Historical, observational basis of Maxwell's Equations

field — force at a distance.

Coulomb's law:

$$\underline{F} = \frac{1}{4\pi\epsilon}\frac{q_1}{r_{12}^2} q_2 \, \underline{u}_{12} \qquad (1)$$

$\underline{u}_{12}$: unit vector

q_1, q_2 are in coulombs

r_{12} in meters (distance between charges)

ϵ permittivity

8.854×10^{-12} farad/meter

or, approximately, $\frac{1}{36\pi} \times 10^{-9}$ farad/meter

It is from Coulomb's law that we get the basic idea of a field due to a single charge, i.e.
This field concept is due to Faraday.

$$\underline{F} = q_2 \underline{E}_{12} \quad \text{or} \quad \underline{E} = \frac{\underline{F}}{q}$$

$$\text{where } \underline{E}_{12} = \frac{1}{4\pi\epsilon}\frac{q_1}{r_{12}^2} \underline{u}_{12} \qquad (2)$$

This is a vector function of charge, distance and direction
In general, for any given charge q_1

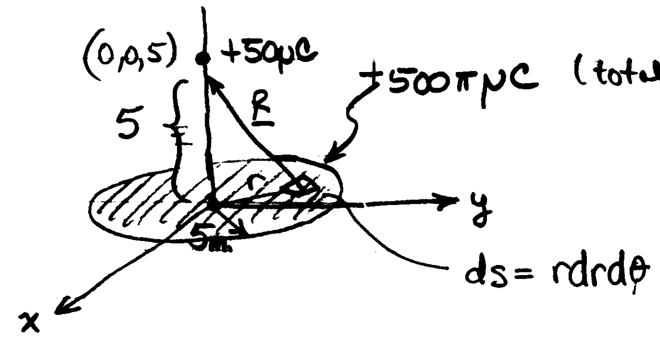
$$\underline{E}_{12} = \underline{f}\,(x_1, y_1, z_1, x_2, y_2, z_2) \qquad (3)$$

As important as Coulomb's law is principle of superposition which allows us to sum the fields from many charges into one resultant field.

$$E_{tot} = \sum_i E_i \quad \text{where } E_i \text{ is the field from a single charge} \qquad (4)$$

Coulombs law example: (Schaumm's outline 2.4)

Find the force on a point charge of 50 μC (micro coulombs) at (0,0,5) due to a charge of $500\pi\ \mu C$ that is uniformly distributed over the circular disk $r \le 5$ meters, $z = 0$ meters.



$$\rho_s = \frac{Q}{A} = \frac{500\cancel{\pi} \times 10^{-6}}{\cancel{\pi}(5)^2} = 20 \times 10^{-6}\ C/m^2$$

do this problem in cyclindrical coordinates because of symmetry.

The vector $\underline{a_R}$ pointing from a differential surface ds with charge dq is given by.

$$\underline{R} = -r\underline{a_r} + 5\underline{a_z} \Rightarrow \underline{a_R} = \frac{-r\underline{a_r} + 5\underline{a_z}}{\sqrt{r^2+25}}$$

Note the − sign on $\underline{a_r}$ since we are pointing inward as shown.

What is the force at (0,0,5) due to ds?

$$\underline{dF} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{R^2}\underline{a_R} \qquad \text{where } q_2 = \rho_s\, ds$$

$$= \frac{1}{4\pi\left(\frac{1}{36\pi}\times 10^{-9}\right)} \frac{(50\times10^{-6})(20\times10^{-6}\cdot r\,dr\,d\theta)}{(r^2+25)}\left(\frac{-r\underline{a_r} + 5\underline{a_z}}{\sqrt{r^2+25}}\right)$$

To get the total force we must integrate $\underline{dF}$ over the entire surface area of the disk. Note that the $\underline{a_R}$ components cancel due to symmetry but the $\underline{a_z}$'s add so only an integration over $\underline{a_z}$ is necessary.

$$\underline{F} = \iint \frac{9(50\times10^{-6})(20\times10^{-6})\, 5r\underline{a_z}\, dr\,d\theta}{10^{-9}\,(r^2+25)^{3/2}} = 45\underline{a_z}\int_0^{5}\int_0^{2\pi}\frac{r\,dr\,d\theta}{(r^2+25)^{3/2}} = 45\underline{a_z}\cdot 2\pi\int_0^5 \frac{r\,dr}{(r^2+25)^{3/2}}$$

$$= 90\pi\,\underline{a_z}\ \left.\frac{-1}{(r^2+25)^{1/2}}\right|_0^5 = 90\pi\,\underline{a_z}\left[-\frac{1}{\sqrt{50}} + \frac{1}{5}\right] = 16.56\,\underline{a_z}$$

Maxwell's Equations: Historically, these were derived on an observational basis

In integral form, Maxwell's equations are:

$$\oint_C \underline{E}\cdot d\underline{l} = -\frac{d}{dt}\int_S \underline{B}\cdot\underline{n}\, da \qquad (5)$$

This is Faraday's Law of induction

$$\oint \underline{H}\cdot d\underline{l} = \int_S \underline{J}\cdot\underline{n}\, da + \frac{d}{dt}\int \underline{D}\cdot\underline{n}\, da \qquad (6)$$

This is Ampere's Law of Magnetic circuits and is the result of observing magnetic circuits. The second integral on the right hand side is actually a current (displacement current) due to the time rate of change of the electric flux

$$\oint_S \underline{D}\cdot\underline{n}\, da = \int_V \rho\, dv \qquad (7)$$

$$\oint_S B\cdot n\, da = 0 \qquad (8)$$

These are Gauss' Laws for electric and magnetic fields.

The above four equations are derived from observation and describe any electromagnetic field.

$\int \rho\, dv$ in Gauss' law means all free charge in the volume, i.e. volume distribution, surface distribution and line distribution.

$$\int_V \rho\, dv = \underbrace{\int \rho_v\, dV}_{\substack{\text{volume} \\ \text{distribution}}} + \underbrace{\int \rho_s\, ds}_{\substack{\text{surface} \\ \text{distribution}}} + \underbrace{\int \rho_\ell\, d\ell}_{\substack{\text{line} \\ \text{distribution}}} \qquad (9)$$

$\rho_v = \text{coul}/m^3 \qquad \rho_s = \text{coul}/m^2 \qquad \rho_\ell = \text{coul}/m$

To finish the formulation of Maxwell's Equations we need to include the relationships between D and E, B and H, J and E.

$\underline{E}$ = electric field intensity
$\underline{D}$ = electric flux density
} $\underline{D} = D(\underline{E})$

$\underline{H}$ = magnetic field intensity
$\underline{B}$ = magnetic flux density
} $\underline{B} = B(\underline{H})$

$\underline{J}$ = current density } $\underline{J} = J(\underline{E})$

The materials we will study in this course are
- linear — simple multiplicative relationship
- homogeneous — uniform
- isotropic — same in all directions

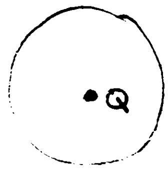
Constitutive relationships

$$\underline{D} = \epsilon \underline{E} \qquad (10a)$$
$$\underline{B} = \mu \underline{H} \qquad (10b)$$
$$\underline{J} = \sigma \underline{E} \qquad (10c)$$

where ϵ, μ, σ are functions of the material

Examples of applying integral Maxwell Equations

1. Electric field from point charge.



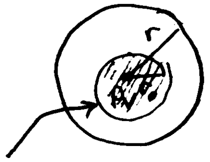
1) Field has spherical symmetry, so $\underline{E} = E_r \underline{a_r}$
2) Use gauss' law $\oint_S \underline{D} \cdot \underline{n} da = \int \rho \, dv$.

$$\oint_S \underline{D} \cdot \underline{n} da = \oint_S D_r \, r^2 \sin\theta \, d\theta \, d\varphi = D_r r^2 \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\varphi}_{4\pi}$$

$$\int \rho \, dv = Q$$

$$\therefore \quad D_r 4\pi r^2 = Q \quad \text{or} \quad D_r = \frac{Q}{4\pi r^2} \qquad E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

2. Electric Field from uniform volume distribution of charge



charge density ρ_v confined to $r < r_o$

1) Field has spherical symmetry whether $r > r_o$ or $r < r_o$ so $\underline{E} = E_r \underline{a_r}$
2) Use Gauss' Law

for $r > r_o$ $\quad \oint_S \underline{D} \cdot \underline{n} \, da = D_r 4\pi r^2$ as in Example 1

$$\int_v \rho_v \, dv = \frac{4}{3}\pi r_o^3 \rho_v$$

$$\therefore \quad D_r 4\pi r^2 = \frac{4}{3}\pi r_o^3 \rho_v \quad \text{or} \quad D_r = \frac{\rho_v r_o^3}{3r^2}$$

$$\underline{E} = \frac{\rho_v r_o^3}{3\epsilon_0 r_o^2} \underline{a_r} \qquad r > r_o$$

for $r < r_o$: $\quad \oint_S D \cdot n \, da = D_r 4\pi r^2$ as before

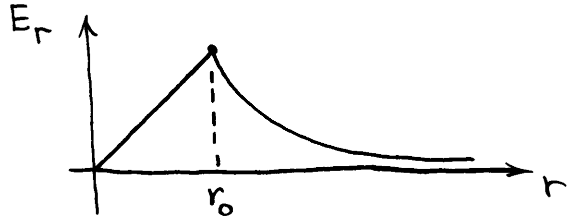
$$\int_v \rho_v \, dv = \int_0^r \int_0^{2\pi} \int_0^{\pi} \rho_v \, r^2 \sin\theta \, d\theta \, d\varphi = 4\pi r^2 \rho_v$$

since ρ_v is a constant and the remaining integral is just the surface area of a sphere of radius r.

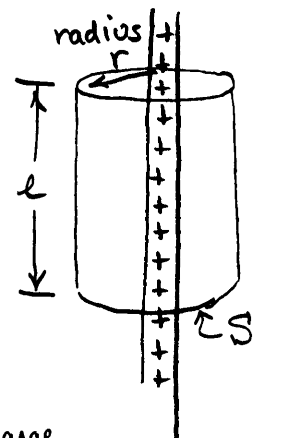
$$\therefore \quad D_r \frac{4\pi r^3}{3} = 4\pi r^2 \rho_v$$

$$\underline{E} = \frac{\rho_v}{3\epsilon_0 r} \underline{a_r}$$

If we plot E_r as a function of r we see that E_r undergoes a change in slope at $r = r_o$ but is continuous. We will study this in detail shortly.



3. Electric field from an infinite line charge



charge density ρ_ℓ coulombs/m.

1) Field now has cyclindrical symmetry so $\underline{E} = E_r \underline{a_r}$

2) Use Gauss Law $\oint \underline{D} \cdot \underline{n}\, da = \int_V \rho\, dv$

We will pick a cyclinder of length ℓ to begin the problem and will later let $\ell \to \infty$. Note That because we picked this surface $D = D_r \underline{a_r}$ is normal to the curved part of S but is parallel to the flat ends of S. As a result there is no contribution in Gauss' Law from the ends.

$$\text{Neglecting the ends } \oint \underline{D} \cdot \underline{n}\, da = \int_0^{\ell}\int_0^{2\pi} D_r \underline{a_r} \cdot \underline{a_r}\, dz\, r\, d\theta = D_r\, 2\pi r \ell$$

$$\int_V \rho\, dv = \rho_\ell\, \ell$$

$$\therefore \quad D_r\, 2\pi r \ell = \rho_\ell\, \ell$$

$$D = \frac{\rho_\ell}{2\pi r} \underline{a_r} \quad \text{independent of } \ell \text{ as } \ell \to \infty.$$

Let's see if there need to be any components of $\underline{B}_{tot}$ in the $\underline{a_r}$ direction. If H_θ satisfies Gauss' Law we do not need a H_r to allow $\underline{B}_{tot}$ to satisify Gauss' Law and we may let $H_r = 0$.

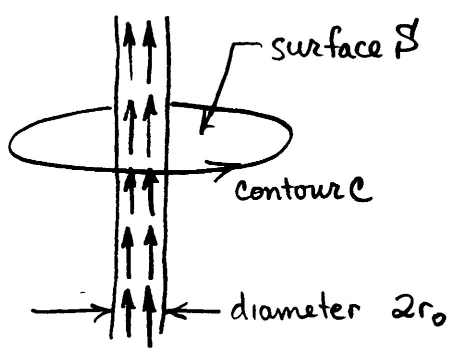
Gauss' Law: $\oint \underline{B} \cdot \underline{n} \, da = 0$ where $\underline{B} = \mu_0 \underline{H}$

as $\underline{n} = \underline{a_z}$ and B_θ is always normal to $\underline{a_z}$

$$\oint B_\theta \underline{a_\theta} \cdot \underline{a_z} \, da \equiv 0$$

and the problem is satisfied by just an H component in the θ direction.

5, <u>magnetic field from infinite uniform current density distributed over radius r_0.</u>



1) Use Ampère's Law (static case)

$$\oint \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{n} \, da$$

2) Cyclindrical symmetry as before. There is again no need to use a radial component to satisify the field conditions. Therefore, we will again use

$$\underline{H} = H_\theta \underline{a_\theta}$$

However, there is a total current I flowing in the wire so the current density must be given by

$$\underline{J} = \begin{cases} \dfrac{I}{\pi r_0^2} \underline{a_z} & r \leq r_0 \\ 0 & r > r_0 \end{cases}$$

For $r > r_0$, by Ampère's Law

$$\oint \underline{H} \cdot \underline{dl} = H_\theta \, 2\pi r$$

$$\int_S \underline{J} \cdot \underline{n} \, da = \int_0^{r_0}\int_0^{2\pi} \frac{I}{\pi r_0^2} \underline{a_z} \cdot \underline{a_z} \, r dr d\theta + \int_{r_0}^{r}\int_0^{2\pi} 0 \cdot \underline{a_z} \, r dr d\theta$$

since there is only a contribution for $r < r_0$. Note that this integral can easily be seen to be just the current I.

Thus, $\int \underline{J} \cdot \underline{n} \, da = I$

$$H_\theta \, 2\pi r = I$$

and $H_\theta = \dfrac{I}{2\pi r}$ as before.

For $r < r_0$, only the first integral contributes to $\int_S \underline{J} \cdot \underline{n} \, da$ and we get

$$\int_S \underline{J} \cdot \underline{n} \, da = \int_0^{r}\int_0^{2\pi} \frac{I}{\pi r_0^2} \underline{a_z} \cdot \underline{a_z} \, r dr d\theta$$

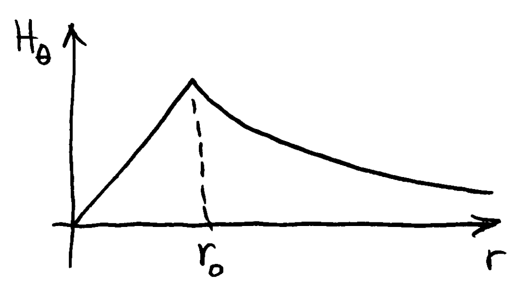
$$= \int_0^r \frac{I}{\pi r_0^2} r dr \, 2\pi = I \frac{r^2}{r_0^2}$$

So, for $r < r_0$

$$H_\theta \, 2\pi r = I \frac{r^2}{r_0^2}$$

$$H_\theta = \frac{I}{2\pi r_0^2} r$$

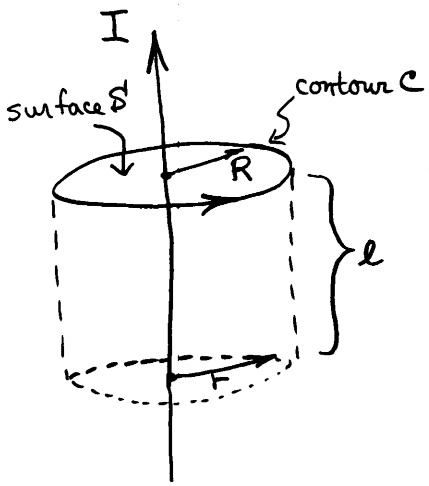
Plotting H_θ as a function of r



4. Magnetic field from infinite straight line currents

1. This is for a static problem so we will use Ampère's Law for $\frac{\partial}{\partial t} \to 0$

$$\oint \underline{H} \cdot \underline{d\ell} = \int_S \underline{J} \cdot \underline{n}\, da$$



2. As in example #3 we will use the cyclindrical symmetry of the problem by choosing a circular loop centered on the conductor with the plane of the loop perpendicular to the conductor.

3. By symmetry components of $\underline{H}$ parallel to $\underline{d\ell}$ cannot depend upon θ. Components of $\underline{H}$ perpendicular to $\underline{d\ell}$ will be discussed below.

$$\oint_C \underline{H} \cdot \underline{d\ell} = \int_0^{2\pi} H_\theta \underline{a_\theta} \cdot \underline{a_\theta}\, r\, d\theta = H_\theta 2\pi r$$

$$\int_S \underline{J} \cdot \underline{n}\, da = \int_0^R \int_0^{2\pi} \left[\frac{I\delta(r)}{2\pi r} \right] \underline{a_z} \cdot \underline{a_z}\, r\, dr\, d\theta$$

But this is simply the total current through S so

$$\int_S \underline{J} \cdot \underline{n}\, da = I$$

and

$$H_\theta 2\pi r = I$$

$$H_\theta = \frac{I}{2\pi r}$$

4. The total H field must satisify Gauss' Law so if there are any components of $\underline{H}$ perdendicular to $\underline{d\ell}$ (i.e. $\underline{a_r}$) we would have.

$$\oint_S \underline{B}_{tot} \cdot \underline{n}\, da = 0$$

?

In all these examples symmetry was used to arrive at a solution. We need to develop more powerful techniques for non-symmetric situations.

Furthermore, we can see that when we look at fields inside a source distribution they were continuous with the external fields at the boundary. The relationships between internal and external fields is not always so simple and will be altered by surface charges or currents.

To continue our study of the above points we must look at the differential form of Maxwell's equations

We can convert these equations to differential form using divergence and Stokes Theorems.

$$\oint_S \underline{D}\cdot\underline{n}\, da = \int_V \rho\, dv$$

From the divergence theorem

$$\oint_S \underline{D}\cdot\underline{n}\, da = \int_V \underline{\nabla}\cdot\underline{D}\, dv$$

The integrands must be equal, thus, $\nabla\cdot\underline{D} = \rho$ (11)

In the same manner,

$$\oint_S B\cdot n\, da = 0$$

becomes $\nabla\cdot\underline{B} = 0$ (12)

Faraday's law can be converted using Stoke's Theorem

$$\oint_C \underline{E}\cdot\underline{dl} = \int_S (\nabla\times\underline{E})\cdot ds$$

or

$$\int_S (\nabla\times\underline{E})\cdot d\underline{s} = -\frac{d}{dt}\int \underline{B}\cdot\underline{ds}$$

$$= -\int \frac{\partial \underline{B}}{\partial t}\cdot\underline{ds}$$

Thus, in differential form Faraday's Law becomes

$$\nabla\times E = -\frac{\partial B}{\partial t} \qquad (13)$$

Amperes' law can be manipulated in the same manner

$$\oint \underline{H} \cdot \underline{dl} = \int_S (\nabla \times \underline{H}) \cdot \underline{dS} = \int_S \underline{J} \cdot \underline{dS} + \int_S \frac{\partial \underline{D}}{\partial t} \cdot \underline{dS}$$

Note that I have casually used the identity $\underline{n}\, da = d\underline{S}$.

Finally, in differential form

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \qquad (14)$$

Maxwell's Equations in differential form

$$\nabla \cdot \underline{D} = \rho \qquad (15a)$$
$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad (15b)$$

$$\nabla \cdot \underline{B} = 0 \qquad (15c)$$

$$\nabla \times \underline{B} = \underline{J} + \frac{\partial D}{\partial t} \qquad (15d)$$

Note that if all fields are static $\left(\frac{\partial}{\partial t} \to 0\right)$

$$\nabla \cdot D = \rho \qquad \nabla \cdot E = \frac{\rho}{\epsilon} \qquad (16a)$$
$$\nabla \times E = 0 \qquad \nabla \times E = 0 \qquad (16b)$$

$$\nabla \cdot B = 0 \qquad \nabla \cdot B = 0 \qquad (16c)$$
$$\nabla \times \underline{B} = \underline{J} \qquad \nabla \times B = \underline{J} \qquad (16d)$$

Note that $\underline{E}$ is a pure irrotational field
$\underline{B}$ is a pure rotational field.

This means that E has discrete sources and field lines terminate on sources/sinks; B, on the other hand, describe's field lines which close on themselves and have no discrete sources.

The static equations are independent, i.e. $\underline{B}=0$ does not require $\underline{E}=0$ and vice versa. This means that static electric or magnetic fields can exist independently. On the other hand, retaining the time dependent terms produces coupling between $\underline{E}$ and $\underline{B}$ fields which leads to waves. It was this coupling and the resultant wave propagation that Maxwell predicted.

Electrostatics

Integral	Differential	
$\oint_S \underline{D}\cdot\underline{n}\,da = \int_V \rho\, dv$	$\nabla\cdot D = \rho$	(1)
$\oint_C \underline{E}\cdot \underline{dl} = 0$	$\nabla \times \underline{E} = 0$	

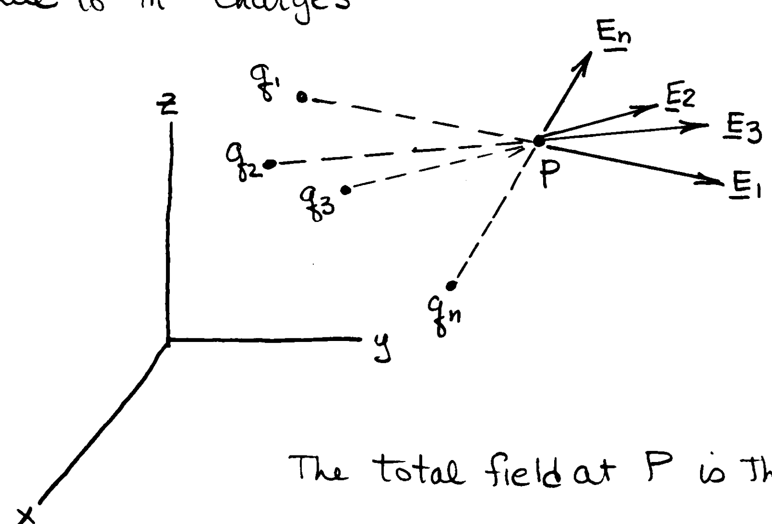
$$\underline{D} = \epsilon \underline{E} \qquad (2a)$$
$$\underline{J} = \sigma \underline{E} \qquad (2b)$$

Note that these are sufficent to describe a pure, irrotational field given by Coulomb's Law

$$\underline{F} = \frac{1}{4\pi\epsilon_0}\,\frac{q_1 q_2}{r_{12}^2}\,\underline{a_{r2}}$$

$$\underline{E} \triangleq \frac{\underline{F}}{q} \qquad \left[\text{Units are volts/meter}\right] \qquad (3)$$

Superposition is implicit in Maxwell's equations because they are linear. Consider the resultant field intensity due to n charges



The total field at P is then $\sum_{i=1}^{n} \underline{E}_i$

or $$\underline{E} = \sum_i \frac{q}{4\pi\epsilon_0 r_i^2}\,\underline{a}_{r_i} \qquad (4)$$

This can be generalized to continuous integrals and charge densities

$$\underline{E} = \int d\underline{E} \tag{5}$$

$$\underline{E} = \int \frac{dq \; \underline{a_r}}{4\pi\epsilon_0 \, r^2} \tag{6}$$

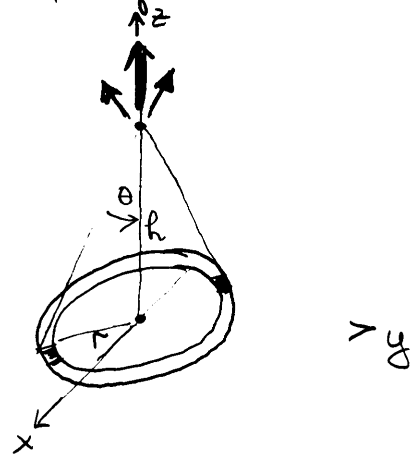
These are variables <u>not</u> constants

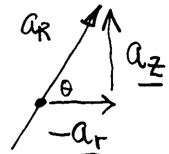
For volume densities $dq = \rho_v \, dv$
surface densities $dq = \rho_s \, ds$
line densities $dq = \rho_\ell \, d\ell$

All calculations of static fields are simply applications (usually complicated algebraically) of (6). This is a complicated process due to the fact that $\underline{a_r}$ is a variable.

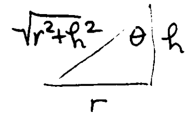
charge use ?
in this example

Example of field from charged plate





where $\theta = \mathrm{Tan}^{-1}\left(\frac{r}{h}\right)$



$$\underline{E} = \int \frac{dq\, \underline{a_r}}{4\pi\epsilon_0 R^2}$$

This problem is simple if we properly exploit symmetry. Each field contribution can be regarded as from a ring symmetric about the z-axis. This is a surface charge density in cylindrical coordinates so we use

$$dq = \rho_s ds = \rho_s\, r dr d\phi$$

or
$$\underline{E} = \int_0^{\infty}\int_0^{2\pi} \frac{\rho_s\, r dr d\phi\ \underline{a}_R(r,\phi)}{4\pi\epsilon_0 (r^2+h^2)}$$

where we noted that $R^2 = r^2+h^2$. However, we still have $\underline{a_r}$ as a function of r and ϕ. We can reduce our result to something simple to evaluate if we note that

$$\underline{a_r} = -\cos\theta\, \underline{a_r} + \sin\theta\, \underline{a_z}$$

Thus, $$\underline{E} = \int_0^{\infty}\int_0^{2\pi} \frac{-\rho_s\, r dr d\phi \cos\theta}{4\pi\epsilon_0 (r^2+h^2)}\underline{a_r} + \int_0^{\infty}\int_0^{2\pi} \frac{\rho_s\, r dr d\phi \sin\theta}{4\pi\epsilon_0 (r^2+h^2)}\underline{a_z}$$

By symmetry the first integral must go to zero because as we integrate along ϕ the $\underline{a_r}$ components π apart cancel each other out.

$$\underline{E} = \int_0^{\infty}\int_0^{2\pi} \frac{\rho_s\, r\,dr\,d\phi \sin\theta}{4\pi\epsilon_0 (r^2 + h^2)}\, \underline{a_z}$$

Looking back at our diagram we can replace $\sin\theta$ by

$$\sin\theta = \frac{r}{\sqrt{r^2 + h^2}}$$

to give

$$\underline{E} = \int_0^{\infty}\int_0^{2\pi} \frac{\rho_s\, r^2 dr\,d\phi}{4\pi\epsilon_0 \,(r^2 + h^2)^{3/2}}\, \underline{a_z}$$

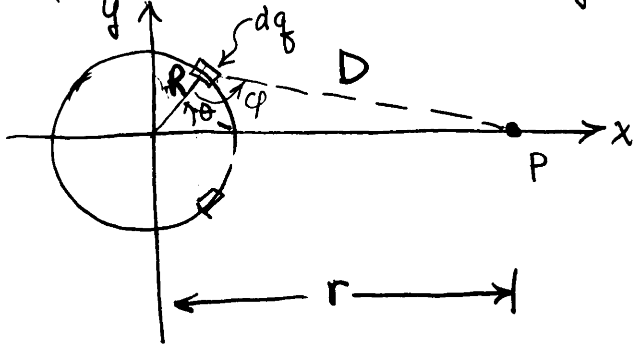
$$= \int_0^{\infty} \frac{2\pi\rho_s}{4\pi\epsilon_0} \frac{r^2 dr}{(r^2 + h^2)^{3/2}}\, \underline{a_z}$$

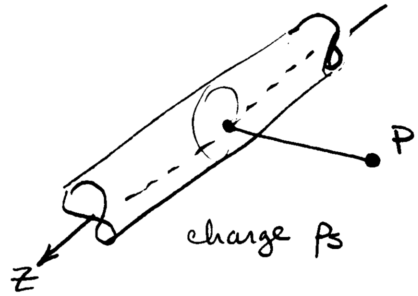
$$= \frac{\rho_s}{2\epsilon_0} \int_0^{\infty} \frac{r^2\, dr}{(r^2 + h^2)^{3/2}}\, \underline{a_z}$$

The value of the integral is 1 so.

$$\underline{E} = \frac{\rho_s}{2\epsilon_0}\, \underline{a_z}$$

Example of Gauss Law from a cylindrical shell of charge, of radius R.





Surface charge density ρ_s
differential surface element $\quad ds = \quad R\, d\theta\, dz$

$$E = \int \frac{dq\ a_D}{4\pi\epsilon_0\ D^2}$$

What is D? $\quad D^2 = R^2 + r^2 - 2Rr\cos\theta$ by law of cosines

$a_D = (r - R\cos\theta)\,\underline{a}_x - (R\sin\theta)\,\underline{a_y}$ a complex expression

what is our convention for a_D? from source to observation point!

By symmetry the $\underline{a_y}$ components are seen to cancel so:

$$\underline{E} = \iiint \frac{\rho_s\ R\,d\theta\,dz\ \left[(r - R\cos\theta)\,a_x + R\sin\theta\ \underline{a_y}\ \right]}{4\pi\epsilon_0\ (R^2 + r^2 - 2Rr\cos\theta)}$$

becomes

to other side

no r integral.

$$\underline{E} = \frac{\rho_s}{4\pi\epsilon_0} \int \frac{R\,d\theta\,dz\ (r - R\cos\theta)\,\underline{a_x}}{R^2 + r^2 - 2Rr\cos\theta}$$

$$\frac{\underline{E}}{\Delta z} = \frac{\rho_s R \cdot 2}{4\pi\epsilon_0} \underbrace{\int_0^{\pi} \frac{(r - R\cos\theta)\ d\theta}{R^2 + r^2 - 2Rr\cos\theta}}_{\text{This we look up because its hard.}}\ \underline{a_x}$$

from symmetry.

$$= \frac{\rho_s R}{2\pi\epsilon_0} \frac{2\pi}{r}\ \underline{a_x} \quad \text{for } D \geq R, \quad 0 \text{ otherwise.}$$

$$\frac{\underline{E}}{\Delta z} = \frac{\rho_s R}{\epsilon_0\, r}\ \underline{a_x}$$

Electrostatic potential

There are simpler ways of evaluating fields. Usually we do not know the charge densities.

How much work is done in moving a charge to a point in a given electrostatic field?

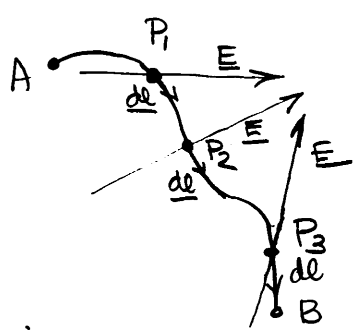
The force on a test charge Δq is given by the Lorentz force law, i.e.

$$\underline{F} = \Delta q \, \underline{E}$$

The incremental work dW done in moving Δq along the contour $d\underline{l}$ is given by

$$dW = -\underline{F} \cdot \underline{dl}$$

Now why does $\underline{E}$ have a minus sign associated with it? Consider moving a positive charge in the field. If the charge is moved against the field the resulting work will be positive not negative because of the $-$ sign count acting the sign of the dot product.



$$dW = -\Delta q \, \underline{E} \cdot \underline{dl}$$

Consider moving $+\Delta Q$ along the contour AB.

At P_1 $\quad \underline{E} \cdot \underline{dl} > 0$ so $dW < 0$

At P_2 $\quad \underline{E} \cdot \underline{dl} = 0$ so no work is being done

At P_3 $\quad \underline{E} \cdot \underline{dl} < 0$ and $dW > 0$

At P_1 the field is moving the charge, whereas at P_3 work is being done in moving the charge against the field.

Just as we defined the field as force/unit charge we can define work/unit charge done moving along a contour as

$$\frac{dW}{\Delta q} = -\underline{E}\cdot \underline{dl}$$

If we define this as the potential we have.

$$d\Phi = -\underline{E}\cdot\underline{dl}$$

or
$$\int_{P_1}^{P_2} d\Phi = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{dl}$$

or
$$\Phi(P_1) - \Phi(P_2) = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{dl}$$

We have seen this result before in Eqn (7) of Ch. 1. If $P_1 = P_2$ the right hand side of the above equation becomes the net circulation of an irrotational field which we already know is zero. Furthermore, we can also say that

$$\underline{E} = -\nabla\Phi$$

as long as ϕ is a single valued function. As long as our field is real no work is done traveling in circles in the field so Φ must be single valued.

This is summed up by saying

$$\oint \underline{E}\cdot\underline{dl} = 0$$

which is called the CONSERVATIVE property of an electric field.

Note the following properties of Φ:

- Φ is independent of path since $\phi = -\int_{P_1}^{P_2} \underline{E} \cdot d\underline{l}$ along any contour.

- Φ is only relative.

  From the conservative property we can only calculate a potential difference, so we must define some point as a reference if we are to do absolute measurements. Often we pick $\Phi(r=\infty) = 0$ as an arbitrary reference.

- Φ is not unique.

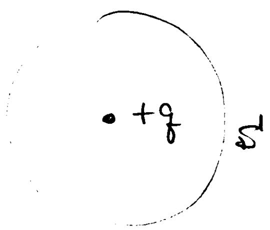
  Let $\Phi' = \Phi + k$ (a constant)

  $$\underline{E} = -\nabla\Phi' = -\nabla\Phi$$

  so both Φ and Φ' give the same $\underline{E}$ field. Hence, Φ cannot be unique.

Example:

Find $\underline{E}$, ϕ of point charge.



From Gauss' Law $\displaystyle\oint_S \underline{E}\cdot d\underline{s} = \frac{Q_{total}}{\epsilon_0} = \frac{q}{\epsilon_0}$

Choose a spherical coordinate system because of symmetry

Assume $\underline{E} = E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi}$

$d\underline{s} = r^2 \sin\theta \, d\theta \, d\phi \; \underline{a_r}$

where we picked a symmetric surface to make the problem easier.

Then $\underline{E}\cdot d\underline{s} = E_r \, r^2 \sin\theta \, d\theta \, d\phi$

and we simply integrate to find $\underline{E}$

$$\oint_S \underline{E}\cdot d\underline{s} = \int_0^\pi \int_0^{2\pi} E_r \, r^2 \sin\theta \, d\theta \, d\phi$$

no integration over r

ϕ integration

θ integration

$$= E_r \, r^2 2\pi \int_0^\pi \sin\theta \, d\theta = E_r \, r^2 2\pi \left(-\cos\theta \Big|_0^\pi \right)$$

$$= 2\pi r^2 E_r \left(-\overset{+1}{\cancel{\cos\pi}} + \overset{+1}{\cancel{\cos 0}} \right)$$

$$= 4\pi r^2 E_r$$

$\therefore \quad 4\pi r^2 E_r = \dfrac{q}{\epsilon_0}$ and $E_r = \dfrac{q}{4\pi\epsilon_0 r^2}$

$$\underline{E} = \frac{q}{4\pi\epsilon_0 r^2} \underline{a_r}$$

We can evaluate Φ directly from our result for $\underline{E}$.

In spherical coordinates

$$\underline{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\underline{a_r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\underline{a_\theta} - \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}a_\phi$$

which only contains an $\underline{a_r}$ term as $\underline{E}$ is only in the $\underline{a_r}$ direction.

$$\underline{E} = -\frac{\partial\Phi}{\partial r}\underline{a_r}$$

Substituting for $\underline{E}$ and integrating over r

$$\frac{q}{4\pi\epsilon_0 r^2}\underline{a_r} = -\frac{\partial\Phi}{\partial r}\underline{a_r}$$

$$= -\frac{d\Phi}{dr}\underline{a_r}$$

since there is no other dependence.

$$\int_{r=a}^{r=b}\frac{q}{4\pi\epsilon_0 r^2}dr = -\int_{r=a}^{r=b} d\Phi$$

$$\Phi(r=a) - \Phi(r=b) = \frac{q}{4\pi\epsilon_0} \left. -\frac{1}{r}\right|_{r=a}^{r=b}$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Note the additive constant $\Phi(r=b)$ here.

$$\Phi(r=a) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \Phi(r=b)$$

Pick $b=\infty$ and define $\Phi(r=\infty) = 0$ so.

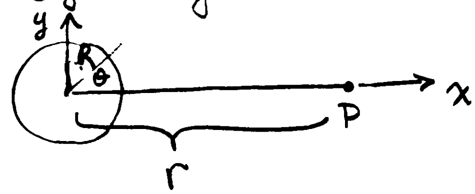
$$\Phi(r=a) = \frac{q}{4\pi\epsilon_0 a}$$

Example: electric potential from a charged cylinder of radius R.

We have previously solved this problem for the electric field (p. 30a) and obtained the result.

$$\underline{E} = \frac{\rho_s R}{\epsilon_0 r}\,\underline{a_x}$$

for the geometry shown below



Since $E = -\nabla\phi$ we can simply integrate this equation to obtain ϕ.

$$E = -\nabla\phi = -\underline{a_r}\frac{\partial\phi}{\partial r} - \underline{a_\phi}\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \underline{a_z}\frac{\partial\phi}{\partial z}$$

Well, $\frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial z} = 0$. As the x-direction was arbitrarily chosen we can let $\underline{a_x} = \underline{a_r}$ for this problem.

$$\therefore \quad E_r = -\frac{\partial\phi}{\partial r} = -\frac{d\phi}{dr} \quad \text{since } \phi \text{ is only a function of } r.$$

Integrating this expression:

$$E_r\,dr = -d\phi$$

$$\int_{r=a}^{r=b} E_r\,dr = -\int_{r=a}^{r=b} d\phi = -\phi(r=b) + \phi(r=a)$$

$$\therefore \quad \phi(r=a) = \phi(r=b) + \int_{r=a}^{r=b} \frac{\rho_s R}{\epsilon_0 r}\,dr = \frac{\rho_s R}{\epsilon_0}\ln r\Big|_{r=a}^{r=b}$$

$$= \phi(r=b) + \frac{\rho_s R}{\epsilon_0}(\ln b - \ln a)$$

$$\phi(r=a) = \phi(r=b) + \frac{\rho_s R}{\epsilon_0}\ln\left(\frac{b}{a}\right)$$

Example: Find the electric field and electric potential of a charged sphere with surface charge density ρ_s.

We can find E from Gauss' Law $\oint_S \epsilon_0 \underline{E} \cdot \underline{ds} = Q_{total}$

In spherical coordinates (because of symmetry)

$$d\underline{S} = r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$$

$$\underline{E} = E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi}$$

Now using a spherical surface of radius R centered on our original sphere:

$$\oint \epsilon_0 \underline{E} \cdot \underline{ds} = \iint \epsilon_0 \left[E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi} \right) \cdot r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$$

$$= \iint \epsilon_0 E_r \, r^2 \sin\theta \, d\theta \, d\phi = \epsilon_0 E_r \underbrace{\int_0^{2\pi}\!\!\int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi}$$

this integral is nothing but the surface of a sphere and is equal to $4\pi r^2$

$$= \epsilon_0 E_r \, r^2 2\pi \int_0^{\pi} \sin\theta \, d\theta = 2\pi \epsilon_0 E_r \, r^2 \left[-\cos\theta \Big|_0^{\pi} \right]$$

$$\oint \epsilon_0 \underline{E} \cdot d\underline{S} = 4\pi r^2 \epsilon_0 E_r$$

The total charge on the sphere is simply the charge density times the surface area:

$$Q_{total} = \rho_s \, 4\pi r_s^2$$

where r_s is the radius of the sphere. Since this must equal $\oint \epsilon_0 \underline{E} \cdot \underline{dS}$ we have

$$4\pi r^2 \epsilon_0 E_r = \rho_s 4\pi r_s^2$$

$$E_r = \frac{\rho_s}{\epsilon_0} \left(\frac{r_s}{r} \right)^2$$

and $$\underline{E} = \frac{\rho_s}{\epsilon_0} \left(\frac{r_s}{r} \right)^2 \underline{a}_r$$

To get the electric potential we must integrate the expression $\underline{E} = -\nabla\phi$. Since $\underline{E}$ has only radial components

$$\underline{E} = E_r \underline{a_r} = -\nabla\phi = -\underline{a_r}\,\frac{d\phi}{dr}$$

$$\therefore \quad d\phi = -E_r\,dr$$

and integrating

$$\int d\phi = -\int E_r dr = -\int \frac{\rho_s}{\epsilon_0}\left(\frac{r_s}{r}\right)^2 dr$$

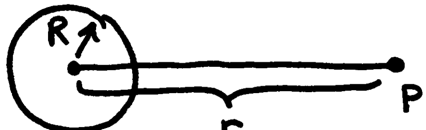
$$\phi(r=b) - \phi(r=a) = -\frac{\rho_s}{\epsilon_0} r_s^2 \left[-\frac{1}{r}\Big|_{r=a}^{r=b}\right]$$

$$= -\frac{\rho_s r_s^2}{\epsilon_0}\left[-\frac{1}{b} + \frac{1}{a}\right]$$

$$\phi(r=b) = \phi(r=a) + \frac{\rho_s r_s^2}{\epsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$$

Example: Find the potential of an infinitely long line charge with linear charge density ρ_ℓ.

We can treat this as a special case of the previously solved problem of a charged cylinder as shown below.



Our result for this charged cylinder was $\frac{\underline{E}}{\Delta z} = \frac{\rho_s R}{\epsilon_0 r}\underline{a_r}$

Note that as $R \to 0$ (a line charge) $\underline{E} \to 0$ because ρ_s is a surface charge. As before

$$\oint \epsilon_0 \underline{E} \cdot d\underline{s} = Q_{total}$$

In cylindrical coordinates

$$\epsilon_0 \oint (E_r \underline{a_r} + E_\phi \underline{a_\phi} + E_z \underline{a_z}) \cdot r\,d\phi\,dz\,\underline{a_r}$$

$$= \epsilon_0 \int_0^{\Delta z}\int_0^{2\pi} E_r\, r\,d\phi\,dz = \epsilon_0 2\pi r \Delta z\, E_r$$

where Δz is the length of a section of the line, we can later let $\Delta z \to \infty$. The total charge on a length Δz of line is $Q_{total} = \rho_\ell \Delta z$ so that Gauss' Law becomes

$$\epsilon_0 2\pi r \Delta z E_r = \rho_\ell \Delta z$$

$$E_r = \frac{\rho_\ell}{2\pi\epsilon_0}\frac{1}{r}$$

As in previous problems we use $\underline{E} = -\nabla\phi$ to solve for ϕ.

$$\underline{E} = E_r \underline{a_r} = -\nabla\phi = -\frac{d\phi}{dr}\underline{a_r}$$

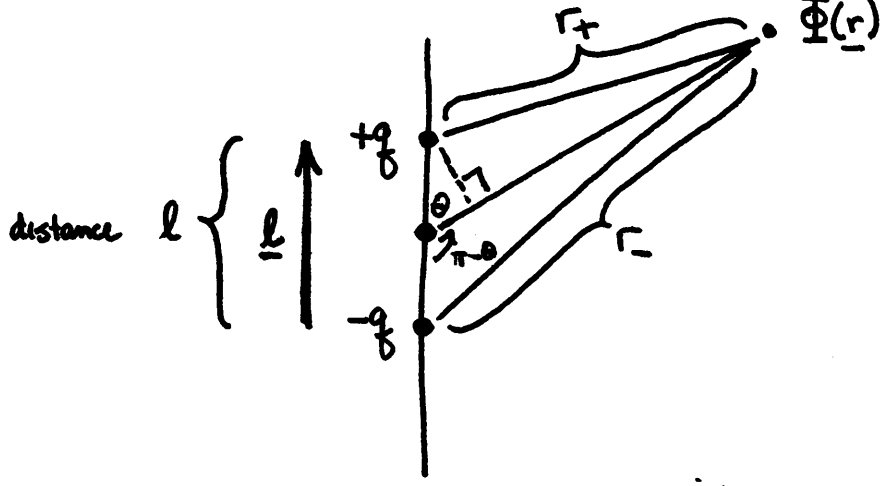
$$d\phi = -E_r\,dr$$

$$\int_{r=a}^{r=b} d\phi = -\int_{r=a}^{r=b} E_r\,dr = -\int_{r=a}^{r=b}\frac{\rho_\ell}{2\pi\epsilon_0}\frac{1}{r}\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\ln r\Big|_{r=a}^{r=b}$$

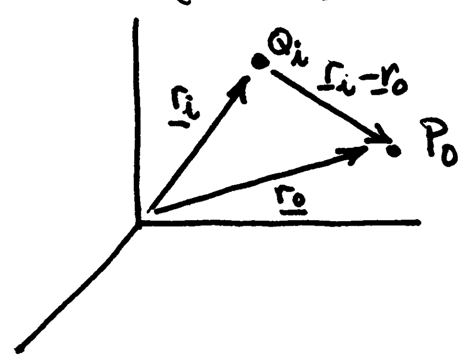
$$\phi(b) - \phi(r=a) = -\frac{\rho_\ell}{2\pi\epsilon_0}\ln(r=b) - \ln(r=a)$$

$$\phi(b) = \phi(a) + \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left(\frac{a}{b}\right)$$

One of the most important electrostatic field problems (because of its application in the study of materials) is the electric dipole. This consists of a positive charge and an equal negative charge separated by a distance d as shown below.



To solve this problem we must consider the superposition of the fields and potentials due to a distribution of discrete charges Q_i.



The potential at point P_0 due to a point charge at location r_i has already been found to be

$$\Phi_i(\underline{r}_0) = \frac{Q_i}{4\pi\epsilon_0 |\underline{r}_i - \underline{r}_0|}$$

where Φ_i is the potential due to charge Q_i and $|\underline{r}_i - \underline{r}_0|$ is the distance between Q_i and P_0. Note that Φ_i is a scalar and is not influenced by other charges; hence, superposition must hold

For a collection of N charges

$$\Phi_i(\underline{r_o}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_o |\underline{r_i} - \underline{r_o}|}$$

This can easily be generalized to a continuous charge distribution by replacing Q_i by the differential element of charge $\rho(r')dv'$ where r_i becomes r', a continuous variable,

$$\Phi(\underline{r_o}) = \int \frac{\rho(r')\,dv'}{4\pi\epsilon_o |\underline{r'} - \underline{r_o}|}$$

This is a very powerful result. For a unity point charge $\rho(r')dv'$ the origin $r'=0$, the associated potential ϕ is

$$\phi(r_o) = \frac{\delta(r)}{4\pi\epsilon_o |r_o|}$$

Rewriting our expression for the continuous distribution in terms of ϕ we get

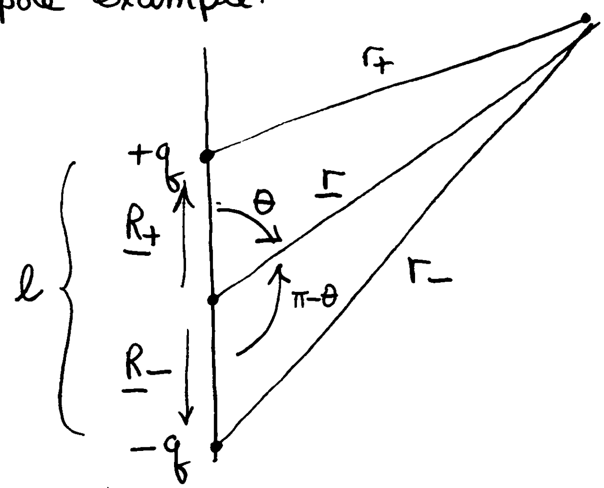
$$\Phi(\underline{r_o}) = \int \phi(r'-r_o)\,\rho(r')\,dv'$$

The potential Φ_i is then simply the convolution of the potential of a single charge with the charge distribution. This means that if we know $\rho(r)$ we can always calculate Φ. This result is very similar to the impulse response of a linear system. The potential $\phi(r'-r_o)$ becomes the impulse response and $\rho(r')$ is the driving function.

The technique is very generally known as a Green's function solution. A Green's function is what mathematicians call an impulse response. By any name it is a superposition integral which is easier to evaluate than Gauss' law for many cases and leads to a direct solution for ϕ without having to integrate $E = -\nabla\phi$.

We will return to this concept of Green's function after using it to solve for the potential and electric field of a dipole charge distribution.

Electric dipole example:



By the superposition principle

$$\Phi(\underline{r}) = \sum_{i=1}^{2} \frac{q_i}{4\pi\epsilon_0 \, |\underline{r}_i - \underline{r}|}$$

$$\Phi(\underline{r}) = \frac{+q}{4\pi\epsilon_0 \, |\underline{R}_+ - \underline{r}|} + \frac{-q}{4\pi\epsilon_0 \, |\underline{R}_- - \underline{r}|}$$

Note this solution for the individual charges assumes $\Phi(\infty) = 0$.

According to our diagram $r_+ = |\underline{R}_+ - \underline{r}|$ and $r_- = |\underline{R}_- - \underline{r}|$.

So that

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_+} - \frac{1}{r_-}\right].$$

Both r_+ and r_- can be written in terms of the position distance $r = |\underline{r}|$ and the polar angle θ using the law of cosines

$$r_+^2 = r^2 + \left(\frac{l}{2}\right)^2 - 2(r)\left(\frac{l}{2}\right)\cos\theta$$

since $|\underline{R}_+| = \frac{l}{2}$ and $|\underline{r}| = r$. Similarily for r_-

$$r_-^2 = r^2 + \left(\frac{l}{2}\right)^2 - 2(r)\left(\frac{l}{2}\right)\cos(\pi-\theta)$$

Looking at the resulting set of equations

$$r_+^2 = r^2 + \frac{l^2}{4} - rl\cos\theta$$

$$r_-^2 = r^2 + \frac{l^2}{4} + rl\cos\theta$$

and solving for r_+ and r_-

$$r_+ = \sqrt{r^2 + \frac{l^2}{4} - rl\cos\theta}$$

$$r_- = \sqrt{r^2 + \frac{l^2}{4} + rl\cos\theta}$$

Factoring r out of the right hand side

$$r_+ = r\left(1 + \frac{l^2}{4r^2} - \frac{l}{r}\cos\theta\right)^{\frac{1}{2}}$$

$$r_- = r\left(1 + \frac{l^2}{4r^2} + \frac{l}{r}\cos\theta\right)^{\frac{1}{2}}$$

We can expand the right hand sides by the Taylor series expansion $(1+x)^n \approx 1 + nx$ if $x \ll 1$. So that

$$r_+ \approx r\left(1 + \frac{l^2}{8r^2} - \frac{1}{2}\frac{l}{r}\cos\theta\right)$$

$$r_- \approx r\left(1 + \frac{l^2}{8r^2} + \frac{1}{2}\frac{l}{r}\cos\theta\right)$$

For any reasonable problem if we are far enough away $\frac{l}{r} \ll 1$ and the above expressions further simplify to

$$r_+ \approx r\left(1 - \frac{1}{2}\frac{l}{r}\cos\theta\right)$$

$$r_- \approx r\left(1 + \frac{1}{2}\frac{l}{r}\cos\theta\right)$$

These can be substited back into the expression for $\Phi(\underline{r})$ to give

$$\Phi(\underline{r}) \approx \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r - \frac{1}{2}l\cos\theta} - \frac{1}{r + \frac{1}{2}l\cos\theta}\right]$$

By a slight re-arrangement

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{1-\frac{l}{2r}\cos\theta} - \frac{1}{1+\frac{l}{2r}\cos\theta}\right].$$

As $\frac{l}{r} << 1$ these terms can be further approximated by

$$\frac{1}{1+\frac{l}{2r}\cos\theta} \approx 1-\frac{l}{2r}\cos\theta \text{ and } \frac{1}{1-\frac{l}{2r}\cos\theta} \approx 1+\frac{l}{2r}\cos\theta$$

Substituting back into the above equation for $\Phi(\underline{r})$

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0 r}\left[\left(1+\frac{l}{2r}\cos\theta\right) - \left(1-\frac{l}{2r}\cos\theta\right)\right]$$

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0 r}\left[+\frac{l}{r}\cos\theta\right]$$

$$\Phi(\underline{r}) = \frac{q l \cos\theta}{4\pi\epsilon_0 r^2}$$

This is the electric potential due to the defined electric dipole. Usually we define the electric dipole moment $\underline{p} = q\underline{l}$ where $\underline{l}$ points from the negative charge to the positive charge.

If $\Phi(\underline{r})$ is known (in spherical coordinates) we can calculate $\underline{E}$ by $\underline{E} = -\nabla\Phi$.

$$-\nabla\Phi = -\underline{a}_r \frac{\partial\Phi}{\partial r} - \underline{a}_\theta \frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \underline{a}_\phi \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

where we note that $\frac{\partial\Phi}{\partial\phi} = 0$

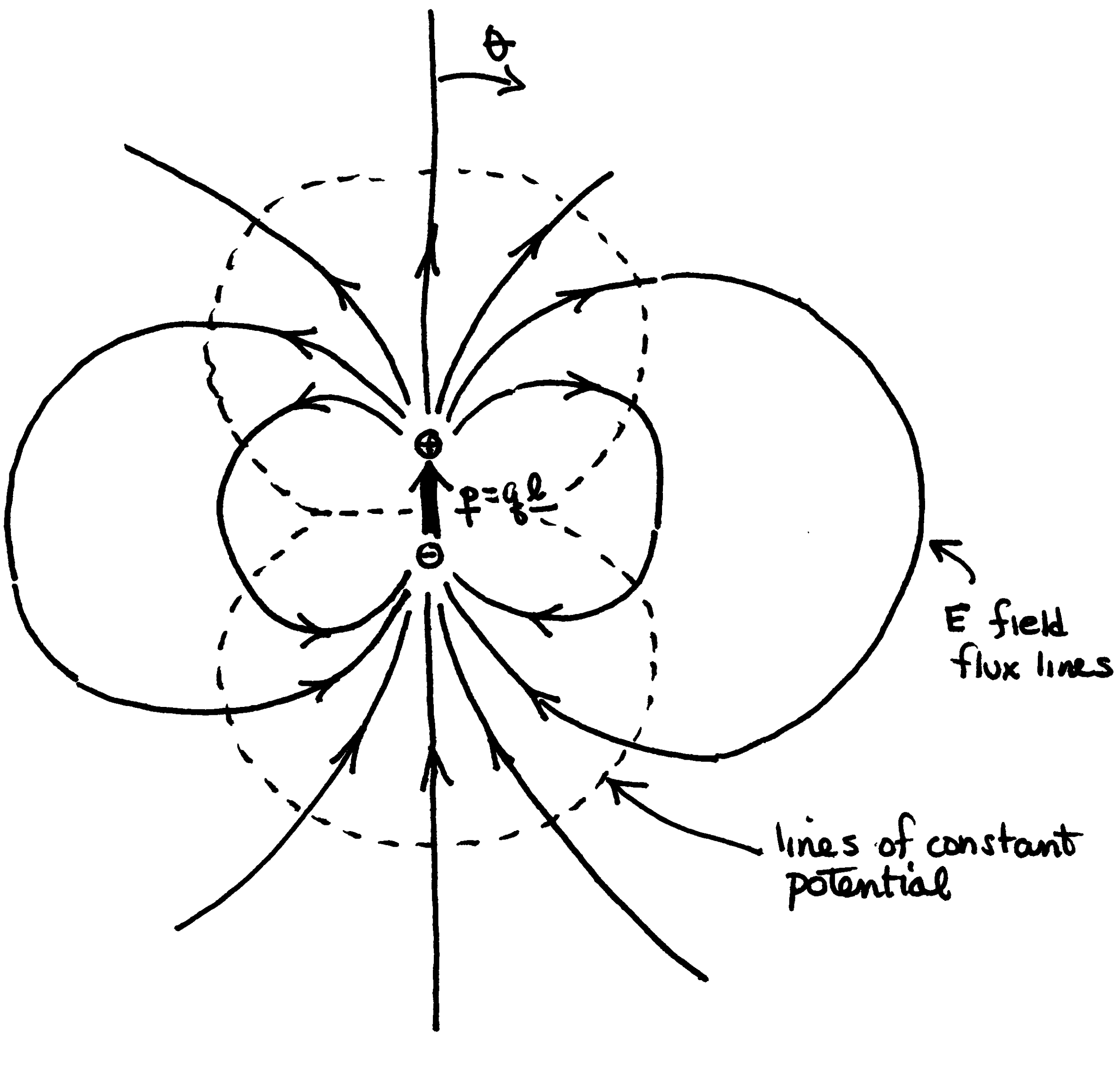
$$\frac{\partial\Phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{q l\cos\theta}{4\pi\epsilon_0 r^2}\right) = -\frac{q l\cos\theta}{4\pi\epsilon_0}\frac{2}{r^3}$$

$$\frac{\partial\Phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{q l\cos\theta}{4\pi\epsilon_0 r^2}\right) = -\frac{q l\sin\theta}{4\pi\epsilon_0 r^3}$$

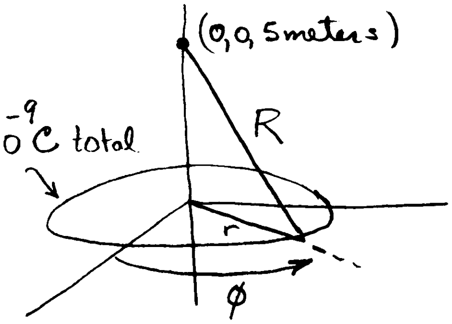
Thus, $\underline{E} = \underline{E}(r,\theta,\phi) = \underline{a_r}\ \frac{q l \cos\theta}{4\pi\epsilon_0}\ \frac{2}{r^3} + \underline{a_\theta}\ \frac{q l \sin\theta}{4\pi\epsilon_0 r^3}$

$$\underline{E} = \frac{q l}{4\pi\epsilon_0 r^3}\left(2\cos\theta\ \underline{a_r} + \sin\theta\ \underline{a_\theta}\right)$$

The resulting field is plotted below



Example: Potential from a continuous charge distribution



r = 2 meters

what is the potential $\qquad \Phi_{tot} = \int d\Phi = \int \frac{\rho_\ell \, d\ell}{4\pi\epsilon_0 R}$

$$\rho_\ell = \frac{\frac{40}{3}\times 10^{-9} C}{2\pi r} = \frac{40/3 \times 10^{-9}}{2\pi(2)} = \frac{10^{-8}}{3\pi} \text{ C/m.}$$

$$R = \sqrt{5^2+2^2} = \sqrt{29} \text{ meters}$$

$$d\ell = r\,d\phi = 2\,d\phi$$

$$\therefore \quad \Phi_{tot} = \int_0^{2\pi} \frac{\frac{10^{-8}}{3\pi}\cdot 2\,d\phi}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\sqrt{29}} = 22.3 \text{ volts.}$$

Suppose all the charge were concentrated at the origin the potential then would be just

$$\Phi = \frac{q}{4\pi\epsilon_0 R} = \frac{40/3\times 10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)(5)} = 24.0 \text{ volts.}$$

Finally, if the charge were spread uniformly over a disk of radius 2 meters.

$$\Phi_{tot} = \int \frac{\rho_s\, ds}{4\pi\epsilon_0 R}$$

where $ds = r\,dr\,d\phi$, $\rho_s = \frac{\frac{40}{3}\times 10^{-9}}{\pi(2)^2} = \frac{10^{-8}}{3\pi}$ C/m², $R = \sqrt{25+r^2}$

$$\Phi_{tot} = \int\int \frac{\frac{10^{-8}}{3\pi}\, r\,dr\,d\phi}{4\pi\left(\frac{10^{-9}}{36\pi}\right)\sqrt{25+r^2}}$$

$$= 23.1 \text{ volts.}$$

These are all solved by the Green's function method.