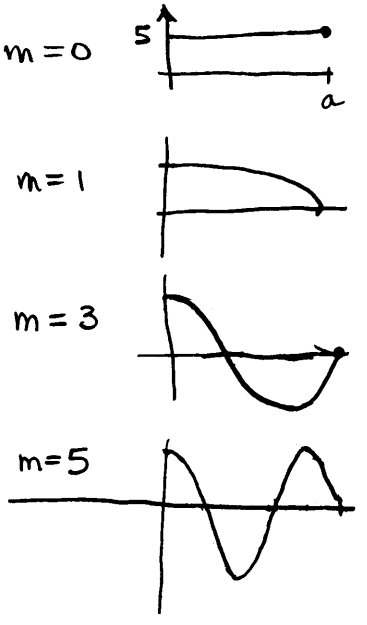
$$5 = \Phi(x,0) = \sum_{n=1,3,5,\ldots}^{\infty} C_n' \cos\left(\frac{n\pi x}{2a}\right)\left[1 - e^{-2k_y b}\right]$$

For the Fourier series approach we multiply both sides of the above equation by $\cos\left(\frac{m\pi x}{2a}\right)$ and integrate from 0 To a.

$$\int_0^a 5\cos\frac{m\pi x}{2a}\,dx = \int_0^a \sum_{n=1,3,5}^{\infty} C_n' \cos\left(\frac{n\pi x}{2a}\right)\cos\left(\frac{m\pi x}{2a}\right)\left[1-e^{-2k_y b}\right]dx$$

$$= \sum_{n=1,3,5}^{\infty} C_n'\left[1-e^{-2k_y b}\right]\int_0^a \cos\left(\frac{n\pi x}{2a}\right)\cos\left(\frac{m\pi x}{2a}\right)dx$$

First, let's examine the left-hand side of the above equation. Our first reaction is that it is zero except for $m=0$. But recall that $m = 1, 3, 5, \ldots$ i.e. odd integers. The cosine is <u>not</u> an odd function over this interval and the resulting integral is <u>not</u> zero.




[This incidentally is not one of the allowed solutions since $m \neq 0$. ]

On the right hand side

$$\int_0^a \cos\left(\frac{n\pi x}{2a}\right)\cos\left(\frac{m\pi x}{2a}\right)dx = a\,\delta_{m,n} \quad \text{where } \delta_{m,n} = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$$

Using these results we have

$$\int_0^a 5\cos\left(\frac{m\pi x}{2a}\right)dx = \sum_{n=1,3,5,\ldots}^{\infty} c_n'\left[1-e^{-2k_y b}\right]a\,\delta_{m,n}$$

$$= c_m'\left[1-e^{-2k_y b}\right]a$$

$$c_m' = \frac{1}{a\left[1-e^{-2k_y b}\right]}\int_0^a 5\cos\left(\frac{m\pi x}{2a}\right)dx$$

Our final and complete solution is

$$\Phi(x,y) = \sum_{n=1,3,5,\ldots}^{\infty} C_n \cos\left(\frac{n\pi x}{2a}\right)\left[e^{-\frac{n\pi}{2a}y} - e^{+\frac{n\pi}{2a}y} e^{-2\frac{n\pi}{2a}b}\right]$$

where $$C_n = \frac{1}{a\left[1-e^{-2\frac{n\pi}{2a}b}\right]} \int_0^a 5\cos\left(\frac{n\pi x}{2a}\right) dx$$

since $$k_y = \frac{n\pi}{2a}$$

Note that $k_x = k_y = \frac{n\pi}{2a}$. Even though $k_x^2 + k_y^2 = 0$ we have already assumed $k_x^2 < 0$ so $k_y^2 > 0$ and $k_y^2 = \frac{n^2\pi^2}{4a^2}$. Note that $k_x$ and $k_y$ are <u>always positive</u> in our solutions because the sign is already included in our assumption that one solution is sinusoidal and the other exponential.

## Laplace's equation in cyclindrical coordinates:

Up to now we have only solved Laplace's equation $\nabla^2\Phi=0$ for the case of rectilinear $(x-y)$ coordinates. Many problems have circular or cyclindrical symmetries which requires the specification of B.C.'s along cyclindrical boundaries. This means that we must specify a solution of $\nabla^2\Phi=0$ of the form $\Phi(r,\phi,z)$.

Let us solve this equation as we did in Cartesian coordinates by separation of variables. Let

$$\Phi(r,\phi,z) = f(r)\,g(\phi)\,h(z).$$

In cyclindrical coordinates

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

Substituting our expression for $\Phi$ and converting to ordinary differentials

$$gh\,\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + fh\,\frac{1}{r^2}\frac{d^2g}{d\phi^2} + fg\,\frac{d^2h}{dz^2} = 0$$

Dividing by $\frac{1}{fgh}$

$$\frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{1}{r^2}\frac{d^2g}{d\phi^2} + \frac{1}{h}\frac{d^2h}{dz^2} = 0.$$

Our first response is to separate out the $h(z)$ term; however, $g(\phi)$ is likely to be periodic in $\phi$, i.e. $g(0)=g(2\pi)$, etc. To exploit this we will do $g(\phi)$ first by multiplying the above equation by $r^2$

$$\frac{1}{f}r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{d^2g}{d\phi^2} + r^2\frac{1}{h}\frac{d^2h}{dz^2} = 0$$

Now, let us examine the term containing $g(\phi)$. As this is the only term with a $\phi$ dependence it must be constant for all values of $\phi$ otherwise it could not cancel the other terms for $r$ and $z$ fixed. For historical reasons we will call this constant $-\nu^2$. Then,

$$\frac{1}{g}\frac{d^2g}{d\phi^2} = -\nu^2$$

$$\frac{d^2g}{d\phi^2} + \nu^2 g = 0$$

As we expect $g(0) = g(2\pi)$ for $g$ to be single-valued and continuous, we pick $\nu^2 > 0$ which gives

$$g(\phi) = c_1 \sin \nu\phi + c_2 \cos \nu\phi$$

For $g(0)$ to equal $g(2\pi)$ $\nu$ must be equal to an integer.

i.e. if $\nu = 1$ $\quad g(0) = g(2\pi)$
if $\nu = 2$ $\quad g(0) = g(4\pi)$
if $\nu = 3$ $\quad g(0) = g(6\pi)$
etc.

As long as $\nu$ is an integer the functions are equal since the functions are periodic in $2\pi$.

Returning to our original equation and substituting $-\nu^2$ for the $\phi$ dependent term we get

$$\frac{1}{f} r \frac{d}{dr}\left(r \frac{df}{dr}\right) + \nu^2 + r^2 \frac{1}{h}\frac{d^2h}{dz^2} = 0.$$

Now, let's multiply through by $\frac{1}{r^2}$ and separate the equations into one for $r$ and one for $z$ dependent terms

$$\frac{1}{f}\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{\nu^2}{r^2} + \frac{1}{h}\frac{d^2h}{dz^2} = 0$$

After separation

$$\frac{d^2h}{dz^2} + k_z^2 h = 0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \left(\frac{\nu^2}{r^2} + k_z^2\right) f = 0$$

The first equation is of the type we have seen before. If $k_z^2 > 0$

$$h(z) = C_3 \sin k_z z + C_4 \cos k_z z$$

and if $k_z^2 < 0$

$$h(z) = C_3 e^{+k_z z} + C_4 e^{-k_z z}$$

It is the second equation that is new and very complicated in appearance. Actually it is Bessel's equation and has several different types of solutions.

If $k_z = 0$ Bessel's equation has the simple solution

$$f(r) = C_5 r^{+\nu} + C_6 r^{-\nu}.$$

This type of solution often occurs when there is no z-dependence. In the more general case we have

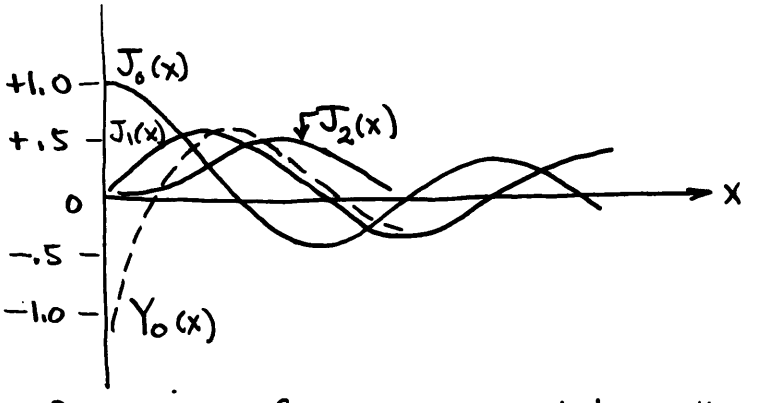
$$k_z^2 < 0 \qquad f(r) = C_5 J_\nu(\Gamma r) + C_6 Y_\nu(\Gamma r)$$

$$k_z^2 > 0 \qquad f(r) = C_5 I_\nu(\Gamma r) + C_6 K_\nu(\Gamma r)$$

where in all cases $\Gamma = |\sqrt{k_z^2}|$. The functions $I_\nu$ and $K_\nu$ are known as modified Bessel functions of the first and second kinds, respectively, and will not be further discussed since they can never go to zero and, thus, cannot satisify many B.C.'s that we will be interested in.

$J_\nu(\Gamma r)$ and $Y_\nu(\Gamma r)$ are tabulated in many references such as the CRC Mathematical Handbook. The function $Y_\nu$ is usually not an acceptable solution since it is singular (i.e. goes to infinity) at $r=0$. $J_\nu$, on the other hand, is well behaved and looks like a damped sinusoid.

Some typical Bessel functions are shown below




The Bessel functions form a complete orthogonal series and allow arbitrary functions to be expanded in a Fourier type series called a Fourier-Bessel series.

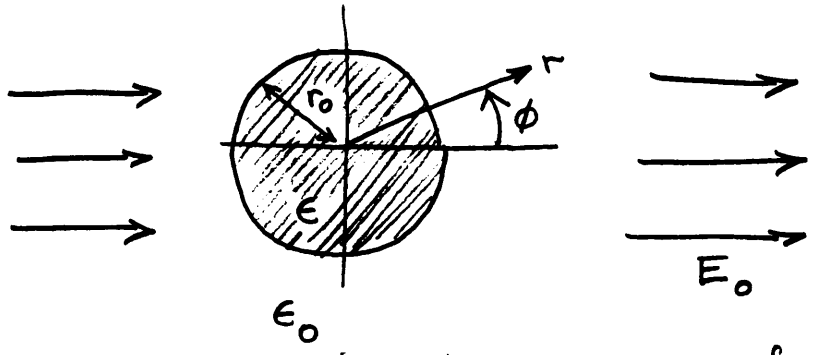
Furthermore,

$$\int_0^a r\, J_n(\Gamma_{nm} r)\, J_n(\Gamma_{ns} r)\, dr = 0 \quad \text{if} \quad m \neq s$$

$$= \frac{r^2}{2}\left[ J_n^2(\Gamma_{nm} r) - J_{n-1}(\Gamma_{nm} r)\, J_{n+1}(\Gamma_{nm} r) \right] \quad m = s.$$

The subscripts $\Gamma_{nm}$, etc. may seem complex, but is a short hand way of saying these are values of $\Gamma a$ that make $J_n(\Gamma a) = 0$ since the zeros of the Bessel function are not simply placed like those of the sine function. Thus, $\Gamma_{mn}$ denotes that $\Gamma_{mn} a$ is the m-th zero of the Bessel function $J_n(r)$.

Example: No z-dependence, Laplace's Equation in Cyclindrical Coordinates




This is a dielectric cyclinder in a uniform Electric field.

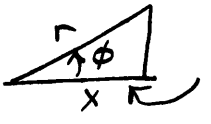
First, we must determine that there is no z-dependence Hence, $k_z = 0$ since we want $h(z) =$ constant.

Since, $k_z = 0$

$$f(r) = c_1 r^{+\nu} + c_2 r^{-\nu}$$

and, in all cases,

$$g(\phi) = c_3 \sin\nu\phi + c_4 \cos\nu\phi$$




$x = r\cos\phi$.

As $E = -\nabla\Phi$ we would expect $\Phi$ to have only an x-depencence. i.e. $\Phi =$ (constant) $x$. So that $E = -$(constant). But in cyclindrical coordinates $x = r\cos\phi$. This agree's with the $\Phi(r,\phi)$ we have already derived above except that the $\sin\nu\phi$ term is not consistent with an $E$ field in the x-direction. Thus, $c_3 = 0$.

Thus, $\Phi$ must be of the form (constant) $r\cos\phi$.

Our general potential must be of the form

$$\Phi_{tot}(r,\phi) = \Phi_{in}(r,\phi) + \Phi_0(r,\phi).$$

$\Phi_0$ is the potential causing the initial electric field $\underline{E_0}$

We have already argued that since $\underline{E} = -\nabla\Phi$ we expect

$$\Phi_0(r,\phi) = E_0 r\cos\phi.$$

Since all induced charges and fields must respond to this applied field $\Phi_{in}$ must be of the same form, as well as $\Phi_{tot}(r,\phi)$.

$\Phi_{in}$ is of the general form

$$\Phi_{in}(r,\phi) = \begin{cases} (C_1 r^{\nu} + C_2 r^{-\nu})\cos\phi & r<r_0 \\ (C_1 r^{\nu} + C_2 r^{-\nu})\cos\phi & r\geq r_0 \end{cases}$$

where we have recognized that $\nu=1$ for $\Phi_{in}$ to be of the same form as $\Phi_0$. Furthermore, we require the potential to be finite for $r<r_0$ and to go to zero as $r\to 0$. This reduces the above solution to

$$\Phi_{in}(r,\phi) = \begin{cases} A\, r\cos\phi & r<r_0 \\ B\, r^{-1}\cos\phi & r\geq r_0 \end{cases}$$

where the coefficents A and B are used to exemplify the two different regions. To find A and B we must make use of the two B.C.'s: that $\Phi_{tot}$ is continuous at $r_0$ and that $D_{normal}$ is continuous at $r_0$ since $\rho_s = 0$ on the cyclinder.

For the first B.C. at $r_o$

$$A r_o \cos\phi + E_o r_o \cos\phi = B r_o^{-1} \cos\phi + E_o r_o \cos\phi$$

$$A r_o = B r_o^{-1}$$

$$B = A r_o^2$$

$D_{normal}$ is the radial component of $-\nabla \underline{\Phi}_{tot}$ [which is just $-\frac{\partial \Phi}{\partial r}_{tot}$] times $\epsilon$ in each region. Thus,

$$(\epsilon) -\frac{\partial}{\partial r} \Phi_{tot}(r_o) \qquad = (\epsilon_o) -\frac{\partial}{\partial r} \Phi_{tot}(r_o)$$

$$-\epsilon \frac{\partial}{\partial r}\left[ A r \cos\phi + E_o r \cos\phi \right] = -\epsilon_o \frac{\partial}{\partial r}\left[ B r^{-1} \cos\phi + E_o r \cos\phi \right]$$

$$-\epsilon \left[ A \cos\phi + E_o \cos\phi \right] = -\epsilon_o \left[ -\frac{B}{r^2} \cos\phi + E_o \cos\phi \right]$$

$$\epsilon \left[ A + E_o \right] = \epsilon_o \left[ -\frac{B}{r_o^2} + E_o \right] \quad \text{at } r_o$$

and since $B = A r_o^2$

$$\epsilon \left[ A + E_o \right] = \epsilon_o \left[ -\frac{A r_o^2}{r_o^2} + E_o \right]$$

$$A + E_o = \frac{\epsilon_o}{\epsilon} \left[ -A + E_o \right]$$

$$A \left[ 1 + \frac{\epsilon_o}{\epsilon} \right] = E_o \left[ \frac{\epsilon_o}{\epsilon} - \epsilon \right]$$

$$A \left[ \epsilon + \epsilon_o \right] = E_o \left[ \epsilon_o - \epsilon \right]$$

$$A = E_o \frac{\epsilon_o - \epsilon}{\epsilon_o + \epsilon}$$

$$\therefore \quad \Phi_{tot} = \begin{cases} \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} E_0 r\cos\phi + E_0 r\cos\phi & r < r_0 \\ \frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} E_0 r_0^2 r^{-1}\cos\phi + E_0 r\cos\phi & r \geq r_0 \end{cases}$$

$$\Phi_{tot} = \begin{cases} \frac{2\epsilon_0}{\epsilon_0 + \epsilon} E_0 r\cos\phi & r < 0 \\ \left[\frac{\epsilon_0 - \epsilon}{\epsilon_0 + \epsilon} \frac{r_0^2}{r} + r\right] E_0 \cos\phi & r \geq r_0 \end{cases}$$