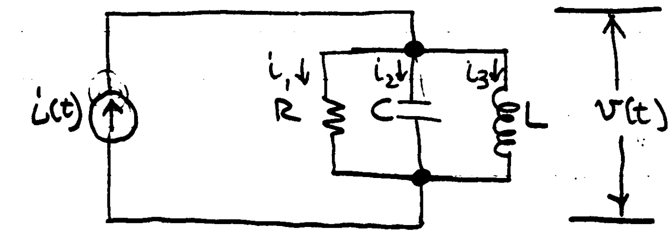
Why study electromagnetic fields?

Answer: lumped versus distributed parameters



This is a standard, discrete element circuit as found in Circuits I. We can quickly write the voltage-current terminal relations for each discrete element as

$$i_1 = \frac{v}{R} \qquad i_2 = C\frac{dv}{dt} \qquad i_3 = \frac{1}{L}\int v\,dt \qquad [1]$$

and for the RLC circuit, using Kirchoff's Law's,

$$i = i_1 + i_2 + i_3 = \frac{v}{R} + C\frac{dv}{dt} + \frac{1}{L}\int v\,dt \qquad [2]$$

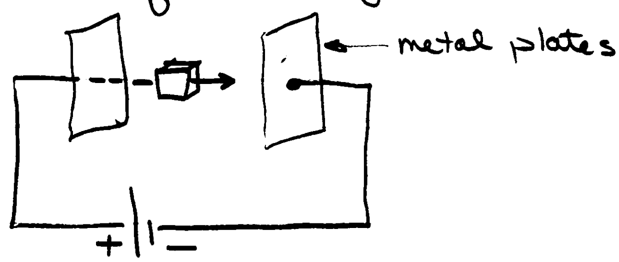
The fundamental questions one can ask are:

1. Under what conditions do the device characteristics given as equation [1] hold?
2. Are R, L and C independent of frequency?
3. How large a voltage can appear across C before it breaks down.
4. How can one construct and inductor, capacitor or resistor?
5. Will this circuit radiate energy?
6. under what conditions will pulses from ~~the se~~ be reflected back into the source

These are just examples of some questions only a course in electromagnetic fields can answer. One can also be concerned about the propagation of waves, or wave phenemona which is a pure fields concept.

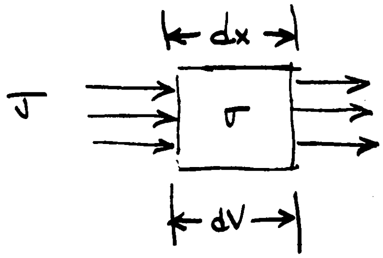
fields describe a system at <u>all</u> points in space rather than at the terminals as we have seen in our circuits courses. As with most physical parameters a field can be either a scalar (point) or vector description.

As an example of a simple scalar field consider the case of current flow through a resistive medium, i.e.



From Ohm's law we can relate the current to the voltage across a differential volume element, i.e.

$$dI = \frac{dV}{R}$$



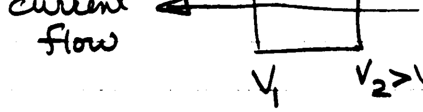
$$\sigma = \frac{\text{mhos}}{\text{meter}} = \frac{1}{\text{ohm-meter}}$$

$$\frac{1}{\sigma} = \text{ohm-meter.}$$

If each element has conductivity σ ($\mho/m$), its resistance is $(\frac{1}{\sigma})dx$. If The current through each element is J, Ohm's Law becomes

$$J = -\sigma \frac{dV}{dx}$$

where the minus sign is due to common conventions, i.e. if $V_2 > V_1$ then current flow is from right to left.



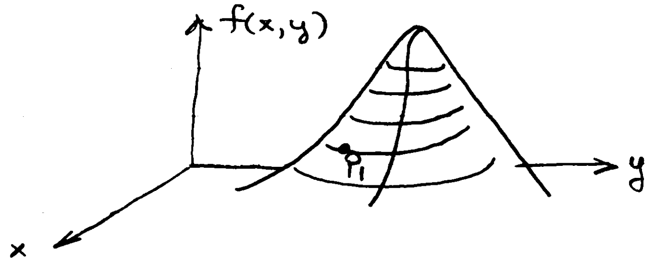
This relationship describes a field which gives the current flow at any point within the resistor. We can define a field

$$\underline{E} = -\frac{dV}{dx}$$

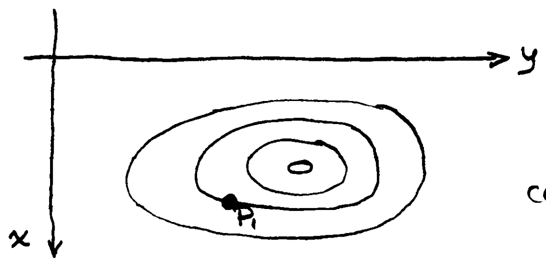
which is related to a scalar potential. This is a fundamental fields property — depending upon the nature of the source of the field, all fields can be represented as functions of scalar or vector potentials. In this case, V is a scalar potential which describes a vector field. We will come back to this example when we study electrostatics in more detail.

Gradient:

the gradient is the directional derivative of a scalar field



three-dimensional plot of f(x,y)



two-dimensional plot of f(x,y)

contours of constant f

Note that a derivative may be defined at P_1 but its value depends upon the direction. To illustrate this consider a small change in f

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad (3)$$

This is true in all directions. However, this looks a lot like a vector dot product – it's a sum of operations in x, y, and z. So one part will be

$$\frac{\partial f}{\partial x} \underline{a_x} + \frac{\partial f}{\partial y} \underline{a_y} + \frac{\partial f}{\partial z} \underline{a_z}$$

which we call grad f or $\underline{\nabla} f$. If we also define

$$d\underline{l} = dx\, \underline{a_x} + dy\, \underline{a_y} + dz\, \underline{a_z}$$

we can write (3) as

$$df = \underline{\nabla} f \cdot d\underline{l} \qquad (4)$$

Note that: ① ∇f is a vector result of a scalar function
② $\underline{dl}$ is a differential $\underline{\underline{\text{unit}}}$ vector

③ the resulting product is a scalar
④ The operation $\underline{\nabla}$ is called the del operator where

$$\underline{\nabla} = \underline{a}_x \frac{\partial}{\partial x} + \underline{a}_y \frac{\partial}{\partial y} + \underline{a}_z \frac{\partial}{\partial z} \qquad (5)$$

Returning to our original concept that the derivative is a function of direction, we can rewrite (4) as

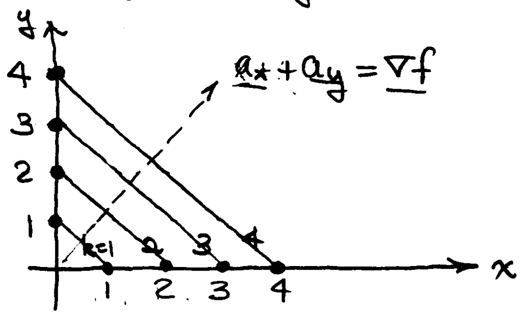
$$df = |\underline{\nabla f}|\,|\underline{dl}| \cos\theta \qquad (6)$$

We can quickly see that because $\theta < \cos\theta < 1$ and $|\underline{dl}| = dl$, ∇f must be the <u>MAXIMUM RATE OF CHANGE OF F</u>.

Going back to our two-dimensional plot of f, if we start at P_1 and follow the contour that P_1 is on it follows that $df = 0$ along this contour. Thus, $\underline{\nabla}f$ must be perpendicular to the contour $\underline{dl}$ lies on.

Example:

$$f(x,y) = x+y = k \quad \text{(a constant)}$$



Note that f is increasing most rapidly along the 45° line between the x and y axes.

What is the direction of ∇f?

$$f = x+y$$

$$\frac{\partial f}{\partial x} = 1$$

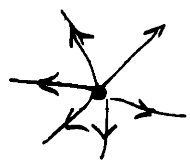
$$\frac{\partial f}{\partial y} = 1$$

$$\frac{\partial f}{\partial z} = 0$$

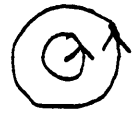
$\underline{\nabla f} = \underline{a_x} + \underline{a_y}$ which is perpendicular to the lines of constant f as expected.

General (mathematical) properties of fields

There are two types of fields: those which originate and terminate at discrete points and those which close upon themselves. These can be classified as rotational and irrotational fields by mathematicians.



irrotational field



rotational field

Mathematically, a vector field $\underline{F}(x,y,z,t)$ is irrotational if it has zero circulation, i.e. no component of it forms a closed loop. We can define

$$\text{net circulation} = \oint_C \underline{F} \cdot \underline{dl} \qquad [7]$$

Obviously, an irrotational field has zero net circulation.

This has an important immediate result. Consider a function Φ which is single valued.

$$\oint_C d\Phi = \Phi(p_1) - \Phi(p_1) \equiv 0$$

The integral of $d\Phi$ around any contour must be identically zero. Re-writing the above equation as a contour (line) integral

$$\oint_C d\Phi = \oint_C \frac{d\Phi}{dl}\, dl = \oint_C \nabla\Phi\, dl \qquad [8]$$

But this is identically zero, so combining [7] and [8] we have that

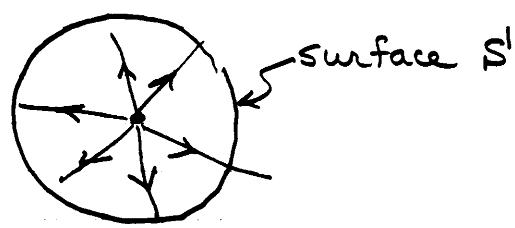
$$\underline{F} = -\nabla\Phi \qquad [9]$$

The (−) sign is chosen to agree with common electromagnetic field sign conventions. This is an important result: it says that any irrotational field $\underline{F}$ can be written as the gradient of some scalar "potential" function Φ.

Irrotational fields terminate in sources. Another property of irrotational fields can be derived by considering the divergence of the field.

By definition $$\nabla \cdot \underline{F} = \lim_{\Delta V \to 0} \frac{\oint \underline{F} \cdot d\underline{S}}{\Delta V}. \qquad (10)$$

This is very fancy mathematics with a simple physical explanation. The integral above represents the net field.



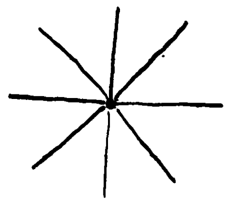
If there is a source within S the integral will be non-zero. If there is no source in S, the integral will be zero.

Thus, for an irrotational field $\nabla \cdot \underline{F}$ cannot be identically zero everywhere since an irrotational field must have a source.

The form of the integral $\int \underline{F} \cdot d\underline{s}$ is rather suggestive.

Since $d\underline{s}$ is a unit area, $\underline{F}$ must represent the field per unit area on some surface. This suggests that $\underline{F}$ is actually a density or flux.

flux plot:

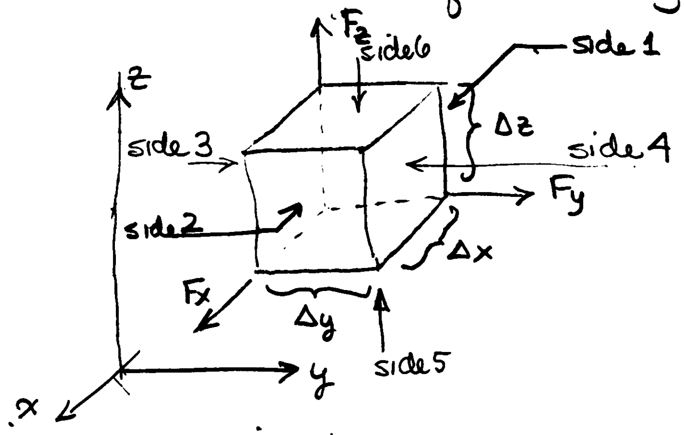


① direction of flux lines correspond with direction of field vectors

② transverse densities are the same as magnitude of field vector.

Thus: $\psi = \int \underline{F} \cdot d\underline{s}$ is the flux through a surface.

let's do the detailed math of computing div F



$\Delta x, \Delta y, \Delta z$ are small

Consider a differential volume of space in a vector field $\underline{F}$.

For side #1, the normal component is $F_x \underline{a_x}$
the normal vector $\underline{n}$ is $-\underline{a_x}$
the differential surface $ds = \Delta y \, \Delta z$

so $$\int_{\text{side 1}} \underline{F} \cdot d\underline{s} = (F_x \underline{a_x}) \cdot (-\underline{a_x}) \, \Delta y \, \Delta z \quad \text{on side 1}$$

For side 2 the normal component is $\left(F_x + \frac{\partial F_x}{\partial x} \Delta x\right) \underline{a_x}$
the normal vector is $+\underline{a_x}$
the differential surface element is $\Delta y \, \Delta z$.

$$\int_{\text{Side 2}} \underline{F} \cdot d\underline{s} = \left(F_x + \frac{\partial F_x}{\partial x} \Delta x\right) \underline{a_x} \cdot (\underline{a_x}) \, \Delta y \, \Delta z$$

Summing these results up

$$\int_{\text{side 1+side 2}} \underline{F} \cdot d\underline{s} = \frac{\partial F_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

We can get similar results for the other sides and add everything up.

$$\int_{\text{side 3+side 4}} \underline{F} \cdot d\underline{s} = \frac{\partial F_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

$$\int_{\text{side 5+side 6}} \underline{F} \cdot d\underline{s} = \frac{\partial F_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

Using these results

$$\operatorname{div} \underline{F} = \lim_{\Delta v \to 0} \frac{\int_{\text{side1} + \text{side2} + \text{side3} + \text{side4} + \text{side5} + \text{side6}} \underline{F} \cdot \underline{ds}}{\Delta v}$$

$$= \lim_{\Delta v \to 0} \frac{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

where we note that $\Delta v = \Delta x \Delta y \Delta z$. Thus,

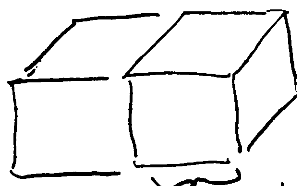
$$\operatorname{div} \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{11}$$

We define $\underline{\nabla} = \frac{\partial}{\partial x} \underline{a_x} + \frac{\partial}{\partial y} \underline{a_y} + \frac{\partial}{\partial z} \underline{a_z}$ and write (12)

$\underline{F} = F_x \underline{a_x} + F_y \underline{a_y} + F_z \underline{a_z}$. Our result is

$$\operatorname{div} \underline{F} = \underline{\nabla} \cdot \underline{F} \tag{13}$$

This result can be generalized to the divergence theorem by summing up differential elements.



$$\mathrm{div}\,\underline{F} = \frac{\int_i \underline{F}\cdot d\underline{s}}{\Delta v_i}$$

for each differential volume

since div $\underline{F}$ is summed over the surfaces and the contributions from common surfaces cancel.

note cross-multiplication of Δv

$$\sum_i \mathrm{div}\,\underline{F}\cdot \Delta v_i = \sum_i \oint_{S_i} \underline{F}\cdot d\underline{s} \qquad (14)$$

Extending to continuous elements, the left hand and right-hand side become integrals

$$\int \mathrm{div}\,\underline{F}\, dv = \oint_S \underline{F}\cdot \underline{ds} \qquad (15)$$

Recalling $\mathrm{div}\,\underline{F} = \underline{\nabla}\cdot\underline{F}$

$$\int_v \underline{\nabla}\cdot\underline{F}\, dv = \oint_S \underline{F}\cdot\underline{ds} \qquad (16)$$

which is called the divergence theorem.

As a final description of irrotational fields consider an irrotational field $\underline{F}$ given by some source function $\rho(x,y,z,t)$. Then,

$$\nabla \cdot \underline{F} = \rho$$

and
$$\nabla \cdot (-\nabla \Phi) = -\nabla^2 \Phi = \rho .$$

$\nabla^2 \Phi$ is the Laplacian and is a generalized second derivative operator which is tabulated for different coordinate systems at the back of our textbook.

$$\nabla^2 \Phi = -\rho \quad \text{is Poisson's Equation} \qquad (17)$$

For a source free region of space $(\rho = 0)$ we have Laplace's equation

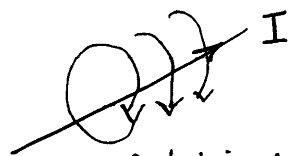
$$\nabla^2 \Phi = 0 . \qquad (18)$$

Note that if we know $\underline{F}$ we know Φ and vice versa. In many cases, Φ is simple to derive, easy to measure, and because it is a scalar function easy to superimpose (superposition).

For these reason's Laplace's and Poisson's equations are fundamental to irrotational fields. An example of an irrotational field is the electric field due to a point charge.

Rotational fields

As you have no doubt guessed, irrotational fields are examined first because they are simpler to understand. Rotational fields because they do not have discrete sources are more difficult to understand. An example of a rotational field is a magnetic field due to a current.



rotational magnetic field.

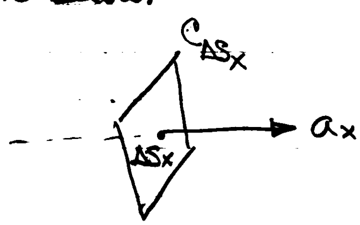
Up to now we crudely defined the net circulation as

$$\oint_C \underline{F} \cdot \underline{dl}$$

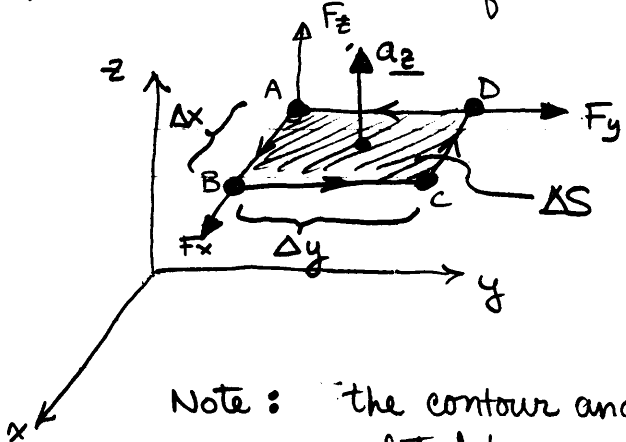
Just as we defined the divergence at the differential level, let us define the curl to be the differential net circulation

$$(\text{curl } \underline{F}) \cdot \underline{a_x} = (\nabla \times F) \cdot \underline{a_x} = \lim_{\Delta S_x \to 0} \frac{\oint_{C_{\Delta S_x}} \underline{F} \cdot \underline{dl}}{\Delta S_x} \quad (20)$$

What we have done is to define a small surface ΔS_x with it's normal parallel to the x-axis and contour $C_{\Delta S_x}$ as shown below.



The curl has a rigorous mathematical interpretation. If we expand $\underline{F}$ in terms of a Taylor series:



Note: the contour and the normal to ΔS are related by the right-hand rule

$$\oint \underline{F}\cdot d\underline{l} = F_x\Big|_{AB} \Delta x + F_y\Big|_{BC} \Delta y + (-F_x)\Big|_{CD} \Delta x + (-F_y)\Big|_{DA} \Delta y \quad (21)$$

The minus signs are due to the dot product of F and the contour, i.e. they are in opposite directions.
If A is at (x, y, z) then

$$F_x\big|_{AB} = F_x$$

$$F_x\big|_{CD} = F_x + \frac{\partial F_x}{\partial y}\Delta y$$

$$F_y\big|_{DA} = F_y$$

$$F_y\big|_{BC} = F_y + \frac{\partial F_y}{\partial x}\Delta x$$

Thus,

$$\oint_c \underline{F}\cdot d\underline{l} = F_x\Delta x + F_y\Delta y + \frac{\partial F_y}{\partial x}\Delta x\Delta y - F_x\Delta x - \frac{\partial F_x}{\partial y}\Delta x\Delta y - F_y\Delta y$$

$$\oint_c \underline{F}\cdot d\underline{l} = \frac{\partial F_y}{\partial x}\Delta x\Delta y - \frac{\partial F_x}{\partial y}\Delta x\Delta y$$

Thus,

$$(\text{curl } F) \cdot \underline{a_z} = \lim_{\Delta S \to 0} \frac{\oint_C \underline{F} \cdot \underline{dl}}{\Delta S}$$

$$= \lim_{\Delta S \to 0} \frac{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \Delta S}{\Delta S}$$

where we recognized that $\Delta S = \Delta x \Delta y$ resulting in

$$\text{curl } \underline{F} \cdot \underline{a_z} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \qquad (22a)$$

This is the z-component of the curl $\underline{F}$. By symmetry the other possible components of $\underline{F}$ are

$$\text{curl } \underline{F} \cdot \underline{a}_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \qquad (22b)$$

$$\text{curl } \underline{F} \cdot \underline{a_y} = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \qquad (22c)$$

In three-dimensions this is

$$\text{curl } \underline{F} = \underline{a_x} \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \underline{a_y} \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \underline{a_z} \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \qquad (23)$$

This can be written in vector form as

$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} \quad \text{where } \underline{\nabla} = \underline{a_x} \frac{\partial}{\partial x} + \underline{a_y} \frac{\partial}{\partial y} + \underline{a_z} \frac{\partial}{\partial z} \qquad (24)$$

and can be summarized in vector format as

$$\underline{\nabla} \times \underline{F} = \begin{bmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \qquad (25)$$

which can be evaluated as an ordinary determinant

This curl expression was evaluated in a rectangular coordinate system. The relevant expressions for other coordinate systems are given on the inside back cover of our text.

To this point we have not considered sources or potentials for rotational fields. Consider what happens if $\underline{F} = -\nabla\Phi$. The curl of $\underline{F}$ becomes

$$\nabla \times \underline{F} = -\underline{\nabla} \times \underline{\nabla\Phi} \equiv 0 \qquad (26)$$

[This is a common vector identity — TRY IT]

This means that all fields represented by <u>scalar</u> potentials must be irrotational with curl $\underline{F} \equiv 0$. For a purely rotational field $\underline{\nabla} \times \underline{F} \neq 0$. This motivates us to consider a field of the form

$$\underline{F} = \nabla \times \underline{A} \qquad (27)$$

where $\underline{A}$ is a <u>vecta</u> potential function. This can be shown to result in a <u>purely</u> rotational field. The divergence of $\underline{F}$ is given by

$$\nabla \cdot \nabla \times \underline{A} \equiv 0$$

indicating that there are no field sources/sinks. The curl of $\underline{F}$ is given by

$$\nabla \times F = \underline{\nabla} \times \nabla \times \underline{A} = \underline{J}$$

$\underline{J}$ is now the vector source of $\underline{A}$. Note that we have not given any physical interpretation to $\underline{J}$, $\underline{A}$ or $\underline{F}$. They are purely mathematical concepts.

This can now be put in a simpler form by using the vector identity

$$\nabla \times \nabla \times \underline{A} = \nabla \nabla \cdot \underline{A} - \nabla^2 \underline{A} \qquad (28)$$

For most fields $\nabla \nabla \cdot A = 0$. This will be explained when we study magnetic fields, but is a consequence of properly selecting a region of space. It turns out that we can always pick $\nabla \cdot \underline{A} = 0$ for a magnetic field.

For $\nabla \cdot \underline{A} \equiv 0$,

$$\nabla \nabla \cdot \underline{A} - \nabla^2 \underline{A} = -\nabla^2 \underline{A} = \underline{J} \qquad (29)$$

This is the vector Poisson equation: $\nabla^2 \underline{A} = -\underline{J}$

To complete our study of vector calculus we need to discuss two equations which relate the divergence and curl of a field $\underline{F}$ to integral quantities.

GAUSS' THEOREM

From the divergence theorem

$$\oint_S \underline{F} \cdot d\underline{S} = \int_V \nabla \cdot \underline{F}\, dv$$

From our study of irrotational fields $\nabla \cdot \underline{F} = \rho$ so that Gauss' theorem

$$\oint_S \underline{F} \cdot d\underline{S} = \int_V \rho\, dv$$

This theorem states that the total outward flux of a vector field through a closed surface S is equal to the integral (sum) of the enclosed sources.

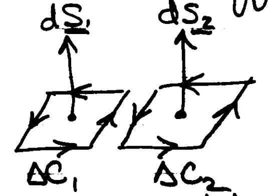
STOKES THEOREM

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S} = \oint \underline{F} \cdot \underline{d\ell} \qquad (30)$$

Consider a differential surface element $d\underline{S}$ with contour ΔC.

$$(\nabla \times F) \cdot d\underline{S} = \oint_C \underline{F} \cdot \underline{d\ell} \qquad (31)$$

This is nothing more than a description of a rotational field. Now place a second differential surface next to our first



Note that the contours add in a scalar manner but the curls (in directions $d\underline{S}_1, d\underline{S}_2$, etc.) add in a vector fashion.

Thus, for two surface elements

$$\int_{\Delta S_1} (\nabla \times \underline{F}) \cdot d\underline{S}_1 + \int_{\Delta S_2} (\nabla \times \underline{F}) \cdot d\underline{S}_2 = \oint_{\Delta C_1} \underline{F} \cdot \underline{dl} + \oint_{\Delta C_2} \underline{F} \cdot \underline{dl}$$

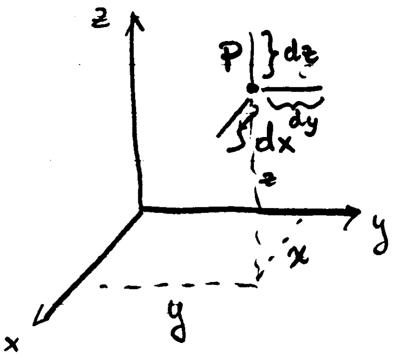
And as the number of elements becomes large

$$\int_S (\nabla \times \underline{F}) \cdot \underline{dS} = \oint_C \underline{F} \cdot \underline{dl} \qquad (32)$$

Differential elements in various coordinate systems

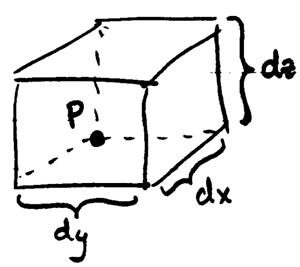
In the course of solving for electromagnetic fields one is often required to perform volume and surface integrals to reach a field solution. We will attempt to review how the various differential surfaces and volumes are arrived at.

Starting with rectangular coordinates because they are intuitively the easiest to visualize and understand let us draw a point P in an xyz space.

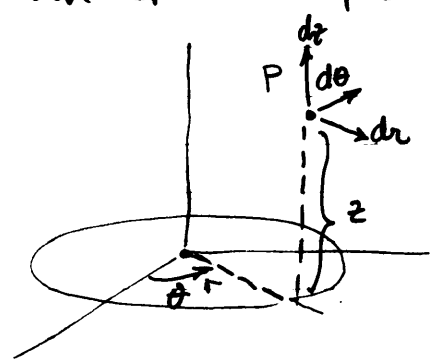


The differential lengths are always taken in the positive directions from the point P. In this coordinate system we can define three surfaces based upon these basic differentials dxdy, dydz and dxdz all of which have different unit vectors associated with their normal.

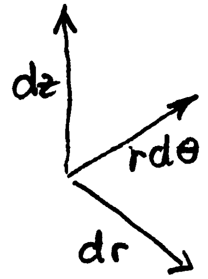
However there is only one volume dx dy dz we can define which forms a small (differential) parallelapiped.



We can repeat this process for cylindrical coordinates



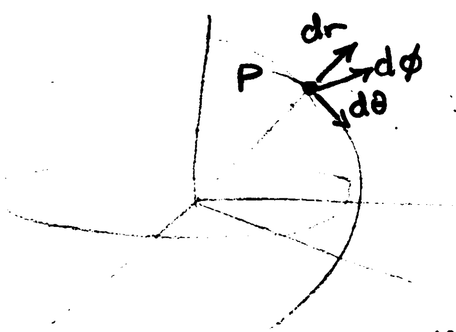
Note that $d\theta$ now points into the paper because it is taken in the direction of positive θ. Several surfaces can be constructed about P; however, we must not be careless and use $d\theta$ as the length of one of the sides of our surface. $d\theta$ is an angle and must be converted to a length, i.e. the differential length ds is $r\,d\theta$.



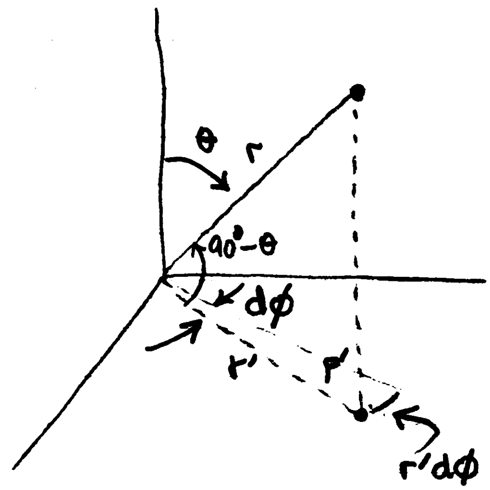
Now we can construct the differential surfaces $r\,dr\,d\theta$ (normal $\underline{a_z}$), $r\,d\theta\,dz$ (normal $\underline{a_r}$) and $dz\,dr$ (normal $\underline{a_\theta}$). Also using these differential lengths we can construct the differential volume

$$dv = (dr)(r\,d\theta)(dz) = r\,dr\,d\theta\,dz$$

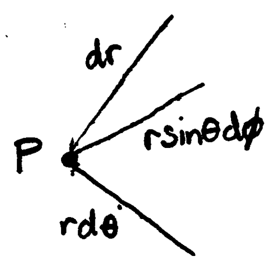
Spherical coordinates can be handled in exactly the same manner.



Note that now two of the differentials are angles and must be converted to lengths. $d\theta$ is no problem as its corresponding differential length is $r\,d\theta$. However, $d\phi$ is a little more complicated because ϕ is only measured in the horizontal plane. This means we cannot use $r\,d\phi$ because the correct radius is the dot product of r onto the horizontal plane as shown below.



Note that $r' = r\sin\theta$ so that the correct differential in the a_ϕ direction is $r\sin\theta\, d\phi$. Our differential lengths now look like



The various surfaces and their corresponding normals are then

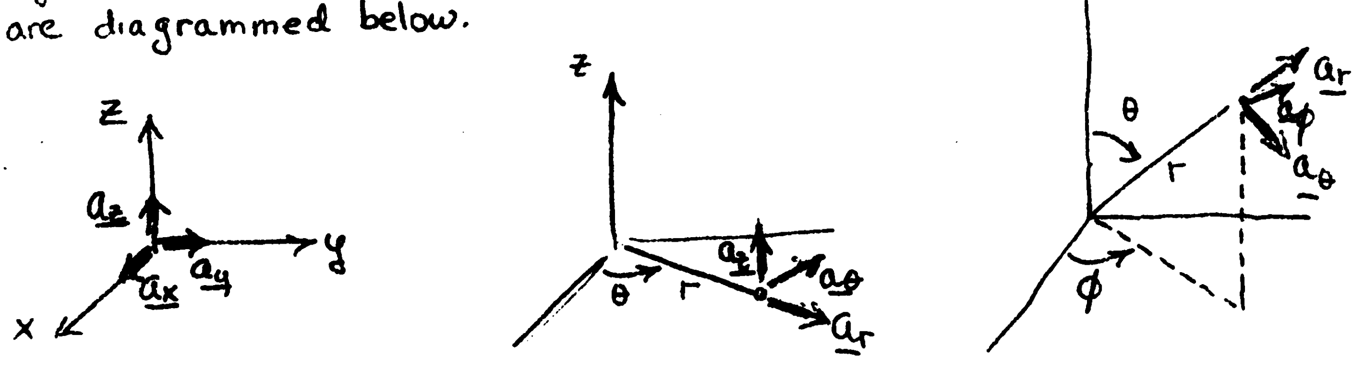
ds differential	normal
$r\,dr\,d\theta$	$\underline{a}_\theta$
$r\sin\theta\, dr\, d\phi$	$\underline{a}_\phi$
$r^2\sin\theta\, d\theta\, d\phi$	$\underline{a}_r$

The only possible volume element becomes

$$dv = (dr)\,(r\sin\theta d\phi)(r d\theta) = r^2\sin\theta\, dr\, d\theta\, d\phi$$

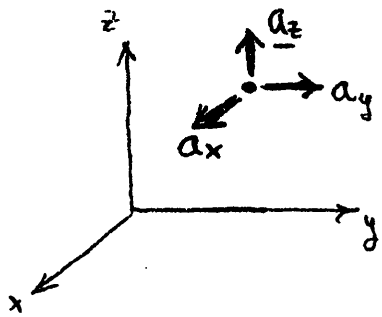
Conversion of Vector Fields Between Coordinate Systems

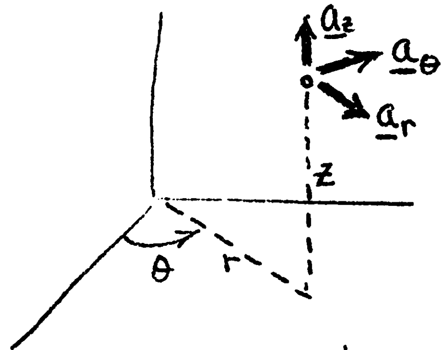
In our study of electromagnetic fields we will often want to convert an expression for a field with components in one set of vectors to an equivalent set of vectors in another coordinate system. For example one might solve a problem in spherical coordinates for field components in the $\underline{a_r}$, $\underline{a_\theta}$ and $\underline{a_\phi}$ directions to a field with $\underline{a_x}$, $\underline{a_y}$ and $\underline{a_z}$ components. The problem of finding the field components might be easier in spherical coordinates but an answer may be needed in $\underline{a_x}$, $\underline{a_y}$, $\underline{a_z}$ (rectangular or Cartesian) coordinates, to add to another previously known field or to combine two fields. As another example suppose you know a field with spherical (r, θ, ϕ) components and another with (r, θ, z) cylindrical coordinates. They cannot be combined easily because the unit vectors in each system do not always point in the same direction; however, if we convert them to rectangular coordinates they can be easily combined. Note that the exact direction of $\underline{a_r}$, $\underline{a_\theta}$ and $\underline{a_\phi}$ in spherical coordinates are a function of the location in the coordinate system. Likewise, a_r & a_θ in cyclindrical coordinates are functions of location — but, $\underline{a_z}$ is not. Hence, the advantage of using functions that have $\underline{a_x}$, $\underline{a_y}$ $\underline{a_z}$ components is that they are simply combined. On the other hand many problems are more easily solved due to symmetry arguements in other coordinate systems. The specific techniques of expressing vector fields in other coordinate systems is best illustrated by examples. For reference, the three major coordinate systems are diagrammed below.



rectangular (Cartesian) cyclindrical. spherical.

Example: Convert a vector field $F(r,\theta,z) = F_r \underline{a}_r + F_\theta \underline{a}_\theta + F_z \underline{a}_z$ to a field $F(r,\theta,z) = G_x \underline{a}_x + G_y \underline{a}_y + G_z \underline{a}_z$ with rectilinear components. Note that we are not changing the function to one of x, y, z; rather, we are only changing the direction of the vector components.





By inspection we would expect the z-components to remain the same so we will start with the $\underline{a}_z$ component of $\underline{F}$

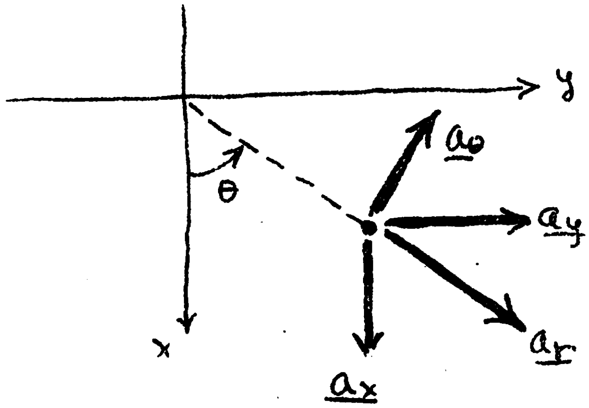
$$\underline{F} \cdot \underline{a}_z = F_r \underline{a}_r \cdot \underline{a}_z + F_\theta \underline{a}_\theta \cdot \underline{a}_z + F_z \underline{a}_z \cdot \underline{a}_z = F_z$$

$$= G_x \underline{a}_x \cdot \underline{a}_z + G_y \underline{a}_y \cdot \underline{a}_z + G_z \underline{a}_z \cdot \underline{a}_z = G_z$$

Therefore, $G_z = F_z$ as expected. However, finding G_x and G_y is not as simple. Consider $\underline{F} \cdot \underline{a}_x$

$$\underline{F} \cdot \underline{a}_x = F_r \underline{a}_r \cdot \underline{a}_x + F_\theta \underline{a}_\theta \cdot \underline{a}_x + F_z \underline{a}_z \cdot \underline{a}_x = G_x$$

Only $\underline{a}_z \cdot \underline{a}_x = 0$ in this case. We must therefore find expressions for the other dot products. Note first that $\underline{a}_x$, $\underline{a}_y$, $\underline{a}_r$ and $\underline{a}_\theta$ all lie in the same plane. We can look at the plane from the $\underline{a}_z$ direction to make interpreting the vector relationships a bit clearer.



To figure out what $\underline{a_r} \cdot \underline{a_x}$ is we must return to the definition of the dot product, i.e.

$$\underline{a_r} \cdot \underline{a_x} = |\underline{a_r}||\underline{a_x}|\cos\varphi$$

where φ is the angle included between the two vectors. As $|\underline{a_r}| = |\underline{a_x}| = 1$ because these are unit vectors

$$\underline{a_r} \cdot \underline{a_x} = \cos\varphi$$

What is φ? From the drawing the angle between $\underline{a_r}$ and $\underline{a_x}$ is θ. Thus, $\underline{a_r} \cdot \underline{a_x} = \cos\theta$. In a similar fashion we can compute $\underline{a_\theta} \cdot \underline{a_x}$

$$\underline{a_\theta} \cdot \underline{a_x} = \cos\varphi$$

where φ is now the angle between $\underline{a_\theta}$ and $\underline{a_x}$. From the drawing we see that $\varphi = 90° + \theta$ as $\underline{a_\theta}$ is perpendicular to $\underline{a_r}$.

$$\underline{a_\theta} \cdot \underline{a_x} = \cos(90° + \theta) = \cancel{\cos 90°}^{\,0} \cos\theta - \cancel{\sin 90°}^{\,1} \sin\theta$$

$$= -\sin\theta$$

Combining these two results:

$$\underline{F} \cdot \underline{a_x} = F_r \cos\theta - F_\theta \sin\theta = G_x$$

Repeating the same process for the $\underline{a_y}$ direction:

$$\underline{F} \cdot \underline{a_y} = F_r\, \underline{a_r} \cdot \underline{a_y} + F_\theta\, \underline{a_\theta} \cdot \underline{a_y} + F_z\, \underline{a_z} \cdot \underline{a_y} = G_y$$

$$\underline{a_z} \cdot \underline{a_y} = 0$$

$$\underline{a_\theta} \cdot \underline{a_y} = |\underline{a_\theta}||\underline{a_y}|\cos\varphi = \cos(\theta)$$

Note that the angle between $\underline{a_y}$ and $\underline{a_r}$ must be $90° - \theta$, but the angle between $\underline{a_r}$ and $\underline{a_\theta}$ is $90°$; hence, that between $\underline{a_\theta}$ and $\underline{a_y}$ is θ.

$$\underline{a_r} \cdot \underline{a_y} = |\underline{a_r}||\underline{a_y}|\cos\varphi$$

We already know from the above arguement that the angle between $\underline{a_r}$ and $\underline{a_y}$ is $90° - \theta$ so $\cos\varphi = \cos(90° - \theta) = \cos 90° \cos\theta + \sin 90° \sin\theta$

or

$$\underline{a_r} \cdot \underline{a_y} = \sin\theta$$

Thus,

$$\underline{F} \cdot \underline{a_y} = F_r \sin\theta + F_\theta \cos\theta = G_y$$

We now have our desired result, namely $\underline{F}$ in rectangular components:

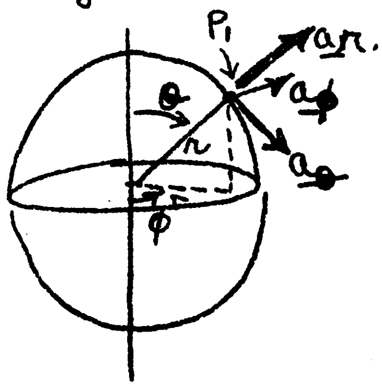
$$\underline{F}(r,\theta,z) = (F_r\cos\theta - F_\theta\sin\theta)\,\underline{a_x} + (F_r\sin\theta + F_\theta\cos\theta)\,\underline{a_y} + F_z\,\underline{a_z}$$

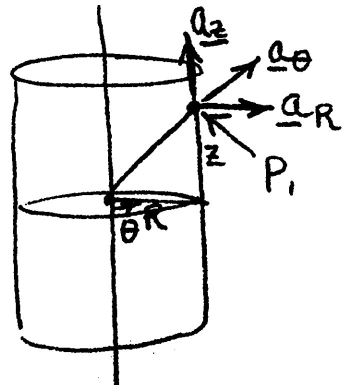
Note that $\underline{F}$ remains a function of r, θ, z ; we have only changed the components to an equivalent representation. This was a generally easy example.

Example: Convert a function $\underline{F}(r,\phi,\theta)$ in spherical coordinates with components F_r, F_ϕ, F_θ to an equivalent function with components F_R, F_θ, F_z, i.e. cyclindrical components.

This problem is not much more difficult than the other except for the complexity of imaging (or visualizing) the relationships between the unit vectors.

Drawing the coordinate system





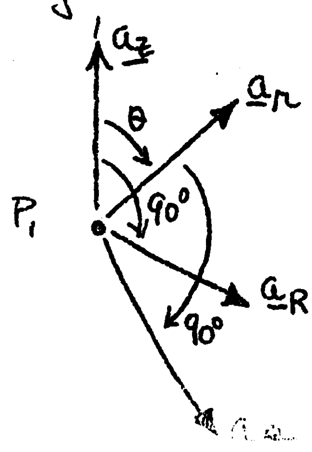
Note that r and θ were used for spherical coordinates to emphasize that they are <u>not</u> the same as R and θ in cyclindrical coordinates. Note also that $\underline{a}_\phi$ and $\underline{a}_\theta$ point <u>into</u> the paper — this is difficult to illustrate in a drawing.

$$\underline{F}(r,\phi,\theta) = F_r\,\underline{a}_r + F_\phi\,\underline{a}_\phi + F_\theta\,\underline{a}_\theta$$

$$\underline{F}(r,\phi,\theta) = G_R\,\underline{a}_R + G_\theta\,\underline{a}_\theta + G_z\,\underline{a}_z$$

$$\underline{F}\cdot\underline{a}_R = F_r\,\underline{a}_r\cdot\underline{a}_R + F_\phi\,\underline{a}_\phi\cdot\underline{a}_R + F_\theta\,\underline{a}_\theta\cdot\underline{a}_R = G_R$$

In this conversion all three dot products are non-zero. To aid us in determining the necessary dot products we need a detailed drawing of the various unit vectors at P_1



$\underline{a}_r$, $\underline{a}_R$ and $\underline{a}_\theta$ are all in the same plane. $\underline{a}_\theta$ and $\underline{a}_\phi$ are in a plane perpendicular to this plane.

By examination of the drawing

$$\underline{a}_r \cdot \underline{a}_R = |\underline{a}_r||\underline{a}_R|\cos(90^\circ - \theta) = \sin\theta$$

$$\underline{a}_\theta \cdot \underline{a}_R = |\underline{a}_\theta||\underline{a}_R|\cos(\theta) = \cos\theta$$

Since $\underline{a}_\Theta$ is perpendicular to this plane

$$\underline{a}_\Theta \cdot \underline{a}_R = |a_\Theta||\underline{a}_R|\cos 90^\circ = 0$$

Therefore, $G_R = F_r \sin\theta + F_\theta \cos\theta$

In a similar fashion

$$\underline{F} \cdot \underline{a}_\theta = F_r\, \underline{a}_r \cdot \underline{a}_\theta + F_\phi\, \underline{a}_\phi \cdot \underline{a}_\theta + F_\Theta\, \underline{a}_\Theta \cdot \underline{a}_\theta = G_\theta$$

$\underline{a}_r \cdot \underline{a}_\theta = 0$ since they are perpendicular

$\underline{a}_\phi \cdot \underline{a}_\theta = 1$ since they are parallel

$\underline{a}_\Theta \cdot a_\theta = 0$ since they are perpendicular

Therefore, $G_\theta = F_\phi$

Again, in a similar fashion

$$\underline{F} \cdot \underline{a}_z = F_r\, \underline{a}_r \cdot \underline{a}_z + F_\phi\, \underline{a}_\phi \cdot \underline{a}_z + F_\Theta\, \underline{a}_\Theta \cdot \underline{a}_z = G_z$$

$$\underline{a}_r \cdot \underline{a}_z = |a_r||a_z|\cos\theta = \cos\theta$$

$$\underline{a}_\phi \cdot \underline{a}_z = |\underline{a}_\phi||\underline{a}_z|\cos 90^\circ = 0$$

$$\underline{a}_\Theta \cdot \underline{a}_z = |\underline{a}_\Theta||\underline{a}_z|\cos(90^\circ + \theta) = \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$$

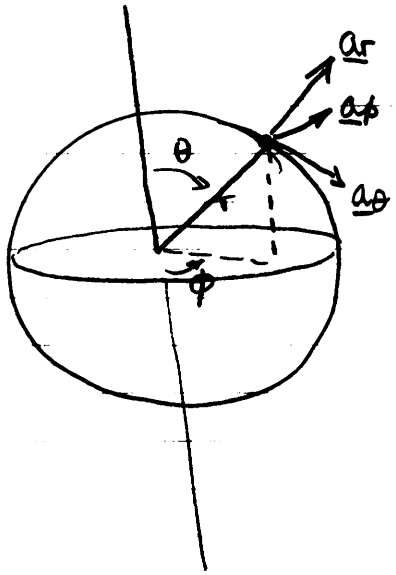
$$= -\sin\theta$$

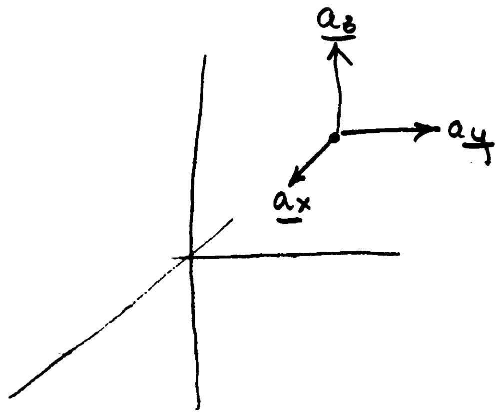
Therefore, $G_z = F_r \cos\theta - F_\Theta \sin\theta$

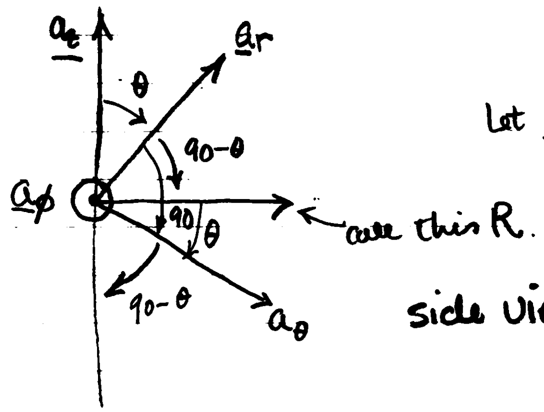
Our desired result is then

$$\underline{F}(r, \phi, \Theta) = (F_r \sin\theta + F_\Theta \cos\theta)\,\underline{a}_R + F_\phi\, \underline{a}_\theta + (F_r \cos\theta - F_\Theta \sin\theta)\,\underline{a}_z$$

Example: conversion from spherical to rectangular coordinates



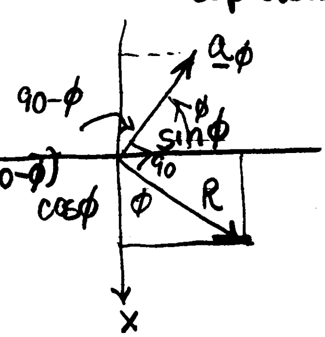




Let $\underline{F}(r,\theta,\phi) = F_r \underline{a_r} + F_\theta \underline{a_\theta} + F_\phi \underline{a_\phi}$

$$\underline{F}\cdot\underline{a_z} = F_r \underline{a_r}\cdot\underline{a_z} + F_\theta \underline{a_\theta}\cdot\underline{a_z} + F_\phi \underline{a_\phi}\cdot\underline{a_z}$$
$$= F_r \cos\theta + F_\theta \cos(90^\circ+\theta) + F_\phi \cos(90^\circ)$$
$$= F_r \cos\theta - F_\theta \sin\theta$$

top view



$$\underline{F}\cdot\underline{a_x} = F_r \underline{a_r}\cdot\underline{a_x} + F_\theta \underline{a_\theta}\cdot\underline{a_x} + F_\phi \underline{a_\phi}\cdot\underline{a_x}$$
$$= F_r \cos(90-\theta)\cos\phi + F_\theta \cos\theta\cos\phi + F_\phi \cos(90-\phi)$$
$$= F_r \sin\theta\cos\phi + F_\theta \cos\theta\cos\phi - F_\phi \sin\phi$$

$$\underline{F}\cdot\underline{a_y} = F_r \underline{a_r}\cdot\underline{a_y} + F_\theta \underline{a_\theta}\cdot\underline{a_y} + F_\phi \underline{a_\phi}\cdot\underline{a_y}$$
$$= F_r \cos(90-\theta)\sin\phi + F_\theta \cos\theta\sin\phi + F_\phi \cos\phi$$
$$= F_r \sin\theta\sin\phi + F_\theta \cos\theta\sin\phi + F_\phi \cos\phi$$