These past problems bring us to two fundamental limits of our approach so far:

1. free space — what if $\epsilon \neq \epsilon_0$
2. boundaries - what happens <u>between</u> differing media.

conductors - electrically neutral materials which <u>possess a large number of free (mobile) charges and</u>, in some <u>sense</u>, are <u>similar to an electron gas</u>.

In metals, free electrons are due to conduction electrons in the unfilled <u>outer atomic orbits</u>

electric current - movement of electronic charge

current-density vector ($\underline{J}$) - amperes/m^2 or coulombs/

Ohm's Law: $\underbrace{\underline{J}}_{\frac{\text{amperes}}{m^2}} = \sigma \underbrace{\underline{E}}_{\frac{\text{volts}}{m}}$

$$\therefore \quad \sigma = \frac{\text{amperes}}{m^2} \cdot \frac{m}{\text{volts}}$$

$$= \frac{\text{siemens}}{m} = \frac{1}{\text{ohm}-m}$$

The common form of Ohm's law is

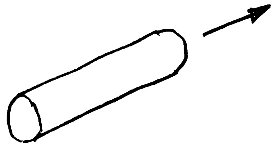
$$I = \frac{E}{R} \quad \text{(a scalar representation of } \underline{J} = \sigma \underline{E})$$

$$I = \int_S \underline{J} \cdot \underline{ds}$$

in general σ is a function of the material and frequency

Example: copper wire

$I = 0.2$ Amperes.



(12 ga wire) $r = 1.025\,\text{mm}$

$$I = \int_S \underline{J} \cdot d\underline{s}$$

we have no reason to assume $\underline{J}$ is not uniform so this integral becomes

$$= J\,(\pi r_w^2)$$

$$\therefore\ J = \frac{I}{\pi r_w^2} = \frac{0.2}{\pi (1.025 \times 10^{-3})^2} = 6.06 \times 10^{4}\ \frac{\text{Amperes}}{m^2}$$

watch units

what's the electric field? Ohm's law $\underline{J} = \sigma \underline{E}$

$$E = \frac{J}{\sigma} = \frac{6.06 \times 10^{4}\ \text{amperes}/m^2}{5.8 \times 10^{7}\ \text{Siemens}/m} = 1.05 \times 10^{-3}\ \frac{\text{volts}}{m}$$

what is the conventional resistance?

$$R = \frac{V}{I} = \frac{1.05 \times 10^{-3}\ \text{volts}/m}{0.2\ \text{Amperes}} = 5.25 \times 10^{-3}\ \Omega/\text{meter}.$$

objective : dielectrics in terms of polarization

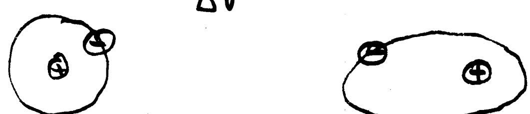
dielectrics – electrically insulating materials in which electrons are tightly bound to the nucleus. Even though the material is electrically neutral an externally applied field may cause separation of positive and negative charge at the microscopic level. These will act as electrical dipoles and the polarization is defined as the dipole moment per unit volume.

polarization (types)

$$\underline{P} \triangleq \lim_{\Delta v \to 0} \frac{\sum_i \underline{p}_i}{\Delta v} = N \underline{p}$$

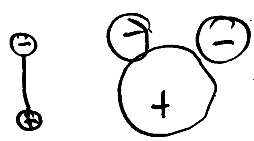
number density

– electronic



motion of electron cloud.

– orientational



H_2O molecule

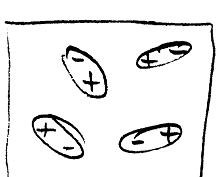
these are in general random, but can line up with an external field to give a net polarization

– ionic

motion of – ions relative to each other

in general, polarization is a combination of the above effects.

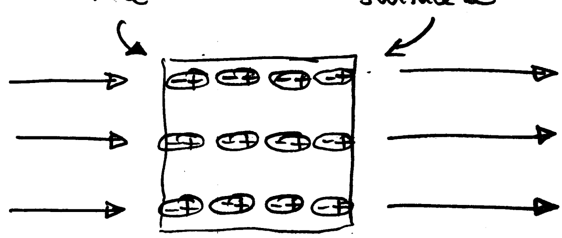
Polarization charge



no applied $\underline{E}$ field
random orientation

$\underline{P} = 0$

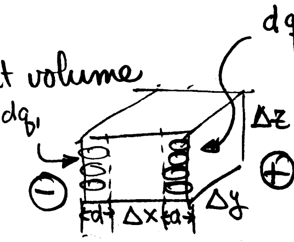
surface 1 surface 2



applied $\underline{E}$ field
charges cancel within the volume but net surface charge.

only those dipoles within a distance d (where $p = qd$) will contribute a net charge.

for the applied field $\underline{E}$, N dipoles/unit volume

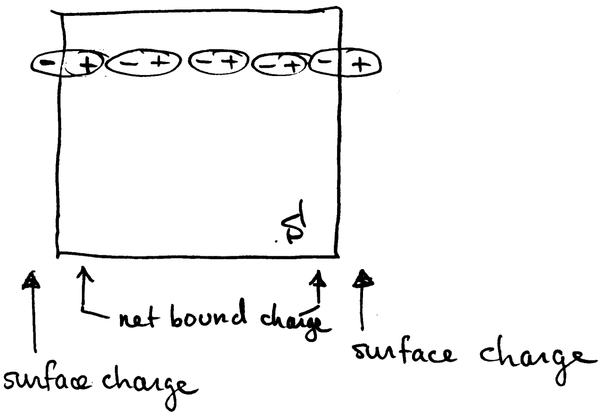


net charge on the surface 1 is:

$$dq_1 = -(N\, q d_x)\big|_x \,\Delta y\, \Delta z$$

$$dq_2 = +(N q\, d_x)\big|_{x+\Delta x} \,\Delta y\, \Delta z.$$

$$\boxed{\text{The net bound charge is that inside the volume and is in the opposite direction}}$$



Note point out this is P_x because applied field is in E_x direction

$\therefore$ net bound charge is

$$dq_1' = +(Nqx)\big|_x \Delta y \Delta z = +P_x(x)\, \Delta y \Delta z$$

$$dq_2' = -(Nqx)\big|_{x+\Delta x} \Delta y \Delta z = -P_x(x+\Delta x)\, \Delta y \Delta z$$

Note that $\underline{P}$ is defined within the volume, not on the surface.

The total charge enclosed within the volume is

$$dq_T = dq_1' + dq_2' = P_x(x)\, \Delta y\, \Delta z - P_x(x+\Delta x)\, \Delta y\, \Delta z$$

$$= -\frac{P_x(x+\Delta x) - P_x(x)}{\Delta x}\, \Delta x\, \Delta y\, \Delta z$$

$\therefore$ the polarization volume charge density ρ_b within S is:

$$\rho_b = \lim_{\Delta v \to 0} -\frac{\frac{P_x(x+\Delta x) - P_x(x)}{\Delta x}\, \cancel{\Delta v}}{\cancel{\Delta v}} = -\frac{dP_x}{dx}$$

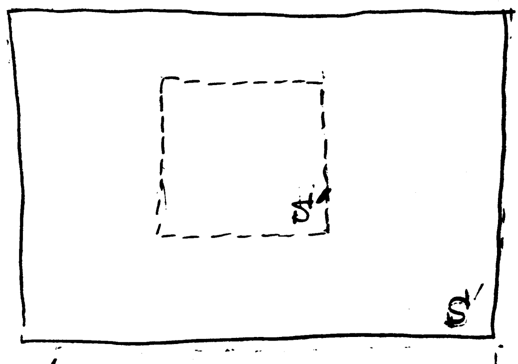
this result can be generalized to three dimensions by applying a field sequentially in the $\underline{a_x}$, $\underline{a_y}$ and $\underline{a_z}$ directions so that

$$\rho_b = -\frac{\partial P_x}{\partial x} - \frac{\partial P_y}{\partial y} - \frac{\partial P_z}{\partial z}$$

$$= -\underline{\nabla} \cdot \underline{P} \qquad \text{(this is within the dielectric)}$$

What happens at the surface of S? Clearly $\underline{P}$ is discontinuous there.

To conveniently express the result re-examine Gauss' Law.



imagine volume S' which is inside S.

Gauss' Law $\qquad \oint_S \underline{E} \cdot \underline{ds} = \frac{Q_{total}}{\epsilon_0}$

but what is Q_{total}? Suppose we consider S' to be a region of open space. There will be a charge density due to free charges and a charge density due to bound charges. i.e.

$$\rho_T = \rho_f + \rho_b$$

and $Q_{total} = \int_V (\rho_f + \rho_b)\, dv$

$$\therefore \quad \oint_S \underline{E} \cdot \underline{ds} = \frac{\int_V (\rho_f + \rho_b)\, dV}{\epsilon_0}$$

Divergence Theorem: (volume integral of source = net flow).

but $$\oint_S \underline{E}\cdot d\underline{s} = \int_V \underline{\nabla}\cdot\underline{E}\, dv$$

$$\therefore \quad \underline{\nabla}\cdot\underline{E} = \frac{\rho_f + \rho_b}{\epsilon_0}$$

$$= \frac{\rho_f - \underline{\nabla}\cdot\underline{P}}{\epsilon_0}$$

combining terms $$\nabla\cdot(\epsilon_0\underline{E} + \underline{P}) = \rho_f$$

ρ_f is the free charge in space

$\epsilon_0\underline{E} + \underline{P}$ is the electric flux density due to those free charges

define $\underline{D} = \epsilon_0\underline{E} + \underline{P}$

$\underline{D} \triangleq$ electric flux density vector.

Note that in real free space $\underline{P} \to 0$ and $\underline{D} = \epsilon_0\underline{E}$

In fact,

$$\oint_S \underline{D}\cdot d\underline{s} = Q_{total}$$ is the most general form of Gauss' Law.

As $\underline{P}$ is some function of $\underline{E}$, more accurately, $\underline{P}$ is some function of $\epsilon_0\underline{E}$, the flux density vector since $\underline{P}$ is due to bound charge.

For <u>most materials</u>, this is a simple relationship

$$\underline{P} = \underbrace{\chi_e}_{\text{susceptibility}}\epsilon_0\underline{E}$$

then, $$\underline{D} = \epsilon_0\underline{E} + \underline{P} = \epsilon_0\underline{E} + \chi_e\epsilon_0\underline{E} = \epsilon_0(1+\chi_e)\underline{E}$$

We can define $\epsilon = \epsilon_0(1+\chi_e)$ this is the permittivity

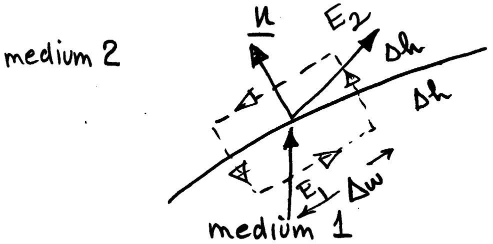
or K or $\epsilon_r = \dfrac{\epsilon}{\epsilon_0} = 1+\chi_e$. This is the dielectric constant.

Boundary conditions

We just examined bound charges due to an external applied field. How about the surface charges and the relationship between the internal and external field?

internal $\qquad \underline{D} = \epsilon_0 \underline{E} + \underline{P}$

external $\qquad \underline{D} = \epsilon_0 \underline{E}$



We showed earlier that

$$\oint \underline{E} \cdot d\underline{l} = 0$$

since $V(P_1) = V(P_1)$ i.e. $\underline{E}$ field is conservative

Choosing the contour as indicated above and decomposing

$$\underline{E}_1 = \underline{E}_{1t} + \underline{E}_{1n}$$

$$\underline{E}_2 = \underline{E}_{2t} + \underline{E}_{2n}$$

and picking $\underline{n}$ to point from 1 to 2.

$$\cancel{(\Delta h)E_{1n}} + (\Delta h)E_{2n} + E_{2t}(\Delta w) - (\Delta h)E_{2n} - \cancel{(\Delta h)E_{1n}} + (E_{1t})(\Delta w) = 0$$

cancelling terms $\qquad E_{1t} - E_{2t} = 0$

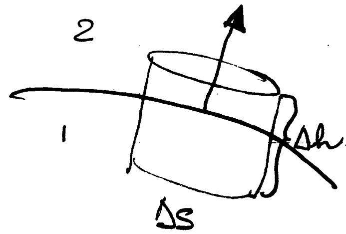
or $\quad E_{1t} = E_{2t}$

The tangential component is continuous...

This can be written in vector form as $n \times (\underline{E}_2 - \underline{E}_1) = 0.$

How about the normal componet?

Use Gauss' Law



$$\oint \underline{D} \cdot \underline{ds} = \rho_f$$

as $\Delta h \to 0$ There will be no contribution from the sides

so the surface integral reduces to

$$\int \underline{D}_1 \cdot \underline{n} + \underline{D}_2 \cdot \underline{n} ds = -D_{n1} \Delta S + D_{n2} \Delta S$$

what is ρ_f? As $\Delta h \to 0$ $\rho_f \to \rho_s$, the surface charge density, and the total surface charge is $\rho_s \Delta S$

$\therefore$

$$D_{n2} \Delta S - D_{n1} \Delta S = \rho_s \Delta S$$

$$D_{n2} - D_{n1} = \rho_s$$

$$\underbrace{\underline{n} \cdot (\underline{D}_2 - \underline{D}_1)}_{\text{note signs are important}} = \rho_s$$

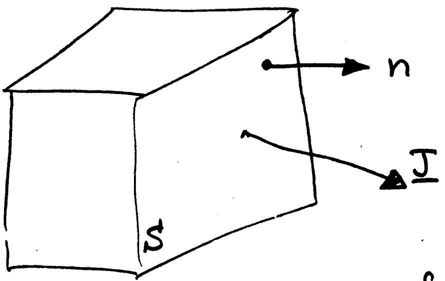
to complete our study of materials, we must examine a conductor.

In a conductor currents are due to the movement of free charge.

By charge conservation, the <u>net</u> current through a closed surface gives the <u>rate of change</u> of free charges, i.e.

$$\oint_S \underline{J} \cdot \underline{ds} = - \frac{d}{dt} Q \longrightarrow - \int \frac{d\rho}{dt} dv. \quad (1)$$

sign due to fact charge is moving out of S



If $\underline{n} \cdot \underline{J} > 0$ then there is a net flow of charge out of S causing Q to decrease

Using the divergence Theorem

$$\oint_S \underline{J} \cdot \underline{ds} = \int_V \underline{\nabla} \cdot \underline{J} \, dv. \quad (2)$$

Equating (1) and (2) $\quad \underline{\nabla} \cdot \underline{J} = - \frac{\partial \rho}{\partial t}$ this is the law of conservation of charge.

For a conductor $\underline{J} = \sigma \underline{E}$

$$\underline{\nabla} \cdot \sigma \underline{E} = -\frac{\partial \rho}{\partial t}$$

But, from Gauss' Law

$$\underline{\nabla} \cdot \epsilon \underline{E} = \rho$$

If σ and ϵ are linear and isotropic

$$-\frac{1}{\sigma} \frac{\partial \rho}{\partial t} = \frac{\rho}{\epsilon}$$

or $$\frac{\partial \rho}{\partial t} + \left(\frac{\sigma}{\epsilon}\right) \rho = 0$$

The solution is $\rho = \rho_0 e^{-\left(\frac{\sigma}{\epsilon}\right)t}$

To interpret, this suppose I put charge density ρ_0 into S' at $t=0$. For a perfect conductor $\sigma = \infty$, and, hence, instantly $\rho \equiv 0$. Therefore, by Gauss' Law there can be <u>NO</u> electric field inside a perfect conductor ∎

For a material such as Copper $\epsilon = \epsilon_0 = \left(\frac{1}{36\pi} \times 10^{-9} \frac{\text{farads}}{\text{m}}\right) \cong 10^{-12}$

$\sigma = 5.8 \times 10^{7} \frac{}{\text{ohm-m}}$

$$\therefore \quad \frac{\epsilon}{\sigma} \cong 1.5 \times 10^{-19} \text{ seconds}$$

and no field exists after that time (approximately).

for two media : $D_{n2} - D_{n1} = \rho_s$

$$E_{t2} = E_{t1}$$

I, Suppose, medium 1 is a perfect conductor,

In 1 $\quad E_{t1} \to 0$

$\therefore D_{n1} \to 0$

and $\quad E_{t2} = 0$

$$D_{n2} = \rho_s$$

Interpreting this ① the tangential component of $\underline{E}$ is always zero at a conducting surface

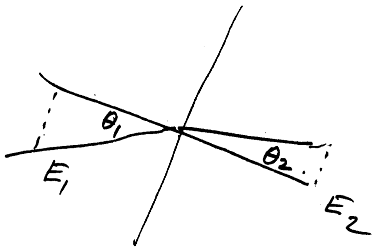
② The surface charge density is the normal component of $\underline{D}$ at the conductor surface.

II, If media 1 and 2 are perfect dielectrics how can one have a ρ_s?

In general, $\rho_s = 0$ unless deliberately put there.

So, $\quad \epsilon_2 E_{n2} - \epsilon_1 E_{n1} = 0$

$$E_{t2} = E_{t1}$$



In terms of angles, relative to the surface.

$$E_1 \cos\theta_1 = E_2 \cos\theta_2$$

$$\frac{E_2}{E_1} = \frac{\cos\theta_1}{\cos\theta_2}.$$

$$\epsilon_2 E_2 \sin\theta_2 = \epsilon_1 E_1 \sin\theta_1$$

$$\frac{\epsilon_2 E_2}{\epsilon_1 E_1} = \frac{\sin\theta_1}{\sin\theta_2}$$

combining these results.

$$\frac{E_2}{E_1} = \frac{\cos\theta_1}{\cos\theta_2} = \frac{\epsilon_1 \sin\theta_1}{\epsilon_2 \sin\theta_2}$$

$$\tan\theta_2 = \frac{\epsilon_1}{\epsilon_2}\tan\theta_1 .$$

and $$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_1}{\epsilon_2}$$

III. If the regions have finite conductivity?

use Ohm's law $\underline{J} = \sigma \underline{E}$ where σ_1, σ_2

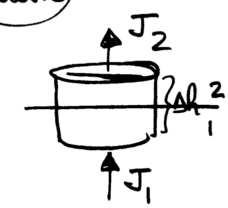
Recall that: $$\oint \underline{J} \cdot d\underline{s} = -\frac{d}{dt}\int_v \rho \, dv$$

$$\parallel$$

$$\int_v \underline{\nabla} \cdot \underline{J} \, dv$$

If the currents are static then $\frac{d}{dt} \to 0$

what is a static current...



let $\Delta h \to 0$ and then $J_{1n} = J_{2n}$.

Furthermore, $E_{2t} = E_{1t}$ or as $J = \sigma E$

$$\frac{J_{2t}}{\sigma_2} = \frac{J_{1t}}{\sigma_1}$$

We don't know what ρ_s is, since charge flow is permitted.

In general, $D_{n2} - D_{n1} = \rho_s$

$$\epsilon_2 E_{n2} - \epsilon_1 E_{n1} = \rho_s$$

but $\sigma_2 E_{2n} = \sigma_1 E_{1n}$

or $$\epsilon_2 \frac{\sigma_1}{\sigma_2} E_{1n} - \epsilon_1 E_{1n} = \rho_s$$

and if I know E in 1 or 2 I can compute the surface charge.

mathematically,

$$2W_T = Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32}).$$

$$V_i \triangleq \sum_{\substack{j=1\\ i\neq j}}^{3} V_{ij}$$

and $$W_T = \frac{1}{2}\sum_{i=1}^{3} Q_i V_i$$

Now, extend this result to a continious distribution of charge.

$$W_e = \frac{1}{2}\int_v \rho_v V \, dv$$

This is the energy due to putting the electric field in place.

Use Gauss' Law to re-write it as $\underline{\nabla}\cdot\underline{D} = \rho_v$

$$W_e = \frac{1}{2}\int_v (\underline{\nabla}\cdot\underline{D})\, V \, dv$$

We use pure math from this point.

$$(\underline{\nabla}\cdot\underline{D})V = \underline{\nabla}\cdot(V\underline{D}) - \underline{D}\cdot(\nabla V)$$

$-\underline{E}$

$$W_e = \frac{1}{2}\int_v \underline{\nabla}\cdot(V\underline{D})\, dv - \frac{1}{2}\int_v \underline{D}\cdot(\nabla V)\, dv$$

this is a source integral, convert to a flux

$$= \frac{1}{2}\oint_S V\underline{D}\cdot d\underline{s} - \frac{1}{2}\int_V \underline{D}\cdot(-\underline{E})\, dv.$$

for point charges $V \propto \frac{1}{r}$ $\qquad |V| = \frac{q}{4\pi\epsilon_0 r}$

$|\underline{D}| \propto \frac{1}{r^2}$ $\qquad |D| = \epsilon_0 E = \epsilon_0 \frac{q}{4\pi\epsilon_0 r^2}$

$ds \propto r^2$ $\qquad |ds| = r^2 \sin\theta \, d\theta \, d\phi.$

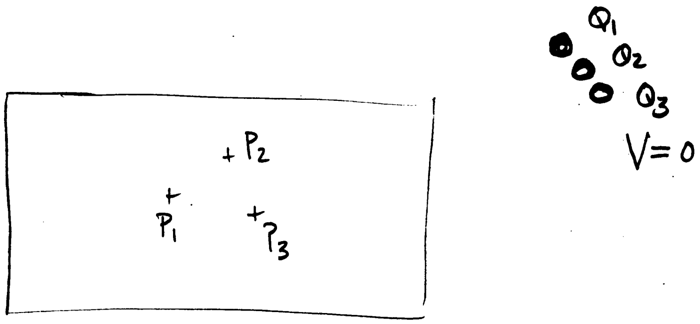
so as $r \to \infty$, i.e. all space, the first integral $\to 0$

$$W_e \to \frac{1}{2}\int_v \underline{D}\cdot\underline{E}\, dv = \frac{1}{2}\int_v \epsilon \underline{E}\cdot\underline{E}\, dv = \frac{1}{2}\int_v \epsilon |E|^2 dv.$$

energy density and the electric field

From earlier lecture incremental work per charge is

$$dV = \frac{dW}{\Delta Q} = -\underline{E} \cdot \underline{dl}$$



putting in one charge is <u>no</u> work.

~~putting~~ in Q_2 given Q_1's field will be given by

$$dW = dV \Delta Q$$

$$W_{21} = V_{21}\, Q_2$$

where W_{21} = work required to move Q_2 against Q_1's field

V_{21} is the potential ~~at the final~~ position of Q_2.

putting in Q_3 given Q_1 <u>and</u> Q_2's fields will be given by.

$$W_{31} + W_{32} = V_{31}\, Q_3 + V_{32}\, Q_3$$

$\therefore$ total energy is $W_T = W_{21} + W_{31} + W_{32} = V_{21} Q_2 + V_{31} Q_3 + V_{32} Q_3$

this is also the same as

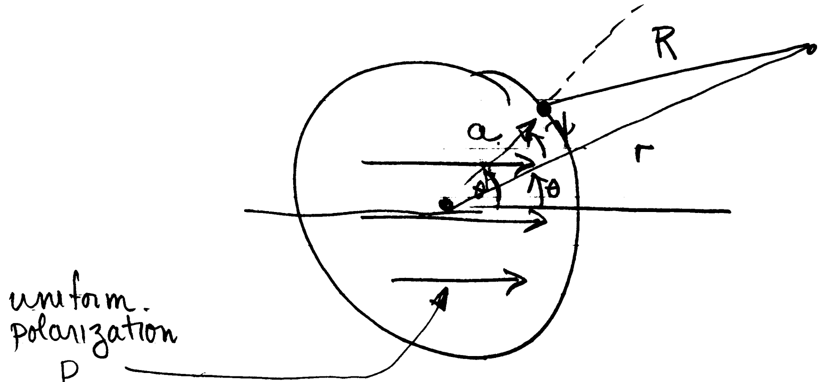
$$W_T = W_{12} + W_{13} + W_{23} = V_{12}\, Q_1 + V_{13}\, Q_1 + V_{23}\, Q_2$$

i.e. bring in Q_3 first, then Q_2, then Q_1

so, the order of bringing them together doesn't matter.

Now, add these results together

$$2W_T = V_{12} Q_1 + V_{13} Q_1 + V_{23} Q_2 + V_{21} Q_2 + V_{31} Q_3 + V_{32} Q_3$$



$\nabla \cdot \underline{P} = 0$

surface $\rho_{sp} = \bar{P} \cdot \bar{n} = (P_2 - P_1)\cos\theta.$

normal component is charge density Eqn. 33

$D = \epsilon_0 E + P$

here $E = 0$

so $n \cdot (P_2 - P_1) = \rho_s$

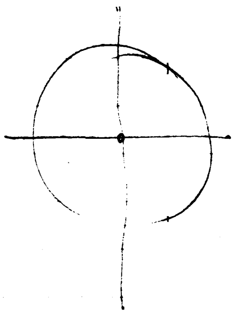
this is a bound charge.
but exists as a surface charge density.

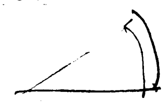
<u>ferro electric material</u>

way to solve this is potential.

$$\Phi = \frac{1}{4\pi\epsilon_0} \int_V \frac{\boxed{\rho\, dV}}{R}. \quad \rho_s\, d\ell.$$

given in class.





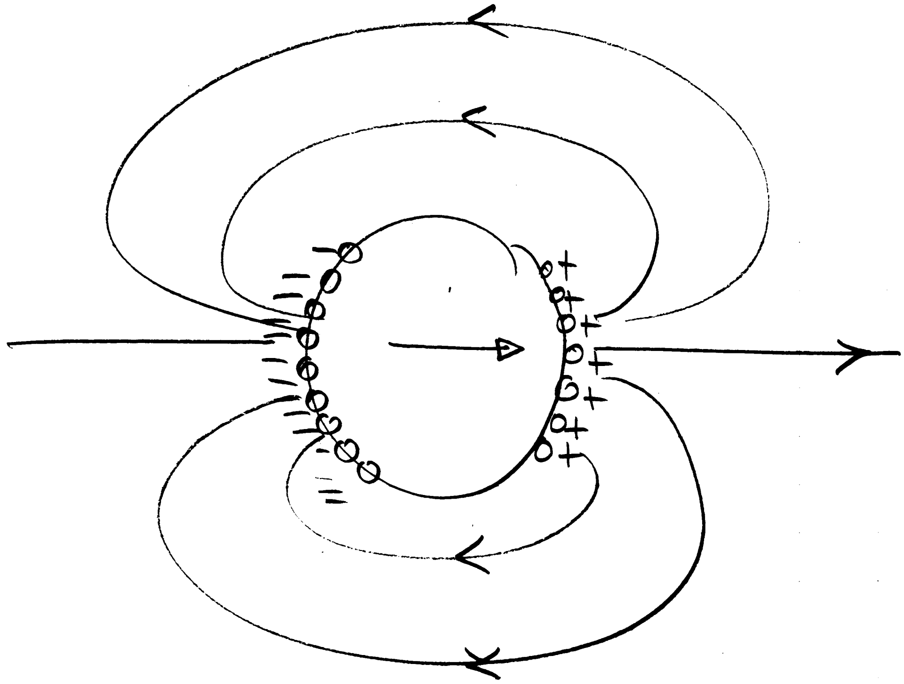
$$= \frac{1}{4\pi\epsilon_0} \iint \frac{(P_2 - P_1)\cos\theta \;\; a \;\; d\theta}{|\underline{a} - \underline{r}|}$$

$$|\underline{a} - \underline{r}|^2 = |\underline{a}|^2 + |\underline{r}|^2 - 2|\underline{a}||\underline{r}|\cos(\theta - \theta')$$

$r \gg a$

$$= a^2 + r^2 - 2ar\cos(\theta - \theta')$$

$$\Phi = \frac{a}{4\pi\epsilon_0} (P_2 - P_1) \int_0^{2\pi} \frac{\cos\theta \;\; d\theta}{|\underline{a} - \underline{r}|}$$



if you got to here I was happy, this equation is hard to solve unless $r \gg a$ then

$$|\underline{a}-\underline{r}|^2 = r^2\left[\left(\frac{a}{r}\right)^2 + 1 - 2\left(\frac{a}{r}\right)\cos(\theta-\theta')\right]$$

$$\cong r^2\left[1 - \frac{2a}{r}\cos(\theta-\theta')\right].$$

$$|\underline{a}-\underline{r}| \cong r\left[1 - 2\frac{a}{r}\cos(\theta-\theta')\right]^{1/2}$$

$$|\underline{a}-\underline{r}| \approx r\left[1 - 2\,\frac{1}{2}\,\frac{a}{r}\cos(\theta-\theta')\right]$$

$$= r\left[1 - \frac{a}{r}\cos(\theta-\theta')\right]$$

$$\Phi = \frac{a}{4\pi\epsilon_0}(P_2-P_1)\int_0^{2\pi}\frac{\cos\theta\, d\theta}{r\left[1-\frac{a}{r}\cos(\theta-\theta')\right]}$$

$$\approx \frac{a}{4\pi\epsilon_0}(P_2-P_1)\int_0^{2\pi}\left[\frac{\cos\theta\, d\theta}{r}\left(1+\frac{a}{r}\cos(\theta-\theta')\right)\right]$$

$$\approx \frac{a}{4\pi\epsilon_0}(P_2-P_1)\left[\int_0^{2\pi}\frac{\cos\theta\, d\theta}{r} + \frac{a}{r^2}\int_0^{2\pi}\cos\theta\cos(\theta-\theta')\,d\theta\right]$$

$$\Phi \approx \frac{a}{4\pi\epsilon_0}(P_2-P_1)\left[\frac{1}{r}\int_0^{2\pi}\cancel{\cos\theta\, d\theta}^{\,0} + \frac{a}{r^2}\int_0^{2\pi}\cos\theta\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\right]d\theta\right]$$

$$\approx \frac{a}{4\pi\epsilon_0}(P_2-P_1)\left[\frac{a}{r^2}\int_0^{2\pi}\underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)}\cos\theta'\, d\theta + \frac{a}{r^2}\int_0^{2\pi}\cancel{\underbrace{\cos\theta\sin\theta}_{\frac{1}{2}\sin 2\theta}\sin\theta' d\theta}^{\,0}\right]$$

$$\approx \frac{a}{4\pi\epsilon_0}(P_2-P_1)\frac{a}{r^2}\int_0^{2\pi}\frac{1}{2}\cos\theta'\, d\theta$$

$$\approx \frac{a^2}{4\cancel{\pi}\epsilon_0 r^2}(P_2-P_1)\,\cancel{\pi}\cos\theta'$$

$$\approx \frac{a^2}{4\epsilon_0 r^2}(P_2-P_1)\cos\theta'$$

This is an interesting result.

When we did the electric dipole

we got $\Phi = \dfrac{ql\cos\theta'}{4\pi\epsilon_0 r^2}$.

so that our result looks like a big electric dipole.

If I add the $\underline{E}$ field it does not change the bound surface charges; however, it everywhere adds to the P vector

$$D = \underline{\epsilon_0 E} + \underline{P}$$

Laplace & Poisson's Equations

$$\underline{E} = -\underline{\nabla}\underline{\Phi}$$

$$\nabla \cdot \underline{E} = -\underline{\nabla} \cdot \underline{\nabla}\underline{\Phi}$$

$$\downarrow$$

$$\underline{\nabla} \cdot \left(\frac{1}{\epsilon}\underline{D}\right) = -\underline{\nabla} \cdot \underline{\nabla}\underline{\Phi}$$

$$\frac{1}{\epsilon} \underbrace{\underline{\nabla} \cdot \underline{D}}_{\text{Gauss' law}} = -\underline{\nabla} \cdot \underline{\nabla}\underline{\Phi}$$

$$\frac{\rho_v}{\epsilon} = -\underbrace{\underline{\nabla} \cdot \underline{\nabla}\underline{\Phi}}_{\text{known as } \nabla^2 \underline{\Phi}}$$

general solution $\Phi = \int \frac{\rho \, dv}{4\pi\epsilon_0 R}$.

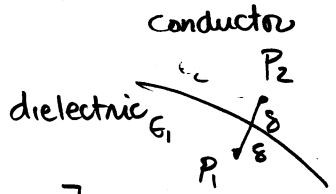
$$\nabla^2 \underline{\Phi} = -\frac{\rho_v}{\epsilon}$$ Poisson's Equation.

If the region contains no charge, this becomes

$$\nabla^2 \underline{\Phi} = 0$$ Laplace's Equation

What are boundary conditions on $\underline{\Phi}$?

$$\int_{P_1}^{P_2} d\underline{\Phi} = -\int_{P_1}^{P_2} \underline{E} \cdot \underline{dl}$$



$$\underline{\Phi}(P_2) - \underline{\Phi}(P_1) = -\left[E_{2n}\delta - E_{1n}\delta\right] = -\delta\left[E_{2n} - E_{1n}\right]$$

but what is $E_{2n} - E_{1n}$? From boundary conditions on $\underline{D}$,

$$\cancel{D_{n2}}^{\,0} - D_{n1} = \rho_s \quad \text{or} \quad -\epsilon_1 E_{1n} = \rho_s$$

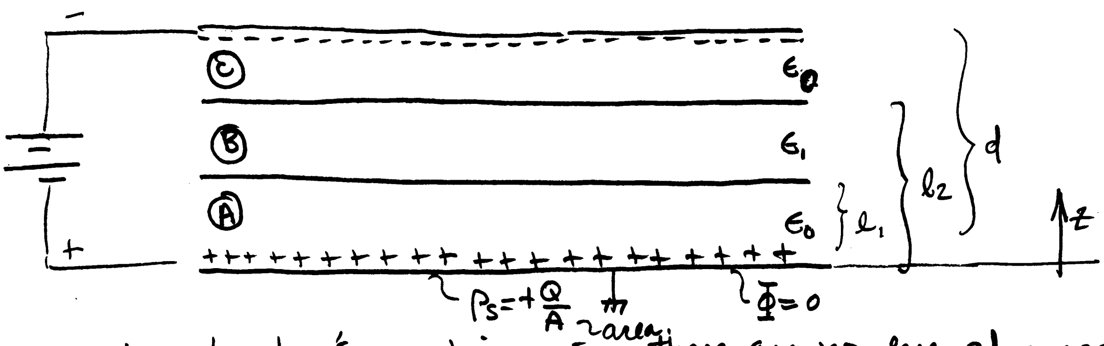
If ρ_s is finite – as it must be for any real problem.

$$\lim_{\delta \to 0} -\delta\left[\cancel{E_{2n}}^{\,0} - E_{1n}\right] = \lim_{\delta \to 0} -\delta\left[\frac{\rho_s}{\epsilon}\right] = 0$$

$\therefore \quad \Phi(P_2) - \Phi(P_1) = 0$ and potential is continuous across boundary.

Similar results can be shown for dielectric-dielectric interfaces

Parallel plate capacitor



use Laplace's equation since there are no free charges.

- use $\nabla^2 \Phi = 0$
- Assume plates infinitely large in x & y so $\frac{\partial}{\partial x} \to 0$ $\frac{\partial}{\partial y} \to 0$
- $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \to \frac{d^2 \Phi}{dz^2}$ under above assumption's
- if $\frac{d\Phi}{dz} = 0$ then $\Phi(z) = c_1 z + c_2$
- assume c_1 & c_2 different for each region
- find c_1 & c_2 using boundary conditions

Region A $\quad \Phi_A(z) = c_1 z + c_2$

at $z = 0$ $\Phi = 0$ so $c_2 = 0$.

How to get c_1? as $E = -\nabla\Phi$ and $\underline{D} = \epsilon \underline{E}$ use B.C. on $\underline{D}$

$$\underline{E} = -\nabla\Phi = -\nabla(c_1 z) = -\underline{a}_z c_1$$

$$\underline{D} = \epsilon_0 \underline{E} = -\epsilon_0 \underline{a}_z c_1$$

$D_{2n} - \cancelto{0}{D_{1n}} = \rho_s$ conductor, so $\underline{E} \to 0$

$\therefore \quad D_{2n} = -\epsilon_0 c_1 = \rho_s = \frac{Q}{A}$

so $C_1 = -\frac{Q}{\epsilon_0 A}$

and $\Phi_A(z) = -\frac{Q}{\epsilon_0 A} z$

how about A - B boundary?

as Φ is continuous.

$$\Phi_A(\ell_1) = \Phi_B(\ell_1) = C_{1B} z + C_{2B}$$

$$\| $$

$$-\frac{Q}{\epsilon_0 A}\ell_1 = C_{1B}\ell_1 + C_{2B}.$$

– need to get a second equation... so as A and B are dielectrics

$D_{2n} - D_{1n} = \cancel{\rho_s} \to 0$ (no charge on boundary, insulators)

$$\therefore \quad D_{2n} = \epsilon_1 \left(-\frac{d\Phi_B}{dz}\bigg|_{\ell_1} \right) = -\epsilon_1 C_{1B}$$

$$D_{1n} = \epsilon_0 \left(-\frac{d\Phi_A}{dz}\bigg|_{\ell_1} \right) = -\epsilon_0 \left(-\frac{Q}{\epsilon_0 A} \right) = +\frac{Q}{A}$$

$$\therefore \quad -\epsilon_1 C_{1B} - \frac{Q}{A} = 0$$

$$\text{or } C_{1B} = -\frac{Q}{\epsilon_1 A}$$

and $-\frac{Q}{\epsilon_0 A}\ell_1 = -\frac{Q}{\epsilon_1 A}\ell_1 + C_{2B}$

$$\frac{Q}{A}\ell_1\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_0}\right) = C_{2B}$$

$\therefore$ in B

$$\Phi_B(z) = -\frac{Q}{\epsilon_1 A} z + \frac{Q}{A}\ell_1\left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_0}\right)$$

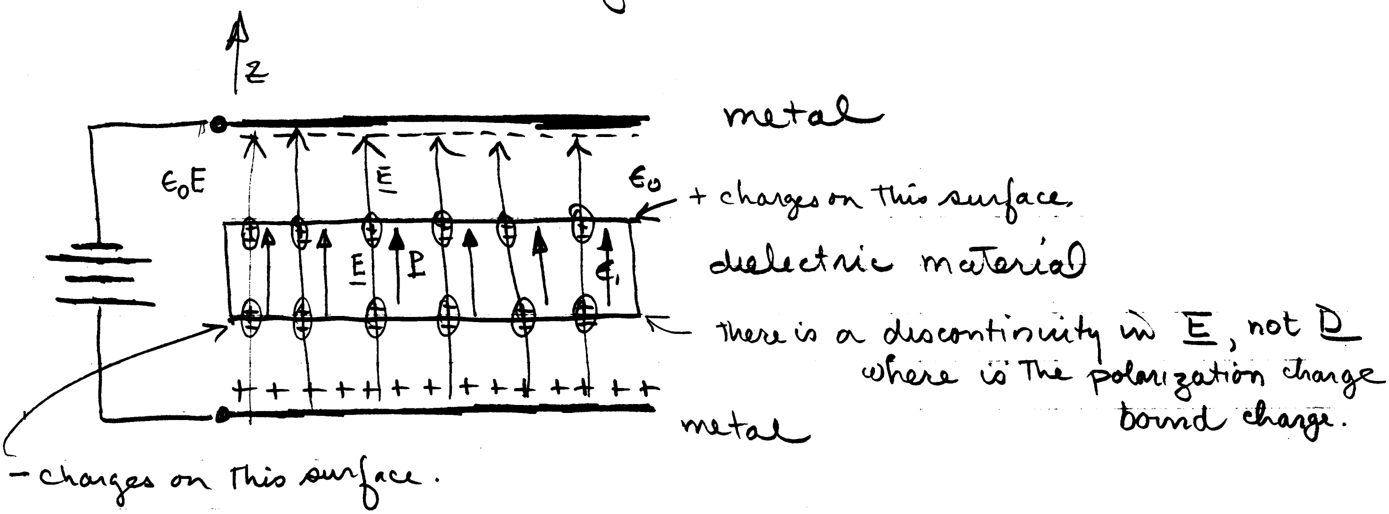
at B - C boundary. $\Phi_C(\ell_2) = \Phi_B(\ell_2)$ Φ continuous as before.

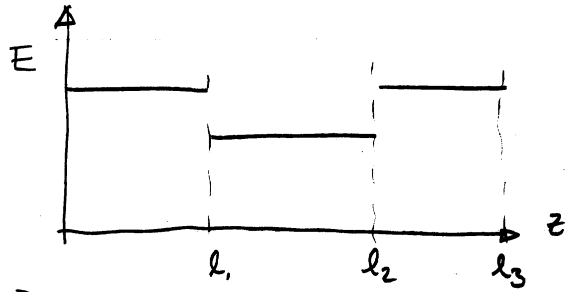
$$C_{1C}\ell_2 + C_{2C} = -\frac{Q\ell_2}{\epsilon_1 A} + \frac{Q\ell_1}{A\epsilon_1} - \frac{Q\ell_1}{A\epsilon_0}$$

as before $D_{Cn} = D_{Bn}$

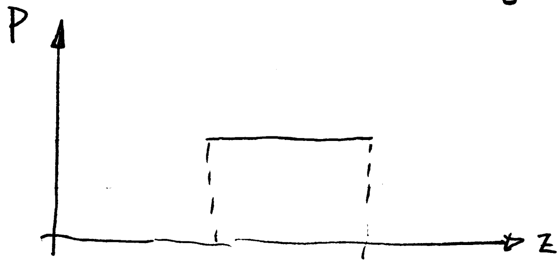
$$\epsilon_2 \left(-\frac{\partial \Phi_C}{\partial z}\right)\bigg|_{\ell_2} = \epsilon_1 \left(-\frac{\partial \Phi_B}{\partial z}\right)\bigg|_{\ell_2}$$

review the boundary

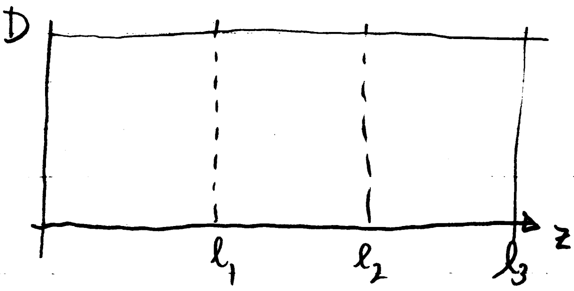




E is due to charge of all types



P is due to bound charges



D is due to free charge

watch signs

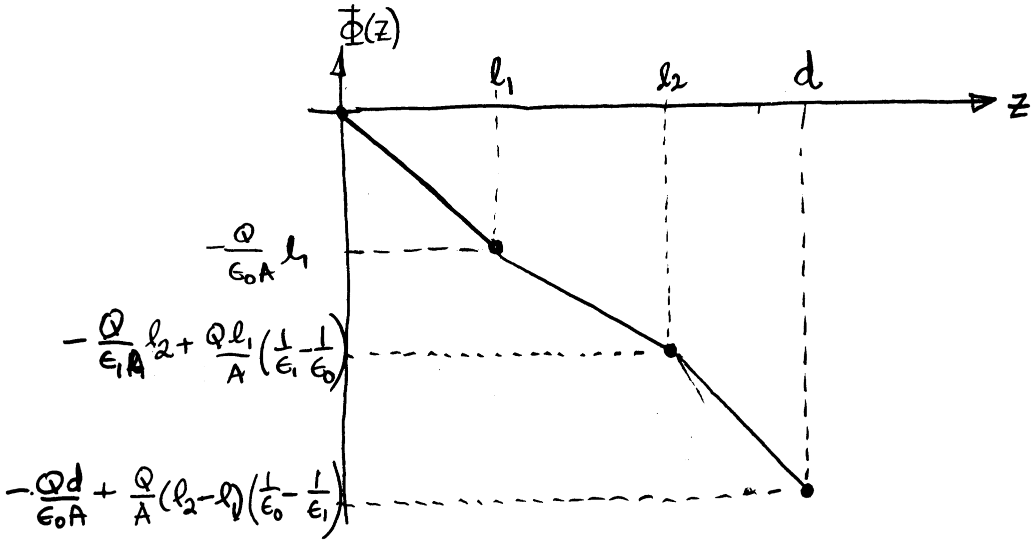
$$\epsilon_0(-C_{1c}) = \cancel{\epsilon_1}\left(+\frac{Q}{\cancel{\epsilon_1} A}\right)$$

$$\therefore C_{1c} = \frac{-Q}{\epsilon_0 A}$$

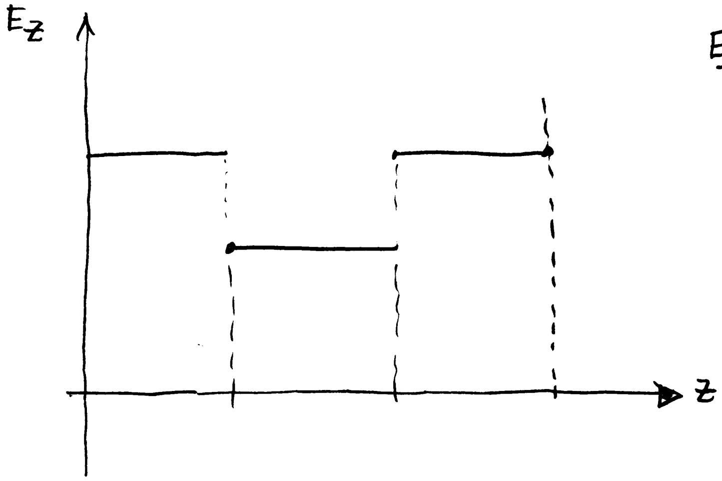
$$-\frac{Q}{\epsilon_0 A}\ell_2 + C_{2c} = -\frac{Q\ell_2}{\epsilon_1 A} + \frac{Q\ell_1}{A\epsilon_1} - \frac{Q\ell_1}{A\epsilon_0}$$

$$C_{2c} = \frac{Q\ell_2}{\epsilon_0 A} - \frac{Q\ell_2}{\epsilon_1 A} + \frac{Q\ell_1}{\epsilon_1 A} - \frac{Q\ell_1}{\epsilon_0 A} = \frac{Q}{A}(\ell_2 - \ell_1)\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_1}\right)$$

$$\therefore \Phi_c(z) = -\frac{Q}{\epsilon_0 A} z + \frac{Q}{A}(\ell_2 - \ell_1)\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_1}\right)$$

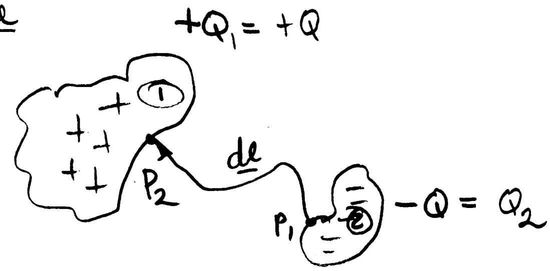


notice that as V was given we could solve for $\rho_s = \frac{Q}{A}$.



$$E_z = -\frac{\partial \Phi}{\partial z}$$

Capacitance



- Consider two arbitrarily shaped conductors with equal total charges $+Q$ and $-Q$.

- surface charge densities are different...

$$Q_1 = \int_{S_1} \rho_{s1}\, ds_1 \qquad Q_2 = \int_{S_2} \rho_{s2}\, ds_2$$

- as each surface is conducting $E_{tan}=0$, $\underline{E}$ is normal to each surface, and each surface is equipotential (no tangential field on surface) as. $\Phi_2 - \Phi_1 = -\int_{P_1}^{P_2} \underline{E}\cdot \underline{dl}$

- potential differences between surfaces is $V = \Phi_2 - \Phi_1 = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{dl}$

- for any point P_0 not in 1 or 2

$$\Phi_1(P_0) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_{s1}}{r_{10}}\, ds_1$$

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\underline{r}')}{|\underline{r}'-\underline{r}_0|}\, dv'$$

$$\Phi_2(P_0) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_{s2}}{r_{20}}\, ds_2$$

back when we looked at the electric dipole.

$$\Phi_i(\underline{r}) = \frac{Q_i}{4\pi\epsilon_0|\underline{r_i} - \underline{r_0}|}$$

this was for a charge at r_i and the field at r.

This is the volume density analog.

- Now, double Q_1 and Q_2. $\Rightarrow$ ρ_{s1} and ρ_{s2} are doubled

$\Rightarrow$ Φ_1 and Φ_2 are doubled

$\Rightarrow$ $V = \Phi_2 - \Phi_1$ is doubled.

Thus, one can conclude that $V \propto Q$ or $\frac{Q}{V} = \text{constant}$

$C \equiv \frac{Q}{V}$ capacitance.

For the example we just did, parallel plates...

$$\mathcal{C} = \frac{Q}{V} = \frac{Q}{+\frac{Qd}{\epsilon_0 A} - \frac{Q}{A}(l_2 - l_1)\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_1}\right)}$$

this is because
$V = \Phi(+) - \Phi(-)$
$\Phi(+) = 0$ in this problem, $\Phi(-) = \Phi(d)$

$$= \frac{A}{+\frac{d}{\epsilon_0} + (l_2 - l_1)\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon_1}\right)}$$