Boundary value problems in cylindrical coordinates

$$\nabla^2 \Phi = 0$$

as we discovered in rectangular coordinates the trick is to let $\Phi = f(r)\, g(\phi)\, h(z)$ in

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

we can substitute Φ into this equation and divide by fgh to get

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{g}\frac{d^2 g}{d\phi^2} + r^2\left(\frac{1}{h}\frac{d^2 h}{dz^2}\right) = 0$$

Note new complications: r^2 on last term.

The second term is the easiest: it must be a function of ϕ only and be a constant

$$\frac{1}{g}\frac{d^2 g}{d\phi^2} = -\nu^2$$

$$\frac{d^2 g}{d\phi^2} + \nu^2 g = 0$$

Solutions are $g = B_1 \sin n\phi + B_2 \cos n\phi$

why: function must be periodic or 2π

ϕ must be single valued

i.e. $\Phi(0) = \Phi(2\pi)$

Now put $-\nu^2$ into (1) and see what happens

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \nu^2 + r^2 \frac{1}{h}\left(\frac{d^2 h}{dz^2}\right) = 0$$

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{\nu^2}{r^2} + \underbrace{\frac{1}{h}\frac{d^2 h}{dz^2}}_{-k_z^2} = 0$$

$\therefore$

$$\frac{1}{rf}\frac{d}{dr}\left(r\frac{df}{dr}\right)-\frac{\nu^2}{r^2}-k_z^{\,2}=0$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)-\left(\frac{\nu^2}{r^2}+k_z^2\right)f=0$$

this is the equation for Bessel functions

$$\frac{1}{h}\frac{d^2h}{dz^2}=-k_z^{\,2}$$

$$\frac{d^2h}{dz^2}+k_z^2h=0$$

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this one has sin or exponential solutions depending on whether k_z is real or imaginary

$$h= c_1\sin k_zz+C_2\cos k_zz$$

or

$$c_1\sinh k_zz+C_2\cosh k_zz$$

this one depends upon the values of k_z and ν

if $k_z=0$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)-\frac{\nu^2}{r^2}f=0$$

general solution is $f=A_0\ln r+A_1$ if $\nu=0$ no ϕ dependence

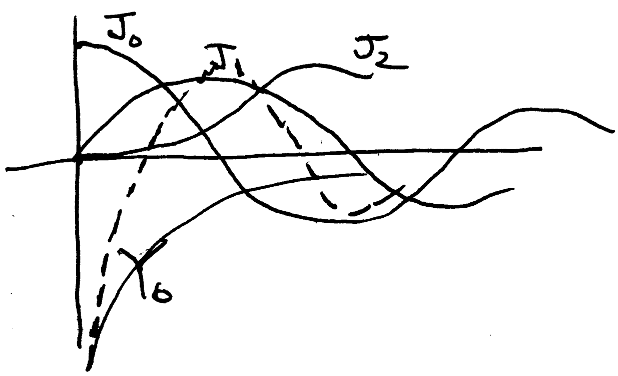
$f=A_0r^n+A_1r^{-n}$ if $\nu\neq0$ ϕ dependence.

if $k_z\neq0$

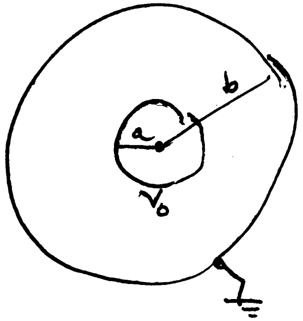
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\Gamma^2-\frac{\nu^2}{r^2}\right)f=0$$

$$k_z=j\Gamma$$

solutions are $\underbrace{J_n(\Gamma r)}_{\text{finite at } r=0}$ $\underbrace{Y_n(\Gamma r)}_{\text{infinite at } r=0}$



Examples:



cross section of a coaxial cable.

no z-dependence $\therefore$ $k = $ constant at most and $k_z^2 = 0$.

$$\Phi = g(\phi)\, f(r)$$

where $g(\phi) = B_1 \sin n\phi + B_2 \cos n\phi$

$f(r) = A_0 \ln r + A_1$ $\quad \nu = 0$ solution,

no ϕ dependence $\Rightarrow$ $\Phi = f(r)$ and match B.C.'s.

$V(b) = 0$

$V(a) = V_0$

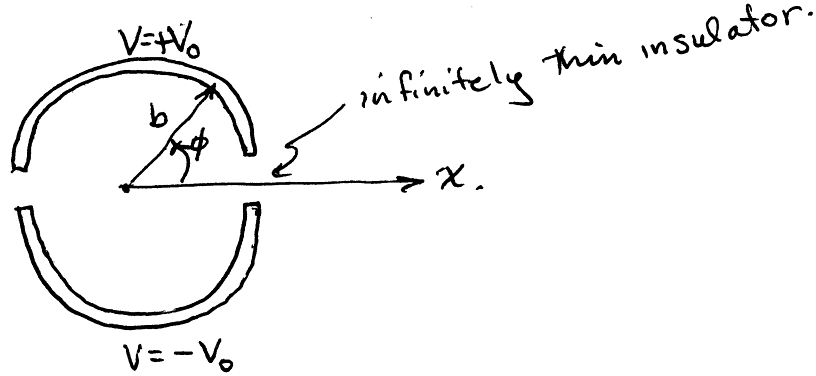
$f(b) = A_0 \ln b + A_1 = 0$

$f(a) = A_0 \ln a + A_1 = V_0$

solving $\quad A_0 = -\dfrac{V_0}{\ln(b/a)} \qquad A_1 = \dfrac{V_0 \ln b}{\ln(b/a)}$

$$\Phi(r) = \frac{-V_0 \ln r + V_0 \ln b}{\ln(b/a)} = \frac{V_0 \ln(b/r)}{\ln(b/a)}$$

more sophisticated problem



obtain inside and outside separately. Again $k_z=0$.

For $r<b$ ϕ dependent

$$\therefore \quad g = B_1 \sin n\phi + B_2 \cos n\phi$$
$$f = A_0 r^n + A_1 r^{-n}$$

write general solution as.

$$\Phi = \sum_{n=1}^{\infty} r^n (B_1 \sin n\phi + B_2 \cos n\phi) + r^{-n}(B_3 \sin n\phi + B_4 \cos n\phi)$$

and examine possible solutions.

obviously $B_3 = B_4 = 0$ since Φ must be finite at $r=0$.

furthermore ϕ is an odd function of ϕ [Fourier series expansion all even coefficients $=0$]

$$\therefore \quad \Phi = \sum_{n=1}^{\infty} r^n B_1 \sin n\phi$$

$$\therefore \quad \sum_{n=1}^{\infty} r^n B_1 \sin n\phi = \begin{cases} V_0 & 0<\phi<\pi \\ -V_0 & \pi<\phi<2\pi \end{cases}$$

to find our coefficients multiply through by $\sin m\phi$ and integrate

$$\int_0^{2\pi} \sum_{n=1}^{\infty} \underbrace{b^n}_{r=b} B \sin n\phi \sin m\phi \, d\phi = \int_0^{2\pi} \Phi^{\sin m\phi} d\phi$$

$$= \int_0^{\pi} V_0 \sin m\phi \, d\phi - \int_{\pi}^{2\pi} V_0 \sin m\phi \, d\phi$$

$$\sum_{n=1}^{\infty} b^n B_n \underbrace{\int_0^{2\pi} \sin n\phi \sin m\phi \, d\phi}$$

$$= V_0 \left[-\frac{1}{m} \cos m\phi \Big|_0^{\pi} + \frac{1}{m} \cos m\phi \Big|_{\pi}^{2\pi} \right]$$

$$\sum_{n=1}^{\infty} b^n B_n \left[\frac{\phi}{2} - \frac{1}{4m} \sin 2m\phi \right]_0^{2\pi} \delta_{mn} = -\frac{V_0}{m}\left[\cos^m \pi - \cancel{\cos m 0}^{1}\right] + \frac{V_0}{m}\left[\cancel{\cos 2\pi^m}^{1} - \cos \ldots\right]$$

$$= \frac{-\cos m\pi + 1 + 1 - \cos m\pi}{m} V_0$$

$$b^m B_n \pi = \frac{2(1-\cos m\pi)}{m} V_0$$

$$\therefore \quad B_m = \frac{2 b^{-m} (1-\cos m\pi)}{m\pi} V_0$$

if $m\pi$ is even $\cos m\pi = +1$ $\quad B_m = 0$

$m\pi$ is odd. $\cos m\pi = -1$ $\quad B_m = \frac{2b^{-m} 2V_0}{m\pi}$

$$= \frac{4b^{-m} V_0}{m\pi}$$

$$\therefore \quad \Phi = \sum_{\substack{n=1 \\ \text{odd only}}}^{\infty} r^n \frac{4b^{-n} V_0}{n\pi} \sin n\phi = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4V_0}{\pi} \frac{1}{n} \left(\frac{r}{b}\right)^n \sin n\phi$$

$$= \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{1}{n} \left(\frac{r}{b}\right)^n \sin n\phi \qquad r<b$$

How about outside?

same reasoning for $\sin n\phi$ term
but exponentials must be r^{-n} so $\Phi \to 0$ as $r \to \infty$

$$\therefore \int_0^{2\pi} \sum_{n=1}^{\infty} b^{-n} B_n \sin n\phi \sin m\phi \, d\phi = \int_0^{2\pi} \Phi \sin m\phi \, d\phi$$

same problem so $B_m = \dfrac{4 b^{m} V_0}{m\pi}$ change in power of b only.

$$\Phi = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ \text{odd only}}}^{\infty} \frac{1}{n} \left(\frac{b}{r}\right)^n \sin n\phi \qquad r > b$$

spherical coordinates : the worst.

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0.$$

assume $\Phi = f(r)\, g(\theta)\, h(\phi)$.

$$\frac{\sin^2\theta}{f}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\sin\theta}{g}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) + \underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial \phi^2}}_{-n^2} = 0$$

$$\frac{\partial^2 h}{\partial \phi^2} + n^2 h = 0 \qquad \therefore h(\phi) = C_1 \cos n\phi + C_2 \sin n\phi$$

$$\frac{\sin^2\theta}{f}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{\sin\theta}{g}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) - n^2 = 0$$

$$\underbrace{\frac{1}{f}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right)}_{r \text{ only}} + \underbrace{\frac{1}{g \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) - \frac{n^2}{\sin^2\theta}}_{\theta \text{ only.}} = 0$$

$+m(m+1)$ $\qquad$ $-m(m+1)$

$$\therefore \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = m(m+1) f \qquad \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial g}{\partial \theta}\right) + \left[m(m+1)\sin\theta - \frac{n^2}{\sin\theta}\right] g = 0$$

solutions are

$$f(r) = B_1 r^m + B_2 r^{-(m+1)}$$

Legendre's equation.

functions are

$P_m^n(\cos\theta)$ and $Q_m^n(\sin\theta)$

unless $m(m+1)$ where $m=0, 1, \ldots$
solutions blow up at $\theta = 0, \pi$
which is bad.
If m is an integer
P_m^0 is always finite $\qquad$ Q_m^0 is infinite, so exclude.

$$P_0^0 = 1$$

$$P_1^0 = \cos\theta$$

$$P_2^0 = \frac{3}{4}\cos 2\theta + \frac{1}{4}$$

$$P_3^0 = \frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right)$$

$$P_1^1 = \sin\theta$$

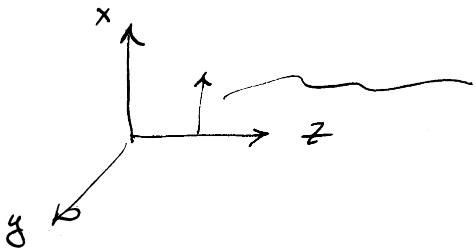
$$P_2^1 = \frac{3}{2}\sin 2\theta$$

$$P_m^n = 0 \quad \text{if } n > m.$$

general solution

$$\Phi(r,\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}\left(A_n\cos n\phi + B_n\sin n\phi\right)\left(C_m r^m + D_m r^{-(m+1)}\right) P_m^n(\cos\theta).$$

6.2 Uniform Plane Waves



let $\hat{\underline{E}} = \hat{E}_x(z)\underline{a_x}$ and see what happens.

From Faraday's Law. $\nabla \times \hat{\underline{E}} = -j\omega\mu\hat{\underline{H}}$

$$\nabla \times \hat{\underline{E}} = \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \hat{E}_x & 0 & 0 \end{vmatrix} = \left(\underline{a_y}\frac{\partial \hat{E}_x}{\partial z}\right) - \left(\underline{a_z}\frac{\partial \hat{E}_x}{\partial y}\right)$$

but $\hat{E}_x$ is not a function of y

$$\nabla \times \hat{\underline{E}} = +\underline{a_y}\frac{\partial \hat{E}_x}{\partial z} = -j\omega\mu\hat{\underline{H}}$$

$$\therefore \hat{\underline{H}} = \hat{H}_y(z)\,\underline{a_z}$$

Note that $\hat{\underline{E}} \cdot \hat{\underline{H}} = 0. \ \forall \underline{z}, \underline{x}, \underline{y},$

Since these satisfied Faraday's Law, they must satisify the wave equation and all of Maxwell's equations

$$\nabla^2\hat{\underline{E}} = \gamma^2\hat{\underline{E}} \qquad \nabla^2\hat{\underline{H}} = \gamma^2\hat{\underline{H}}$$

$$\nabla^2\hat{E}_x = \gamma^2\hat{E}_x \qquad \nabla^2\hat{H}_y = \gamma^2\hat{H}_y$$

what is γ^2? Recall $\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) = \alpha + j\beta$

$$\frac{d^2}{dz^2}\hat{E}_x = \gamma^2\hat{E}_x \qquad \frac{d^2}{dz^2}\hat{H}_y = \gamma^2\hat{H}_y$$

just because the constants are complex is no reason to think these are new equations. These are our old friends with solutions $C_1e^{\gamma z} + C_2e^{-\gamma z}$.

Because I know the results of this analysis call these.

$$\hat{E}_x = \hat{E}_m^+ e^{-\gamma z} + \hat{E}_m^- e^{+\gamma z}$$
$$= \hat{E}_m^+ e^{-\alpha z}e^{-j\beta z} + \hat{E}_m^- e^{+\alpha z}e^{j\beta z}.$$

$$\hat{H}_y = \hat{H}_m^+ e^{-\gamma z} + \hat{H}_m^- e^{+\gamma z}$$

$$\hat{H}_y = \hat{H}_m^+ e^{-\alpha z} e^{-j\beta z} + \hat{H}_m^- e^{+\alpha z} e^{+j\beta z}$$

These results are 2 equations in 4 unknowns. There are actually only two. Use Faraday's law again.

$$\nabla \times \hat{\underline{E}} = -j\omega\mu \hat{\underline{H}}$$

$$\frac{d\hat{E}_x}{dz} = -j\omega\mu \hat{H}_y$$

$$\left[\hat{E}_m^+ (-\gamma) e^{-\alpha z - j\beta z} + \gamma \hat{E}_m^- e^{\alpha z + j\beta z}\right] = -j\omega\mu \left[\hat{H}_m^+ e^{-\alpha z - j\beta z} + \hat{H}_m^- e^{+\alpha z + j\beta z}\right].$$

Equating exponents:

$$\hat{E}_m^+ (-\gamma) = -j\omega\mu H_m^+ \quad \text{and} \quad \gamma \hat{E}_m^- = -j\omega\mu \hat{H}_m^-$$

$$\therefore \frac{\gamma}{j\omega\mu} = \frac{\hat{E}_m^+}{\hat{H}_m^+} \qquad -\frac{\gamma}{j\omega\mu} = \frac{\hat{E}_m^-}{\hat{H}_m^-}$$

define $\hat{\eta} = \frac{\gamma}{j\omega\mu}$.

$$\hat{\eta} = \frac{\hat{E}_m^+}{\hat{H}_m^+} \qquad -\hat{\eta} = \frac{\hat{E}_m^-}{\hat{H}_m^-}$$

$$\hat{E}_x = \hat{E}_m^+ e^{-\alpha z - j\beta z} + \hat{E}_m^- e^{+\alpha z + j\beta z}$$

$$\hat{H}_y = \frac{\hat{E}_m^+}{\hat{\eta}} e^{-\alpha z - j\beta z} \underbrace{- \frac{\hat{E}_m^-}{\hat{\eta}}}_{\text{What's the significance of this sign?}} e^{\alpha z + j\beta z}.$$

$$\hat{E}_m^+ = E_m^+ e^{j\theta_+}$$

re-write
$$\frac{\hat{E}_m^+}{\hat{\eta}} = \frac{E_m^+ e^{j\theta_+}}{\eta\, e^{j\theta_\eta}} = \frac{E_m^+}{\eta} e^{j(\theta_+ - \theta_\eta)}$$

$$\hat{E}_m^- = E_m^- e^{j\theta_-}$$

$$-\frac{\hat{E}_m^-}{\hat{\eta}} = -\frac{E_m^- e^{j\theta_-}}{\eta\, e^{j\theta_\eta}} = -\frac{E_m^-}{\eta} e^{j(\theta_- - \theta_\eta)}$$

Write results:

$$\hat{E}_x = E_m^+ e^{-\alpha z - j\beta z + j\theta_+} + E_m^- e^{\alpha z + j\beta z + j\theta_-}$$

$$\hat{H}_x = \frac{E_m^+}{\eta} e^{-\alpha z - j\beta z + j\theta_+ - j\theta_\eta} - \frac{E_m^-}{\eta} e^{\alpha z + j\beta z + j\theta_- - j\theta_\eta}$$

Time-dependent forms:

Recall $\mathcal{E}_x = \mathcal{R}e\left\{\hat{E}_x e^{j\omega t}\right\} =$

$$\boxed{\begin{aligned} \therefore\quad \mathcal{E}_x &= E_m^+ e^{-\alpha z}\cos(\omega t - \beta z + \theta_+) + E_m^- e^{\alpha z}\cos(\omega t + \beta z + \theta_-) \\ \mathcal{H}_y &= \frac{E_m^+}{\eta} e^{-\alpha z}\cos(\omega t - \beta z + \theta_+ - \theta_\eta) - \frac{E_m^-}{\eta} e^{\alpha z}\cos(\omega t + \beta z + \theta_- - \theta_\eta) \end{aligned}}$$

What do these equation's mean?

γ, η and θ_η contain all the material properties.

For a lossless medium $\sigma = 0$. i.e. no conduction.

$$\gamma^2 = j\omega\mu(\underset{\downarrow 0}{\sigma} + j\omega\epsilon)$$

then $\gamma^2 = -\omega^2\mu\epsilon = \quad \gamma = j\omega\sqrt{\mu\epsilon} = \alpha + j\beta$

$$\therefore\quad \beta = \omega\sqrt{\mu\epsilon} \qquad \alpha = 0$$

what is $\hat{\eta}$ in this case?

$$\hat{\eta} = \frac{\gamma}{j\omega\mu} = \frac{\alpha + j\beta}{j\omega\mu} = \frac{j\omega\sqrt{\mu\epsilon}}{j\omega\mu} = \sqrt{\frac{\epsilon}{\mu}}$$

$\approx 377\,\Omega$ for free space.

This reduces $\mathcal{E}_x$ and $\mathcal{H}_y$ to:

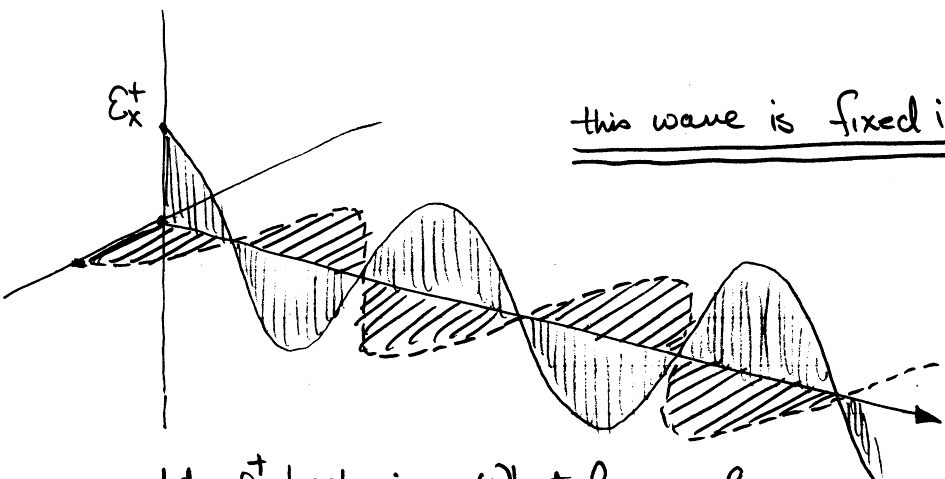
$$\mathcal{E}_x = E_m^+ \cos(\omega t - \beta z + \theta^+) + E_m^- \cos(\omega t + \beta z + \theta^-)$$

$$\mathcal{H}_y = \frac{E_m^+}{\eta} \cos(\omega t - \beta z + \theta^+) - \frac{E_m^-}{\eta} \cos(\omega t + \beta z + \theta^-)$$

What do these results mean?

neglect the θ^+ for the moment and let $t = 0$.

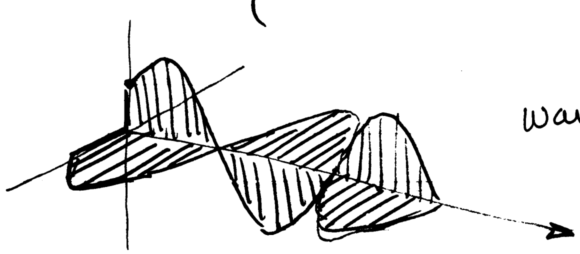
$$\mathcal{E}_x^+ = E_m^+ \cos(\omega t - \beta z) = E_m^+ \cos \beta z$$
$$\mathcal{H}_y^+ = \frac{E_m^+}{\eta} \cos(\omega t - \beta z) = \frac{E_m^+}{\eta} \cos \beta z$$



this wave is fixed in space

now add θ^+ back in: What happens?

$$\mathcal{E}_x^+ = E_m^+ \cos(-\beta z + \theta^+) = E_m^+ \cos(\beta z - \theta^+) \quad \frac{\pi}{4}$$
$$\mathcal{H}_y^+ = \frac{E_m^+}{\eta} \cos(-\beta z + \theta^+) = E_m^+ \cos(\beta z - \theta^+) \quad \frac{\pi}{4}$$



wave is shifted forward by θ^+

As I add the time dependence back in, ωt is of the same sign as θ^+ and I see the wave being shifted forward in space.

Now, let us interpret some of our parameters.

① By inspection, for $t=0$, β is the spatial wavelength. We define $\beta\lambda = 2\pi$ i.e. λ is the spatial distance over which the waveform repeats, and β is the spatial frequency. By simple math,

$$\beta = \frac{2\pi}{\lambda}$$

② Consider $\omega t - \beta z + \theta^+$. As t increases, any particular point on the wave moves forward. For purposes of discussion $\varphi \triangleq \omega t - \beta z + \theta^+$ and is called the wave phase function. $\varphi =$ constant defines a point of constant phase on the wave and can be seen to be the forward movement of the wave. As all parts of the wave move forward with a constant velocity, pick the constant to be zero for simplicity. Then,

$$\varphi = \omega t - \beta z + \theta^+ = 0$$

$$\beta z = \omega t + \theta^+$$

or $$z = \frac{\omega}{\beta} t + \frac{\theta^+}{\beta}$$

This represents the spatial movement of a point of constant phase with time. In general, this is called the **phase velocity**

$$v_\varphi \triangleq \frac{dz}{dt} = \frac{\omega}{\beta} \quad \text{from above.}$$

③ If we returned to our phase function for the other component of $\mathcal{E}_x$ and $\mathcal{H}_y$ we get.

$$\phi = \omega t + \beta z + \theta^-$$

and, as before,

$$\omega t + \beta z + \theta^- = 0$$

$$\beta z = -(\omega t + \theta^-)$$

$$z = -\frac{\omega}{\beta}t - \frac{\theta^-}{\beta}$$

$$v_\phi = \frac{dz}{dt} = -\frac{\omega}{\beta}$$

$\therefore$ this corresponds to a wave traveling in the $-z$ direction.

④ What happens to u_g, etc. if not free space?

formula	lossless $\sigma = 0$	lossy
$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$	$\gamma^2 = -\omega^2\mu\epsilon$	
$\gamma = \alpha + j\beta$	$\gamma = j\omega\sqrt{\mu\epsilon} = j\beta$	$\alpha + j\beta$ where α gives rise to attenuating terms
$\hat{\eta} = \frac{\alpha + j\beta}{j\omega\mu}$	$\hat{\eta} = \frac{j\omega\sqrt{\mu\epsilon}}{j\omega\mu} = \sqrt{\frac{\epsilon}{\mu}} = 377\Omega$	
β, λ	$\beta = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$	β is more complex but has the same interpretation. usually β is larger. $\phi = \omega t - \beta z + \theta^+$ as before so $v_\phi = \frac{\omega}{\beta}$ decreases.
$v_\phi = \frac{dz}{dt}$	$\frac{\omega}{\beta}$	

and if we interpret v_ϕ in terms of the speed of light, it slows down.

(5) How about the θ_η that has been neglected?

recall $\hat{\eta} = \frac{\alpha + j\beta}{j\omega\mu}$ for lossy media $\theta_\eta \neq 0$. in general.

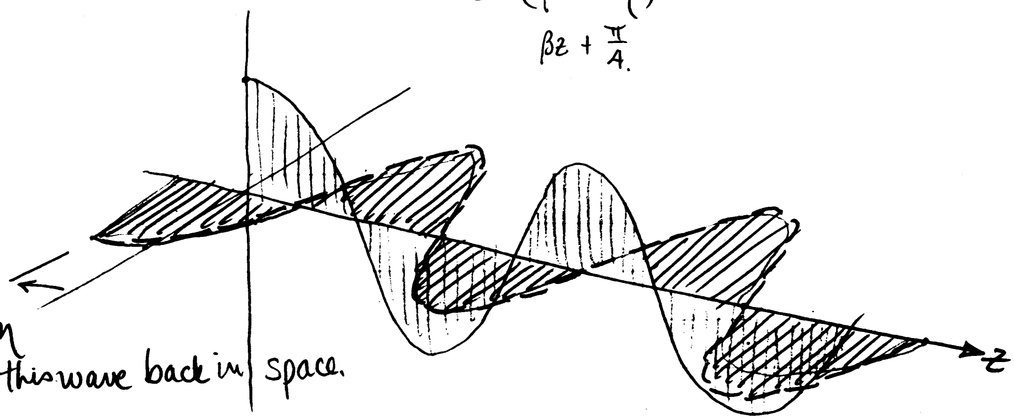
$$\mathcal{E}_x^+ = E_m^+ e^{-\alpha z} \cos(\omega t - \beta z + \theta^+)$$

$$\mathcal{H}_y^+ = \frac{E_m^+}{\eta} e^{-\alpha z} \cos(\underbrace{\omega t - \beta z + \theta^+}_{\phi - \theta_\eta} - \theta_\eta)$$

real

$\cos(\beta z + \theta_\eta)$

$\beta z + \frac{\pi}{4}$.



<u>Now</u>, $\mathcal{E}$ and $\mathcal{H}$ are <u>NOT</u> in phase and the power decreases since $\hat{S} = \frac{1}{2} \hat{E} \times \hat{H}^*$

$$\hat{S}_m^+ = \frac{1}{2} E_m^+ e^{-\alpha z} e^{j\phi} \frac{E_m^{+*}}{\eta} e^{-\alpha z} e^{-(j\phi - j\theta_\eta)}$$

$$= \frac{1}{2} \frac{|\hat{E}_m|^2}{\eta} e^{-2\alpha z} \underbrace{e^{j\theta_\eta}}_{(\cos\theta_\eta + j\sin\theta_\eta)}$$

represents reactive power. ....

picked $d\underline{S}$ <u>INTO</u> <u>volume</u>

$$\oint_S \hat{\underline{E}} \times \hat{\underline{H}}^* \cdot d\underline{S} = -j\omega \int_V (\epsilon^* \hat{\underline{E}} \cdot \hat{\underline{E}}^* - \mu \hat{\underline{H}} \cdot \hat{\underline{H}}^*) \, dV + \int_V \sigma \hat{\underline{E}} \cdot \hat{\underline{E}}^* dV$$

time averaged → $U_e = \frac{1}{4} \mathrm{Re}\, \epsilon \hat{\underline{E}} \cdot \hat{\underline{E}}^*$ $\quad U_m = \frac{1}{4} \mathrm{Re}\, \mu \hat{\underline{H}} \cdot \hat{\underline{H}}^*$

electric polarization damping losses, etc. $\qquad$ conduction current losses

<u>(we will do this later)</u>

(6) phase velocity = velocity of a phase front and is frequency dependent
group velocity is the **average** phase velocity

Example: $D(t) = \cos(\underbrace{\omega_1 t - \beta_1 z}_{x}) + \cos(\underbrace{\omega_2 t - \beta_2 z}_{y})$

use $\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$

$$D(t) = 2\cos\underbrace{\left(\frac{\omega_1+\omega_2}{2}t - \frac{\beta_1+\beta_2}{2}z\right)}_{\text{this is a very high frequency.}}\cos\left(\underbrace{\frac{\omega_1-\omega_2}{2}}_{\text{this is a very low frequency modulation}}t - \frac{\beta_1-\beta_2}{2}z\right)$$

this is the frequeny at which information propagates.

$$\phi_{mod} = \frac{\omega_1-\omega_2}{2}t - \frac{\beta_1-\beta_2}{2}z$$

for constant ϕ_{mod} $\quad \frac{\omega_1-\omega_2}{2}t - \frac{\beta_1-\beta_2}{2}z = 0$

$$\frac{\beta_1-\beta_2}{\cancel{2}}z = \frac{\omega_1-\omega_2}{\cancel{2}}t$$

$$z = \frac{\omega_1-\omega_2}{\beta_1-\beta_2}t$$

$$v_g \triangleq \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{d\omega}{d\beta}$$

This is subtly different than the phase velocity....

Example: vacuum where $\quad \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$.

or $\frac{\omega}{c} = \beta$

$$\omega = \beta c$$

$$v_\phi = \frac{\omega}{\beta} = \frac{\beta c}{\beta} = c \qquad v_g = \frac{d\omega}{d\beta} = c$$

Nothing strange.

Example: vacuum

$$\beta = \frac{\omega}{c}$$

then $v_\phi = \dfrac{\omega}{\beta} = c$

$$v_g = \frac{d\omega}{d\beta} = c \quad \text{(also)}$$

Example: ionosphere

$$\omega^2 = \omega_p^2 + \beta^2 c^2$$

$f_p \approx 20\,\text{mHz}$
plasma reflection
frequency of ionosphere.

$$v_\phi = \frac{\omega}{\beta} = \frac{\sqrt{\omega_p^2 + \beta^2 c^2}}{\beta} = \sqrt{\frac{\omega_p^2}{\beta^2} + c^2} \geq c$$

$v_g = \dfrac{d\omega}{d\beta}$ go to original

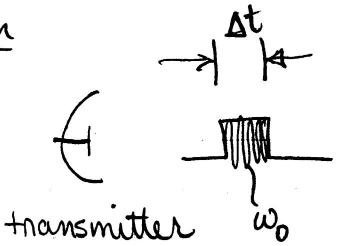
$$\cancel{2}\omega\, d\omega = 0 + \cancel{2}\beta\, d\beta\, c^2$$

$$\frac{d\omega}{d\beta} = \frac{\beta c^2}{\omega}$$

Note that $v_\phi v_g = \dfrac{\omega}{\beta} \cdot \dfrac{\beta c^2}{\omega} = c^2$ always true

?

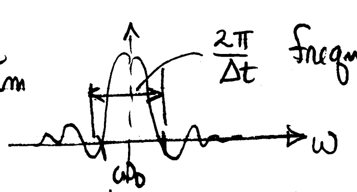
Problem



frequency center propagates with group velocity v_g.

$$v_g = c^2 \frac{\beta_o}{\omega_o}$$

Fourier transform



contains many frequencies

obviously, if the individual components spread too much the signal will not be recoverable. A general rule of thumb for this limit is $\Delta t \approx \frac{2\pi}{\Delta\omega}$ this is the time for all phase information to become lost.

$$\text{i.e.}\quad \underbrace{(\omega_2 - \omega_1)}_{\text{frequency components}}\ \underset{\uparrow\ \text{length of pulse}}{t_1} < 2\pi$$

$$\Delta\omega\, \Delta t < 2\pi$$

$\therefore$ if $\Delta t = 1\times10^{-9}$ seconds

$$\Delta\omega < \frac{2\pi}{\Delta t} = 6.28\times10^{9}$$

what carrier frequency must I use.

$$v_g = \frac{\Delta\omega}{\Delta\beta} \quad \text{or} \quad \Delta\omega = v_g \Delta\beta$$

$$v_g \Delta\beta = 6.28\times10^{9}$$

amplitude modulating my signal

3cm

at 3cm.

$$c\sqrt{1+\frac{4\pi^2(20\times10^6)^2}{\left(3\times10^{10}\frac{cm}{sec}\right)^2\left(\frac{2\pi}{3cm}\right)^2}}$$

$$c\sqrt{1+\frac{\cancel{4\pi^2}\;400\times10^{12}}{\cancel{4\pi^2}\;\frac{9\times10^{20}}{\cancel{9}}}}$$

$$c\sqrt{1+400\times10^{12}\times10^{-20}}$$

$$c\sqrt{1+400\times10^{-8}}$$

$$c\sqrt{1+4\times10^{-6}}$$

$$\approx c\left(1+\tfrac{1}{2}4\times10^{-6}\right)$$

$$\approx c\left[\left(1+2\times10^{-6}\right)\right]$$

at $3+\Delta\lambda$

$$c\sqrt{1+\frac{4\pi^2(20\times10^6)^2}{(3\times10^{10})^2\frac{4\pi^2}{(3+\Delta\lambda)^2}}}$$

$$c\sqrt{1+\frac{400\times10^{12}}{9\times10^{20}}(3+\Delta\lambda)^2}$$

$$c\sqrt{1+\frac{4}{9}\times10^{14}\times10^{-20}(3+\Delta\lambda)^2}$$

$$c\sqrt{1+\frac{4}{9}\times10^{-6}(3+\Delta\lambda)^2}$$

$$\approx c\left[1+\frac{2}{9}\times10^{-6}(3+\Delta\lambda)^2\right]$$

$$\approx c\left[1+2\times10^{-6}\left(\frac{3+\Delta\lambda}{3}\right)^2\right]$$

$$= c\left[1+2\times10^{-6}\left(1+\frac{\Delta\lambda}{3}\right)^2\right]$$

this wave propagates at a slightly faster velocity than this one.

the difference in velocities is about

$$(3\times10^{10})(2\times10^{-6})\left[\left(1+\frac{\Delta\lambda}{3}\right)^2-1\right]\cong\left[6\times10^4\right]\frac{cm}{sec}\left[\cancel{1}+\frac{2\Delta\lambda}{3}+\frac{\Delta\lambda^2}{9}\cancel{-1}\right]$$

if $\Delta\lambda = 0.3$cm. $\frac{\Delta\lambda}{3}=\frac{0.3}{3}=\frac{1}{10}$

this is Δv_ϕ. $\Delta v \approx (6\times10^4)\frac{cm}{sec}\;(0.2) = 1.2\times10^4\frac{cm}{sec}.$

Conductors and dielectrics re-visited

$$\nabla \times \underline{\hat{H}} = j\omega\epsilon \underline{\hat{E}} + \sigma \underline{\hat{E}}$$

this term is due to polarization damping, i.e. it takes a while for $\underline{P}$ to catch up with $\underline{E}$

$$= j\omega(\epsilon' - j\epsilon'') \underline{\hat{E}} + \sigma \underline{\hat{E}}$$

$$= j\omega\epsilon' \underline{\hat{E}} + (\omega\epsilon'' + \sigma) \underline{\hat{E}}$$

$$= j\omega \left[\epsilon' + \frac{\omega\epsilon'' + \sigma}{j\omega} \right] \hat{E}$$

$$= j\omega \underbrace{\left[\epsilon' - j\left(\epsilon'' + \frac{\sigma}{\omega}\right) \right]}_{\hat{\epsilon}} \hat{E}$$

this is the displacement current.

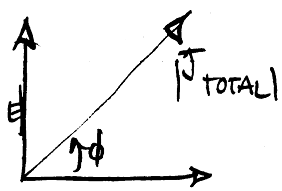
this part is due to a conduction current

the ϵ'' is due to movement of the dipole moments creating currents.

same thing for μ

$$\nabla \times \underline{\hat{E}} = -j\omega\mu \underline{\hat{H}} = -j\omega(\mu' - j\mu'')\hat{H}$$

the ratio of $\frac{J_c}{J_d}$ comes in $|J_d| = |\omega\epsilon' E|$



$|J_c| = |(\sigma + \omega\epsilon'') E|$

the only thing ϕ is good for is to determine whether the media is a conductor or dielectric

$$\tan\phi = \frac{\sigma + \omega\epsilon''}{\omega\epsilon'} \begin{array}{l} \leftarrow \text{effective conductivity} \\ \leftarrow \text{effective permittivity.} \end{array}$$

in general all are functions of frequency.

re-look at Poynting vectr.

total energy $W_e = \frac{1}{2}\int \epsilon |E|^2 dv.$

energy density $W_e = \int w_e\, dv$ where $w_e = \frac{1}{2}\epsilon |E|^2.$

How about for a time dependent field?

let $E = E_0 \cos\omega t$

$$w_e = \frac{1}{2}\epsilon\, E_0\cos\omega t\, E_0\cos\omega t = \frac{1}{2}\epsilon E_0^2 \cos^2\omega t$$

$$= \frac{1}{2}\epsilon E_0^2 \frac{1}{2}\left[1+\cos 2\omega t\right]$$

and if I time average $w_e = \frac{1}{4}\epsilon E_0^2$

Now, look at Poyntings vector

$$\nabla\times\underline{\hat{E}} = -j\omega\mu\underline{\hat{H}} \qquad \nabla\times\underline{\hat{H}}^* = -j\omega\epsilon^*\underline{\hat{E}}^* + \sigma\hat{E}^*$$

$$\underline{\nabla}\cdot(\underline{\hat{E}}\times\underline{\hat{H}}^*) = \underline{\hat{H}}^*\cdot\underline{\nabla}\times\underline{\hat{E}} - \underline{\hat{E}}\cdot\underline{\nabla}\times\underline{H}^*$$

$$\underline{\nabla}\cdot\frac{1}{2}(\underline{\hat{E}}\times\underline{\hat{H}}^*) = \frac{1}{2}\underline{\hat{H}}^*\cdot(-j\omega\mu\underline{\hat{H}}) - \frac{1}{2}\underline{\hat{E}}\cdot(-j\omega\epsilon^*\underline{\hat{E}}^* + \sigma\underline{\hat{E}}^*)$$

now integrate over a volume v.

$$\int\nabla\cdot\Big[\underbrace{\frac{1}{2}\hat{E}\times\underline{\hat{H}}^*}_{=\underline{S}}\Big]dv = -j\omega\int\frac{1}{2}\mu|\underline{\hat{H}}|^2 dv + j\omega\int\frac{1}{2}\epsilon^*|\underline{\hat{E}}|^2 dv - \int\frac{1}{2}\sigma|\underline{\hat{E}}|^2 dv.$$

$\Downarrow$

$\oint \underline{S}\cdot d\underline{s}$

$$\epsilon = \epsilon' - j\epsilon'' \qquad \mu = \mu' - j\mu''$$

$$\mathrm{Re}\oint\underline{S}\cdot d\underline{s} = \frac{\omega}{2}\underbrace{\int\left[\epsilon''|E|^2 - \mu|H|^2\right]dv}_{\text{polarization damping terms.}} - \underbrace{\frac{1}{2}\int\sigma|E|^2 dv.}_{\text{conduction losses.}}$$

conduction terms

$$\mathrm{Im}\oint\underline{S}\cdot d\underline{s} = \omega\int\left[\epsilon'|E|^2 - \mu'|H|^2\right]dv$$

displacement terms.

energy stored in the fields

A quicker and easier way to get our result is to use the defin. of $\hat{S}$

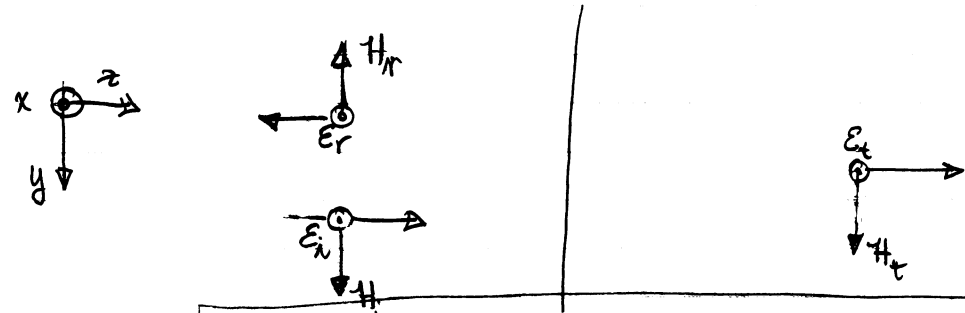
$$\underline{\hat{S}} = \frac{1}{2} \underline{\hat{E}} \times \underline{\hat{H}}^{*}$$

but for plane waves. $\hat{E} = \frac{\gamma}{j\omega\mu} H = \hat{\eta} \hat{H}$

$$\underline{\hat{S}} = \frac{1}{2} \underline{E} \times \left(\frac{\underline{\hat{E}}^{*}}{\hat{\eta}^{*}} \underline{a}_H \right) = \frac{1}{2} \frac{|\hat{E}|^2}{\eta^{*}} \underline{a}_n$$

where $\underline{a}_n$ = direction $\underline{\hat{E}} \times \underline{\hat{H}}$

incident normal



$$\hat{\underline{E}}_i = \hat{E}_i e^{-\gamma_1 z} \underline{a_x} \qquad \hat{\underline{E}}_t = \hat{E}_t e^{-\gamma_2 z} \underline{a_x}$$

$$\hat{\underline{E}}_r = \hat{E}_r e^{+\gamma_1 z} \underline{a_x}$$

$$\hat{\underline{H}}_i = \frac{\hat{E}_i}{\hat{\eta}_1} e^{-\gamma_1 z} \underline{a_y} \qquad \hat{\underline{H}}_t = \frac{\hat{E}_t}{\hat{\eta}_2} e^{-\gamma_2 z} \underline{a_y}$$

$$\hat{\underline{H}}_r = -\frac{\hat{E}_r}{\hat{\eta}_1} e^{+\gamma_1 z} \underline{a_y}$$

direction

recall $\hat{\eta} = \eta e^{j\theta_\eta}$

$\gamma_1 = \alpha_1 + \beta_1$

$\gamma_2 = \alpha_2 + \beta_2$

$$\hat{\underline{E}}_i = \hat{E}_i e^{-\alpha_1 z - j\beta_1 z} \underline{a_x}$$

$$\hat{\underline{H}}_i = \frac{\hat{E}_i}{\eta} e^{-\alpha_1 z - j\beta_1 z - j\theta_{\eta_1}} \underline{a_y}$$

$$\hat{\underline{E}}_t = \hat{E}_t e^{-\alpha_2 z - j\beta_2 z} \underline{a_x}$$

$$\hat{\underline{H}}_t = \frac{\hat{E}_t}{\eta} e^{-\alpha_2 z - j\beta_2 z - j\theta_{\eta_2}} \underline{a_y}$$

$$\hat{\underline{E}}_r = \hat{E}_r e^{\alpha_1 z + j\beta_1 z}$$

$$\hat{\underline{H}}_r = -\frac{\hat{E}_r}{\eta} e^{+\alpha_1 z + j\beta_1 z - j\theta_{\eta_1}} \underline{a_y}$$

because of direction.

what are the boundary conditions?

tangential $\vec{E}$ and H <u>must</u> be continuous!

$\downarrow$

no surface current.

$\therefore$ @ $z=0$

$$\hat{E}_i + \hat{E}_r = \hat{E}_t \quad \text{and} \quad \frac{\hat{E}_i}{\eta_1} e^{-j\theta_{\eta_1}} - \frac{\hat{E}_r}{\eta_1} e^{-j\theta_{\eta_1}} = \frac{\hat{E}_t}{\eta_2} e^{-j\theta_{\eta_2}}$$

$$\text{or} \quad \frac{\hat{E}_i}{\hat{\eta}_1} - \frac{\hat{E}_r}{\hat{\eta}_1} = \frac{\hat{E}_t}{\hat{\eta}_2}$$

$$\hat{E}_i + \hat{E}_r = \hat{E}_t \qquad \hat{E}_i - \hat{E}_r = \frac{\hat{\eta}_1}{\hat{\eta}_2} \hat{E}_t$$

add.

$$2\hat{E}_i = \left(1 + \frac{\hat{\eta}_1}{\hat{\eta}_2}\right)\hat{E}_t = \frac{\hat{\eta}_1 + \hat{\eta}_2}{\hat{\eta}_2} \hat{E}_t$$

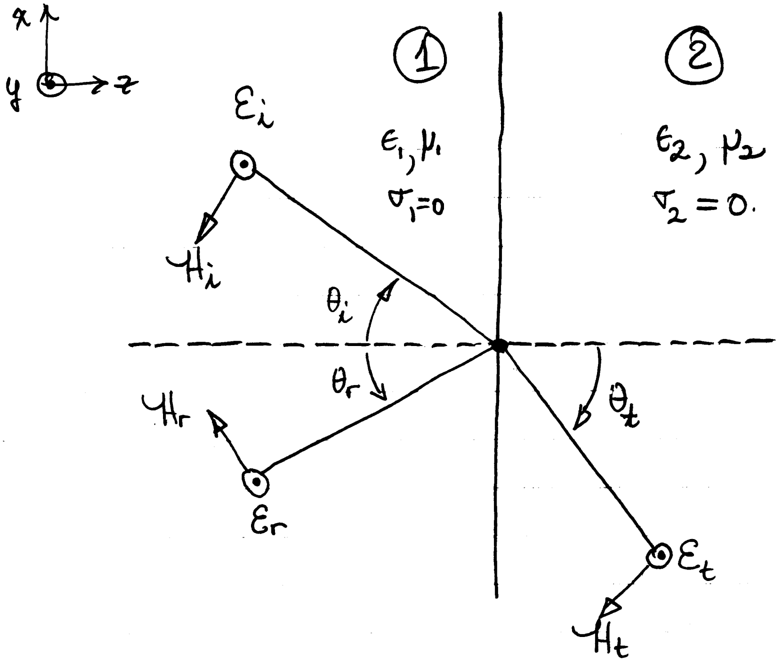
$$\hat{T} \equiv \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\hat{\eta}_2}{\hat{\eta}_1 + \hat{\eta}_2}$$

$$\frac{\hat{E}_i}{\hat{E}_i} + \frac{\hat{E}_r}{\hat{E}_i} = \frac{\hat{E}_t}{\hat{E}_i}$$

$$1 + \frac{\hat{E}_r}{\hat{E}_i} = \frac{\hat{E}_t}{\hat{E}_i} = \hat{T}$$

$$\hat{\Gamma} = \frac{\hat{E}_r}{\hat{E}_i} = \hat{T} - 1 = \frac{2\hat{\eta}_2}{\hat{\eta}_1 + \hat{\eta}_2} - 1 = \frac{2\hat{\eta}_2 - \hat{\eta}_1 - \hat{\eta}_2}{\hat{\eta}_1 + \hat{\eta}_2}$$

$$\hat{\Gamma} = \frac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1}$$



this is perpendicular polarization

$$\underline{\hat{E}}_i = \hat{E}_i e^{-\gamma_1 z} \underline{a_y}$$

$$\underline{\hat{E}}_t = \underline{\hat{E}}_t e^{-\gamma_2 z} \underline{a_y}$$

$$\underline{\hat{E}}_r = \underline{\hat{E}}_r e^{-\gamma_1 z} \underline{a_y}$$

conduction current $J_c = \sigma E = \sigma \hat{E}$

displacement current $J_d = \frac{\partial D}{\partial t} = j\omega\epsilon \hat{E}$

to illustrate the differences between dielectrics and conductors

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon)$$

$$\gamma = \alpha + j\beta$$

look at a Taylor series expansion of γ^2

$$\gamma^2 = -\omega^2\mu\epsilon + j\omega\mu\sigma$$

this term goes to zero if $\sigma = 0$, i.e. a perfect dielectric

$$= -\omega^2\mu\epsilon\left(1 + \frac{\sigma}{j\omega\epsilon}\right)$$ if $\frac{\sigma}{\omega\epsilon} \ll 1$: $\gamma^2 \approx -\omega^2\mu\epsilon$, $\gamma \approx j\omega\sqrt{\mu\epsilon}$

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{\frac{1}{2}}$$ if $\frac{\sigma}{\omega\epsilon} \gg 1$: $\gamma^2 \approx j\omega\mu\sigma$, $\gamma \approx \sqrt{j\omega\mu\sigma}$

good dielectric $\left(\frac{\sigma}{\omega\epsilon} \ll 1\right)$

$$\gamma \approx j\omega\sqrt{\mu\epsilon}\left[1 + \frac{1}{2}\frac{\sigma}{j\omega\epsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\epsilon}\right)^2 + \dots\right]$$

$$\Downarrow$$

$$\beta = \omega\sqrt{\mu\epsilon}\left[1 + \boxed{\frac{1}{8}\left(\frac{\sigma}{\omega\epsilon}\right)^2}\right]$$

$$\alpha = \boxed{\frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$$

$$\hat{\eta} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{j\omega\sqrt{\mu\epsilon}}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{-\frac{1}{2}}$$

$$= \sqrt{\frac{\mu}{\epsilon}}\left(1 - \frac{1}{2}\frac{\sigma}{j\omega\epsilon}\right)$$

good conductor $\left(\frac{\sigma}{\omega\epsilon} \gg 1\right)$

$$\gamma \approx j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}}$$

$$= \sqrt{j}\sqrt{\omega\mu\sigma}$$

$$\downarrow$$

$$e^{j\pi/2} \quad \sqrt{e^{j\frac{\pi}{2}}} = e^{j\frac{\pi}{4}} = \frac{1+j}{\sqrt{2}}$$

$$\therefore \gamma = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$$\therefore \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

$$\hat{\eta} = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j}\sqrt{\omega\mu\sigma}} = \sqrt{j}\sqrt{\frac{\omega\mu}{\sigma}}$$

$$= (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}$$

attenuation depth.

$$\underline{e^{-\alpha z}}$$

$$\underline{e^{-1} = e^{-\alpha\delta}} \leftarrow \text{skin depth.}$$

$$\therefore \underline{\delta = \frac{1}{\alpha}}$$

for good conductors $\underline{\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}}$

$$\underline{\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \text{ where } \sigma >> 1}$$

$\therefore$ waves do <u>NOT</u> propagate into conductors.

remember what we did the last time.

$$E = E^+ e^{-j\beta z} e^{-\alpha z} + E^- e^{+j\beta z} e^{+\alpha z}.$$

defined the phase function.

$$\varphi(z,t) = \omega t - \beta z$$

looked at the velocity of this phase function.

$$\omega t - \beta z = \text{constant}. \qquad \therefore \; z = \frac{\omega t - \text{constant}}{\beta}$$

Phase velocity $\quad u_p = \dfrac{dz}{dt} = \dfrac{\omega}{\beta}$

The group velocity is much more complicated. Consider the case of two waves of slightly different frequencies, $\omega_0 \pm \Delta\omega$ where $\Delta\omega \ll \omega_0$. Corresponding to these will be $\beta_0 \pm \Delta\beta$.

neglect α
$$\hat{E}(z,t) = E_0 e^{-j(\beta+\Delta\beta)z} + E_0 e^{-j(\beta-\Delta\beta)z},$$

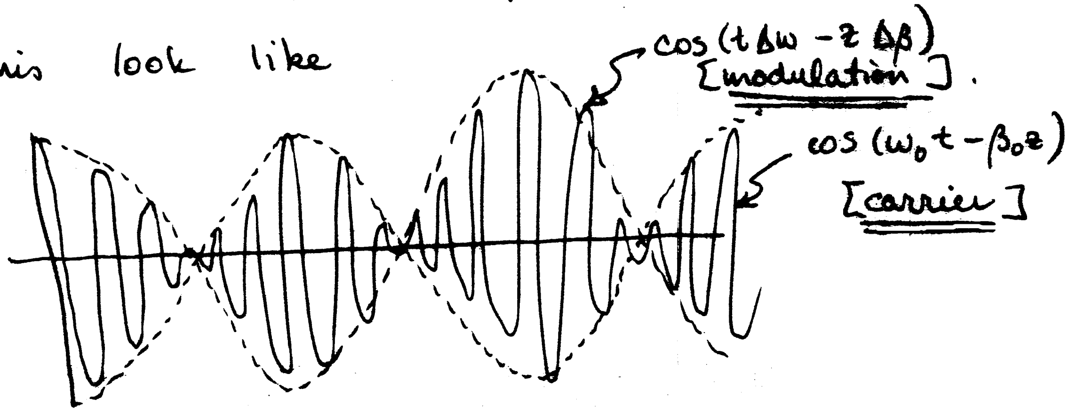
$$\mathcal{E}(z,t) = Re\left[E_0 e^{-j(\beta+\Delta\beta)z + j(\omega+\Delta\omega)t}\right] + Re\left[E_0 e^{-j(\beta-\Delta\beta)z + j(\omega-\Delta\omega)t}\right]$$

$$= E_0 \cos\left[(\omega+\Delta\omega)t - (\beta+\Delta\beta)z\right] + E_0 \cos\left[(\omega-\Delta\omega)t - (\beta-\Delta\beta)z\right]$$

use sum & difference formula.

$$= 2E_0 \cos(t\Delta\omega - z\Delta\beta)\cos(\omega_0 t - \beta_0 z)$$

what does this look like



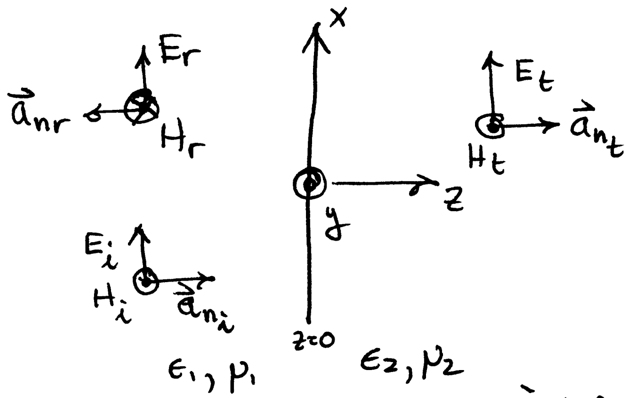
Note that u_p is the velocity of the carrier.

$$\underline{u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}}$$

The velocity of the modulation is

$$\underline{u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} \longrightarrow \frac{d\omega}{d\beta}} \quad \text{as } \Delta\beta \to 0.$$

normal incidence at a plane dielectric boundary



Note: All E's in same direction (this is polarization with E H ⊥ to interface (not relevant).

we will assume $\sigma = 0$. so $\eta = \sqrt{\frac{\mu}{\epsilon}}$ in each region.

incident wave:

$$\boxed{\begin{aligned} \hat{E}_i &= E_i e^{-j\beta_1 z} \\ \hat{H}_i &= \frac{E_i}{\eta_1} e^{-j\beta_1 z} \end{aligned}}$$

reflected wave

$$\boxed{\begin{aligned} \hat{E}_r &= E_r e^{+j\beta_1 z} \\ \hat{H}_r &= -\frac{E_r}{\eta_1} e^{j\beta_1 z} \end{aligned}}$$

transmitted wave

$$\boxed{\begin{aligned} \hat{E}_t &= E_t e^{-j\beta_2 z} \\ \hat{H}_t &= \frac{E_t}{\eta_2} e^{-j\beta_2 z} \end{aligned}}$$

two unknowns E_r and E_t so we need two equations in two unknowns. These will be the boundary conditions.

tangential E is continuous

tangential H $\quad H_{1t} - H_{2t} = J_s$ but no J_s

$\therefore$ tangential H is continuous

$$\therefore \quad E_i e^{-j\beta_1 z} + E_r e^{j\beta_1 z} = E_t e^{-j\beta_2 z}$$

and

$$\frac{E_i}{\eta_1} e^{-j\beta_1 z} - \frac{E_r}{\eta_1} e^{j\beta_1 z} = \frac{E_t}{\eta_2} e^{-j\beta_2 z} \quad \Bigg|_{\text{at } z=0}$$

$$\therefore \quad E_i + E_r = E_t$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

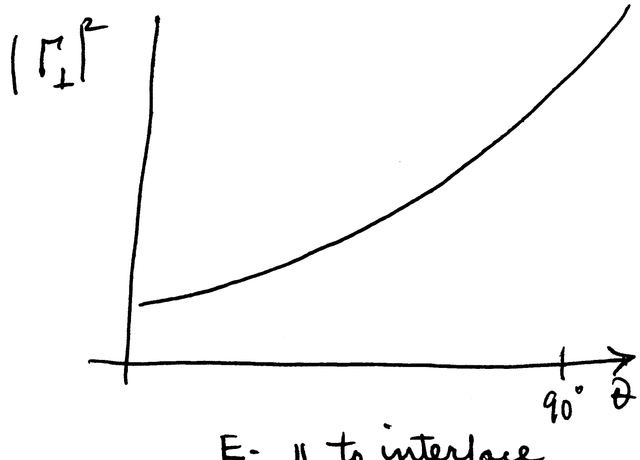
These are simple to solve giving

$$E_r = \underbrace{\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}_{\text{defined as } \hat{\Gamma}} E_i$$

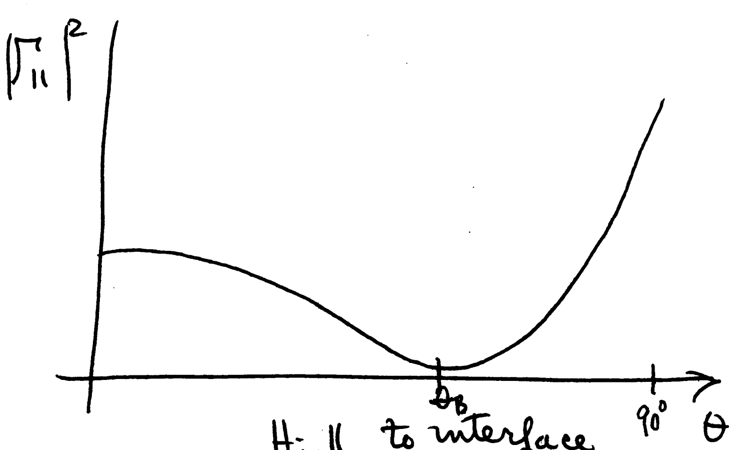
$$E_t = \underbrace{\frac{2\eta_2}{\eta_2 + \eta_1}}_{\text{defined as } \hat{T}} E_i$$

Always: $\quad 1 + \hat{\Gamma} = \hat{T}$

not yet



E_i ∥ to interface



H_i ∥ to interface

Consider what happens if medium 2 is a perfect conductor.

η_1 remains the same

$\eta_2 \Rightarrow 0$ so $\Gamma \to -1$, $T \to 0$

$\hat{\eta} = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$ as $\sigma \to \infty$, $\hat{\eta} \to 0$

$\therefore$ no fields in perfect conductors.

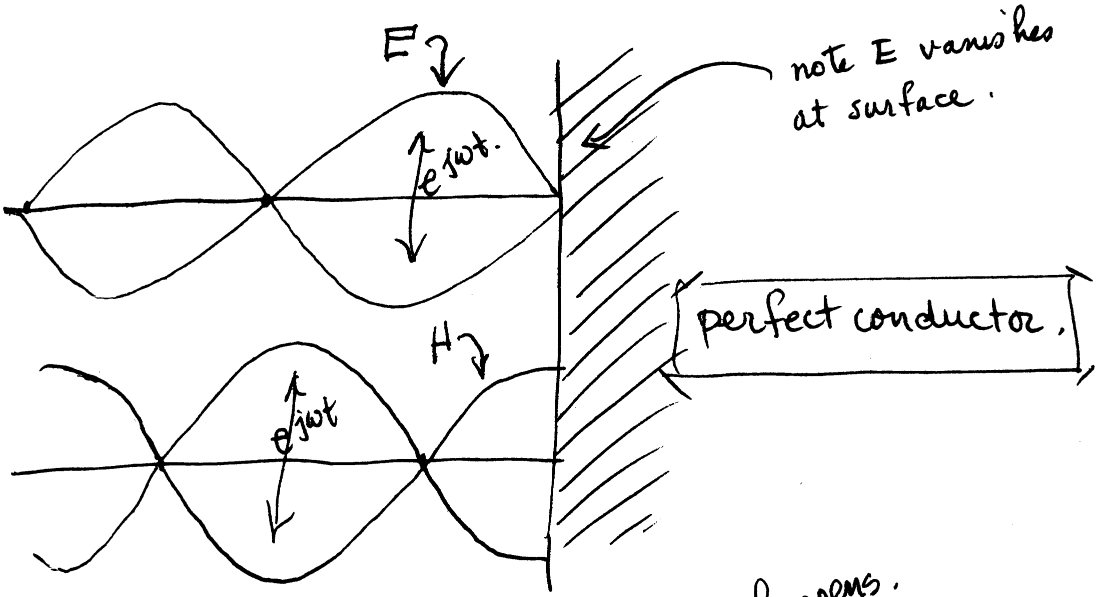
This produces a standing wave — what is a standing wave.

$$\hat{E}(z,t) = \hat{E}_i\left(e^{-j\beta_1 z} + \frac{E_r}{E_i} e^{j\beta_1 z}\right) \to \hat{E}_i\left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right)$$

incident and reflected add coherently.

$$\hat{H}(z,t) = \frac{\hat{E}_i}{\eta_1}\left(e^{-j\beta_1 z} - \frac{\hat{E}_r}{\hat{E}_i} e^{j\beta_1 z}\right) \to \hat{H}_i\left(e^{-j\beta_1 z} + e^{j\beta_1 z}\right)$$

$$\therefore \quad \boxed{\begin{aligned} \hat{E}(z,t) &\to -2j\,\hat{E}_i \sin\beta_1 z \\ \hat{H}(z,t) &\to 2\frac{E_i}{\eta}\cos\beta_1 z. \end{aligned}}$$

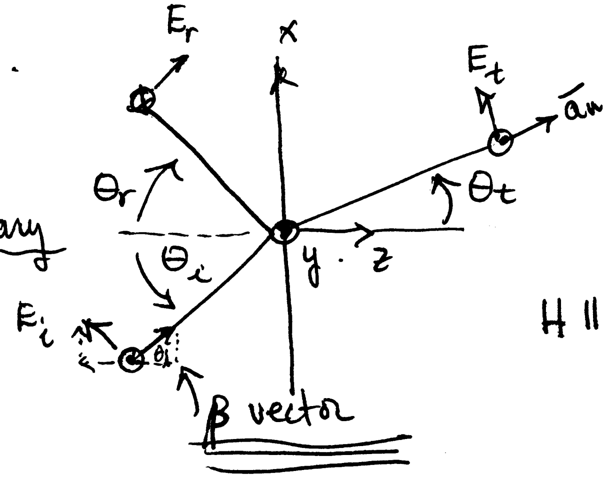


picture of what happens.

what happens to <u>oblique</u> incidence?

polarization.

H parallel to boundary



H ∥ to interface

incident wave

$$\hat{E}_i = E_i\left(\underline{a_x}\cos\theta_i - \underline{a_z}\sin\theta_i\right)e^{-j\beta_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

important!

how to describe the propagation in a direction not along the axis

$$\underline{x}\cdot\underline{\beta} = \left(x\underline{a_x} + z\underline{a_z}\right)\cdot\left(\underline{a_x}\beta_x + \underline{a_z}\beta_z\right)$$

$$= \left(x\underline{a_x} + z\underline{a_z}\right)\cdot\left(\underline{a_x}\beta_1\sin\theta_i + \underline{a_z}\beta_1\cos\theta_i\right)$$

$$= x\beta_1\sin\theta_i + z\beta_1\cos\theta_t$$

$$H_i = a_y\frac{E_i}{\eta_1}a_y e^{-j\beta_1\left(x\sin\theta_i + z\cos\theta_i\right)}$$

Note this stays the same.

reflected

$$E_r = \hat{E}_r\left(a_x\cos\theta_r + a_z\sin\theta_r\right)e^{-j\beta_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$H_r = -\frac{E_r}{\eta_1}a_y e^{-j\beta_1\left(x\sin\theta_r - z\cos\theta_r\right)}$$

components. direction

transmitted

$$\hat{E}_t = E_t\left(\underline{a_x}\cos\theta_t - \underline{a_z}\sin\theta_t\right)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\hat{H}_t = \underline{a_y}\,\frac{E_t}{\eta_2}\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

what happens at $z=0$.

	$\underline{E_i}, \underline{H_i}$	$\underline{E_r}, \underline{H_r}$	$\underline{E_t}, \underline{H_t}$
E:	$E_i(\underline{a_x}\cos\theta_i - \underline{a_z}\sin\theta_i)e^{-j\beta_1 x\sin\theta_i}$	$E_r(\underline{a_x}\cos\theta_r + \underline{a_z}\sin\theta_r)e^{-j\beta_1 x\sin\theta_r}$	$E_t(\underline{a_x}\cos\theta_t - \underline{a_z}\sin\theta_t)\, e^{-j\beta_2 x\sin\theta_t}$
H:	$\left(\frac{E_i}{\eta_1}\underline{a_y}e^{-j\beta_1 x\sin\theta_i}\right.$	$-\frac{E_r}{\eta_1}\underline{a_y}e^{-j\beta_1 x\sin\theta_r}$	$\frac{E_t}{\eta_2}e^{-j\beta_2 x\sin\theta_t}$.

what do we equate:

<u>tangential components must be ~~equal~~ continuous:</u>

$\therefore$ E_x cont.

H_y cont.

$$E_i\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_r\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = E_t\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

$$\frac{E_i}{\eta_1}e^{-j\beta_1 x\sin\theta_i} - \frac{E_r}{\eta_1}e^{-j\beta_1 x\sin\theta_r} = \frac{E_t}{\eta_2}e^{-j\beta_2 x\sin\theta_t}$$

<u>since this must be true for **all** x</u>

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t$$

Snell's Laws

since $\beta_1 = \beta_1$ — reflection

$$\sin\theta_i = \sin\theta_r \implies \theta_i = \theta_r$$

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{\beta_1}{\beta_2}$$

thus $$E_i \cos\theta_i + E_r \cos\overset{\theta_i}{\theta_r} = E_t \cos\theta_t$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$E_i \cos\theta_i + E_r \cos\theta_i = E_t \cos\theta_t$$

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

solutions are:

$$\Gamma_{\parallel} = \frac{E_r}{E_i} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

Fresnel Equation

$$T_{\parallel} = \frac{E_t}{E_i} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

$$\Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$T_{\perp} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

plots of these functions