$$\therefore \quad \frac{\partial^2 V^+}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial s^2}$$

$$\frac{\partial V^+}{\partial t^2} = \frac{\partial^2 V^+}{\partial s^2}$$

Hemholtz equation

$$\nabla^2 \mathcal{E} = \gamma^2 \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$\text{or} \quad u^2 \frac{\partial^2 V^+}{\partial z^2} = \frac{\partial^2 V^+}{\partial t^2}$$

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} = \gamma^2 \cdot \frac{\partial^2 \mathcal{E}}{\partial t^2}$$

$$\text{or} \quad \frac{\partial V^+}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V^+}{\partial t^2}$$

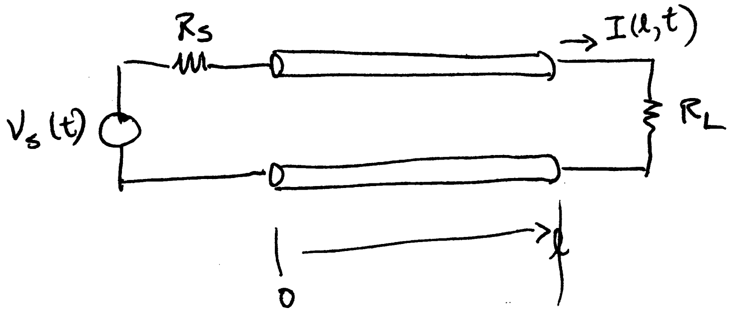
just as for traveling waves V and I must be related

$$I^+\left(t-\frac{z}{u}\right) = \frac{V^+\left(t-\frac{z}{u}\right)}{R_c}$$

$$I^-\left(t+\frac{z}{u}\right) = -\frac{V^-\left(t-\frac{z}{u}\right)}{R_c}$$

where $R_c = \sqrt{\frac{L}{C}}$ $\}$ will not prove this.

consider the problem of a line connected to a load.



what is $V(l,t)$

$$V(l,t) = R_L I(l,t)$$

in general this gives rise to forward & backward traveling waves, why,

$$\left.\begin{array}{ll} \text{forward only} & V^+(t - \frac{l}{u}) = R_c\, I^+(t - \frac{l}{u}) \\ \text{backward only} & V^-(t + \frac{l}{u}) = -R_c\, I^-(t + \frac{l}{u}) \end{array}\right\} \text{transmission line}$$

$$V(l,t) = R_L I(l,t) \quad \Big\} \text{ load}$$

Neither line equation describes the load unless $R_L = R_c$

If $R_L = R_c$ forward satisfies the load equation, but not the reverse.

just as for plane waves we could set this up and solve

transmitted

$$V(l,t) = V^+(t - \tfrac{l}{u}) + V^-(t + \tfrac{l}{u})$$

$$= V^+(t - \tfrac{l}{u})\,(1 + \Gamma_L) \quad \text{where } \Gamma_L = \frac{V^-(t + \frac{l}{u})\ \text{ref.}}{V^+(t - \frac{l}{u})\ \text{inc.}}$$

$$I(l,t) = I^+(t - \tfrac{l}{u}) + I^-(t + \tfrac{l}{u})$$

$$= \frac{V^+(t - \frac{l}{u})}{R_c} - \frac{V^-(t + \frac{l}{u})}{R_c}$$

$$= \frac{1}{R_c} V^+(t - \tfrac{l}{u}) \left[1 - \Gamma_L\right]$$

and we could even define a current reflection equation.

if $V(l,t) = R_L \; I(l,t)$

where $V(l,t) = V^+(t-\frac{l}{u})\,(1+\Gamma_L)$

$$I(l,t) = \frac{1}{R_c} V^+(t-\frac{l}{u})(1-\Gamma_L)$$

then $\cancel{V^+(t-\frac{l}{u})}(1+\Gamma_L) = R_L \frac{1}{R_c} \cancel{V^+(t-\frac{l}{u})}(1-\Gamma_L)$

$$(1+\Gamma_L) = \frac{R_L}{R_c}(1-\Gamma_L)$$

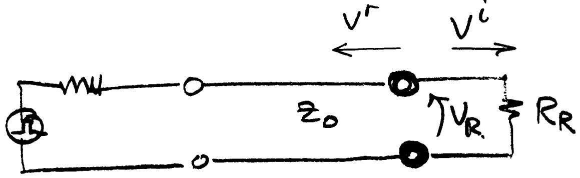
$$R_c + R_c\Gamma_L = R_L - R_L\Gamma_L$$

$$R_c\Gamma_L + R_L\Gamma_L = R_L - R_c$$

$$\Gamma_L = \frac{R_L - R_c}{R_L + R_c}$$

for current Γ is of the opposite sign

since $$\frac{I^-(t+\frac{l}{u})}{I^+(t-\frac{l}{u})} = \frac{\frac{1}{R_c} V^-(t+\frac{l}{u})}{-\frac{1}{R_c} V^+(t-\frac{l}{u})} = -\Gamma$$



resistor. $V_R = V_i + V_r$

$$I_R = I_i + I_r = \frac{V_i - V_r}{Z_0}$$

impedance for negative traveling waves

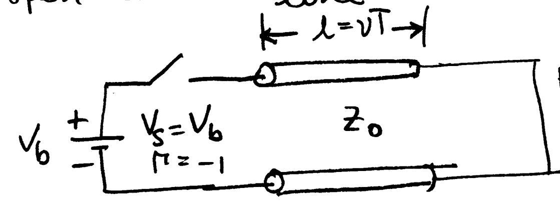
i measured on line subtract.
v measured to line.

$$\therefore \quad R_R = \frac{V_R}{I_R} = \frac{V_i + V_r}{I_i + I_r} = \frac{V_i + V_r}{V_i - V_r} Z_0$$

$$\Gamma_V = \frac{V_r}{V_i} = \frac{R_R - Z_0}{R_R + Z_0} \qquad \Gamma_I = -\Gamma_V$$

$$T = \frac{V_R}{V_i} = 1 + \Gamma = \frac{2R_R}{R_R + Z_0}.$$

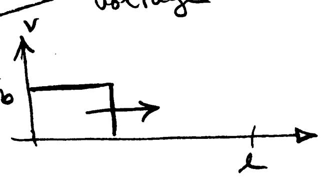
Example: open circuit line

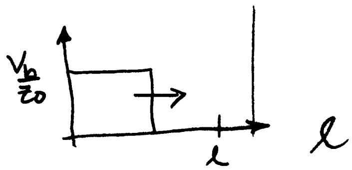


$R_R = 0$
$\Gamma = -1$
$V_R = 0$
$T = 0$

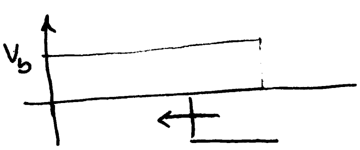
anywhere.
voltage and current on the line

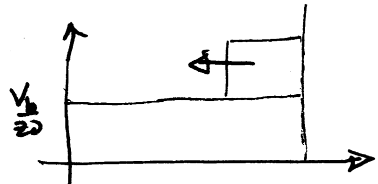
pulse moves out onto line.

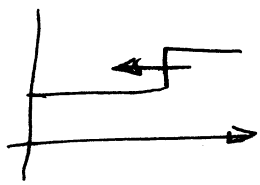




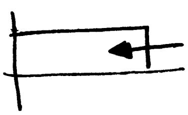
as reflection $\Gamma = -1$ so pulse reflects inverted

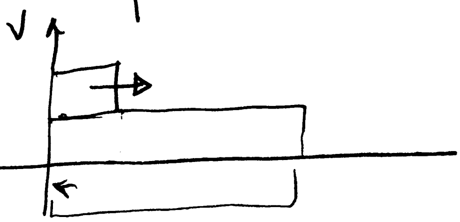


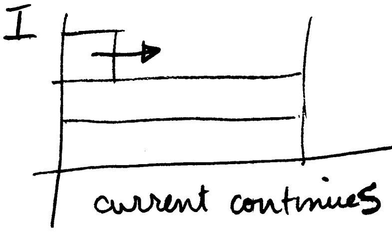




resultant

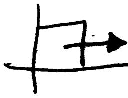






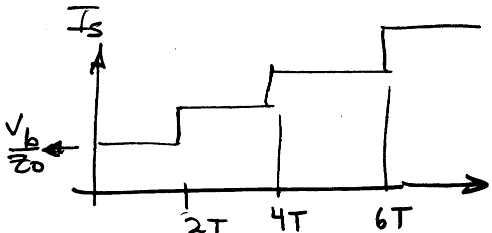
reflection at this end, battery is short $\Gamma = -1$ cancels all pulses.

current continues to add up.

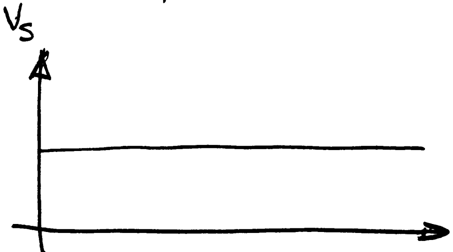


voltage goes between 0 and V_0

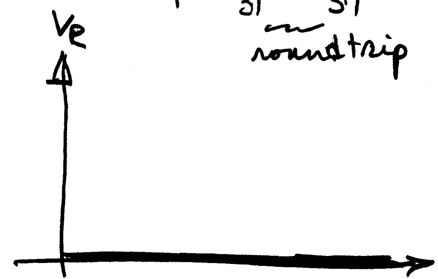
what happens at each end.



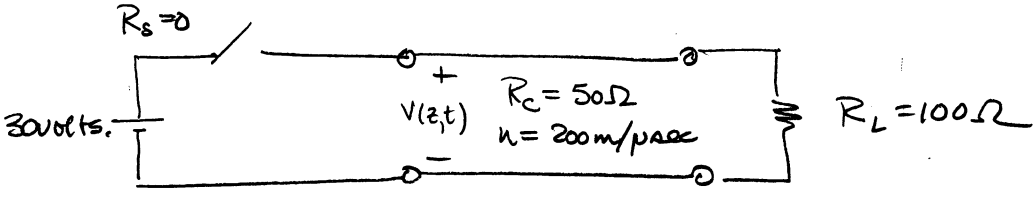
roundtrip



$\Gamma = -1$ also keeps voltages constant.



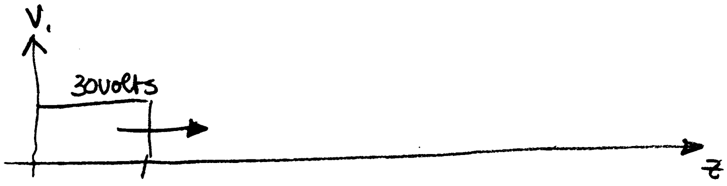
$\Gamma_R = -1$ always forced V_R to zero.



$\Gamma_s = \frac{0-50}{0+50} = -1$ $\qquad$ $\Gamma_L = \frac{100-50}{100+50} = \frac{50}{150} = \frac{1}{3}.$

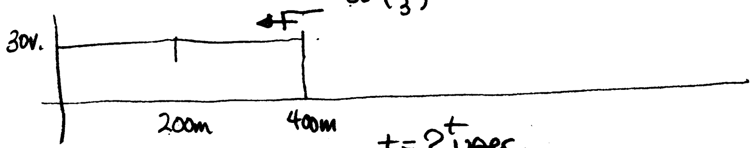
$\leftarrow \mathcal{L} = 400m \rightarrow$

$\Gamma_s = -1$ $\qquad$ $\Gamma_L = \frac{1}{3}$



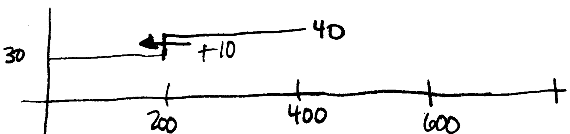
200m.
$t = 1\,\mu sec$

$30\left(\frac{1}{3}\right) = 10$ volts.

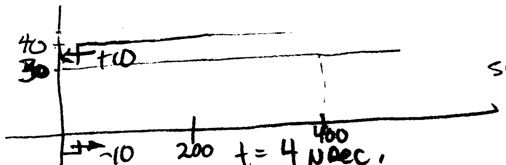


$t = 2^{+}\mu sec$

at each point in time the total line voltage is the sum of the waves present on the line.



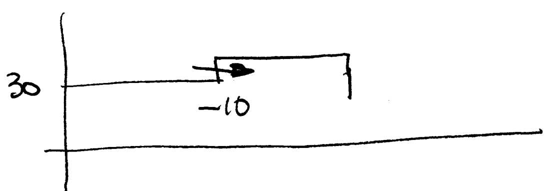
$t = 3\,\mu sec$



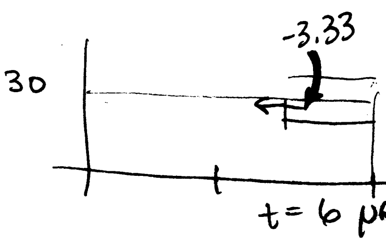
so remains constant at 30

$t = 4\,\mu sec.$

$\Gamma = -1(10)$
what is incoming pulse not total magnitude



$t = 5\,\mu sec.$



$t = 6\,\mu sec^{+}$

−10V comming in
$\Gamma = \frac{1}{3}$ ∴ −3.33 going back.

Eventually This steadys down to 30 volts.

transmission line equations

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

suppose $V_s(t) = V_s \cos\omega t = \mathrm{Re}\left(\hat{V}_s e^{j\omega t}\right)$

if we assume $V(z,t) = \hat{V}(z)e^{j\omega t}$ the transmission line eqns become:

$$\frac{d\hat{V}(z)}{dz} = -j\omega L\,\hat{I}(z)$$

$$\frac{d\hat{I}(z)}{dz} = -j\omega C\,\hat{V}(z)$$

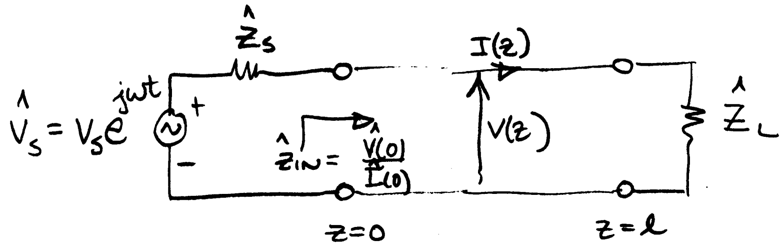
or $$\frac{d^2\hat{V}(z)}{dz^2} = -j\omega L\frac{d\hat{I}}{dz} = -j\omega L\left(-j\omega C\,\hat{V}(z)\right)$$

$$\frac{d^2\hat{V}(z)}{dz^2} = -\omega^2 LC\,\hat{V}(z) = -\underbrace{\frac{\omega^2}{u^2}}_{\beta^2 \text{ lossless.}}\hat{V}(z)$$

$$\hat{V}(z) = \hat{V}^+ e^{-j\frac{\omega}{u}z} + \hat{V}^- e^{+j\frac{\omega}{u}z}$$

$$\hat{I}(z) = \hat{I}^+ e^{-j\frac{\omega}{u}z} + \hat{V}^- e^{+j\frac{\omega}{u}z}$$

$$= \frac{\hat{V}^+}{R_c}e^{-j\frac{\omega}{u}z} - \frac{\hat{V}^-}{R_c}e^{+j\frac{\omega}{u}z}$$



← diagram

in general, at any point on the line $$\boxed{\hat{Z}(z) = \frac{\hat{V}(z)}{\hat{I}(z)}}$$

in general, because of line - load - source discontinuities in impedance, we have forward and backward waves.

$$\hat{\Gamma}(z) = \frac{\hat{V}^-(z)}{\hat{V}^+(z)} \quad \frac{\text{backward}}{\text{forward.}}$$ *

$$\hat{\Gamma}(z) = \frac{\hat{V}^- e^{+j\beta z}}{\hat{V}^+ e^{-j\beta z}} = \frac{\hat{V}^-}{\hat{V}^+} e^{j2\beta z}$$

using this definition we can write the waves (V & I) everywhere as:

$$\hat{V}(z,t) = \hat{V}^+ e^{-j\beta z} + \hat{V}^- e^{+j\beta z}$$

$$= \hat{V}^+ e^{-j\beta z}\left[1 + \frac{\hat{V}^- e^{+j\beta z}}{\hat{V}^+ e^{-j\beta z}}\right]$$

$$\boxed{\hat{V}(z,t) = \hat{V}^+ e^{-j\beta z}\left[1 + \hat{\Gamma}(z)\right]}$$ *

for current.

$$\hat{I}(z,t) = \frac{\hat{V}^+}{R_c} e^{-j\beta z} - \frac{\hat{V}^-}{R_c} e^{+j\beta z}$$

$$\boxed{\hat{I}(z,t) = \frac{\hat{V}^+}{R_c} e^{-j\beta z}\left[1 - \frac{\hat{V}^-}{\hat{V}^+}\frac{e^{+j\beta z}}{e^{-j\beta z}}\right]}$$ *

since we have $\hat{V}$ and $\hat{I}$

$$\hat{Z}_{IN}(z) = \frac{\hat{V}(z)}{\hat{I}(z)} = \frac{V^+ e^{-j\beta z}\left[1 + \hat{\Gamma}(z)\right]}{\frac{V^+}{R_c} e^{-j\beta z}\left[1 - \hat{\Gamma}(z)\right]}$$

$$\boxed{\hat{Z}_{IN}(z) = R_c \frac{1 + \hat{\Gamma}(z)}{1 - \hat{\Gamma}(z)}}$$ *

why is $\hat{Z}_{IN}$ not equal to just R_c

because waves are distributed on the line. *

$$\boxed{\hat{\Gamma}(z) = \frac{\hat{V}^-}{\hat{V}^+} e^{j2\beta z}}$$

and
$$\boxed{Z_{IN}(z) = R_c \frac{1+\hat{\Gamma}(z)}{1-\hat{\Gamma}(z)}}$$

are what we need to solve most problems as they relate the wave magnitudes, impedances and reflection coefficeints. In fact they can be used to relate different points on the line.

suppose $z=\ell$ (i.e, at the load)

$$Z_{IN}(z=\ell) = \boxed{\underset{\uparrow}{\hat{Z}_L} = R_c\left[\frac{1+\hat{\Gamma}(z=\ell)}{1-\hat{\Gamma}(z=\ell)}\right].}$$

define Z_L as impedance looking into load.

complex impedance of load

this allows us to solve for $\hat{\Gamma}(z=\ell)$

define $\hat{\Gamma}(z=\ell) \equiv \hat{\Gamma}_L$

then
$$\boxed{\hat{\Gamma}_L = \frac{\hat{Z}_L - \hat{R}_c}{\hat{Z}_L + \hat{R}_c}}$$

but
$$\hat{\Gamma}(z) = \frac{\hat{V}^-}{\hat{V}^+} e^{j2\beta z}$$

$$\hat{\Gamma}_L = \left[\frac{\hat{V}^-}{\hat{V}^+}\right] e^{2j\beta\ell}$$

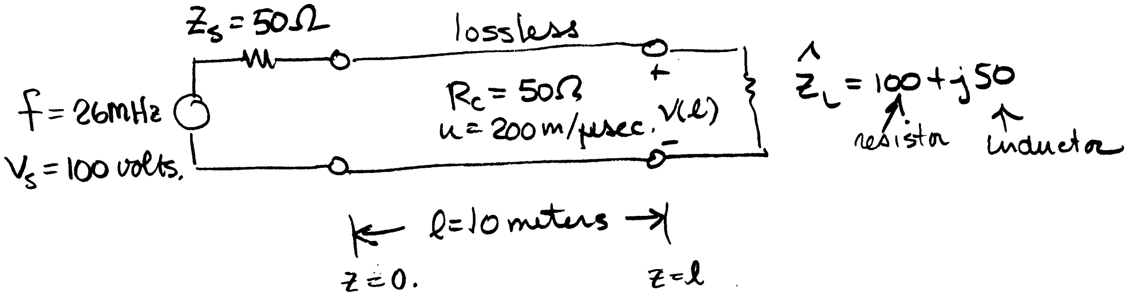
$$= \left[\hat{\Gamma}(z)\, e^{-j2\beta z}\right] e^{j2\beta\ell}$$

$$\therefore \boxed{\hat{\Gamma}(z) = \hat{\Gamma}_L\, e^{j2\beta(z-\ell)}}$$

important result $|\hat{\Gamma}(z)| = |\hat{\Gamma}_L|$

relates reflection coefficient at any point on line to that at load

Example:



Problem: determine the input impedance to the line and the phasor voltages at $z=0$ and $z=l$.

① need for later

$$\boxed{\begin{aligned} \beta &= \frac{\omega}{u} = \frac{2\pi \times 26\times10^6 \text{ rad/sec}}{200 m/\mu sec} = 8.2\times10^{-1} \text{ rad/meter.} \\ 2\beta l &= (2)(8.2\times10^{-1}\tfrac{\text{rad}}{m})(10 \text{ meters}) = 16.34 \text{ meters} \\ &= 936^\circ = 936 - \left[\tfrac{936^\circ}{360}\right] = 216^\circ \end{aligned}}$$

electrical length → $2\beta l$

② what is $\hat{\Gamma}_L$ at load

$$\hat{\Gamma}_L = \frac{\hat{Z}_L - \hat{R}_c}{\hat{Z}_L + \hat{R}_c} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50}$$

$$= 0.45 e^{j27^\circ}$$

③ this can be related to $\hat{\Gamma}$ at $z=0$

$$\hat{\Gamma}(z) = \hat{\Gamma}_L e^{+j2\beta(z-l)}$$

$$\hat{\Gamma}(z=0) = \hat{\Gamma}_L e^{+j2\beta l}$$

$$= 0.45 e^{j27^\circ} e^{-j216^\circ}$$

$$= 0.45 e^{-j189^\circ}$$

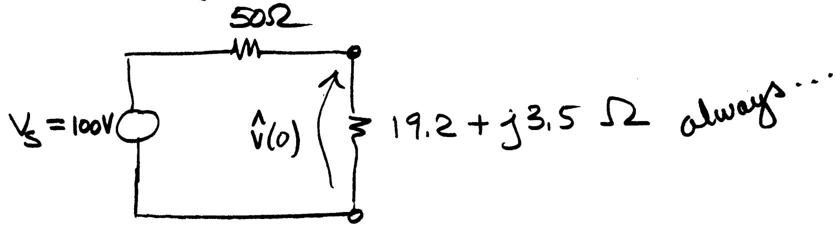
④ if we know Γ at any point we know $\hat{Z}_{IN}$

$$\hat{Z}_{IN}(z) = R_c \frac{1 + \Gamma(z=l)}{1 - \Gamma(z=l)}$$

$$\boxed{\hat{Z}_{IN}(z=0) = R_c \frac{1 + \Gamma(z=0)}{1 - \Gamma(z=0)}} = 50 \frac{1 + 0.45 e^{-j189^\circ}}{1 - 0.45 e^{-j189^\circ}}$$

$$= 19.53 e^{j10.4^\circ} = 19.2 + j3.5\Omega.$$

circuit generator sees is



$$\hat{V}(0) = \left[\frac{19.2 + j3.5}{50 + 19.2 + j3.5}\right] 100 = 28.2\, e^{j7.5°}$$

to get $\hat{V}(z=\ell)$ we need to know either V^+ or V^- at the load, as we know $\hat{\Gamma}(z=0)$

at $z=0$ $\quad \hat{V}(z) = \hat{V}^+ e^{-j\beta z}\left[1 + \hat{\Gamma}(z)\right]$

$$28.2e^{j7.5°} = \hat{V}^+ (1)\left[1 + 0.45\, e^{-j189°}\right]$$

5 find V^+ at $z=0$ $\quad V^+ = 50\, e^{j0.033°}$

$$\therefore \quad \hat{V}(z=\ell) = \hat{V}^+ e^{-j\beta\ell}\left[1 + \hat{\Gamma}(z=\ell)\right]$$

$$= 50 e^{j0.033°} e^{-j108°}\left[1 + 0.45\, e^{j27°}\right]$$

6 then find $\hat{V}^+(z=\ell)$ $\quad = 70.7\, e^{-100°}$

problem illustrated <u>complexity of calculations</u>.

<u>need a faster way of relating line impedances.</u>

combine $\hat{Z}_{IN}(z=\ell) = \hat{Z}_L = R_c \dfrac{1+\hat{\Gamma}(z=\ell)}{1-\hat{\Gamma}(z=\ell)}$

$$\hat{\Gamma}_L = R_c \frac{1+\Gamma_L}{1-\Gamma_L}$$

$$\hat{\Gamma}_L = \hat{\Gamma}_L(z)\, e^{j2\beta(\ell-z)}$$

to get $\boxed{\hat{Z}_{IN}(z) = R_c \dfrac{\hat{Z}_L + jR_c \tan\beta(\ell-z)}{R_c + j\hat{Z}_L \tan\beta(\ell-z)}}$

so than in one step we can get

$$\boxed{\hat{Z}_{IN}(z=0) = R_c \frac{\hat{Z}_L + jR_c \tan\beta\ell}{R_c + j\hat{Z}_L \tan\beta\ell}}$$

this is very easy to use if ℓ is in wavelengths.

if $Z_L = 0$ a short

$$Z_{IN}(z=0) = R_c \frac{j\cancel{R_c}\tan\beta\ell}{\cancel{R_c}} = jR_c \tan\beta\ell$$

but $\beta = \dfrac{2\pi}{\lambda}$

if $\boxed{\ell = \dfrac{\lambda}{4}}$ $\Rightarrow \beta\ell = \dfrac{2\pi}{\lambda}\dfrac{\lambda}{4} = \dfrac{\pi}{2}$

$Z_{IN}(z=0) = \boxed{jR_c \tan\dfrac{\pi}{2}} \to \infty$ <u>an open</u>! (ac only)

$j\omega L$

pick $Z_L = \infty$

$Z_{IN}(z=0) = R_c \dfrac{1}{j\tan\beta\ell} = \boxed{-jR_c \dfrac{1}{\tan\beta\ell}} \to 0$

$\boxed{\ell = \dfrac{\lambda}{4}}$

a short! (ac only)

$$\frac{1}{j\omega C} = -\frac{j}{\omega C}$$

a quarter-wavelength of line transforms everything

suppose $\boxed{Z_L = j\omega L}$ inductive load.

lines can act as impedance transformers.

$$Z_{IN}(z=0) = R_c \frac{j\omega L + jR_c \tan\frac{\pi}{2}}{R_c + j(j\omega L)\tan\frac{\pi}{2}} \longrightarrow R_c \frac{jR_c}{-\omega L}$$

$\lambda/4$

$$\boxed{= -j\frac{R_c^2}{\omega L}} \quad \text{(a capacitor)}$$

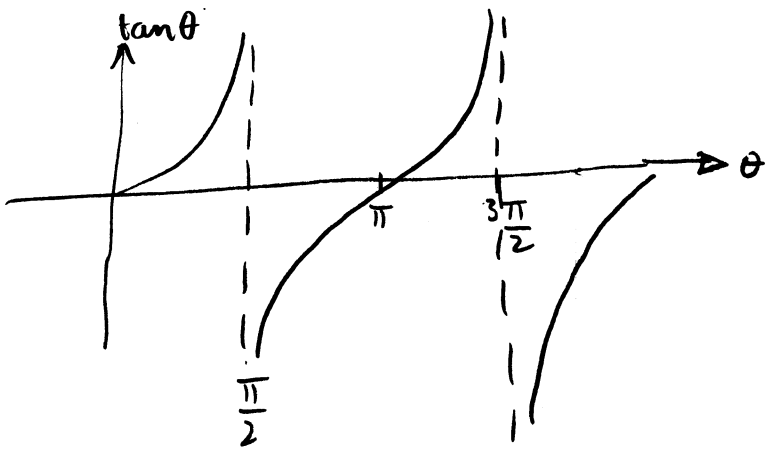
Not just a quarter wave line does this, any multiple of $\frac{\lambda}{2} + \frac{\lambda}{4}$

since $$Z_{IN}(z=0) = R_c \frac{\hat{Z}_L + jR_c \tan\beta l}{R_c + jZ_L \tan\beta l}$$

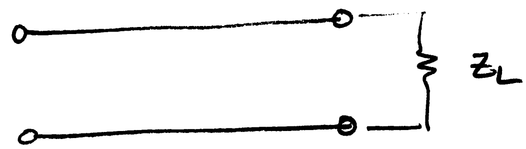
obviously, Z_{in} repeats every $\frac{\lambda}{2}$.

since $\beta l = \frac{2\pi}{\lambda} \cdot n\frac{\lambda}{2} = n\pi$

because

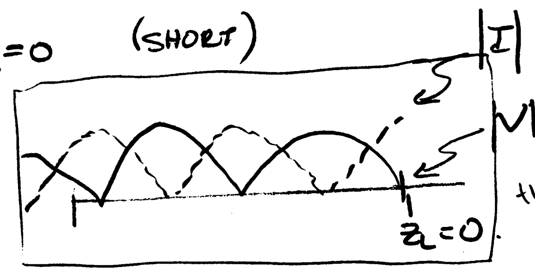


this can also be seen in the line voltages.



if $Z_L = 0$ (SHORT)

line voltages also repeat every $\frac{\lambda}{2}$. Will show this later.



there are $e^{-j\beta z}$ terms here...

are voltage and current 90° out of phase? YES

if $Z_L = 0$

$$\Gamma_L = \frac{0 - R_c}{0 + R_c} = -1$$

$$\therefore \hat{V}(z,t) = \hat{V}^+ e^{-j\beta z}\left[1 + \Gamma(z)\right]$$

$$\hat{V}(\ell,t) = \hat{V}^+ e^{-j\beta z}\underbrace{\left[1 - 1\right]}_{0}$$

on the other hand

$$\hat{I}(z,t) = \frac{\hat{V}^+}{R_c} e^{-j\beta z}\left[1 - \Gamma(z)\right]$$

$$\text{or } \hat{I}(\ell,t) = \frac{\hat{V}^+}{R_c} e^{-j\beta \ell}\left[1 + 1\right].$$

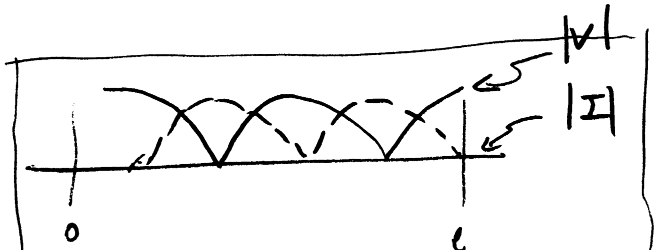
$f \cdot \lambda = c$

$\lambda = \frac{c}{f} = \frac{3 \times 10^{10}\ \text{cm/sec}}{2 \times 10^{9}/\text{sec}}$

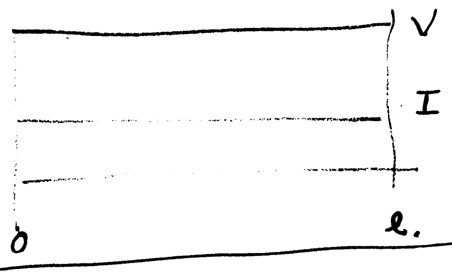
$= 1.5 \times 10\ \text{cm}$

$= 15\ \text{cm}$

if $Z_L = \infty$ (OPEN)



suppose $Z_L = R_c$. They are matched.



how to prove this, easy.

$$\Gamma_L = \frac{R_c - R_c}{R_c + R_c} = 0.$$

$$\therefore \quad \hat{V}(l,t) = \hat{V}^{+} e^{-j\beta l}$$

$$\hat{I}(l,t) = \frac{\hat{V}^{+}}{R_c} e^{-j\beta l}.$$

$$\hat{V}(z) = \hat{V}^{+} e^{-j\beta z} \qquad \hat{I}(z) = \frac{\hat{V}^{+}}{R_c} e^{-j\beta z}.$$

$$\hat{V}(z) = \hat{V}(l,t) e^{+j\beta l} e^{-j\beta z} \qquad \hat{I}(z) = \frac{\hat{V}(l,t) e^{j\beta l}}{R_c} e^{-j\beta z}$$

$$|\hat{V}(z)| = |\hat{V}(l,t)| \qquad |\hat{I}(z)| = \frac{|\hat{V}(l,t)|}{R_c}$$

just as we expected. ∎

crank diagram

recall $\hat{V}(z) = \hat{V}^+ e^{-j\beta z}\left[1 + \hat{\Gamma}(z)\right]$

we were plotting $|\hat{V}(z)|$

voltage on line

$$\boxed{\therefore\ |\hat{V}(z)| = |\hat{V}^+|\,|1 + \hat{\Gamma}(z)|}$$

this is independent of z.

want to look at z-dependence of this.

recall $\hat{\Gamma}(z) \triangleq \dfrac{\hat{V}^-}{\hat{V}^+} e^{+j2\beta z}$

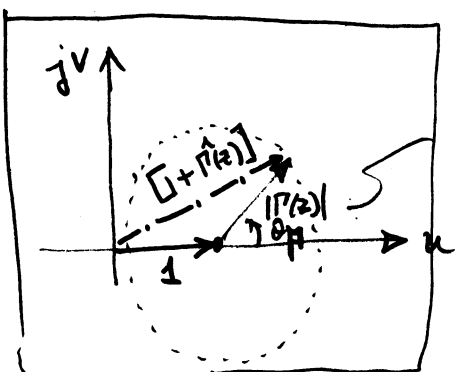
so write $\hat{\Gamma}(z) = |\hat{\Gamma}(z)|\, e^{j\theta_\Gamma}$

$$|\hat{\Gamma}(z)| = \left|\frac{V_m^- e^{+j\theta^-}}{\hat{V}_m^+ e^{+j\theta^+}}\right| = \frac{V_m^-}{V_m^+}$$

how about θ_Γ $\quad e^{j\theta_-} e^{-j\theta+} e^{j2\beta z} = e^{j\theta_\Gamma}$

$$\therefore\ \theta_\Gamma = 2\beta z + (\theta- - \theta+)$$

plot $(1 + \hat{\Gamma}(z))$ in $u + jv$ plane



From definition $\Gamma(z)$ is periodic in z, i.e. every time

$2\beta z = n\cancel{2}\pi$

or $\beta z = n\pi$.

$\frac{2\pi}{\lambda} z = n\cancel{\pi}$

$z = n\frac{\lambda}{2}$.

$\therefore$ irregardless of $\hat{Z}_L$

$\boxed{\text{voltage is periodic at } \frac{\lambda}{2}}$

same for current.

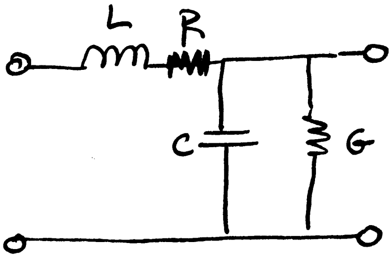
exact location is dependent on Z_L

show voltage is periodic.

A systems look at a transmission line.

$$\frac{\partial V}{\partial z} = -RI - L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -GV - C\frac{\partial V}{\partial t}$$



suppose $\quad V(0,t) = \begin{cases} V_0 & 0<t<T \\ 0 & t>T \end{cases}$

Pick $\quad R = G = 0.$

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

boundary conditions

$$V(x,0) = I(x,0) = 0 \qquad V(0,t) = \begin{cases} V_0 & 0<t<T \\ 0 & t>T \end{cases}$$

let $\tilde{V}(x,s) = \mathcal{L}\{V(x,t)\}$ $\quad \tilde{I}(x,s) = \mathcal{L}\{I(x,t)\}$

$$\frac{d\tilde{V}}{dz} = -L\{s\tilde{I} - \cancel{I(x,0)}^{\,0}\} \qquad \frac{d\tilde{I}}{dz} = -C\{s\tilde{V} - \cancel{V(x,0)}^{\,0}\}$$

initial value theorem.

$$\therefore \quad \frac{d\tilde{V}}{dz} = -Ls\tilde{I} \qquad \frac{d\tilde{I}}{dz} = -Cs\tilde{V}$$

$$\frac{d^2\tilde{V}}{dz^2} = -Ls\frac{d\tilde{I}}{dz} = -Ls(-Cs\tilde{V}) = +LCs^2\tilde{V}$$

what is the general solution of this equation...?

$$\frac{d^2\tilde{V}}{dz^2} = \gamma^2\tilde{V} \quad \text{where } \gamma^2 = LCs^2.$$

general soution is $\tilde{V}(x,s) = Ae^{-\gamma x} + Be^{+\gamma x}$

and A and B can be found from the boundary conditions.

suppose boundedness picks $B=0$, i.e limit magnitude

write $V(0,t) = p(t)$.

Then $\mathcal{L}[V(0,t)] = \tilde{V}(0,s) = p(s)$.

$$\tilde{V}(x,s) = A e^{-\sqrt{LCs^2}\,x}$$

$$\tilde{V}(0,s) = A = p(s)$$

$$\therefore \quad \tilde{V}(x,s) = p(s)\, e^{-\sqrt{LCs^2}\,x}$$

$$\mathcal{L}^{-1}\left[e^{-\sqrt{LC}\,sx}\right]$$

$$= \mathcal{L}^{-1}\left[e^{-\frac{sx}{u}}\right]$$

but this is simply a delay of $\frac{x}{u}$.

not necessarily a delay if R or $G \neq 0$.

so $v(x,t) = p\left(t - \frac{x}{u}\right)$

degree of mismatch

voltage standing wave ratio

why is this useful. for a perfect line (matched)

$$|\hat{V}|_{max} = |\hat{V}|_{min}$$

for any other situation

$$|\hat{V}|_{max} \neq |\hat{V}|_{min}$$

∴ use this as a convenient indicator of mismatch...

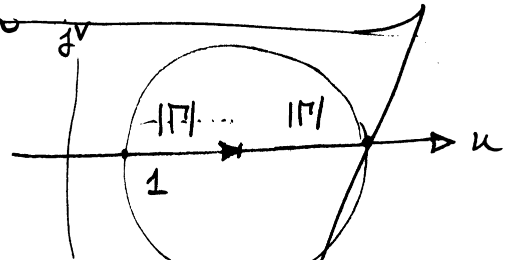
$$\boxed{VSWR = \frac{|\hat{V}|_{max}}{|\hat{V}|_{min}}}$$

and VSWR between 1 and ∞.

for a matched line $VSWR = 1$.

for $R_L = 0$ or $R_L = \infty$ the minimum is zero and $VSWR \to \infty$

look at "crank" diagram



obviously $|\hat{V}|_{max} = 1 + |\Gamma|$

$$|\hat{V}|_{min} = 1 - |\Gamma|$$

and $VSWR = \dfrac{1+|\Gamma|}{1-|\Gamma|}$

but this becomes very easy to measure if we recall

$$\hat{\Gamma}(z) = \hat{\Gamma}_L e^{j2\beta(z-\ell)}$$

then $|\hat{\Gamma}(z)| = |\hat{\Gamma}_L|$

$$\therefore \quad VSWR = \frac{1+|\hat{\Gamma}_L|}{1-|\hat{\Gamma}_L|}$$

VSWR is often easily measured as a indicator of Γ_L

power on a transmission line.

in the + z direction time averaged power

$$P_{av+z} = \frac{1}{2} Re[\hat{V}(z) I^*(z)].$$

this is in both directions.

$$\begin{cases} \hat{V}(z) = \hat{V}_m^+ e^{-j\beta z}[1+\hat{\Gamma}(z)] \\ I(z) = \dfrac{\hat{V}^+}{R_c} e^{-j\beta z}[1-\hat{\Gamma}(z)] \end{cases} \qquad I^*(z) = \frac{\hat{V}^{+*}}{R_c} e^{+j\beta z}[1-\hat{\Gamma}^*(z)]$$

$$P_{av+z} = \frac{1}{2} Re\left[(\hat{V}^+ \cancel{e^{-j\beta z}})(1+\hat{\Gamma}(z)) \frac{\hat{V}^{+*}}{R_c} \cancel{e^{j\beta z}} (1-\Gamma^*(z))\right]$$

$$= \frac{1}{2} Re\left[\frac{|\hat{V}^+|^2}{R_c}[1+\Gamma(z)][1-\Gamma^*(z)]\right]$$

$$= \frac{1}{2}\frac{|\hat{V}^+|^2}{R_c} Re\left[1 + \underbrace{\hat{\Gamma}(z) - \hat{\Gamma}^*(z)}_{\text{imaginary only}} - \hat{\Gamma}(z)\hat{\Gamma}^*(z)\right]$$

consider, if $\Gamma = a + jb$

$\Gamma^* = a - jb$

and $\Gamma - \Gamma^* = j2b$

$$\therefore P_{av+z} = \frac{1}{2}\frac{|\hat{V}^+|^2}{R_c}\left(1 - |\hat{\Gamma}|^2\right)$$

but $|\hat{\Gamma}|^2 = |\hat{\Gamma}_L|^2$

$$\boxed{P_{avg+z} = \frac{1}{2}\frac{|V^+|^2}{R_c}\left(1-|\Gamma_L|^2\right)}$$

there are imaginary terms which were ignored since $P_{avg} = \frac{1}{2} Re[E \times H^*]$

depending on the magnitude of Γ_L we can have no power delivered to the load.

if $R_L = 0$ $\quad \Gamma_L = -1$

$R_L = \infty$ $\quad \Gamma_L = +1$

then $P_{AVG+z} = 0$

max power transfer ONLY if $\Gamma_L = 0$.

---

suppose we want the power delivered to the load. (if matched).

if matched Ohm's Law.

$$\boxed{P_{AVG+z} = \frac{1}{2}\frac{|V_m^+|^2}{R_c}}$$ if $R_L = R_c$.

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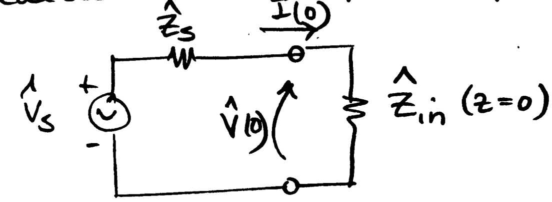
how do we find power delivered to load for general case.

① calculate line input impedance Z_{in}



$$P_{in} = \frac{1}{2}\,\mathrm{Re}\left[\hat{V}(0)\,\hat{I}^*(0)\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\hat{V}(0)\,\frac{\hat{V}^*(0)}{\hat{Z}_{in}^*(z=0)}\right]$$

$$P_{in} = \frac{1}{2}\,|\hat{V}(0)|^2\;\mathrm{Re}\left[\frac{1}{\hat{Z}_{in}^*(z=0)}\right]$$

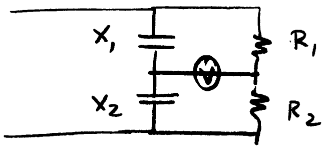
② $$\hat{V}(0) = \frac{\hat{Z}_{in}(z=0)}{\hat{Z}_{in}(z=0) + \hat{Z}_s}\,\hat{V}_s$$

③ if the line is lossless

$$P_{in} = P_{LOAD}.$$

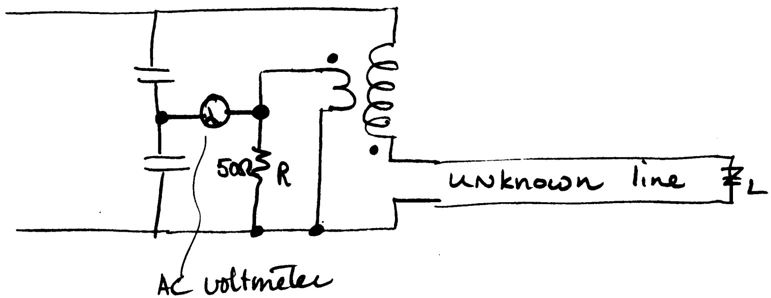
How do you measure VSWR

A.C. bridge technique



if $\frac{X_2}{X_1+X_2} = \frac{R_2}{R_1+R_2}$ then $V=0$.

a.c. bridge ...

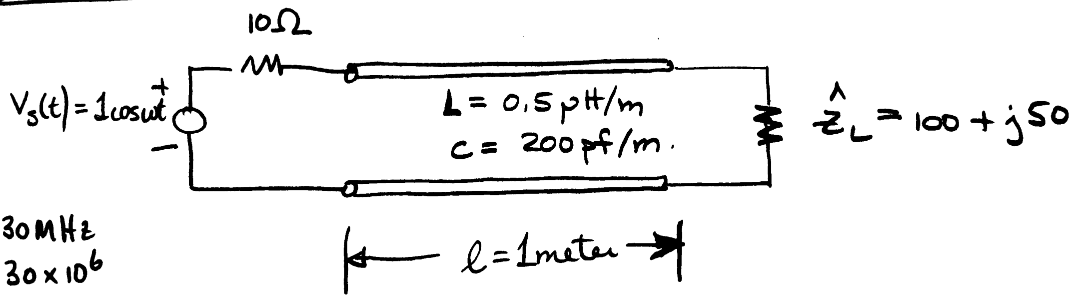


note R is real only.

if Z_L is real voltage across R will be 180° out of phase with voltage across bridge and $V=0$ indicating match.

if Z_L is complex then $V \neq 0$

Example:



$f = 30\,MHz$
$= 30 \times 10^6$

① what is R_c and u?

$$R_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.5 \times 10^{-6}}{200 \times 10^{-12}}} = 50\,\Omega.$$

$$u = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times 10^{-6} \times 200 \times 10^{-12}}} = \frac{1}{\sqrt{10^2 \times 10^{-18}}} = \frac{1}{\sqrt{10^{-16}}}$$

$$= 10^8\ m/sec \text{ or } 100 \text{ meters}/\mu sec.$$

② what is the electrical length of the line?

first, what is λ.

$$\beta = \frac{2\pi}{\lambda} = \frac{\omega}{u}. \qquad \therefore \quad \lambda = \frac{2\pi u}{\omega} = \frac{2\pi u}{2\pi f} = \frac{u}{f}$$

$$= \frac{10^8\ m/sec}{30 \times 10^6 /sec.} = \frac{100}{30} \text{ meters}$$

$$= 3.333 \text{ meters}.$$

$$\therefore\ l = \frac{1 \text{ meter}}{3.3333 \text{ etc.}} = 0.3\,\lambda$$

③ what is $\hat{\Gamma}_L$?

$$\hat{\Gamma}_L = \frac{\hat{Z}_L - R_c}{\hat{Z}_L + R_c} = \frac{100 + j50 - 50}{100 + j50 + 50} = \frac{50 + j50}{150 + j50} = 0.45\,e^{j27^\circ}$$

$$\beta l = \frac{2\pi}{\cancel{\lambda}} \cdot 0.3\cancel{\lambda} = 0.6\pi$$

to get the power to the load ... we need the input impedance.

we can get the input impedance by relating Γ_L to the input reflection coefficient.

$$\hat{\Gamma}(z\ \ ) = \hat{\Gamma}_L e^{j2\beta(z-l)}$$

$$\therefore\ \hat{\Gamma}(0) = 0.45 e^{j27°} e^{-j2\beta l}$$

$$= 0.45 e^{j27°} e^{-j2\beta l}.$$

$$2\beta l = 2 \cdot \frac{2\pi}{\cancel{\lambda}} \cdot 0.3\cancel{\lambda} = 4\pi(0.3) = 1.2\pi = 216°.$$

$$\hat{\Gamma}(0) = 0.45 e^{j27°} e^{-j216°} = 0.45 e^{-189°}$$

$$\hat{Z}_{in}(z=0) = R_c \frac{1+\Gamma(z=0)}{1-\Gamma(z=0)}$$

to get the input impedance.

$$\hat{Z}_{in}(z) = R_c \frac{Z_L + jR_c \tan\beta(l-z)}{R_c + jZ_L \tan\beta(l-z)}$$

at $z=0$ $$\hat{Z}_{in}(0) = R_c \frac{Z_L + jR_c \tan\beta l}{R_c + jZ_L \tan\beta l}$$

$$= (50\Omega)\frac{100 + j50 + j50\tan(216^\circ)}{50 + j(100+j50)\tan(216^\circ)}$$

$$= 19.5\, e^{j10^\circ}$$

what is the input voltage?

$$\hat{V}(0) = \frac{\hat{Z}_{in}(0)}{\hat{Z}_s + \hat{Z}_{in}(0)} \hat{V}_s = \frac{19.5e^{j10}}{10 + 19.5e^{j10}} \, 1 \text{ volt.} = 0.66\, e^{j3.5^\circ}$$

$$\hat{I}(0) = \frac{\hat{V}(0)}{\hat{Z}_{in}(0)} = \frac{0.66\, e^{j3.5}}{19.5\, e^{j10}} = 0.034\, e^{-j6.5^\circ}$$

what is our input power.

$$P_{AV+z} = \frac{1}{2}\mathrm{Re}\left[\hat{V}(0)\hat{I}^*(0)\right] = \frac{1}{2}\mathrm{Re}\left[(0.66)(0.034)\, e^{-j3.0^\circ}\right]$$

$$= 0.011 \text{ watts.}$$

and as long as the line is lossless

$$P_{LOAD} = 0.011 \text{ watts.}$$

Back in electrostatics we learned that $\nabla^2\phi = -\frac{\rho}{\epsilon}$

For most problems, $\rho = 0$ simplifying the problem.

There is a special class of problems for which $\nabla^2\phi = 0$ and either ϕ or $\frac{\partial\phi}{\partial n}$ is specified on the boundaries. This is called a boundary value problem with Dirichlet (ϕ), Neumann ($\frac{\partial\phi}{\partial n}$) or mixed boundary conditions. There is a very powerful method called separation of variables to solve such problems.

Consider $\nabla^2\phi$ in rectangular coordinates.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \qquad (1)$$

To solve this problem we assume that ϕ can be written as

$$\phi = f(x)\, g(y)\, h(z). \qquad (2)$$

This can be applied in any problem where the boundaries coincide with constant coordinate surfaces. Substituting (2) into (1)

$$gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} = 0 .$$

Dividing through by fgh.

$$\frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = 0$$

Each term is a function of only one variable so it is reasonable that each is a constant, a different constant, all of which must sum to zero.

$$k_x^2 \qquad k_y^2 \qquad k_z^2$$

k_x^2, k_y^2 and k_z^2 are called seperation constants because They result in three separate equations.

$$\left.\begin{aligned} &\frac{d^2f}{dx^2} + k_x^2 f = 0 \\ &\frac{d^2g}{dy^2} + k_y^2 g = 0 \\ &\frac{d^2h}{dz^2} + k_z^2 h = 0 \end{aligned}\right\}$$

we will pick k_x, k_y and k_z to reflect the boundary condition after we determine the solutions of these equations.

use p as an example

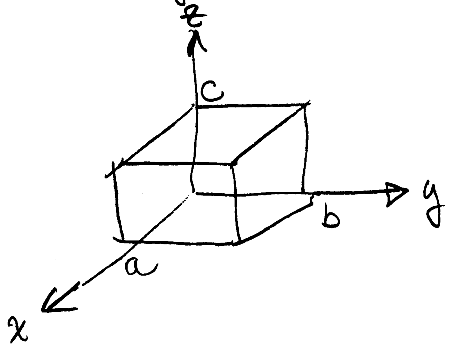
① if $k^2 = 0$ $\quad \frac{d^2p}{dx^2} = 0 \quad \frac{dp}{dx} = c_1 \quad p(x) = c_1 x + c_2$ where c_1, c_2 are constants

② if $k^2 > 0$ $\quad \frac{d^2p}{dx^2} + k^2 p = 0 \quad p(x) = c_1 \sin kx + c_2 \cos kx$

③ if $k^2 < 0$ $\quad \frac{d^2p}{dx^2} + k^2 p = 0 \quad p(x) = c_1 e^{-kx} + c_2 e^{+kx}$

picking say $k_x = 0$ will result in $k_y^2 + k_z^2 = 0$ and the equations for $g(y)$ and $h(z)$ will result in functions looking like the above. where we can pick either g to be exponential, or h to be exponential by choosing the sign of k_y^2 or k_z^2.

Consider the rectangular dirichlet Boundary-value problem.



boundary conditions

Let ϕ be 0 volts on all sides except the top which is set to some $V(x,y)$

at $x=0$, $x=a$ $\quad \phi \equiv 0 \quad \forall y, z \quad \therefore \quad f(x) =$ constant on these surfaces.

if $k_x = 0$ easy to meet this criteria if c_1 & $c_2 = 0$.

$k_x > 0 \leftarrow$ this is the proper choice

$k_x < 0$ only if c_1 & $c_2 = 0$

$\therefore$ pick $k_x > 0$ and see what happens.

$$\therefore \quad f(x) = c_1 \sin k_x x + c_2 \cos k_x x$$

as $f(0) = 0 \quad \therefore \quad c_2 = 0.$

but $f(a) = 0 = c_1 \sin k_x a$

but $\sin k_x a = 0$ for $k_x a = n\pi$ where $n = 1, 2, 3, \dots$

$$\therefore \quad k_x = \frac{n\pi}{a}$$

This type of problem has the general solution

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi}{a} x$$

We could also pick $k_y > 0$ giving

$$g(y) = \sum_{m=1}^{\infty} c_m \sin \frac{m\pi}{b} y$$

if I pick k_x^2 and k_y^2, k_z^2 is determined

$$k_x^2 + k_y^2 + k_z^2 = 0$$

$$k_z^2 = -(k_x^2 + k_y^2) = -\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right).$$

equation becomes $\frac{d^2h}{dz^2} - k^2 p = 0$

solutions $e^{kx} + e^{-kx}$

$$\therefore \; k_z = \pm j\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]$$

what solution is this? $h(z) = C_1 e^{-k_{nm} z} + C_2 e^{+k_{nm} z}$

since we often need $h(z)=0$ for some z we instead pick the forms.

never goes to zero

$$h(z) = C_1 \cosh k_z z + C_2 \sinh k_z z$$

goes to zero

where $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

In our problem $h(0)=0$ so $\therefore$ $C_1 = 0$.

$\therefore$ our general solution is

$$\phi(x,y,z) = f(x)\,g(y)\,h(z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n B_m \sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}\sinh k_z z$$

and the C_2 for $h(z)$ was incorporated into $A_n B_m$.

Now, how do we finish the problem.

<u>WE CHEAT</u> and use the result

$$\int_0^a \sin\frac{n\pi x}{a}\sin\frac{s\pi x}{a}\,dx = 0 \quad n \neq s \qquad = \frac{a}{2} \text{ if } n=s$$

$$\int_0^b \sin\frac{m\pi y}{b}\sin\frac{s\pi y}{b}\,dy = 0 \quad m \neq s. \qquad = \frac{b}{2} \text{ if } m=s.$$

what is our final B.C.'s

at $z=c$
$$V(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n B_m \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \sinh \Gamma_{mn} c$$

$$\Gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

what we do is a 2-D Fourier transform of the left & right.

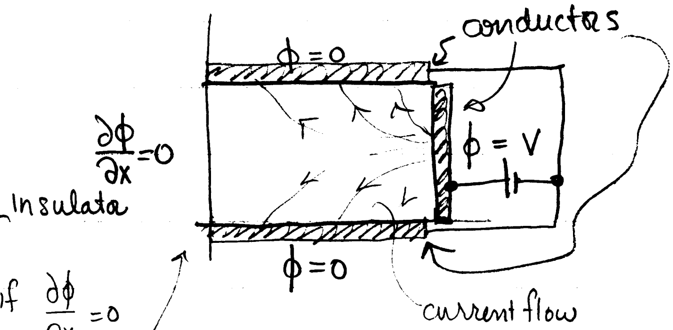
$$\int_0^a \int_0^b V(x,y) \sin \frac{s\pi x}{a} \sin \frac{t\pi y}{b}\, dx\, dy = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_n B_m \left[\int_0^a \sin \frac{n\pi x}{a} \sin \frac{s\pi x}{a}\, dx\right]$$

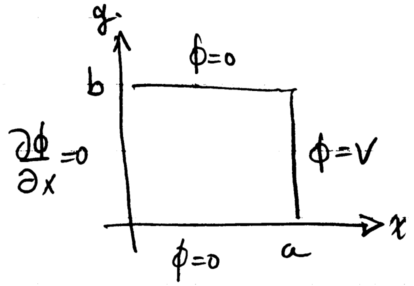
$$\left[\int_0^b \sin \frac{m\pi y}{b} \sin \frac{t\pi y}{b}\, dy\right] \sinh \Gamma_{mn} c.$$

$$= A_s B_t \frac{a}{2} \frac{b}{2} \sinh \Gamma_{st} c.$$

this is the 2-D Fourier transform and if $V(x,y)$ is known we just plug away varying s and t to get the constant $A_s B_t \sinh \Gamma_{st} c$

Let's do a funny problem in two-dimensions!





if $\frac{\partial\phi}{\partial x}=0$

lines do not want to terminate here.

what does $\frac{\partial\phi}{\partial x}=0$. well, $E_x=-\frac{\partial\phi}{\partial x}$ and $J=\sigma E$

$\therefore \quad J_x=-\sigma\frac{\partial\phi}{\partial x}\rightarrow 0$ along $x=0$

What is the solution

want $\phi=0$ at both $y=0$ and $y=b$ so pick sin & cos, i.e. $k_y^2>0$ solution.

$$\nabla^2\phi=0$$

$$\nabla^2 fg=0$$

$$g\frac{d^2f}{dx^2}+f\frac{d^2g}{dy^2}=0$$

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{k_x^2}+\underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{k_y^2}=0. \Rightarrow \quad \frac{d^2f}{dx^2}-k_x^2 f=0.$$

$$\frac{d^2g}{dy^2}+k_y^2 g=0.$$

since $\phi=0$ in y direction I want g to be sinusoids,

$$g= A\sin k_y y+B\cos k_y y.$$

$g(0)=0 \quad \therefore \quad B=0$

$g(b)=0 \quad \therefore \qquad k_y b=n\pi \quad$ or $k_y=\frac{n\pi}{b}$.

$$g(y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b} y$$

what is f ?

$$f = A \sinh k_x x + B \cosh k_x x$$

$$\frac{\partial f}{\partial x} = A k_x \cosh k_x x + B k_x \sinh k_x x$$

$\frac{\partial f}{\partial x} = 0$ at $x = 0$ since cosh never goes to zero then $A = 0$.

and $f(x) = B \cosh k_x x$

and it remains to satisify the B.C. at $x = a$.

$$\therefore \quad V(x,y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b} y \cosh \frac{n\pi}{b} x$$

$$V(a,y) = V_0 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{b} y \cosh \frac{n\pi a}{b}$$

multiply through by $\sin \frac{m\pi}{b} y$ and integrate $\int_0^b$.

$$\int_0^b V_0 \sin \frac{m\pi}{b} y \, dy = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi a}{b} \underbrace{\int_0^b \sin \frac{n\pi}{b} y \sin \frac{m\pi y}{b} \, dy}_{\frac{b}{2}\delta_{mn}}$$

$$V_0 \left[\frac{-\cos \frac{m\pi}{b} y}{\frac{m\pi}{b}} \right]_0^b = V_0 \left[-\cos \frac{m\pi b}{b} + 1 \right]$$

$$= V_0 [1 - \cos m\pi].$$

$$\therefore \quad V_0 \frac{[1 - \cos m\pi]}{\frac{m\pi}{b}} = A_m \cosh \left(\frac{m\pi a}{b} \right) \frac{b}{2}$$

$$A_m = \frac{2V_0 [1 - \cos m\pi]}{b \left[\cosh \frac{m\pi a}{b} \right] \frac{m\pi}{b}} = \frac{2V_0}{m\pi} \left[\frac{1 - \cos m\pi}{\cosh \frac{m\pi a}{b}} \right]$$