# EEAP 210

## ELECTROMAGNETIC FIELD THEORY

### HOMEWORK SET # 1
### DUE JANUARY 24, 1983

2-2 A vector $\underline{A}(x,y,z)$ is defined in a rectangular coordinate system as being directed from $[0,-1,3]$ to $[5,1,-2]$. Determine:

(a) an expression for $\underline{A}$;
(b) the distance between the two points; and
(c) a unit vector pointing in the direction of $\underline{A}$.

2-3 Determine the distance between two points $P_1$ and $P_2$ when these are expressed:

(a) in a cyclindrical coordinate system as $P_1 = [2, \frac{3\pi}{2}, 1]$ and $P_2 = [3, \pi, 0]$;
(b) in a spherical coordinate system as $P_1 = [1, \frac{\pi}{2}, \frac{3\pi}{2}]$ and $P_2 = [2, \frac{\pi}{2}, \frac{\pi}{4}]$.

2-7 If two vectors are expressed in cyclindrical coordinates as

$$\underline{A} = 2\underline{a_r} + \pi\,\underline{a_\phi} + \underline{a_z}$$

$$\underline{B} = -\underline{a_r} + \frac{3\pi}{2}\underline{a_\phi} - 2\underline{a_z}$$

compute:

(a) $\underline{A} + \underline{B}$
(b) $\underline{A} \cdot \underline{B}$
(c) $\underline{A} \times \underline{B}$
(d) $\underline{a_B}$ (unit vector in direction of $\underline{B}$)
(e) $|\underline{A}|$
(f) the smallest angle between $\underline{A}$ and $\underline{B}$
(g) a unit vector perpendicular to the plane containing $\underline{A}$ and $\underline{B}$.

2-12 Sketch the following scalar fields by showing contours of constant value of the field for $f = 0, 1, 2$:

(b) $f(x,y,z) = x^2 + y^2$
(d) $f(r,\phi,z) = r$ [Compare with (b). Are they equivalent?]

EEAP 210 Problem Set #1

2-18 Determine the rate of change of the scalar field $f(x,y,z) = xy + 2z^2$ at [1, 1, 1] in the direction of the vector $\underline{a_x} - 2\underline{a_y} + \underline{a_z}$.

2-21 Evaluate the line integral of the vector field $\underline{E} = \underline{a_x} + 2\underline{a_y} + \underline{a_z}$ along a circullar path of unity radius from [1,0,1] to [0,1,1].

2-25 Determine the net flux of the vector field

$$\underline{F}(x,y,z) = 2x^2y\,\underline{a_x} + z\,\underline{a_y} + y\,\underline{a_z}$$

(b) A cube defined by [1,0,0], [1,1,0], [1,0,1], [1,1,1], [2,0,0], [2,0,1], [2,1,0], [2,1,1].

Check your result with the divergence theorem.

2-31 Verify Stokes' theorem for a flat rectangular surface in the xy plane bounded by [0,0,0], [1,0,0], [1,1,0] [0,1,0] when the vector field is

(b) $\underline{F} = 2xy\,\underline{a_x} - y\,\underline{a_z}$

# Homework #1 Solution

2-2.

a) $\underline{A} = (5-0)\,\underline{a_x} + (1+1)\,\underline{a_y} + (-2-3)\,\underline{a_z}$

$= \boxed{5\,\underline{a_x} + 2\,\underline{a_y} + (-5)\,\underline{a_z}}$

b) $|\underline{A}| = \sqrt{5^2 + 2^2 + (-5)^2} = \sqrt{54} = \boxed{3\sqrt{6}}$

c) $\underline{a_A} = \dfrac{\underline{A}}{|\underline{A}|} = \boxed{\dfrac{1}{3\sqrt{6}}\left(5\,\underline{a_x} + 2\,\underline{a_y} - 5\,\underline{a_z}\right)}$

2-3.

a) The easiest way to find the distance between $P_1$ and $P_2$ is to convert the coordinates in cylindrical system to Cartesian system. See below.

Cylindrical $\rightarrow$ Cartesian

$P_1 = \left(2, \frac{3\pi}{2}, 1\right) \rightarrow P_1 = \left(2\cos\frac{3\pi}{2}, 2\sin\frac{3\pi}{2}, 1\right)$

$P_2 = (3, \pi, 0) \rightarrow P_1 = (3\cos\pi, 3\sin\pi, 0)$

$\therefore D = \sqrt{\left(3\cos\pi - 2\cos\frac{3\pi}{2}\right)^2 + \left(3\sin\pi - 2\sin\frac{3\pi}{2}\right)^2 + (0-1)^2}$

$= \boxed{\sqrt{14}}$

(b) By the same method, we convert the spherical coordinates to Cartesian coordinates.

If $P = (r, \phi, \theta)$ in spherical coordinates, then

$P = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$ in Cartesian

coordinates.

Bearing this in mind, we found

Spherical $\longrightarrow$ Cartesian

$P_1 = \left(1, \frac{\pi}{2}, \frac{3\pi}{2}\right) \rightarrow P_1 = (0, -1, 0)$

$P_2 = \left(2, \frac{\pi}{2}, \frac{\pi}{4}\right) \rightarrow P_2 = (0, \sqrt{2}, \sqrt{2})$

Therefore

$$D = \sqrt{(0-0)^2 + (\sqrt{2}+1)^2 + (\sqrt{2}-0)^2} = \boxed{\sqrt{5+2\sqrt{2}}}$$

2-7.

$$\underline{A} = 2\,\underline{a_r} + \pi\,\underline{a_\phi} + \underline{a_z}$$

$$\underline{B} = -\underline{a_r} + \frac{3\pi}{2}\,\underline{a_\phi} - 2\,\underline{a_z}$$

(a) $\underline{A}+\underline{B} = (2-1)\,\underline{a_r} + \left(\pi + \frac{3\pi}{2}\right)\underline{a_\phi} + (1-2)\,\underline{a_z}$

$= \boxed{\underline{a_r} + \frac{5\pi}{2}\,\underline{a_\phi} - \underline{a_z}}$

(b) $\underline{A}\cdot\underline{B} = 2\cdot(-1) + \pi\cdot\frac{3\pi}{2} + 1\cdot(-2) = \boxed{\frac{3\pi^2}{2} - 4}$

(c) $\underline{A}\times\underline{B} = \begin{vmatrix} \underline{a_r} & \underline{a_\phi} & \underline{a_z} \\ 2 & \pi & 1 \\ -1 & \frac{3\pi}{2} & -2 \end{vmatrix}$

$= \underline{a_r}\left(-2\pi - \frac{3\pi}{2}\right) + \underline{a_\phi}(-1+4) + \underline{a_z}(3\pi+\pi)$

$= \boxed{-\frac{7\pi}{2}\underline{a_r} + 3\underline{a_\phi} + 4\pi\underline{a_z}}$

(d) $\underline{a_B} = \frac{\underline{B}}{|\underline{B}|} = \frac{1}{\sqrt{(-1)^2+(3\pi/2)^2+(-2)^2}}\left(-\underline{a_r} + \frac{3\pi}{2}\underline{a_\phi} - 2\underline{a_z}\right)$

$= \boxed{\frac{1}{\sqrt{5+9\pi^2/4}}\left(-\underline{a_r} + \frac{3\pi}{2}\underline{a_\phi} - 2\underline{a_z}\right)}$

(e) $|\underline{A}| = \sqrt{2^2+\pi^2+1^2} = \boxed{\sqrt{5+\pi^2}}$

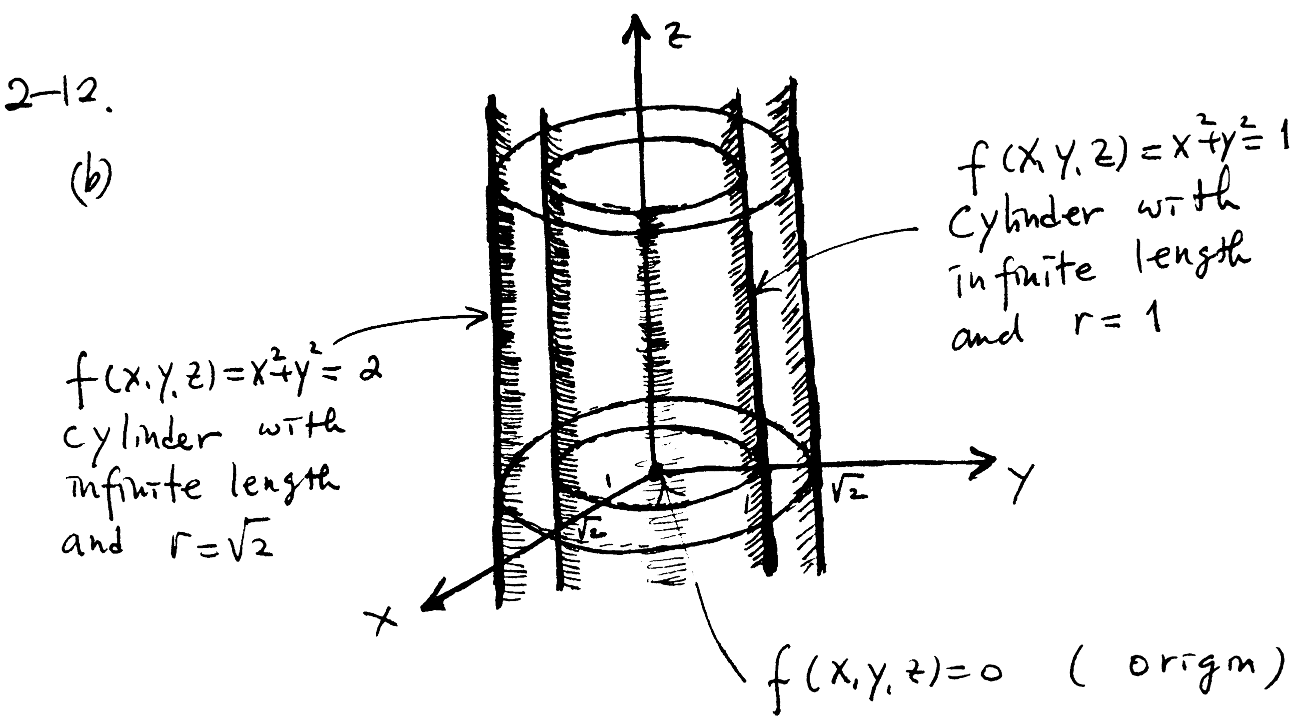
(f) $\theta_{AB} = \cos^{-1}\frac{\underline{A}\cdot\underline{B}}{|\underline{A}|\cdot|\underline{B}|} = \cos^{-1}\frac{\left(\frac{3\pi^2}{2}-4\right)}{\sqrt{5+\pi^2}\sqrt{5+9\pi^2/4}}$

$= \boxed{1.00372 \text{ radian or } 57.51^\circ}$

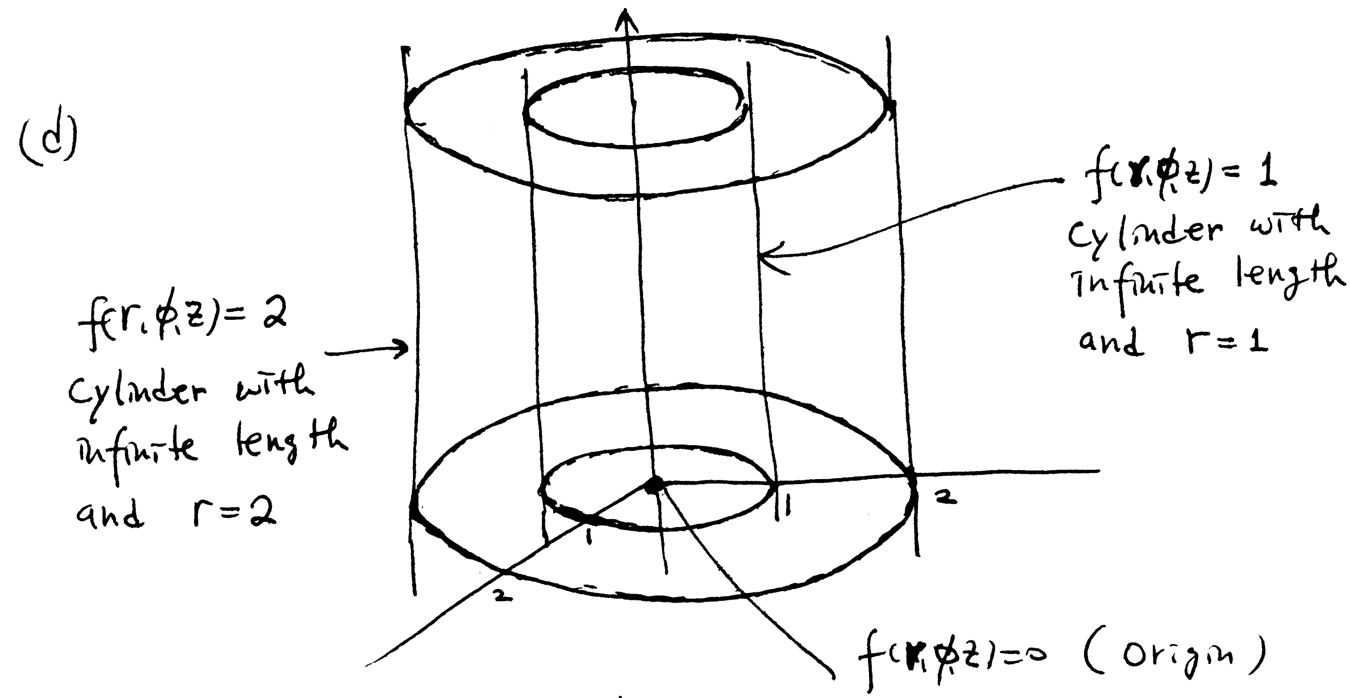
(g) $\underline{n} = \dfrac{\underline{A} \times \underline{B}}{|\underline{A} \times \underline{B}|} = \boxed{\dfrac{\left(-\frac{7\pi}{2}\underline{a_r} + 3\underline{a_\phi} + 4\pi \underline{a_z}\right)}{\sqrt{\left(\frac{7\pi}{2}\right)^2 + 3^2 + (4\pi)^2}}}$

2-12.

(b)




(d)




$\therefore$ (b) $\neq$ (d)

2-18 $\quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (y, x, 4z)$

So $\quad \nabla f\Big|_{(1,1,1)} = (1, 1, 4)$

Now a unit vector in the direction of the vector $\underline{V} = \underline{a}_x - 2\underline{a}_y + \underline{a}_z$ is

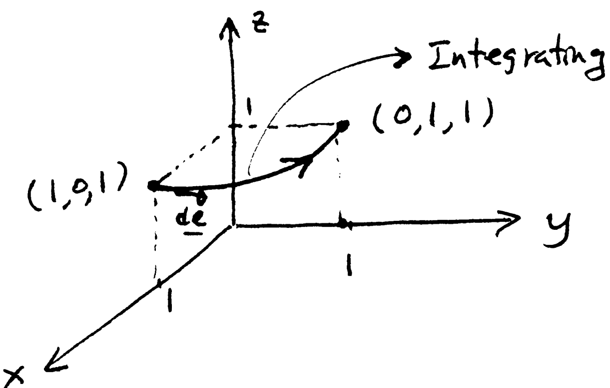
$$\underline{a}_v = \frac{\underline{V}}{|\underline{V}|} = \frac{\underline{a}_x - 2\underline{a}_y + \underline{a}_z}{\sqrt{(1^2 + (-2)^2 + 1^2)}} = \frac{1}{\sqrt{6}}\left(\underline{a}_x - 2\underline{a}_y + \underline{a}_z\right)$$

Hence the rate of change of $f$ in that direction now becomes,

$$\frac{\partial f}{\partial \ell} = \nabla f\Big|_{(1,1,1)} \cdot \underline{a}_v = (1, 1, 4) \cdot \frac{1}{\sqrt{6}}(1, -2, 1)$$

$$= \frac{1}{\sqrt{6}}\left(1\cdot 1 + 1\cdot(-2) + 4\cdot 1\right) = \boxed{\frac{3}{\sqrt{6}}}$$

2-21.




Now $\quad I = \int_c \underline{F} \cdot d\underline{\ell} = \int_0^{\frac{\pi}{2}} \underline{F} \cdot \underline{a}_\phi \, d\phi \quad (\because d\underline{\ell} = \underline{a}_\phi d\phi)$

$$= \int_0^{\frac{\pi}{2}} (-\sin\phi + 2\cos\phi)\, d\phi = \boxed{1}$$

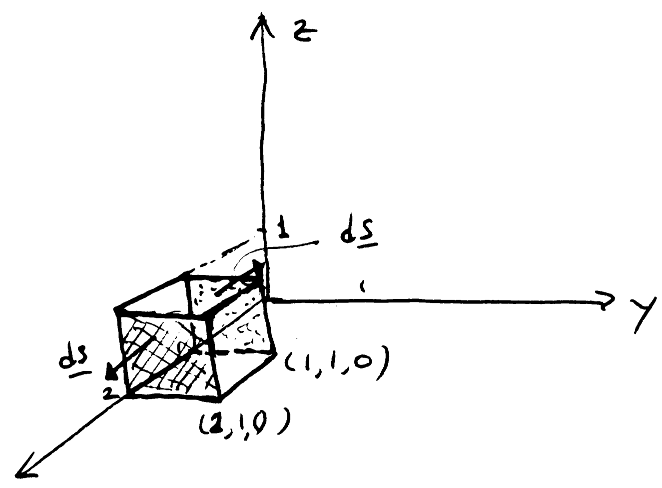
In calculating the above integral, we used following relations

i) $\underline{a}_\phi = (-\sin\phi, \cos\phi, 0)$ in Cartesian coordinates.

ii) $\underline{F} \cdot \underline{a}_\phi = (1, 2, 1) \cdot (-\sin\phi, \cos\phi, 0)$

$= \underline{\underline{-\sin\phi + 2\cos\phi}}$

2-25. $\underline{F} = (2x^2y)\,\underline{a}_x + z\,\underline{a}_y + y\,\underline{a}_z$




Now for the cross-hatched side of a cube, we have,

$$\int \underline{F} \cdot d\underline{s} = \int_0^1 \int_0^1 2x^2y \Big|_{x=2} dy\,dz \quad (\because d\underline{s} = \underline{a}_x\, dx\, dy)$$

$$= \int_0^1 \int_0^1 8y\, dy\, dz = \left[4y^2\right]_0^1 = 4$$

Likewise for the dotted side of a cube, we have,

$$\int \underline{F} \cdot d\underline{s} = \int_0^1 \int_0^1 (-2x^2y) \Big|_{x=1} dy\,dz = -1$$

By the same manner, we get

$$\Psi = \int_{\text{on a cube}} \underline{F} \cdot d\underline{s} = \int_0^1 \int_0^1 2x^2 y \Big|_{x=2} dy\,dz + \int_0^1 \int_0^1 (-2x^2 y)\Big|_{x=1} dy\,dz$$

$$+ \underbrace{\int_0^1 \int_1^2 z\Big|_{y=1} dx\,dz + \int_0^1 \int_1^2 (-z)\Big|_{y=0} dx\,dz}_{\text{cancels each other}}$$

$$+ \underbrace{\int_0^1 \int_1^2 y\Big|_{z=1} dx\,dy + \int_0^1 \int_1^2 (-y)\Big|_{z=0} dx\,dy}_{\text{cancells each other}}$$

$$= 4 - 1 = \boxed{3}$$

Using the divergence theorem, we have

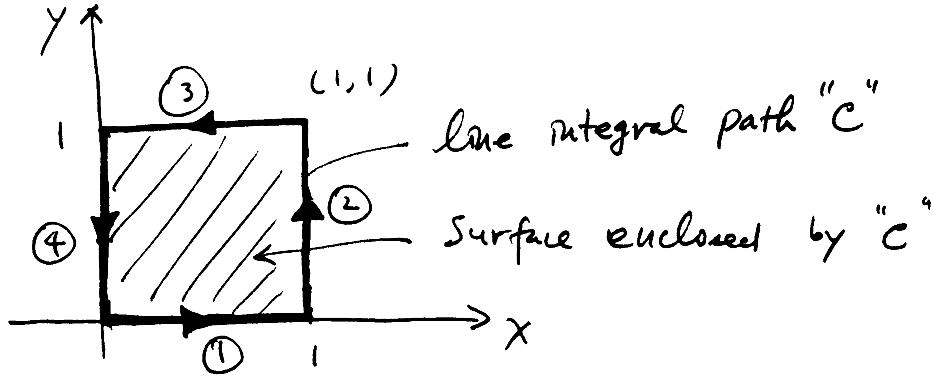
$$\Psi = \int_v \nabla \cdot \underline{F}\, dv = \int_0^1 \int_0^1 \int_1^2 (4xy + 0 + 0)\, dx\,dy\,dz$$

$$= \int_0^1 \int_1^2 4xy\,dx\,dy = \int_0^1 \left[2x^2 y\right]_1^2 dy$$

$$= \int_0^1 6y\,dy = \left[3y^2\right]_0^1 = \boxed{3}$$

Hence two seperate calculations yield the same results.

2-31.




Since $\underline{F} = 2xy\,\underline{a_x} - y\,\underline{a_z}$, we get

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 0 & -y \end{vmatrix} = -\underline{a_x} - 2x\,\underline{a_z}$$

1) Line integral evaluation

$$\oint_C \underline{F} \cdot d\underline{\ell} = \underbrace{\int_0^1 2xy\Big|_{y=0} dx}_{1} + \underbrace{\int_0^1 0 \cdot dy}_{2} + \underbrace{\int_1^0 2xy\Big|_{y=1} dx}_{3}$$

$$+ \underbrace{\int_1^0 0 \cdot dy}_{4} = \int_1^0 2x\,dx = \left[x^2\right]_1^0 = \boxed{-1}$$

2) By Stoke's theorem

$$\int_s \nabla \times \underline{F} \cdot d\underline{s} = \int_0^1\int_0^1 (-\underline{a_x} - 2x\underline{a_z})\,\overbrace{\underline{a_z}\,dx\,dy}^{d\underline{s}} = \int_0^1\int_0^1 (-2x)\,dx\,dy$$

$$= \left[-x^2\right]_0^1 = \boxed{-1}$$

Therefore we proved the Stoke's theorem.

— The End —

# EEAP 210

## ELECTROMAGNETIC FIELDS

## HOMEWORK ASSIGNMENT NO. 2

DUE: FEBRUARY 7, 1983.

PROBLEMS: 3-7
3-11
3-12 (Use Gauss' Law only)
3-19 (all parts)

Reading Assignment:

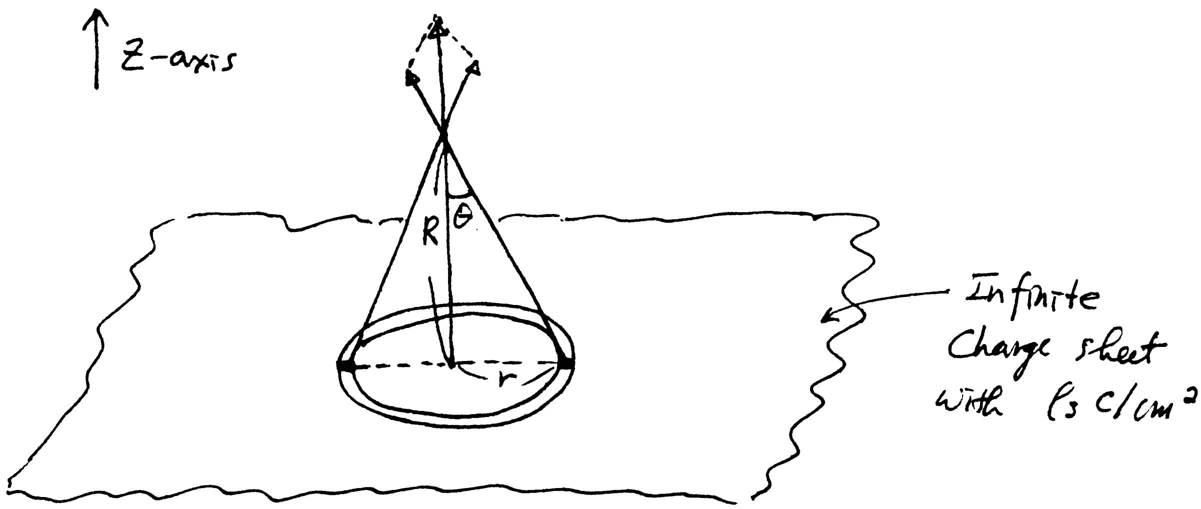
3.6.3 - 3.6.4
3.7
3.8
3.9
3.10

# EEAP 210

# HOMEWORK #2

Feb 8, 1983

3-7.




Let's consider the contribution from the ring desribed above. From the geometrical symmetry we can say that the resultant $\underline{E}$ has no horizontal component.

Now the contribution from the ring is

$$d\underline{E} = \int_0^{2\pi} \frac{\rho_s \, r dr d\phi}{4\pi\epsilon(r^2+R^2)} \cdot \cos\theta \; \underline{a_z} \quad \text{--- (1)}$$

Since $r = R\tan\theta$, we have $dr = R\sec^2\theta\, d\theta$

and Eq (1) becomes

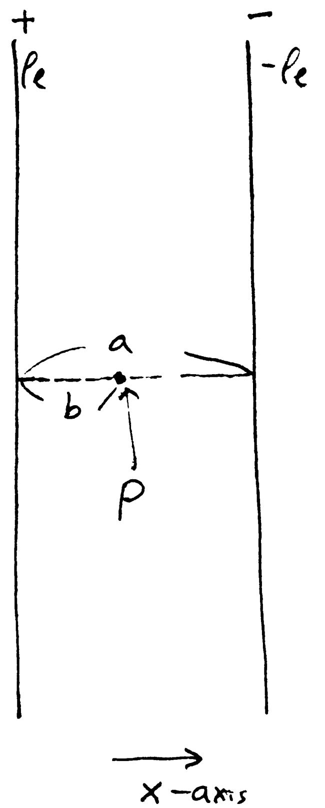
$$d\underline{E} = \frac{2\pi\rho_s (R\tan\theta)(R\sec^2\theta)}{4\pi\epsilon R^2(1+\tan^2\theta)} \cdot \cos\theta \, d\theta \, \underline{a_z}$$

$$= \frac{\rho_s}{2\epsilon} \sin\theta \, d\theta \; \underline{a_z} \quad \text{--- (2)}$$

Hence

$$\underline{E} = \int d\underline{E} = \frac{\rho_s}{2\epsilon}\underline{a_z}\int_0^{\frac{\pi}{2}} \sin\theta\, d\theta = \boxed{\frac{\rho_s}{2\epsilon}\underline{a_z}}$$

3-11.




Let's denote

$\underline{E}_+$ be the contribution from + charge

& $\underline{E}_-$ be the contribution from − charge,

Then

$$\underline{E}_+ = \frac{\rho_e}{2\pi\epsilon b}\,\underline{a}_x \quad (\because \text{Text Ex. 3-4})$$

$$\underline{E}_- = \frac{(-\rho_e)}{2\pi\epsilon(a-b)}\,(-\underline{a}_x)$$

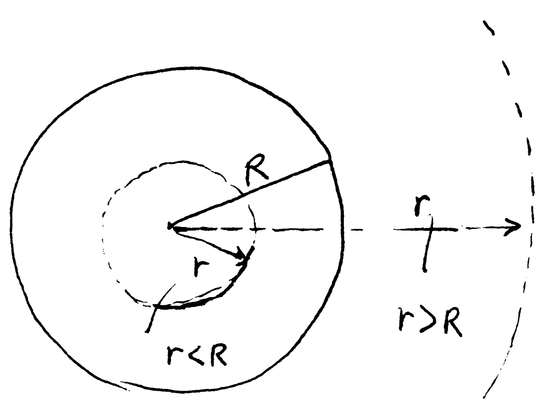
Hence the total electric field $\underline{E}$ at point P

is $\underline{E} = \underline{E}_+ + \underline{E}_-$

$$= \underline{a}_x \frac{\rho_e}{2\pi\epsilon}\left(\frac{1}{b} + \frac{1}{a-b}\right)$$

$$= \boxed{\underline{a}_x \frac{\rho_e}{2\pi\epsilon}\left(\frac{a}{b(a-b)}\right)}$$

3-12.



i) When $0 \le r \le R$ ( Inside sphere )

From the sperical geometry, we can assert that $\underline{E}$ has only r component.

Then by Gauss's law we have

$$\epsilon \cdot E_r \cdot 4\pi r^2 = \underbrace{\frac{4}{3}\pi r^3 \cdot \rho_v}$$

↳ total charge inside the sphere with radius r. It varies as r varies.

$$\therefore \quad E_r = \frac{r}{3\epsilon}\rho_v$$

And

$$\boxed{\underline{E} = \frac{r\rho_v}{3\epsilon}\underline{a_r}}$$

ii) When $R \le r$ ( Outside sphere )

By the same manner we have

$$\epsilon \cdot \underline{E_r} \cdot 4\pi r^2 = \underbrace{\frac{4}{3}\pi R^3 \cdot \rho_v}$$

↳ total charge is a fixed quantity for this case.

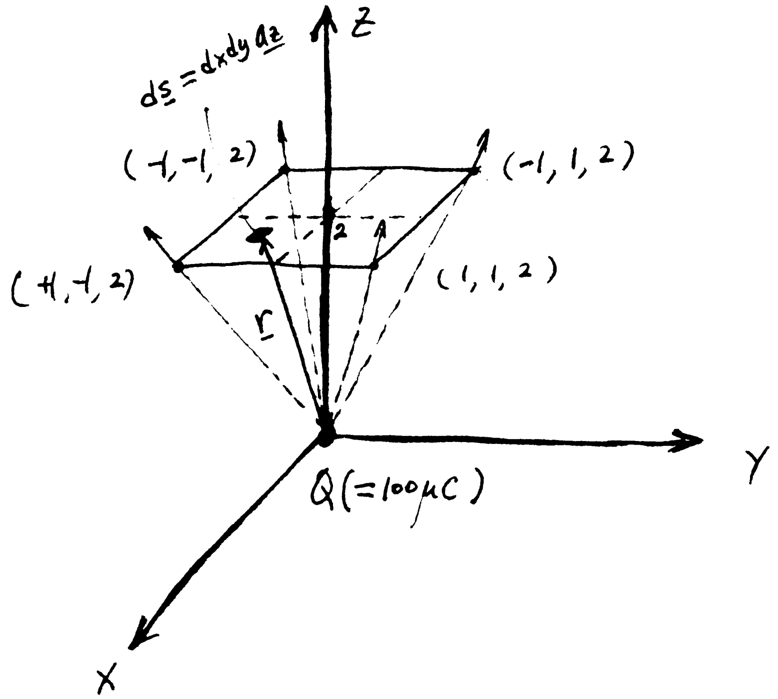
$$\therefore \quad E_r = \frac{R^3\rho_v}{3\epsilon r^2}$$

And

$$\boxed{\underline{E} = \frac{R^3\rho_v}{3\epsilon r^2}\underline{a_r}}$$

3-19.

a)




z=2

$$\left( \begin{aligned} \underline{r} &= (x, y, 2) \\ \underline{a_z} &= (0, 0, 1) \\ \underline{r}\cdot\underline{a_z} &= 2 \end{aligned} \right.$$

Since $\underline{E} = \dfrac{Q}{4\pi\epsilon r^3}\,\underline{r} \quad \left( \equiv \dfrac{Q}{4\pi\epsilon r^2}\,\underline{a_r} \right)$,

We have

$$\underline{E}\cdot d\underline{s} = \frac{Q}{4\pi\epsilon r^3}\,\underline{r}\cdot(\underline{a_z}\,dx\,dy)$$

$$= \frac{2Q\,dx\,dy}{4\pi\epsilon\,(x^2+y^2+2^2)^{3/2}} \quad (\because \underline{r}\cdot\underline{a_z} = 2)$$

Hence the flux passing through this square surface

$$\Phi = \int_{-1}^{1}\int_{-1}^{1} \frac{2Q\,dx\,dy}{4\pi\epsilon\,(x^2+y^2+4)^{3/2}} = \boxed{\frac{Q}{2\pi\epsilon}\int_{-1}^{1}\int_{-1}^{1}\frac{dx\,dy}{(x^2+y^2+4)^{3/2}}}$$

— ①

Before evaluating this integral, let's consider the general case. If the given square surface is centered at $z = a$, then the total flux $\Phi$ would be

$$\Phi = \frac{Q}{4\pi\epsilon} \int_{-1}^{1}\int_{-1}^{1} \frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}} \qquad (\because\ \underline{r}\cdot \underline{a_z} = a)$$

Now the integral

$$I = \int_{-1}^{1}\int_{-1}^{1} \frac{a\,dx\,dy}{(x^2+y^2+a^2)^{3/2}} = \boxed{4\tan^{-1}\frac{1}{a\sqrt{a^2+2}}}\ ^{*}$$

So, if $a = 2$ (our case) the flux $\Phi$ will be

$$\Phi = \frac{Q}{4\pi\epsilon}\cdot 4\tan^{-1}\frac{1}{2\sqrt{6}} = \boxed{\frac{Q}{\pi\epsilon}\tan^{-1}\frac{1}{2\sqrt{6}}} \leftarrow \text{Answer}$$

$$\text{or} \quad \cong \boxed{0.0641\,\frac{Q}{\epsilon}}$$

When $a = 1$, the flux $\Phi$ is given by

$$\Phi = \frac{Q}{\pi\epsilon}\tan^{-1}\frac{1}{\sqrt{3}} = \frac{Q}{\pi\epsilon}\cdot\frac{\pi}{6} = \boxed{\frac{1}{6}\cdot\frac{Q}{\epsilon}}$$

as expected.

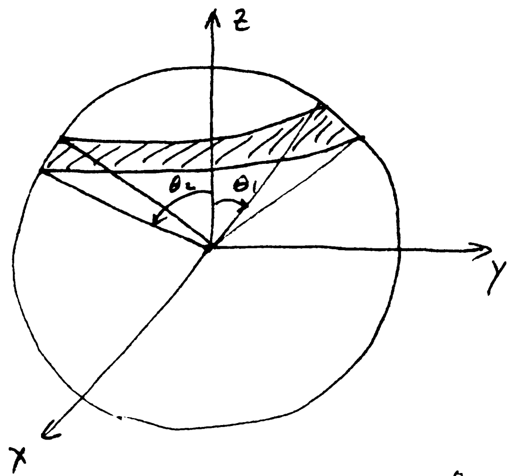
*(Note) For the evaluation of the integral, consult any standard math. handbook.

(b) Total flux $\boxed{\Phi = \frac{1}{2}\frac{Q}{\epsilon}}$ by inspection

Or, if $\epsilon = \epsilon_0$ $(= 8.854 \times 10^{-12}\ F/m)$, then

$$\Phi = \frac{1}{2} \cdot \frac{100 \times 10^{-6}\ C}{8.854 \times 10^{-12}\ F/m} \cong 5.647 \times 10^{6}\ (C\,m/F)$$

(c)




$$\Phi(\theta_1 \le \theta \le \theta_2) = \int_{\theta_1}^{\theta_2} \underline{E} \cdot d\underline{s}$$

$$= \int_{\theta_1}^{\theta_2} \frac{Q \cdot \underline{a_r}}{4\pi\epsilon R^2} \cdot \underbrace{\underline{a_r}\ 2\pi R^2 \sin\theta\, d\theta}_{d\underline{s}}$$

$$= \frac{Q}{2\epsilon}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta$$

$$= \boxed{\frac{Q}{2\epsilon}(\cos\theta_1 - \cos\theta_2)}$$

EEAP 210

ELECTROMAGNETIC FIELDS

# HOMEWORK ASSIGNMENT No. 3

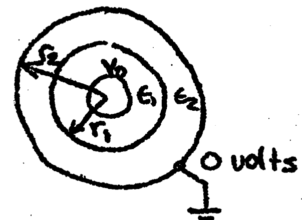
DUE FEBRUARY 14, 1983

3.31 Consider two concentric spheres. The inner sphere has radius $a$ and charge $+Q$ distributed uniformly over its outer surface. The outer sphere has radius $b$ and charge $-Q$ is distributed uniformly over its inner surface. Determine the potential difference between the two spheres and the capacitance of the structure.

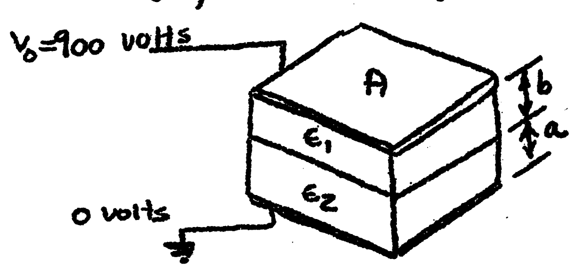
3.33 A coaxial power cable, having a core (conductor) radius of $r_1$, is filled with two concentric layers of dielectrics $\epsilon_1$ and $\epsilon_2$ as shown in Figure P3-33.

(a) Determine the capacitance of the cable per unit length.

(b) If the conductor is at a potential $V_0$ and the outer shield is grounded, determine the maximum potential gradient in each dielectric from the following data: $V_0 = 1200$ volts, $\epsilon_{r1} = 1.5$; $\epsilon_{r2} = 4.5$, $r_3 = 2r_2 = 4r_1 = 4$ cm.




3.37 A parallel-plate capacitor has two layers of dielectrics as shown in Figure P3-37. For a 900 volt potential difference applied between the plates, calculate how this voltage is divided across the dielectrics; that is, find $V_1$ and $V_2$. Given: $\epsilon_{r2} = \epsilon_{r1} = 6\epsilon_0$; $A = 100\,cm^2$; $b = 2a = 2mm$.

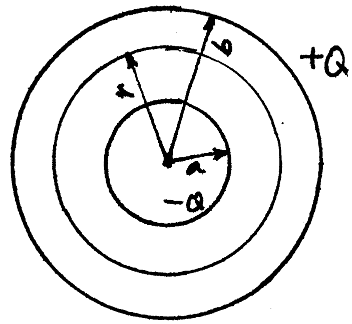



(Not from text) A permanently polarized cyclinder $P_2\,\underline{a_x}$ of radius $a$ is placed within a polarized medium $P_1\,\underline{a_x}$ of infinite extent. A uniform electric field $E_0\,\underline{a_x}$ is applied at infinity. There is NO free charge on the cyclinder. What are the potential and electric field distributions.

# EEAP 210

## HOMEWORK SOLUTION #3 2/14/83

3-31.




Applying Gauss's law, we can obtain

$$\underline{E} = \frac{-Q}{4\pi\epsilon r^2}\,\underline{a_r}$$

Then the potential difference between two spheres

$$V = -\int \underline{E}\cdot d\underline{l}$$

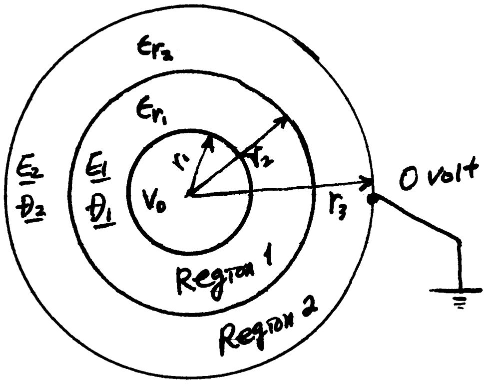
$$= -\int_a^b \frac{-Q}{4\pi\epsilon r^2}\,\underline{a_r}\cdot \overbrace{\underline{a_r}\,dr}^{d\underline{l}}$$

$$= \int_a^b \frac{Q\,dr}{4\pi\epsilon r^2} = \boxed{\frac{Q}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)}$$

Now the capacitance

$$C = \frac{Q}{V} = \boxed{\frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}} \text{ or } = \boxed{\frac{4\pi\epsilon ab}{b-a}}$$

3-33.




$V_0 = 1200\ V$
$r_1 = 0.01\ m$
$r_2 = 0.02\ m$
$r_3 = 0.04\ m$
$\epsilon_{r_1} = 1.5$
$\epsilon_{r_2} = 4.5$

This problem can be solved in two ways. Let's consider the two methods seperately.

< 1st - Method : E-field Approach >

See note in page 6

a) Since the electric displacement vector $\underline{D}$ is continuous regardless of the dielectric media**, we can write

$$2\pi r \cdot D_r = \rho_\ell \quad (\text{line charge density})$$

in both region 1 & 2. We will find $\rho_\ell$ later.

Hence we have $\underline{D} = D_r \underline{a_r} = \dfrac{\rho_\ell}{2\pi r} \underline{a_r}$ --- ①

From ①, we can calculate the E-field in region 1 & 2, that is :

$$\underline{E} = \begin{cases} \dfrac{\underline{D}}{\epsilon_1} = \dfrac{\rho_\ell}{2\pi\epsilon_1 r} \underline{a_r} & (\text{Region } 1) \\[2ex] \dfrac{\underline{D}}{\epsilon_2} = \dfrac{\rho_\ell}{2\pi\epsilon_2 r} \underline{a_r} & (\text{Region } 2) \end{cases}$$

Now, in order to find $\rho_\ell$, we calculate the total potential difference as follows.

$$\underbrace{\int_{r_1}^{r_2} \frac{\rho_\ell}{2\pi\epsilon_1 r}\,dr}_{\text{Voltage applied at dielectric 1}} + \underbrace{\int_{r_2}^{r_3} \frac{\rho_\ell}{2\pi\epsilon_2 r}\,dr}_{\text{Voltage applied at dielectric 2}} = V_0 \quad \cdots \text{②}$$

Evaluating ② , we get

$$\rho_\ell = \frac{2\pi V_0}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)} \quad \cdots\cdots \text{③}$$

Then from ③ , we obtain the capacitance per unit length

$$C_\ell = \frac{\rho_\ell}{V_0} = \boxed{\frac{2\pi}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)}} \leftarrow \text{Answer}$$

(b) Potential gradient is the electric field strength. So, we have to find the maximum value of $\underline{E}$ in both region 1 & 2.

i) Region 1 $\quad (r_1 \le r \le r_2)$

$$E_1 = \frac{\rho_\ell}{2\pi\epsilon_1 r} = \frac{V_0}{\ln\left(\frac{r_2}{r_1}\right) + \frac{\epsilon_1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)} \cdot \frac{1}{r}$$

It is obvious that E will have its maximum at $r = r_1$ and

$$E_{1\,max} = \frac{1200}{\ln 2 + \frac{1}{3}\ln 2} \cdot \frac{1}{0.01} \cong \boxed{129.843\ \text{KV/M} = 1.29843\ \text{KV/cm}}$$

ii) Region 2 $(r_2 \le r \le r_3)$

$$E_2 = \frac{\rho_e}{2\pi\epsilon_2 r} = \frac{V_0}{\frac{\epsilon_2}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \ln\left(\frac{r_3}{r_2}\right)} \cdot \frac{1}{r}$$

So, at $r = r_2 = 0.02$ meter

$$E_{2max} = \frac{1200}{3\ln 2 + \ln 2} \cdot \frac{1}{0.02} = \boxed{21.64\,KV/m \text{ or } 216.4\,V/cm}$$

< 2nd - Approach : Potential approach >

In this method, we will find the expression for the potential $\Phi$ by solving the Laplace's equation.

In cylindrical coordinates, the solution for $\Phi$ will be (see Text pp 125-126)

$$\Phi = \begin{cases} c_1 \ln r + d_1 & (\text{Region } 1) \leftarrow \Phi_1 \\ c_2 \ln r + d_2 & (\text{Region } 2) \leftarrow \Phi_2 \end{cases}$$

To determine the coefficients $c_1, c_2, d_1$, and $d_2$, we apply the Boundary conditions, namely;

I $\Phi(r=r_1) = c_1 \ln r_1 + d_1 = V_0$ --- ①

II $\Phi(r=r_3) = c_2 \ln r_3 + d_2 = 0$ --- ②

III At $r = r_2$ we have to have $\Phi_1 = \Phi_2$

$\therefore \; c_1 \ln r_2 + d_1 = c_2 \ln r_2 + d_2$ --- ③

and $\epsilon_1 \frac{\partial \Phi_1}{\partial r} = \epsilon_2 \frac{\partial \Phi_2}{\partial r}$ or

$\epsilon_1 c_1 / r_2 = \epsilon_2 c_2 / r_2$ --- ④

Solving this simultaneous equations, we obtain

$$\Phi = \begin{cases} V_0\left(1 - \dfrac{\frac{1}{\epsilon_1}\ln\left(\frac{r}{r_1}\right)}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)}\right) = \Phi_1 , & r_1 \le r \le r_2 \\ \dfrac{-V_0 \frac{1}{\epsilon_2}\ln\left(\frac{r}{r_3}\right)}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)} = \Phi_2 , & r_2 \le r \le r_3 \end{cases}$$

a) Capacitance calculation

To calculate the capacitance $C_\ell$, we have to find $\rho_\ell$ as before.

$$\rho_\ell = -\epsilon_1 \frac{\partial \Phi_1}{\partial r}\bigg|_{r=r_1} \times 2\pi r_1 \left( = -\epsilon_2 \frac{\partial \Phi_2}{\partial r}\bigg|_{r=r_2} \times 2\pi r_2 \right)$$

$$= \boxed{\frac{2\pi V_0}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)}}$$

And $$C_\ell = \frac{\rho_\ell}{V_0} = \boxed{\frac{2\pi}{\frac{1}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \frac{1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)}}$$

, which is the same result as before.

b) Potential gradient calculation.

i) Region 1 $( r_1 \le r \le r_2 )$

$$\left|\frac{\partial \Phi_1}{\partial r}\right| = \frac{V_0}{\ln\left(\frac{r_2}{r_1}\right) + \frac{\epsilon_1}{\epsilon_2}\ln\left(\frac{r_3}{r_2}\right)} \cdot \frac{1}{r}$$

and $$\left|\frac{\partial \Phi_1}{\partial r}\right|_{max} = \frac{1200}{\ln 2 + \frac{1}{3}\ln 2} \cdot \frac{1}{0.01} = \boxed{129.843\,\mathrm{KV/M}}$$

ii) Region 2 $( r_2 \le r \le r_3 )$

$$\left|\frac{\partial \Phi_2}{\partial r}\right| = \frac{V_0}{\frac{\epsilon_2}{\epsilon_1}\ln\left(\frac{r_2}{r_1}\right) + \ln\left(\frac{r_3}{r_2}\right)} \cdot \frac{1}{r}$$

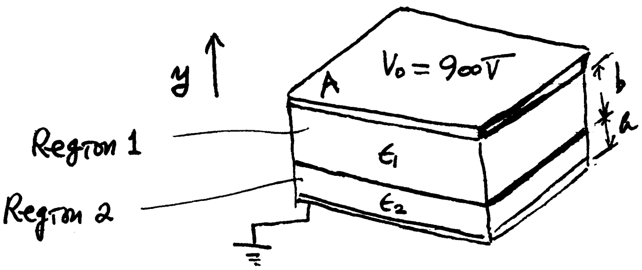
and $$\left|\frac{\partial \Phi_2}{\partial r}\right|_{max} = \frac{1200}{3\ln 2 + \ln 2} \cdot \frac{1}{0.02} = \boxed{21.64\,\mathrm{KV/m}}$$

(Note) As you see from the above calculations, E-field approach is much easier. It is, however, not generally true. When a system has simple and symmetrical geometry, E-field approach (by using the Gauss's law) maybe easier.

Solution by solving the Laplace's equation works at any case and we usually resort to this method when a system has complex geometry and the Gauss's law does not work well.

(note)** This statement is true only when the boundary surface is normal to $\underline{D}$ vector. More precisely speaking, $D_n$ (normal component of $D$ at the boundary) should be continuous. In our case $\underline{D}$ has no tangential component at the boundary and $D_{n1} = D_{n2}$ is the same as $\underline{D}_1 = \underline{D}_2$. So in general case you should be careful.

3–37.




$\epsilon_1 = \epsilon_2 = 6\epsilon_0$
$A = 100\ cm^2$
$b = 2a = 2mm$

Let's consider the general case and find $\underline{D}$ first. Neglecting the fringing field at the plate edge, we get

$$D \cdot A = Q \quad \text{(total charge on a plate)}.$$

$$\therefore \quad D = \frac{Q}{A}, \quad \underline{D} = -\frac{Q}{A}\underline{a_y}$$

Hence we get

$$\underline{E} = \begin{cases} \dfrac{\underline{D}}{\epsilon_1} = \dfrac{-Q}{A\epsilon_1}\underline{a_y} & \text{(Region 1)} \\[2ex] \dfrac{\underline{D}}{\epsilon_2} = \dfrac{-Q}{A\epsilon_2}a_y & \text{(Region 2)} \end{cases}$$

Then we have to calculate Q as follows

$$\overbrace{\int_a^{a+b} \frac{Q}{A\epsilon_1}dy}^{V_1} + \overbrace{\int_0^a \frac{Q}{A\epsilon_2}dy}^{V_2} = V_0$$

$$\therefore \quad \overbrace{\frac{bQ}{A\epsilon_1}}^{V_1} + \overbrace{\frac{aQ}{A\epsilon_2}}^{V_2} = V_0$$

and

$$Q = \frac{A V_0}{\dfrac{b}{\epsilon_1} + \dfrac{a}{\epsilon_2}}$$

Therefore

$$V_1 = \frac{bQ}{A\epsilon_1} = \boxed{\frac{V_0}{1+\frac{a}{b}\frac{\epsilon_1}{\epsilon_2}}}$$

and

$$V_2 = \frac{aQ}{A\epsilon_2} = \boxed{\frac{V_0}{1+\frac{b}{a}\frac{\epsilon_2}{\epsilon_1}}}$$

The above results are valid for general case.

In our case $V_0 = 900V$ and $a = 1mm$, $b = 2mm$, $\epsilon_1 = \epsilon_2 = 6\epsilon_0$, so we finally obtain

$$V_1 = \frac{900V}{1+\frac{1}{2}\cdot 1} = \boxed{600V} \leftarrow \text{Answer}$$

$$V_2 = \frac{900V}{1+\frac{2}{1}\cdot 1} = \boxed{300V} \leftarrow \text{Answer}$$

As a matter of fact, we don't have to follow these lengthy derivations if $\epsilon_1 = \epsilon_2$. In this case we have

$$E_1 = \frac{V_1}{b} = \frac{V_2}{a} = E_2 \quad \text{--- ①}$$

and

$$V_1 + V_2 = V_0 \quad \text{--- ②}$$

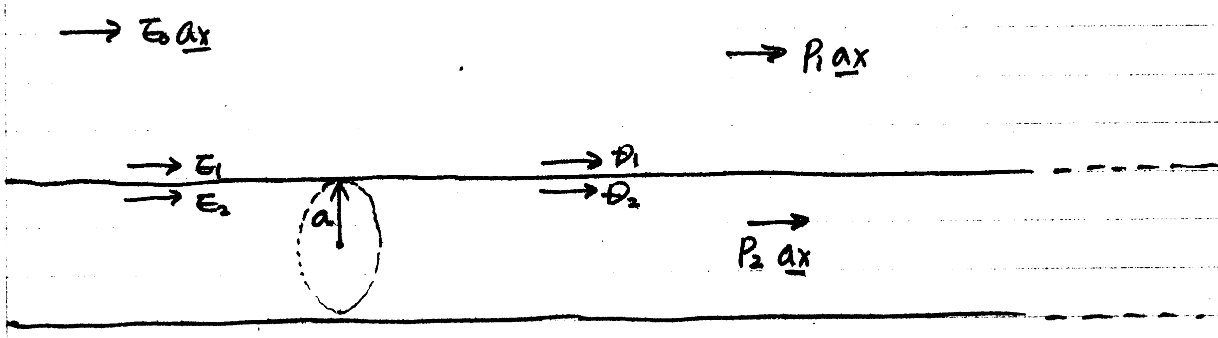
From ① & ②

$$V_1 = \frac{V_0}{1+a/b} = 600V$$

$$V_2 = \frac{V_0}{1+b/a} = 300V$$

— 9/10 —

< Last problem in Homework #3 >




From the given condition, we can say that $\underline{E_1}$ and $\underline{E_2}$ has only X component. Furthermore we can assert that

$$\underline{E_1} = \underline{E_2} = \underline{E_0} = E_0 \underline{a_x} \quad \text{---} \; ①$$

$\underline{E_1}$ shoud be equal to $\underline{E_2}$ because $\underline{E_1}, \underline{E_2}$ has only the tangential component at the boundary between two medium and $E_{1t} = E_{2t}$ should be met by the boundary condition between dielectric media.

Also we know that $\underline{E}$ field depends only on the free charge and that there is no free charge in medium 1 & medium 2 initially. Therefore $\underline{E_1} = \underline{E_2} = \underline{E_0}$. In other words, $\underline{E_0}$ is the only source of electric field for both medium 1 & 2.

Let's go on for the potential distribution. Since we know that $V = -\int \underline{E} \cdot d\underline{\ell}$, it is evident that $V$ depends only on $\underline{E}$ field.

Hence the potential distribution in medium 1 and medium 2 should be the same, and it is a linear function on $X$. So if we assume $V\big|_{X=-\infty} = V_0$

, then $V = V_0 - E_0 X$.

only normal component at the boundary between two media. And the requirement $D_{1n} = D_{2n}$ is equal

2/18/83

EENP [illegible]

ELECTROMAGNETIC FIELDS

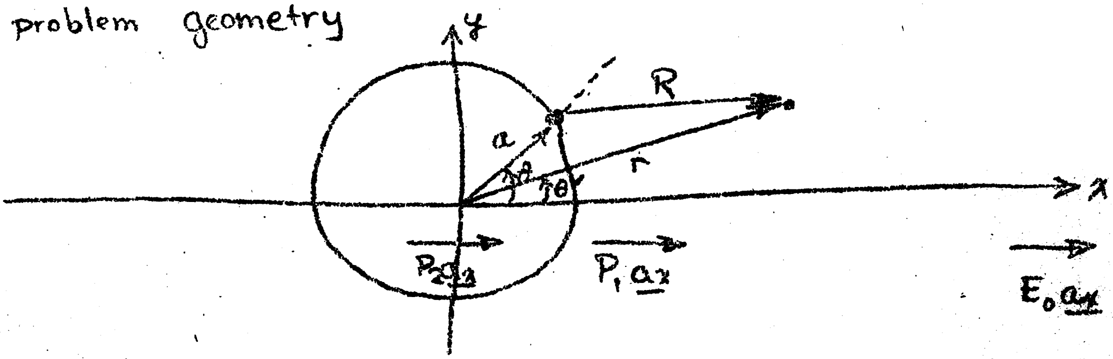
SOLUTION

HW # 3 PROBLEM # 4

A permanently polarized cylinder $P_2\,\underline{a_x}$ of radius $a$ is placed within a polarized medium $P_1\,\underline{a_x}$ of infinite extent. A uniform electric field $E_0\,\underline{a_x}$ is applied at infinity. There is NO free charge on the cylinder. What are the potential and electric field distributions?

This is a classic problem illustrating many basic principles.
Note: a permanently polarized material is called a ferroelectric.

The problem geometry




This problem is easiest to solve if we let the $\underline{E}$ field be zero at first and just examine the polarized materials. Recall the normal component of $\underline{D}$ is discontinuous by an amount equal to the surface charge density; however, $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$ where $\underline{E}$ is zero for the moment. Then $\underline{D} = \underline{P}$. But

$$n \cdot (D_2 - D_1) = \rho_s$$

where we note that $D_{2n} = P_1 \cos\theta$ and $D_{1n} = P_2 \cos\theta$ and the subscript $n$ indicates the normal component. By this substitution

$$\rho_s = D_{2n} - D_{in} = P_1 \cos\theta - P_2 \cos\theta = (P_1 - P_2)\cos\theta.$$

<u>WHAT IS THIS CHARGE DENSITY</u>? It is the surface bound charge density. This is USUALLY included in the D vector by letting $\underline{D} = \epsilon_0 \underline{E} + \underline{P} = \epsilon \underline{E}$

Recall, that $\nabla \cdot (\epsilon_0 \underline{E}) = \rho_f + \rho_b$

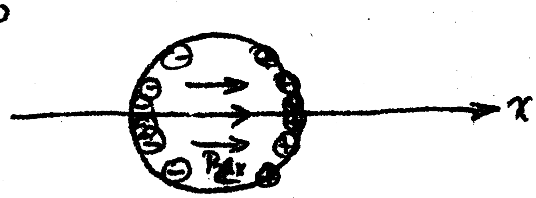
$$= \rho_f - \nabla \cdot P$$

and we then defined $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$.

We did not explicitly consider P and $\rho_b$ because
1) they were due to the external field $\underline{E}$
2) they we included in $\underline{D} = \epsilon \underline{E}$.

Where $\underline{E} = 0$ and polarization exists the bound charge MUST be explicitly considered. In this case, the permanently polarized material gives rise to bound surface charge densities and associated electric fields.

we have found $\rho_s = (P_1 - P_2)\cos\theta$ and this will look like the figure below




what is the field due to $\rho_s$?

I COULD use Gauss' law but I would have a lot of vector additions. Whenever I know the charge density I should use the ~~vector~~ electric potential, either as

$$\nabla^2 \Phi = 0$$

$$\text{or} \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon}.$$

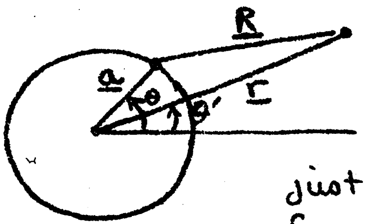
I don't know how to solve these very well yet but the general solution can be obtained from the superposition of the potentials from point charges. This was shown in class to be

$$\Phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho \, dv}{R}$$

In this case our charge density is a SURFACE charge density and the integral becomes a line integral, i.e.

$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_b \, dl}{R}$$

Once we have reached this step the remainder of the problem is just the evaluation of this integral.




$$\Phi = \frac{1}{4\pi\epsilon_0}\int \frac{(P_1 - P_2)\cos\theta \, a \, d\theta}{|\underline{a} - \underline{r}|}$$

just as in the problem of finding the electric potential from a dipole

$$|\underline{a} - \underline{r}|^2 = |\underline{a}|^2 + |\underline{r}|^2 - 2|\underline{a}||\underline{r}|\cos(\theta - \theta')$$

$$= a^2 + r^2 - 2ar\cos(\theta - \theta')$$

$$\Phi(\underline{r}) = \frac{a}{4\pi\epsilon_0}(P_1 - P_2)\int_0^{2\pi} \frac{\cos\theta \, d\theta}{|\underline{a} - \underline{r}|}$$

If you got this far, you're a fields whip!

To solve at this point consider the case where $r \gg a$.

[An exact solution is possible, but not by this method.]

Then

$$|\underline{a} - \underline{r}| = \sqrt{r^2}\left[\left(\frac{a}{r}\right)^2 + 1 - 2\left(\frac{a}{r}\right)\cos(\theta - \theta')\right]^{1/2}$$

$$\cong r\left[1 - 2\left(\frac{a}{r}\right)\cos(\theta - \theta')\right]^{1/2} \quad \text{as } \left(\frac{a}{r}\right) \ll 1$$

$$\cong r\left[1 - \frac{1}{2}\cdot 2\,\frac{a}{r}\cos(\theta - \theta')\right]$$

~~binom~~ Taylor expansion. Binomial expansion, etc.

and

$$\frac{1}{|\underline{a} - \underline{r}|} = \frac{1}{r\left[1 - \frac{a}{r}\cos(\theta - \theta')\right]} \approx \frac{1 + \frac{a}{r}\cos(\theta - \theta')}{r}$$

Now expand $\cos(\theta-\theta')$.

$$\cos(\theta-\theta') = \cos\theta\cos\theta' + \sin\theta\sin\theta'$$

Combining our results

$$\Phi(\underline{r}) = \frac{a}{4\pi\epsilon_0}(P_1-P_2)\left[\cancelto{0}{\int_0^{2\pi}\frac{\cos\theta}{r}\,d\theta} + \frac{a}{r^2}\int_0^{2\pi}\cos\theta\cos(\theta-\theta')\,d\theta\right]$$

$$= \frac{a}{4\pi\epsilon_0}(P_1-P_2)\frac{a}{r^2}\int_0^{2\pi}\Big(\overbrace{\cos^2\theta}^{\frac{1}{2}(1+\cos 2\theta)}\cos\theta' + \overbrace{\cos\theta\sin\theta}^{\frac{1}{2}\sin 2\theta}\sin\theta'\Big)\,d\theta$$

$$= \frac{a}{4\pi\epsilon_0}(P_1-P_2)\frac{a}{r^2}\int_0^{2\pi}\frac{1}{2}\cos\theta'\,d\theta$$

$$= \frac{a^2}{4\epsilon_0 r^2}(P_1-P_2)\cos\theta'$$

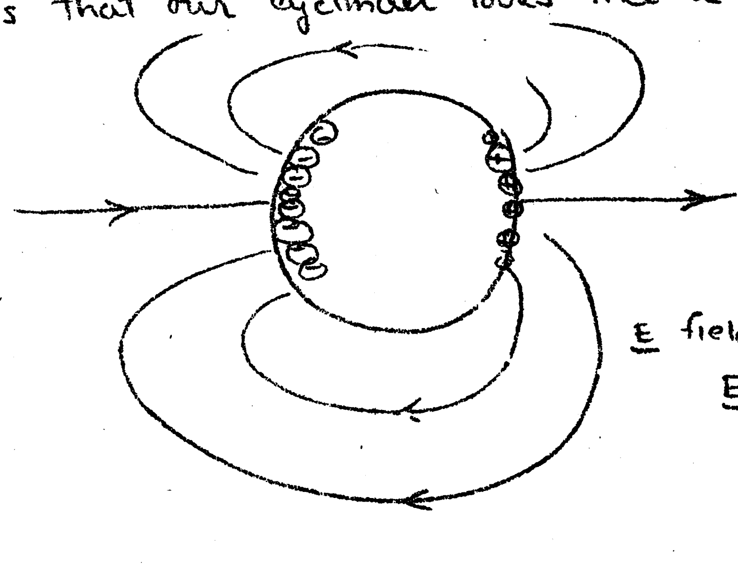
This is a very interesting result. We have seen it before as the field of the electric dipole.

For this orientation of the dipole we got

$$\Phi_{dipole} = \frac{q\ell\cos\theta'}{4\pi\epsilon_0 r^2}$$

Aside from the constants, the potentials are identical. This means that our cylinder looks like a big electric dipole.

i.e.



$\underline{E}$ field found from

$$\underline{E} = -\nabla\Phi$$

How about the forgotten $\underline{E}$ field? Easy. As we are using ~~vector po~~ electric potentials which are scalars we can find $\Phi$ due to $\underline{E}$ and add the results together.

$$\text{If} \quad \underline{E} = E_0 \underline{a_x} = -\nabla \Phi$$

Then

$$E_0 = -\frac{d\Phi}{dx}$$

$$\text{and} \quad \Phi = -E_0 x + C_1$$

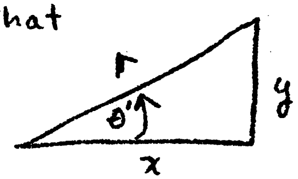
where $C_1$ is a constant. As we have no further information about $\Phi$, i.e. is $\Phi$ zero anywhere, let's just pick $C_1 = 0$ to make the problem easier.

Our potentials add

$$\Phi_{\text{ferroelectric}_L} = \frac{a^2}{4\epsilon_0 r^2}(P_1 - P_2)\cos\theta'$$

$$\Phi_{\text{applied field}} = -E_0 x$$

Note that




$$x = r\cos\theta'$$

so that we can write

$$\Phi_{\text{total}} = \frac{a^2}{4\epsilon_0 r^2}(P_1 - P_2)\cos\theta' - E_0 r\cos\theta'$$

and $\underline{E}_{\text{total}} = -\nabla \Phi_{\text{total}}$. (which is just more math).

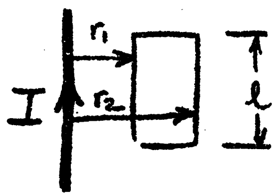
# Homework Assignment #4
## EEAP 210
## ELECTROMAGNETIC FIELDS

From Paul & Nasar, Ch. 4) DUE FEB. 28, 1983

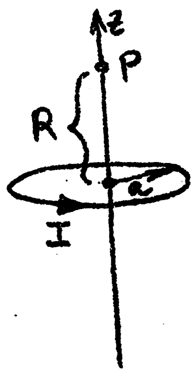
4.2 Using the Biot-Savart Law, find the magnetic field intensity $\underline{H}$ at a point on the axis of a circular loop of radius a carring a current I. The point is a distance h (on the a… from the center of the loop.

4.4 Within a cyclindrical conductor of radius a, the current density exponentially decreases with the radius such that $\underline{J} = Ae^{-kr}\,\underline{a_z}$, where A and k are constants. Determine the resulting magnetic field intensity everywhere.

4.6 A rectangular loop is placed in the field of a very long straight conductor carrying a current I as shown below which also shows the various dimensions. What is the total magnetic flux passing through the loop?




4.12 A circular loop of radius a carries a current I. Determine the magnetic vector potential at a point P which is at a distance R from the center of the loop. [Assume R>>a]

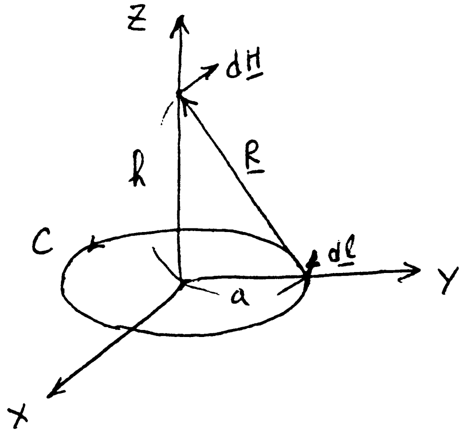


# EEAP 210

## HOMEWORK SOLUTION (#4)

2/28/83

[4-2]




$$\begin{cases} d\underline{l} = (a\,d\phi)\,\underline{a_\phi} \\ \underline{R} = h\,\underline{a_z} - a\,\underline{a_r} \\ R = (h^2 + a^2)^{1/2} \end{cases}$$

Referred to the above figure, we have

$$d\underline{l} \times \underline{R} = (a\,d\phi)\,\underline{a_\phi} \times (h\,\underline{a_z} - a\,\underline{a_r})$$

$$= ah\,d\phi\,\underline{a_r} + a^2\,d\phi\,\underline{a_z}$$

Now $$d\underline{H} = \frac{I\,d\underline{l} \times \underline{a_R}}{4\pi R^2} = \frac{I\,d\underline{l} \times \underline{R}}{4\pi R^3} = \frac{I}{4\pi R^3}\left(ah\,d\phi\,\underline{a_r} + a^2 d\phi\,\underline{a_z}\right)$$

$$\therefore\ \underline{H} = \frac{I}{4\pi R^3}\left[\oint_c ah\,d\phi\,\underline{a_r} + \oint_c a^2 d\phi\,\underline{a_z}\right]$$

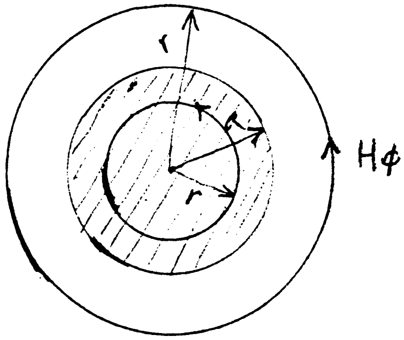
Here $$\oint_c ah\,d\phi\,\underline{a_r} = ah \oint_c \underline{a_r}\,d\phi \equiv 0$$, and

$$\oint_c a^2 d\phi\,\underline{a_z} = 2\pi a^2\,\underline{a_z}\ .$$

Hence we have

$$\underline{H} = \frac{I a^2 \cdot 2\pi}{4\pi R^3}\,\underline{a_z} = \boxed{\frac{I a^2}{2(h^2+a^2)^{3/2}}\,\underline{a_z}}$$

[4-4]




From the geometrical symmetry we can assert that $\underline{H}$ has only $\phi$-component, say $H\phi$.

Then using Ampere's circuital law we get

$$\oint \underline{H}\, d\underline{l} = \int_s \underline{J}\cdot d\underline{s} = 2\pi r H\phi \quad \cdots ①$$

1) When $0 \le r \le a$

$$I = \int_s \underline{J}\cdot d\underline{s} = \int_0^r \int_0^{2\pi} (Ae^{-kr}\underline{a_z})\cdot \overbrace{(r dr d\phi\, \underline{a_z})}^{d\underline{s}}$$

$$= 2\pi A \int_0^r r e^{-kr} dr$$

$$= \frac{2\pi A}{k^2}\left[1 - e^{-kr} - re^{-kr}\right] \quad \cdots ②$$

From ① & ② we have

$$\boxed{\underline{H} = \frac{A}{k^2 r}\left[1 - e^{-kr} - re^{-kr}\right]\underline{a_\phi}} \quad (0 \le r \le a)$$

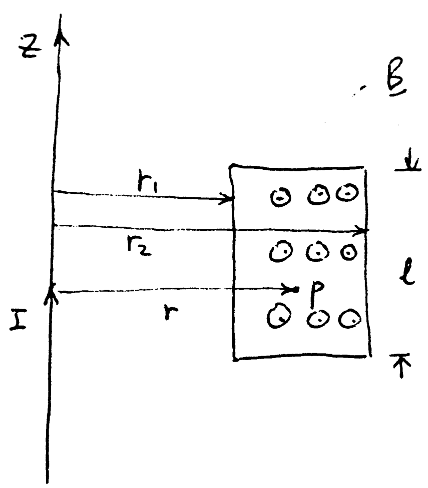
2) When $r \ge a$

From ② $\quad I = \dfrac{2\pi A}{k^2}\left[1 - e^{-ka} - ae^{-ka}\right] \quad \cdots ③$

Equating ① to ② we obtain

$$\boxed{\underline{H} = \frac{A}{k^2 r}\left[1 - e^{-ka} - ae^{-ka}\right]\underline{a_\phi}} \quad (r \ge a)$$

[4-6]




$$d\underline{s} = l\,dr\,\underline{a_\phi}$$

$\underline{B}$ field at point P can be obtained by Ampere's la

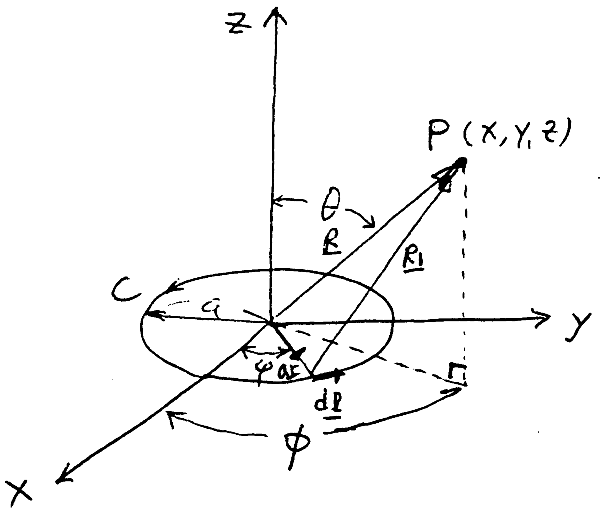
$$\underline{B} = \frac{\mu_0 I}{2\pi r}\,\underline{a_\phi}$$

Hence the total flux passing through this rectangle given by

$$\Phi = \int \underline{B} \cdot d\underline{s} = \int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r}\, l\, dr = \frac{\mu_0 I l}{2\pi} \ln \frac{r_2}{r_1}$$

$$\therefore \boxed{\Phi = \frac{\mu_0 I l}{2\pi} \ln \frac{r_2}{r_1}}$$

[4-12]




$$d\underline{l} = a\,d\varphi\,\underline{a_\varphi}$$
$$= a\,d\varphi\,(-\sin\varphi\,\underline{a_x} + \cos\varphi\,\underline{a}$$ --- ①

From the above configuration we have

$$\underline{R_1} = \underline{R} - a\,\underline{a_r} \quad , \text{ and}$$

$$R_1 = (\underline{R_1} \cdot \underline{R_1})^{1/2} = \left[ (\underline{R} - a\underline{a_r})(\underline{R} - a\underline{a_r}) \right]^{1/2}$$
$$= \left[ R^2 - 2a\underline{R}\cdot\underline{a_r} + a^2 \right]^{1/2}$$

Now $\underline{R}\cdot\underline{a_r} = (x, y, z)\cdot(\cos\varphi, \sin\varphi, 0)$
$$= x\cos\varphi + y\sin\varphi$$

Therefore

$$R_1^{-1} = R^{-1}\left[ 1 - \frac{2a}{R^2}(x\cos\varphi + y\sin\varphi) + \underbrace{\frac{a^2}{R^2}}_{0 \text{ (if } R \gg a)} \right]^{-1/2}$$

$$\cong R^{-1}\left[ 1 + \frac{ax}{R^2}\cos\varphi + \frac{ay}{R^2}\sin\varphi \right] \quad (\text{when } R \gg a) \cdots$$

Vector potential $\underline{A}$ is given by

$$\underline{A} = \frac{\mu_0 I}{4\pi} \oint_c \frac{d\underline{l}}{R_1}$$

From ① & ② we get ( when $R \gg a$ )

$$\underline{A} \cong \frac{\mu_0 I a}{4\pi R} \oint_c d\varphi \left(-\sin\varphi \, \underline{a_x} + \cos\varphi \, \underline{a_y}\right)\left(1 + \frac{ax}{R^2}\cos\varphi + \frac{ay}{R^2}\sin\varphi\right)$$

Here

$$\left.\begin{aligned} &\int_0^{2\pi} d\varphi \left(-\sin\varphi \, \underline{a_x} + \cos\varphi \, \underline{a_y}\right) \equiv 0 \\ &\int_0^{2\pi} d\varphi \left(-\sin\varphi \, \underline{a_x} + \cos\varphi \, \underline{a_y}\right)\cos\varphi = \pi \, \underline{a_y} \\ &\int_0^{2\pi} d\varphi \left(-\sin\varphi \, \underline{a_x} + \cos\varphi \, \underline{a_y}\right)\sin\varphi = -\pi \, \underline{a_x} \end{aligned}\right\} \cdots ③$$

From ③ we obtain

$$\boxed{\underline{A} = \frac{\pi \mu_0 I a^2}{4\pi R^3}\left(-y \, \underline{a_x} + x \, \underline{a_y}\right)}$$ ← Cartesian coordina[tes]

Or converting it to spherical coordinates we get

$$\boxed{\underline{A} = \frac{\pi \mu_0 I a^2}{4\pi R^2} \sin\theta \, \underline{a_\phi}}$$ ← Spherical coordinat[es]

See back side of this page for the conversion ⟶

over

Cartesian $\longleftrightarrow$ Spherical conversion

$$\begin{cases} X = R\sin\theta\cos\phi \\ Y = R\sin\theta\sin\phi \\ \\ \underline{a_X}\cdot\underline{a_R} = \sin\theta\cos\phi \qquad \underline{a_Y}\cdot\underline{a_R} = \sin\theta\sin\phi \\ \underline{a_X}\cdot\underline{a_\theta} = \cos\theta\cos\phi \qquad \underline{a_Y}\cdot\underline{a_\theta} = \cos\theta\sin\phi \\ \underline{a_X}\cdot\underline{a_\phi} = -\sin\phi \qquad \underline{a_Y}\cdot\underline{a_\phi} = \cos\phi \end{cases}$$

Hence

$$-Y\underline{a_X} = -R\sin\theta\sin\phi\left[\sin\theta\cos\phi\,\underline{a_R} + \cos\theta\cos\phi\,\underline{a_\theta} - \sin\phi\,\underline{a_\phi}\right]$$

$$X\underline{a_Y} = R\sin\theta\cos\phi\left[\sin\theta\sin\phi\,\underline{a_R} + \cos\theta\sin\phi\,\underline{a_\theta} + \cos\phi\,\underline{a_\phi}\right]$$

And

$$-Y\underline{a_X} + X\underline{a_Y} = R\sin\theta\,\underline{a_\phi}$$

Therefore

$$\underline{A} = \frac{\pi\mu_0 I a^2}{4\pi R^3}\cdot R\sin\theta\,\underline{a_\phi}$$

$$= \boxed{\frac{\pi\mu_0 I a^2}{4\pi R^2}\sin\theta\,\underline{a_\phi}}$$ in spherical coordinates

EEAP 210

ELECTROMAGNETIC FIELDS

HOMEWORK NO. 5

TEXT:

4.7 FOR THE AIR CORE TOROID SHOWN IN FIGURE P4-7:

(a) DETERMINE THE CORE FLUX

(b) IF THE CORE FLUX DENSITY IS ASSUMED TO BE UNIFORM AND EQUAL TO ITS VALUE AT THE ARITHMETIC MEAN RADIUS, WHAT PERCENT ERROR WOULD BE MADE IN THE COMPUTATION OF THE FLUX BY THIS APPROXIMATION, AS COMPARED TO YOUR RESULT FOR (a).

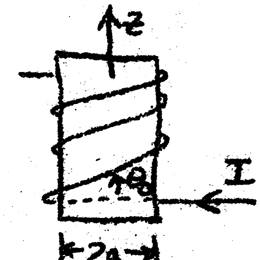
4.15 MAGNETIC FLUX LINES ENTER AT AN ANGLE OF 45° FROM FREE SPACE INTO A FERROMAGNETIC REGION HAVING A RELATIVE PERMEABILITY $20\mu_0$. DETERMINE THE ANGLE AT WHICH THE FLUX LINES WOULD EMERGE FROM THE INTERFACE. THERE IS NO SURFACE CURRENT.

EXTERNAL

1. CLOSELY SPACED WIRES ARE WOUND ABOUT AN INFINITELY LONG CORE AT PITCH ANGLE $\theta_0$. A CURRENT FLOWING IN THE WIRES THEN APPROXIMATES A SURFACE CURRENT

$$\underline{K} = K_0 \left( \underline{a_z} \sin\theta_0 + \underline{a_\phi} \cos\theta_0 \right)$$

WHAT IS THE MAGNETIC FIELD EVERYWHERE.

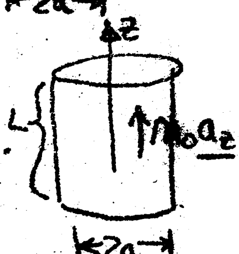



2. SUPPOSE A CYCLINDER OF RADIUS $a$ AND LENGTH $L$ IS PERMANENTLY MAGNETIZED AS $\underline{M} = M_0 \underline{a_z}$.

(a) WHAT ARE THE $\underline{B}$ AND $\underline{H}$ FIELDS EVERYWHERE ALONG THE Z-AXIS. THERE IS NO APPLIED FIELD.

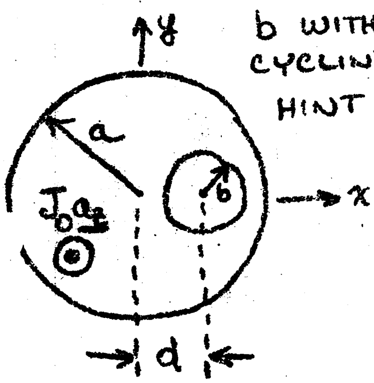
(b) WHAT ARE THE FIELDS FAR FROM THE MAGNET $(r \gg a,\ r \gg L)$

HINT: CONSIDER THE EQUIVALENT CURRENT $\underline{J_m} = \underline{\nabla} \times \underline{M}$ AS THE SOURCE OF YOUR FIELD.




3. AN INFINITELY LONG CYCLINDER OF RADIUS $a$ CARRYING A UNIFORM CURRENT $J_0 \underline{a_z}$ HAS AN OFF-AXIS HOLE OF RADIUS $b$ WITH CENTER A DISTANCE $d$ FROM THE CENTER OF THE CYCLINDER. WHAT IS THE MAGNETIC FIELD IN THE HOLE?

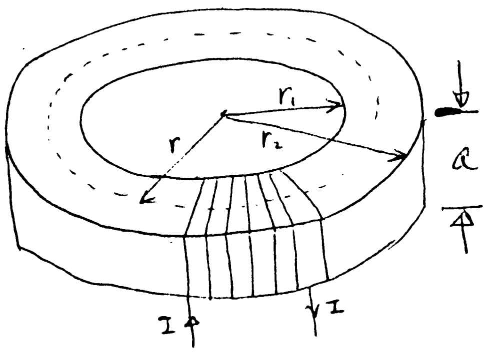
HINT: FIND THE FIELD IN THE REGION OF THE HOLE ASSUMING THE HOLE ISN'T THERE. THEN FIND THE FIELD DUE TO A ~~EQUAL~~ BUT OPPOSITELY DIRECTED CURRENT JUST IN THE HOLE. SUPERIMPOSE YOUR RESULTS. USE CARTESIAN COORDINATES TO COMBINE YOUR RESULTS



# EEAP 210

## HOMEWORK SOLUTION (#5) 3/15/83

(4-7)




(a) Invoking Ampere's law, we get

$$B_\phi = \frac{\mu N I}{2\pi r}$$

Then the flux $\Phi$ is calculated by

$$\Phi = \int \underline{B} \cdot d\underline{s} = \int_{r_1}^{r_2} \frac{\mu N I}{2\pi r}\, a\, dr = \boxed{\frac{\mu N I a}{2\pi} \ln \frac{r_2}{r_1}}$$

(b) Since the arithmetic mean radius

$$\bar{r} = \frac{r_1 + r_2}{2}$$

we have the average B-field

$$\bar{B} = \frac{\mu N I}{\pi (r_1 + r_2)}$$

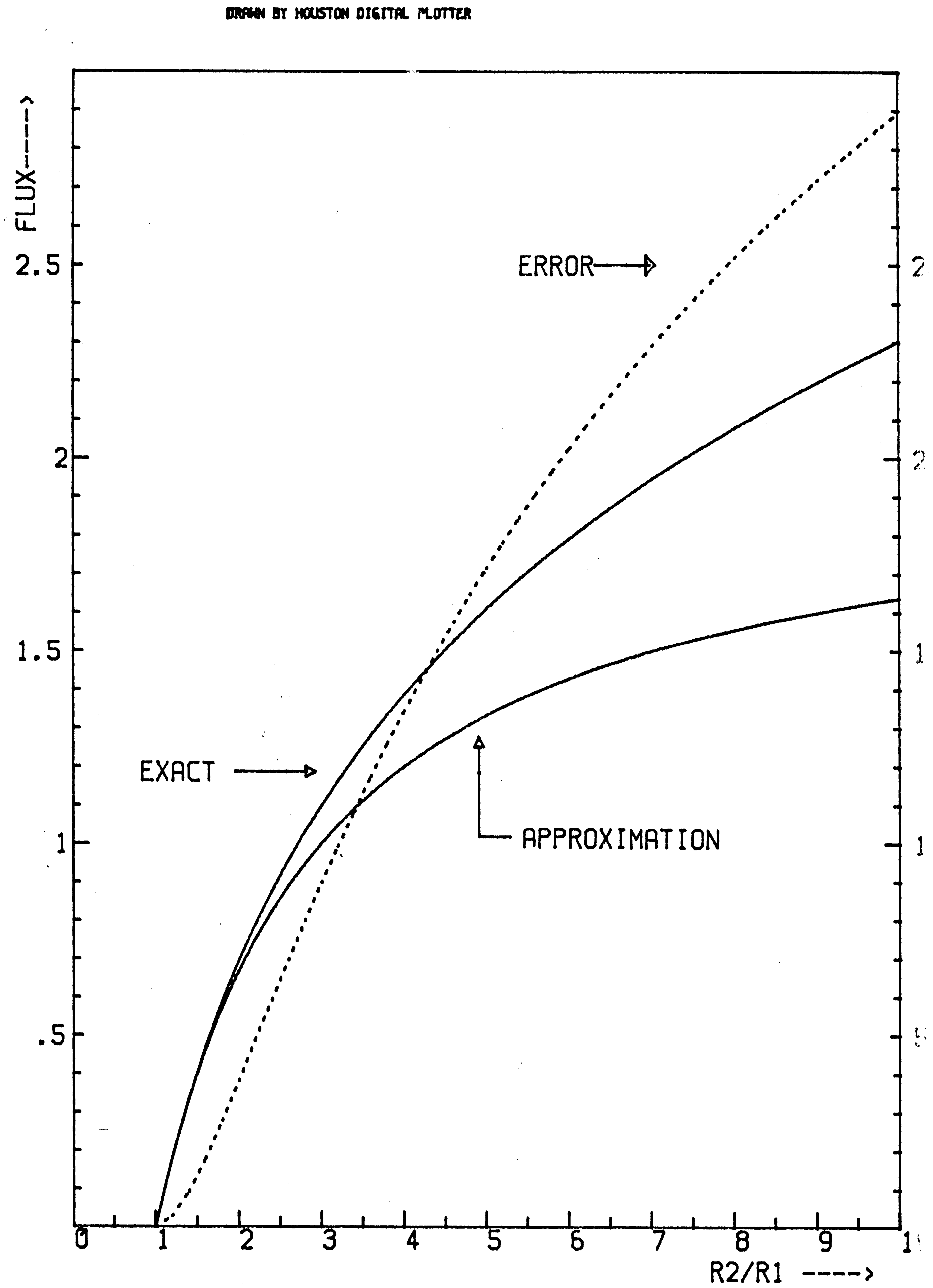
And the average flux $\bar{\Phi}$ now becomes

$$\bar{\Phi} = \frac{\mu N I}{\pi (r_1 + r_2)} \cdot a(r_2 - r_1) = \boxed{\frac{\mu N I a (r_2 - r_1)}{\pi (r_1 + r_2)}}$$

The percentage error can be calculated by

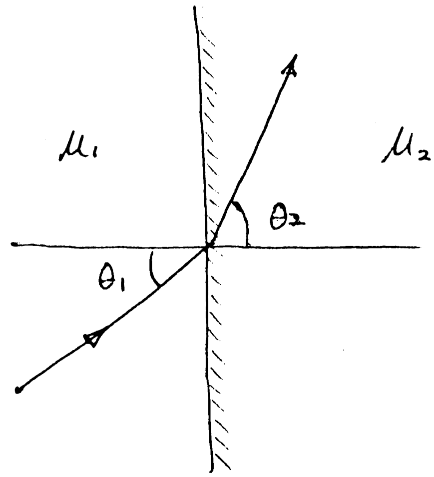
$$\text{Error}(\%) = \left( 1 - \frac{2\left(\frac{r_2 - r_1}{r_2 + r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)} \right) \times 100$$

Curves for $\Phi$, $\bar{\Phi}$, and Error (%) are shown in the next page. For simplicity, $\frac{\mu N I a}{2\pi}$ is set to be unity.



DRAWN BY HOUSTON DIGITAL PLOTTER
FLUX----->
2.5
2
1.5
1
.5
0
1
2
3
4
5
6
7
8
9
R2/R1 ----->
ERROR
EXACT
APPROXIMATION

(4-15)




Boundary conditions for $\underline{B}$ and $\underline{H}$ are

$$B_{1n} = B_{2n}$$

$$\frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2}$$

In accord with the above conditions, we have

$$B_1 \cos\theta_1 = B_2 \cos\theta_2$$

$$\mu_2 B_1 \sin\theta_1 = \mu_1 B_2 \sin\theta_2$$

By taking the quotient of the two equations, we end up with

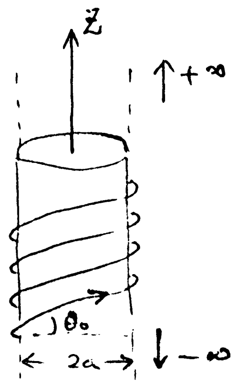
$$\boxed{\tan\theta_2 = \frac{\mu_2}{\mu_1} \tan\theta_1} .$$

If $\theta_1 = 45°$ and $\mu_2 = 20\mu_1$, then

$$\boxed{\theta_2 = \tan^{-1} 20 \cong 87.14°} .$$

## External

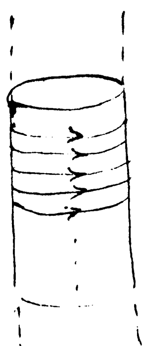
[1.]




Surface current

$\underline{K} = K_0 (\underline{a_z} \sin\theta_0 + \underline{a_\phi} \cos\theta_0)$

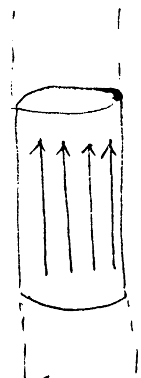
This problem can be viewed as a superposition of following two cases



<case 1>

(Infinite solenoid with surface current $\underline{J_{s_1}} = K_0 \cos\theta_0 \, \underline{a_\phi}$)

+



<case 2>

(Infinite cylinder with surface current $\underline{J_{s_2}} = K_0 \sin\theta_0 \, \underline{a_z}$)

Let's examine each cases.

1) Case 1

Since the solenoid is infinitely long the field inside the solenoid is uniform and is given by

$$B_{i1} = \mu_0 J_{s_1} = \mu_0 K_0 \cos\theta_0$$

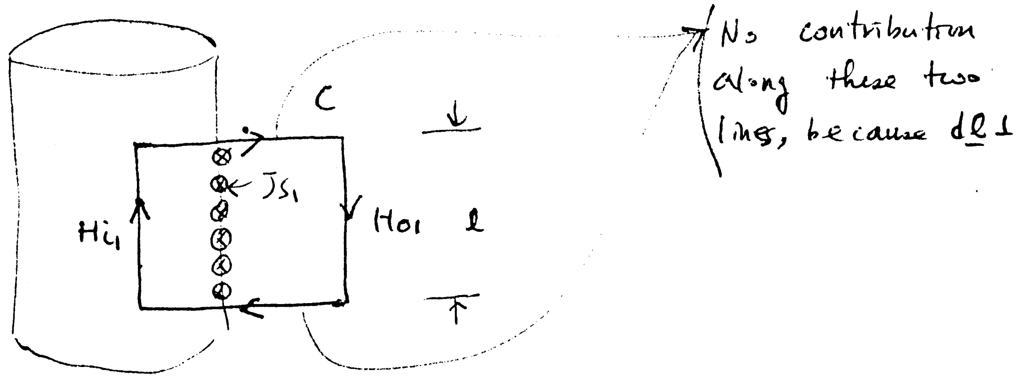
or $\boxed{\underline{B_{i1}} = \mu_0 K_0 \cos\theta_0 \, \underline{a_z}}$ ------- ①

And the field outside the solenoid is simply null.

$\boxed{\underline{B_{o1}} = 0}$ ------- ②

This is due to the following reasoning.

Since the solenoid is infinitely long, the magnetic field outside will have only a z-component, if any.. Then, if we construct a path as shown below and apply Ampere's law, we have




$$H_{i1}\, l + H_{o1}\, l = J_{s1}\, l .$$

$$\therefore \quad H_{i1} + H_{o1} = J_{s1}$$

or

$$\boxed{B_{i1} + B_{o1} = \mu_0 J_{s1}}$$

From the previous result, we know that $B_{i1} = \mu_0 J_{s1}$ and $B_{o1} \equiv 0$ follows.

2) Case 2

In this case, we can use Ampere's law again. B-field outside the cylinder is calculated by

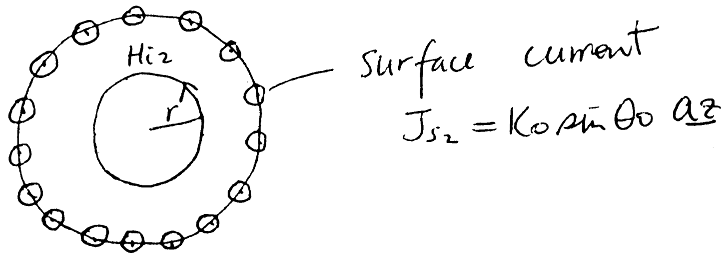
$$B_{o2} \cdot 2\pi r = \mu_0 2\pi a\, J_{s2} = \mu_0 2\pi a K_0 \sin\theta_0$$

$$\therefore \quad \boxed{\underline{B_{o2}} = \frac{\mu_0 a}{r} K_0 \sin\theta_0 \; \underline{a_\phi}} \quad \text{- - - - ③}$$

And the field inside the cylinder is simply null.

$$\boxed{\underline{B_{i2}} = 0} \quad \text{- - - - - ④}$$

$B_{i2}$ should be zero, because we expect that $B_{i2}$ has a $\phi$-component only and if we set up an integrating path as shown below




and apply Ampere's law, then we will end up with

$$2\pi r H_{iz} \equiv 0 \quad (\because \text{ No current inside })$$

$$\therefore \quad H_{iz} = 0 \quad \text{or} \quad B_{iz} = 0 \quad **$$

Combining the above results we finally have

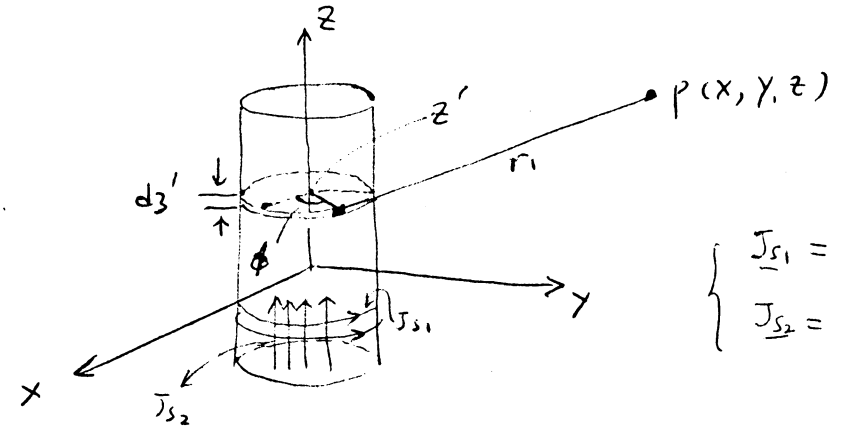
$$\underline{B} = \begin{cases} \mu_o K_o \cos\theta_o \, \underline{a_z} & (\text{inside the cylinder}) \\ \\ \dfrac{\mu_o a}{r} K_o \sin\theta_o \, \underline{a_\phi} & (\text{outside the cylinder}) \end{cases}$$

, where $r = \sqrt{x^2 + y^2}$

Same results can be obtained by mathematical calculations. Mathematical derivations are presented here for those who have interest in mathematical proof.

(Note)** Strictly speaking, we have to say that $\phi$-component of $H_{iz}$ is zero. In this case, however, $H_{iz}$ has only a $\phi$-component and $H_{iz} = 0$ is equivalent to saying $H_{iz\phi} = 0$.

< Mathematical Derivation >




$$\begin{cases} \underline{J_{s_1}} = K_0 \cos\theta_0 \, \underline{a_\phi} \\ \underline{J_{s_2}} = K_0 \sin\theta_0 \, \underline{a_z} \end{cases}$$

Applying Biot-Savart's law we have

$$\underline{B} = \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \oint_c \frac{I\underline{dl} \times \underline{r_1}}{r_1^3} .$$

Here $\quad \underline{r_1} = (x - a\cos\phi)\underline{a_x} + (y - a\sin\phi)\underline{a_y} + (z - z')\underline{a_z}$

and $\quad I\underline{dl} = \underline{J_{s_1} dz' \, a\,d\phi \, a_\phi} + \underline{J_{s_2} \, a\,d\phi \, dz' \, a_z}$

$$= a J_{s_1} d\phi \, dz' (-\sin\phi \, \underline{a_x} + \cos\phi \, \underline{a_y}) + a J_{s_2} d\phi \, dz' \, \underline{a_z} .$$

Now

$I\underline{dl} \times \underline{r_1}$

$= \underline{a_x} \left( J_{s_1} a \cos\phi \, (z - z') - J_{s_2} ( y - a\sin\phi ) a \right)$

$+ \underline{a_y} \left( J_{s_1} a \sin\phi \, (z - z') + J_{s_2} ( x - a\cos\phi ) a \right)$

$+ \underline{a_z} \left( J_{s_1} a ( a - x\cos\phi - y\sin\phi ) \right)$

It can be proved, without much difficulties, that

$$\int_{-\infty}^{\infty} \frac{(z-z')dz'}{r_1^3} = \int_{-\infty}^{\infty} \frac{(z-z')\,dz'}{[(x-a\cos\phi)^2+(y-a\sin\phi)^2+(z-z')^2]^{3/2}} \equiv 0 \quad \text{---} (1$$

and
$$\int_{-\infty}^{\infty} \frac{dz'}{r_1^3} = \frac{2}{(x-a\cos\phi)^2+(y-a\sin\phi)^2} . \quad \text{----} (2$$

Using (1) & (2) we have

$$B_x = \frac{a\mu_0 \cdot J_{s_2}}{4\pi} \int_{-\pi}^{\pi} \frac{-2(y-a\sin\phi)\,d\phi}{(x-a\cos\phi)^2+(y-a\sin\phi)^2} \quad \text{---} (3)$$

$$B_y = \frac{a\mu_0 J_{s_2}}{4\pi} \int_{-\pi}^{\pi} \frac{2(x-a\cos\phi)d\phi}{(x-a\cos\phi)^2+(y-a\sin\phi)^2} \quad \text{----} (4)$$

$$B_z = \frac{a\mu_0 J_{s_1}}{4\pi} \int_{-\pi}^{\pi} \frac{2(a-x\cos\phi-y\sin\phi)}{(x-a\cos\phi)^2+(y-a\sin\phi)^2}\,d\phi \quad \text{---} (5)$$

Integrals in (3), (4), (5) has the following basic form

$$I = \int_{-\pi}^{\pi} \frac{A+B\cos\phi+C\sin\phi}{\alpha+\beta\cos\phi+\gamma\sin\phi}\,d\phi$$

This integral has the following evaluation.

$$I = \frac{B\beta + C\gamma}{\beta^2+\gamma^2} \cdot 2\pi + \left(A - \frac{B\beta + C\gamma}{\beta^2+\gamma^2}\alpha\right) \frac{2\pi}{\sqrt{\alpha^2-\beta^2-\gamma^2}} \quad \cdots \textcircled{6}$$

(Refer to a mathematics handbook for this result)

Using the relation ⑥, we finally have

$$B_x = \begin{cases} a\mu_0 J_{s_2} \cdot \left(\dfrac{-y}{x^2+y^2}\right) & (x^2+y^2 > a^2) \\ 0 & (x^2+y^2 < a^2) \end{cases}$$

$$B_y = \begin{cases} a\mu_0 J_{s_2} \cdot \left(\dfrac{x}{x^2+y^2}\right) & (x^2+y^2 > a^2) \\ 0 & (x^2+y^2 < a^2) \end{cases}$$

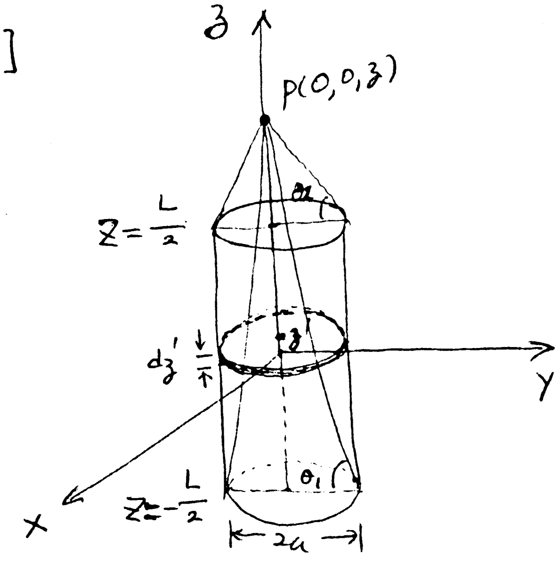
$$B_z = \begin{cases} 0 & (x^2+y^2 > a^2) \\ a\mu_0 J_{s_1} \cdot \left(\dfrac{1}{a}\right) & (x^2+y^2 < a^2) \end{cases}$$

In other words

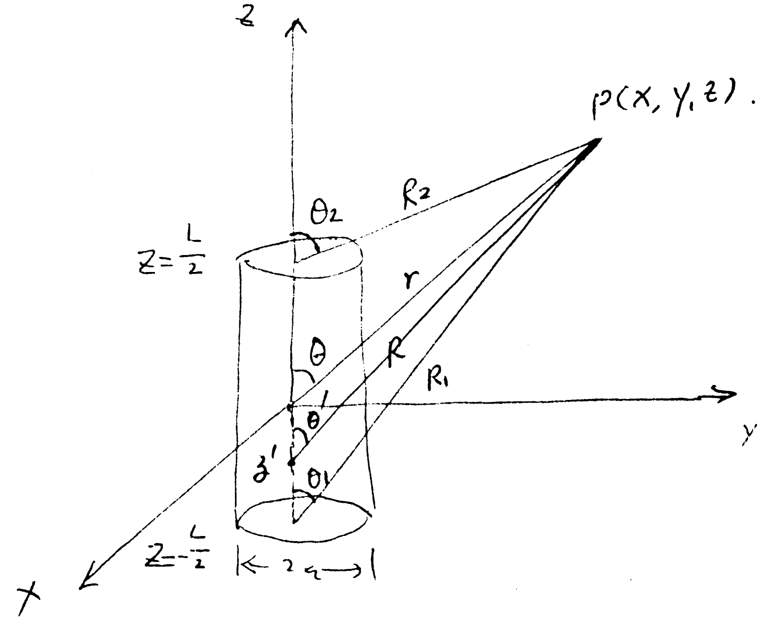
$$\boxed{\underline{B} = \begin{cases} \mu_0 K_0 \cdot \cos\theta_0 \, \underline{a_z} & (\text{inside}) \\ \dfrac{\mu_0 K_0 a}{r} \sin\theta_0 \, \underline{a_\phi} & (\text{outside}) \end{cases}}$$

External

[2.]




⟨Fig 1⟩ Field along the axis




⟨Fig 2⟩ Far-field Calculation

The field produced by a bar magnet may be computed in terms of equivalent polarization current.

Since $\underline{M} = M_0 \underline{a}_z$, volume density of polarization current inside the magnet $J_m = \nabla \times \underline{M} \equiv 0$.

Along the boundary, there is surface current which is given by

$$J_{ms} = \underline{M} \times \underline{a}_r = M_0 \underline{a}_\phi .$$

This current flows circumferentially around the magnet. And it is now apparent that the field from the bar magnet will be the same as that from an equivalent solenoid having an effective surface current of $M_0$ amp/meter. Our calculations is completely unnecessary if we already knew the field solution from a solenoid.

It is, however, reproduced here for those who are interested in calculation.

(a) Field along the axis

Referred to Fig 1., we can write

$$d\underline{B} = \underline{a_z} \frac{\mu_0 a^2 M_0}{2((z-z')^2+a^2)^{3/2}} dz' \quad \text{(See Homework \#4 (4-}$$

Hence

$$\underline{B} = \int d\underline{B} = \underline{a_z} \frac{\mu_0 a^2 M_0}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz'}{[(z-z')^2+a^2]^{3/2}}$$

Putting $z-z' = a\tan\theta$, we have $dz' = -a\sec^2\theta\, d\theta$ an

$$\underline{B} = \underline{a_z} \frac{\mu_0 M_0}{2} \int_{\theta_1}^{\theta_2} (-\cos\theta)\, d\theta$$

$$= \underline{a_z} \frac{\mu_0 M_0}{2} (\sin\theta_1 - \sin\theta_2)$$

Here $\sin\theta_1 = \dfrac{z+L/2}{\sqrt{(z+L/2)^2+a^2}}$ and

$$\sin\theta_2 = \frac{z-L/2}{\sqrt{(z-L/2)^2+a^2}}$$

Therefore

$$\boxed{\underline{B} = \underline{a_z} \frac{\mu_0 M_0}{2} \left[ \frac{z+L/2}{\sqrt{(z+L/2)^2+a^2}} - \frac{z-\frac{L}{2}}{\sqrt{(z-L/2)^2+a^2}} \right]}$$

(b) Far – field calculation

Refered to Fig 2, we can write

$$d\underline{A} = \underline{a_\phi} \frac{\mu_0 M_0 a^2}{4R^2} \sin\theta' \, dz' \quad \left( \leftarrow \text{See H.W \#4 solution to (4-12)} \right)$$

Here we have following relationships.

$$\begin{cases} z - z' = R\cos\theta' = r\sin\theta \cot\theta' \\ R = r\sin\theta \operatorname{cosec}\theta' \\ dz' = r\sin\theta \operatorname{cosec}^2\theta' d\theta' \end{cases}$$

So,

$$d\underline{A} = \underline{a_\phi} \frac{\mu_0 M_0 a^2}{4} \cdot \frac{\sin\theta'}{r^2\sin^2\theta \operatorname{cosec}^2\theta'} \cdot r\sin\theta \operatorname{cosec}^2\theta' \, d\theta'$$

$$= \underline{a_\phi} \frac{\mu_0 M_0 a^2}{4 r\sin\theta} \sin\theta' \, d\theta'$$

Hence

$$\underline{A} = \int d\underline{A} = \underline{a_\phi} \cdot \frac{\mu_0 M_0 a^2}{4r\sin\theta} \int_{\theta_1}^{\theta_2} \sin\theta' \, d\theta'$$

$$= \underline{a_\phi} \frac{\mu_0 M_0 a^2}{4r\sin\theta} (\cos\theta_1 - \cos\theta_2)$$

, where $\cos\theta_1 = \dfrac{z + L/2}{R_1}$ , and $\cos\theta_2 = \dfrac{z - L/2}{R_2}$ .

Again we need approximation for $R_1$ and $R_2$ as done in H.W. #4 (Prob. 4-12).

(if $r \gg L$)

$$R_1^{-1} = r^{-1}\left(1 + \frac{L}{r}\cos\theta + \underbrace{\left(\frac{L}{2r}\right)^2}_{\to 0}\right)^{-1/2} \approx r^{-1}\left(1 + \frac{-L}{2r}\cos\theta\right)$$

$$R_2^{-1} = r^{-1}\left(1 + \frac{-L}{r}\cos\theta + \left(\frac{L}{2r}\right)^2\right)^{-1/2} \approx r^{-1}\left(1 + \frac{L}{2r}\cos\theta\right)$$

Then

$$\cos\theta_1 - \cos\theta_2 = (z + L/2)/R_1 - (z - L/2)/R_2$$

$$\simeq \frac{1}{r}\left[(z+L/2)\left(1 - \frac{L}{2r}\cos\theta\right) - (z - L/2)\left(1 + \frac{L}{2r}\cos\theta\right)\right]$$

$$= \frac{1}{r}\left(L - \frac{Lz}{r}\cos\theta\right)$$

$$= \frac{1}{r} L \sin^2\theta \qquad (\because\ z = r\cos\theta)$$

And

$$\boxed{\underline{A} = \underline{a_\phi}\ \frac{\mu_0 M_0 a^2 L}{4r^2} \sin\theta}$$

Finally, $\underline{B}$ field can be obtained by

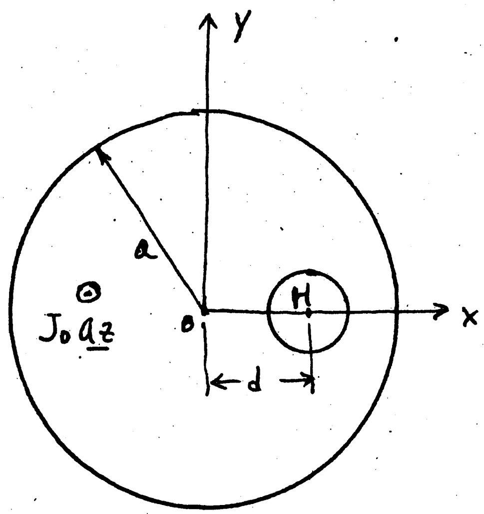
$$\boxed{\underline{B} = \nabla \times \underline{A} = \frac{\mu_0 M_0 a^2 L}{4}\left[\frac{2\cos\theta}{r^3}\underline{a_r} + \frac{\sin\theta}{r^3}\underline{a_\theta}\right]}$$

If we denote M as the total moment of the magnet, then $M = \pi a^2 L M_0$ and we have

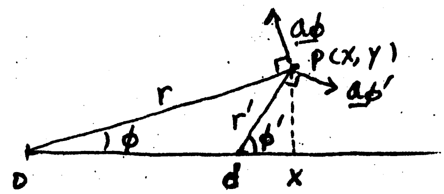
$$\boxed{\underline{B} = \frac{\mu_0}{4\pi} M \left(\underline{a_r}\ \frac{2\cos\theta}{r^3} + \underline{a_\theta}\ \frac{\sin\theta}{r^3}\right)}.$$

External

[3.]




< Fig 1. >




< Fig 2. >

Let $H_+$ be the H-field inside the cylinder without a hole and $H_-$ be the H-field inside a conductor that has same size of diameter as the hole carrying $-J_0 \underline{a_z}$.

Then we can write $H = H_+ + H_-$, where

$$\underline{H_+} = \frac{r}{2} J_0 \underline{a_\phi} = \frac{r}{2} J_0 \left( -\sin\phi \, \underline{a_x} + \cos\phi \, \underline{a_y} \right)$$

$$= \frac{J_0}{2} \left( -y \, \underline{a_x} + x \, \underline{a_y} \right) \quad \text{, and}$$

$$\underline{H_-} = \frac{-r' J_0}{2} \underline{a_\phi'} = \frac{-r' J_0}{2} \left( -\sin\phi' \, \underline{a_x} + \cos\phi' \, \underline{a_y} \right)$$

$$= \frac{J_0}{2} \left( y \, \underline{a_x} - (x-d) \, \underline{a_y} \right)$$

Hence H-field inside the hole now becomes

$$\boxed{\underline{H} = \underline{H_+} + \underline{H_-} = \frac{J_0 d}{2} \underline{a_y}}$$ ← Answer

EEAP 210
HOMEWORK # 6
DUE APRIL 11, 1983

INDUCTANCE, RELUCTANCE, & MAGNETIC CIRCUITS

4-16
4-17
4-26
4-27

FARADAY'S LAW

5-6
* 5-7

DISPLACEMENT CURRENT

5-11

POYNTING VECTOR

5-26
5-34

TIME DEPENDENT FIELDS

6-2
6-6
* 6-27

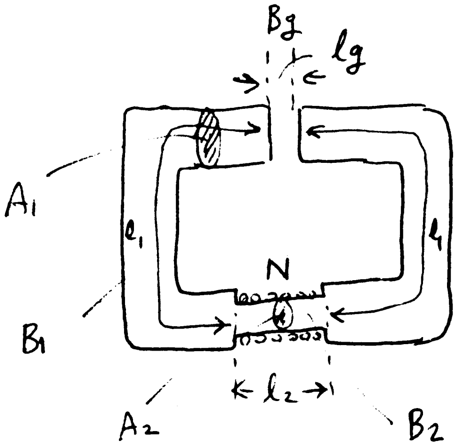
* THINK BEFORE DOING

THIS IS A LOT OF PROBLEMS BUT RECALL THAT HOMEWORK MAY APPEAR ON THE EXAM.

# EEAP 210
# HOMEWORK #6

APR 12, 1983

**(4-16.)**




$N = 100$

$l_1 = 40\ cm$

$l_2 = 10\ cm$

$A_1 = 10\ cm^2$

$A_2 = 5\ cm^2$

$l_g = 2\ mm.$

$B_g = 0.6T$

Total mmf $\quad \mathcal{F} = NI = (R_g + 2R_1 + R_2)\psi_m$

, Here

$$R_g = \frac{l_g}{\mu_0 A_1} = \frac{2\times10^{-3}}{4\pi\times10^{-7}\times10\times10^{-4}} = 1.59155\times10^{6}\ At/w$$

$$\psi_m = B_g \times A_1 = 0.6\times10\times10^{-4} = 6\times10^{-4}\ wb$$

$B_1 = B_g = 0.6\,T$ , From Fig 4-16(b) $\quad H \cong 100\ At/m$

$$\therefore\ \mu_1 = \frac{0.6}{100} = 6\times10^{-3}\ Henry/m$$

$B_2 = 2B_g = 1.2\,T$ . From Fig 4-16(b) $\quad H \cong 375\ At/m$

$$\therefore\ \mu_2 = \frac{1.2}{375} = 3.2\times10^{-3}\ Henry/m.$$

Therefore

$$R_1 = \frac{l_1}{\mu_1 A_1} = \frac{40\times10^{-2}}{6\times10^{-3}\times10\times10^{-4}} \cong 6.67\times10^{4}\ At/wb$$

$$R_2 = \frac{l_2}{\mu_2 A_2} = \frac{10\times10^{-2}}{3.2\times10^{-3}\times5\times10^{-4}} \cong 6.25\times10^{4}\ At/wb$$

$$\therefore\ I = \frac{(R_g + 2R_1 + R_2)\psi_m}{N} = \frac{(1.592\times10^{6} + 1.33\times10^{5} + 6.25\times10^{4})\times6\times10^{-4}}{100} = 10.7$$

$$\boxed{I \approx 10.73\ Ampere}$$

(4-17.)

(a) $$W_{m\,steel} = \frac{B_1^2}{2\mu_1} \times A_1 \times l_1 \times 2 + \frac{B_2^2}{2\mu_2} \times A_2 \times l_2$$

$$\cong \frac{0.6^2}{2 \times 6 \times 10^{-3}} \times 10 \times 10^{-4} \times 40 \times 10^{-2} \times 2 + \frac{1.2^2}{2 \times 3.2 \times 10^{-3}} \times 5 \times 10^{-4} \times 10 \times 10^{-2}$$

$$\cong 0.035 \text{ Joule}$$

$$\therefore \boxed{W_{m\,steel} \approx 0.035 \text{ Joule}}$$

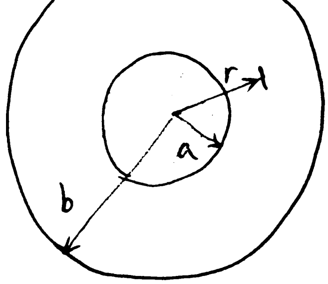
(b) $$W_{m\,air} = \frac{B_g^2}{2\mu_0} \times A_1 \times l_g$$

$$\cong \frac{0.6^2 \times 10 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} \times 2 \times 10^{-3}$$

$$\cong \boxed{0.286 \text{ Joule}}$$

$$\therefore \boxed{W_{m\,air} \cong 0.286 \text{ Joule}}$$

(4-26)




We already knew that

$$H = \begin{cases} \dfrac{Ir}{2\pi a^2}, & 0<r<a \\ \dfrac{I}{2\pi r}, & a<r<b \end{cases}$$

Hence, the total magnetic energy stored in this coaxial conductors system is given by

$$W_{m_{total}} = W_{m_{\substack{inside\\conductor}}} + W_{m_{gap}}$$

$$= \frac{1}{2}\mu_0 \int_0^a \left(\frac{Ir}{2\pi a^2}\right)^2 \cdot 2\pi r\,dr + \frac{1}{2}\mu_0 \int_a^b \left(\frac{I}{2\pi r}\right)^2 2\pi r\,dr$$

$$= \frac{1}{2}\left[\frac{\mu_0 I^2}{8\pi} + \frac{\mu_0 I^2}{2\pi} \ln\left(\frac{b}{a}\right)\right] \quad \cdots ①$$

Also we have to have $W_{m_{total}} = \frac{1}{2} L I^2$ --②

Comparing ① with ②, we get the inductance/unit length

$$\boxed{L = \underbrace{\frac{\mu_0}{8\pi}}_{\substack{\text{Internal Inductance}\\\text{of the center conductor}}} + \underbrace{\frac{\mu_0}{2\pi} \ln \frac{b}{a}}_{\text{External Inductance}}} \leftarrow \text{Answer.}$$

(4-27) Since the relation between $i$ and $\psi_m$ is nonlinear, we cannot use the usual force relations such as $f = \frac{1}{2} I^2 \frac{dL}{dx}$.

Instead we use

$$F = \frac{\partial}{\partial x} W_{mc} \Big|_{i = const} \qquad (W_{mc} : \text{magnetic coenergy})$$

, where $W_{mc} = \int_0^{I(\psi)} \psi_m \, di$

Now

$$\psi_m = \frac{-(x-b)^2 + \sqrt{(x-b)^4 + 4ai}}{2a}$$

$$\therefore \quad W_{mc} = \int_0^{Ni} \frac{-(x-b)^2 + \sqrt{(x-b)^4 + 4ai}}{2a} \, di$$

$$= \frac{-(x-b)^2}{2a} + \frac{1}{12a^2} \left[ \left\{ (x-b)^4 + 4aNi \right\}^{3/2} - (x-b)^6 \right]$$

Therefore

$$F = \frac{\partial}{\partial x} W_{mc} \Big|_{i=const} = -\frac{(x-b)}{a} Ni + \frac{(x-b)^3}{2a^2} \left[ \sqrt{(x-b)^4 + 4aNi} - (x-b)^2 \right]$$

And

$$\boxed{F \Big|_{x=g} = -\frac{(g-b)}{a} Ni + \frac{(g-b)^3}{2a^2} \left[ \sqrt{(g-b)^4 + 4aNi} - (g-b)^2 \right]}$$

(5-6.) $\Psi_m = B_0 L \int_0^t u\,dt$

$$= 0{,}01 \times L \int_0^t 100 \cos 10t \, dt = \frac{L}{10} \sin 10t$$

$$\therefore \boxed{V = -\frac{\partial \Psi_m}{\partial t} = -L \cos 10t}$$

(5-7.) $\Psi_m = BLW \cos 2t$

$$\therefore \quad 0{,}02\, i = -\frac{\partial \Psi_m}{\partial t} = 2BLW \sin 2t$$

Therefore

$$i = \frac{2}{0{,}02} \times 0{,}01 \times 0{,}01 \times 0{,}02 \sin 2t$$

$$= \boxed{0{,}2 \sin 2t \ (mA)}$$

(5-11.) $|E| = \dfrac{50\,V}{2 \times 10^{-3}\,m} = 2{,}5 \times 10^4$ Volt/m

Hence the magnitude of the displacement current is

$$|I_d| = A\omega\epsilon|E| = 10 \times 10^{-4} \times 2\pi \times 10^6 \times 8{,}854 \times 10^{-12} \times 2{,}5 \times 10^4$$

$$\cong 2\pi \times 2{,}214 \times 10^{-4} \text{ Ampere}$$

$$= 2\pi \times 0{,}2214 \ mA$$

$$= 1.39 \ mA$$

(5-26 a) $\underline{\mathcal{E}} = \frac{10}{r}\sin\theta\cos\left(\omega t - \frac{4\pi}{3}r\right)\underline{a_\theta}$ V/m

$\underline{\mathcal{H}} = \frac{10}{120\pi r}\sin\theta\cos\left(\omega t - \frac{4\pi}{3}r\right)\underline{a_\phi}$ A/m

Poynting Vector $\underline{\mathcal{S}}$ is calculated by

$$\boxed{\underline{\mathcal{S}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = \frac{5}{6\pi r^2}\sin^2\theta\cos^2\left(\omega t - \frac{4\pi}{3}r\right)\underline{a_r} \quad \text{W/m}^2}$$

Then the total instantaneous power leaving spherical closed region of radius $R$ is

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} \underline{\mathcal{S}} \cdot d\underline{s}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{5}{6\pi R^2}\sin^2\theta\cos^2\left(\omega t - \frac{4\pi R}{3}\right) R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{20}{9}\cos^2\left(\omega t - \frac{4\pi R}{3}\right) \text{ Watt}$$

Then the average power now becomes

$$\overline{P_{rad}} = \frac{1}{T}\int_0^T P_{rad} = \boxed{\frac{10}{9} \text{ Watt}}$$

$\overline{P_{rad}}$ is independent of the radius of the spherical surface.

(5-34)

$$\begin{cases} \hat{\underline{E}} = -j\dfrac{\omega\mu_0 a}{\pi}\sin\dfrac{\pi x}{a}\, e^{-j\beta z}\, \underline{a_y} \\ \hat{\underline{H}} = j\dfrac{\beta a}{\pi}\sin\dfrac{\pi x}{a}\, e^{-j\beta z}\,\underline{a_x} + \cos\dfrac{\pi x}{a}\, e^{-j\beta z}\,\underline{a_z} \end{cases}$$

$$\hat{\underline{H}}^* = -j\frac{\beta a}{\pi}\sin\frac{\pi x}{a}\, e^{j\beta z}\,\underline{a_x} + \cos\frac{\pi x}{a}\, e^{j\beta z}\,\underline{a_z}$$

So, we now have

$$\hat{\underline{S}} = \hat{\underline{E}} \times \hat{\underline{H}}^* = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ 0 & -j\frac{\omega\mu_0 a}{\pi}\sin\frac{\pi x}{a} e^{-j\beta z} & 0 \\ -j\frac{\beta a}{\pi}\sin\frac{\pi x}{a} e^{+j\beta z} & 0 & \cos\frac{\pi x}{a} e^{j\beta z} \end{vmatrix}$$

$$= \left(-j\frac{\omega\mu_0 a}{\pi}\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\right)\underline{a_x}$$

$$+ \left(\frac{\omega\mu_0\beta a^2}{\pi^2}\sin^2\frac{\pi x}{a}\right)\underline{a_z}$$

$$\therefore \boxed{\hat{\underline{S}} = \left(-j\frac{\omega\mu_0 a}{\pi}\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\right)\underline{a_x} + \left(\frac{\omega\mu_0\beta a^2}{\pi^2}\sin^2\frac{\pi x}{a}\right)\underline{a_z}}$$

(6-20)

$$\begin{cases} \underline{\mathcal{E}_1} = E_{10}\cos(\omega t - \beta z)\,\underline{a_x}, & \beta = \omega\sqrt{\mu\epsilon} \\ \underline{\mathcal{H}_1} = \dfrac{E_{10}}{\eta}\cos(\omega t - \beta z)\,\underline{a_y}, & \eta = \sqrt{\dfrac{\mu}{\epsilon}} \end{cases}$$

(Proof)

① $\nabla \times \underline{\mathcal{E}_1} = \dfrac{\partial \mathcal{E}_x}{\partial z}\underline{a_y} = \beta E_{10}\sin(\omega t - \beta z)\,\underline{a_y}$

$\mu \dfrac{\partial \underline{\mathcal{H}_1}}{\partial t} = -\dfrac{\mu\omega E_{10}}{\eta}\sin(\omega t - \beta z)\,\underline{a_y}$

$\therefore \boxed{\nabla \times \underline{\mathcal{E}_1} = -\mu \dfrac{\partial \underline{\mathcal{H}_1}}{\partial t}}$

② $\nabla \times \underline{\mathcal{H}_1} = -\dfrac{\partial H_y}{\partial z}\underline{a_x} = \dfrac{-\beta E_{0}}{\eta}\sin(\omega t - \beta z)\,\underline{a_x}$

$\epsilon \dfrac{\partial \underline{\mathcal{E}_1}}{\partial t} = -\omega\epsilon E_{10}\sin(\omega t - \beta z)\,\underline{a_x}$

$\therefore \boxed{\nabla \times \underline{\mathcal{H}_1} = \epsilon \dfrac{\partial \underline{\mathcal{E}_1}}{\partial t}}$

③ $\nabla \cdot \underline{\mathcal{E}_1} = \dfrac{\partial \mathcal{E}_x}{\partial x} + \dfrac{\partial \mathcal{E}_y}{\partial y} + \dfrac{\partial \mathcal{E}_z}{\partial z} = 0$

$\therefore \boxed{\nabla \cdot \underline{\mathcal{E}_1} = 0}$

④ $\nabla \cdot \underline{\mathcal{H}_1} = \dfrac{\partial H_x}{\partial x} + \dfrac{\partial H_y}{\partial y} + \dfrac{\partial H_z}{\partial z} = 0$

$\therefore \boxed{\nabla \cdot \underline{\mathcal{H}_1} = 0}$

Therefore the given set of equations satisfy the Maxwell's Equations.

By the same manner you can prove the second set of equations.

(6-60) $\sigma = 2\, S/m$, $\epsilon_r = 9$, $\mu_r = 16$, $f = 10^9\, Hz$

① $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$

$= \sqrt{j \cdot 2\pi \times 10^9 \times 4\pi \times 16 \times 10^{-7} \times (2 + j \times 2\pi \times 10^9 \times 9 \times 8.854 \times 10^{-12})}$

$= 314.012 + j\, 402.312 = 510 \angle 52.03^\circ$

$\therefore\ \alpha = 314.012\ /m$

$\beta = 402.312\ rad/m$

② $\hat{\eta} = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}} = \sqrt{\dfrac{j \times 2\pi \times 10^9 \times 4\pi \times 16 \times 10^{-7}}{2 + j \times 2\pi \times 10^9 \times 9 \times 8.854 \times 10^{-12}}}$

$\cong 247.537 \angle 37.97^\circ$

$= 195.134 + j\, 152.306$

(note) ① $\gamma^2 = \omega\mu\sqrt{\sigma^2 + (\omega\epsilon)^2} \angle -\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right) + \pi$

$\therefore\ \gamma = \sqrt{\omega\mu\sqrt{\sigma^2 + (\omega\epsilon)^2}} \angle \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)$

② $\hat{\eta} = \sqrt{\dfrac{\omega\mu}{\sqrt{\sigma^2 + (\omega\epsilon)^2}}} \angle \frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)$

③ $\measuredangle\gamma + \measuredangle\hat{\eta} = 90^\circ$, $|\gamma| \cdot |\hat{\eta}| = \omega\mu$.

(6-27) $\mu_r = 1$, $\epsilon_r = 79$, $\sigma = 3$ S/m, $E_0 = 1$ V/m

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$$

$\boxed{a) \quad f = 20\text{kHz}}$

$$\gamma = \sqrt{j \times 2\pi \times 20 \times 10^3 \times 4\pi \times 10^{-7} (3 + j \times 2\pi \times 20 \times 10^3 \times 79 \times 8.854 \times 10^{-12})}$$

$$= 0.486686 + j\,0.486701$$

Hence, the attenuation constant $\alpha = 0.486686$

Then, the maximum depth $d$ can be found by

$$E_0 e^{-\alpha d} = 10 \times 10^{-6}$$

, or $\boxed{d = \frac{5}{\alpha} \ln 10 \cong 23.66 \text{ meter}}$

At this frequency we have the ratio of $|I_d / I_c|$;

$$\left|\frac{I_d}{I_c}\right| = \left|\frac{\omega\epsilon}{\sigma}\right| = \frac{2\pi \times 20 \times 10^3 \times 79 \times 8.854 \times 10^{-12}}{3} \cong 1.465 \times 10^{-6} \ll 1$$

Therefore, we can neglect the Displacement current at this frequency.

$\boxed{b) \quad f = 20\text{ GHz}}$

By the same way, we get $\boxed{d \cong 18 \text{ cm}}$

Also, $\left|\frac{I_d}{I_c}\right| \cong 1.465$. Therefore we cannot neglect the conduction current.

– The End –

# EEAP 210
# HOMEWORK #7

4/27/83

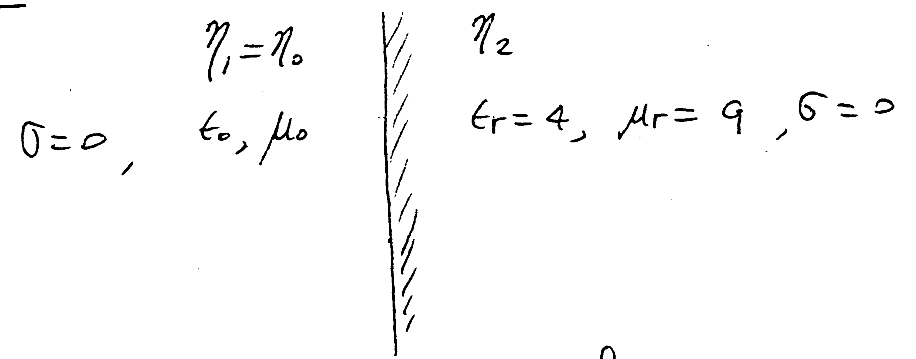
## 6-14.

Total electric field is zero at the distance $d = \frac{n\lambda}{2}$ away from the boundary. Hence, we have $\lambda = 2d/n$ and $f = c/\lambda = nc/2d$ $(n=1,2,\cdots)$.

Therefore $f_{min} = \frac{c}{2d} = \frac{3\times10^8\,\text{m/sec}}{2\times1\ \text{m}} = \boxed{150\ \text{MHz.}}$

## 6-17.

$\omega = 4\pi\times10^8$ rad/sec




For this problems we have

1) $\beta_1 = \omega\sqrt{\mu_0\epsilon_0} = \boxed{\frac{4\pi}{3}\ \text{rad/m}}$, $\beta_2 = \omega\sqrt{9\mu_0\cdot4\epsilon_0} = \boxed{8\pi\ \text{rad/m}}$

2) $\eta_1 = \eta_0 = \boxed{120\pi\,\Omega}$, $\eta_2 = \sqrt{\frac{9\times\mu_0}{4\times\epsilon_0}} = \boxed{180\pi\,\Omega}$

3) $\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \boxed{0.2}$, $T = 1+\Gamma = \boxed{1.2}$

Since $\mathcal{H}_i = \cos(\omega t - \beta_1 y)\,\underline{a_z}$ , we have

$$\hat{\underline{H_i}} = e^{-j\beta_1 y}\,\underline{a_z} \quad \text{and} \quad \hat{\underline{E_i}} = \frac{1}{j\omega\epsilon}\nabla\times\hat{\underline{H_i}} = \boxed{-120\pi\, e^{-j\beta_1 y}\,\underline{a_x}}$$

Then by straightforward calculations we obtain

(← Text Eqs(84) P267)

$$\begin{cases} \underline{\mathcal{E}_i} = -120\pi\cos(\omega t - \beta_1 y)\,\underline{a_x} \\ \underline{\mathcal{H}_i} = \cos(\omega t - \beta_1 y)\,\underline{a_z} \end{cases}$$

··· Incident fields.

$$\begin{cases} \underline{\mathcal{E}_r} = -24\pi\cos(\omega t + \beta_1 y)\,\underline{a_x} \\ \underline{\mathcal{H}_r} = -0.2\cos(\omega t + \beta_1 y)\,\underline{a_z} \end{cases}$$

··· Reflected fields

$$\begin{cases} \underline{\mathcal{E}_t} = -144\pi\cos(\omega t - \beta_2 y)\,\underline{a_x} \\ \underline{\mathcal{H}_t} = 0.8\cos(\omega t - \beta_2 y)\,\underline{a_z} \end{cases}$$

···· Transmitted fields

And the average power crossing $5\,m^2$ area is

$$P_{transmitted} = \frac{1}{2}\cdot\frac{|E_t|^2}{\eta_2}\times 5$$

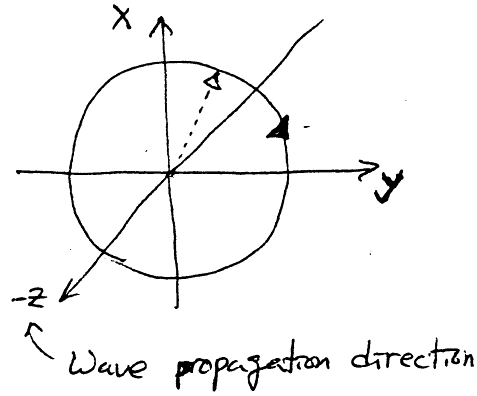
$$= \frac{1}{2}\times\frac{(144\pi)^2}{180\pi}\times 5$$

$$\cong \boxed{904.8\ \text{Watt}}$$

6-24.

(a) $\underline{\mathcal{E}} = \cos(\omega t + \beta z)\underline{a_x} + \sin(\omega t + \beta z)\underline{a_y}$

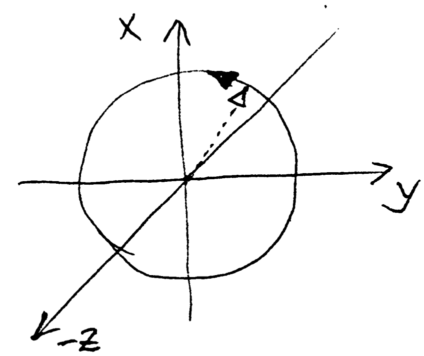
$= \cos(\omega t + \beta z)\,\underline{a_x} + \cos(\omega t + \beta z - 90^\circ)\underline{a_y}$




$\therefore$ Circular polarization with left-hand polarization when looking in the $-z$ direction

(b) $\underline{\mathcal{E}} = \cos(\omega t + \beta z)\underline{a_x} - \sin(\omega t + \beta z)\underline{a_y}$

$= \cos(\omega t + \beta z)\underline{a_x} + \cos(\omega t + \beta z + 90^\circ)\underline{a_y}$

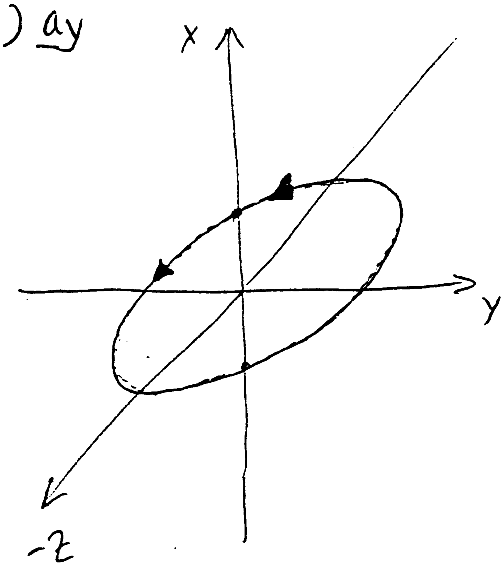



$\therefore$ Circular polarization with right-hand polarization when looking in the $-z$ direction

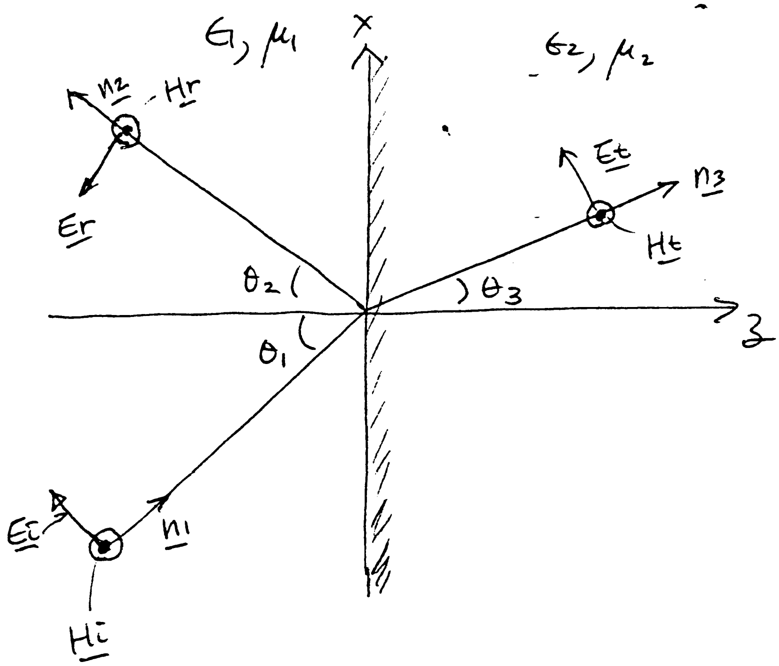
(c) $\underline{\mathcal{E}} = \cos(\omega t + \beta z)\underline{a_x} - 2\sin(\omega t + \beta z - 45^\circ)\underline{a_y}$

$= \cos(\omega t + \beta z)\,\underline{a_x} + 2\cos(\omega t + \beta z + 45^\circ)\,\underline{a_y}$

$\therefore$ Elliptic polarization



6-29. < Parallel polarization >




Let the incident wave be

$$(I)\begin{cases} \underline{E_i} = \underline{E_1}\, e^{-j\beta_1 \underline{n_1}\cdot \underline{r}}, & \underline{H_i} = \dfrac{1}{\eta_1}\underline{n_1}\times \underline{E_1}\, e^{-j\beta_1 \underline{n_1}\cdot\underline{r}} \\ \underline{E_r} = \underline{E_2}\, e^{-j\beta_1 \underline{n_2}\cdot \underline{r}}, & \underline{H_r} = \dfrac{1}{\eta_1}\underline{n_2}\times \underline{E_2}\, e^{-j\beta_1 \underline{n_2}\cdot\underline{r}} \\ \underline{E_t} = \underline{E_3}\, e^{-j\beta_2 \underline{n_3}\cdot \underline{r}}, & \underline{H_t} = \dfrac{1}{\eta_2}\underline{n_3}\times \underline{E_3}\, e^{-j\beta_2 \underline{n_3}\cdot\underline{r}} \end{cases}$$

From the geometry as shown above we have

$$(II)\begin{cases} \underline{n_1} = \underline{a_x}\sin\theta_1 + \underline{a_z}\cos\theta_1 \\ \underline{n_2} = \underline{a_x}\sin\theta_2 - \underline{a_z}\cos\theta_2 \\ \underline{n_3} = \underline{a_x}\sin\theta_3 + \underline{a_z}\cos\theta_3 \\ \underline{r} = x\underline{a_x} + y\underline{a_y} + z\underline{a_z} \end{cases}$$

And we also have;

$$(\mathrm{II})\begin{cases} \underline{E_1} = E_1 \cos\theta_1\, \underline{a_x} - E_1 \sin\theta_1\, \underline{a_z} \\ \underline{E_2} = -E_2 \cos\theta_2\, \underline{a_x} - E_2 \sin\theta_2\, \underline{a_z} \\ \underline{E_3} = E_3 \cos\theta_3\, \underline{a_x} - E_3 \sin\theta_3\, \underline{a_z} \end{cases}$$

Using (II) and (III) we can rewrite the fields as follows

$$(\mathrm{IV\text{-}A})\begin{cases} \underline{E_i} = \left( E_1 \cos\theta_1\, \underline{a_x} - E_1 \sin\theta_1\, \underline{a_z} \right) e^{-j\beta_1 (x\sin\theta_1 + z\cos\theta_1)} \\ \underline{H_i} = \dfrac{E_1}{\eta_1}\, \underline{a_y}\, e^{-j\beta_1 (x\sin\theta_1 + z\cos\theta_1)} \end{cases}$$

$$(\mathrm{IV\text{-}B})\begin{cases} \underline{E_r} = \left( -E_2 \cos\theta_2\, \underline{a_x} - E_2 \sin\theta_2\, \underline{a_z} \right) e^{-j\beta_1 (x\sin\theta_2 - z\cos\theta_2)} \\ \underline{H_r} = \dfrac{E_2}{\eta_1}\, \underline{a_y}\, e^{-j\beta_1 (x\sin\theta_2 - z\cos\theta_2)} \end{cases}$$

$$(\mathrm{IV\text{-}C})\begin{cases} \underline{E_t} = \left( E_3 \cos\theta_3\, \underline{a_x} - E_3 \sin\theta_3\, \underline{a_z} \right) e^{-j\beta_2 (x\sin\theta_3 + z\cos\theta_3)} \\ \underline{H_t} = \dfrac{E_3}{\eta_2}\, \underline{a_y}\, e^{-j\beta_2 (x\sin\theta_3 + z\cos\theta_3)} \end{cases}$$

At the boundary plane ($z=0$), tangential component of $\underline{E}$ and $\underline{H}$ field should be continuous, so

$$(\mathrm{V})\begin{cases} \dfrac{E_1}{\eta_1} + \dfrac{E_2}{\eta_1} = \dfrac{E_3}{\eta_2} \\ E_1 \cos\theta_1 - E_2 \cos\theta_2 = E_3 \cos\theta_3 \end{cases}$$

Also the phase variation should be the same, that is;

$$\beta_1 \sin\theta_1 = \beta_1 \sin\theta_2 = \beta_2 \sin\theta_3$$

or

$$\text{(VI)} \quad \begin{cases} \theta_1 = \theta_2 \\ \dfrac{\sin\theta_1}{\sin\theta_3} = \dfrac{\beta_2}{\beta_1} \end{cases}$$ (← Snell's law of refraction)

Solving (V) and (VI) simultananeously we obtain

$$\Gamma_{\parallel} = \frac{E_2}{E_1} = \frac{\eta_1 \cos\theta_1 - \eta_2 \cos\theta_3}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_3}$$

and $$T_{\parallel} = \frac{E_3}{E_1} = \frac{2\eta_2 \cos\theta_1}{\eta_1 \cos\theta_1 + \eta_2 \cos\theta_3}$$

In order to get a total transmission (or $\Gamma_{\parallel} = 0$)

$$\eta_1 \cos\theta_1 = \eta_2 \cos\theta_3$$

or

$$\eta_1^2 (1 - \sin\theta_1) = \eta_2^2 (1 - \sin^2\theta_3)$$

Eliminating $\sin\theta_3$ by Snell's law (Eq VI above), we end up with

$$\sin^2\theta_1 = \frac{\mu_2\epsilon_1/\mu_1\epsilon_2 - 1}{(\epsilon_1/\epsilon_2)^2 - 1} \quad \therefore \theta_1 = \sin^{-1}\sqrt{\frac{\mu_2\epsilon_1/\mu_1\epsilon_2 - 1}{(\epsilon_1/\epsilon_2)^2 - 1}}$$

Under the normal condition that $\mu_1 \cong \mu_2 = \mu_0$, we get the expression for Brewster angle $\theta_B$, namely

$$\theta_B = \sin^{-1} \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \quad \text{or} \quad \theta_B = \tan^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

7-1.

a) $$u = \frac{1}{\sqrt{\ell c}} = \frac{1}{\sqrt{1\times10^{-6}\times 20\times10^{-12}}} \cong \boxed{2.236\times10^{8}\ \mathrm{m/sec}}$$

$$R_c = \sqrt{\frac{\ell}{c}} = \sqrt{\frac{1\times10^{-6}}{20\times10^{-12}}} = \boxed{223.6\ \Omega}$$

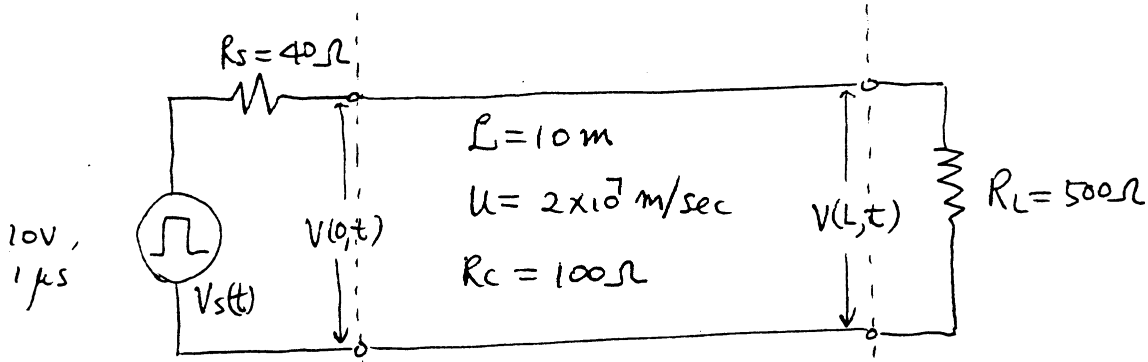
b) $c = 10\ \mathrm{pF/m}$ , $\epsilon_r = 2.25$ , $\mu_r = 1$

$$\therefore\ \ell_e = \frac{\mu\epsilon}{c} = \frac{4\pi\times10^{-7}\times 2.25\times 8.854\times10^{-12}}{10\times10^{-12}} \cong 2.50\ \mu\mathrm{H/m}$$

$$\therefore\ u = \frac{1}{\sqrt{2.50\times10^{-6}\times 10\times10^{-12}}} \cong \boxed{2\times10^{8}\ \mathrm{m/sec}}$$

$$R_c = \sqrt{\frac{25\times10^{-6}}{10\times10^{-12}}} \cong \boxed{500\ \Omega}$$

7.4.

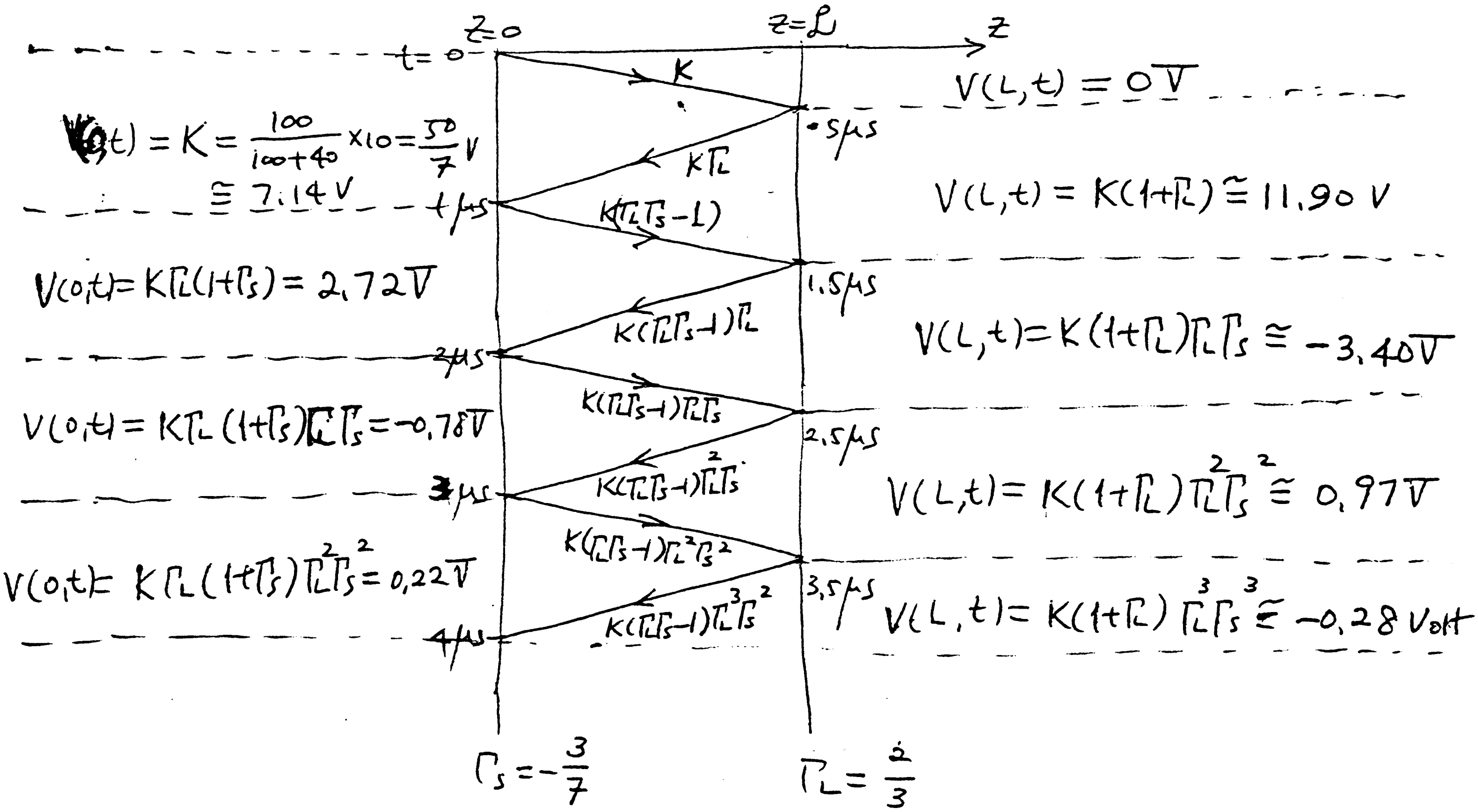



$$\Gamma_s = \frac{40-100}{40+100} = -\frac{3}{7}$$

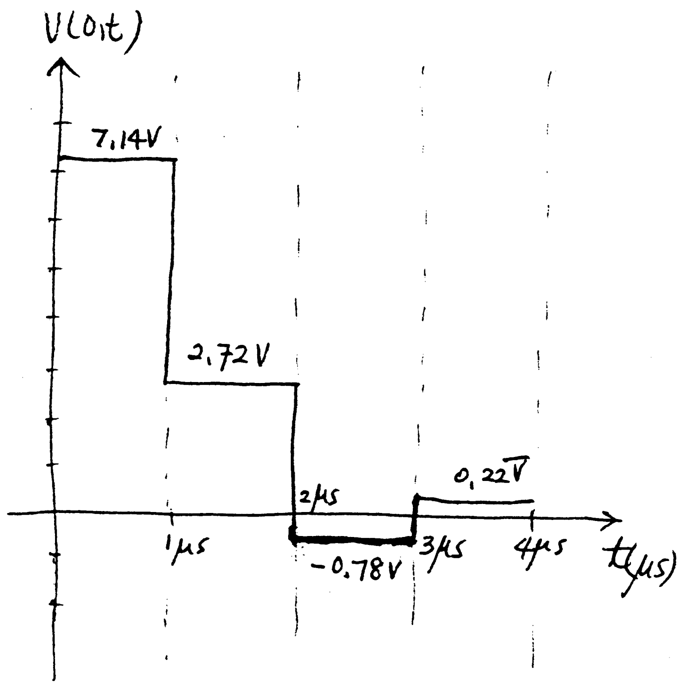
$$\mathcal{L}/u = 0.5\mu\mathrm{sec}$$

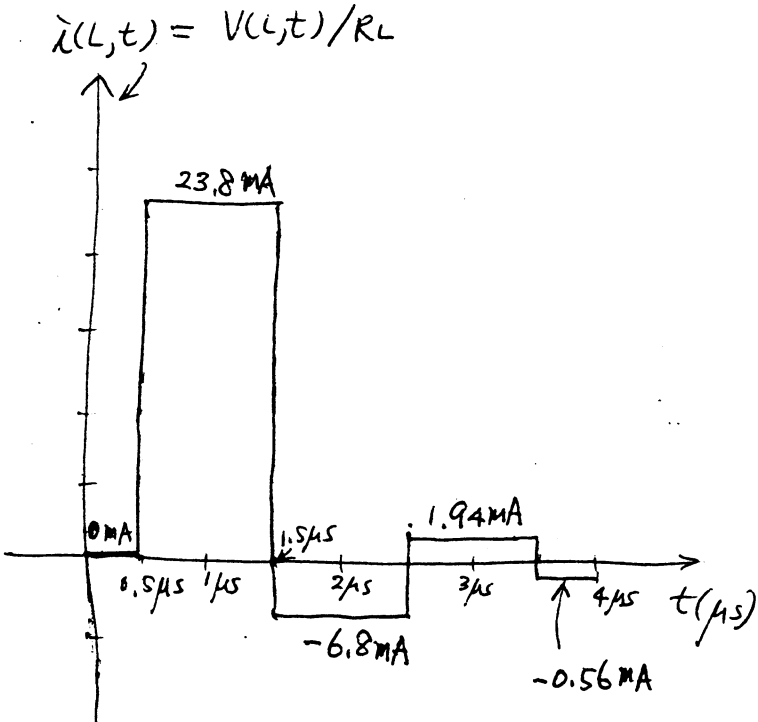
$$\Gamma_L = \frac{500-100}{500+100} = \frac{2}{3}$$

The lattice diagram looks like following.

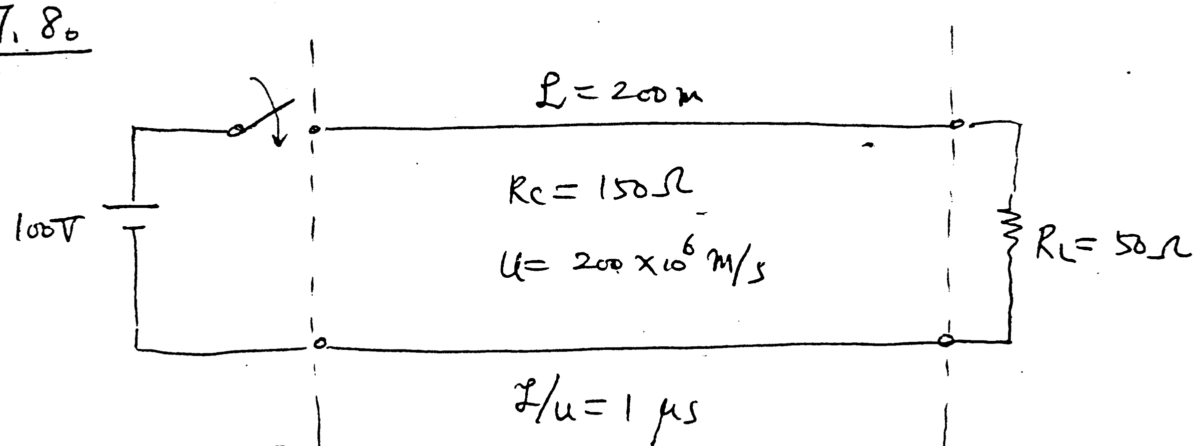



From this diagram, we obtain the required waveforms shown below.



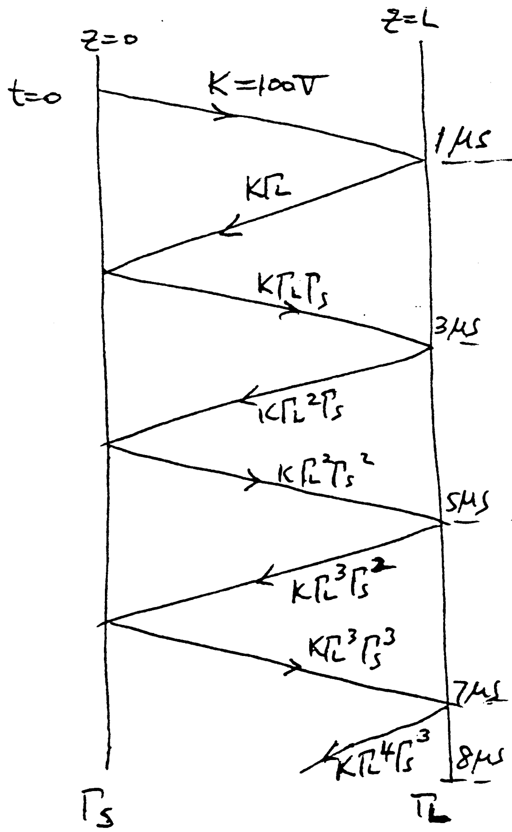



## 7.8.




$\Gamma_s = -1$ $\qquad$ $\Gamma_L = \frac{50-150}{50+150} = -\frac{1}{2}$

Lattice diagram is shown below.




$V(L,t) = 0$ Volt

$V(L,t) = K(1+\Gamma_L) = 50\,V \qquad (n=1)$

$V(L,t) = K(1+\Gamma_L)(1+\Gamma_L\Gamma_s) = 75\,V \qquad (n=2)$

$V(L,t) = K(1+\Gamma_L)(1+\Gamma_L\Gamma_s+(\Gamma_L\Gamma_s)^2) = 87.5 \qquad (n=3)$

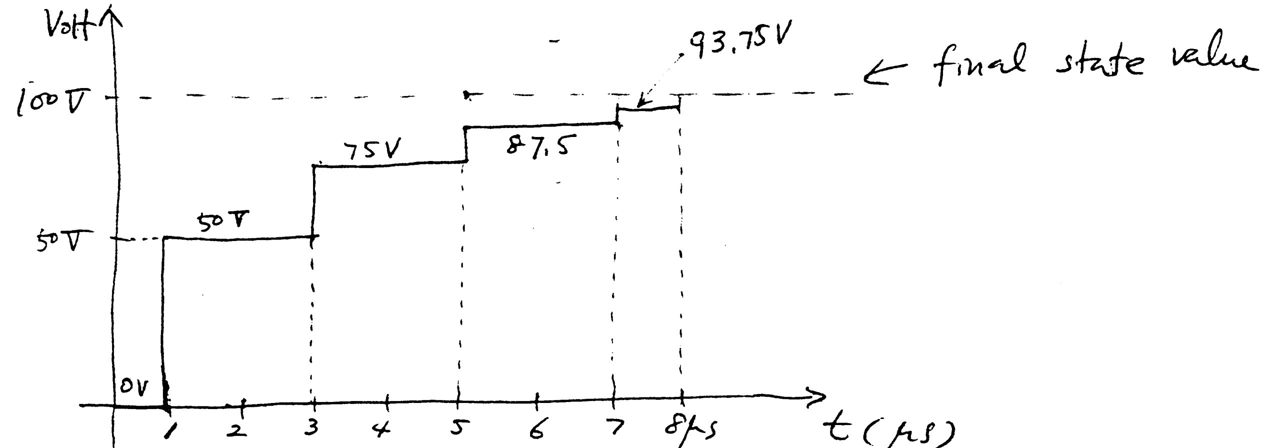
$V(L,t) = K(1+\Gamma_L)(1+\Gamma_L\Gamma_s+(\Gamma_L\Gamma_s)^2+(\Gamma_L\Gamma_s)^3) = 93.75\,V \quad (n=4)$

In general, for $(2n-1)_{\mu s} \le t \le (2n+1)\mu s$, we have

$$V(L,t) = K(1+\Gamma_L) \times \frac{1-(\Gamma_L\Gamma_s)^n}{1-\Gamma_L\Gamma_s} \qquad (n = 1, 2, \cdots)$$

$$= 100\left(1-\left(\tfrac{1}{2}\right)^n\right)$$

Voltage waveform is sketched below




Final value for the $V(L,t) = 100V$ and the time required for $V(L,t)$ to reach 99% of the battery voltage can be found by

$$100\left(1-\left(\frac{1}{2}\right)^n\right) = 99 \text{ (volt)}$$

$$\therefore \quad n = \frac{2}{\log_{10} 2} = 6.64$$

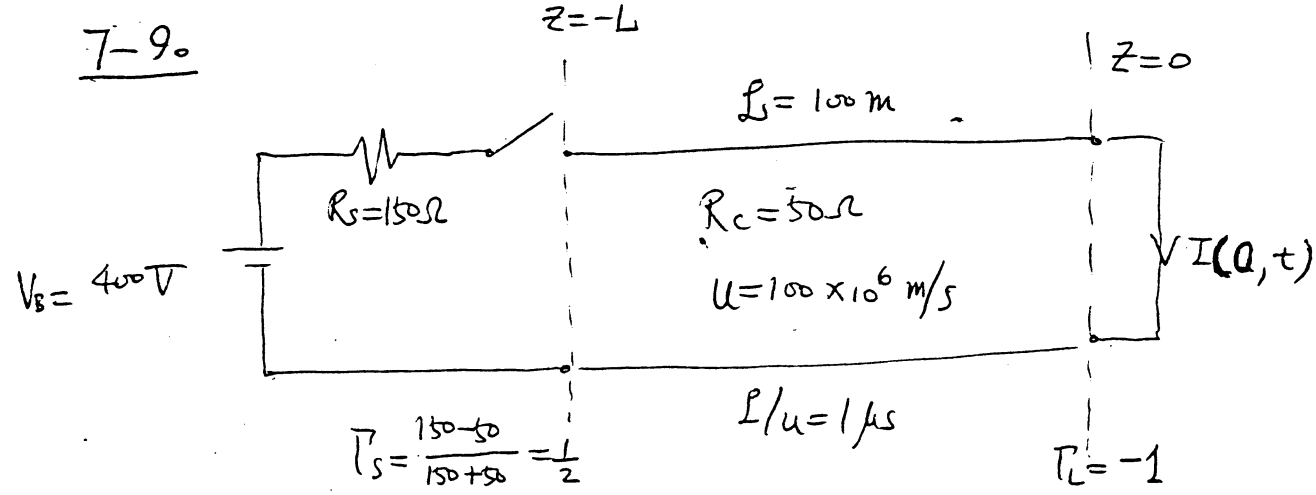
Since $n$ is an integer we have to take

$n = 7$ or $\boxed{t = (2n-1)_{\mu s} = 13\,\mu sec}$ ← Answer

For $11\mu \le t < 13\mu s$, we have $V(L,t) = 100\left(1-\left(\frac{1}{2}\right)^6\right) = \underline{98.44 \text{ Volt}}$

and for $13\mu s \le t < 15\mu s$, we have $V(L,t) = 100\left(1-\left(\frac{1}{2}\right)^7\right) = \underline{99.22 \text{ Volt}}$

## 7-9.



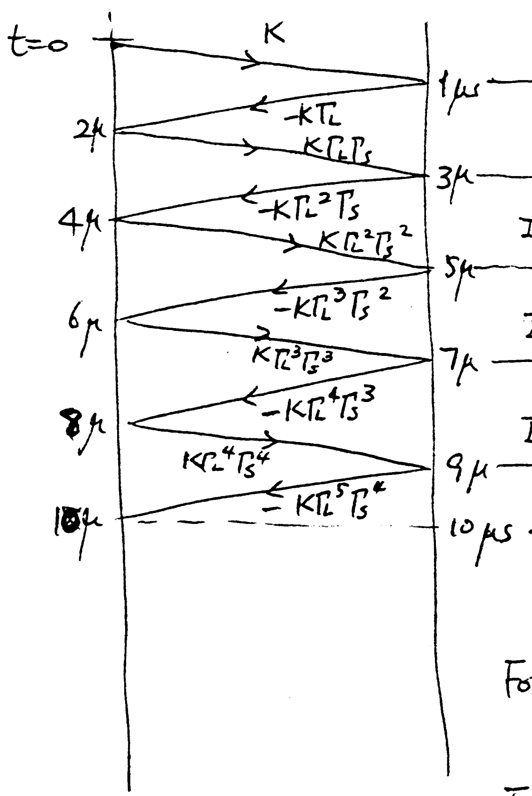

If we take $z=0$ as the load plane (or shorted terminal) then

$$I(0,t) = \frac{1}{R_c} V^+(t) - \frac{1}{R_c} V^-(t) \qquad (\because \text{ text p301, (19b)})$$

$$= \frac{V^+}{R_c}(1-\Gamma_L) = I^+(1-\Gamma_L) \quad \text{in general.}$$

Our lattice diagram is shown below

$$K = \frac{V^+}{R_c} = \left(\frac{1}{R_c}\right) V_B \times \frac{R_c}{R_s + R_c} = \frac{V_B}{R_s+R_c} = \frac{400}{200} = 2A$$




$I(0,t) = 0$

$I(0,t) = K(1-\Gamma_L) = 2K = 4A$

$I(0,t) = K(1-\Gamma_L)(1+\Gamma_L\Gamma_s) = 2A$

$I(0,t) = K(1-\Gamma_L)(1+\Gamma_L\Gamma_s + \Gamma_L^2\Gamma_s^2) = 3A$

$I(0,t) = K(1-\Gamma_L)(1+\Gamma_L\Gamma_s + \Gamma_L^2\Gamma_s^2 + \Gamma_L^3\Gamma_s^3) = 2.5A$

$I(0,t) = K(1-\Gamma_L)(1+\Gamma_L\Gamma_s + \Gamma_L^2\Gamma_s^2 + \Gamma_L^3\Gamma_s^3 + \Gamma_L^4\Gamma_s^4) = 2.75A$

⋮

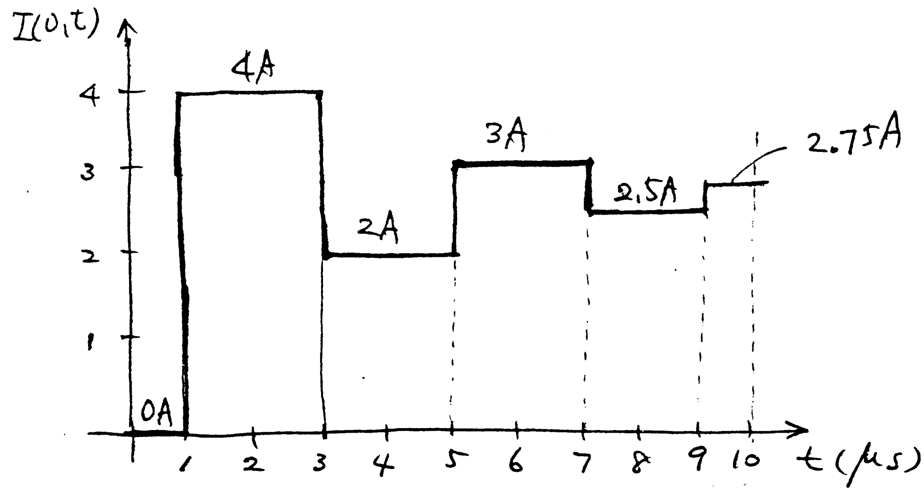
For $(2n-1)\mu s \le t \le (2n+1)\mu s$, $(n = 1, 2, \ldots)$

$$I(0,t) = K(1-\Gamma_L)\frac{1-(\Gamma_L\Gamma_s)^n}{1-\Gamma_L\Gamma_s} = \frac{8}{3}\left(1-\left(-\frac{1}{2}\right)^n\right)A$$

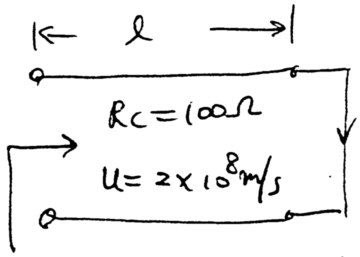
From this expression, we obtain the final value of $I(0,t)$, that is

$$I(0,\infty) = \frac{8}{3}A \quad \left( = \frac{V_B}{R_S} = \frac{400\,V}{150\,\Omega} \right)$$

The sketch for $I(0,t)$ follows.




7-13.




$f = 10\,MHz$

Since $Z_{in} = j\cdot R_c \tan\beta l = j R_c \tan\frac{2\pi f}{u} l = j\cdot 100 \tan\frac{l\pi}{10}\ \Omega$ --①

i) When $Z_{in} = -j\left(\frac{1}{\omega C}\right) = -j\left(\frac{1}{2\pi\times 10^{7}\times 100\times 10^{-12}}\right)$

$$\simeq -j\,159.2\,\Omega \quad \text{--②}$$

From ①=②, $\quad l_{min} = \frac{10}{\pi}\tan^{-1}(-1.592) \simeq \boxed{6.785\,m}$

ii) When $Z_{in} = j\omega L = j\times 2\pi\times 10^{7}\times 10^{-6} = j\cdot 62.83\,\Omega$ --③

From ①=③, $\quad l_{min} = \boxed{1.786\,m}$

## 7-14.

Since VSWR = 4.5, we have

$$|\Gamma_L| = \frac{VSWR-1}{VSWR+1} = \frac{4.5-1}{4.5+1} = \frac{3.5}{5.5} = \frac{7}{11} \quad \text{--} ①$$

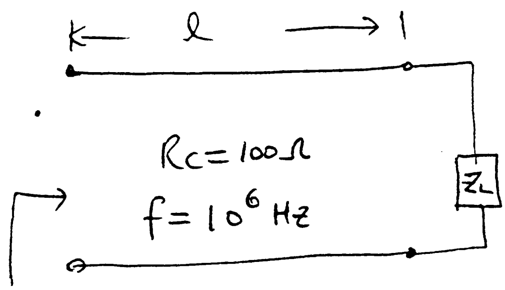
And we know that

$$\Gamma_L = \frac{Z_L - R_C}{Z_L + R_C} = \frac{50+jX-100}{50+jX+100} = \frac{-50+jX}{150+jX} \quad \text{---} ②$$

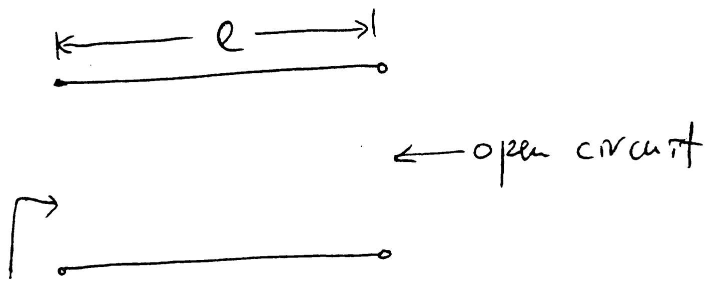
$$\therefore |\Gamma_L| = \sqrt{\frac{(-50)^2+X^2}{150^2+X^2}} \quad \text{----------} ③$$

① = ③ gives $\boxed{X \cong \pm 105.41\ \Omega}$

## 7-20.




①: $Z_{in} = 10 - j50\ \Omega$




②: $Z_{in} = -j75\ \Omega$

From condition ②,

$Z_{in} = -jR_c \cot\beta l = -j75\Omega$, and $\cot\beta l = 0.75$

$\therefore \tan\beta l = 4/3$

Now from condition ①, we have

$$Z_L = R_c \cdot \frac{Z_{in} - jR_c \tan\beta l}{R_c - jZ_{in}\tan\beta l} \quad (\text{where } Z_{in} = 10 - j50\,\Omega)$$

$$= 100 \, \frac{(10-j50) - j100 \cdot \frac{4}{3}}{100 - j(10-j50)\cdot\frac{4}{3}}$$

$$\cong 215.5 + j(-463.8)\ \Omega$$

$$\therefore \boxed{Z_L = 215.5 - j463.8\ \Omega} \leftarrow \text{Answer}$$

The End.