EEAP 210

ELECTROMAGNETIC FIELDS

SPRING SEMESTER 1983

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TEXT: PAUL & NASAR, INTRODUCTION TO ELECTROMAGNETIC FIELDS

COURSE OUTLINE (Approximately – SUBJECT TO SOME CHANGE)

LECTURES	TOPIC	CHAPTERS IN TEXT
3	VECTOR ANALYSIS	1, 2
1	Exam	—
6	Electrostatics	3
1	Exam	—
6	Magnetostatics	4
1	Exam	—
8	Maxwell's Equations	5
1	Exam	—
6	Waves and Radiation	6
1	Exam	—
6	Transmission Lines	7
1	Exam	—
3	Waveguides	8
2	Potential Functions	10
	Final Exam	

EEAP 210
REVISED SYLLABUS
&
GRADING POLICY

FRIDAY'S MEETING WITH THE <u>COMMITTEE OF FOUR</u> PRODUCED THE REVISED GRADING POLICY SHOWN BELOW:

HOMEWORK	18	POINTS
EXAMS (5)	59	POINTS
FINAL	23	POINTS
	100	POINTS TOTAL

THE FINAL TWO (2) EXAMS WILL BE:

FRIDAY APRIL 15TH
FRIDAY APRIL 29TH.

THESE EXAMS WILL BE EQUALLY WEIGHTED WITH THE PREVIOUS 3 EXAMS.

SYLLABUS

DATE	LECTURE
4/4	Time-dependent Maxwell's Equations, displacement current
4/6	Sinusoidal-steady state, poynting vector
4/8	Wave equation, plane waves
4/11	Traveling waves, group & phase velocity, impedance
4/13	Conductors & dielectrics, reflection & transmission of plane waves
4/15	**EXAM #4**
4/18	TEM waves, "transmission line" equations, traveling waves
4/20	pulse transmission on a "transmission" line
4/22	reflection coefficent, impedance, load transformation
4/25	"lumped" versus "distributed" transmission lines
4/27	waveguides <u>OR</u> quasi-statics
4/29	**EXAM #5**

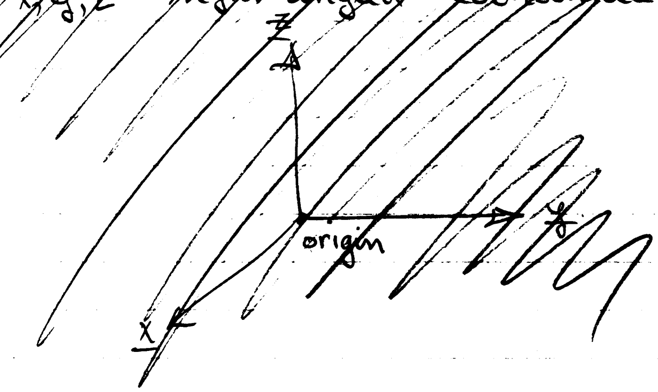
THERE WILL BE EVENING LECTURES APRIL 12TH and 26TH, PROBABLY AT 7:00 IN GLENNAN, PROBABLY IN 713(?).

Lecture #1 1/9/82

Purpose: a. vector definitions & scalar operations ✓
b. Cartesian coordinates ✓
c. vector operations
d. coordinate systems
e. scalar vs. vector fields

Cartesian (rectangular) coordinates

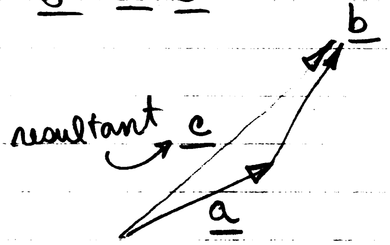
x, y, z right angled coordinates



vector – basic magnitude (quantity) with direction

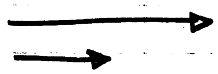
scalar operations on vectors operate in a algebraic manner (addition, subtraction, scalar multiplication)

addition $\underline{c} = \underline{a} + \underline{b}$



representations of vectors

magnitude by length



direction by direction and arrow head.

unit vectors The magnitude is one
indicate direction only.

For example:
$$\underline{a} = a\,\underline{a_u}$$
$$\underline{b} = \underbrace{b}_{\text{magnitude}}\;\underbrace{\underline{b_u}}_{\text{direction}}$$

vector components - represent vectors by sums of unit vectors

$$\underline{c} = \underline{a} + \underline{b} = a\,\underline{a_u} + b\,\underline{b_u} = [a, b]$$

where it is assumed we know what the unit vectors are.

$$\underline{a} = a\,\underline{a_u} = [a, 0]$$
$$\underline{b} = b\,\underline{b_u} = [0, b]$$
$$\underline{c} = a\underline{a_u} + b\underline{b_u} = [a, 0] + [0, b] = [a, b].$$

scalar multiplication

$$\underline{d} = k\underline{a} = \underbrace{ka}\,\underline{a_u} + \ldots$$

increases magnitude by k, does not change direction

$$\underline{d} = k\underline{c} = k[\underline{a} + \underline{b}] = k\underline{a} + k\underline{b} = [ka, 0] + [0, kb]$$
$$= [ka, kb]$$

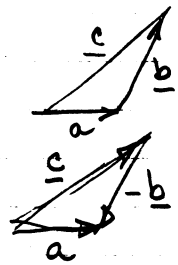
commutative

subtraction

if $\underline{c} = \underline{a} + \underline{b}$

$\underline{a} = \underline{c} - \underline{b}$

$[a, 0]$



subtraction reverses the direction of $\underline{b}$

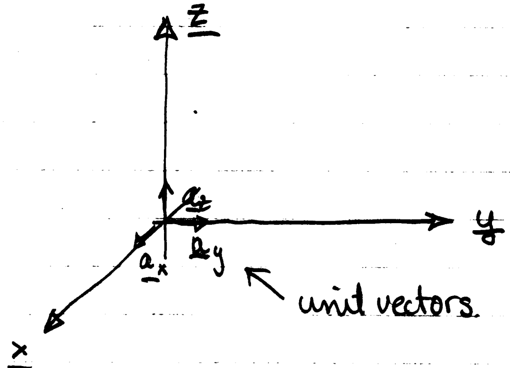
$\underline{c} = [a, b]$

$\underline{b} = [0, b]$

$-\underline{b} = [0, -b]$

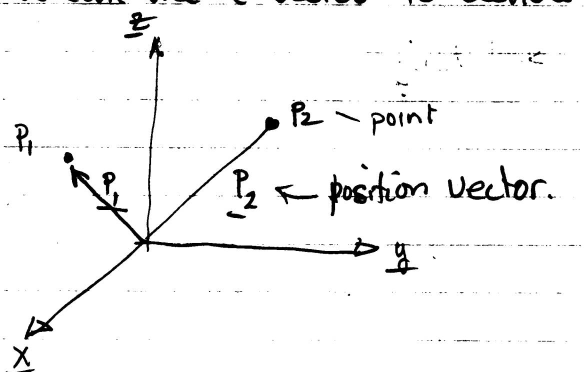
$$\therefore \quad \underline{c} - \underline{b} = [a, b] + [0, -b] = [a, 0] = \underline{a}$$

Cartesian coordinates



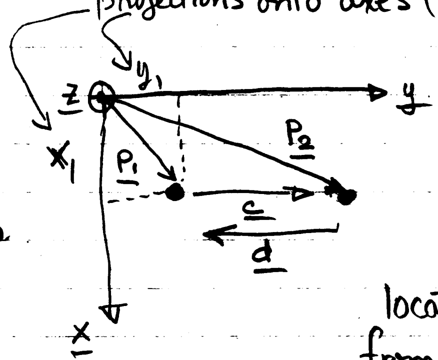
axes (unit vectors) at right angles.

we can use a vector to denote only position, location, distance, etc.



just
x, y
for example.

projections onto axes (can be mathematically defined)



as seen
from the top

located
from origin

$$\underline{P_1} = [x_1, y_1, z_1] = x_1 \underline{a_x} + y_1 \underline{a_y} + z_1 \underline{a_z}$$

$$\underline{P_2} = [x_2, y_2, z_2] = x_2 \underline{a_x} + y_2 \underline{a_y} + z_2 \underline{a_z}$$

since the vector indicates the location of an object

what is difference in locations between P_1 and P_2.

Since this is a vector, the ~~position~~ ~~from~~ $\underline{P_1}$ ~~from~~ $\underline{P_2}$

is **NOT** the same as $\underline{P_2}$ from $\underline{P_1}$

the distance (magnitude & direction) from P_1 to P_2 is

$$\underline{P_1} + \underline{c} = \underline{P_2} \quad \text{or} \quad \underline{P_2} + \underline{d} = \underline{P_1}$$

$$\underline{c} = \underline{P_2} - \underline{P_1} \qquad \underline{d} = \underline{P_1} - \underline{P_2}$$

$$= -(\underline{P_2} - \underline{P_1})$$

$$= -\underline{c}$$

as we expected.

in vector notation

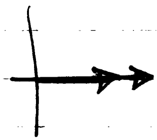
$$\underline{c} = \underline{P_2} - \underline{P_1} = [x_2, y_2, z_2] - [x_1, y_1, z_1] = [x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1]$$

$\underline{c}$ represents the vector distance from $\underline{P_1}$ to $\underline{P_2}$

what is the scalar distance

if $\underline{c}$ lay along one of the axes, say $\underline{a_x}$, only

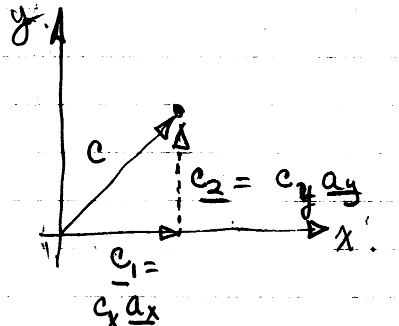
then $\underline{c} = [0, 0, c]$



and by inspection $|\underline{c}| = c$

means scalar magnitude

if $\underline{c}$ lay in the plane, say $\underline{a_x}$ and $\underline{a_y}$



$$\underline{c} = \underline{c_1} + \underline{c_2}$$
$$= c_x \underline{a_x} + c_y \underline{a_y}$$

what is $|\underline{c}|$? by inspection of graph. $|c| = \sqrt{c_x^2 + c_y^2}$

using Pythagorans' theorem.

expanding to three dimensions

$$|\underline{c}| = \sqrt{c_x^2 + c_y^2 + c_z^2}$$

differential lengths

an incremental distance in the x direction would be dx

magnitude since direction is assumed.

the distance from x_1 to x_2 must be.

$$d = \int_{x_1}^{x_2} dx = x\Big|_{x_1}^{x_2} = x_2 - x_1$$

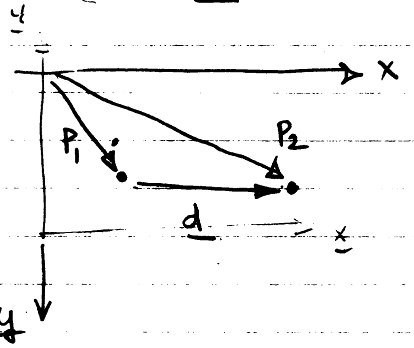
the distance from $\underline{P_1}$ to $\underline{P_2}$

$$\underline{d} = \int_{\underline{P_1}}^{\underline{P_2}} dl$$

what is dl?

dl is a differential magnitude as we move along a vector from $\underline{P_1}$ to $\underline{P_2}$

Assume the $\underline{P_1}$ and $\underline{P_2}$ we had before



Δl_1 Δl_2 Δl_3 Δl_4 etc.

consider $\underline{d} = \Delta \underline{l}_1 + \Delta \underline{l}_2 + \Delta \underline{l}_3 + \Delta \underline{l}_4 + \ldots$

then $\underline{d} = \sum_{i=1}^{N} \Delta \underline{l}_i$

as $|\Delta \underline{l}_i| \to 0$

$$\underline{d} = \int_{P_1}^{P_2} d\underline{l}$$

what is $d\underline{l}$? $d\underline{l}$ is the differential vector along $\underline{d}$ and is not necessarily pointing in the same direction all the time.

however, in this case

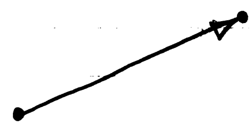
$$d\underline{l} = dx\,\underline{a_x} + dy\,\underline{a_y} + dz\,\underline{a_z}$$

suppose we simply wanted the distance from P_1 to P_2

no vector.

i.e. $|\underline{d}|$

then $$|\underline{d}| = \int_{\underline{P_1}}^{\underline{P_2}} |d\underline{l}| = \int_{\underline{P_1}}^{\underline{P_2}} \sqrt{dx^2 + dy^2 + dz^2}$$



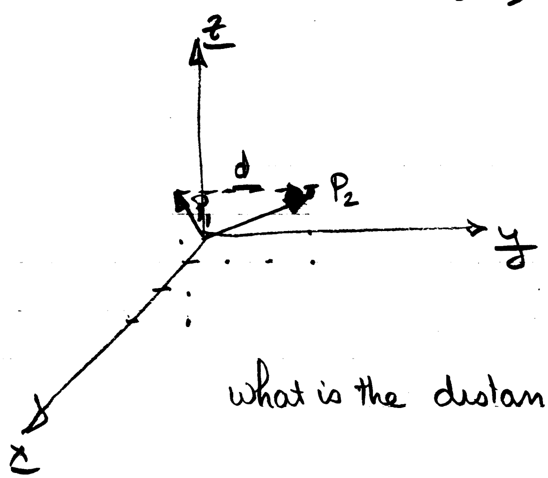
if $\underline{d}$ is a straight line then we can calculate

$|d|$ by $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

however, if we need the length along a specific contour we must use the above equation.

Example:

$$\left.\begin{array}{l} P_1 = [3, 1, 3] \\ P_2 = [1, 3, 2] \\ \quad\;\; x_2 \;\; y_2 \;\; z_2 \end{array}\right\}$$ rectangular coordinate system



what is the distance between P_1 and P_2

since there is no contour

$$|\underline{P_2} - \underline{P_1}| = |\underline{d}| = \sqrt{(1-3)^2 + (3-1)^2 + (2-3)^2}$$

$$= \sqrt{(-2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

by contour integral

what is the vector from P_1 to P_2

$$\underline{d} = \underline{P_2} - \underline{P_1} = (\underline{a_x} + 3\underline{a_y} + 2\underline{a_z}) - (3\underline{a_x} + \underline{a_y} + 3\underline{a_z})$$

$$= -2\underline{a_x} + 2\underline{a_y} - \underline{a_z}$$

the problem in evaluating the integral this way is that we <u>must</u> write the equation of the line ($\underline{d}$) between $\underline{P_1}$ and $\underline{P_2}$.

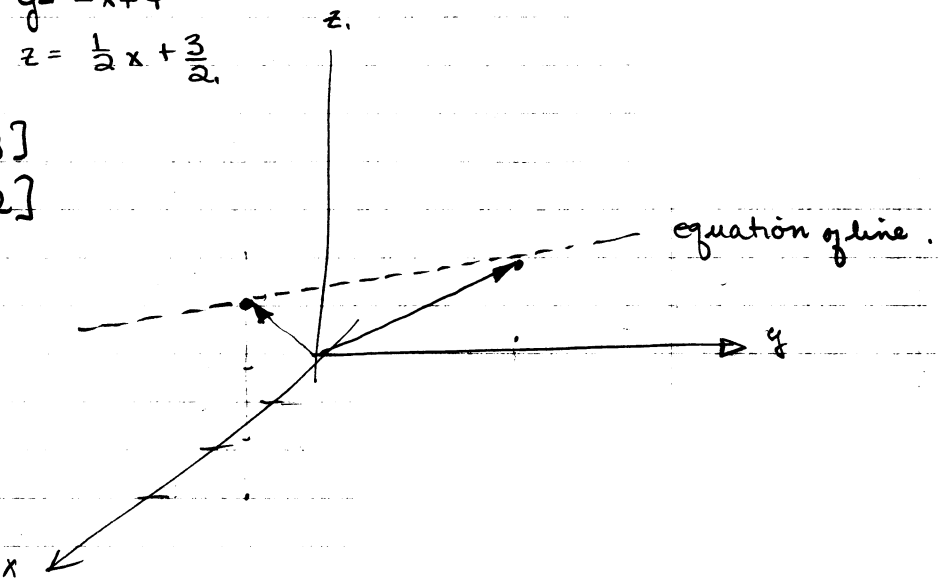
the vector $\underline{d} = -2\underline{a_x} + 2\underline{a_y} - \underline{a_z}$

$y = -x + 4$

$z = \frac{1}{2}x + \frac{3}{2}$

$[x_1, y_1, z_1] = [3, 1, 3]$

$[x_2, y_2, z_2] = [1, 3, 2]$



$\boxed{\text{two-point form}}$ (see a text on analytic geometry)

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 1}{x - 3} = \frac{3 - 1}{1 - 3}$$

$$\frac{y-1}{x-3} = \frac{2}{-2} = -1$$

$$y - 1 = -x + 3$$

$$\boxed{y = -x + 4}$$

$$\frac{z - z_1}{x - x_1} = \frac{z_2 - z_1}{x_2 - x_1}$$

$$\frac{z - 3}{x - 3} = \frac{2 - 3}{1 - 3} = \frac{-1}{-2} = \frac{1}{2}$$

$$z - 3 = \frac{1}{2}(x - 3)$$

$$z - 3 = \frac{1}{2}x - \frac{3}{2}$$

$$z = \frac{1}{2}x - \frac{3}{2} + \frac{6}{2} = \boxed{\frac{1}{2}x + \frac{3}{2}}$$

this is the equation of the line.

example (cont.)

$$y = -x + 4 \qquad z = \tfrac{1}{2}x + \tfrac{3}{2}$$

$$dy = -dx \qquad dz = +\tfrac{1}{2}\,dx.$$

$$\therefore\ d\ell = \sqrt{dx^2 + dy^2 + dz^2}$$

$$= \sqrt{dx^2 + (-dx)^2 + (\tfrac{1}{2}dx)^2}$$

$$= \sqrt{dx^2 + dx^2 + \tfrac{1}{4}dx^2}$$

$$= \sqrt{(1+1+\tfrac{1}{4})\,dx^2} = \sqrt{\tfrac{9}{4}\,dx^2} = \tfrac{3}{2}\,dx$$

$$|\underline{d}| = \int_{P_1}^{P_2} d\ell = \int_{x=3}^{x=1} \tfrac{3}{2}\,dx = \tfrac{3}{2}x\Big|_3^1 = \tfrac{3}{2}\big[1-3\big] = -3.$$

Why is this answer wrong?

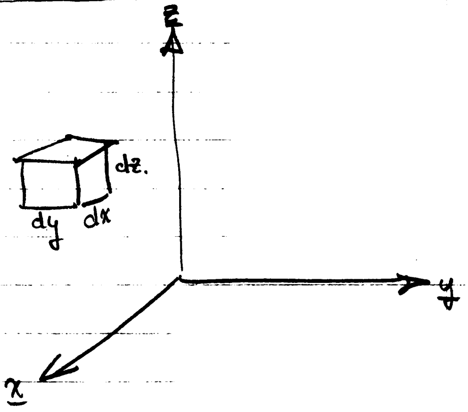
because $d\ell = \pm\sqrt{dx^2 + dy^2 + dz^2}$

and in this case we choose the minus sign

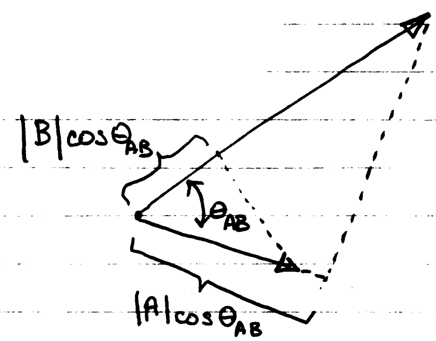
because $|\underline{d}|$ must <u>always</u> be positive or zero.

differential volume.

$dV = dx\,dy\,dz.$



~~Vector~~ operations (these appear in electromagnetic field equations)



dot product $\qquad \underbrace{\underline{A}\cdot\underline{B}}_{\text{this is read A dot B.}} \triangleq |\underline{A}||\underline{B}| \cos\theta_{AB}$

the result is a scalar and this is often called the scalar product.

① The best geometric interpretation of $\underline{A}\cdot\underline{B}$ is that it is the projection of $\underline{A}$ onto $\underline{B}$.

② A valuable property of the dot product is that we can, if we, know $\underline{A}$ & $\underline{B}$, compute the included angle.

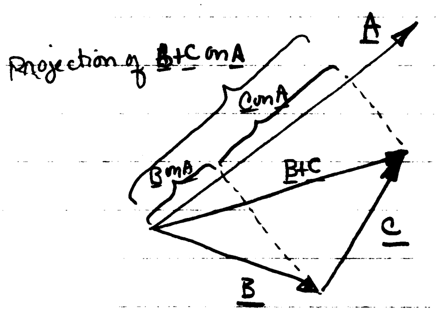
If $\quad \underline{A}\cdot\underline{B} = |\underline{A}||\underline{B}| \cos\theta_{AB}$

$$\theta_{AB} = \cos^{-1}\left[\frac{\underline{A}\cdot\underline{B}}{|\underline{A}|\,|\underline{B}|}\right]$$

This works if we know how to compute $\underline{A}\cdot\underline{B}$ by another method.

to develop how we can operate in terms of unit vectors consider the dot product of $\underline{A}$ and the sum of two vectors $\underline{B}$ and $\underline{C}$

$$\underline{A}\cdot(\underline{B}+\underline{C}) = \underline{A}\cdot\underline{B} + \underline{A}\cdot\underline{C}$$



extending this result

$$\underline{A}\cdot(\underline{B}+\underline{C}+\underline{D}) = \underline{A}\cdot\underline{B} + \underline{A}\cdot\underline{C} + \underline{A}\cdot\underline{D}$$

$$(\underline{A}+\underline{E})\cdot(\underline{B}+\underline{C}+\underline{D}) = \underline{A}\cdot(\underline{B}+\underline{C}+\underline{D}) + \underline{E}\cdot(\underline{B}+\underline{C}+\underline{D})$$

$$= \underline{A}\cdot\underline{B} + \underline{A}\cdot\underline{C} + \underline{A}\cdot\underline{D} + \underline{E}\cdot\underline{B} + \underline{E}\cdot\underline{C} + \underline{E}\cdot\underline{D}$$

etc.

suppose $\underline{A} = A_x\,\underline{a_x} + A_y\,\underline{a_y} + A_z\,\underline{a_z}$

$\underline{B} = B_x\,\underline{b_x} + B_y\,\underline{b_y} + B_z\,\underline{b_z}$

$$\underline{A}\cdot\underline{B} = (A_x\underline{a_x} + A_y\underline{a_y} + A_z\underline{a_z})\cdot(B_x\underline{b_x} + B_y\underline{b_y} + B_z\underline{b_z})$$

$$= A_xB_x\,\underline{a_x}\cdot\underline{b_x} + A_yB_x\,\underline{a_y}\cdot\underline{b_x} + A_zB_x\,\underline{a_z}\cdot\underline{b_x}$$

$$A_xB_y\,\underline{a_x}\cdot\underline{b_y} + A_yB_y\,\underline{a_y}\cdot\underline{b_y} + A_zB_y\,\underline{a_z}\cdot\underline{b_y}$$

$$A_xB_z\,\underline{a_x}\cdot\underline{b_z} + A_yB_z\,\underline{a_y}\cdot\underline{b_z} + A_zB_z\,\underline{a_z}\cdot\underline{b_z}$$

but $\underline{a_i}\cdot\underline{b_j} = \delta_{ij}$ (use projections

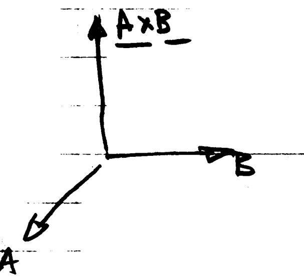
$$\therefore \underline{A}\cdot\underline{B} = A_xB_x + A_yB_y + A_zB_z$$

can I write $\underline{A} \cdot \underline{B} \cdot \underline{C}$

No! $\underline{A} \cdot \underline{B}$ is not a vector

and $(\underline{A} \cdot \underline{B}) \cdot C$ makes no sense.

cross product $\underline{A} \times \underline{B} = |A||B| \sin \theta_{AB}\, \underline{a}_N$



$\underline{a}_N \perp$ to plane containing $\underline{A}$ and $\underline{B}$

Right hand rule

Take your right hand, extend it along a and rotate your hand from $\underline{A}$ to $\underline{B}$, your thumb will point in the direction of $\underline{A} \times \underline{B}$

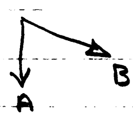
(this is also the way a screw operates, often operations in the plane produce operations $\perp$ to the plane).

from the definition

$$\theta_{AB} = \sin^{-1} \frac{\underline{A} \times \underline{B}}{|\underline{A}|\,|\underline{B}|}$$

This is a much more difficult operation to write in terms of unit vectors.

First; $\underline{A} \times \underline{B} = -\underline{B} \times \underline{A}$



this can be seen in that the angle θ_{AB} changes sign and the sine changes accordingly

$$\underline{A} \times (\underline{B}+\underline{C}) = \underline{A}\times\underline{B} + \underline{A}\times\underline{C}$$

$$(\underline{A}+\underline{D}) \times (\underline{B}+\underline{C}) = \underline{A}\times(\underline{B}+\underline{C}) + \underline{D}\times(\underline{B}+\underline{C})$$

$$= \underline{A}\times\underline{B} + \underline{A}\times\underline{C} + \underline{D}\times\underline{B} + \underline{D}\times\underline{C}$$

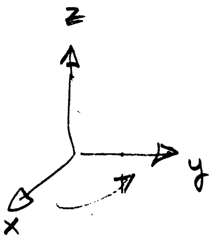
etc.

$$\underline{A}\times\underline{B} = (A_x \underline{a_x} + A_y \underline{a_y} + A_z \underline{a_z}) \times (B_x \underline{b_x} + B_y \underline{b_y} + B_z \underline{b_z})$$

$$= A_x B_x\, \underline{a_x}\times\underline{a_x} + A_y B_x\, \underline{a_y}\times\underline{a_x} + A_z B_x\, \underline{a_z}\times\underline{a_x}$$

$$A_x B_y\, \underline{a_x}\times\underline{a_y} + A_y B_y\, \underline{a_y}\times\underline{a_y} + A_z B_y\, \underline{a_z}\times\underline{a_y}$$

$$A_x B_z\, \underline{a_x}\times\underline{a_z} + A_y B_z\, \underline{a_y}\times\underline{a_z} + A_z B_z\, \underline{a_z}\times\underline{a_z}$$



any vector crossed with itself will be zero as $\Theta_{AB} \to 0$.

$$\underline{a_y}\times\underline{a_x} = -\underline{a_z}$$

$$\underline{a_z}\times\underline{a_x} = \underline{a_y}$$

$$\underline{a_x}\times\underline{a_y} = \underline{a_z}$$

$$\underline{a_z}\times\underline{a_y} = -\underline{a_x}$$

$$\underline{a_x}\times\underline{a_z} = -\underline{a_y}$$

$$\underline{a_y}\times\underline{a_z} = \underline{a_x}$$

$$\underline{A}\times\underline{B} = \underline{a_x}(A_yB_z - A_zB_y) + \underline{a_y}(A_zB_x - A_xB_z) + \underline{a_z}(A_xB_y - A_yB_x)$$

$$= \begin{vmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

be careful $\underline{A}\cdot(\underline{B}+\underline{C}) = \underbrace{\underline{A}\cdot\underline{B}} \times \underbrace{\underline{B}\cdot\underline{C}}$

these are dot products!

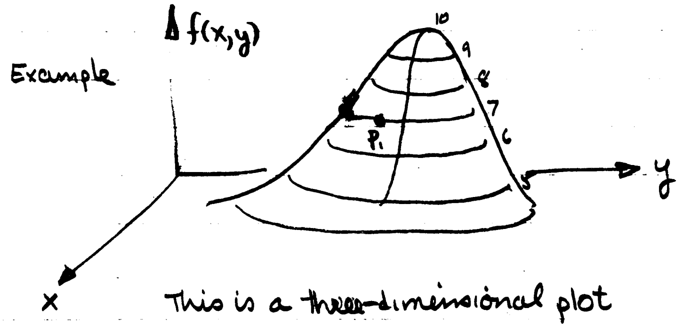
Objective: basic concepts of vector calculus
1. gradient
2. divergence
3. curl
4. Stoke's Theorem.

As will be shown in later lectures, the form of vector fields tells us about the sources of the fields. Basically, Maxwell's equations are ~~the~~ complete relationship between sources and fields. It turns out that the divergence and curl are the form that the source-field relationship takes.

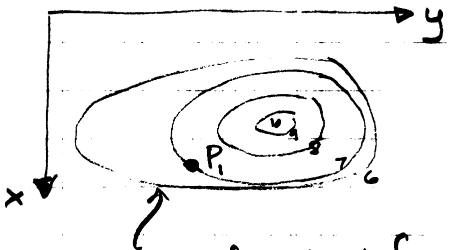
GRADIENT – the gradient is the directional derivative of a scalar field.

A scalar field is one that is a simple function of position; the dependent variable is not a vector. Examples: temperature, pressure, humidity, etc.

Example



This is a three-dimensional plot of $f(x,y)$.



contours of constant f

a two-dimensional plot of $f(x,y)$

Note that the derivative may be defined at P_1, but its value depends upon the direction. In general, a small change in f will be given by

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \qquad \text{This is in any direction.} \tag{1}$$

Lets re-write this in a vector form and look at its directional properties. The above looks a lot like a vector dot product — it's a sum of ~~vector~~ operations in x, y, then z. So one part must be

$$\frac{\partial f}{\partial x}\underline{a_x} + \frac{\partial f}{\partial y}\underline{a_y} + \frac{\partial f}{\partial z}\underline{a_z} \tag{2}$$

which we will call $\underline{\text{grad} f}$ or $\underline{\nabla f}$

If we define the rest of (1) to be

$$d\underline{l} \triangleq dx\,\underline{a_x} + dy\,\underline{a_y} + dz\,\underline{a_z} \qquad (3)$$

we can write (1) as:

$$df = \underline{\nabla f} \cdot d\underline{l} \qquad (4)$$

Note that :
1. $\underline{\nabla f}$ is a vector result of a scalar function
2. $d\underline{l}$ is a differential unit vector
3. the resulting dot product is a scalar, as we expected.
4. The operation $\underline{\nabla}$ is often called the del operator where
$$\underline{\nabla} = \underline{a_x}\frac{\partial}{\partial x} + \underline{a_y}\frac{\partial}{\partial y} + \underline{a_z}\frac{\partial}{\partial z}$$

Returning to the original concept that the derivative is a function of direction, rewrite (4) as

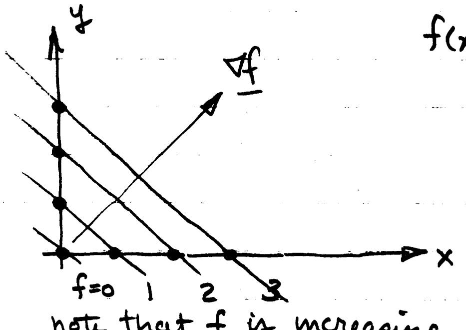
$$df = |\nabla \underline{f}|\,|d\underline{l}|\cos\theta$$

Note that $\nabla \underline{f}$ then gives the MAXIMUM RATE OF CHANGE OF F.

Starting at P_1 if we follow the contour $df = 0$ and hence $\underline{\nabla f}$ must be perpendicular to the contour $d\underline{l}$ lies on.

Example:

For $f(x,y) = f$ (a constant) show $\underline{\nabla f}$ is normal to lines of constant $\underline{f}$.



$f(x,y) = x + y = k$ (a constant)

$x + y = 0$
$x + y = 1$
$x + y = 2$

note that f is increasing most rapidly along the 45° line between x and y

what is the direction of $\underline{\nabla f}$

$$f(x,y,z) = x + y$$

$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial y} = 1 \quad \frac{\partial f}{\partial z} = 0 \Rightarrow \underline{\nabla f} = \underline{a_x} + \underline{a_y}$ and $\underline{\nabla f}$ perpendicular to the lines of constant f as ~~expected~~.

The fact that $df = \underline{\nabla f} \cdot \underline{dl}$ is a function of the contour, i.e. $\underline{dl}(x, y, z)$ leads to the idea of a contour integral, i.e.

$$f = \int_{P_1}^{P_2} df = \int_{P_1}^{P_2} \underline{\nabla f} \cdot \underline{dl} \qquad (5)$$

The most common example of this equation is Work where $\underline{W} = \underline{F} \cdot \underline{s}$ (force x distance). Unfortunately, the form of (5) corresponding to work is

$$W = \int_{P_1}^{P_2} \underline{\nabla f} \cdot \underline{dl}$$

We must identify $\underline{F} = \underline{\nabla f}$ and $\underline{s} = \underline{dl}$. The Second is simple, but what is the first. It turns out that for work f must be the energy but that is not important here. We can write

$$W = \int_{P_1}^{P_2} \underline{F} \cdot \underline{dl}$$

A very important result comes from (5) if we let $P_2 = P_1$. This is a special integral known as a contour integral. and is denoted

$$\int_{P_1}^{P_1} \underline{\nabla f} \cdot \underline{dl} = \oint_C \underline{\nabla f} \cdot \underline{dl} \qquad (6)$$

where the circle indicates a closed contour, i.e. no endpoints.

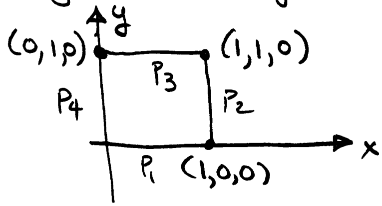
Recall, however, that $\int_{P_1}^{P_2} \underline{\nabla f} \cdot \underline{dl} = \int_{P_1}^{P_1} df = f(P_1) - f(P_2)$.

As long as f is a single-valued function (usually) we have shown that

$$\oint_C \underline{\nabla f} \cdot \underline{dl} = 0 \qquad (7)$$

for any closed contour C.

Example: let $f(x,y,z) = 2xy+3$ and evaluate the contour integral along the rectangular path shown below.



$$\underline{\nabla f} = \frac{\partial f}{\partial x}\underline{a_x} + \frac{\partial f}{\partial y}\underline{a_y} + \frac{\partial f}{\partial z}\underline{a_z} = 2y\,\underline{a_x} + 2x\,\underline{a_y} + 0\cdot\underline{a_z}$$

$$\underline{\nabla f}\cdot\underline{dl} = 2y\,dx + 2x\,dy$$

path P_1

$$\int_{(0,0,0)}^{(1,0,0)} \underline{\nabla f}\cdot\underline{dl} = \int_{x=0}^{x=1} 2y\,dx + \cancelto{0}{\int_{y=0}^{y=0} 2x\,dy} = 2y\,[1-0] = 2y \rightarrow 0 \text{ as } y=0 \text{ here}$$

path P_2

$$\int_{(1,0,0)}^{(1,1,0)} \underline{\nabla f}\cdot\underline{dl} = \cancelto{0}{\int_{x=1}^{x=1} 2y\,dx} + \int_{y=0}^{y=1} 2x\,dy = 0 + 2x\,[1-0] \rightarrow 2 \text{ as } x=1 \text{ here.}$$

path P_3

$$\int_{1,1,0}^{0,1,0} \underline{\nabla f}\cdot\underline{dl} = \int_{x=1}^{x=0} 2y\,dx + \cancelto{0}{\int_{y=1}^{y=1} 2x\,dy} = 2y\,[0-1] + 0 = -2 \text{ as } y=1 \text{ here.}$$

path P_4

$$\int_{0,1,0}^{0,0,0} \underline{\nabla f}\cdot\underline{dl} = \cancelto{0}{\int_{x=0}^{x=0} 2y\,dx} + \int_{y=1}^{y=0} 2x\,dy = 0 + 2x\,[0-1] = 0 \text{ as } x=0 \text{ here}$$

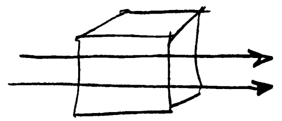
$$\oint \underline{\nabla f}\cdot\underline{dl} = \int_{P_1} + \int_{P_2} + \int_{P_3} + \int_{P_4} = 0 + 2 - 2 + 0 = 0.$$

DIVERGENCE

It is useful to think of the net flux entering or leaving a region of space. This is very useful to fields problems as the sources of flux are charges or currents and knowing fluxes allowes us to locate field sources, or vice versa.

Example:

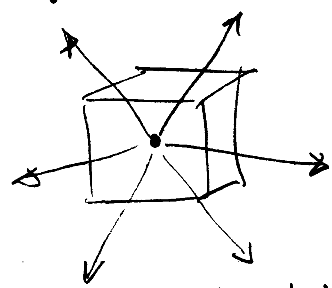
Electric field going through box



net in = net out

sum of flux = 0

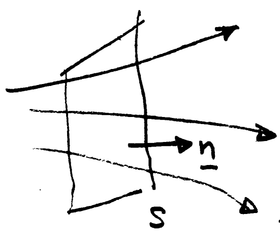
source in box



net out ≠ net in

source (or sink) in box.

How can I define the net flux through a surface S.



flux (vector field $\underline{F}$)

scalar fields dont have flux

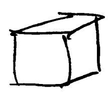
Looking at S we see that the transverse components of $\underline{F}$ on $\underline{S}$ do <u>NOT</u> contribute to the net flux only the normal components do. Transverse components are movements on S, not through $\underline{S}$.

$$\therefore \quad \psi = \int_S \underline{F} \cdot \underline{n} \; ds$$

differential element of surface area

To get to a box we use a closed surface and denote that integral by

$$\psi = \oint_S \underline{F} \cdot \underline{n} \; ds \qquad (8)$$



Now we have not defined the direction of $\underline{n}$ does it point <u>in or out</u> of the box. Convention is to pick $\underline{n}$ pointing out so

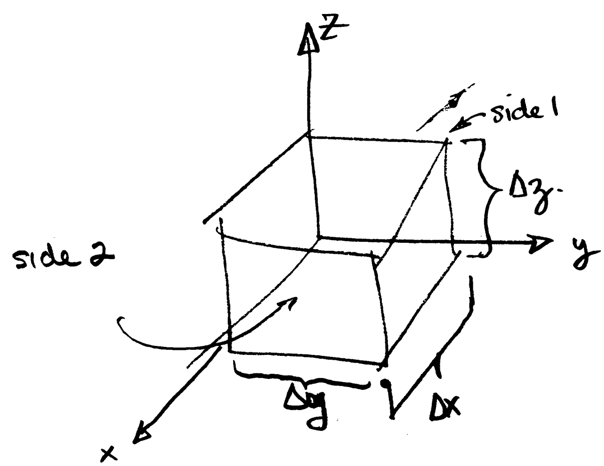
flux out > 0 and flux in < 0.

start by →

Equation (8) is very common sense and tells us a lot about a large volume of space. On the other hand, can we localize just one source in a very tiny box. This is important in making a charge (a bunch of electrons) look mathematically like a bunch of electrons. This very tiny box operation has a special mathematical name, the DIVERGENCE, and is defined as

$$\operatorname{div} \underline{F} = \lim_{\Delta v \to 0} \frac{\oint_s \underline{F} \cdot d\underline{s}}{\Delta v} \qquad (9)$$

This is a nice definition but impractical to use. We can relate $\operatorname{div} \underline{F}$ to the vector field $\underline{F}$ in a easier to use manner by returning to our box, computing $\oint_s \underline{F} \cdot d\underline{s}$ and then taking the limit in (9).



$\underline{F}$ is a vector so write it as $F_x^{(x,y,z)}\underline{a_x} + F_y^{(x,y,z)}\underline{a_y} + F_z^{(x,y,z)}\underline{a_z})$

We picked a special box where $\underline{n}$ can only be $\underline{a_x}$, $\underline{a_y}$ or $\underline{a_z}$ depending upon the side.

suppose the center of our coordinate system is at (x, y, z) and that Δx, Δy, and Δz are small. As $\Delta x, \Delta y, \Delta z \to 0$

F will eventually become constant over that side.

For side #1, the normal component is $F_x \underline{a_x}$

the normal vector $\underline{n}$ is $-\underline{a_x}$

the differential surface ds is $\Delta y \Delta z$

so, the [redacted] flux ψ_1 through side 1 is $F_x \underline{a_x} \cdot -\underline{a_x} \Delta y \Delta z = -F_x \Delta y \Delta z$

For side #2, the normal component is $\left(F_x + \frac{\partial F_x}{\partial x} \Delta x\right) \underline{a_x}$

the normal vector is $+\underline{a_x}$

the differential surface area is $\Delta y \Delta z$

the flux ψ_2 through side 2 is $\left(F_x + \frac{\partial F_x}{\partial x} \Delta x\right) \Delta y \Delta z$.

The <u>net</u> flux in the x direction is

$$\psi_1 + \psi_2 = -F_x \Delta y \Delta z + F_x \Delta y \Delta z + \frac{\partial F_x}{\partial x} \Delta x \Delta y \Delta z$$

$$= \frac{\partial F_x}{\partial x} \Delta x \Delta y \Delta z$$

The same result can be done for the remaining sides

$$\psi_3 + \psi_4 = \frac{\partial F_y}{\partial y} \Delta x \Delta y \Delta z$$

$$\psi_5 + \psi_6 = \frac{\partial F_z}{\partial z} \Delta x \Delta y \Delta z.$$

so that we can re-write Eqn (9) as.

$$\operatorname{div} \underline{F} = \lim_{\Delta V \to 0} \frac{\oint_S \underline{F} \cdot \underline{ds}}{\Delta V} = \lim_{\Delta V \to 0} \frac{\psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6}{\Delta V}.$$

$$\text{div}\,\underline{F} = \lim_{\Delta V \to 0} \frac{\frac{\partial F_x}{\partial x}\Delta x \Delta y \Delta z + \frac{\partial F_y}{\partial y}\Delta x \Delta y \Delta z + \frac{\partial F_z}{\partial z}\Delta x \Delta y \Delta z}{\Delta V}$$

but $\Delta x \Delta y \Delta z = \Delta V$ so

$$\text{div}\,\underline{F} = \lim_{\Delta V \to 0} \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

This can be written as a vector operation by recalling our del operator ∇

$$\underline{\nabla} = \frac{\partial}{\partial x}\underline{a_x} + \frac{\partial}{\partial y}\underline{a_y} + \frac{\partial}{\partial z}\underline{a_z}$$

If $\underline{F} = F_x \underline{a_x} + F_y \underline{a_y} + F_z \underline{a_z}$ as we assumed, then

$$\underline{\nabla} \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

> The general result is $\text{div}\,\underline{F} = \underline{\nabla} \cdot \underline{F}$ and is a scalar representing net flux in a differential volume.

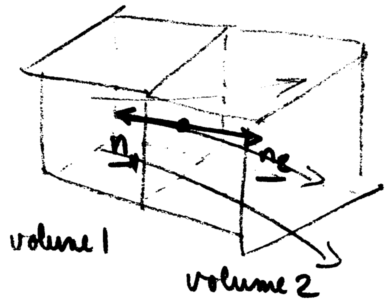
How about the net flux through a macroscopic volume, i.e. a volume $\mathcal{V}$ enclosed by a surface S. If div $\underline{F}$ is the net flux from a differential volume we can guess that the total net flux must be given by

$$\int_V \underline{\nabla} \cdot \underline{F}\, dv$$

but the total net flux is $\oint_S \underline{F} \cdot \underline{ds}$

> The result is the divergence theorem...
>
> $$\oint_S \underline{F} \cdot \underline{dS} = \int_V \underline{\nabla} \cdot \underline{F}\, dv$$

Consider two ~~small~~ differential volumes



what happens in $\oint_{S_1} \underline{F}\cdot d\underline{s} + \oint_{S_2} \underline{F}\cdot d\underline{s}$

the contributions over the common surface cancel out so that the sum equals

$$\oint_{S_1+S_2} \underline{F}\cdot d\underline{S} = \int_{V_1} \underline{\nabla}\cdot\underline{F}\,dv + \int_{V_2} \underline{\nabla}\cdot\underline{F}\,dv = \int_{V_1+V_2} \underline{\nabla}\cdot\underline{F}\,dv$$

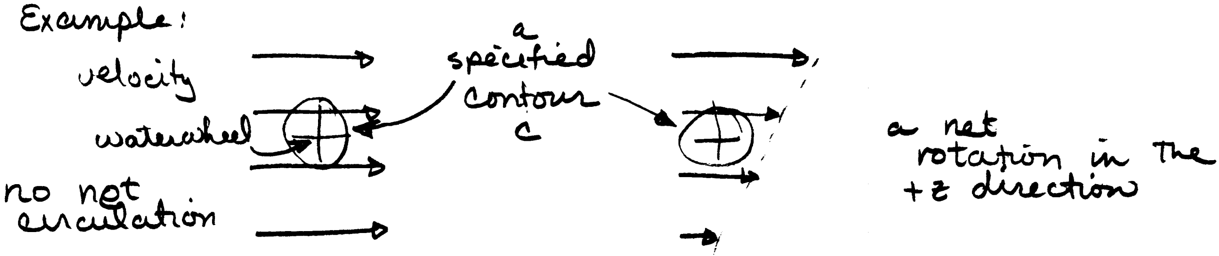
so that we can generalize to arbitrary volumes simply summing

$$\oint_{\Sigma S_i} \underline{F}\cdot d\underline{S} = \int_{\Sigma V_i} \underline{\nabla}\cdot\underline{F}\,dV$$

$$\oint_{S} \underline{F}\cdot d\underline{S} = \int_{V} \underline{\nabla}\cdot\underline{F}\,dV$$

what is circulation? circulation indicates that a ~~vector~~ rotational force is present to give a net non-zero result in the direction specified.

Example:



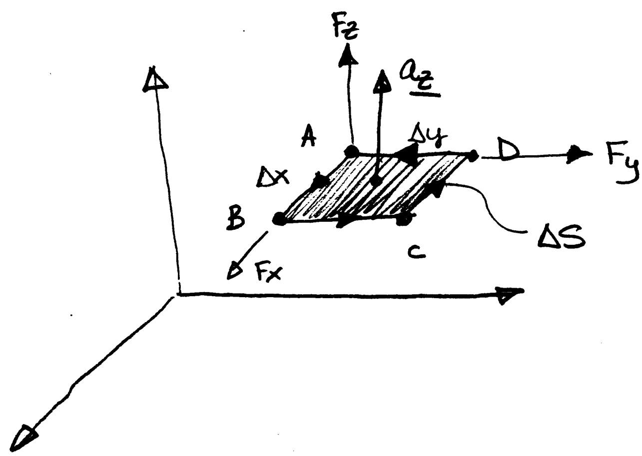
what is the net circulation of a vector field $\underline{F}$ about some contour c? It is the integral of F about the contour c.

the curl c $\triangleq \oint_c \underline{F} \cdot \underline{dl}$ ← only a force around c gives a contribution

How about the curl of $\underline{F}$ at a microscopic point. Let c enclose a surface Δs and let the contour and Δs go to zero.

then
$$\text{curl } \underline{F} \cdot \underline{a_n} = \lim_{\Delta s \to 0} \frac{\oint_c \underline{F} \cdot \underline{dl}}{\Delta s}$$

↑ the direction of c and $\underline{a_n}$ are related by the right-hand rule



we can evaluate this just as we did the divergence.

evaluate $\oint_C \underline{F} \cdot d\underline{l}$

what is F along AB, BC, CD, DA? $\qquad F_x \underline{a_x} \cdot (-\underline{a_x})$

$$\oint_C \underline{F} \cdot d\underline{l} = F_x\Big|_{AB} \Delta x + F_y\Big|_{BC} \Delta y + (-F_x)\Big|_{CD} \Delta x + (-F_y)\Big|_{DA} \Delta y$$

$$= \cancel{F_x} \Delta x + \left[\cancel{F_y} + \frac{\partial F_y}{\partial x} \Delta x\right] \Delta y - \left[\cancel{F_x} + \frac{\partial F_x}{\partial y} \Delta y\right] \Delta x - \cancel{F_y \Delta y}$$

$$= \frac{\partial F_y}{\partial x} \Delta x \Delta y - \frac{\partial F_x}{\partial y} \Delta x \Delta y$$

$$\lim_{\Delta s \to 0} \frac{\oint_C \underline{F} \cdot d\underline{l}}{\Delta s} = \lim_{\Delta s \to 0} \frac{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \Delta x \Delta y}{\Delta x \Delta y} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

this is the z component of curl $\underline{F}$ as given by right-hand rule

the x and y components are.

$$(\text{curl } \underline{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

$$(\text{curl } \underline{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}$$

in three-dimensions $\quad \text{curl } \underline{F} = \underline{a_x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \underline{a_y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \underline{a_z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$

inspecting this result

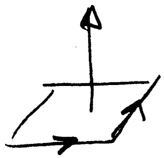
$$\text{curl } \underline{F} = \underline{\nabla} \times \underline{F} \quad \text{where} \quad \underline{\nabla} = \underline{a_x}\frac{\partial}{\partial x} + \underline{a_y}\frac{\partial}{\partial y} + \underline{a_z}\frac{\partial}{\partial z}$$

to show this write

$$\nabla \times \underline{F} = \begin{bmatrix} \underline{a_x} & \underline{a_y} & \underline{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix}$$

Stokes' Theorem

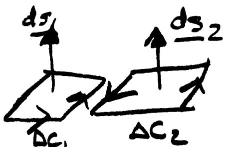
Stokes' Theorem is simply the evaluation of curl $\underline{F}$ over an open surface



For an infinitesimal surface element $d\underline{s}$ with contour Δc we have

$$(\nabla \times \underline{F}) \cdot d\underline{s} = \oint_{\Delta c} \underline{F} \cdot d\underline{l}$$

Suppose we generate a second $d\underline{s}$



note that the contours add $\Delta c_1 + \Delta c_2$ in a scalar manner
the curls (in directions $d\underline{s}_1$ and ds_2) add in a vector manner, i.e. a vector sum.

so

$$\int_{\Delta s_1} (\nabla \times \underline{F}) \cdot ds_1 + \int_{\Delta s_2} (\nabla \times \underline{F}) \cdot d\underline{s}_2 = \oint_{\Delta c_1} \underline{F} \cdot d\underline{l} + \oint_{\Delta c_2} \underline{F} \cdot d\underline{l}$$

as the number of Δs and Δc becomes large,

$$\int_S (\nabla \times \underline{F}) \cdot d\underline{s} = \oint_{C} \underline{F} \cdot d\underline{l}$$

(1.) Electromagnetism is study of interactions among charges and currents. Original concept — action at a distance ⇒ Coulomb's Law

$$\underline{F} \triangleq K \frac{Q_1}{r_{12}^2} Q_2 \, \underline{a}_{r_{12}}$$

= force on Q_2 by Q_1 (directed along radius)

history

— Faraday suggested considering charge q_1 to be surrounded by "lines of force". Lines of force, or force field, indicate everywhere the direction and magnitude of force on unit positive charge if that charge were present.

field in absence of charge

— In Faraday concept, charge produces force field and field acts on other charges. This way effects can be discussed even if second charge is not present.

definition

— define the electric field intensity $\underline{E}(x,y,z)$

$$\underline{E}(x,y,z) = \frac{\underline{F}}{Q}$$

Q = magnitude of test charge.

$\underline{E}$ = force / unit test charge. (assumed positive)

$$\underline{E} = K \frac{Q_1}{r^2} \underline{a}_r \quad \text{for single source charge } q_1$$

— in MKS units $K = \dfrac{1}{4\pi\epsilon_0}$

2. Lorentz force law

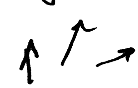
- charge in motion can be acted on by two types of electromagnetic forces.

$$\underline{F} = \underbrace{q\underline{E}}_{\substack{\text{coulomb}\\ \text{force in terms}\\ \text{of } \underline{E} \text{ field}}} + q\,(\underbrace{\underline{v} \times \underline{B}}_{\text{velocity dependent force.}})$$

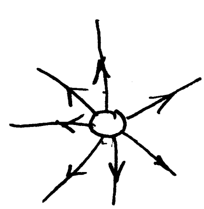
- second term defines magnetic flux density $\underline{B}$ in terms of force on moving charge.

3 Any vector field can be represented graphically in one of two ways

a) array of vectors of appropriate lengths & directions



b) flux plot -- system of lines, called flux lines, drawn by two rules.
   1) direction of flux lines correspond with direction of field vectors
   2) transverse densities are same as magnitude of field vector.



flux lines for point charge.

Net flux thru closed surface is

$$\Psi = \oint_S \underline{E} \cdot d\underline{s}$$

④ Gauss' Law & The electric flux density vector

displacement, or electric flux density

$$\oint_S \underline{D} \cdot d\underline{s} = Q_{\cancel{total}} \text{ enclosed by } S$$

closed surface.

proof: $\underline{E} \cdot d\underline{s} = \dfrac{Q}{4\pi\epsilon_0} \, \underline{a}_r \cdot d\underline{s}$

$ds\,\underline{a}_n$ normal to surface

from solid geometry $d\Omega = \dfrac{ds \cos\theta}{R^2}$

differential solid angle.

$$\oint_S \underline{E} \cdot d\underline{s} = \oint_S \underbrace{\frac{Q}{4\pi\epsilon_0 R^2}}_{E \text{ field.}} \underbrace{\frac{d\Omega\, R^2}{\cos\theta}}_{ds} \cdot \cos\theta$$

dot product

$$= \oint_S \frac{Q}{4\pi\epsilon_0} d\Omega \Rightarrow \frac{Q}{4\pi\epsilon_0} \cdot 4\pi$$

but, in solid geometry, $\oint_S d\Omega = 4\pi$

$\therefore \oint \underline{E} \cdot d\underline{s} = \dfrac{Q}{\epsilon_0}$

and re-writing $\oint_S (\epsilon_0 \underline{E}) \, d\underline{S} = Q$

dimensions = $\dfrac{\text{coulombs}}{m^2}$

Furthermore, recall if $\Psi = \int_S \underline{F} \cdot d\underline{s} > 0$ source

$\Psi = \int_S \underline{F} \cdot d\underline{s} < 0$ sink

where Ψ was a ~~flux~~ so F must be flux density

Electrostatic potential

electrostatic potential

how much work is done moving a charge in a static electric field?

- force on a test charge ΔQ is given by Lorentz force law

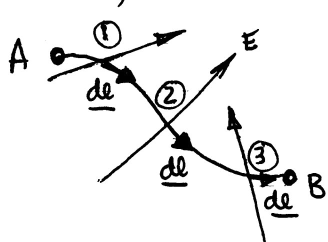
$$\underline{F} = \frac{Q(\Delta Q)}{4\pi\epsilon_0 R^2}\,\underline{a_R} \qquad (1)$$

- what is the incremental work dW done in moving ΔQ a distance $\underline{dl}$

$$dW = -\underline{F}\cdot\underline{dl} \qquad (2)$$

why a minus sign? The sign is negative so that if a positive test charge is moved against the field, the work done will be positive.

move ΔQ along AB



at ① $\underline{F}\cdot\underline{dl} > 0$

② $\underline{F}\cdot\underline{dl} = 0$

③ $\underline{F}\cdot\underline{dl} < 0$

At ① the field is moving the charge, so the field is losing energy. At ② no work is being done. At ③ the charge is being moved against the field so real work is being done. Thus, we want case ③ to end up being a positive quantity of work.

Combining (1) and (2),

$$dW = -\frac{Q\,(\Delta Q)}{4\pi\epsilon_0 R^2}\,\underline{a_R} \cdot \underline{dl}$$

Note: this is work done in field of point charge.

- incremental work per charge moved is then

$$\frac{dW}{\Delta Q} = -\frac{Q}{4\pi\epsilon_0 R^2}\,\underline{a_R} \cdot \underline{dl}$$

But the first quantity on the right is $\underline{E}$, so.

$$\frac{dW}{\Delta Q} = -\underline{E} \cdot \underline{dl}$$

- define the voltage V [or electrostatic potential Φ] to be the incremental work / charge moved.

$$dV = \frac{dW}{\Delta Q} = -\underline{E} \cdot \underline{dl} \qquad (3)$$

- The work done in moving ΔQ from P_1 to P_2 is given by integrating (3).

$$\int_{P_1}^{P_2} dU = \int_{P_1}^{P_2} -\underline{E} \cdot \underline{dl}$$

$$V(P_2) - V(P_1) = -\int_{P_1}^{P_2} \underline{E} \cdot \underline{dl} \qquad (4)$$

- In passing, note that if $P_2 \to P_1$ and $V(P)$ is single valued (a requirement of a real field), then

$$-\int_{P_1}^{P_1} \underline{E} \cdot \underline{dl} = -\oint \underline{E} \cdot \underline{dl} = V(P_1) - V(P_1) = 0$$

this is the CONSERVATIVE PROPERTY OF AN ELECTRIC FIELD

$$\oint \underline{E} \cdot \underline{dl} = 0 \qquad (5)$$

Applying Stoke's Theorem to ~~this~~ (5)

$$\oint_C \underline{E} \cdot \underline{dl} = \int_S (\underline{\nabla} \times \underline{E}) \cdot \underline{ds}$$

But, as $\oint \underline{E} \cdot \underline{dl} = 0$, this implies that

$$\underline{\nabla} \times \underline{E} = 0 \tag{6}$$

<u>This is actually one of Maxwell's equations.</u>

- Returning to (3), let us rewrite dV.

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \tag{7}$$

From (3), we can also write

$$\begin{aligned} dV &= -\underline{E} \cdot \underline{dl} \\ &= -(E_x \underline{a_x} + E_y \underline{a_y} + E_z \underline{a_z}) \cdot (dx\, \underline{a_x} + dy\, \underline{a_y} + dz\, \underline{a_z}) \\ &= -(E_x dx + E_y dy + E_z dz) \end{aligned} \tag{8}$$

Equating (7) and (8) we can make the identifications

$$\frac{\partial V}{\partial x} = -E_x \qquad \frac{\partial V}{\partial y} = -E_y \qquad \frac{\partial V}{\partial z} = -E_z$$

In vector form, this can be written

$$\begin{aligned} \underline{E} &= -\underline{\nabla} V \\ &= -\underline{\nabla} \Phi \end{aligned} \tag{9}$$

- Is Φ dependent upon the path?

No, because of the conservative property.

- Φ is only relative.

From the conservative property we can only calculate a potential difference, so we must define some point as zero potential. We often pick $\underline{r} = \infty$ as the reference point and let $\Phi = 0$ there.

- Is Φ unique?

Consider. Let $\Phi' = \Phi +$ constant

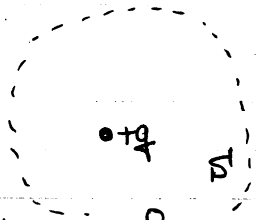
$$\underline{E} = -\nabla\Phi' = -\nabla\left(\Phi + \text{constant}\right) = -\nabla\Phi$$

Thus, both Φ and Φ' give the same $\underline{E}$. Φ is NOT unique; it is only defined to an additive constant

- methodology to solving electrostatic field problems.

I. Use Gauss' Law to find E
II. Choose an appropriate coordinate system
III. pick an appropriate surface.
IV. Integrate to get $\underline{E}$

Example: Find $\underline{E}$, Φ of point charge.



I. Gauss' law $\quad \oint \underline{E} \cdot d\underline{s} = \dfrac{Q_{total}}{\epsilon_0} = \dfrac{q}{\epsilon_0}$

II. choose a spherical coordinate system because of spherical symmetry, i.e.

$$\underline{E} = E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi}$$

$$d\underline{s} = r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$$

$$\underline{E} \cdot d\underline{s} = E_r r^2 \sin\theta \, d\theta \, d\phi$$

III. Pick appropriate surface.
Actually we picked the surface already by selecting $d\underline{s}$.

IV. Integrate.

$$\oint \underline{E} \cdot d\underline{s} = \underbrace{\int_0^{\pi} \underbrace{\int_0^{2\pi} E_r r^2 \sin\theta \, d\theta \, d\phi}_{\phi \text{ integration}}}_{\theta \text{ integration}}$$

$$= E_r r^2 \, 2\pi \int_0^{\pi} \sin\theta \, d\theta = E_r r^2 \, 2\pi \left(-\cos\theta \Big|_0^{\pi} \right)$$

$$= 2\pi r^2 E_r \left(-\overset{-1}{\cos\pi} + \overset{1}{\cos 0} \right)$$

$$= 4\pi r^2 E_r$$

$$\therefore \quad 4\pi r^2 E_r = \frac{q}{\epsilon_0} \; ; \quad E_r = \frac{q}{4\pi r^2 \epsilon_0}$$

$$\underline{E} = \frac{q}{4\pi\epsilon_0 r^2} \underline{a_r}$$ which is consistent with our earlier results and definitions.

Now that we have $\underline{E}$, how about Φ?

$$\underline{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\,\underline{a_r} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\,\underline{a_\theta} - \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\,\underline{a_\phi}$$

only the first derivative for $\underline{a_r}$ can be non-zero.
Equating terms in $\underline{a_r}$

$$\frac{q}{4\pi\epsilon_0 r^2} = -\frac{\partial\Phi}{\partial r} = -\frac{d\Phi}{dr} \quad \text{since no other dependence.}$$

Integrating

$$\int_{P_a}^{P_b} \frac{q}{4\pi\epsilon_0}\,\frac{dr}{r^2} = -\int_{P_a}^{P_b} d\Phi$$

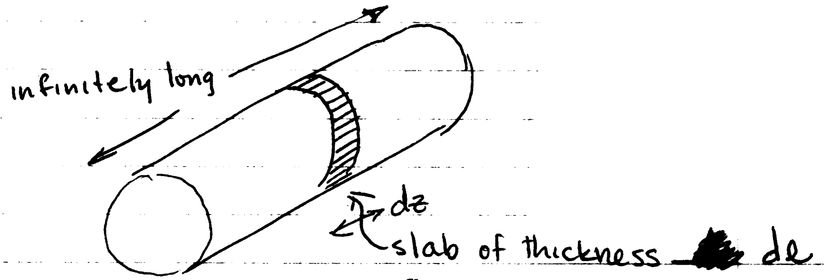
$$\frac{q}{4\pi\epsilon_0}\,\frac{r^{-1}}{-1}\bigg|_{r=a}^{r=b} = -\Phi(r=b) + \Phi(r=a)$$

$$\therefore \quad \Phi(r=a) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) + \Phi(r=b)$$

Note the additive constant here. So, define $\Phi(r=\infty) = 0$
and pick $b = \infty$

Then $$\Phi(r=a) = \frac{q}{4\pi\epsilon_0 a} \qquad ∎$$

Example: cyclinder, charged on surface.



finite charge on each slab.

① since the cyclinder is 'infinitely long its total charge is infinite; however, the charge on a short length of cyclinder is finite

② as the cyclinder is uniform, let the charge in any length of the cyclinder be uniformly distributed. Let this charge density be ρ_s coulombs/m^2

[what is a coulomb? unit of charge
$q = 1.6 \times 10^{-19}$ C on an electron
6.25×10^{18} electrons / coul.

③ the total charge on any slab will be $\Lambda = \rho_s 2\pi r_c \Delta l$

where r_c = the radius, $2\pi r_c$ The circumference, and Δl The thickness

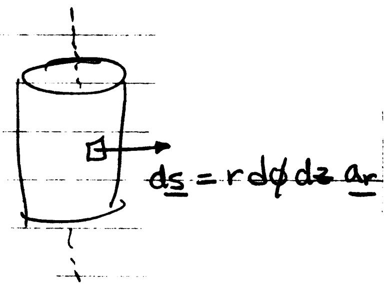
I. To find $\underline{E}$ use Gauss' Law (given ρ, find $\underline{E}$)

II. Choose the appropriate geometry because of symmetry. [pick a cylindrical volume, this means the source of charge and the surface integral can be described by a common set of coordinates making the integral easier.]

III. Write Gauss' Law in the cyclindrical coordinate system and calculate Q_{total}, $\underline{E}$

cylinder (cont.)

1. $$\oint_S \underline{E} \cdot d\underline{s} = \frac{Q_{total}}{\epsilon_0}$$



2. Pick cylindrical coordinates

$$d\underline{s} = r\,d\phi\,dz\,\underline{a_r}$$

$$\underline{E} = E_r\,\underline{a_r} + E_\phi\,\underline{a_r} + E_z\,\underline{a_z}$$

3. What is Q total?

Pick a length of the cylinder dz.

$$Q_t = \rho_s \cdot 2\pi r_c \cdot dz$$

How about the surface integral.

$$\oint_S \underline{E} \cdot d\underline{s} = \iint (E_r\underline{a_r} + E_\phi\underline{a_\phi} + E_z\underline{a_z}) \cdot r\,d\phi\,dz\,\underline{a_r}$$

0 – 2π

$$= \int_0^{\Delta l} \int_0^{2\pi} E_r\, r\, d\phi\, dz$$

0 – Δl.

THINK: If the problem is radially symmetric it makes sense that all E_r's are the same so assume E_r = constant.

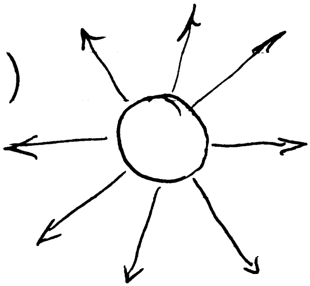
$$\oint_S \underline{E} \cdot d\underline{s} = E_r r \int_0^{\Delta l} \int_0^{2\pi} d\phi\, dz = 2\pi\,\Delta l\, r\, E_r$$

This must equal the total enclosed charge divided by ϵ_0

$$E_r\, r\, \cancel{2\pi} \cancel{\Delta l} = \frac{\rho_s\, \cancel{2\pi}\, r_c\, \cancel{\Delta l}}{\epsilon_0}$$

$$\therefore\ \underline{E} = \frac{\rho_s}{\epsilon_0}\left(\frac{r_c}{r}\right)\underline{a_r}$$

cylinder (cont.)



flux density goes down as $\frac{1}{2\pi r}$

How about Φ?

$$\underline{E} = -\nabla\Phi = -\underline{a}_r\frac{\partial\Phi}{\partial r} - \underline{a}_\phi\frac{1}{r}\frac{\partial\Phi}{\partial\phi} - \underline{a}_z\frac{\partial\Phi}{\partial z}$$

$$\frac{\rho_s}{\epsilon_0}\left(\frac{r_c}{r}\right)\cancel{\underline{a}_r} = -\cancel{\underline{a}_r}\frac{\partial\Phi}{\partial r} = -\frac{d\Phi}{dr}\cancel{\underline{a}_r}$$

$$\int_a^b \frac{\rho_s}{\epsilon_0} r_c \frac{dr}{r} = -\int_a^b d\Phi$$

what limits, a & b? obviously $a > r_c$

$$\frac{\rho_s}{\epsilon_0} r_c \ln b - \ln a = -\Phi(b) + \Phi(a)$$

$$\Phi(a) = \Phi(b) + \frac{\rho_s}{\epsilon_0} r_c \ln\left(\frac{b}{a}\right)$$

Example: a charged sphere
(same as last problem except for geometry)

given: surface charge density ρ_s
total surface charge is $(4\pi r_s^2)\rho_s$.

I. $\oint_S \underline{E}\cdot \underline{ds} = \frac{Q_{total}}{\epsilon_0}$

II. use spherical coordinates

$\underline{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$ (Figure 2.10)

$\underline{E} = E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi}$

III. use a sphere of radius r

$$\oint \underline{E}\cdot\underline{ds} = \iint (E_r \underline{a_r} + E_\theta \underline{a_\theta} + E_\phi \underline{a_\phi}) \cdot r^2 \sin\theta \, d\theta \, d\phi \, \underline{a_r}$$

$$= \iint E_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= E_r \cdot \underbrace{\int_0^\pi \int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi}_{\text{surface of sphere}}$$

$$= E_r r^2 2\pi \int_{-\pi}^{\pi} \sin\theta \, d\theta = 2\pi E_r r^2 \left[-\cos\theta \Big|_0^\pi \right]$$

$$= 2\pi E_r r^2 \left[-\cos(\pi) + \cos(0) \right] = 2\pi E_r r^2 [1+1].$$

$$= 4\pi r^2 E_r$$

$$\cancel{4\pi} r^2 E_r = \frac{\cancel{4\pi} r_s^2 \rho_s}{\epsilon_0}$$

$$\boxed{\underline{E} = \frac{\rho_s}{\epsilon_0}\left(\frac{r_s}{r}\right)^2 \underline{a_r}}$$

as before $\underline{E} = -\nabla\phi = -\underline{a_r}\frac{d\Phi}{dr}$ since only E_r component.

$$\frac{\rho_s r_s}{\epsilon_0} r^{-2} = -\frac{d\Phi}{dr}$$

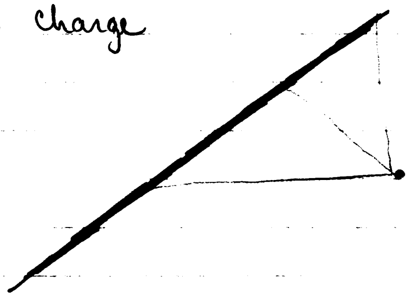
$$\leftarrow \frac{\rho_s r_s}{\epsilon_0} \frac{r^{-1}}{-1}\Bigg|_{r_a}^{r_b} = -\Phi(b) + \Phi(a)$$

$$\frac{\rho_s r_s}{\epsilon_0}\left(\frac{-1}{r_b} + \frac{1}{r_a}\right) = -\Phi(b) + \Phi(a)$$

pick $r_b = \infty$ then $\frac{\rho_s r_s}{\epsilon_0} \frac{1}{r_a} + \underset{\substack{\Downarrow \\ \emptyset}}{\Phi(\infty)} = \Phi(r_a)$

⑥

line charge



do in cyclindrical coordinates

result for charged cyclinder $\underline{E}_r = \frac{\rho_s}{\epsilon_0}\left(\frac{r_c}{r}\right) \underline{a_r}$

as $r_c \to 0$ $\underline{E} \to 0$ because we used a surface charge density

use a linear charge density ρ_l

I. $\oint \underline{E} \cdot \underline{ds} = \frac{Q_{total}}{\epsilon_0}$

II. use cyclindrical coordinates

III. $\oint (E_r \underline{a_r} + E_\phi \underline{a_\phi} + E_z \underline{a_z}) \cdot r\, d\phi\, dz\, \underline{a_r}$

$$= \int_0^{\Delta l}\int_0^{2\pi} E_r\, r\, d\phi\, dz = E_r\, r\, 2\pi \Delta l$$

total charge: $Q_{total} = \rho_l \Delta l.$

$$E_r r 2\pi \cancel{\Delta l} = \frac{\rho_l \cancel{\Delta l}}{\epsilon_0}$$

$$\underline{E} = \frac{\rho_l}{2\pi\epsilon_0} \frac{1}{r}$$

as before $\underline{E} = -\nabla \Phi = -\frac{d\Phi}{dr} \underline{a_r}$

$$\frac{\rho_l}{2\pi\epsilon_0} \frac{1}{r} = -\frac{d\Phi}{dr}$$

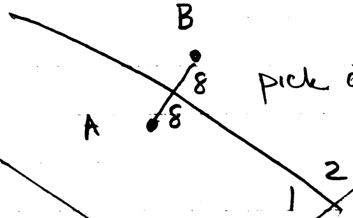
$$\frac{\rho_l}{2\pi\epsilon_0} \int_{r_a}^{r_b} \frac{dr}{r} = -\int_{r_a}^{r_b} d\Phi$$

$$\frac{\rho_l}{2\pi\epsilon_0} \ln(r_b) - \ln(r_a) = -\Phi(r_b) + \Phi(r_a)$$

$$\Phi(r_a) = \frac{\rho_l}{2\pi\epsilon_0} \ln\left(\frac{r_b}{r_a}\right) + \Phi(r_b)$$

in many problems it is impossible to pick a simple $\Phi(b)$ as shown in the cases of the cylinder and line. This is due to the charge being of infinite extent.

Boundary condition for potential



pick $\underline{dl}$ ⊥ to boundary.

potential difference between A and B is

$$\Phi(B) - \Phi(A) = -\int_A^B \underline{E}\cdot\underline{dl}$$

$$= -\left(E_{2n}\delta - E_{1n}\delta\right)$$

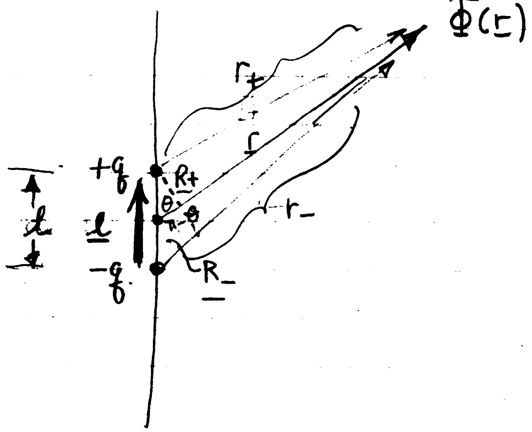
neglect components ∥ to boundary as $\underline{E}_{\parallel}\cdot\underline{\delta} = 0$.

now, let $\delta \to 0$

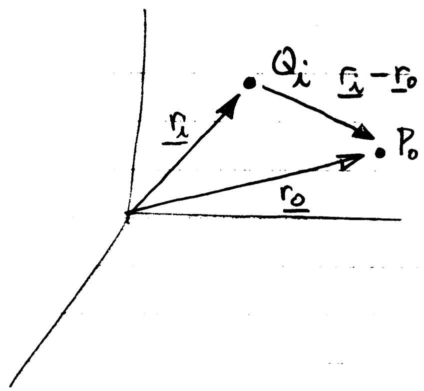
$$\Phi(B) - \Phi(A) = \lim_{\delta\to 0} -\left[E_{2n} - E_{1n}\right]\delta$$

Φ, E for electric dipole

This is a very important and special example.



we can find this problem by considering the electric potential for a charge distribution of discrete charges Q_i



Φ_i is associated with each Q_i

each $\Phi_i(\underline{r_0}) = \dfrac{Q_i}{4\pi\epsilon_0 |\underline{r_i} - \underline{r_0}|}$

at r_0

since Φ_i is a scalar, and not influenced by other charges. the Φ_i's must add.

$$\Phi(P_0) = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_0 |\underline{r_i} - \underline{r_0}|} \longrightarrow \int_V \frac{\rho(r')\, dV'}{4\pi\epsilon_0 |r' - r_0|}$$

for $\rho(r')$ a volume charge density

this gives us a powerful technique for solving problems such as the electric dipole in terms of previously solved solutions.

consider $\Phi(r) = \dfrac{+q}{4\pi\epsilon_0 |\underline{R_+} - \underline{r}|} + \dfrac{-q}{4\pi\epsilon_0 |\underline{R_-} - \underline{r}|}$

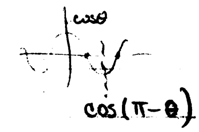
this assumes $\Phi(\infty) = 0$.

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

make this simpler: by the law of cosines.

$$|\underline{r_+}|^2 = |\underline{r}|^2 + |\underline{R_+}|^2 + 2|\underline{r}||\underline{R_+}|\cos\theta$$

$$|\underline{r_-}|^2 = |\underline{r}|^2 + |\underline{R_-}|^2 + 2|\underline{r}||\underline{R_-}|\cos(\pi-\theta) \xrightarrow{} -\cos\theta.$$



$$r_+^2 = r^2 + \frac{\ell^2}{4} + 2r\frac{\ell}{2}\cos\theta$$

$$r_-^2 = r^2 + \frac{\ell^2}{4} - 2r\frac{\ell}{2}\cos\theta$$

$$r_+ = \left(r^2 + \frac{\ell^2}{4} + r\ell\cos\theta\right)^{\frac{1}{2}} \qquad r_- = \left(r^2 + \frac{\ell^2}{4} - r\ell\cos\theta\right)^{\frac{1}{2}}$$

$$r_+ = r\left(1 + \frac{\ell^2}{4r^2} + \frac{\ell}{r}\cos\theta\right)^{\frac{1}{2}} \qquad r_- = r\left(1 + \frac{\ell^2}{4r^2} - \frac{\ell}{r}\cos\theta\right)^{\frac{1}{2}}$$

expand this: $(1+x)^n \approx 1 + nx$

$$r_+ \approx r\left(1 - \frac{\ell^2}{8r^2} - \frac{\ell}{2r}\cos\theta\right) \qquad r_- \approx r\left(1 - \frac{\ell^2}{8r^2} + \frac{\ell}{2r}\cos\theta\right)$$

it is reasonable that $\frac{\ell}{r} \ll 1$ so neglect second order terms.

$$r_+ \approx r\left(1 - \frac{\ell}{2r}\cos\theta\right) \qquad r_- \approx r\left(1 + \frac{\ell}{2r}\cos\theta\right)$$

$$\approx r - \frac{\ell}{2}\cos\theta \qquad \approx r + \frac{\ell}{2}\cos\theta$$

substituting

$$\Phi(\underline{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \approx \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r - \frac{\ell}{2}\cos\theta} - \frac{1}{r + \frac{\ell}{2}\cos\theta}\right)$$

$$\Phi(\underline{r}) \approx \frac{q}{4\pi\epsilon_0}\left(\frac{\cancel{r}+\frac{l}{2}\cos\theta - \cancel{r}+\frac{l}{2}\cos\theta}{(r-\frac{l}{2}\cos\theta)(r+\frac{l}{2}\cos\theta)}\right) \quad r^2 \text{ since } \frac{l}{2} \ll r$$

$$\approx \frac{q\, l \cos\theta}{4\pi\epsilon_0 r^2}$$

this is the electric potential due to the electric dipole

define the electric dipole moment $\underline{p} = +q\underline{l} = +q l\, \underline{a_z}$

(vector points to + charge)

if $\Phi(\underline{r}) = \dfrac{1}{4\pi\epsilon_0 r^2}\, \underline{a_r} \cdot \underline{p}$ for $|\underline{r}| \gg l$

$\underline{E} = -\nabla\Phi$

use spherical coordinates

$$\nabla\Phi = \underline{a_r}\frac{\partial\Phi}{\partial r} + \underline{a_\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \cancel{\underline{a_\phi}\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}}$$

axisymmetric

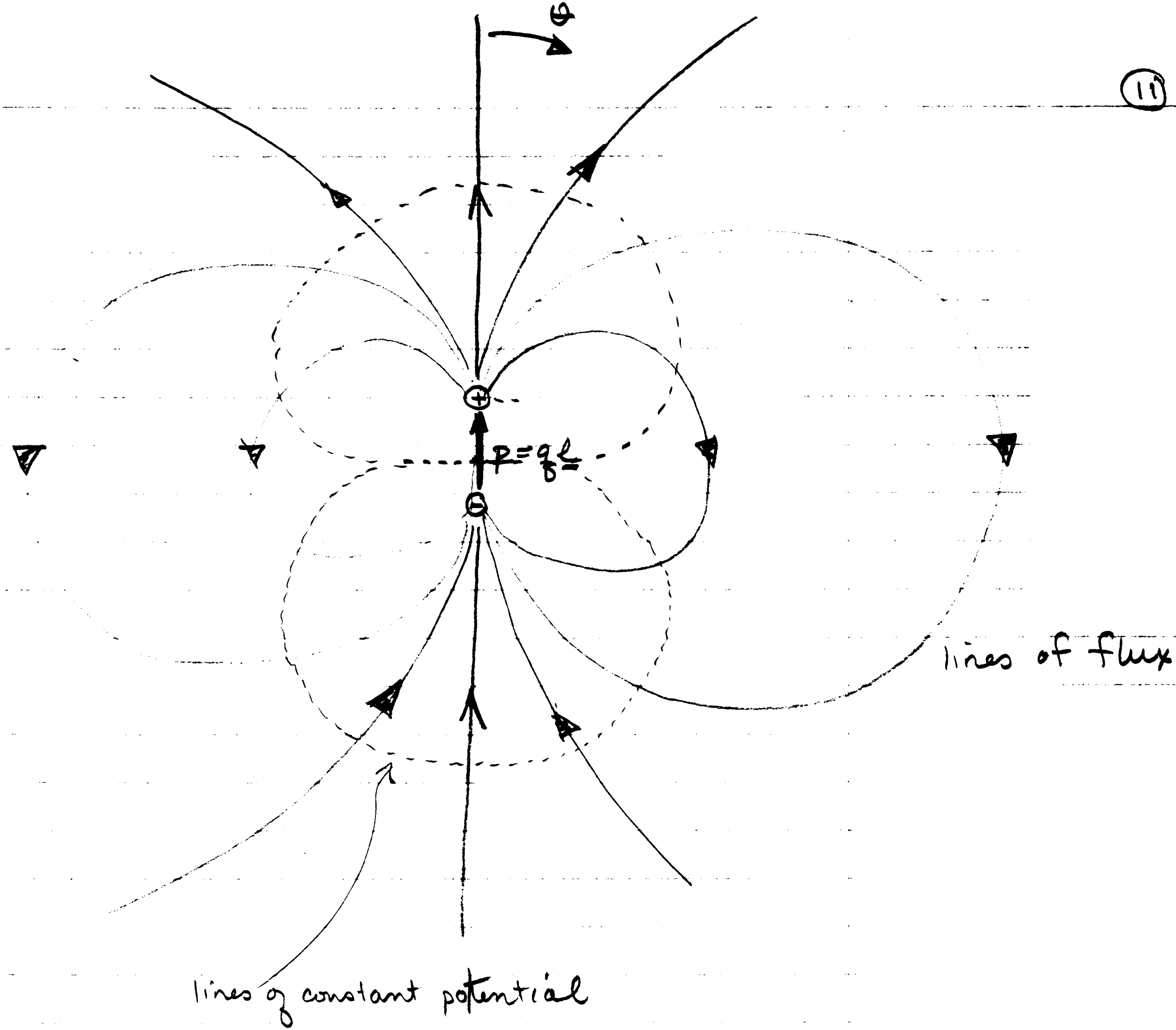
but a θ dependence

$$\frac{\partial\Phi}{\partial r} = -\frac{p\cos\theta}{4\pi\epsilon_0}\frac{2}{r^3}$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$\therefore \underline{E}(\underline{r}) = \underline{E}(r,\theta,\phi) = +\frac{p\cos\theta}{4\pi\epsilon_0}\frac{2}{r^3}\underline{a_r} + \frac{p\sin\theta}{4\pi\epsilon_0 r^3}\underline{a_\theta}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\, \underline{a_r} + \sin\theta\, \underline{a_\theta}\right)$$


θ
+
p = ql
−
lines of flux
lines of constant potential

topics in electrostatics

(1)
electric field intensity
Lorentz force law
flux plots
Gauss' law

electrostatic potential
conservative property – ~~divergence theorem~~
work in electrostatic fields
Example – E & Φ for point charge
Example – ∞ long cylinder
Example – charged sphere
Example – line charge
Example – electric dipole

conductors
Ohm's law
Example – fields in a wire

dielectrics
polarizability
electric flux density vector

boundary conditions
charge relaxation
conductors (revisited) – boundary conditions
energy density & electric field.

Laplace & Poisson's Eqns.
parallel plate capacitor
capacitance

Mike Nacc