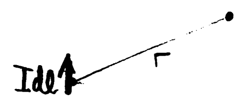
return to Lorentz Force law.

$$\underline{F} = q\underline{E} + \underbrace{q\underline{v} \times \underline{B}}_{\text{force due to charge in motion.}}$$

basically (without proof) the <u>Biot - Savart Law</u>

$$d\underline{H} = \frac{I\, dl \times a_r}{4\pi r^2}$$

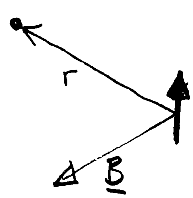


H is the magnetic field intensity; it corresponds to E. There also is a $\underline{B}$ which corresponds to D.

① Ampere' showed that currents exert forces on magnets.

② the next result was that electric currents exert forces on each other - and that a magnet could be replaced by a current.

③ the Biot - Savart Law involved determining the magnetic field from a magnet and an equivalent current.



current element.

④ the magnetic field intensity at $\underline{r}$ is given mathematically by

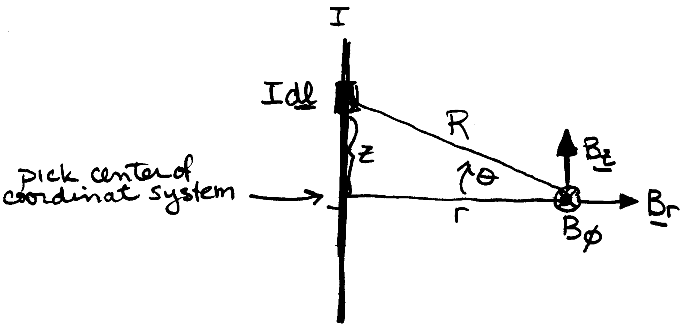
$$\underline{H}(\underline{r}) = \frac{I\, dl \times a_r}{4\pi r^2}$$

5 in general, I flows in a complex direction so we use the linear nature of the Biot - Savart Law and add up The contributions from many current elements.

$$H_{tot}(\underline{r}) = \int d\underline{H}(\underline{r}) = \underbrace{\int \frac{I\, \underline{dl} \times \underline{a_r}}{4\pi r^2}}_{\text{this is a contour integral}}$$

Three examples:

1 current in a wire



a use cylindrical coordinates

b define The angle θ as shown so that

$$\underline{B} = B_r \underline{a_r} + B_\phi \underline{a_\phi} + B_z \underline{a_z}$$

c Biot Savart law

$$\underline{H}(r, \phi, z) = \int \frac{I\, \underline{dl} \times \underline{a_R}}{4\pi R^2}$$

what is $I\underline{dl}$? $\quad I\, dz\, \underline{a_z}$

what is $\underline{a_R}$?

let. $\underline{a_R} = C_\phi \underline{a_\phi} + C_r \underline{a_r} + C_z \underline{a_z}$

dotting with a_r: $\underbrace{\underline{a_R} \cdot \underline{a_r}}_{\underbrace{|a_R|}_{1}\underbrace{|a_r|}_{1}\cos\theta} = C_\phi \cancelto{0}{\underline{a_\phi} \cdot \underline{a_r}} + c_r \underbrace{\underline{a_r} \cdot \underline{a_r}}_{1} + C_z \cancelto{0}{\underline{a_z} \cdot \underline{a_r}}$

$\therefore \quad \cos\theta = c_r$

dotting with $\underline{a_\phi}$:

$$\underline{a_R}\cdot\underline{a_\phi} = C_\phi \underbrace{\underline{a_\phi}\cdot\underline{a_\phi}}_{1} + C_r \cancelto{0}{\underline{a_r}\cdot\underline{a_\phi}} + C_z \cancelto{0}{\underline{a_z}\cdot\underline{a_\phi}}$$

$$\underbrace{|\underline{a_R}||a_\phi|\cos\frac{\pi}{2}}$$

this can also be seen to be zero as $\underline{a_R}$ and $\underline{a_\phi}$ are perpendicular.

$\therefore C_\phi = 0.$

dotting with $\underline{a_z}$:

$$\underline{a_R}\cdot\underline{a_z} = C_\phi \cancelto{0}{a_\phi\cdot\underline{a_z}} + C_r \cancelto{0}{\underline{a_r}\cdot\underline{a_z}} + C_z \cancelto{1}{\underline{a_z}\cdot\underline{a_z}}$$

$$|\underbrace{\underline{a_R}}_{1}||\underbrace{\underline{a_z}}_{1}|\cos\left(\frac{\pi}{2}+\theta\right)$$

$$\cancelto{0}{\cos\frac{\pi}{2}}\cos\theta - \cancelto{1}{\sin\frac{\pi}{2}}\sin\theta$$

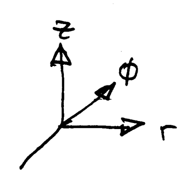
$\therefore \quad c_z = -\sin\theta.$

and $\quad \underline{a_R} = \cos\theta\,\underline{a_r} - \sin\theta\,\underline{a_z}$

<u>What is R?</u> $\quad R = \sqrt{z^2+r^2}.$

(d) · reduce integral

$$\underline{H}(r,\phi,z) = \int \frac{I\,dz\,\underline{a_z}\times(\cos\theta\,\underline{a_r} - \sin\theta\,\underline{a_z})}{4\pi(z^2+r^2)}$$



$$= \int \frac{I\,dz\,\cos\theta\,\underline{a_\phi}}{4\pi(z^2+r^2)} - \int \frac{I\,dz\,\sin\theta\,\cancelto{0}{\underline{a_z}\times\underline{a_z}}}{4\pi(z^2+r^2)}$$

note that $\cos\theta = \frac{r}{R}$ and write integral only in terms of z and r.

i.e. $\quad \cos\theta = \frac{r}{R} = \frac{r}{\sqrt{z^2+R^2}}$

$$\underline{H}(r,\phi,z) = \int \frac{I \; r \; a_\phi \; dz}{4\pi \, (z^2+r^2)^{1/2} \, (z^2+r^2)}$$

$$= \int \frac{I \, r \, \underline{a_\phi} \; dz}{4\pi \, (z^2+r^2)^{3/2}}$$

$$= \frac{I r \underline{a_\phi}}{4\pi} \int \frac{dz}{(z^2+r^2)^{3/2}}$$

(e) Now determine the exact contour and evaluate the integral. There is a trick here.

Let the wire extend from $-\frac{L}{2}$ to $+\frac{L}{2}$.

$$\int_{-\frac{L}{2}}^{+\frac{L}{2}} \frac{dz}{(z^2+r^2)^{3/2}} = \left. \frac{z}{r^2 (z^2+r^2)^{1/2}} \right|_{-\frac{L}{2}}^{+\frac{L}{2}}$$

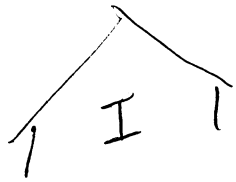
$$= \frac{\frac{L}{2}}{r^2\left(\frac{L^2}{4}+r^2\right)^{\frac{1}{2}}} - \frac{-\frac{L}{2}}{r^2\left(\frac{L^2}{4}+r^2\right)^{1/2}}$$

$$= \frac{L}{r^2\left(\frac{L^2}{4}+r^2\right)^{\frac{1}{2}}}$$

$$\therefore \; \underline{H}(r,\phi,z) = \frac{I \cancel{r} \underline{a_\phi}}{4\pi} \frac{L}{\cancel{r^2}\left(\frac{L^2}{4}+r^2\right)^{1/2}}$$

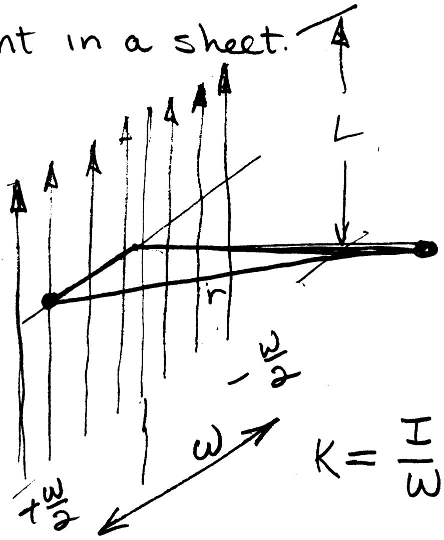
$$= \frac{I}{4\pi r} \underline{a_\phi} \frac{L}{\left(\frac{L^2}{4}+r^2\right)^{1/2}}$$

Note that if $L \to 0$ $\quad \underline{H}(r,\phi,z) \to \frac{I}{2\pi r} \underline{a_\phi}$



(2) current in a sheet.

a sheet looks like a lot of line currents.



$$K = \frac{I}{w}$$

do this problem in x, y, z coordinates
Why? easy to add vectors at point R.

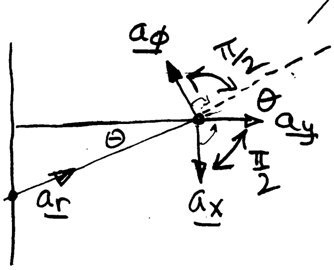
we know $H(r,\phi,z) = \frac{I}{4\pi r} \cdot \underline{a_\phi}$ as $L \to \infty$

what is this in Cartesian coordinates

$$H_\phi \underline{a_\phi} = H_x \underline{a_x} + H_y \underline{a_y} + H_z \underline{a_z}$$

$$H_\phi \cdot \underline{a_\phi} \cdot \underline{a_x} = H_x \cancelto{1}{\underline{a_x} \cdot \underline{a_x}} + H_y \cancelto{0}{\underline{a_y} \cdot \underline{a_x}} + H_z \cancelto{0}{\underline{a_z} \cdot \underline{a_x}}$$

top view



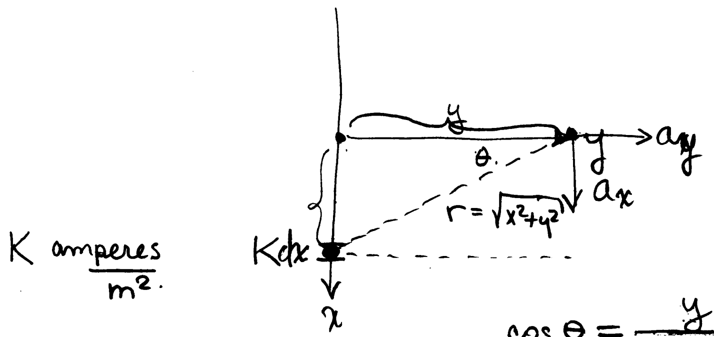
$$\underline{a_\phi} \cdot \underline{a_x} = |a_\phi||\underline{a_x}| \cos\left(\frac{\pi}{2} + \frac{\pi}{2} + \theta\right)$$
$$= \cos(\pi + \theta)$$
$$= \cancelto{-1}{\cos\pi} \cos\theta - \cancelto{0}{\sin\pi} \sin\theta$$
$$= -\cos\theta$$

$$H_\phi \underline{a_\phi} \cdot \underline{a_y} = H_x \cancelto{0}{\underline{a_x} \cdot \underline{a_y}} + H_y \cancelto{1}{\underline{a_y} \cdot \underline{a_y}} + H_z \cancelto{0}{\underline{a_z} \cdot \underline{a_y}}$$

$$\underline{a_\phi} \cdot \underline{a_y} = |\underline{a_\phi}||\underline{a_y}| \cos\left(\frac{\pi}{2} + \theta\right)$$
$$= \cancelto{0}{\cos\frac{\pi}{2}} \cos\theta - \cancelto{1}{\sin\frac{\pi}{2}} \sin\theta = -\sin\theta$$

$$\therefore \quad H_x = \frac{I}{2\pi r}(-\cos\theta)$$

$$H_y = \frac{I}{2\pi r}(-\sin\theta)$$



7

$$\cos\theta = \frac{y}{\sqrt{x^2+y^2}}, \quad \sin\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$H_x = \int \frac{(K\,dx)}{2\pi r}\left(-\frac{y}{(x^2+y^2)^{1/2}}\right) = \int -\frac{Ky}{2\pi}\frac{dx}{(x^2+y^2)}$$

also $\sqrt{x^2+y^2}$

$$= -\frac{Ky}{2\pi}\int_{-\frac{w}{2}}^{+\frac{w}{2}} \frac{dx}{(x^2+y^2)} = -\frac{Ky}{2\pi}\,\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)\Bigg|_{-\frac{w}{2}}^{+\frac{w}{2}}$$

math errors in integral!

$$= -\frac{K}{2\pi y}\;\underbrace{\tan^{-1}\left(\frac{w}{2y}\right) - \tan^{-1}\left(\frac{-w}{2y}\right)}_{\text{this is equivalent to}}$$

$\tan^{-1}(\theta_{max}) - \tan^{-1}(-\theta_{max})$.

$2\tan^{-1}(\theta_{max})$

and as $\theta_{max} \rightarrow \frac{\pi}{2}$ $\quad H_x \rightarrow \infty$

$$H_y = \int \frac{K\,dx}{2\pi r}\left(-\frac{x}{\sqrt{x^2+y^2}}\right) = \frac{-K}{2\cdot 2\pi}\int_{-\frac{w}{2}}^{+\frac{w}{2}} \frac{2x\,dx}{x^2+y^2} = \frac{-K}{4\pi}\left[(x^2+y^2)^{-2}\right]\Bigg|_{-\frac{w}{2}}^{+\frac{w}{2}}$$

$$= \frac{-K}{4\pi}\,\frac{1}{(x^2+y^2)^2}\Bigg|_{-\frac{w}{2}}^{+\frac{w}{2}} \Rightarrow 0.$$

Ampère Law (this is the magnetic equivalent of Gauss' law)

the derivation of Ampère's Law follows from the Biot-Savart law

Biot-Savart Law

$$d\underline{H} = \frac{I\, d\underline{l} \times \underline{a_r}}{4\pi r^2}$$

let I become a volume distribution of current
not $I\,d\underline{l}$
but $\underline{J}\,dv$

$$d\underline{H}(r') = \frac{\underline{J}(r')\, dv' \times \underline{a_r}}{4\pi r^2}$$

include source & field coordinates

integrating

$$\underline{H}(r) = \frac{1}{4\pi}\int_{v'} \frac{J(r')\,dv' \times \underline{a_r}'}{r'^2}$$

$$= \frac{1}{4\pi}\int \frac{\underline{J}(r') \times \underline{a_r}'}{r'^2}\, dv'$$

note that $\quad \dfrac{\underline{a_r}'}{r'^2} = \underline{\nabla}'\left(\dfrac{1}{r'}\right) = \dfrac{\partial}{\partial r}(r'^{-1}) = -r'^{-2}$

where ∇ operates on source coordinates

$$\underline{H}(r) = \frac{-1}{4\pi}\int \underline{J}(r') \times \underline{\nabla}\left(\frac{1}{r}\right) dv'$$

taking the divergence of both sides

$$\underline{\nabla}\cdot\underline{H} = -\frac{1}{4\pi}\int \underbrace{\underline{\nabla}}_{\text{operates on } r,\ \text{not } r'}\cdot\left(\underline{J}(r') \times \underline{\nabla}\left(\frac{1}{r}\right)\right) dv'$$

use the vector identity

$$\nabla\cdot(\underline{A}\times\underline{B}) = \underline{B}\cdot(\underline{\nabla}\times\underline{A}) - \underline{A}\cdot(\nabla\times\underline{B})$$

rewrite the integrand as.

$$\underline{\nabla}\cdot\left(\underline{J}\times\underline{\nabla}\left(\frac{1}{r}\right)\right) = \underline{\nabla}\left(\frac{1}{r}\right)\cdot(\underline{\nabla}\times\underline{J}) - \underline{J}\cdot\underbrace{\underline{\nabla}\times\underline{\nabla}\left(\frac{1}{r}\right)}_{\underline{\nabla}\times\underline{\nabla}f \equiv 0 \text{ for any } f}$$

$$\underline{\nabla} \times \underline{H} = -\frac{1}{4\pi} \int \underline{\nabla}^2 \left(\frac{1}{r}\right) \underline{J} \, dv$$

This is a special integral.

$$\int \underline{\nabla}^2 \left(\frac{1}{r}\right) \underline{J}(r') \, dr' =$$

$\nabla^2 \left(\frac{1}{r}\right) = 0$ for all finite values of r

thus, if (x,y,z) is outside a finite source region
r is always > 0 and $\nabla^2 \left(\frac{1}{r}\right) \equiv 0$ so that $\underline{\nabla} \times \underline{H} \equiv 0$.

If (x,y,z) is in a source region, then
r is sometimes is zero and $\nabla^2 \left(\frac{1}{r}\right) \neq 0$.

Basically $\nabla^2 \left(\frac{1}{r}\right)$ is a delta function of r and r'

$$\int 4\pi \delta(\underline{r} - \underline{r}') \, J(\underline{r}') \, dv' = -4\pi J(\underline{r})$$

$$\underline{\nabla} \times \underline{H} = -\frac{1}{4\pi} \cdot -4\pi J(\underline{r})$$

$$\nabla \times H(\underline{r}) = J(\underline{r})$$

Applying Stokes Theorem

$$\int_S \underline{\nabla} \times \underline{H} \cdot d\underline{S} = \oint_C \underline{H} \cdot d\underline{l}$$

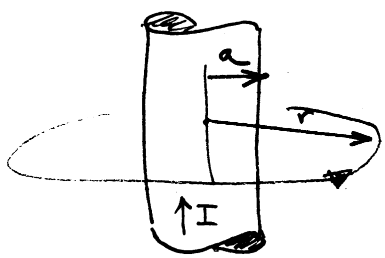
$$\int_S \underline{\nabla} \times \underline{H} \cdot d\underline{s} = \int_S \underline{J} \cdot d\underline{s}$$

or $$\oint \underline{H} \cdot d\underline{l} = \int_S \underline{J} \cdot d\underline{s}$$

Now that we have Ampère' Law how can we use it.

<u>Just like Gauss Law</u>

<u>Example 1</u> Field from an infinite wire of finite radius.



consider $\underline{H}\cdot d\underline{l}$

pick a contour C outside any sources.

$\underline{H} = H_r \underline{a_r} + H_\phi \underline{a_\phi} + H_z \underline{a_z}$

$d\underline{l} = r\,d\phi\,\underline{a_\phi}$ pick a circular contour

$$\underline{H}\cdot d\underline{l} = H_\phi\, r\, d\phi$$

$$\oint_C \underline{H}\cdot d\underline{l} = \int_0^{2\pi} H_\phi\, r\, d\phi = 2\pi r H_\phi \quad (7)$$

$\int \underline{J}\cdot d\underline{S} = I \quad \therefore \quad 2\pi r H_\phi = I$ and $H_\phi = \dfrac{I}{2\pi r}$ for $\geq a$

suppose $r<0$

then only a part of I $\displaystyle\int \underline{J}\cdot d\underline{s} = \int_0^r \frac{I}{\pi a^2}\, r\,dr\,d\phi = \frac{I\cdot\frac{r^2}{\cancel{2}}\cdot\cancel{2\pi}}{\cancel{\pi}a^2}$

$$= I\frac{r^2}{a^2}$$

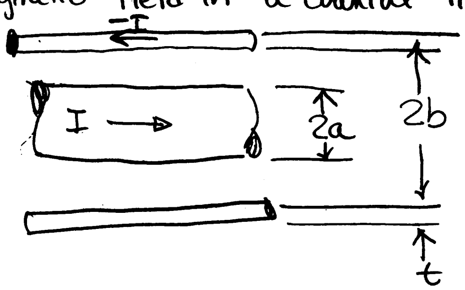
$$2\pi r H_\phi = I\frac{r^2}{a^2}$$

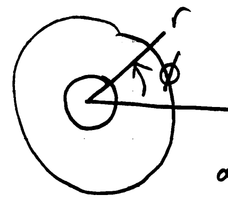
$$H_\phi = \frac{I}{2\pi}\frac{r}{a^2} \qquad r<a$$

$$= \frac{I}{2\pi r} \qquad r>a$$

Example #2

Magnetic field in a coaxial line.





as Before

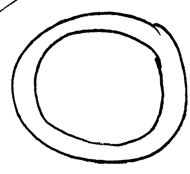
the field is The same as befre for $r<b$

$$H_\phi = \begin{cases} \dfrac{I}{2\pi r}\,\dfrac{r}{a^2} & r<a \\ \dfrac{I}{2\pi r} & r>a \end{cases}$$

consider now what happens between b and b+t.

$$\int \underline{J}\cdot d\underline{S} = \underbrace{\int_0^a J\cdot dS}_{I} + \int_b^r \frac{-I}{\pi(2bt+t^2)}\cdot r\,dr\,d\phi$$

$$J = \frac{-I}{\pi(2bt+t^2)}$$



$$\begin{aligned} \text{area} &= \pi(b+t)^2 - \pi(b)^2 \\ &= \pi(b^2+2bt+t^2-b^2) \\ &= \pi(2bt+t^2) \end{aligned}$$

$$= I - I\frac{1}{\pi(2bt+t^2)}\int_b^r r\,dr\,d\phi$$

$$= I\left(1 - \frac{1}{\pi(2bt+t^2)}\,\cancel{2}\pi\,\frac{r^2}{\cancel{2}}\Bigg|_b^r\right)$$

$$= I\left(1 - \frac{\cancel{\pi}}{\cancel{\pi}}\,\frac{r^2-b^2}{(b+t)^2-b^2}\right)$$

$$= I\left(1 - \frac{r^2-b^2}{(b+t)^2-b^2}\right)$$

$$2\pi r H_\phi = I\left(\frac{(b+t)^2 - b^2 - r^2 + b^2}{(b+t)^2 - b^2}\right)$$

$$H_\phi = \frac{I}{2\pi r}\left[\frac{(b+t)^2 - r^2}{(b+t)^2 - b^2}\right] \qquad b \leq r < b+t$$

For $\quad r > b+t \qquad \int \underline{J} \cdot d\underline{S} = I - I = 0$

$$H_\phi = 0 \qquad b+t \leq r$$

Now that we have Ampère's law in integral form we need to think about the differential form.

$$\oint_C \underline{H} \cdot \underline{dl} = \int_S J \cdot ds$$

Recall that $\quad \text{curl} F \triangleq \lim_{\Delta s \to 0} \dfrac{\oint_c \underline{F} \cdot \underline{dl}}{\Delta s} \underline{a_n} = \underline{\nabla} \times \underline{F}$

i.e. the contribution to a vector circulation from a differential surface.

consider for Ampère's Law.

$$\lim_{\Delta s \to 0} \frac{\oint \underline{H} \cdot \underline{dl}}{\Delta s} \underline{a_n} = \text{curl } \underline{H} = \lim_{\Delta s \to 0} \frac{\int \underline{J} \cdot \underline{da}}{\Delta s} \underline{a_n}$$

also

$$= \lim_{\Delta s \to 0} \frac{\underline{J} \Delta s}{\Delta s} \underline{a_n}$$

$$= \underline{J} \cdot \underline{a_n}$$

the curl and J are in the same direction

$$\underline{\nabla} \times \underline{H} = \underline{J}$$

Question: can we define any functions (potentials) for the magnetic field that are easier to find and from which we can obtain the field vectors.

Yes, both vector and scalar.

For a source-free region of space $\nabla \times \underline{H} = J \rightarrow 0$

$\nabla \times (-\nabla \mathcal{F}) = 0$ is true for any scalar field $\mathcal{F}$.

So let $\underline{H} = -\nabla \mathcal{F}$

$\oint H \cdot dl$ then becomes $\oint_{P_1}^{P_2} -\nabla \mathcal{F} = \mathcal{F}(P_2) - \mathcal{F}(P_1) = 0$

just as for our conservative electric field.

$\mathcal{F}$ is called the magnetomotive potential

finally, note that the flux

$$\Phi_m = \oint_S \mu_0 \underline{H} \cdot \underline{ds} = 0 \quad \text{over any closed surface.}$$

why? because there is no magnetic monopole.

a sink or source of magnetic flux.

$$\oint_S \underline{H} \cdot \underline{ds} = \int_V \underline{\nabla} \cdot \underline{H} \, dv = 0$$

since $\underline{\nabla} \cdot \underline{H} = \rho_m$ a magnetic charge density

Magnetic vector potential

The magnetomotive potential was defined for a source free region of space. Suppose $\underline{J} \neq 0$.

we noted that $\underline{B} = \mu_0 \underline{H}$ just as $\underline{D} = \epsilon_0 \underline{E}$ defines a flux density

Let us examine a vector potential.

starting with $\underline{\nabla} \cdot \underline{B} = 0$

$$\nabla \cdot \nabla \times A = 0$$

so let $\underline{B} = \underline{\nabla} \times \underline{A}$
magnetic vector potential

Ampère's Law

$$\nabla \times \underline{H} = \underline{J}$$

becomes $\underbrace{\underline{\nabla} \times \mu_0 \underline{H}}_{\underline{B}} = \mu_0 \underline{J}$

Consider our definition and taking the ~~divergence~~ curl of both sides

$$\underline{\nabla} \times \underline{B} = \underline{\nabla} \times \underline{\nabla} \times \underline{A}$$

$$= \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}$$

To simplify our results let's require $\underline{\nabla} \cdot \underline{A} = 0$. This is not arbitrary — it's related to something called the Lorentz condition — and as we have not specified $\underline{A}$ is just another condition:

$$\therefore \quad \underline{\nabla} \times \underline{B} = -\nabla^2 \underline{A}$$

$$\nabla^2 A = -\mu J \quad \& \quad \underline{\nabla} \cdot \underline{A} = 0$$

The vector potential is then specified by

$$-\nabla^2 \underline{A} = \mu \underline{J}$$

(Eqn. 31 p. 150 should have a minus sign in it)

$$\underline{\nabla}^2 \underline{A} = -\mu J$$

$$\underline{\nabla} \cdot \underline{A} = 0$$

How to solve this set of equations in integral from.

Recall that the electric potential obeyed a similar equation

$$\nabla^2 \Phi = 0$$

and had a general solution $\quad \Phi = \int_v \frac{\rho_v \, dv}{4\pi\epsilon r}$

By analogy, we shall use the same solution

$$\underline{A} = \int \frac{\mu \underline{J} \, dv}{4\pi r}$$

We can then operate on this equation to show that the Biot-Savart and Ampere's law can be derived from this solution.

Properties of the vector potential:

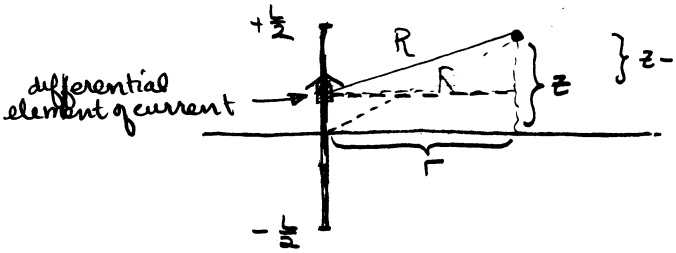
- A is not unique $\quad$ let $A' = \underline{A} + \nabla f$

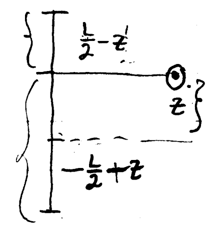
$$\underline{B} = \underline{\nabla} \times \underline{A}' = \underline{\nabla} \times (\underline{A} + \underline{\nabla} f) = \underline{\nabla} \times \underline{A} + 0$$

as $\quad \underline{\nabla} \times \underline{\nabla} f \equiv 0$

Examples of vector and scalar potential

Example 1: Vector potential of a finite length line current.





$\nabla^2 \Phi = -\frac{\rho}{\epsilon}$ Poisson's Eqn.

$\underline{A} = \int \frac{\mu \underline{J}}{4\pi R}\, dv$ this is almost identical to defin for $\Phi = \int \frac{\rho}{4\pi\epsilon_0 R}\, dv$

$\nabla^2 \underline{A} = -\mu \underline{J}$ vector potential

$\underline{A} = \int \frac{\mu \underline{J}\, dv}{4\pi R}$

Note: $\underline{J}\, dv \rightarrow \underline{K}\, dS \rightarrow I\, d\underline{\ell} \rightarrow q\underline{v}$

These are all just current elements.

$$\underline{A} = \int \frac{\mu I\, d\underline{\ell}}{4\pi R} = \int \frac{\mu I\, dz'\, \underline{a_z}}{4\pi \sqrt{(z-z')^2 + r^2}}$$

dz' are the coordinates of the current element.

$$= \frac{\mu I}{4\pi} \underline{a_z} \int_{z'=-\frac{L}{2}}^{z'=+\frac{L}{2}} \frac{dz'}{\sqrt{(z-z')^2 + r^2}} = \frac{\mu I}{4\pi} \underline{a_z}$$

let $z'' = z - z'$ ← limits are $\pm\frac{L}{2}$

$dz'' = -dz'$

$z'' = z - \frac{L}{2}$

$z +$

$$\int_{z+\frac{L}{2}}^{z-\frac{L}{2}} \frac{-dz''}{\sqrt{z''^2 + r^2}} = + \int_{z-\frac{L}{2}}^{z+\frac{L}{2}} \frac{dz''}{(z''^2 + r^2)^{\frac{1}{2}}} = \ln\left(z'' + \sqrt{z''^2 + r^2}\right)\Big|_{z-\frac{L}{2}}^{z+\frac{L}{2}}$$

$$= \ln\left(z + \frac{L}{2} + \sqrt{\left(z+\frac{L}{2}\right)^2 + r^2}\right)$$

$$- \ln\left(z - \frac{L}{2} + \sqrt{\left(z-\frac{L}{2}\right)^2 + r^2}\right)$$

$$= \ln r \left\{\frac{z+\frac{L}{2}}{r} + \sqrt{\left(\frac{z+\frac{L}{2}}{r}\right)^2 + 1}\right\} - \ln r \left\{\left(\frac{z-\frac{L}{2}}{r}\right) + \sqrt{\frac{\left(z-\frac{L}{2}\right)^2}{r} + 1}\right\}$$

$$= \cancel{\ln r} + \ln\left(\frac{z+\frac{L}{2}}{r}\right) + \sqrt{\left(\frac{z+\frac{L}{2}}{r}\right)^2 + 1} - \cancel{\ln r} - \ln\left(\frac{z-\frac{L}{2}}{r}\right)^2 + \sqrt{\frac{\left(z-\frac{L}{2}\right)^2}{r} + 1}$$

$$A = \frac{\mu I}{4\pi}\,\underline{a_z}\left[\ln\left|\left(\frac{z+\frac{L}{2}}{r}\right)+\sqrt{\left(\frac{z+\frac{L}{2}}{r}\right)^2+1}\right| - \ln\left|\left(\frac{z-\frac{L}{2}}{r}\right)^2+\sqrt{\left(\frac{z-\frac{L}{2}}{r}\right)^2+1}\right|\right]$$

$$= \frac{\mu I}{4\pi}\,\underline{a_z}\left[\sinh^{-1}\left(\frac{z+\frac{L}{2}}{r}\right) - \sinh^{-1}\left(\frac{z-\frac{L}{2}}{r}\right)\right]$$

$$\underline{B} = \underline{\nabla}\times\underline{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\underline{a_r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\underline{a_\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial\phi}\right)\underline{a_z}$$

need $\nabla\times A$ $\rightarrow$

$$= -\frac{\partial A_z}{\partial r}\,\underline{a_\phi}$$

recalling $\frac{\partial}{\partial x}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{\partial u}{\partial x}$

$$= -\frac{\mu I}{4\pi}\,\underline{a_\phi}\left[\frac{1}{\sqrt{\left(\frac{z+\frac{L}{2}}{r}\right)^2+1}}\left(z+\frac{L}{2}\right)\left(\frac{-1}{r^2}\right) - \frac{1}{\sqrt{\left(\frac{z-\frac{L}{2}}{r}\right)^2+1}}\left(z-\frac{L}{2}\right)\left(-\frac{1}{r^2}\right)\right]$$

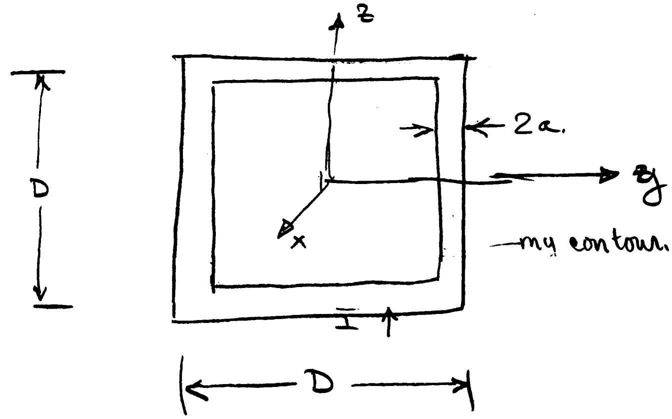
$$= -\frac{\mu I}{4\pi}\,\underline{a_\phi}\left[\frac{\left(z+\frac{L}{2}\right)\left(-\frac{1}{r^2}\right)}{\frac{1}{r}\sqrt{\left(z+\frac{L}{2}\right)^2+r^2}} - \frac{\left(z-\frac{L}{2}\right)\left(-\frac{1}{r^2}\right)}{\frac{1}{r}\sqrt{\left(z-\frac{L}{2}\right)^2+r^2}}\right]$$

$$= \frac{\mu I}{4\pi r}\,\underline{a_\phi}\left[\frac{z+\frac{L}{2}}{\sqrt{\left(z+\frac{L}{2}\right)^2+r^2}} - \frac{z-\frac{L}{2}}{\sqrt{\left(z-\frac{L}{2}\right)^2+r^2}}\right]$$

if $\frac{L}{2}\rightarrow\infty$

$$\underline{B} \rightarrow \frac{\mu I}{4\pi r}\,\underline{a_\phi}\left[\frac{\frac{L}{2}}{\frac{L}{2}} - \frac{-\frac{L}{2}}{\frac{L}{2}}\right] = \frac{\mu I}{2\pi r}\,\underline{a_\phi}$$

Example 2: Flux through a square loop.



Use the result for a finite length current along the z-axis.

$$\underline{A} = \frac{\mu I}{4\pi} \underline{a_z} \left[\sinh^{-1}\left(\frac{z+\frac{L}{2}}{r}\right) - \sinh^{-1}\left(\frac{z-\frac{L}{2}}{r}\right) \right]$$

To find the flux we recall that

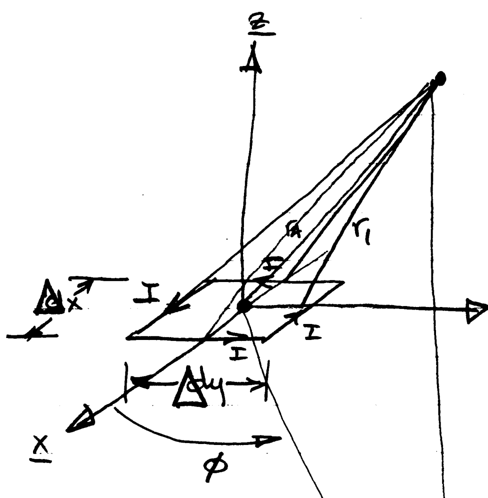
$$\Phi = \int_S \underline{B} \cdot \underline{ds} = \underbrace{\int_S \underline{\nabla \times A} \cdot \underline{ds} = \oint_C \underline{A} \cdot \underline{dl}}_{\text{by stokes theorem.}}$$

Note that the contour integral consists of four parts – all identical, All currents are in the same direction as the contour so all add.

$$\Phi = 4 \int_{-(\frac{D}{2}-a)}^{(\frac{D}{2}-a)} A_z \, dz$$

$$= 4 \int_{-(\frac{D}{2}-a)}^{(\frac{D}{2}-a)} \frac{\mu I}{4\pi} \left[\sinh^{-1}\left(\frac{z+\frac{D}{2}}{a}\right) - \sinh^{-1}\left(\frac{z-\frac{D}{2}}{a}\right) \right] dz.$$

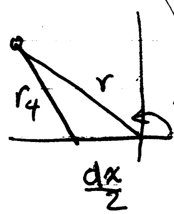
Example 3 Magnetic dipole (square)

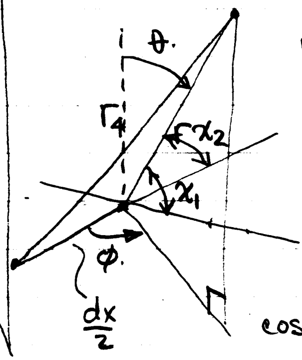


$$\underline{A} = \int \frac{\mu J}{4\pi r}\, dv.$$

$$\frac{\underline{A}}{4} = \frac{\mu_0 I\, dy\, a_y}{4\pi} \frac{1}{r_4}$$

use Δx
Δy.



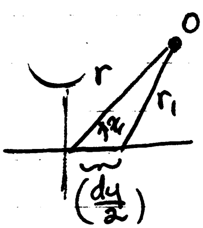


where

$$r_4^2 = r^2 + \left(\frac{dx}{2}\right)^2 - 2(r)\left(\frac{dx}{2}\right)\cos\left(\pi - \frac{x_2}{2}\right)$$

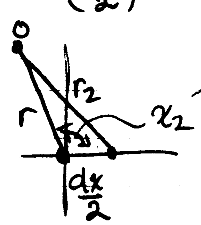
$$= r^2 + \frac{dx^2}{4} + r\,dx\cos x_2$$

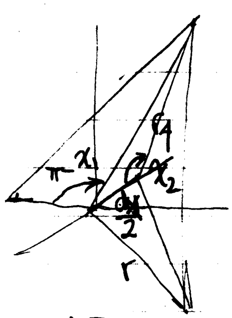
$$\cos\left(\pi - \frac{x}{2}\right) = \overset{-1}{\cos\pi}\cos x_2 + \overset{0}{\sin\pi}\sin x_2$$

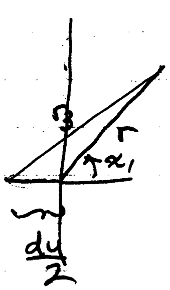


$$\underline{A}_1 = -\frac{\mu_0 I\, dx\, a_x}{4\pi}\frac{1}{r_1} \quad \text{where } r_1 = r^2 + \left(\frac{dy}{2}\right)^2 - 2(r)\left(\frac{dy}{2}\right)\cos x_1$$

$$\underline{A}_2 = -\frac{\mu_0 I\, dy\, a_y}{4\pi}\frac{1}{r_2} \quad \text{where } r_2 = r^2 + \left(\frac{dx}{2}\right)^2 - 2(r)\left(\frac{dx}{2}\right)\cos x_2$$







$$\underline{A}_3 = \frac{\mu_0 I\, dx\, a_x}{4\pi}\frac{1}{r_3} \quad \text{where } r_3 = r^2 + \left(\frac{dy}{2}\right)^2 - 2(r)\left(\frac{dy}{2}\right)\cos(\pi - x_1)$$

$$= r^2 + \frac{dy^2}{4} + r\,dy\cos x_1$$

consider $\lim_{\substack{\frac{dx}{r}\to 0 \\ \frac{dy}{r}\to 0}} \underline{A}$

consider A_1 in this limit

$$\frac{dx}{\sqrt{r^2+\left(\frac{dy}{2}\right)^2 - r\,dy\cdot\cos x_1}} = \frac{dx}{r\left[1+\left(\frac{dy}{2dr}\right)^2 - \frac{dy}{r}\cos x_1\right]^{1/2}}$$

$$\approx \frac{\frac{dx}{r}}{1-\frac{1}{2}\frac{dy}{r}\cos x_1}$$

$$\approx \frac{dx}{r}\left(1+\frac{1}{2}\frac{dy}{r}\cos x_1\right)$$

$\therefore$ in the limit $\underline{A}_1 \approx -\frac{\mu_0 I}{4\pi}\underline{a_x}\left[\frac{dx}{r}+\frac{dx\,dy}{2r^2}\cos x_1\right]$

for A_3.
$$\frac{dx}{r^2+\left(\frac{dy}{2}\right)^2 + r\,dy\cos x_1} = \frac{dx}{r\left[1+\left(\frac{dy}{2}\right)^2+\frac{dy}{r}\cos x_1\right]^{1/2}}$$

$$\approx \frac{\frac{dx}{r}}{1+\frac{1}{2}\frac{dy}{r}\cos x_1} = \frac{dx}{r}\left[1-\frac{dy}{2r}\cos x_1\right].$$

$$\underline{A}_3 \approx \frac{\mu_0 I}{4\pi}\underline{a_x}\,\frac{dx}{r}\left[1-\frac{dy}{2r}\cos x_1\right]$$

$$\underline{A}_1+\underline{A}_3 = \frac{\mu_0 I}{4\pi}\underline{a_x}\left[-\cancel{\frac{dx}{r}}-\frac{1}{2}\frac{dx\,dy}{r^2}\cos x_1+\cancel{\frac{dx}{r}}-\frac{1}{2}\frac{dx\,dy}{r^2}\cos x_1\right]$$

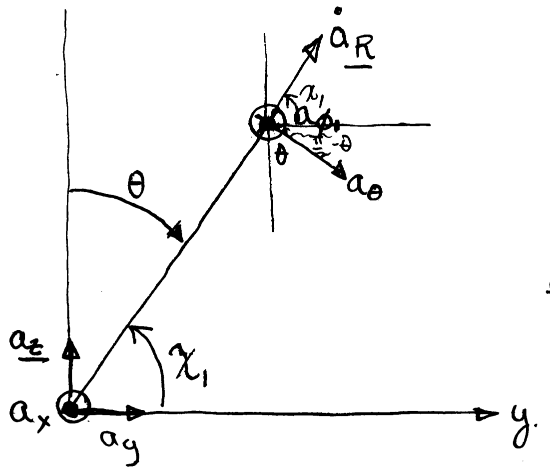
$$= -\frac{\mu_0 I}{4\pi r^2}\left[dx\,dy\cos x_1\ \underline{a_x}\right].$$

Similarily

$$\underline{A_2}+\underline{A_4} = -\frac{\mu_0 I}{4\pi r^2}\,dx\,dy\cos x_2\ \underline{a_y}$$

$$\underline{A} = -\frac{\mu_0 I}{4\pi r^2}\, dx\, dy \left[\cos\chi_1 \underline{a_x} + \cos\chi_2 \underline{a_y} \right]$$

lets put this in spherical coordinates.



this is in plane.

$$\underline{a_R} \cdot \underline{a_y} = |\underline{a_R}||\underline{a_y}| \cos\chi_1 = \cos\chi_1$$

$$\underline{a_R} \cdot \underline{a_x} = |\underline{a_R}||\underline{a_y}| \underbrace{(-\cos\chi_2)}_{\text{recall direction of } \chi_2} = -\cos\chi_2$$

by inspection $\cos\chi_1 = \cos\left(\frac{\pi}{2} - \theta\right)$

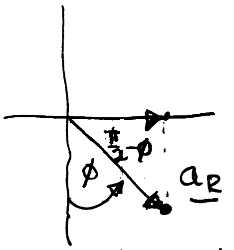
$$= \cancel{\cos\frac{\pi}{2}}^{\,0} \cos\theta + \cancel{\sin\frac{\pi}{2}}^{\,1} \sin\theta$$

$$= +\sin\theta$$

This is however only for $\phi = \frac{\pi}{2}$.

For $\phi \neq \frac{\pi}{2}$ this is reduced by $\sin\phi$.

so $\cos\chi_1 = +\sin\theta\,\sin\phi$.



this dot product is

$|1||1|\cos\left(\frac{\pi}{2}-\phi\right)$

$\Downarrow$

$\sin\phi$

similarly

$+\cos\chi_2 = -\sin\theta\cos\phi$.

sign.

$$\therefore \quad \underline{A} \rightarrow -\frac{\mu_0 I}{4\pi r^2}\, dx\,dy\,\left[\cos\chi_1\, \underline{a_x} + \cos\chi_2\, \underline{a_y}\right].$$

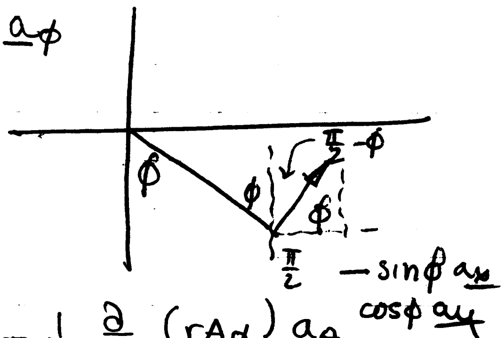
but $\underline{a_R} \cdot \underline{a_y} = \cos\chi_1 = \sin\theta \sin\phi$

$\underline{a_R} \cdot \underline{a_x} = -\cos\chi_2 = +\sin\theta\cos\phi.$

$$\underline{A} \doteq -\frac{\mu_0 I}{4\pi r^2}\, dx\,dy\,\left[\sin\theta\sin\phi\, \underline{a_x} + \sin\theta\cos\phi\, \underline{a_y}\right]$$

$$= \frac{\mu_0 I}{4\pi r^2}\, \underbrace{dx\,dy}_{dS}\, \sin\theta\, \left[\underbrace{-\sin\phi\, \underline{a_x} + \cos\phi\, \underline{a_y}}_{\underline{a_\phi}}\right]$$

$$\underline{A} = \frac{\mu_0 I}{4\pi r^2}\, dS\, \sin\theta\, \underline{a_\phi}$$



$$\underline{B} = \underline{\nabla} \times \underline{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi \sin\theta)\, \underline{a_r} - \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\, \underline{a_\theta}$$

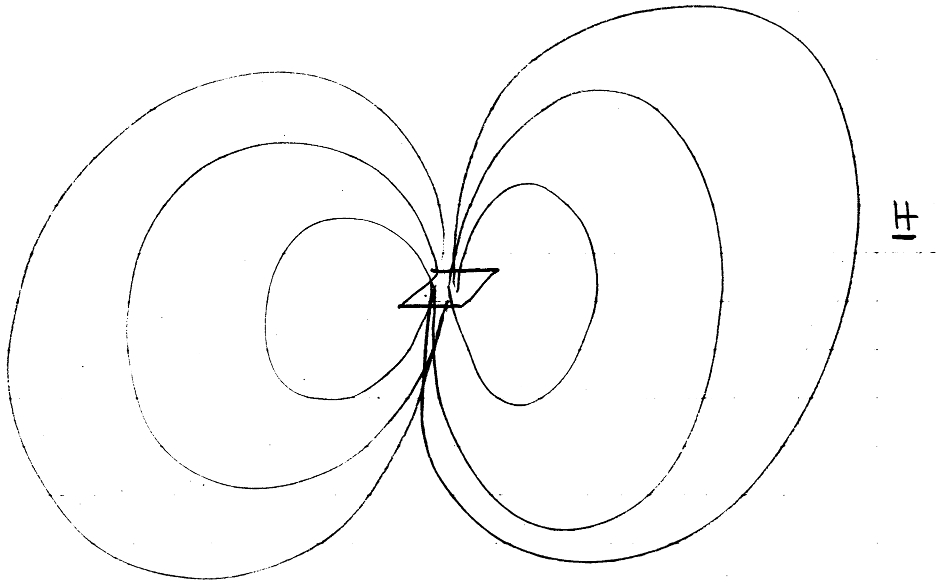
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{\mu_0 I}{4\pi r^2}\, dS\, \sin^2\theta\, \underline{a_r}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi}\, r^{-1}\, dS\sin\theta\right]\underline{a_\theta}$$

$$= \frac{\mu_0 I\, dS}{4\pi}\left[\frac{1}{r\sin\theta}\, \frac{1}{r^2}\, 2\sin\theta\cos\theta\, \underline{a_r} - \frac{1}{r}\, \frac{(-1)}{r^2}\, \sin\theta\, a_\theta\right]$$

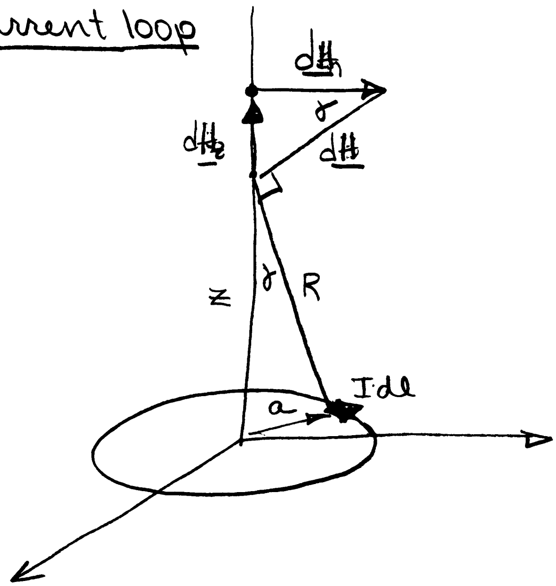
$$= \frac{\mu_0 I\, dS}{4\pi}\left[\frac{2}{r^3}\cos\theta\, \underline{a_r} + \frac{1}{r^3}\sin\theta\, \underline{a_\theta}\right]$$

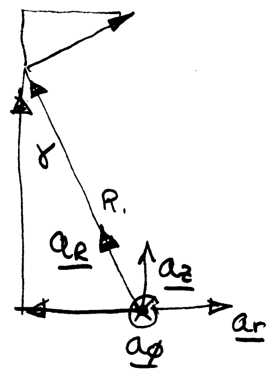
$$= \frac{\mu_0 I}{4\pi r^3}\, dS\, \left[2\cos\theta\, \underline{a_r} + \sin\theta\, \underline{a_\theta}\right].$$

This is the same field as for the electric dipole.



<u>single current loop</u>





$$d\underline{H} = \frac{I\,\underline{d\ell} \times \underline{a}_r}{4\pi R^2}$$

$$I\,\underline{d\ell} = I\,a\,d\phi\,\underline{a}_\phi$$

$$\underline{a}_R = c_1\,\underline{a}_\phi + c_2\,\underline{a}_z + c_3\,\underline{a}_r$$

$$\underline{a}_R \cdot \underline{a}_\phi = 0 \qquad \therefore \quad c_1 = 0$$

$$c_2 = \underline{a}_R \cdot \underline{a}_z = |\underline{a}_R||\underline{a}_z|\cos\gamma = \cos\gamma$$

$$c_3 = \underline{a}_R \cdot \underline{a}_r = |\underline{a}_R||\underline{a}_r|\cos(\pi/2+\gamma) = \cancel{\cos\tfrac{\pi}{2}}^{0}\cos\gamma - \sin\tfrac{\pi}{2}\sin\gamma = -\sin\gamma$$

$$\underline{a}_R = \cos\gamma\,\underline{a}_z - \sin\gamma\,\underline{a}_r$$

$$R^2 = a^2 + z^2$$

$$dH = \frac{I a\,d\phi\,\underline{a}_\phi \times (\cos\gamma\,\underline{a}_z - \sin\gamma\,\underline{a}_r)}{4\pi(a^2+z^2)}$$

$$dH = \frac{I a \, d\phi}{4\pi (a^2+z^2)} \cdot \left(\cos\gamma \, \underline{a_r} + \sin\gamma \, \underline{a_z} \right)$$

$$= \frac{Ia}{4\pi} \left[\frac{\cos\gamma}{a^2+z^2} \, d\phi \, \underline{a_r} + \frac{\sin\gamma}{a^2+z^2} \, d\phi \, \underline{a_z} \right]$$

$$\cos\gamma = \frac{z}{(a^2+z^2)^{1/2}} \qquad \sin\gamma = \frac{a}{(a^2+z^2)^{1/2}}$$

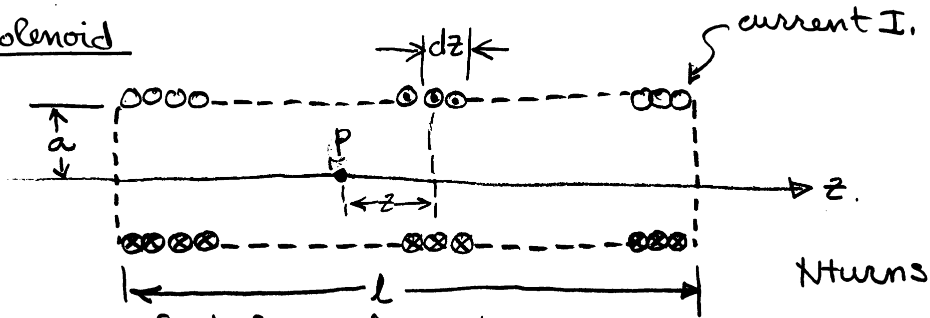
$$H_r = \int dH_r = \frac{Ia}{4\pi} \int_0^{2\pi} \frac{z \, d\phi}{(a^2+z^2)^{1/2}} \, \underline{a_r}$$

I cannot do this because the $\underline{a_r}$'s are not the Same, They will cancel.

$$H_z = \int dH_z = \frac{Ia}{4\pi} \; \frac{a}{(a^2+z^2)^{3/2}} \cdot 2\pi$$

$$= \frac{I a^2}{2 (a^2+z^2)^{3/2}}$$

Solenoid



N turns total.

want to find field along the axis.

what is the field from a single loop?

$$\# \text{ of } \frac{\text{turns}}{\text{length}} \times \frac{\text{current}}{\text{turn}} \times \text{length}$$

$$dH_z = \frac{(dI)\,a^2}{2(a^2+z^2)^{3/2}} = \frac{a^2}{2(a^2+z^2)^{3/2}}\underbrace{\left(\frac{N}{l} I\, dz\right)}_{dI}$$

what is the total field at point P.

$$H_z = \int \frac{a^2}{2(a^2+z^2)^{3/2}} \frac{N}{l} I\, dz$$

$$= \frac{a^2 N I}{2l} \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \frac{\cancel{a^2} N I}{2l}\left[\frac{z}{\cancel{a^2}(z^2+a^2)^{1/2}}\Bigg|_{-\frac{l}{2}}^{+\frac{l}{2}}\right]$$

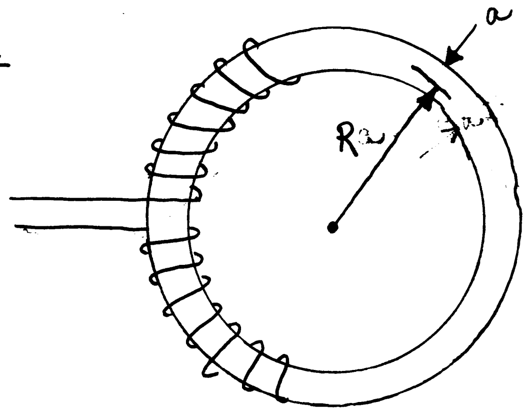
$$H_z = \frac{NI}{2l}\frac{\frac{l}{2}-\left(-\frac{l}{2}\right)}{\left(\left(\frac{l}{2}\right)^2+a^2\right)^{1/2}} = \frac{NI\cancel{l}}{2\cancel{l}\left(\frac{l^2}{4}+a^2\right)^{1/2}} = \frac{NI}{2\left(\frac{l^2}{4}+a^2\right)^{1/2}}$$

Note that if $l \gg a$ this becomes

$$H_z \cong \frac{NI}{2\frac{l}{2}} = \frac{NI}{l}$$

in fact as long as $\frac{l}{a} > 4$ this is a pretty good result.

toroid



a toroid is simply a solenoid in which the ends are brought together.

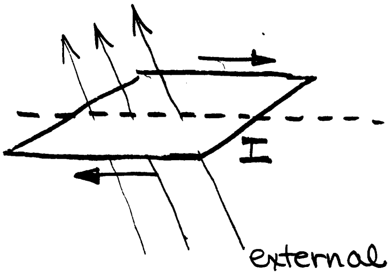
what is the length of this solenoid

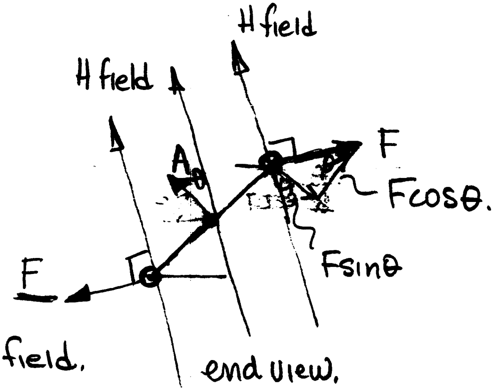
$$l = 2\pi R$$

$$\therefore \quad H_\phi = \frac{NI}{2\left(\frac{4\pi^2 R^2}{4} + a^2\right)^{1/2}} = \frac{NI}{2(\pi^2 R^2 + a^2)^{1/2}}$$

materials

torque also exists





recall $F = q\underline{v} \times \underline{B} \approx \underline{I} \times \underline{B}$

note that F will pull the loop to align it perpendicular with the field.

this is like an electric dipole moment; however, because of the cross products we define the magnetic moment to be

$$\underline{m} = I\underline{A}$$

the force acting on the loop is then

$$F = |\underline{I} \times \underline{B}l|$$

torque $= 2(F \times \text{moment arm})$ since two arms.

$$T = 2\,F\,IBl\,\frac{d}{2}\sin\theta. \quad \leftarrow \text{actually } F\sin\theta.$$

in general we define

$$\underline{m} = I \underline{A}$$

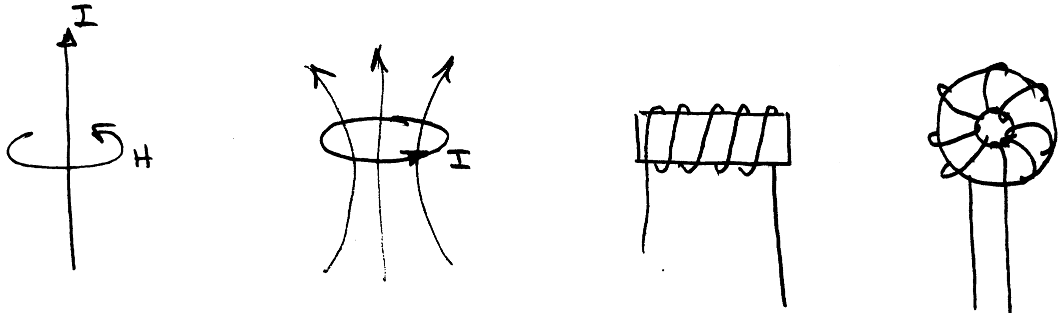
and $$\underline{T} = \underline{m} \times \underline{B}$$

Recalling. $$\underline{H} = \frac{I}{4\pi r^3} dS \left[2\cos\theta \, \underline{a_r} + \sin\theta \, \underline{a_\theta} \right]$$

for a magnetic dipole.

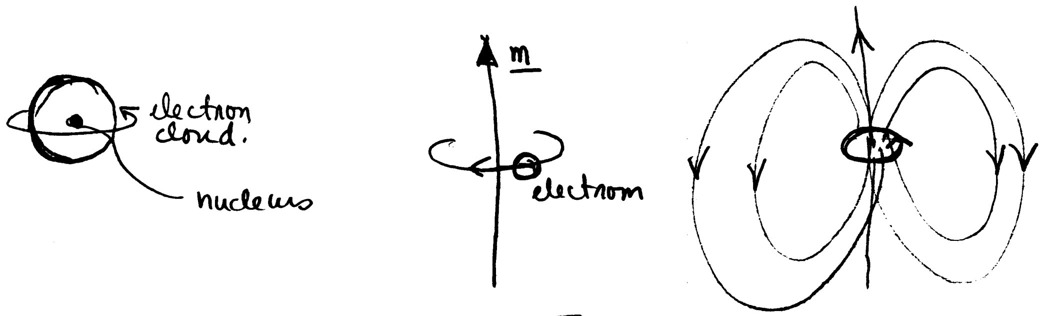
Note that $\underline{m} = I \, \underline{dS}$

common shapes which produce magnetic fields.



wire loop solenoid toroid.

matter has several natural dipole moments.

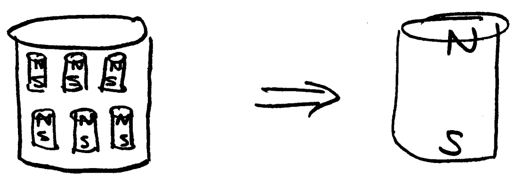


just as we defined $\underline{P} = \lim_{\Delta v \to 0} \dfrac{\sum_i \underline{p}_i}{\Delta v}$.

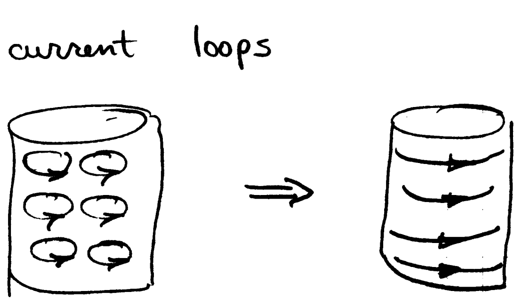
$$\underline{M} = \lim_{\Delta v \to 0} \frac{\sum_i \underline{m}_i}{\Delta v}.$$

the situation for $\underline{M}$ is very complex

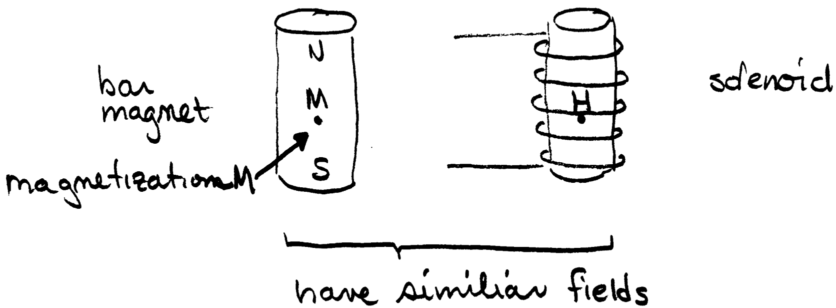
① small bar magnets



② small current loops



equivalent currents



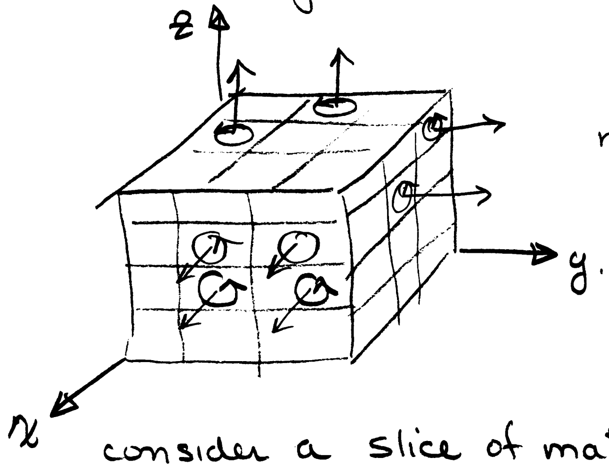
field is $H = \dfrac{N I_m}{l}$

where I_m is the equivalent magnetic current.

$$\therefore \quad M = \frac{N I_m}{l}.$$

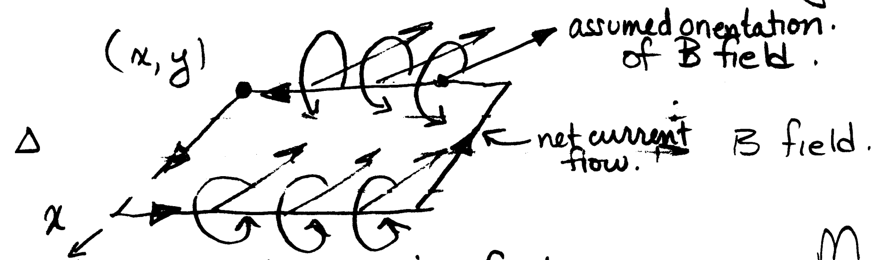
magnetization of matter

consider matter as composed of a number of magnetic dipoles. These dipoles may be permanent or induced by the applied magnetic field.



notice these are **not** random.

consider a slice of matter. Why? $\oint_C \underline{H}\cdot d\underline{l} = \int_S \underline{J}\cdot d\underline{s}$



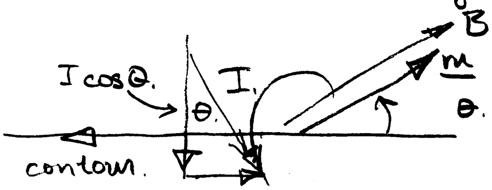
dipoles tend to align with applied field.

$$M = \lim_{\Delta v \to 0} \frac{\sum_i m_i}{\Delta v}$$

we define the dipole moment as $\underline{m} = I\, d\underline{S}$ due to individual atoms

the overall magnetization $\underline{M} = N\underline{m} = NI\,d\underline{S}$

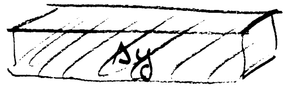
what is the normal component of $\underline{I}$ relative to the contour C? Ans: zero in the interior. Only net contribution is from the layer near the edge of the loop.



z direction of current is

$I\,dS\cos\theta/_x$

if field not uniform θ is a function of x.



need to know # of dipoles along contour.

$\swarrow$ # of dipoles / volume.

length of contour.

$$N \underbrace{\left(-I\,dS\cos\theta\right|_x)}_{\text{current due to one dipole}} \Delta y = I_z$$

– sign because of direction

only that part <u>inside</u> loop contributes.

but by our earlier definitions this is also.

$$-\underbrace{M_y}_{\text{the } y \text{ component of } \underline{M} \text{ at } x}\Big|_x \Delta y = I_z$$

at the $x+\Delta x$ edge of contour.

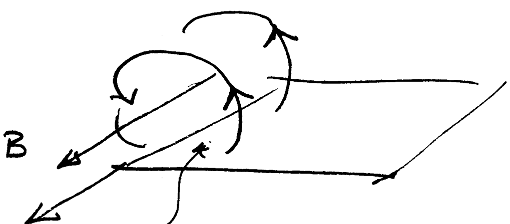
$$+M_y\big|_{x+\Delta x}\,\Delta y = I_z$$

If the field is aligned along the x axis we can get contributions along the x-axis in the same manner.

$$+\ M_x\big|_y\,\Delta x = I_z$$

$$-\ M_x\big|_{y+\Delta y}\,\Delta x = I_z$$

B



net is $+\underline{a_z}$

overall I_z total $= \left(M_y\big|_{x+\Delta x} - M_y\big|_x\right)\Delta y - \left(M_x\big|_{y+\Delta y} - M_x\big|_y\right)\Delta x$

$$= \frac{M_y|_{x+\Delta x} - M_y|_x}{\Delta x}\,\Delta x\Delta y - \frac{M_x|_{y+\Delta y} - M_x|_y}{\Delta y}\,\Delta x\Delta y$$

$$\frac{I_z \text{ total}}{\Delta x\Delta y} = \left[\frac{M_y|_{x+\Delta x} - M_y|_x}{\Delta x} - \frac{M_x|_{y+\Delta y} - M_x|_y}{\Delta y}\right]$$

taking limits as $\Delta x\,\Delta y \to 0$

$$J_z \text{ total} = \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial y}.$$

what do we do with this?

note that $(\nabla \times m)_z = \underline{a_z} \cdot \begin{vmatrix} a_x & a_y & \cancel{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ m_x & m_y & m_z \end{vmatrix} = \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}$

$J_z = (\nabla \times m)_z$

In general, I could orient my loop in all directions and get

$\underline{J} = \underline{\nabla} \times \underline{m}$

Tie this in with the magnetic field intensity. To do so go to historical form of Ampère's Law.

$\oint \underline{H} \cdot \underline{dl} = \int_S \underline{J} \cdot \underline{dS}$

$\oint \underline{E} \cdot \underline{dS} = \frac{Q}{\epsilon_0}$

$\int_S (\nabla \times H)\, d\underline{S} = \int_S J\, d\underline{S}$

$\nabla \times \underline{H} = \underline{J}$

~~$B = \mu_0 H$ free space~~

this is not natural, actually.

$\nabla \times \underline{B} = \mu_0 \underline{J}$

$\nabla \times \left(\frac{B}{\mu_0}\right) = \underline{J}$

we have studied free space & magnetic materials

$\underline{J_f} + \underline{J_m}$

free currents, sources of fields

magnetic, either permanent or induced

Then, $\nabla \times \left(\frac{B}{\mu_0}\right) = \underline{J_f} + \underline{J_m} = \underline{J_f} + \underline{\nabla} \times \underline{m}$

total magnetic flux density due to free & magnetic terms

$$\nabla \times \left(\frac{B}{\mu_0}\right) = \underline{J_f} + \underline{\nabla} \times \underline{m}$$

$$\nabla \times \underbrace{\left(\frac{B}{\mu_0} - m\right)}_{\text{magnetic intensity due to free currents.}} = J_f$$

$$H = \frac{B}{\mu_0} - M$$

$$\therefore \underline{B} = \mu_0 (\underline{H} + \underline{m})$$

the magnet flux density vector $\underline{B}$

Recall $\underline{D} = \epsilon \underline{E}$

In an analogous manner

$$\underline{B} = \mu \underline{H}$$

$\underline{B}$: $\frac{\text{webers}}{\text{meter}^2}$ $\underline{H}$: $\frac{\text{amperes}}{\text{meter}}$

for free space $\mu_0 = 4\pi \times 10^{-7}$ $\frac{\text{Henrys}}{\text{meter}}$.

The relationship between $\underline{H}$ and $\underline{M}$ is not as simple as that given since

$$\underline{B} = \mu(H)\,\underline{H}$$, i.e. μ is not necessarily linear.

The above results can be shown very accurately to describe materials

Depending on the material, there are three general forms of magnetization

diamagnetic $\mu \lesssim \mu_0$ due to orbital motion of electrons

paramagnetic $\mu \gtrsim \mu_0$ } spin motion of electron itself

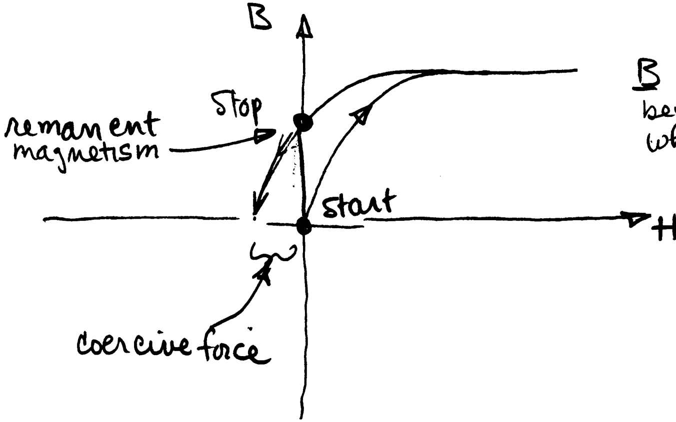
ferrimagnetic and ferromagnetic $\mu \gg \mu_0$

ferrimagnetic: ferrites
- hard
- soft. contain metallic iron. ferrites, have nice high freq. μ characteristics.

ferromagnetic:
- hard: permanent magnets, speaker magnets
- soft: iron, etc.

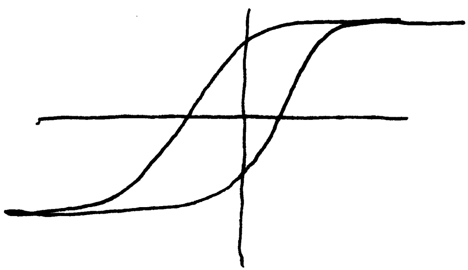
F.

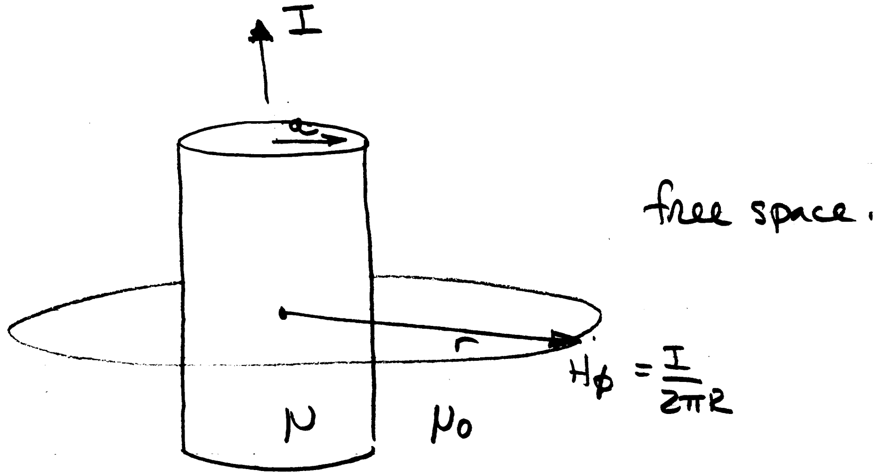
unusual property of ferromagnet's ⟶ HYSTERISIS



B field saturates when ~~H~~ goes beyond a certain limit, when H is removed B is not zero.

overall, B, H curve is cyclic and closed





current flows in a cylinder of radius a and permeability μ.

What are $\underline{B}$, $\underline{H}$ and $\underline{M}$ everywhere

use Ampère's law just as before.

$$\oint \underline{H}\cdot d\underline{l} = H_\phi \, 2\pi r = I \quad \Rightarrow \quad H_\phi = \frac{I}{2\pi r}.$$

does the magnetic flux differ. You bet!

$$H_\phi = \begin{cases} \dfrac{I}{2\pi r} & r > a \\ \dfrac{I}{2\pi r}\,\dfrac{\pi r^2}{\pi a^2} = \dfrac{I}{2\pi r}\left(\dfrac{r}{a}\right)^2 & 0 < r < a \end{cases}$$

$$B_\phi = \frac{\mu_0 I}{2\pi r} \quad r > a \qquad\qquad B_\phi = \mu \frac{I}{2\pi r}\left(\frac{r}{a}\right)^2 \quad 0 < r < a$$

How about the magnetization?

$$B = \mu_0 (H + M) = \mu H. \qquad H + M = \frac{\mu H}{\mu_0}$$

$$M = \left(\frac{\mu}{\mu_0} - 1\right) H$$

$$M = \left(\frac{\mu}{\mu_0} - 1\right) H_\phi \qquad 0 < r < a$$

$$= 0 \qquad r > a$$

$$\therefore \quad \underline{J_m} = \underline{\nabla} \times \underline{M} = -\frac{\partial M_\phi}{\partial z}\,\underline{a_r} + \frac{1}{r}\frac{\partial}{\partial r}(r M_\phi)\,\underline{a_z}$$

$$\underline{M} = \left(\frac{\mu}{\mu_0}-1\right)\frac{I}{2\pi r}\left(\frac{\cancel{\pi} r^2}{\cancel{\pi} a^2}\right)\underline{a_\phi}$$

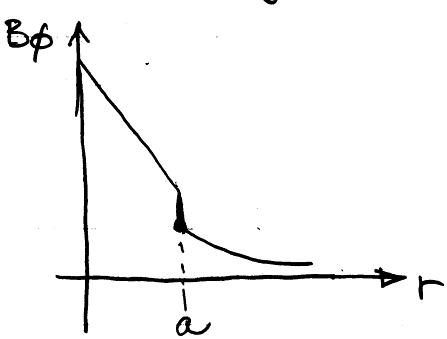
$$\frac{\partial M_\phi}{\partial z} = 0$$

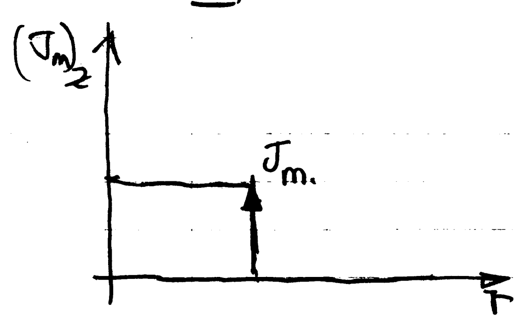
$$\frac{1}{r}\frac{\partial}{\partial r}\left[r M_\phi\right] = \frac{1}{r}\frac{\partial}{\partial r}\left[\left(\frac{\mu}{\mu_0}-1\right)\frac{I}{2\pi a^2}r^2\right] = \frac{1}{\cancel{r}}\left(\frac{\mu}{\mu_0}-1\right)\frac{I}{\cancel{2}\pi a^2}\cancel{2r}$$

$$= \left(\frac{\mu}{\mu_0}-1\right)\frac{I}{\pi a^2}$$

$$\therefore \quad \underline{J_m} = \left(\frac{\mu}{\mu_0}-1\right)\frac{I}{\pi a^2}\,\underline{a_z}$$

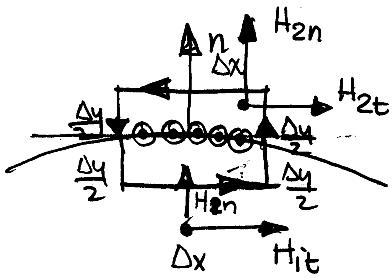
This says that the discontinuity in the B field is due to the magnetization current $\underline{J_m}$.





Magnetic Boundary Conditions

(done just like E fields) tangential & normal components.



Apply Ampère's Law $\oint \underline{H}\cdot d\underline{l} = \int \underline{J_f}\cdot d\underline{S}$

for emphasis

$$\oint \underline{H}\cdot d\underline{l} = H_{1t}\,\Delta x + \underbrace{H_{1n}\frac{\Delta y}{2} + H_{2n}\frac{\Delta y}{2}}_{\text{cancel}} - H_{2t}\,\Delta x - \underbrace{H_{2n}\frac{\Delta y}{2} - H_{1n}\frac{\Delta y}{2}}_{\text{cancel}}$$

$$= (H_{1t} - H_{2t})\,\Delta x.$$

$$\oint \underline{J_f}\cdot d\underline{S} = J_n\,\Delta x\,\Delta y$$

$$\therefore\quad J_n\,\Delta y = H_{1t} - H_{2t}$$

as $\Delta y \to 0$ $\quad J_n\,\Delta y \to K_s$ the surface current

$$\therefore\quad (H_{1t} - H_{2t}) = K_n \quad \text{or} \quad \underline{n}\times(\underline{H_2} - \underline{H_1}) = \underline{K}$$

vector

- How about J_m? Exactly the same except replace $\underline{H}$ by $\underline{m}$!

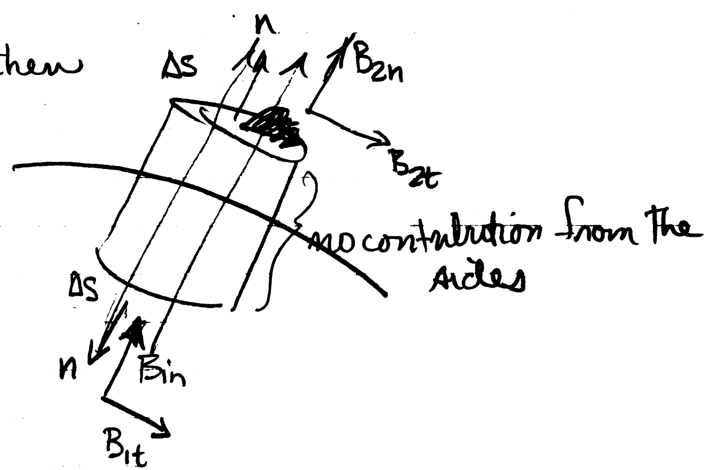
$$(M_{1t} - M_{2t}) = (K_m)_n \quad \text{or} \quad \underline{n}\times(\underline{m_2} - \underline{M_1}) = \underline{K_m}$$

How about the normal components?

to do this we need to recall $\Psi_m = \oint \underline{B} \cdot \underline{n}\, dS = 0$ for a closed surface.

Why? There is no point charge on which field lines can terminate.

If $\oint \underline{B} \cdot \underline{n}\, dS = 0$. then



$$\oint \underline{B} \cdot \underline{dS} = -(B_{1n})\Delta S + (B_{2n})\Delta S = 0$$

$\therefore\ B_{2n} = B_{1n}$ and normal B is continuous.

Scalar magnetic potential

lumped parameter analysis
distributed parameter analysis

for a source-free region

$$\underline{\nabla} \times \underline{H} = 0$$

$$\underline{\nabla} \times \underline{\nabla} \mathfrak{F} = 0 \qquad \text{this is a general vector identity for scalar fields}$$

$$\therefore \; H = -\nabla \mathfrak{F}$$

choose the $-$ sign only to make this the same as for electric potential

we can integrate this to get

$$\mathfrak{F}_2 - \mathfrak{F}_1 = -\int_{P_1}^{P_2} \underline{H} \cdot \underline{dl}$$

units of amperes.

this is often called the magnetomotive force or mmf.

by doing more math I can show that $\nabla^2 \mathfrak{F} = 0$ in a source-free region

$$\Psi_m = \int_S \underline{B} \cdot \underline{ds}$$

this is similar to the electric ~~potential~~ flux density.

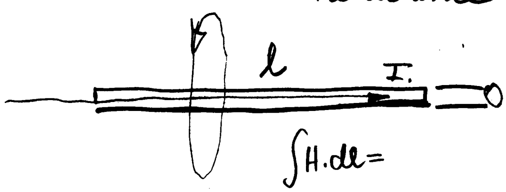
write "ohm's law" for magnetic circuits

$$\mathfrak{F} = \mathcal{R} \Psi_m \qquad \mathcal{R} = \frac{\mathfrak{F}}{\Psi_m}$$

$\mathfrak{F}$ ← magnetomotive force

Ψ_m ← flux

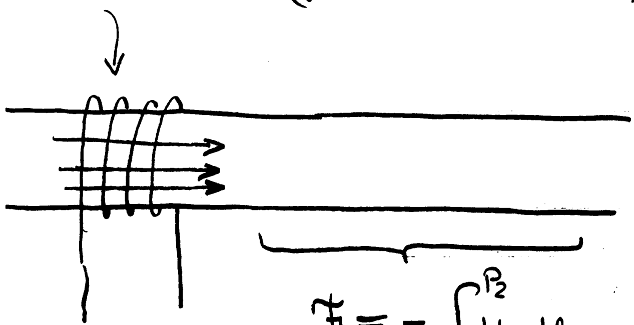
reluctance



$ds = A.$

$\int H \cdot dl =$

source of flux (just like current)



magnetic circuit

$$\mathcal{F} = -\int_{P_1}^{P_2} \underline{H} \cdot \underline{dl}$$

this is really the beginning of Ohm's Law

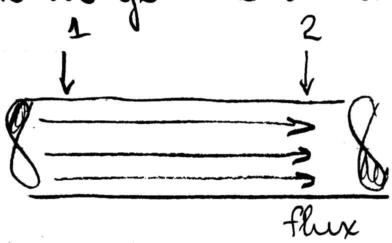
If we define $\Phi_m = \int \underline{B} \cdot \underline{ds}$

then we have $\mathcal{F} = \mathcal{R}\Phi$

$$[v = iR]$$

what is $\mathcal{R}$? the reluctance

how can we get the Reluctance

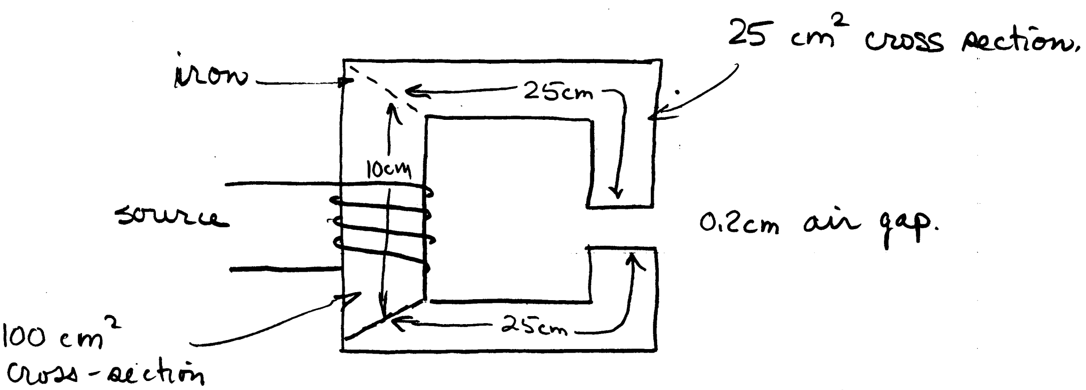


for a simple uniform geometry

$$\Phi_m = \int B \cdot dS = \int \mu H \, dS = \mu H A$$

$$\mathcal{F} = \int_1^2 H \cdot dl = Hl.$$

$$\therefore \mathcal{R} = \frac{\mathcal{F}}{\Phi_m} = \frac{Hl}{\mu H A} = \frac{l}{\mu A} \quad \blacksquare$$



given the field I want in the gap how many turns do I need to get a field of $1\ Wb/m^2$.

air - gap reluctance

$$\mathcal{R}_g = \frac{l}{\mu_0 A} = \frac{2\times 10^{-3}}{4\pi \times 10^{-7} \times 25\times 10^{-4}} = 6.36\times 10^{5}\ \frac{A\text{-}t}{Wb.}$$

H = amperes/m.

B = webers/m^2

μ = henrys/m

$\mathcal{R}$ = $\frac{\text{~~amperes~~(turns)}}{\text{weber. not really}}$

$\mathcal{F}$ - ampere turns.

φ = webers.

air-gap flux

$$\Phi_m = \int B \cdot dS = \int 1 \frac{Wb}{m^2}\, ds = 1 \frac{Wb}{m^2} \cdot 25\times 10^{-4}\ m^2$$

$$= 25\times 10^{-4}\ wb.$$

by "Ohm's Law"

$$\mathcal{F}_g = \varphi_m \mathcal{R}_g = 6.36\times 10^{5} \times 25 \times 10^{-4}$$

$$= 1.590 \times 10^{3}\ \text{Ampere turns.}$$

How about the rest of the circuit?

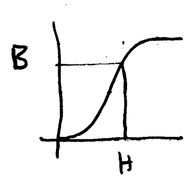
for The two "steel arms"

$$\mathcal{F}_{steel} = (25cm + 25cm) \cdot H$$

but what is H?

<u>we know B</u> $\quad B = 1 \frac{wb}{m^2}$ since normal B is continuous

From chart



$B = 1 \frac{wb}{m^2} \Rightarrow 200\ \frac{\text{Amperes (turns)}}{m}$

$$\mathcal{F}_{steel} = (0.50\text{ meters}) \cdot 200 \frac{\text{Amperes - turns}}{\text{meter}} = 100 \text{ At}$$

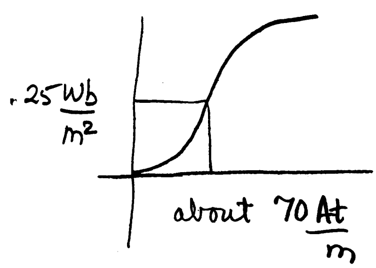
How about the steel transformer section?

$$\mathcal{F} = (0.10 \text{ meters})(H)$$

but what is H? $\Phi = 25 \times 10^{-4}$ webers as before.

but $\Phi = \int B\, dS = B \cdot 100 \text{ cm}^2$

$$B = \frac{25 \times 10^{-4}}{.01 \text{ m}^2} = 0.25 \frac{\text{Wb}}{\text{m}^2}.$$



$$\therefore \mathcal{F} = (0.10\text{m})(70 \frac{\text{At}}{\text{m}}) = 7.0 \text{ A-t}.$$

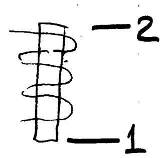
$\therefore$ we have the total circuit mmf.

$$\mathcal{F}_{total} = \mathcal{F}_{gap} + \mathcal{F}_{steel} + \mathcal{F}_{transformer}.$$

$$= 1590 + 100 + 7$$

$$= 1697 \text{ A-t}.$$

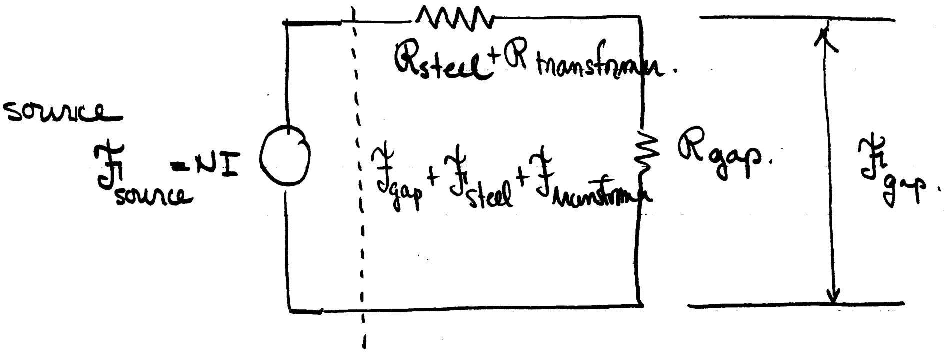
How about the source? This is a solenoid.



Recall $H = \frac{NI}{l}$.

$$\mathcal{F}_{source} = \int_1^2 H \cdot dl = \frac{NI}{l} \cdot l = NI.$$

This problem now has a solution



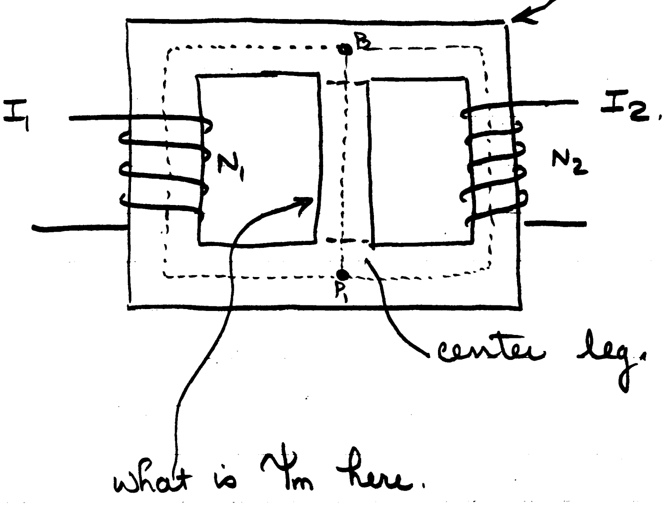
given $\mathcal{F}_{gap}$, what was needed $\mathcal{F}_{source}$?

$\therefore$ $\mathcal{F}_{source} = 1697 \text{ A}\cdot\text{t} = NI$.

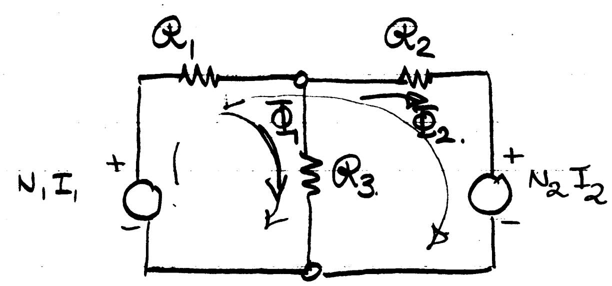
If $I = 10$ A. $N = 169.7$

hence the use of the name Ampere-turns.

Example #2: μ; cross-sectional area A.



draw electrical equivalent.



$\mathcal{R}_1 \triangleq$ reluctance thru coil 1 from P_1 to P_2.

$$= \frac{l_1}{\mu A}$$

$\mathcal{R}_2 \triangleq$ " coil 2 "

$$= \frac{l_2}{\mu A}$$

$\mathcal{R}_3 \triangleq$ reluctance thru center leg. $= \dfrac{l_3}{\mu A}$.

loop equations:

$$N_1 I_1 = \mathcal{R}_1 (\Phi_1 + \Phi_2) + \mathcal{R}_3 \Phi_1$$

$$N_1 I_1 - N_2 I_2 = \mathcal{R}_1 (\Phi_1 + \Phi_2) + \mathcal{R}_2 \Phi_2$$

$$(\mathcal{R}_1+\mathcal{R}_3)\,\Phi_1 + \mathcal{R}_1\,\Phi_2 = N_1 I_1$$
$$\mathcal{R}_1\,\Phi_1 + (\mathcal{R}_1+\mathcal{R}_2)\,\Phi_2 = N_1 I_1 - N_2 I_2 .$$

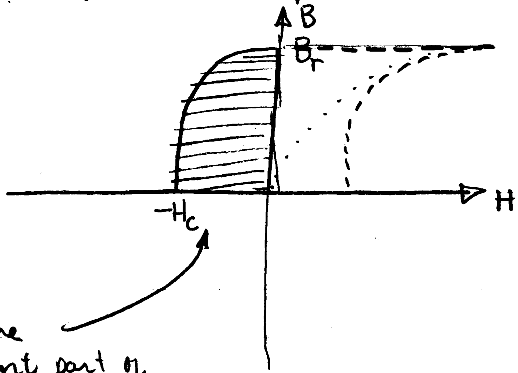
$$\Phi_1 = \frac{\begin{bmatrix} N_1 I_1 & \mathcal{R}_1 \\ N_1 I_1 - N_2 I_2 & \mathcal{R}_1+\mathcal{R}_2 \end{bmatrix}}{\begin{bmatrix} \mathcal{R}_1+\mathcal{R}_3 & \mathcal{R}_1 \\ \mathcal{R}_1 & \mathcal{R}_1+\mathcal{R}_2 \end{bmatrix}}$$

$$= \frac{\cancel{(N_1 I_1)\mathcal{R}_1} + \mathcal{R}_2 N_1 I_1 - \cancel{\mathcal{R}_1 N_1 I_1} + N_2 I_2 \mathcal{R}_1}{\left[\cancel{\mathcal{R}_1^2} + \mathcal{R}_1\mathcal{R}_3 + \mathcal{R}_1\mathcal{R}_2 + \mathcal{R}_2\mathcal{R}_3 - \cancel{\mathcal{R}_1^2}\right]}$$

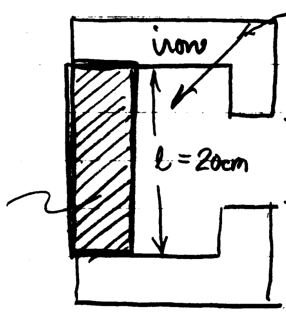
$$\Phi_1 = \frac{\mathcal{R}_2 N_1 I_1 + \mathcal{R}_1 N_2 I_2}{\mathcal{R}_1\mathcal{R}_3 + \mathcal{R}_1\mathcal{R}_2 + \mathcal{R}_2\mathcal{R}_3}.$$

How about a Permanent magnet.

No real problem except we must use a second quadrant curve.

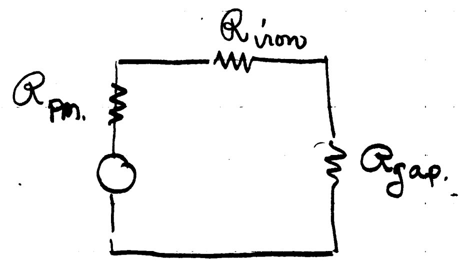


this is the permanent part of the magnet.
(this is actually the demagnetization curve)



A_g = 2.5cm²

as before



How about the magnet?

$$\mathcal{F}_{PM} = \int_1^2 H \cdot dl = Ml$$

$$\Phi_{PM} = \int B \cdot dS = BA$$

$$\mathcal{R}_{PM} = \frac{\mathcal{F}_{PM}}{\Phi_{PM}} = \frac{Ml}{Bl \cdot t} = \frac{M}{Bt} = \frac{M}{\mu M t} = \frac{1}{\mu t}$$

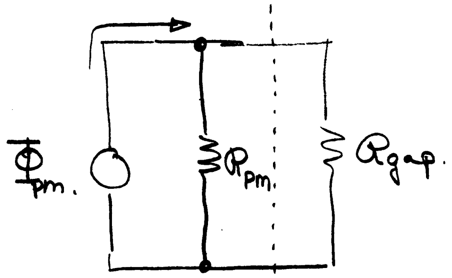
typically. t = 1cm.

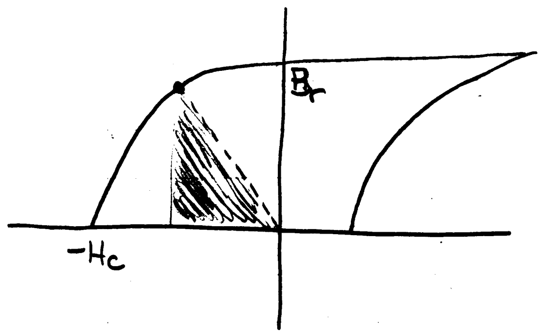
$$\mu = \frac{B}{H} = \frac{1\ Wb/m^2}{25000\ A/m} \cdot \frac{m}{A} \quad \frac{1}{25} \times 10^{-3} \frac{W}{A-m}$$

$$= 4\times 10^{-5} \frac{W}{A-m}.$$

$$\mathcal{R}_{pm} \approx \frac{1}{4\times10^{-5}\frac{W}{A-\cancel{m}} \times 10^{-2}\,\cancel{m}} = \frac{1}{4\times10^{-7}}\frac{A}{W} \approx 10^{7}\frac{A}{W}.$$

neglecting the reluctanance of iron.





this is a current source

$\mathcal{R}_{pm} \gg$ current $\mathcal{R}_{gap}$ and can be neglected.

$$\Phi_{pm} \approx \frac{\mathcal{F}_{gap}}{\mathcal{R}_{gap}}.$$

$$\mathcal{R}_{gap} = \frac{l_g}{\mu_0 A_g} = \frac{0.5\times10^{-2}\,m}{4\pi\times10^{-7}\frac{Hy}{m}\cdot \cancel{2}5\times10^{-4}\,m^2} = \frac{A\text{-}t}{Weber}$$

$$= \frac{1}{48\pi\cdot\frac{5}{\cancel{2}}\times10^{-11}}\times10^{2} = \frac{1}{20\pi}\times10^{9}\ \frac{A\text{-}t}{weber}.$$

the magnet is at $\quad B = 0.95\ Wb/m^2$

$\qquad H_{(m)} = -24{,}000\ A/m.$

$$\begin{array}{r} .95 \\ 3 \\ \hline 2.85 \end{array}$$

$$\Phi = \int B\cdot dS = 0.95\,\frac{Wb}{m^2} \times 3\times10^{-4}\,m^2 = 2.85\times10^{-4}\ Wb.$$

$$\mathcal{F}_{gap} = \Phi_{pm}\mathcal{R}_{gap} = 2.85\times10^{-4}\,Wb\cdot\frac{10^{-1}}{2\pi}\times10^{9}$$

$$= \frac{2.85}{2\pi}\times10^{4} \approx 3\times10^{4}\ \frac{A\text{-}t}{weber}.$$