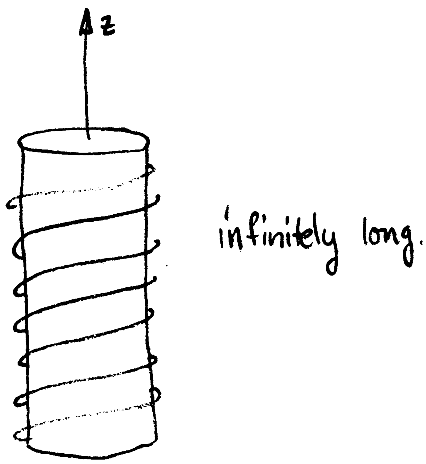
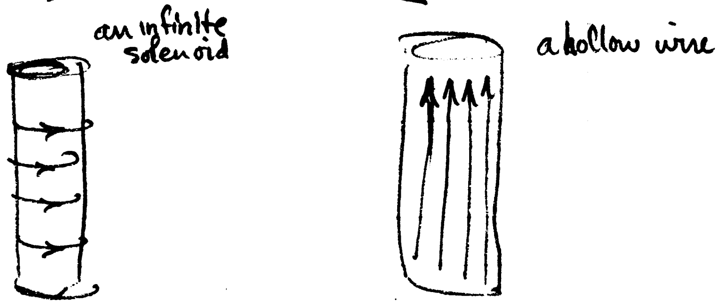


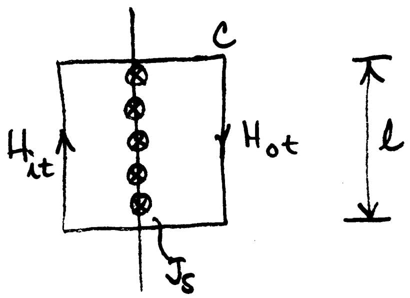
superposition



case 1 $\qquad\qquad$ case 2.

$\underline{J_s} = K_0 \cos\theta_0 \, \underline{a_\phi}$ $\qquad\qquad$ $\underline{J_s} = K_0 \sin\theta_0 \, \underline{a_z}$

<case 1>



for an infinitely long cyclinder there will be no radial fields.
What does this mean?

$$\oint \underline{H} \cdot d\underline{l} = \int \underline{J} \cdot d\underline{S} \qquad \frac{\text{amperes}}{\text{meter}}.$$

$$H_{it}\, l - H_{ot}\, l = J_s l.$$

$$H_{it} - H_{ot} = J_s$$

$$B_{it} - B_{ot} = \mu_0 J_s.$$

Arguement: vary only B_{it} or B_{ot} radially the other is constant hence both must be constant.

$\frac{\text{ampers}}{m.}$

I.

from our previous studies of the finite length solenoid we know that the field at the center is $H = \boxed{\frac{NI}{l}}$

$\frac{\#\text{ wires}}{\text{meter}} \times \frac{\text{current}}{\text{wire}} = \frac{\text{current}}{\text{meter}}.$

$\therefore$ we know that here the current/meter $= J_s$

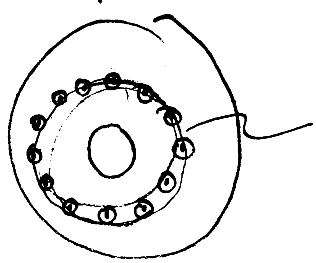
so that for this case
H is simply the current flowing on the surface.

i.e. $H = K_0 \cos\theta_0$.

but if $H_{1t} = J_s \Rightarrow H_{0t} = \phi$. which we could have argued anyway because how could we have an exterior field.

<case 2>

again use Ampère's Law



surface current $J_s = K_0 \sin\theta_0$

$$\oint H \cdot dl = \int J \cdot ds$$

for $r < R$ $\quad H = 0 \Rightarrow B = 0$

outside $r > R$ $\quad H_{0\phi} \cdot 2\pi r = K_0 \sin\theta_0 \, 2\pi R$

$$B_{0\phi} = \mu_0 K_0 \sin\theta_0 \frac{R}{r}$$

Energy stored in the electric field

as we assemble a field, work is done – hence, it is an energy storage device.

simple proof: ① bring Q_1 in from infinity — no work done.

② now bring in charge Q_2 — work done $= Q_2 V_{21} \triangleq W_{21}$

[Recall: voltage $\triangleq \frac{\text{work}}{\text{unit charge}}$.]

③ bring in charge Q_3 — work done $= Q_3 V_{31} + Q_3 V_{32}$.

$$= W_{31} + W_{32}.$$

④ In general, $W_e = W_{21} + W_{31} + W_{32} + \cdots$

⑤ Order is not important so assemble it in another order.

$$W_e = W_{12} + W_{13} + W_{23} + \cdots$$

bring in charge Q_2 with Q_3 first
then bring in Q_1 with Q_2 and Q_3 present.

⑥ Add together

$$2W_e = W_{12} + W_{21} + W_{31} + W_{13} + W_{32} + W_{23} + \cdots$$

$$= Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32}) + \cdots$$

$$= \sum_{i=1}^{3} Q_i V_i \quad \text{where } V_i = \underbrace{\sum_{\substack{j=1 \\ i\neq j}}^{3} V_{ij}}_{\text{total potential on that charge.}}$$

discretely: $W_e = \frac{1}{2}\sum_{i=1}^{3} Q_i V_i$

continuously: $W_e = \frac{1}{2}\int \rho V \, dv.$

for electrostatic fields

$$\nabla \cdot D = \rho.$$

$$W_e = \frac{1}{2} \int (\underline{\nabla} \cdot \underline{D}) V \, dv$$

use vector identity $(\nabla \cdot D) V = \nabla \cdot (V\underline{D}) - \underline{D} \cdot (\underline{\nabla} V)$

sort of the chain rule for the gradient

$$W_e = \frac{1}{2} \underbrace{\int \nabla \cdot (V\underline{D}) \, dv}_{\text{goes to zero}} - \underbrace{\int D \cdot (\nabla V) \, dv}_{\text{goes to } -\epsilon |E|^2}$$

arguement consider point charges

$$E \sim \frac{1}{r^2}$$

$$D \sim \frac{1}{r^2}$$

$$V \sim \frac{1}{r}$$

and convert to a surface integral

$$\oint_S V\underline{D} \cdot dS$$

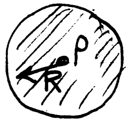
where $S \sim 4\pi r^2$

then $\int V D \cdot dS \sim \int \frac{1}{r} \cdot \frac{1}{r^2} r^2 = \frac{1}{r} \to 0$ as $r \to \infty$

The magnetic field is exactly the same, except we use $\underline{A} \cdot \underline{J}$ in assemblying the fields to give. $W_m = \frac{1}{2} \int_v \underline{A} \cdot \underline{J} \, dv$

$$W_m = \frac{1}{2} \int \mu |H|^2 \, dv$$

Example



energy stored in a uniformly charged sphere.

$$\underline{E} = \begin{cases} \dfrac{r\rho}{3\epsilon} & 0 \le r \le R \\[2ex] \dfrac{R^3\rho}{3\epsilon r^2} & r > R \end{cases}$$

$$W_e = \frac{1}{2}\epsilon \int |E|^2 \, dv$$

$$= \frac{1}{2}\epsilon \left\{ \int_0^R \int_0^\pi \int_0^{2\pi} \frac{r^2\rho^2}{9\epsilon^2} r^2 \sin\theta \, d\theta \, d\phi \, dr + \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{R^6\rho^2}{9\epsilon^2 r^4} r^2 \sin\theta \, d\theta \, d\phi \, dr \right\}$$

$$= \frac{1}{2}\epsilon \left\{ \int_0^R \frac{r^2\rho^2}{9\epsilon^2} 4\pi \, dr \, r^2 + \int_R^\infty \frac{R^6\rho^2}{9\epsilon^2 r^4} 4\pi \, dr \, r^2 \right\}$$

$$= \frac{1}{2}\epsilon \left\{ \frac{4\pi\rho^2}{9\epsilon^2} \int_0^R r^4 \, dr + \frac{4\pi R^6\rho^2}{9\epsilon^2} \int_R^\infty \frac{dr}{r^2} \right\}$$

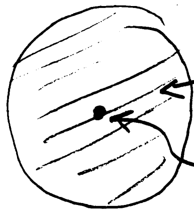
$$= \frac{1}{2}\epsilon \left\{ \frac{4\pi\rho^2}{9\epsilon^2} \frac{R^5}{5} + \frac{4\pi R^6\rho^2}{9\epsilon^2} \frac{1}{R} \right\}$$

$$= \frac{1}{\cancel{2}}\epsilon \frac{\cancel{4}^{\,2}\pi\rho^2}{9\epsilon^2} \left\{ \frac{R^5}{5} + R^5 \right\} = \frac{2\pi\rho^2}{9\epsilon} \left\{ \frac{6R^5}{5} \right\}$$

if we note $Q = \frac{4}{3}\pi R^3 \rho$ $\qquad Q^2 = \frac{16}{9}\pi^2 R^6 \rho$

$$W_e = \frac{Q^2 \, 3}{20\pi\epsilon R}$$

(2) Energy in assemblying an atom?



Assemble electron cloud first? $\qquad W_e = \dfrac{3Q^2}{20\pi\epsilon R}$

How about the positive charge?
What is the energy expended in bringing the $+Q$ in from ∞?
From our original definition

$$W_e = QV$$

What are the fields and potential from a charged sphere?
The $\underline{E}$ fields were given above, so the potential is given by

$$\phi_- = \begin{cases} -\dfrac{3Q}{8\pi\epsilon R^3}\left(R^2 - \dfrac{r^2}{3}\right) & r < R \\[2ex] -\dfrac{Q}{4\pi\epsilon R} & r > R \end{cases}$$

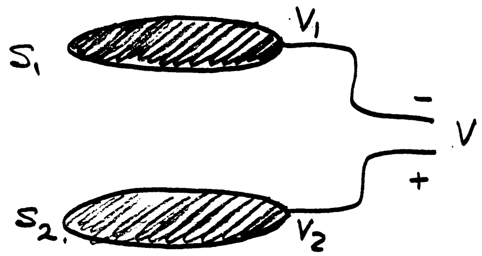
Then $W_e = Q\phi_-(r=0)$ $\qquad$ as $\phi_-(\infty) = 0$

$$= -\frac{3Q^2}{8\pi\epsilon R}$$

$$W_e(\text{total}) = \frac{3Q^2}{20\pi\epsilon R} - \frac{3Q^2}{8\pi\epsilon R} = \frac{3Q^2}{\pi\epsilon R}\left[\frac{1}{20} - \frac{1}{8}\right]$$

$$= \frac{3Q^2}{\pi\epsilon R}\left[\frac{8-20}{160}\right] = \frac{3Q^2}{\pi\epsilon R}\left[-\frac{3}{40}\right] = -\frac{9Q^2}{40\pi\epsilon R}$$

example of capacitor:



best done from definition $\quad C = \dfrac{Q}{V}$

$$W_e = \frac{1}{2}\int \rho V \, dv$$

$$= \frac{1}{2}\int_{S_1} \rho_s V_1 \, dS_1 + \frac{1}{2}\int_{S_2} \rho_s V_2 \, dS_2$$

surfaces are <u>equipotentials</u>

$$= \frac{1}{2} V_1 \underbrace{\int_{S_1} \rho_s \, dS_1}_{-Q} + \frac{1}{2} V_2 \underbrace{\int_{S_2} \rho_s \, dS_2}_{+Q}$$

total charges

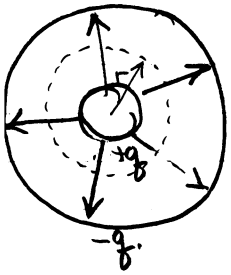
$$= \frac{1}{2}(V_2 - V_1)\, Q = \frac{1}{2} Q V$$

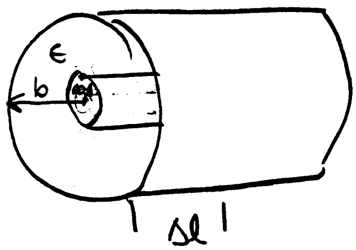
but $C = \dfrac{Q}{V}$

$$\therefore \; W_e = \frac{1}{2} C V^2$$

energy in a capacitor.

a sneaky way of finding capacitance is ~~to~~ use this definitions





$$\oint D \cdot dS = \frac{q}{\Delta l}$$

$$\epsilon E \cdot 2\pi r \cancel{\Delta l} = \frac{q}{\Delta l} \cancel{\Delta l}$$

$$\boxed{E = \frac{q}{\Delta l} \frac{1}{2\pi \epsilon r}}$$

what's the energy stored in the line

$$W_e = \frac{1}{2}\epsilon \int |E|^2 dv = \frac{1}{2}\epsilon \int_a^b \int_0^{\Delta l} \int_0^{2\pi} \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi^2 \epsilon^2} \frac{1}{r^2} r \, dr \, d\phi \, dz$$

$$= \frac{1}{2}\cancel{\epsilon}\left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi^{\cancel{2}}\epsilon^{\cancel{2}}} \quad \cancel{2\pi} \, \Delta l \int_a^b \frac{dr}{r}$$

$$\frac{W_e}{\Delta l} = \left(\frac{q}{\Delta l}\right)^2 \frac{1}{4\pi\epsilon} [\ln b - \ln a]$$

but $\quad W_e = \frac{1}{2} C V^2$

$$\text{if} \quad E = \frac{q}{\Delta l} \frac{1}{2\pi \epsilon_0} \qquad E = -\nabla \phi \qquad \phi = -\int_a^R \underline{E} \cdot \underline{dl}$$

$$\text{so} \quad V = -\int_a^b \frac{q}{\Delta l} \frac{1}{2\pi\epsilon} \frac{dr}{r} = \frac{q}{\Delta l} \frac{1}{2\pi\epsilon} (\ln b - \ln a)$$

$$\therefore \quad \cancel{\left(\frac{q}{\Delta l}\right)^2} \cancel{\frac{1}{4\pi\epsilon}} [\ln b - \cancel{\ln a}] = \frac{1}{2} C \cancel{\left(\frac{q}{\Delta l}\right)^2} \frac{1}{\cancel{4}\pi^{\cancel{2}}\epsilon^{\cancel{2}}} [\ln b - \ln a]^2$$

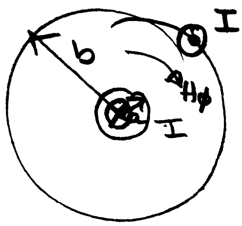
$$\therefore \quad C = \frac{2\pi\epsilon}{\ln b - \ln a}$$

A coaxial cable also carries inductance:

Recall we used Ampere's law in class

to get $H_\phi = \frac{I}{2\pi r}$

just as for the capacitor $\quad W_m = \frac{1}{2} L I^2$



again integrate the field over all space.

$$W_m = \frac{1}{2}\mu \int |H|^2 \, dv$$

$$= \frac{1}{2}\mu \int_a^b \int_0^{\Delta l} \int_0^{2\pi} \frac{I^2}{4\pi^2 r^2} \, r \, dr \, d\phi \, dz$$

$$= \frac{1}{\cancel{2}}\mu \int_a^b \frac{I^2}{4\pi^{\cancel{2}} r^2} \, r \, \cancel{2\pi} \, \Delta l \, dr$$

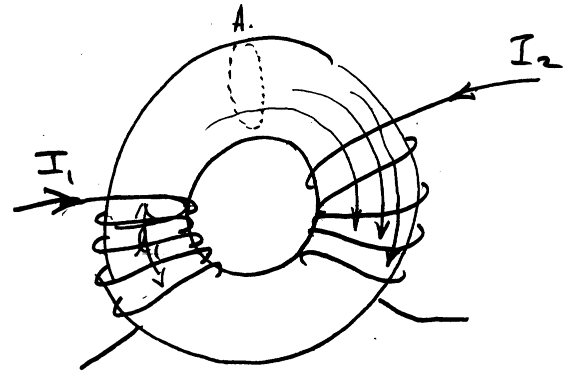
$$= \frac{\mu I^2 \Delta l}{4\pi} \int_a^b \frac{dr}{r}$$

$$\frac{W_m}{\Delta l} = \frac{\mu I^2}{4\pi} (\ln b - \ln a)$$

$$\therefore \quad \frac{W_m}{\Delta l} = \frac{1}{2}\left(\frac{L}{\Delta l}\right) \cancel{I^2} = \frac{\mu \cancel{I^2}}{4\pi} (\ln b - \ln a)$$

$$\left(\frac{L}{\Delta l}\right) = \frac{2\mu}{4\pi} (\ln b - \ln a) = \frac{\mu}{2\pi}[\ln b - \ln a]$$

What is inductance? Inductance refers to flux linkage.



What is the flux due to coil I_1

$$\Phi_m = \frac{\mu N I}{l_1} \, dS$$

the flux linked by a particular coil

the flux linkage is defined as $\lambda = N \Phi_m$.

the inductance is then defined to be. $L = \frac{\lambda}{I}$

For this toroid $L = \frac{\lambda}{I} = \frac{N \Phi_m}{I} = \frac{N \mu N \cancel{I} dS}{\cancel{I} l_1} = \frac{\mu N^2}{l_1} dS$

How about the second coil

$$\lambda_2 = N_2 \Phi_m$$

$$L_{21} = \frac{\lambda_2}{I_1} = \frac{N_2}{I_1} \mu N_1 I_1 \, dS_2 = \mu N_1 N_2 \, dS_2$$

this is usually called the mutual inductance

but L_2 also exists as $L_2 = \frac{\mu N_2^2}{l_1} dS_2$

in general we say $L_{21} = M_{21}$ which is the coupling coefficient.

$$M_{21} = k\sqrt{L_2 L_1}$$

Electromagnetic forces

magnetic system → input electric energy = mechanical work done + increase in stored energy.

$$\underbrace{I\,d\lambda}\ . = F\ dx. + dW_m.$$

how do you know this

this we can't prove yet!

electric system

$$\underbrace{V\,I\,dt}_{\text{input electric energy}} = \underbrace{F\ dx}_{\text{mechanical work done}} + \underbrace{dW_m}_{\text{increase in stored energy}}.$$

$$v = -\frac{d\Phi}{dt}$$

$v\,dt = -d\Phi$ this is energy being taken out of the system

$v\,dt = d\Phi$ putting energy into the system,

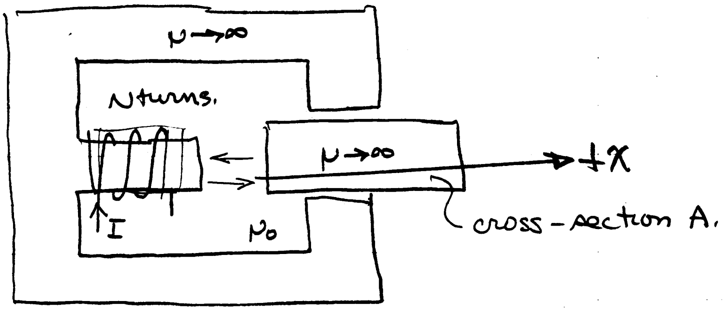
The inductance is handy here:

$$W_m = \tfrac{1}{2} L(x)\, i^2$$

$$dW_m = \tfrac{1}{2} i^2\ L(x)$$

{ usually the current is constant but L, a function of geometry, changes.

(a) relay.



If we let the permeability of the core $\to \infty$ then our magnetic circuit consists of only the source and the gap (a high reluctance).

what is the source $\mathcal{F} = NI$

what is the reluctance of the gap? $R_g = \frac{l}{\mu_0 A} = \frac{x}{\mu_0 A}.$

$$\Phi = \frac{\mathcal{F}}{R_g} = \frac{NI}{x}\mu_0 A$$

flux linked by N turns.

$$\therefore \quad L \triangleq \frac{N\Phi}{I} = \frac{N^2 \mu_0 A}{x}$$

$$W_m = \frac{1}{2} L I^2$$

$$f_x = \frac{dW_m}{dx} = \frac{1}{2} I^2 \frac{dL}{dx} = -\frac{1}{2}\frac{I^2 N^2 \mu_0 A}{x^2}$$

$\therefore$ piece is attracted to the coil

(b) magnetized material

for magnetic material $W_m = \frac{1}{2}\int \mu_0 |M|^2 \, dv$,

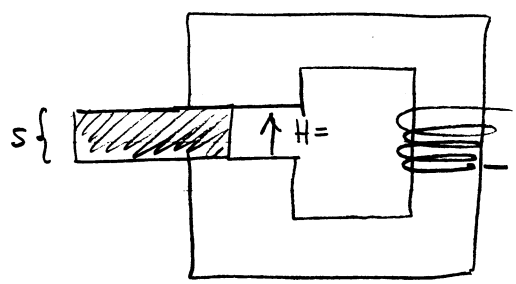
long way.

$\dfrac{\text{force}}{\text{unit volume}}$ $\quad F = \iint \left[\mu_0 \, (M \cdot \nabla) H + \mu_0 \underline{J_f} \times \underline{H} \right] dv.$

this follows from $\quad \nabla \times B = \mu_0 J_f + \nabla \times M$

$$f = i \, d\underline{l} \times \underline{B}$$

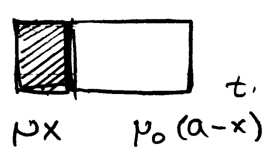
no free currents so $\quad F_{total} = \int \left[\mu_0 \, M \cdot \nabla H \right] dv.$



$\mathcal{F} = NI \qquad R_g = \dfrac{s}{\mu_0 A}.$

$$\Phi = \frac{\mathcal{F}}{R} = \frac{NI\mu_0 A}{s}$$

except μ is not constant.



$R_g = \dfrac{s}{\mu t x} \qquad R_g' = \dfrac{s}{\mu_0 t (a-x)}$

$\Phi = \dfrac{NI(\mu t x)}{s} \qquad \Phi' = \dfrac{NI\mu_0 t(a-x)}{s}$

$$\Phi_{total} = \frac{NIt}{s} \left[\mu x + \mu_0 (a-x) \right]$$

$$L = \frac{N \Phi_{total}}{I} = \frac{N^2 t}{s} \left[\mu x + \mu_0 (a-x) \right]$$

$$f_x = \frac{d}{dx}\left(\frac{1}{2} L I^2\right) = \frac{1}{2} I^2 \frac{N^2 t}{s} \left[\mu - \mu_0 \right].$$

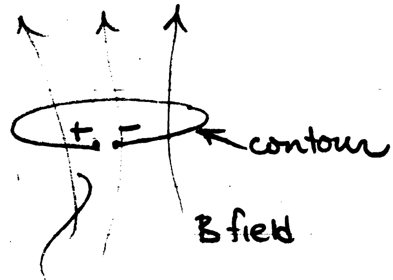
since $\mu > \mu_0$ force pulls magnet into gap.

<u>time dependent fields.</u>

all boundary conditions, Gauss' Law, etc. remain the same.
The big difference is something called the displacement current.
and Faraday's law.

Faraday's Law

$$\text{emf} = -\frac{d\Phi_m}{dt},$$



we use emf specifically because this is really a separation of charge to produce the potential. Because the loop is usually a conductor there is no E field across it and it all appears across the gap.

use right hand rule. $\frac{d\phi}{dt} > 0$ polarity as shown

$$\oint_c \underline{\mathcal{E}} \cdot \underline{dl} = -\frac{d}{dt}\int \underline{B} \cdot dS$$

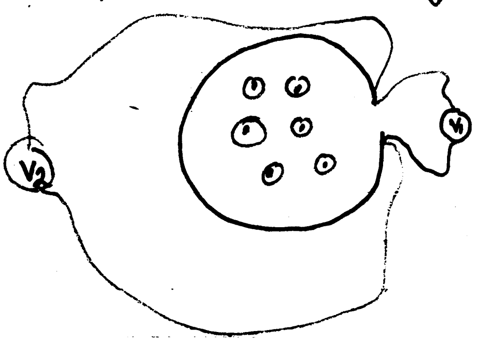
what's really important is that we showed that the E field was conservative for static fields i.e. $\oint_c E \cdot dl = 0$

this is not true. $\oint_c \underline{\mathcal{E}} \cdot \underline{dl} = -\frac{d\phi}{dt} \neq 0.$

specifically $\displaystyle\int_{P_1 \atop C_1}^{P_2} \mathcal{E} \cdot dl + \int_{P_2 \atop C_2}^{P_1} \mathcal{E} \cdot dl \neq 0$

so there are not equal.

to show how important the choice of contour is:



for v_1 $$v_1 + \int_{\text{leads}} \underline{\mathcal{E}}\cdot d\underline{l} + \int_C \underline{\mathcal{E}}\cdot d\underline{l} = -\frac{d}{dt}\left[\int_{\substack{\text{meter}\\\text{loop}}} \underline{\mathcal{B}}\cdot d\underline{s} + \int_S \underline{\mathcal{B}}\cdot d\underline{s}\right]$$

since there is no $\mathcal{E}$ field along a conductor

all $\int \underline{\mathcal{E}}\cdot d\underline{l} \to 0$

$$v_1 = -\frac{d}{dt}\left[\int_{\substack{\text{meter}\\\text{loop}}} \underline{\mathcal{B}}\cdot d\underline{s} + \int_S \underline{\mathcal{B}}\cdot d\underline{s}\right]$$

if the meter loop is small we will get approximately the correct result.

Even more importantly suppose v_1 is as shown then the enclosed flux is what we wanted
if v_2 is as shown then the enclosed flux is zero.

You can convert Faraday's law to point from.

$$\oint_C \underline{\mathcal{E}}\cdot d\underline{l} = -\frac{d}{dt}\int \underline{\mathcal{B}}\cdot d\underline{s}$$

partial because of the spatial coordinates

$$= -\int \frac{\partial}{\partial t}(\mathcal{B}\cdot d\underline{s}) \longleftarrow \text{moving contour.}$$

if s is stationary $\oint \underline{\mathcal{E}}\cdot d\underline{l} = -\int \frac{\partial \mathcal{B}}{\partial t}\cdot d\underline{s}$

use Stokes' Theorem and this becomes

$$\int \nabla\times\mathcal{E}\cdot d\underline{s} = -\int \frac{\partial \mathcal{B}}{\partial t}\cdot d\underline{s}$$

$$\nabla\times\mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t}$$

Faraday's law gives us time varying fields & Maxwell's eqns so let's review everything we've done to put it into a time dependent form.

Gauss' law (no change)

$$\nabla \cdot \underline{D} = \rho \qquad \oint_S \underline{D} \cdot \underline{ds} = \int_v \rho \, dv$$

$$\nabla \cdot \underline{B} = 0 \qquad \oint_S \underline{B} \cdot \underline{ds} = 0$$

Ampère's law

static $\quad \nabla \times \underline{H} = \underline{J}$

time dependent $\nabla \times \underline{\mathcal{H}} \stackrel{?}{=} \mathcal{J}$

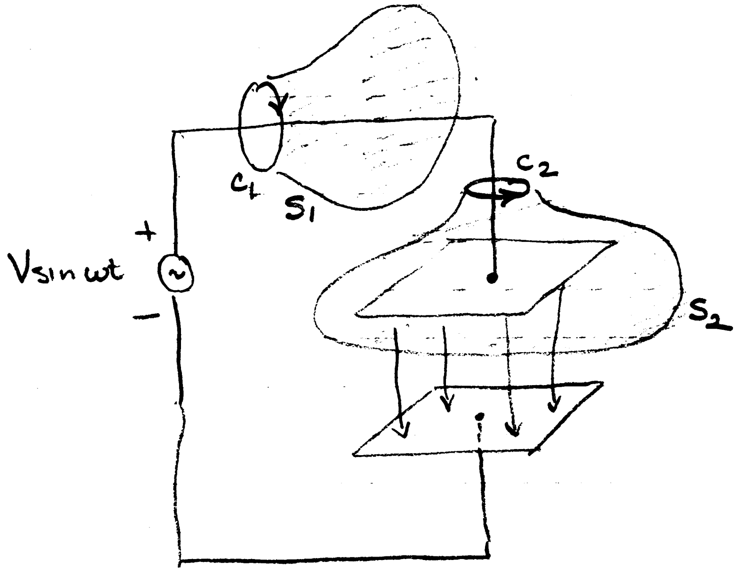
$\underline{\nabla} \cdot \underline{\nabla} \times \underline{\mathcal{H}} = 0$ vector identity $\quad \nabla \cdot \mathcal{J} \neq 0$ as shown below

from the continuity equation $\quad \underline{\nabla} \cdot \mathcal{J} = -\dfrac{\partial \rho}{\partial t}$

but for time dependent fields the charge is not necessarily static, i.e. $\dfrac{\partial \rho}{\partial t} \neq 0$.

There is thus a missing term in this equation.
Maxwell chose to add that term on the right hand side
Displacement current $\quad \dfrac{\partial \mathcal{D}}{\partial t}$

This new term is called the displacement current even though it is not a physical current.



$$\oint \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{ds} + \int_S \frac{\partial \underline{D}}{\partial t} \cdot \underline{ds}$$

$C_1 - S_1$

conduction current J going through S_1 gives rise to H_1 about C_1

$C_2 - S_2$

no conduction current J going through S_2 is there no H_2 around C_2.
but free charge is transferred $\Rightarrow$ there is an H_2
but as this free charge is transferred it changes D creating an induced current on the other plate. $\therefore \frac{\partial D}{\partial t}$ is due to the apparent motion of electrons.

to determine the ratio of these effects.

$$\nabla \times \underline{H} = J + \frac{\partial D}{\partial t}$$

if $\quad \underline{E} = E_0 e^{j\omega t} \qquad \underline{D} = \epsilon \underline{E} = \epsilon E_0 e^{j\omega t}$

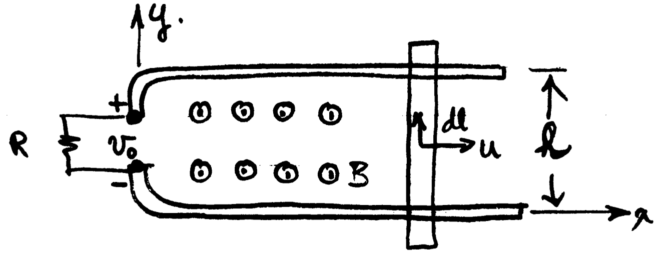
$$J = \sigma E_0 e^{j\omega t} \qquad \frac{\partial D}{\partial t} = \epsilon (j\omega) E_0 e^{j\omega t}$$

$$\therefore \quad \frac{J}{\frac{\partial D}{\partial t}} = \frac{\sigma \cancel{E_0 e^{j\omega t}}}{j\omega\epsilon \cancel{E_0 e^{j\omega t}}} = \frac{\sigma}{j\omega\epsilon}$$

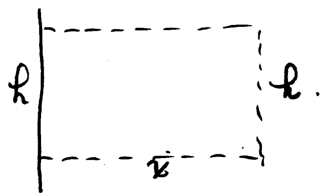
$$\left| \frac{J}{\frac{\partial D}{\partial t}} \right| = \frac{\sigma}{\omega\epsilon} \longleftarrow \text{conduction angle}$$

Example:

$$v_0 = -\frac{d}{dt}\int \underline{B}\cdot d\underline{S}$$



A metal bar slides over a pair of conducting rails in a uniform magnetic field $\underline{B} = B_0 \underline{a_z}$ with a constant velocity u.



(a) $$\int B\cdot dS = Bhx$$

$$v_0 = -\frac{d}{dt}\int B\cdot ds = -\frac{d}{dt}(Bhx) = -Bh\frac{dx}{dt} = -Bhu$$

(b) how much electrical power is developed.

$$I_{OUT} = \frac{v_0}{R} = \frac{Bhu}{R}$$

$$P_{OUT} = I^2R = \frac{(Bhu)^2}{R}$$

properties of the medium

$$\nabla \times \underline{\mathcal{E}} = -\frac{\partial \underline{\mathcal{B}}}{\partial t} \qquad \oint_c \underline{\mathcal{E}} \cdot \underline{dl} = -\frac{d}{dt}\int_s \underline{\mathcal{B}} \cdot \underline{ds}$$

$$\nabla \times \underline{\mathcal{H}} = \underline{\mathcal{J}} + \frac{\partial \underline{\mathcal{D}}}{\partial t} \qquad \oint_c \underline{\mathcal{H}} \cdot \underline{dl} = \int_s \underline{\mathcal{J}} \cdot \underline{ds} + \frac{d}{dt}\int \underline{\mathcal{D}} \cdot \underline{ds}$$

$$\nabla \cdot \mathcal{D} = \rho \qquad \oint_s \underline{\mathcal{D}} \cdot \underline{ds} = \int_v \rho \, dv$$

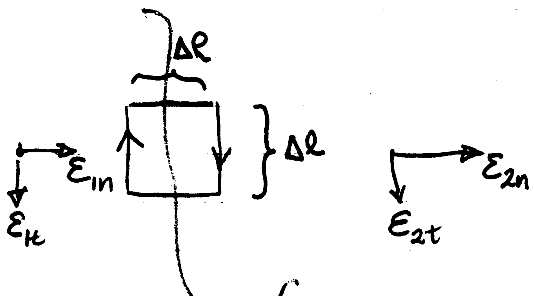
$$\nabla \cdot \mathcal{B} = 0 \qquad \oint_s \underline{\mathcal{B}} \cdot ds = 0$$

$$\underline{F} = q(\underline{\mathcal{E}} + \underline{u} \times \underline{\mathcal{B}})$$

$$\nabla \cdot \underline{\mathcal{J}} = -\frac{\partial \rho}{\partial t}$$

Do any of our boundary conditions change?

the only thing that we added was Faraday's law?
does this change the field B.C.'s.



re-call we used $\int_c \underline{E} \cdot \underline{dl} = 0$ as $\Delta l \Delta h \to 0$

Fields are no longer conservative

$$\oint \underline{\mathcal{E}} \cdot \underline{dl} = -\frac{d}{dt}\int_{sb} \underline{\mathcal{B}} \cdot \underline{ds}$$

if $\mathcal{B}$ is finite then $\mathcal{B}$

$$\therefore \mathcal{E}_{1t} = \mathcal{E}_{2t}$$

although its not a good arguement as the surface $\to 0$ as $\Delta h \to 0$ $\frac{d}{dt}\int \mathcal{B} \cdot ds \to 0$.

How about Ampère's law?

$$\oint_C \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{ds} + \frac{d}{dt} \int_S \underline{D} \cdot \underline{ds}$$

same contour.

again, as $\Delta h \, \Delta l \to 0$

$$\frac{d}{dt} \int_S \underline{D} \cdot \underline{ds} \to 0$$ so results are the same.

$$\underline{H}_{1t} - \underline{H}_{2t} = \underline{K}_s$$

One important consequence of these laws is a perfect conductor.

as before $J_d = \sigma \mathcal{E}$

$$\sigma = \frac{J_d}{\mathcal{E}}$$

if $\sigma = \infty$, then either $J_d = \infty$, or $\mathcal{E} = 0$

Occam's Razor.

not physically realizable

use this

if $\mathcal{E} = 0$ Then $D = \epsilon \mathcal{E} = 0$ also.

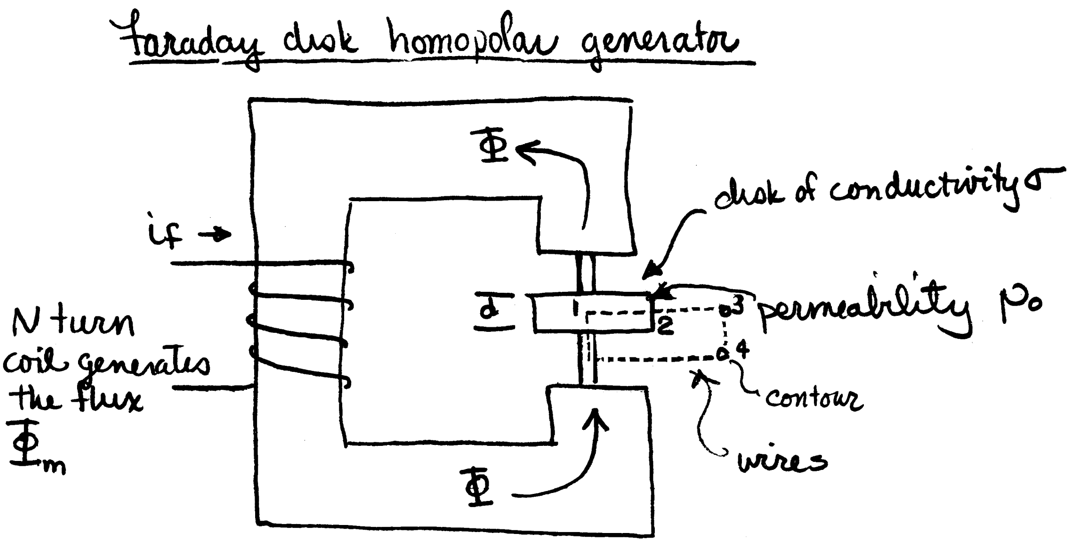
use Faraday's Law $\nabla \times \mathcal{E} = -\frac{\partial B}{\partial t}$

if $\mathcal{E} = 0$, then $\frac{\partial B}{\partial t} = 0$

$\therefore$ there is no time varying magnetic field.

$\therefore$ in a perfect conductor $\to$ no $\mathcal{E}$ fields
only a static H field.

Faraday disk homopolar generator



what is the field across the gap? we did this problem already.

$$\mathcal{F} = NI$$

$$\mathcal{R}_{gap} = \frac{s}{\mu_0 A}.$$

$$\Phi_m = \frac{\mathcal{F}}{\mathcal{R}_{gap}} = \frac{NI}{s/\mu_0 A} = \mu_0 \frac{NIA}{s}$$

$$\downarrow$$

$$B\cancel{A} = \mu_0 \frac{NI\cancel{A}}{s}$$

$$B_{gap.} = \mu_0 \frac{NI}{s}$$

for moving fields

Lorentz $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$: for a charge q moving at velocity v in E, B fields.

If I travel with a velocity $\underline{v}$ I see only a field qE'

where $\cancel{q}E' = \cancel{q}(\underline{E} + \underline{v} \times \underline{B})$

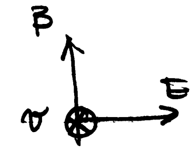
thus $\underline{E}' = \underline{E} + \underline{v} \times \underline{B}$

$$\underline{E}^{(r)} = \underline{E}' - (\underline{v} \times \underline{B})$$

radial field

field on moving disk

angular velocity



angular velocity

$$= \frac{J_r}{\sigma} - \omega r \, B_{gap} \qquad = \frac{i_r}{2\pi r d} \frac{1}{\sigma} - \omega r \, B_{gap}.$$

the current is the same in all frames.

Now use Faraday's law on contour shown.....

$$\oint \underline{E} \cdot d\underline{l} = \int_1^2 E_r \, dr + \underbrace{\int_3^4 \underline{E} \cdot d\underline{l}}_{-v_r} = T \frac{d\theta}{dt} = 0 \text{ since}$$ B is not changing

$$\therefore \quad v_r = \int_1^2 E_r \, dr$$

$$= \int_{R_1}^{R_2} \left[\frac{i_r}{2\pi \sigma d \, r} - \omega r B_{gap} \right] dr$$

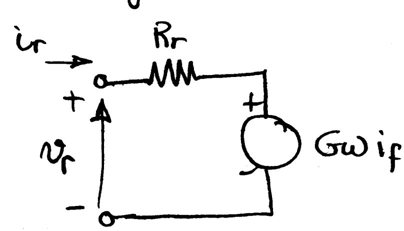
$$= \frac{i_r}{2\pi \sigma d} \left[\ln R_2 - \ln R_1 \right] - \omega B_{gap} \frac{\left[R_2^2 - R_1^2 \right]}{2}$$

rotor resistance

$$\approx i_r R_r - G \omega i_f$$ ← speed coefficient

i_f is the field current.

this is typically a very high current low voltage device.



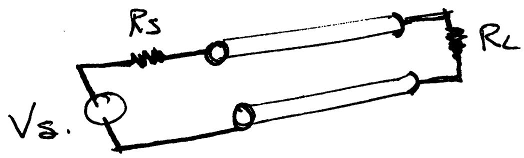
for $\sigma = 6 \times 10^7$ siemens/m
$d = 1mm$
$\omega = [3600 \text{ rpm}] \; 120\pi$ rad/sec.
$R_1 = 1cm$
$R_2 = 10cm$

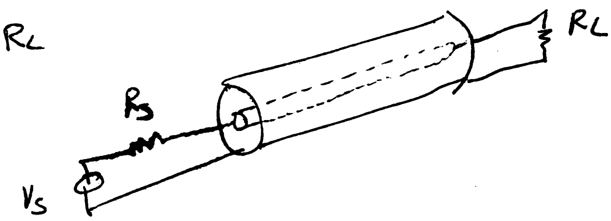
$G\omega i_f = -1.9$ volts.

short circuit current $\approx$ 350,000 Amperes.

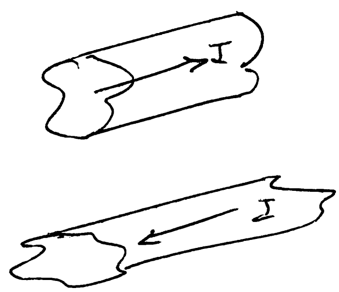
TRANSMISSION LINES

guide propagation of energy from one point to another
typically two conductors, (infinitely long at first)





these are uniform, i.e. cross-section is uniform.
lossless → no dielectric losses so Poynting vector is towards load.



as these are long, fields must only be in transverse coordinates and uniform,
This has important results!.

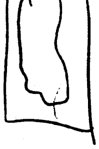
Maxwell's Equations

$$\oint \mathcal{E} \cdot d\underline{l} = -\mu \frac{d}{dt}\int \underline{\mathcal{H}} \cdot d\underline{s}$$

pick contour in transverse plane,
let ds be in transverse plane

$$\oint \mathcal{H} \cdot d\underline{l} = \int_s \underline{\mathcal{J}} \cdot d\underline{s} + \frac{\partial}{\partial t}\int \underline{D} \cdot d\underline{s}$$

i.e.



in plane.

pick x, y coordinates

since $ds = dx\,dy\,\underline{a_z}$

$$\oint (\mathcal{E}_x dx + \mathcal{E}_y dy) = -\mu \frac{d}{dt}\int \mathcal{H}_z\, dx\, dy$$

$$\oint (\mathcal{H}_x dx + \mathcal{H}_y dy) = \int \mathcal{J}_z \cdot dx\, dy + \epsilon \frac{\partial}{\partial t}\int \mathcal{E}_z\, dx\, dy$$

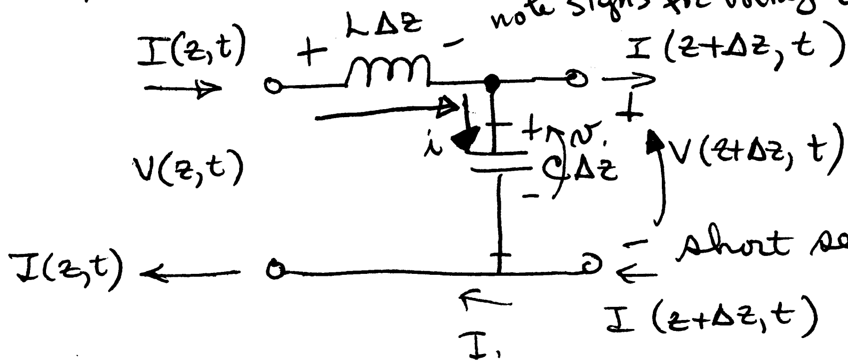
but $\mathcal{H}_z = \mathcal{E}_z = 0$ only transverse coordinates

∴ all time derivates → 0
thus, the static solutions are valid.

If the fields are the static fields we get some interesting results.

1. electric field is conservative
2. per unit length inductance is constant
3. per unit length capacitance is constant

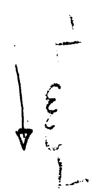
lumped parameter capacitance and inductance of lines.



I can stich these together to make a long line.

Note that these parameters are almost independent of frequency!

what do we know about circuits for inductors $v = L\frac{di}{dt}$

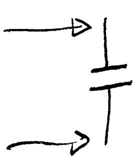


$$\frac{V(z+\Delta z,t) - V(z,t)}{\Delta z} = -L\frac{\partial I}{\partial t} = -\frac{L\Delta z \frac{\partial}{\partial t} I(z,t)}{\Delta z}$$

because of this relationship

$v = \int i\, dt$.

for capacitors $i = C\frac{dv}{dt}$.



$$\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = -\frac{C\frac{\partial V}{\partial t}}{\Delta z} = -\frac{C\Delta z \frac{\partial V(z+\Delta z,t)}{\partial t}}{\Delta z}$$

if $I(z+\Delta z,t) > I$

-sign to indicate current OUT of capacitor

$$\frac{I(z+\Delta z,t) - I(z,t)}{} = -C\Delta z\frac{\partial}{\partial t}\left[V(z,t) - L\Delta z\frac{\partial I(z,t)}{\partial t}\right]$$

$$\frac{I(z+\Delta z,t) - I(z,t)}{\Delta z} = \frac{-C\Delta z\frac{\partial V(z,t)}{\partial t} + LC(\Delta z)^2\frac{\partial^2 I(z,t)}{\partial t}}{\Delta z}$$

if $\Delta z \to 0$ becomes

transmission line equations

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t}$$

$$\frac{\partial I}{\partial z} = -C\frac{\partial V}{\partial t}$$

second term goes to zero.

$$\frac{\partial^2 V}{\partial z^2} = -L\frac{\partial^2 I}{\partial z \partial t} = -L\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial z}\right)$$

$$= -L\frac{\partial}{\partial t}\left(-C\frac{\partial V}{\partial t}\right) = LC\frac{\partial^2 V}{\partial t^2}$$

like wise $\boxed{\frac{\partial^2 I}{\partial z^2} = LC\frac{\partial^2 I}{\partial t^2}}$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}}$$

where $\frac{1}{\sqrt{LC}} = u$

the velocity of propagation

easy way to show this

$$V(z,t) = V^+(\underbrace{t-\tfrac{z}{u}}_{=s^+}) + V^-(\underbrace{t+\tfrac{z}{u}}_{=s^-})$$

$$I(z,t) = \underbrace{I^+\left(t-\tfrac{z}{u}\right) + I^-\left(t+\tfrac{z}{u}\right)}_{\text{don't know the form, but these are just like our phase functions}}$$

look at what happens.

$$\frac{\partial V}{\partial z} = \frac{\partial V^+}{\partial s^+}\frac{\partial s^+}{\partial z} + \frac{\partial V^-}{\partial s^-}\frac{\partial s^-}{\partial z} = \frac{\partial V^+}{\partial s^+}\left(-\frac{1}{u}\right) + \frac{\partial V^-}{\partial s^-}\left(\frac{1}{u}\right)$$

$$\frac{\partial^2 V}{\partial z^2} = -\frac{1}{u}\frac{\partial}{\partial s^+}\left(\frac{\partial V^+}{\partial s^+}\right)\frac{\partial s^+}{\partial z} + \frac{1}{u}\frac{\partial}{\partial s^-}\left(\frac{\partial V^-}{\partial s^-}\right)\frac{\partial s^-}{\partial z} = \frac{1}{u^2}\left[\frac{\partial^2 V}{\partial s^{+2}} + \frac{\partial^2 V}{\partial s^{-2}}\right]$$

$$\frac{\partial^2 V}{\partial t} = \frac{\partial V^+}{\partial s^+}\frac{\partial s^+}{\partial t} + \frac{\partial V^-}{\partial s^-}\frac{\partial s^-}{\partial t} = \frac{\partial V^+}{\partial s^+} + \frac{\partial V^-}{\partial s^-}$$

$$\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V^+}{\partial s^{+2}} + \frac{\partial^2 V^-}{\partial s^{-2}}$$