ENGR 336 F I E L D S I I 15 September 1971

Problem Set # 1

Due: 27 Sept. 1971

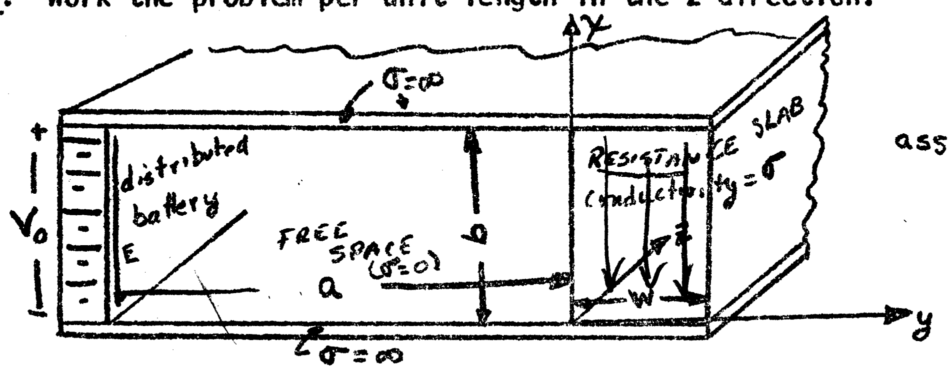
1. (a) ~~Using complex Poynting's theorem find the complex admittance of a system. Use an equivalent parallel R L C circuit as a model.~~
   (b) ~~Find an equation for the admittance near the resonant frequency and determine the Q .~~
   (c) ~~Determine the change in resonant frequency for a small change in the magnetic permeability. Hint: Consider the equivalent inductance to be proportional to the permeability.~~

2. A resistor is fed by a battery as sketched in the accompanying figure.
   (a) Determine the electric and magnetic fields in the resistor (surrounding region has zero conductivity).
   (b) Show the solution for the potential within the space between the battery and the resistor is a solution of Laplace's equation.
   (c) Find and sketch the solution for potential and electric field within the space.
   (d) Find and sketch the magnetic field within the space.
   (e) Sketch Poynting's vector.
   (f) Show that Poynting's theorem is satisfied by your results by investigating it in integral form on the resistor
   (g) Convince yourself that power flows out of the battery <u>into free space</u> to the resistor.

Fano Chu Adler p. 292

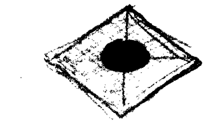
$R = \rho V$

<u>NOTE</u>: Work the problem per unit length in the z-direction.



assume $\frac{\partial}{\partial z} = 0$

3. Find Poynting's vector, electric and magnetic energy density for a uniform plane wave.

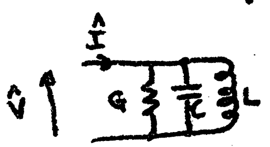


E-336 Problem Set 1 solutions

1 (a) $\frac{1}{2}\hat{V}\hat{I}^* = \langle P_d\rangle + 2j\omega(\langle U_m\rangle - \langle U_e\rangle)$

$$\hat{Y} = \frac{\hat{I}}{\hat{V}} = \frac{1}{2}(\hat{V}\hat{I}^*)^* / \frac{|\hat{V}|^2}{2} = \frac{2}{|V|^2}\left[\langle P_d\rangle + 2j\omega(\langle U_e\rangle - \langle U_m\rangle)\right] = G + jB$$

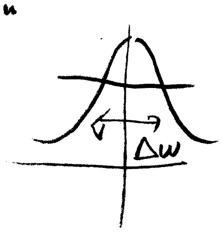
note change of sign



$\langle P_d\rangle = \frac{1}{2}G|\hat{V}|^2$

$\langle U_m\rangle = \frac{1}{4}L|I_L|^2 = \frac{1}{4}L\left|\frac{\hat{V}}{j\omega L}\right|^2 = \frac{1}{4}\frac{|\hat{V}|^2}{\omega^2 L}$

$\langle U_e\rangle = \frac{1}{4}C|\hat{V}|^2$



so $G = 2\langle P_d\rangle/|\hat{V}|^2$

(a) $B = 4\omega/|V|^2\left[\langle U_e\rangle - \langle U_m\rangle\right]$; $\frac{\partial B}{\partial\omega} = \frac{4}{|V|^2}\left[\langle U_e\rangle + \frac{1}{4}\frac{|V|^2}{\omega^2 L}\right] = \frac{4}{|V|^2}\left[\langle U_e\rangle + \langle U_m\rangle\right]$

(b) $Y_{\omega\sim\omega_0} = 2\langle P_d\rangle/|V|^2 + \frac{4}{|V|^2}j\left[\langle U_e\rangle + \langle U_m\rangle\right]_{\omega=\omega_0}(\omega-\omega_0)$

$Q = \omega_0/\Delta\omega$ where

$\left(\frac{\Delta\omega}{2}\right)\times\left[\langle U_e\rangle + \langle U_m\rangle\right]_{\omega=\omega_0} = \frac{1}{2}\langle P_d\rangle$

so

$$Q = \frac{\omega_0\left[\langle U_m\rangle + \langle U_e\rangle\right]_{\omega=\omega_0}}{\langle P_d\rangle} = \frac{2\omega_0\langle U_m\rangle_{\omega=\omega_0}}{\langle P_d\rangle}$$

since $\langle U_m\rangle = \langle U_e\rangle$ at $\omega=\omega_0$

(c) $\Delta Y = \left.\frac{\partial Y}{\partial\omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + \underbrace{\left.\frac{\partial Y}{\partial\mu}\right|_{\mu=\mu_0}\Delta\mu}_{\text{change in } Y \text{ due to change in permeability } \mu}$ but $\frac{\partial Y}{\partial\mu} = \frac{\partial Y}{\partial U_m}\frac{\partial U_m}{\partial\mu}$

$$\frac{\partial}{\partial\mu}\left[\frac{1}{4}\frac{K}{\mu}\right] - \frac{1}{4}\frac{K}{\mu}\frac{1}{\mu}$$

So $\Delta Y = \left.\frac{\partial Y}{\partial\omega}\right|_{\omega=\omega_0}(\omega-\omega_0) + \left(\frac{\partial Y}{\partial U_m}\right)_{\omega=\omega_0}\frac{\partial U_m}{\partial\mu}\Delta\mu$

but $\langle U_m\rangle = \frac{1}{4}\frac{|V|^2}{\omega^2 L} \sim \frac{1}{L} \sim \frac{1}{\mu}$, so $\frac{\partial U_m}{\partial\mu} = -U_m\,\mu$?

$-\frac{U_m}{\mu}$

and $\partial Y/\partial U_m = -\frac{4j\omega_0}{|V|^2}$

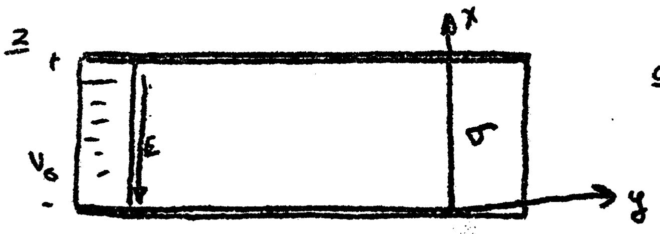
so finally $(\omega-\omega_0)\left.\frac{\partial Y}{\partial\omega}\right|_{\omega=\omega_0} + \frac{4j\omega_0\langle U_m\rangle}{|V|^2}\frac{\Delta\mu}{\mu} = 0$

$$\therefore\ \boxed{\omega-\omega_0 = \Delta\omega} = -\left(\frac{\Delta\mu}{\mu}\right)\frac{\langle U_m\rangle_{\omega=\omega_0}}{(\langle U_m\rangle + \langle U_e\rangle)_{\omega=\omega_0}} = -\frac{\Delta\mu}{2\mu}$$

?

(an increase in inductance decreases the resonant freq.)

what would the answer be if the permeability were changed in only part of the volume ?



a. If the top and bottom plates have infinite conductivity then there can be no tangential E on their surfaces.

$\bar J = \sigma \bar E$, so $\nabla\cdot J = \sigma \nabla\cdot E = -\frac{\partial \rho}{\partial t} = 0$ for $\sigma \neq \sigma(x,y)$

So $\nabla\times E = 0$ and $\nabla\cdot E = 0$ in resistor

therefore we can set $\nabla^2\Phi = 0$ in resistor. Note that at the surfaces $y = w$ and $y = 0$ there can be no normal $\bar E$! because there can be no normal $\bar J$ ($=\sigma\bar E$). If there were, that would imply current flow into regions of zero conductivity (The boundary condition on J is $n\cdot(J_1 - J_2) = 0$ which is the continuity of current across the surface.) The solution (only) possible is in resistor $\bar J = -\underline{a}_x \sigma V_0/b$.

for $\boxed{\bar E = -\underline{a}_x V_0/b}$ here.) Now we use $\nabla\times \underline{H} = \bar J = -a_x \sigma V_0/b$

(but $\frac{\partial}{\partial z} = 0$) $\frac{\partial H_z}{\partial y} - \cancel{\frac{\partial H_y}{\partial z}}^{\,0} = -\sigma V_0/b$, so $H_z = -\sigma V_0/b\, y +$ Constant.

we have assumed that H_z cannot vary with x and this can be found from the y-component of $\nabla\times H$.

Now since $H_z = 0$ for $y = w$ (no fields outside resistor) we find $\boxed{H_z = -\sigma V_0/b\,(y-w) = -\frac{\sigma V_0}{b}(y-w)}$

b. since $\nabla\cdot\epsilon_0 E = \rho = 0$ and $\nabla\times E = -\frac{\partial B}{\partial t} = 0$ we have
$E = -\nabla\phi$ and $\nabla^2\phi = 0$

c. $\nabla^2\phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ and the B.C's are $\phi(0,y) = 0$
perfect conductors are equipotentials $\phi(b,y) = V_0$
as well as $\phi(x,-a) = \boxed{V_0 x/b} = \phi(x,0)$
how?

let $\Phi = X(x)\,Y(y)$ then
$\frac{d^2X}{dx^2} = -k^2$
$\frac{d^2Y}{dy^2} = +k^2$ are solutions

but we note that we have a linear variation of ϕ at the boundaries so we expect linear solutions for X and Y which are obtained for $k = 0$. Therefore

$\Phi = (\cancel{Ax+B)(Cy+D}) (Ax+B)(Cy+D)$

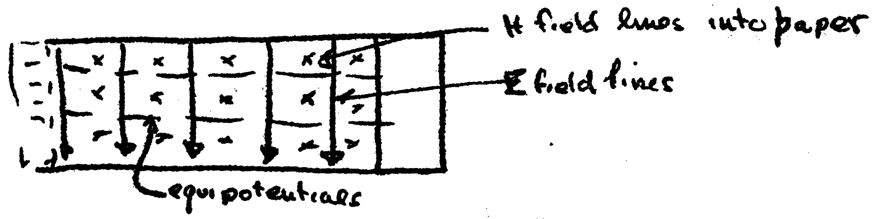
$\Phi(0,y) = 0$ gives $B = 0$

$\Phi(b,y) = (Ab)(Cy+D) = V_0$ sets $C = 0$

Finally $\phi(x,-b) = ADx = V_0/b\, x$ so $AD = V_0/b$

and the solution is

$\Phi = V_0/b\, x$ (we should have been able to guess this)



d since there are no currents in the space $\nabla \times H = 0$

and the B.C.S are $n \times (\bar{H}_1 - \bar{H}_2) = \bar{K}$

on top and bottom there are surface currents

at $x = b, 0$ $|\bar{K}|D = |J| w \times D$ where D is depth into paper (z-direction)

So $|K| = |J| w = \sigma V_0 W/b$

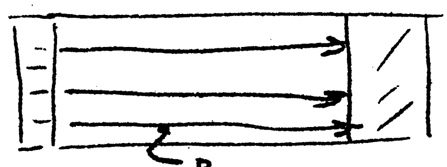
at $x = b$ $\bar{K} = a_y \sigma V_0 W/b$ (current is continuous)

at $x=0$ $K = -a_y \sigma V_0 W/b$

using $H_y = H_x = 0$ and $\nabla \times H = 0$ subject to these B.C.S

as well as $H_z = \frac{\sigma V_0 W}{b}$ at $y=0$ (again because tang H is continuous)

we find $\frac{dH_z}{dy} = 0$ gives $H_z = \sigma V_0 W/b$ ac



e $\underline{P} = \underline{E} \times \underline{H} = (-a_x V_0/b) \times (a_z \frac{V_0 W}{b}) = a_y \frac{\sigma V_0^2 W}{b^2}$

f. $-\oint \underline{P} \cdot d\underline{s} = \int_{vol} (J \cdot E)\, dV$ per unit length in z-direction.

on resistor surface at $y=0$

$-\oint \underline{P} \cdot d\underline{s} = \frac{\sigma V_0^2 W}{b^2} (D \times b) = \frac{\sigma V_0^2 W D}{b}$

$\int_{\text{resistor volume}} (\bar{J} \cdot \bar{E})\, dv = (\sigma V_0/b)(V_0/b)(DWb) = \sigma V_0^2 WD/b$ QED

g since P is non-zero only within space and it begins on the battery and ends within the resistor power flows in the space not in the connecting strips.

3 $\underline{P} = \underline{E} \times \underline{H}$

Plane wave prop. in z-direction

$$\underline{E} = \underline{a_x} E_0 \cos(\omega t - kz)$$

$$\underline{H} = \underline{a_y}\sqrt{\frac{\epsilon_0}{\mu_0}}\, E_0 \cos(\omega t - kz)$$

$$\underline{E} \times \underline{H} = \underline{a_z}\sqrt{\frac{\epsilon_0}{\mu_0}}\,|E_0|^2 \cos^2(\omega t - kz) = \underline{a_z}\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{E_0^2}{2}\left[1 + \cos 2(\omega t - kz)\right]$$

$$w_m = \tfrac{1}{2}\mu_0 H^2 = \tfrac{1}{2}\frac{\epsilon_0}{\cancel{\mu_0}}\cancel{\mu_0}\,\frac{E_0^2}{2}(1 + \cos 2(\omega t - kz))$$

$$w_e = \tfrac{1}{2}\epsilon_0 E^2 = \tfrac{1}{2}\epsilon_0 \frac{E_0^2}{2}(1 + \cos 2(\omega t - kz))$$

so $w_m = w_e$ in a plane wave.

Problem Set 2 Solutions

(1) $\nabla_t^2 \hat{h}_z + (\gamma^2 + \omega^2\mu_0\epsilon_0)\hat{h}_z = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \hat{h}_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \hat{h}_z}{\partial \varphi^2} + \omega^2\mu_0\epsilon_0 \hat{h}_z = 0$$

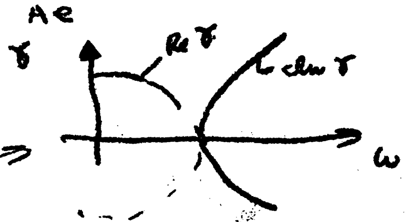
(2) $h_z = f(r)\, g(\varphi)$ so $g(\varphi)\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + f(r)\frac{\partial^2 g}{\partial \varphi^2} + \omega^2 (fg)\mu_0\epsilon_0 = 0$

divide by fg to have

$$\frac{1}{f}\,\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + \frac{1}{r^2}\frac{1}{g}\frac{d^2 g}{d\varphi^2} + \omega^2\mu_0\epsilon_0 = 0$$

for periodicity in φ let $\frac{1}{g}\frac{d^2 g}{d\varphi^2} = -n^2$ n an integer

so $g = A e^{-jn\varphi}$



(3) subs $\frac{1}{g}\frac{d^2 g}{d\varphi^2} = -n^2$ and find

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(\overbrace{\omega^2\mu_0\epsilon_0}^{k_c^2} - \frac{n^2}{r^2}\right)f = 0$$

which is Bessel's equation with solution

$$f = A J_n(k_c r) + B Y_n(k_c r) \qquad B = 0 \text{ since } Y_n(0) \to \infty.$$

represents sources at origin

For the TE_{01} mode $n = 0$ we require that

$$e_t(r=a) = 0 = -\frac{j\omega\mu_0}{\gamma}\,\underline{a_z} \times \hat{h}_t\Big|_{r=a} = \frac{\omega\mu_0}{\omega^2\mu_0\epsilon_0}\,\underline{a_z} \times \nabla_t h_z\Big|_{r=a} = 0$$

4 so $\frac{d J_0(k_c r)}{dr}\Big|_{r=a} = 0$ is the B.C. of interest (see P11 of xerox notes)

5. the root of this equation is called p'_{01} and equals 3.832

i.e. $k_c a = 3.832$

$$\gamma e_t = -j\omega\mu_0\, a_z \times h_t$$

$$\frac{\omega\mu_0}{\omega^2\mu_0\epsilon_0}\, a_z \times \nabla_t h_z = 0$$

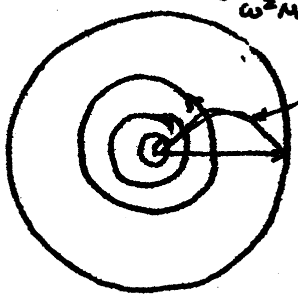
6. $\hat{h}_z = H_0 J_0\left(\frac{3.832}{a} r\right)$

$$\hat{h}_t = -\frac{\gamma}{k_c^2}\nabla_t h_z = -\frac{\gamma}{k_c^2}\underline{a}_r \frac{\partial h_z}{\partial r} = +\frac{H_0\gamma}{k_c^2}\left(\frac{3.832}{a}\right) J_1\left(\frac{3.832}{a} r\right)\underline{a}_r$$

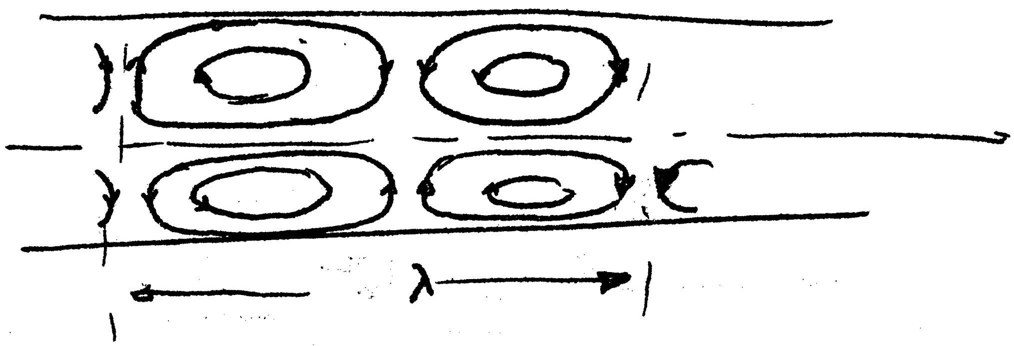
note: $\frac{d}{dx} J_0(x) = -J_1(x)$

$$\hat{\underline{e}}_t = \frac{-j\omega\mu_0}{\gamma}\,\underline{a}_z \times \hat{\underline{h}}_t = -H_0\frac{j\omega\mu_0}{\gamma}\left(\frac{3.832}{a}\right)\frac{\gamma}{k_c^2} J_1\left(\frac{3.832}{a} r\right)\underline{a}_\varphi$$

$$= -H_0\frac{j\omega\mu_0}{\omega^2\mu_0\epsilon_0}\left(\frac{3.832}{a}\right) J_1\left(\frac{3.832}{a} r\right)\underline{a}_\varphi = +\frac{H_0}{j\omega\epsilon_0}\left(\frac{3.832}{a}\right) J_1\left(\frac{3.832}{a} r\right)\underline{a}_\varphi$$

7. 

$\hat{\underline{H}} = \hat{\underline{h}}_z + \hat{\underline{h}}_t$



cut-off occurs for $\beta = 0$ or

$$\omega_{c0}^2\mu_0\epsilon_0 = \left(\frac{3.832}{a}\right)^2$$

$$\omega_{c0} = \left(\frac{3.832}{a}\right)c$$

$$f_{c0} = \left(\frac{1}{2\pi}\right)\left(\frac{3.832}{1}\right)\times 3\times 10^{10} \approx 18\ \text{Gc/s}$$

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Problem Set No. 2

Reading: Xerox Notes

Due: 3 October 1971

Problem

Calculate the fields in a TE_{01} mode cylindric waveguide in the following manner:

(1) Since this is a TE mode write the equation governing $\hat{h}_z$ using cylindric coordinates.

(2) Express $\hat{h}_z$ as a product solution:

$$\hat{h}_z = f(r)\, g(\phi)$$

and separate the equation appropriately by letting the azimuthal dependence be periodic.

(3) Show that $f(r)$ is given by, Bessel's eqn's

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(k_c^2 - \frac{n^2}{r^2}\right) f = 0$$

having a solution which may be written as $f = A\, J_n(k_c r) + B\, Y_n(k_c r)$. Choose the appropriate solution for this problem. (See Sec. 3.25 of the text.) Note: $k_c^2 \equiv \omega^2 \mu_0 \epsilon_0 + \gamma^2$

(4) Find and sketch the dependence of the propagation constant, β, as a function of angular frequency, ω $(\gamma = j\beta)$.

(5) What are the boundary conditions on h_z for perfectly conducting walls? (Remember that the electric field is related to h_z through derivatives.)

(6) Find the fields for the TE_{01} mode. This mode has no azimuthal variation and one-half cycle variation of h_z in the radial direction from the axis to the wall. Determine the cut-off frequency for this mode in terms of the guide radius, $\underline{a}$. If $\underline{a} = 1$ cm, calculate the cut-off frequency.

(7) Sketch the E-field in a z-cross section and the H-fields with z in an equatorial slice. Show at least 1/2 wavelength.

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Problem Set # 3

Reading: Secs. 4.11, 4.12, 5.13, 5.14, 5.15

Secs. 6.07, 6.09, 6.13, 6.15

Due: 15 October 1971

Problems from Text

(1) 4.12e. After working out this problem consider the following: Place some excess charge, q, at the center of a solid conducting sphere of radius $R = 1$ cm (don't ask how we can do this!). Where are the charges in the steady state after we release them? Now calculate the velocity these charges must move at if the material is copper ($\sigma = 5.8 \times 10^{7}$, $\epsilon_0 = \frac{1}{36\pi} \times 10^{-9}$). Comment with some maturity on your results.

how fast they relax

(2) 6.13a

(3) 6.13b

(4) 6.15c

1

$$\nabla \times H = J + \epsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot J + \epsilon_0 \frac{\partial \nabla \cdot E}{\partial t} = 0 \quad (\text{if } \epsilon \neq \epsilon(x,y,z))$$

then write $J = \sigma E$ and if $\sigma \neq \sigma(x,y,z)$

we have $\nabla \cdot E = \rho/\epsilon_0$

$$\sigma/\epsilon_0 \, \rho = -\partial \rho/\partial t \quad \text{whose solution is}$$

$$\rho = \rho(x,y,z)\, e^{-\sigma/\epsilon_0 t}$$

which says that charges arranged with a distribution $\rho(x,y,z)$ decay exponentially with time constant ϵ_0/σ without changing their spatial distribution.

with a charge q at $r = r_0$ then

$$\rho\Big|_{t=0} = \frac{q\,\delta(r)}{4\pi r^2} \quad (\text{integrate } \rho \text{ over volume and see that } \int \rho \, dv = q.)$$

$\frac{\partial}{\partial r}(r J_r) = 0$

$r J_r = \text{const}$

$J_r = \frac{\text{const}}{r}$

$\frac{\partial}{\partial r}(r^2 J) = 0$ hence $\rho(r,t) = \frac{q\,\delta(r)}{4\pi r^2} e^{-\sigma/\epsilon_0 t}$. hence since

$\therefore r^2 J = \text{const}$ within sphere for $r \neq 0$ $\rho = 0$

$\boxed{\nabla \cdot \vec{J}} = -\partial \rho/\partial t = \boxed{0}$ for $r \neq 0$ the current density

$J = \frac{\text{const}}{r^2}$ is divergenceless and angle independent from symmetry $\therefore$

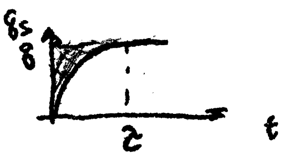
$$J = \left(\frac{R^2}{r^2}\right) J_0 \quad \text{where } R = \text{radius of sphere and } J_0$$

is the current density at its surface.

$$I = \int_{\text{surface of sphere.}} \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int \rho \, dv = \frac{q\sigma}{\epsilon_0} e^{-\sigma/\epsilon_0 t} = J_0 \, 4\pi R^2$$

$$q_s = \text{the charge on the surface} = \int_0^t I \, dt = -q\sigma/\epsilon_0 \frac{1}{\sigma/\epsilon_0} e^{-\sigma/\epsilon_0 t}\Big|_0^t$$

$$= q - q e^{-\sigma/\epsilon_0 t}$$



so after some 2 or 3 time constants, $3\,\epsilon_0/\sigma$ say, most of the charge is on the surface

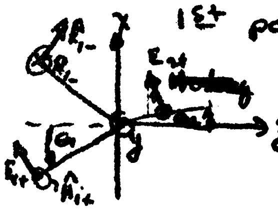
$$\tau = \frac{3}{36\pi} \times 10^{-9} \times \frac{1}{5.8 \times 10^7} \approx 5 \times 10^{-19} \text{ sec.}$$

If we believe that these charges on the surface came from the center then the velocity is $R/\tau \approx 2 \times 10^{18}$ cm/sec!

which would be an impossible velocity for real entities such as electrons.

We of course recognize that the original charges do not appear on the surface but their effects cause the "free" charges in the conduction band to move and it is only some of the outer-most charges on the sphere which appear on its surface - nevertheless we have a manifestation of action at a distance, since charges immediately begin showing up on the surface due to the effect of charge at the center. The answer to the dilemma lies with the model of conductivity, which is valid only for slow changes, being applied to this fast phenomenon. When the period of change (time constant say) is short compared with the mean-free-time between collisions of the free electrons with the lattice then the conductivity must be modified from this "D-C" conductivity given in the tables. In fact the true picture is that of electron density waves of an acoustical nature moving the charges out to the exterior. Closely related to this problem are the discussions of Sec 9.05 through 9.07 of the text.

(2) 6.13 a



1st polarization (E in plane of incidence.)

$$\tau = \frac{E_{2+}\cos\theta_2}{E_{1+}\cos\theta_1}$$ we have $\boxed{\hat{E}_{2+}} = \tau E_{1+}\cos\theta_1/\cos\theta_2$

$\hat{H}_{2+} = \frac{\tau}{\eta_2}\frac{E_{1+}\cos\theta_1}{\cos\theta_2}$ propagation factors for each direction

by components $\hat{E}_2 = a_x \hat{E}_{2+}\cos\theta_2 + a_z E_{2+}\sin\theta_2$

hence So $$\hat{E}_2 = \left[a_x \tau E_{1+}\cos\theta_1 - a_z \tau E_{1+}\cos\theta_1\left(\frac{\sin\theta_2}{\cos\theta_2}\right)\right]\boxed{e^{-j\beta_1\sin\theta_1 x}}\,e^{\boxed{-j\beta_2\cos\theta_2 z}}$$

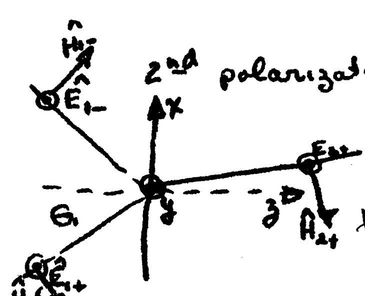
$$\hat{H}_y = a_y\frac{\tau E_{1+}\cos\theta_1}{\eta_2\cos\theta_2}e^{-j\beta_1\sin\theta_1 x}\,e^{-j\beta_2\cos\theta_2 z}$$

this is true only because $\beta_1\sin\theta_1 = \beta_2\sin\theta_2$ Snell's Law.

in general $e^{-j\beta\cdot r}$ direction vector

$r = x a_x + y a_y$

$\beta = \beta_2\sin\theta_1 a_x + \beta_2\cos\theta$



2nd polarization ($E_\perp$ to plane of incidence)

dielectric

Satisfying B.C.'s. $\hat{E}_{1+} + \hat{E}_{1-} = \hat{E}_{2+}$

$\tau = E_{2+}/E_{1+}$

$\rho = E_{1-}/E_{1+}$

$\Rightarrow 1 + \rho = \tau$

$$-\frac{E_{1+}}{\eta_1}\cos\theta_1 + \frac{E_{1-}}{\eta_1}\cos\theta_1 = -\frac{E_{2+}}{\eta_2}\cos\theta_2 \Rightarrow -1+\rho = -\tau\frac{\eta_1\cos\theta_2}{\eta_2\cos\theta_1}$$

now two eqn's in two unknowns solve for τ and ρ in terms of η_1 and η_2, θ_1 & θ_2

this appears to have been developed merely for the hell of it. (3)

so $\boxed{\tau = \dfrac{2Z_2'}{Z_2' + Z_{1\,\mathrm{tan}}'} \quad \text{where } Z_2' = \eta_2/\cos\theta_2, \quad Z_{1\,\mathrm{tan}}' = \eta_1/\cos\theta_1 \qquad \rho = \dfrac{Z_2' - Z_{1\,\mathrm{tan}}'}{Z_2' + Z_{1\,\mathrm{tan}}'}}$

therefore $\hat{\underline{E}}_2 = \underline{a}_y \tau \hat{E}_{1+} e^{-j\beta_1 \sin\theta_1 x} e^{-j\beta_2\cos\theta_2 z}$

$$\hat{\underline{H}}_2 = \boxed{\frac{\tau \hat{E}_{1+}}{Z_2'}\left[-\underline{a}_x + \underline{a}_z \frac{\sin\theta_2}{\cos\theta_2}\right] e^{-j\beta_1\sin\theta_1 x - j\beta_2\cos\theta_2 z}}$$ why is this necessary

$$= \frac{\tau \hat{E}_{1+}}{\eta_2}\left[-\underline{a}_x \cos\theta_2 + \underline{a}_z \sin\theta_2\right] e^{-j\beta_1\sin\theta_1 x - j\beta_2\cos\theta_2 z}$$ this can be developed directly.

6.13(b) in medium ①

for + wave $\hat{\underline{P}}_{1+} = \dfrac{|E_{1+}|^2}{2\eta_1}\left[\underline{a}_z \cos\theta_1 + \underline{a}_x \sin\theta_1\right]$

this is for P_{2-}

$$\hat{\underline{P}}_{1-} = \frac{|E_{1-}|^2}{2\eta_1}\left[-\underline{a}_z\cos\theta_1 + \underline{a}_x \sin\theta_1\right] = \frac{\rho^2 |E_{1+}|^2}{2\eta_1}\left[-\underline{a}_z \cos\theta_1 + \underline{a}_x \sin\theta_1\right]$$

gotten by inspection   by substitution.

in medium ②

$$\hat{\underline{P}}_2 = \frac{|E_2|^2}{2\eta_2}\left[\underline{a}_z \cos\theta_2 + \underline{a}_x \sin\theta_2\right] = \frac{|E_{1+}|^2 \tau^2 \cos^2\theta_1}{2\eta_2 \cos^2\theta_2}\left[\underline{a}_z\cos\theta_2 + \underline{a}_x \sin\theta_2\right]$$

power balance across interface says

$\tau = \dfrac{E_{2+}\cos\theta_2}{E_{1+}\cos\theta_1}$

$$(\hat{\underline{P}}_{1+})_z + (\hat{\underline{P}}_{1-})_z = (\hat{\underline{P}}_2)_z$$ z components only!

or $$\frac{|E_{1+}|^2}{2\eta_1}\left[1 - \rho^2\right]\cos\theta_1 \overset{?}{=} \frac{|E_{1+}|^2 \tau^2}{2\eta_2}\frac{\cos^2\theta_1}{\cos^2\theta_2}\cos\theta_2$$

or $$1 - \rho^2 \overset{?}{=} \tau^2 \frac{\eta_1 \cos\theta_1}{\eta_2\cos\theta_2}$$

from bottom of ②
$$1 + \rho = \tau$$
$$1 - \rho = \tau \frac{\eta_1\cos\theta_1}{\eta_2 \cos\theta_2}$$

multiply both together and get $1 - \rho^2 = \tau^2 \dfrac{\eta_1\cos\theta_1}{\eta_2\cos\theta_2}$

6.15 c for $Z_L > Z_{1\,\mathrm{tan}}$ we find $\rho > 0$, for $Z_L < Z_{1\,\mathrm{tan}}$ $\rho < 0$ ∴ there is a 180° change in phase of the standing wave component as can be seen from eq 14 $e^{-j\beta z} + \rho e^{+j\beta z} = (1-\rho)e^{-j\beta z} + \rho(e^{-j\beta z} + e^{+j\beta z})$
$$= (1-\rho)e^{-j\beta z} + 2\rho \cos\beta z$$

ENGR 336 FIELDS II 15 October 1971

Problem Set No. 4

Due: 22 October 1971

Quiz I: 27 October

1. Show that for an imperfect conductor the integral of the real part of the complex Poynting vector over a unit area of the conductors surface is just equal to $\frac{1}{2}\int_{vol} (\hat{\underline{J}} \cdot \hat{\underline{E}}^*)dv$ where the volume is bounded by the same area and extends to infinity into the conductor.

2. Calculate the ratio of power loss per unit length to power flow in the direction of propagation for the TE_{01} mode circular guide whose solutions you have already found. Use the perturbation approach. use fields from perfect conductor situation and use that to find your loss.

3. Determine the power absorbed relative to power incident on an imperfect conductor ($\sigma >> \omega\epsilon_0$) having a dielectric coating of thickness $\underline{t}$ on its surface with permittivity ϵ. Assume the wave to be launched from free space and propagating normally to the surfaces.

4. What is the skin depth of stainless steel at 1 MHz, 100 MHz, 1 GHz and 10 GHz?

non-magnetic stainless $\mu = \mu_0$

$$\int e^{-\frac{2}{\delta}z}\,dz$$

$$\frac{1}{1+j}$$

$$\frac{e^{-\frac{2}{\delta}z}}{-2/\delta} = \frac{\delta}{2}$$

$$Re \frac{1}{1+j} \cdot \frac{1-j}{1-j} = Re\frac{1-j}{2} = \frac{1}{2}$$

FIELDS II - Problem Set 4

· Solutions

① $\hat{P} = \boxed{\frac{\hat{E} \times \hat{H}^*}{2}}$ average power

time average power $= \frac{\hat{E} \times \hat{H}^*}{4}$

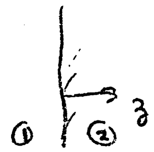
$$\hat{E} = a_x E_0 e^{-(1+j)z/\delta} \qquad \delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$$

$$\hat{H} = a_y \boxed{\frac{E_0}{(1+j)R_s}} e^{-(1+j)z/\delta} \qquad R_s = \frac{1}{\sigma\delta}$$

easily derived if recall $\eta_2 = (1+j) R_s$

so $\hat{P} = a_z \frac{E_0^2}{2(1+j)R_s} e^{-2z/\delta}$

error:



at interface ($z=0$) $\quad \underline{\underline{E_{0_1} = E_{0_2}}}$ (tangential E is continuous

$\underline{\underline{H_{0_1} = H_{0_2}}}$ (" H → continuous because surface currents exist only for perfect conductors)

hence $Re\,\hat{P}\boxed{(z=0)} = a_z \boxed{\frac{E_0^2}{4R_s}} = (\text{power/area.}) \times a_z$ (unit area!)

now inside material $\hat{J} = \sigma \hat{E} = a_x \sigma E_0 e^{-(1+j)z/\delta}$

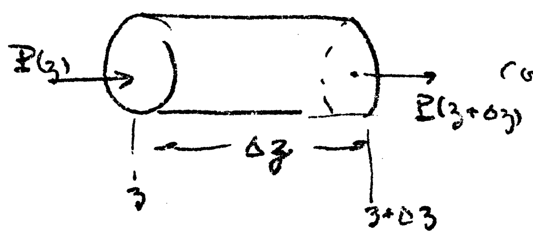
So $\frac{\hat{J} \cdot \hat{E}^*}{2} = \frac{\sigma E_0^2}{2} e^{-2z/\delta}$

and $\int_{vol} \frac{\hat{J} \cdot \hat{E}^*}{2} dv = \text{Area} \times \frac{\sigma E_0^2}{2} \int_0^\infty e^{-2z/\delta} dz = \text{Area} \times \frac{\sigma E_0^2}{2} \times \frac{\delta}{2} = \frac{E_0^2}{4R_s}$

which checks.

---

② The recipe for calculation of power loss in walls of good conductor is that Power dissipated/area $= \frac{1}{2} R_s |H_{tan}|^2$ where H_{tan} is the magnetic field calculated on the basis of having a perfect conductor.



consider a length of waveguide Δz the power flow into the volume is $Re \int_0^{2\pi}\int_0^a a_z \cdot \hat{P}(z)\, r dr d\phi = P_i$

and the power flow out is $Re \int_0^{2\pi}\int_0^a a_z \cdot \hat{P}(z+\Delta z)\, r dr d\phi = P_o$

hence the difference $\lim_{\Delta z \to 0} P_i - P_o = Re\left\{-\frac{\partial}{\partial z}\left[\int_0^{2\pi}\int_0^a a_z \cdot \hat{P}(z)\, r dr d\phi\right]\right\} \Delta z$

which must be the power dissipated in the walls which can be found by integrating $\frac{1}{2}R_s|H_{tan}|^2$ over the surface of the volume

$$P_i - P_o = \left[\frac{1}{2}R_s\int_0^{2\pi}|H_{tan}|^2\, \overset{\text{guide radius}}{a}\, d\phi\right]\Delta z = -\frac{\partial}{\partial z} Re\left\{\int_0^{2\pi}\int_0^a a_z\cdot \hat{P}(z)\, r\,dr\,d\phi\right\}\Delta z$$

from the previous problem set or Text p 430

$$\hat{H}_z = B\, J_0(3.83\, r/a)\, e^{-\alpha z}\, e^{-j\beta z}$$ ← note we have added the attenuation factor $e^{-\alpha z}$ to the fields.

$$\hat{E}_\phi = -j\eta_0 \frac{f}{f_c} B\, J_1(3.83\, r/a)\, e^{-\alpha z} e^{-j\beta z}$$

$$\hat{H}_r = -\frac{E_\phi}{Z_{TE}} \quad \text{where } Z_{TE} = \eta_0\left[1-(f_c/f)^2\right]^{-1/2}$$

$$\hat{P} = \frac{1}{2}\hat{E}\times\hat{H}^* = \frac{1}{2}\frac{|E_\phi|^2}{Z_{TE}}\, a_z = \frac{1}{2}\eta_0^2 \frac{f^2}{f_c^2}\frac{B^2}{Z_{TE}} J_1^2(3.83\, r/a)\, e^{-2\alpha z}$$

then $$P_i - P_o = e^{-2\alpha z}\left[2\alpha\, \Delta z \int_0^{2\pi} d\phi \int_0^a r\, J_1^2(3.83\, r/a)\, dr\right]\frac{1}{2}\eta_0^2 \frac{f^2}{f_c^2}\frac{B^2}{Z_{TE}}$$

and $$P_{diss} = e^{-2\alpha z}\,\frac{1}{2}R_s\, B^2 J_0^2(3.83)\times 2\pi a\,\Delta z$$

thus

$$\alpha = \frac{\frac{1}{2}R_s B^2 J_0^2(3.83)\; 2\pi a}{2\cdot\frac{1}{2}\eta_0^2\, f^2/f_c^2\; B^2/Z_{TE}\; 2\pi\int_0^a r J_1^2(3.83\, r/a)\,dr}$$

the Bessel function integral can be found on p432 of Text

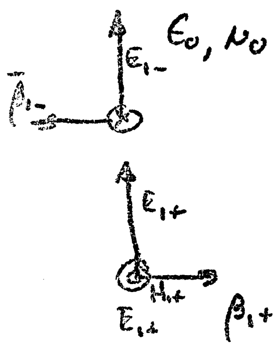
$$\int_0^a J_1^2(k_c r)\, r\,dr = \frac{a^2}{2}\left[J_1'^2(k_c a)\right] = \frac{a^2}{2}\left[J_0 - \frac{1}{k_c a}J_1(k_c a)\right]^2 \quad \text{(see p 2.31)}$$

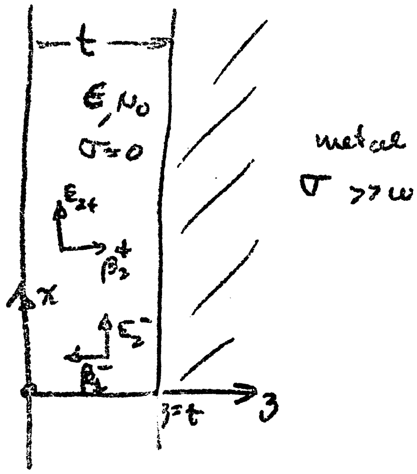
$$= a^2/2\; J_0^2(k_c a)$$

thus ~~final~~ finally

$$\alpha = \frac{R_s Z_{TE}}{\eta_0^2\, f^2/f_c^2\; a} = \frac{R_s}{a\eta_0}\,\frac{f_c^2}{f^2\sqrt{1-(f_c/f)^2}} \quad \text{(see Table 8.04)}$$

(5) (3)





$\beta_1 = \omega\sqrt{\mu_0\epsilon_0}$ $\quad \eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \eta_0$

$\beta_2 = \omega\sqrt{\mu_0\epsilon}$ $\quad \eta_2 = \sqrt{\frac{\mu_0}{\epsilon}}$

We work the problem by assuming that the conductor has $\sigma = \infty$ then find H_{tan} and thence $P_{abs}/area = \frac{1}{2}R_s|H_{tan}|^2$

Set up coordinate system at air-dielectric interface and find the fields in the dielectric by satisfying the B.C at $z=t$ that

$$E_{tan} = 0 = E_{2+}e^{-j\beta_2 z}\Big|_{z=t} + E_{2-}e^{+j\beta_2 z}\Big|_{z=t} \quad \text{so } E_{2-} = -E_{2+}e^{-2j\beta_2 t}$$

hence in the dielectric the fields are

$$\hat{\underline{E}}_2 = \underline{a}_x\left(\hat{E}_{2+}e^{-j\beta_2 z} + E_{2-}e^{+j\beta_2 z}\right) = \underline{a}_x\hat{E}_{2+}\left(e^{-j\beta_2 z} - e^{-2j\beta_2 t}e^{+j\beta_2 z}\right)$$

and $$\hat{\underline{H}}_2 = \underline{a}_y\,\eta_2^{-1}\left(\hat{E}_{2+}e^{-j\beta_2 z} - \hat{E}_{2-}e^{+j\beta_2 z}\right) = \underline{a}_y\,\eta_2^{-1}E_{2+}\left(e^{-j\beta_2 z} + e^{-2j\beta_2 t}e^{+j\beta_2 z}\right)$$

now at the interface we must find $\hat{\underline{H}}_1$ & $\hat{\underline{E}}_1$

$$\hat{\underline{E}}_1(0) = a_x\left(E_{1+}e^{-j\beta_1 z} + E_{1-}e^{+j\beta_1 z}\right) = \underline{a}_x\hat{E}_{1+}(1+\rho) = \underline{E}_2(0)$$

$$\underline{H}(0) = \eta_1^{-1}a_y\left(E_{1+}e^{-j\beta_1 z} - E_{1-}e^{+j\beta_1 z}\right) = \underline{a}_y\hat{E}_{1+}\eta_1^{-1}(1-\rho) = \underline{H}_2(0)$$

(let $\tau = E_{2+}/E_{1+}$ as always) $z=0$

so $$1+\rho = \tau\left(1 - e^{-2j\beta_2 t}\right)$$

$$\eta_1^{-1}(1-\rho) = \eta_2^{-1}\tau\left(1 + e^{-2j\beta_2 t}\right) \qquad \text{solve for } \tau = \frac{2j\eta_2\tan\beta_2 t}{\eta_1 + j\eta_1\tan\beta_2 t} = \frac{E_{2+}}{E_{1+}}\Big|_{z=0}$$

$$\underline{H}_2(z=t) = a_y\,\eta_2^{-1}E_{2+}\left(e^{-j\beta_2 t} + e^{-j\beta_2 t}\right) = 2\underline{a}_y\,\eta_2^{-1}E_2^+e^{-j\beta_2 t}$$

$$= 2\tau E_{1+}\eta_2^{-1}\underline{a}_y\,e^{-j\beta_2 t}$$

(4)

$$P_{inc} = \frac{1}{2}\eta_1^{-1}|E_{1+}|^2$$

$$P_{abs} = \frac{1}{2}R_s|H_{tan}|^2 = \frac{1}{2}R_s\,4|\tau|^2|E_{1+}|^2/\eta_2^2 = \frac{2R_s|E_{1+}|^2}{\eta_2^2}\,\frac{4\eta_2^2\tan^2\beta_2 t}{\eta_1^2+\eta_2^2\tan^2\beta_2 t}$$

hence $$P_{abs}/P_{inc} = \eta_1\left(4R_s\right)\frac{4\tan^2\beta_2 t}{\eta_1^2+\eta_2^2\tan^2\beta_2 t} = \eta_1\frac{16R_s\tan^2\beta_2 t}{\eta_1^2+\eta_2^2\tan^2\beta t}$$

---

4 The relative resistivity (ratio of ~~resistance~~ resistivity $(\frac{1}{\sigma})$ of stainless steel to that of Cu) is ~ 52.8

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \delta_{cu}\times\sqrt{\frac{\sigma_{cu}}{\sigma_{s-s}}} = \delta_{cu}\sqrt{52.8} \approx 7\,\delta_{cu}$$

now from chart on p 252 of text we have

~~δ | 1 MHz | 100 MHz | 1 GHz | 10 GHz~~
~~δ_{cu} | 6.5×10^{-3} | 2×10^{-3}~~

in cm

	1 MHz	100 MHz	1 GHz	10 GHz
δ_{cu} in m	6.5×10^{-3}	6.5×10^{-4}	2×10^{-4}	6.5×10^{-5}
δ_{s-s}	4.5×10^{-2}	4.5×10^{-3}	1.4×10^{-3}	4.5×10^{-4}

ENGR 336 F I E L D S I I 22 October 1971

Problem Set # 5

Due: 29 October 1971

Reading: Sections 1.13 through 1.19,
Section 7.11

Problems:

1.16a

1.16b

1.17a

1.18c

Problem Set # 5

Solutions

Errata for Problem Set # 1 solutions

on P4 the helmholtz equation

$\nabla_t^2 \hat{h}_z + \omega^2 \mu_0 \epsilon_0 \hat{h}_z = 0$ is incorrect and should have read

$\nabla_t^2 \hat{h}_z + \underbrace{(\gamma^2 + \omega^2 \mu_0 \epsilon_0)}_{k_c^2} \hat{h}_z = 0$ and this mistake was carried through,

although did not effect the construction of the solution

Problem # 1

1.16 a

$$\rho = \frac{R_L - Z_0}{R_L + Z_0}, \quad \tau = \frac{2R_L}{R_L + Z_0}$$

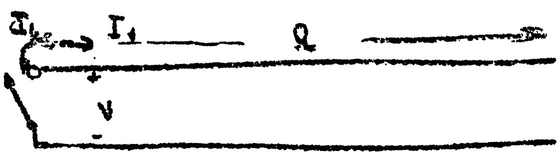
$$P_{refl}/P_{inc} = V_-^2/V_+^2 = \rho^2 = \left(\frac{R_L - Z_0}{R_L + Z_0}\right)^2$$

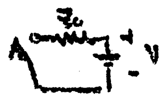
$$P_{trans}/P_{inc} = V_L^2/R_L \times \frac{1}{V_+^2/Z_0} = \frac{Z_0}{R_L} \times \tau^2 = \frac{4R_L Z_0}{(R_L+Z_0)^2}$$

1.16 b

R_L	ρ	τ	P_{refl}/P_{inc}	P_{trans}/P_{inc}
0	-1	0	1	0
$\frac{1}{2}Z_0$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{9}$	$\frac{8}{9}$
Z_0	0	1	0	1
$2Z_0$	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{9}$	$\frac{8}{9}$
∞	1	0	1	0

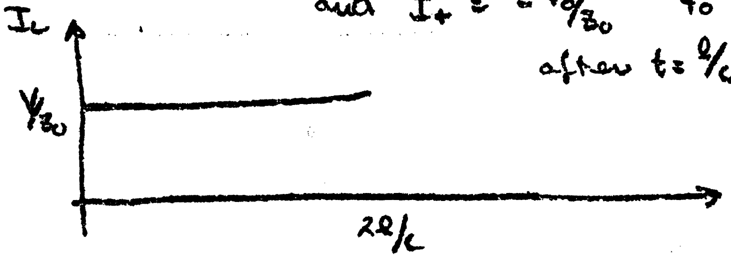
1.17a



equivalent ckt 

$I(0^+) = V/Z_0$,

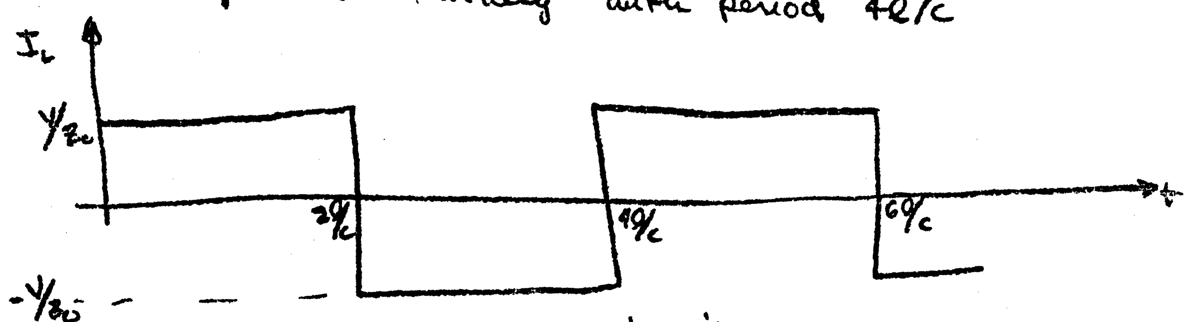
and $I_+ = -V/Z_0$ to conserve current



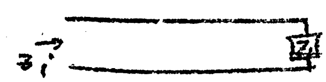
after $t = \ell/c$ a reflected wave $I_- = +V/Z_0$ ~~comes~~ starts back from open end of line with voltage $V_- = -V$ and when this voltage reaches the input after $t = 2\ell/c$ it requires a $V_+ = +V$ so as to have zero net voltage so

$I_+ + I_- = I_{total} = -I_L = V/Z_0 + V/Z_0$

or $I_L = -2V/Z_0$ which when added to the V/Z_0 already flowing gives a net of $-V/Z_0$. We can then expect this process to repeat periodically with period $4\ell/c$

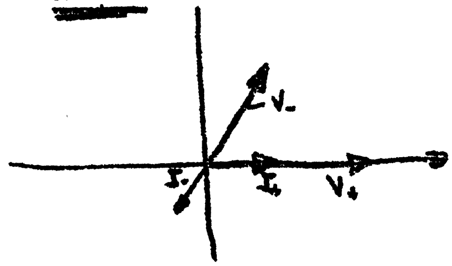


1.18c $\hat{\rho} = \dfrac{Z_L - Z_0}{Z_L + Z_0} = \dfrac{j100}{200 + j100} = \dfrac{j(2-j)}{5} = \dfrac{1+2j}{5}$

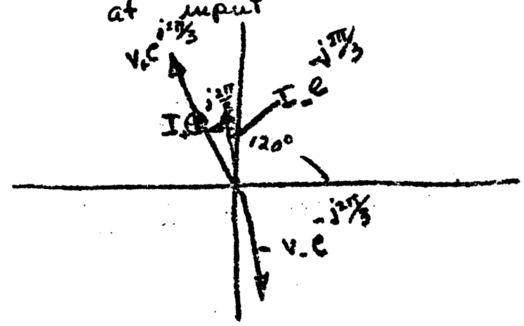


$$Z_i = Z_0 \left[\frac{(100 + j100)\cos\frac{2\pi}{3} + j100 \sin\frac{2\pi}{3}}{100 \cos\frac{2\pi}{3} + j(100 + j100)\sin\frac{2\pi}{3}} \right] = 38 - j2.4$$

at load



at input

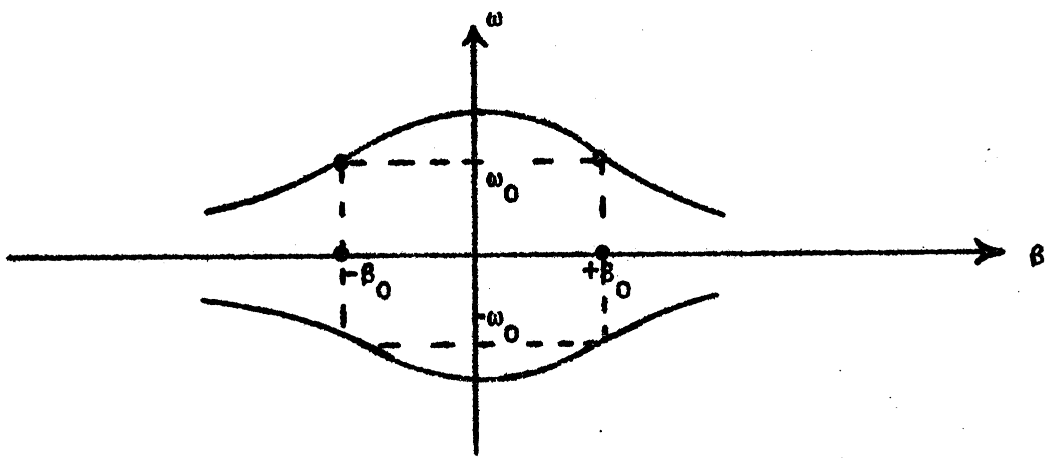


ENGR 336 FIELDS II 29 October 1971

Problem Set # 6

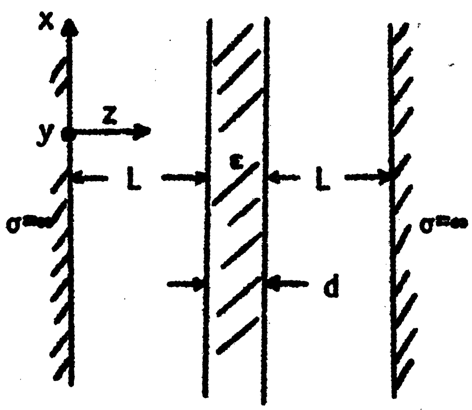
Due: 5 November 1971

1. A one ohm lossless transmission line has the ω-β diagram shown. A source drives a one ohm inductive reactance load ¼ wavelength away. If a one volt amplitude signal at frequency ω_0 appears at the load.
   (a) Find the incident and reflected voltage and current at the load.
   (b) Sketch the phasor diagram for (a).
   (c) Sketch the phasor diagram for the incident reflected voltages and currents at the source.
   (d) What circuit element does the line appear to be at the source?



2. An ideal voltage source feeds a $10\ \Omega$ coaxial line with a ten volt step at $t = 0$. The line is 300 meters long and open-circuited. Plot and dimension the voltage and current at the load and the source as a function of time for $0 \leq t \leq 10\ \mu sec$.

3. Determine the resonant frequencies of the system shown below for the electric field in the x-direction and the magnetic field in the y-direction. The system is infinite in the x-y plane. The center dielectric is lossless.
Hint: Consider the load impedances looking both ways from the center of the system.



Problem Set #6 Solutions

1.

(a)

$Z_L = j\omega L = j$ ← ω_0?

$\hat{V}_L = 1 \text{ volt} = 1\angle 0$

$\hat{I}_L = \frac{1}{Z_L} = -j = 1\angle -90^\circ$

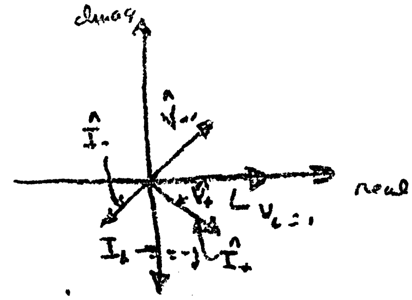
$\hat{V}_+ + \hat{V}_- = 1$ or $2V_+ = 1-j$

$\frac{1}{Z_0}(\hat{V}_+ - \hat{V}_-) = -j$; $Z_0 = 1$

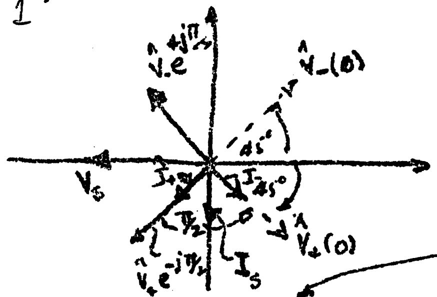
$V_+ = \frac{1-j}{2}$, $\hat{I}_+ = \frac{1-j}{2}$

so $V_- = \frac{1+j}{2}$ $\quad \hat{I}_- = -\frac{(1+j)}{2}$

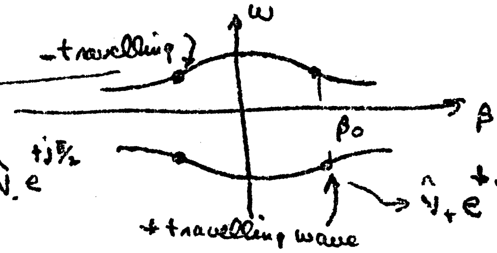
b)



c)



note that $\partial\beta/\partial\omega < 0$ so this is a backward wave line



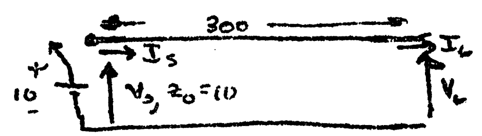
at source $\ell = \lambda/4$, $z = -\ell$

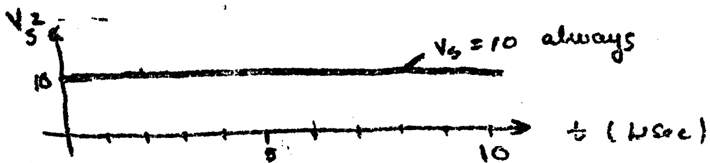
$\hat{V}_- e^{-j\beta_0 z} = \hat{V}_- e^{+j\beta_0 \ell} = \hat{V}_- e^{+j\pi/2}$

$\hat{V}_+ e^{+j\beta_0 z}\Big|_{z=-\ell} = \hat{V}_+ e^{-j\pi/2}$

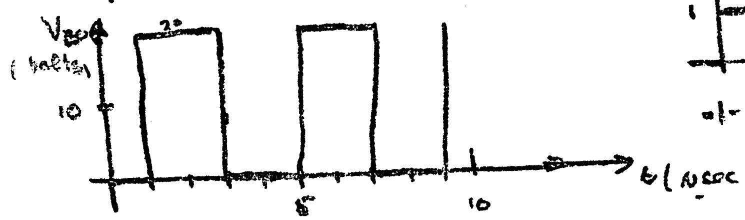
d) since I_s leads V_s the load appears capacitive and $Z_i = -j$

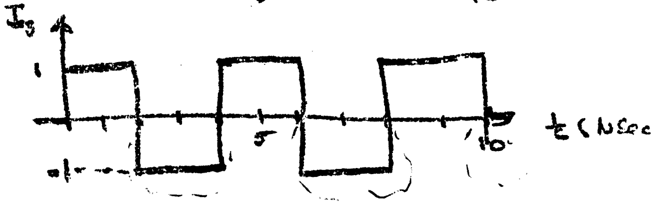
2.

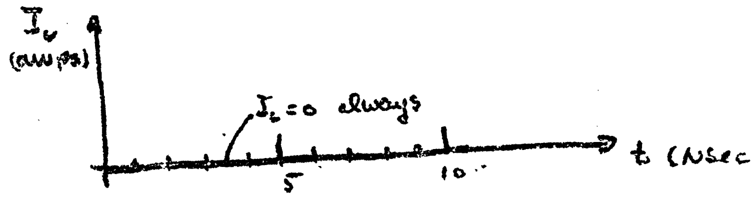


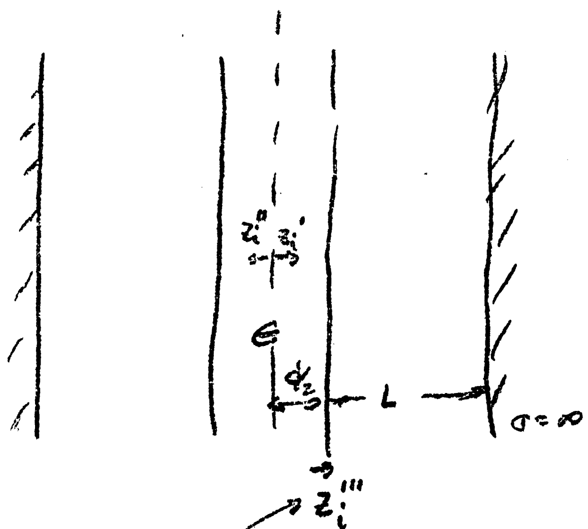


one way time $= \frac{300}{3\times10^8} = 10^{-6}$ sec









resonance can occur in either of two ways →

① $Z_i' = Z_i'' = 0$ that is there is a E field ~~zero~~ zero at the mid-plane

or ② $Z_i' = Z_i'' = \infty$ or an H-field zero at the mid plane.

call Z_i''' the impedance looking into free space at the dielectric interface

$$Z_i''' = +jZ_0 \tan\beta_0 L \qquad \beta_0 = \omega\sqrt{\mu_0\epsilon_0}, \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

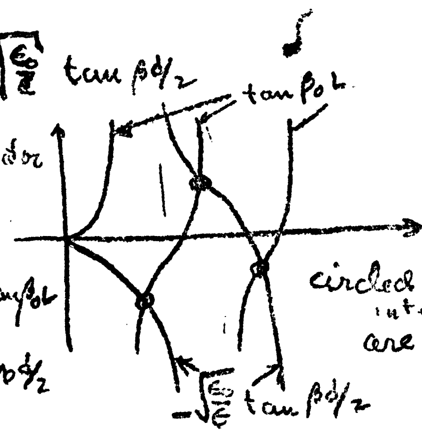
$$\left.\frac{Z_i''' \cos\beta d/2 + jZ\sin\beta d/2}{Z\cos\beta d/2 + jZ_i'''\sin\beta d/2}\right] \qquad \beta = \omega\sqrt{\mu_0\epsilon}, \quad Z = \sqrt{\frac{\mu_0}{\epsilon}}$$

hence we find case ① ($Z_i' = 0$) if

$$Z_i''' \cos\beta d/2 + jZ\sin\beta d/2 = 0 \quad \text{or}$$

$$jZ_0 \tan\beta_0 L \cos\beta d/2 = -jZ\sin\beta d/2$$

$$\text{so} \quad \tan\beta_0 L = -Z/Z_0 \tan\beta d/2 = -\sqrt{\frac{\epsilon_0}{\epsilon}}\tan\beta d/2$$

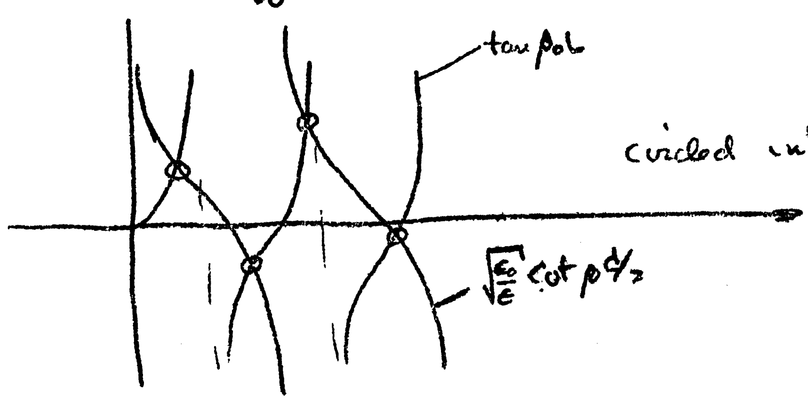


circled intersections are solutions

case ② $Z_i' = \infty$ is found by setting denominator equal to zero

$$\text{or} \quad Z\cos\beta d/2 = -jZ_i''\sin\beta d/2 = +Z_0\sin\frac{\beta d}{2}\tan\beta_0 L$$

$$\text{hence} \quad Z/Z_0 \cot\beta d/2 = \tan\beta_0 L = \sqrt{\frac{\epsilon_0}{\epsilon}}\cot\beta d/2$$



circled intersections are solutions for case ②

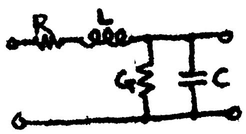
ENGR 336 FIELD II 5 November 1971

Problem Set #7
Due: 19 Nov. 1971
Reading: Sections 6.04, 6.05

1. Problem 1.20a

2. Problem 1.20d

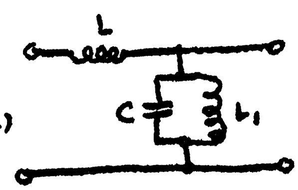
3. For a transmission line whose equivalent circuit is for a differential length is given in the figure, find the complex propagation constant, γ, for the case where $R \ll j\omega L$ and the shunt conductance $G \ll j\omega C$. Show that if



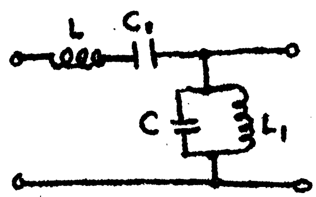
$\gamma = \alpha + j\beta$, $\beta \approx \omega\sqrt{LC}$ and $\alpha \approx \frac{R}{2Z_0} + \frac{GZ_0}{2}$ where $Z_0 = \sqrt{\frac{L}{C}}$

4. Find and sketch the dispersion relation (ω,β diagram) for each of the incremental equivalent circuits shown below. Sketch the group velocity vs. ω for each.

(a)



(b)



for both $LC_1 \lessgtr L_1C$

5. Sketch the ω,β diagram for the dispersion relation

$$\beta^2 = \beta_0^2\left(\frac{\omega_p^2}{(\omega - \beta v_0)^2 - \omega_c^2} - 1\right)$$

where β_0, ω_p, v_0 and ω_c are constants and $\omega_c > \omega_p$. Sketch the group velocity vs. ω for the above waves.

ENGR 336 FIELDS II 10 November 1971

Problem Set #7 Addendum

Reading: Sections 10.01-10.06, 10.08

Problem 6

Using your solutions for the TE_{01} mode cylindric waveguide (Problem Set No. 2 and, Problem Set No. 4, Problem 2). Calculate the fields (E & H) and the Q for the TE_{011} mode cylindric cavity with copper walls. Sketch the field and show how you might use the cavity as a filter by proper placement of coupling devices.

Solutions for P.S #7

1.20 a)

use Smith Chart

normalized impedance $Z_L/Z_0 = 1.5 - j1.38$ enter smith chart and read

$$\hat{\rho} = .51\, e^{-j.725} \qquad .725 \Rightarrow 41^\circ$$

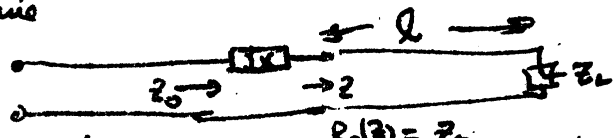
3.5 meters to generator $\rightarrow$ $3.5/c/f = \frac{3.5 \text{ meters}}{6 \text{ meters } (\lambda)} = .583$

this advances the phase of ρ by $(1.166)\,2\pi$ radians in a counter-clockwise manner – one complete revolution plus 60° or 101°. In going around we pass by the Z_{min} position (and then the Z_{max} position). Here Voltage is a minimum (and current a maximum). Z_{min} occurs for the angle of the reflection coeffecient equal to -180° which is 139° away from the load. The $\measuredangle$ of the reflection coeffecient $= 2\beta l$ hence $2\beta l = 139/180 \times \pi = 4\pi l/\lambda$, $l/\lambda = .193$ or $l = 1.16$ meters.

the input $Z_i = (.5 - j.7)50 = 25 - j35$, $Z_{max}/Z_0 = SWR = 3.1$

1.20 d normalized $Z = \frac{50+j10}{70} = .714 + j.143$

now we move towards the Source until we reach a position where the real part of the normalized $Z = 1$. At this position we add a series reactance which is the negative of the reactance of the line



From the smith chart we find this position to be $(.143 - .04)\lambda$ or $.103\lambda$. Here $Z = 70 + j28$ hence a capacitive reactance of $28\,\Omega$ would produce a perfect match at this position.

3

$$\gamma = \sqrt{ZY} \qquad Z = R + j\omega L$$

$$= \sqrt{(R+j\omega L)(G+j\omega C)} \qquad Y = G + j\omega C$$

$$= j\sqrt{\omega^2 LC + \frac{G\omega L}{j} + \frac{R\omega C}{j}} \approx j\omega\sqrt{LC}\left(1 + \frac{G}{2j\omega C} + \frac{R}{2j\omega L}\right)$$

$$\alpha + j\beta \approx j\omega\sqrt{LC} + \frac{R}{2Z_0} + \frac{GZ_0}{2} \qquad \text{where } Z_0 = \sqrt{L/C}$$

So $\beta = \omega\sqrt{LC}$

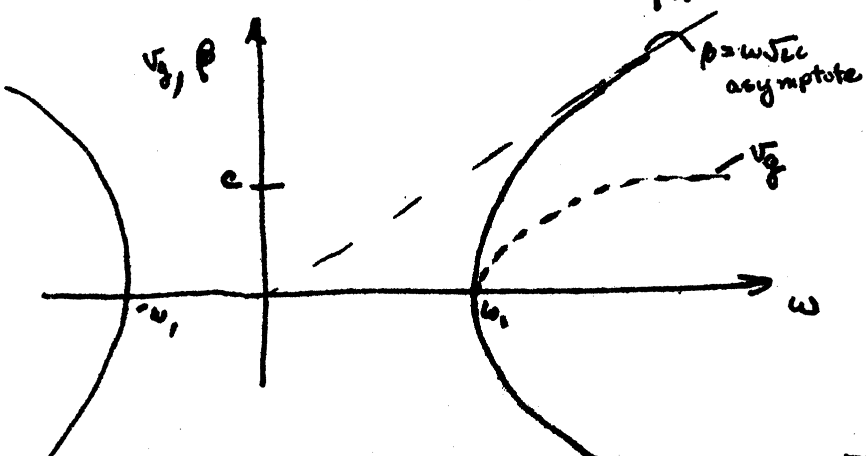
$\alpha = \frac{R}{2Z_0} + \frac{GZ_0}{2}$

4.

(a) $\gamma = \sqrt{ZY} = \sqrt{j\omega L\left(j\omega C + \frac{1}{j\omega L_1}\right)} = \cancel{\sqrt{j\omega C}}$

$$= \sqrt{-\omega^2 LC\left(1 - \frac{1}{\omega^2 LC_1}\right)} = \sqrt{-\omega^2 LC\left(1 - \frac{\omega_1^2}{\omega^2}\right)} \qquad \text{where } \omega_1^2 = \frac{1}{LC_1}$$

So $\beta^2 = \omega^2 LC\left(1 - \omega_1^2/\omega^2\right)$

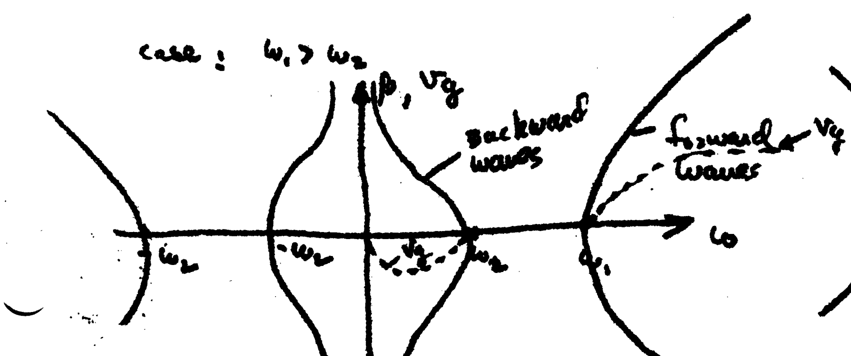


(b) $\gamma = \sqrt{\left(j\omega L + \frac{1}{j\omega C_1}\right)\left(j\omega C + \frac{1}{j\omega L_1}\right)} = \sqrt{j\omega L\left(1 - \frac{\omega_1^2}{\omega^2}\right) j\omega C\left(1 - \frac{\omega_2^2}{\omega^2}\right)}$

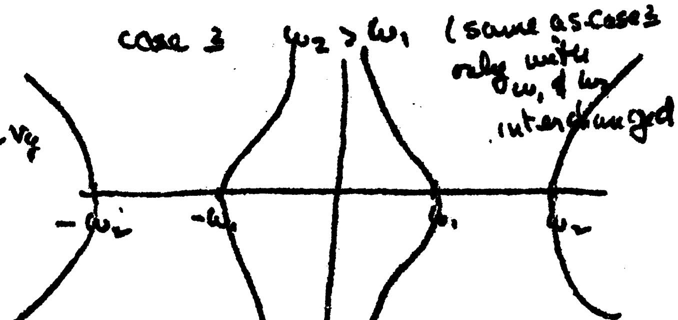
where $\omega_1^2 = \frac{1}{LC_1}$ and $\omega_2^2 = \frac{1}{L_1 C}$

hence $\beta^2 = \omega^2 LC\left(1 - \frac{\omega_1^2}{\omega^2}\right)\left(1 - \omega_2^2/\omega^2\right)$

case 1: $\omega_1 > \omega_2$

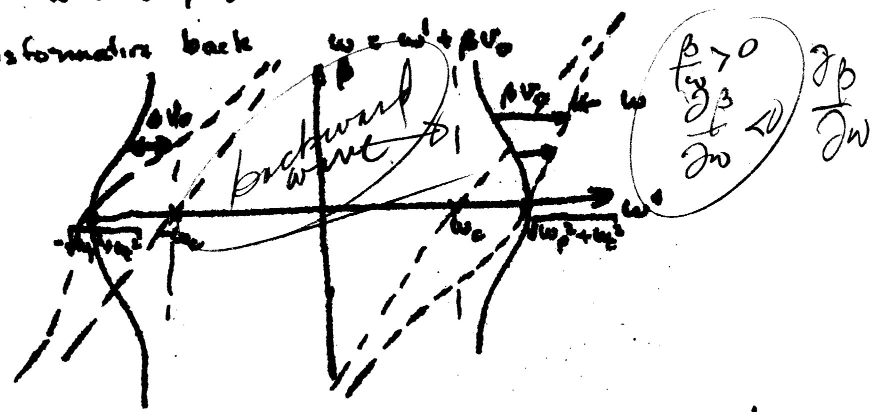


case 3 $\omega_2 > \omega_1$ (same as case 1 only with ω_1 & ω_2 interchanged.)

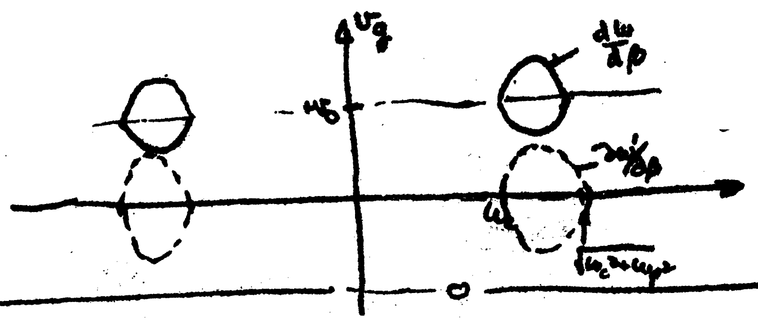


5 $\beta^2 = \beta_0^2\left(\dfrac{\omega_p^2}{(\omega-\beta v_0)^2-\omega_c^2} - 1\right)$

let $\omega' = \omega - \beta v_0$ then plot β vs ω' followed by the transformation back $\omega = \omega' + \beta v_0$



$$v_g = \frac{d\omega}{d\beta} = \frac{\partial \omega'}{\partial \beta}\cdot\frac{\partial \omega}{\partial \omega'} + \frac{\partial \omega}{\partial \beta} = \frac{\partial \omega'}{\partial \beta} + v_0$$



Note that if v_0 is small enough we can retain the backward wave branch.

6

TE$_{01}$ Fields

$H_z = B_z' J_0(k_c r)\, e^{\pm j\beta z}$

$H_r = -E_\varphi / Z_{TE}\; e^{\pm j\beta z}$

$E_\varphi = j\eta \, f/f_c \, B_z \, J_0'(k_c r)\, e^{\pm j\beta z}$

$\Rightarrow$

TE$_{011}$ cavity fields ($0 < z < L$)

$H_z = B_0 J_0(k_c r) \sin \beta z$ ($\beta = \pi/L$)

$E_\varphi = j\eta \, f/f_c \, B_0 J_0'(k_c r) \sin \beta z$

$H_r = -E_\varphi / Z_{TE} \, \cos\beta z / \sin \beta z$

$$Q = \frac{2\omega_0 \langle W_e \rangle}{\langle P_d \rangle}$$

$$\langle P_d \rangle = \tfrac{1}{2} R_s \int_{\text{surface of cavity}} |H_{\tan}|^2 ds = \tfrac{1}{2} R_s \left[\int_{\text{cylindric walls}} |H_z|^2 ds + \int_{\text{end walls}} |H_r|^2 ds \right]$$

c continued

$$\int_{\substack{\text{cylindric}\\ \text{walls}}} |H_z|^2 ds = 2\pi R B_0^2 J_0^2(k_c R) \int_0^L \sin^2 \beta z \, dz = \pi R L B_0^2 J_0^2(k_c R) \qquad \frac{L}{2}$$

$$\int_{\text{end walls}} |H_r|^2 ds = \frac{4\pi \eta^2 f^2 B_0^2}{Z_{TE}^2 f_c^2} \int_0^R r J_0'^2(k_c r) dr \qquad (\text{note } J_0' = -J_1)$$

$$= \frac{2\pi \eta^2 f^2 B_0^2}{Z_{TE}^2 f_c^2} R^2 \left[J_1^2(k_c R) - \cancel{J_0(k_c R) J_2(k_c R)} \right] \qquad \text{by eq 22 p 214 of text}$$

but $J_2(k_c R) = \frac{2}{k_c R} J_1(k_c R) - J_0(k_c R)$ by eq 16 of p 213.

boundary condition on h

hence

$$\int |H_r|^2 ds = \frac{2\pi \eta^2 f^2}{Z_{TE}^2 f_c^2} B_0^2 R^2 J_0^2(k_c R)$$

so

$$\langle P_d \rangle = \frac{R_s}{2} B_0^2 J_0^2(k_c R) \left[\pi R L + \frac{2\pi R^2 \eta^2 f^2}{Z_{TE}^2 f_c^2} \right]$$

$$\langle W_e \rangle = \frac{1}{4}\epsilon_0 \int |E_\varphi|^2 dv = \frac{\epsilon_0}{4} \frac{\eta^2 f^2}{f_c^2} B_0^2 \int_0^R J_0'^2(k_c r) 2\pi r dr \int_0^L \sin^2 \beta z \, dz \qquad \beta = \pi/L$$

thus

$$\langle W_e \rangle = \frac{\epsilon_0}{4} \frac{\eta^2 f^2}{f_c^2} B_0^2 \left[\left(\frac{R^2}{2} J_0^2(k_c R) 2\pi \right) (L/2) \right].$$

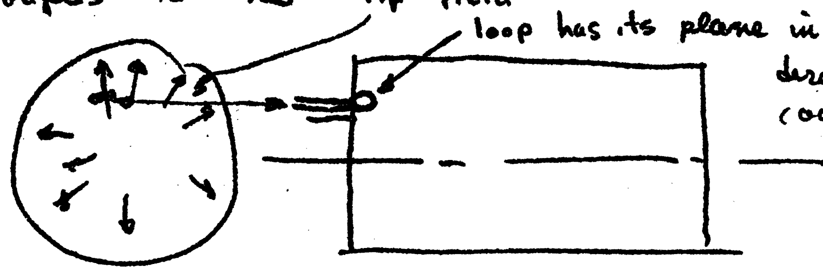
and

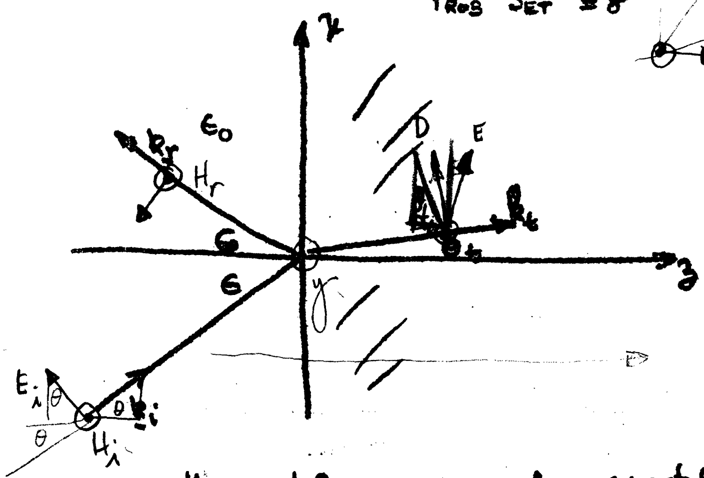
$$Q = \frac{2\omega_0 \frac{\eta^2 f^2}{f_c^2} \frac{\epsilon_0}{4}}{R_s/2} \cdot \frac{\left(\frac{\pi R^2 L}{2} \right)}{\left(\pi R L + \frac{2\pi R^2 \eta^2 f^2}{Z_{TE}^2 f_c^2} \right)} = \frac{\omega_0 \epsilon_0 / R_s}{\left(\frac{2}{R} \frac{f_c^2}{f^2 \eta^2} + \frac{4}{L Z_{TE}^2} \right)}$$

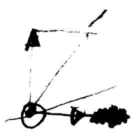
with $Z_{TE}^2 = \eta^2 / \left(1 - \frac{f_c^2}{f^2}\right)$ and $R_s = \frac{1}{\sigma \delta}$, $\delta = \sqrt{\frac{2}{\omega \mu_0 \sigma}}$

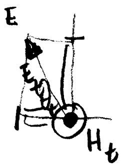
Coupling is best accomplished using a loop in the end wall which couples to the H_r field

loop has its plane in the azimuthal (ϕ) direction for maximum coupling with H_r.









$$\underline{\underline{\epsilon}} = \begin{bmatrix}\epsilon_\perp & 0 & 0\\ 0 & \epsilon_\perp & 0\\ 0 & 0 & \epsilon_{33}\end{bmatrix} = \epsilon_0\begin{bmatrix}K_\perp & 0 & 0\\ 0 & K_\perp & 0\\ 0 & 0 & K_{33}\end{bmatrix}$$

the problem proceeds exactly as with isotropic materials but the propagation constant k_t may depend upon θ_t and hence upon θ (this depends upon the polarization of the incident wave)

done in terms of H_t

a

$$\underline{H}_i = H_+ \underline{a}_y e^{-jk_0\sin\theta x - jk_0\cos\theta z}$$

$$\underline{E}_i = \sqrt{\frac{\mu_0}{\epsilon_0}} H_+ [\underline{a}_x\cos\theta - \underline{a}_z\sin\theta] e^{-jk_0\sin\theta x - jk_0\cos\theta z}$$

$$\underline{H}_r = H_- \underline{a}_y e^{-jk_0\sin\theta x + jk_0\cos\theta z}$$

$$\underline{E}_r = \sqrt{\frac{\mu_0}{\epsilon_0}} H_- [-\underline{a}_x\cos\theta - \underline{a}_z\sin\theta] e^{-jk_0\sin\theta x + jk_0\cos\theta z}$$

and the transmitted fields are

break H into components $\perp$ to optical axis then multiply by Z

$$\underline{H}_t = H_t \underline{a}_y e^{-jk_t\sin\theta_t x - jk_t\cos\theta_t z}$$

$$\underline{E}_t = H_t [Z_\perp \underline{a}_x\cos\theta_t - Z_3\underline{a}_z\sin\theta_t] e^{-jk_t\sin\theta_t x - jk_t\cos\theta_t z}$$

now we can see that I have made provision for a new twist, the wave impedance differs in the two directions. This can be seen from Maxwell's equation

$$-j\underline{k}_t\times\underline{H} = j\omega\underline{D} = j\omega[\epsilon_\perp E_x\underline{a}_x + \epsilon_{33}E_z\underline{a}_z] = +jk_t H_t\begin{bmatrix}\underline{a}_x\cos\theta_t \\ -\underline{a}_z\sin\theta_t\end{bmatrix}$$

so we see (no J) $\quad Z_\perp = \left(\frac{E_t}{H_t}\right)_x = \frac{k_t}{\omega\epsilon_\perp}, \quad Z_3 = \frac{k_t}{\omega\epsilon_{33}}$

Now we use boundary conditions to determine the fields in terms of the incident wave.

① $H_+ + H_- = H_t$

② $\cos\theta\sqrt{\frac{\mu_0}{\epsilon_0}}(H_+ - H_-) = Z_\perp H_t\cos\theta_t$

$D_{(1)} = D_{(2)}$

③ $-\epsilon_0\sqrt{\frac{\mu_0}{\epsilon_0}}(H_- + H_+)\sin\theta = -\epsilon_{33}Z_3 H_t\sin\theta_t$

④ also $k_0\sin\theta = k_t\sin\theta_t$

③ & ④ lead to each other so ③ is redundant!

$$\frac{E_t}{H_t\cos\theta_t} = Z_\perp \qquad \omega\epsilon_\perp E_x = k_t H_t\cos\theta_t$$

using the def. $\rho_{(H)} = H_- / H_+$, $\tau_{(H)} = H_t / H_+$ ① & ② lead to

$$1 + \rho_H = \tau_H \qquad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$$1 - \rho_H = Z_\perp \cos\theta_t / \eta_0 \cos\theta \; \tau_H$$

thus $$\rho_H = \frac{-Z_\perp \cos\theta_t + \eta_0 \cos\theta}{Z_\perp \cos\theta_t + \eta_0 \cos\theta} \; ; \; \tau_H = \frac{2\eta_0 \cos\theta}{\eta_0 \cos\theta + Z_\perp \cos\theta_t}$$

Thus $$\hat{\underline{H}}_r = -\frac{Z_\perp \cos\theta_t + \eta_0 \cos\theta}{Z_\perp \cos\theta_t + \eta_0 \cos\theta} H_+ \underline{a}_y \left[e^{-jk_0(\sin\theta x - \cos\theta z)} \right]$$

$$\hat{\underline{E}}_r = \eta_0 H_+ \rho_H \left[-\underline{a}_x \cos\theta - \underline{a}_z \sin\theta \right] \left[\quad " \quad \right]$$

⑤ $$\hat{\underline{H}}_t = 2\eta_0 \cos\theta / (\eta_0 \cos\theta + Z_\perp \cos\theta_t) \; H_+ \underline{a}_y \left[e^{-jk_t[\sin\theta_t x + \cos\theta_t z]} \right]$$

⑥ $$\hat{\underline{E}}_t = \frac{2\eta_0 \cos\theta}{\eta_0 \cos\theta + Z_\perp \cos\theta_t} H_+ \left[Z_\perp \cos\theta_t \, \underline{a}_x - Z_z \sin\theta_t \underline{a}_z \right] \left[\quad " \quad \right]$$

but we still must determine k_t (and from ④ $\sin\theta_t = \frac{k_0 \sin\theta}{k_t}$) for a fully specified problem solution! This can be found from $\nabla \times \underline{E}_t = -j\omega\mu_0 \underline{H}_t$ (since E_t was found from $\nabla \times H_t$ on p. 1 this is equivalent to using the wave equation to find k_t — the dispersion relation will fall out but now θ_t is specified)

⑦ $$\cancel{-j}\, \underline{k}_t \times \hat{\underline{E}}_t = \cancel{-j}\omega\mu_0 \hat{\underline{H}}_t$$

$$\underline{k}_t = k_t(\underline{a}_x \sin\theta_t + \underline{a}_z \cos\theta_t) = k_x \underline{a}_x + k_z \underline{a}_z$$

$$\underline{k}_t \times \underline{E}_t = \begin{bmatrix} \underline{a}_x & \underline{a}_y & \underline{a}_z \\ k_x & \cancel{k_y}^{0} & k_z \\ E_x & \cancel{E_y}^{0} & E_z \end{bmatrix} = \underline{a}_y (k_z E_x - k_x E_z) + \cancel{\underline{a}_z(0)} + \cancel{\underline{a}_y(0)}$$

but $E_z = -H_t Z_z \sin\theta_t$

and $E_x = H_t Z_\perp \cos\theta_t$ thus ⑦ becomes

⑧ $$H_t k_t (Z_\perp \cos^2\theta_t + Z_z \sin^2\theta_t) = \omega\mu_0 H_t$$

so $$k_t = \frac{\omega\mu_0}{(Z_z \sin^2\theta_t + Z_\perp \cos^2\theta_t)} = k_0 \sin\theta / \sin\theta_t \leftarrow \text{from ④}$$

now using the definitions of $Z_\perp$, Z_z

we have $k_t^2 = \dfrac{\omega^2\mu_0\epsilon_0}{\dfrac{\sin^2\theta_t}{K_{33}} + \dfrac{\cos^2\theta_t}{K_\perp}}$ which is the dispersion relation for the extraordinary wave

but we have $k_0^2\sin^2\theta = k_t^2\sin^2\theta_t$, $\cos^2\theta_t = 1-\sin^2\theta_t = 1-\dfrac{k_t^2}{k_0^2}\sin^2\theta$

with these substitutions one can determine k_t in terms of the incident $\not\theta$, K_{33} and $K_\perp$!

---

b E field $\perp$ plane of incidence — here we will find that the ordinary wave is launched since only one component of $\underline{\underline{\epsilon}}$ is needed ($\epsilon_\perp$)

$$\underline{E_i} = E_+ \underline{a_y}\, e^{-jk_0\sin\theta\, x - jk_0\cos\theta\, z}$$

$$\underline{H_i} = \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_+\left[-\underline{a_x}\cos\theta + \underline{a_z}\sin\theta\right] e^{-(\qquad)}$$

$$\underline{E_r} = E_- \underline{a_y}\, e^{-jk_0\sin\theta\, x + jk_0\cos\theta\, z}$$

$$\underline{H_r} = \sqrt{\frac{\epsilon_0}{\mu_0}}\, E_-\left[\underline{a_x}\cos\theta + a_z\sin\theta\right] e^{-jk_0\sin\theta\, x + jk_0\cos\theta\, z}$$

$$\underline{E_t} = E_t \underline{a_y}\, e^{-jk_t\sin\theta_t\, x - jk_t\cos\theta_t\, z}$$

$$\underline{H_t} = \frac{-j\underline{k_t}\times\underline{E_t}}{-j\omega\mu_0} = \frac{k_t E_t}{\omega\mu_0}\left[-\cos\theta_t\,\underline{a_x} + \sin\theta_t\, a_z\right]$$

now matching E_{tan} and H_{tan} we get

$$E_+ + E_- = E_t$$

$$\frac{1}{\eta_0}(-E_+ + E_-)\cos\theta = -\frac{k_t}{\omega\mu_0}\cos\theta_t\, E_t \qquad \text{with } \rho = E_-/E_+,\quad \tau = E_t/E_+$$

$$1+\rho = \tau$$

$$1-\rho = \tau\,\frac{\eta_0 k_t}{\omega\mu_0}\,\frac{\cos\theta_t}{\cos\theta} \qquad \text{thus } \rho = \frac{\frac{\omega\mu_0}{k_t}\cos\theta - \eta_0\cos\theta_t}{\frac{\omega\mu_0}{k_t}\cos\theta + \eta_0\cos\theta_t}\ ;\ k_t\sin\theta_t = k_0\sin\theta$$

$$\tau = \frac{2\,\frac{\omega\mu_0}{k_t}\cos\theta}{\frac{\omega\mu_0}{k_t}\cos\theta + \eta_0\cos\theta_t}$$

Now we must find k_t, as before we use $\nabla \times \underline{H}_t = j\omega \underline{\epsilon} \cdot \underline{E} = j\omega\epsilon_0 \kappa_\perp E_y \underline{a_y}$

$$-j\underline{k}_t \times \underline{H}_t = \cancel{j}\omega\epsilon_0 \kappa_\perp E_t \underline{a_y}$$

$$-(\underline{k}_t \times \underline{H}_t) = \begin{bmatrix} a_x & a_y & a_z \\ H_{tx} & 0 & H_{tz} \\ k_t \sin\theta_t & 0 & k_t\cos\theta_t \end{bmatrix} = \underline{a_y}\left(H_{tz} k_t \sin\theta_t - H_{tx} k_t \cos\theta_t\right) = \cancel{j}\omega\epsilon_0 \kappa_\perp E_t \underline{a_y}$$

but $H_{tz} = \frac{k_t E_t}{\omega\mu_0}\sin\theta_t$; $H_{tx} = -\frac{k_t E_t}{\omega\mu_0}\cos\theta_t$ so finally we have

$$k_t^2 E_t \left(\frac{\sin^2\theta_t}{\omega\mu_0} + \frac{\cos^2\theta_t}{\omega\mu_0}\right) = \omega\epsilon_0 \kappa_\perp E_t$$

or $k_t^2 = \omega^2\mu_0\epsilon_0 \kappa_\perp$ ← the ordinary wave dispersion relation.

note also $\omega\mu_0/k_t = \sqrt{\frac{\mu_0}{\epsilon_0}}\frac{1}{\sqrt{\kappa_\perp}} = \eta_0/\sqrt{\kappa_\perp}$

Excite the waves by using an E-field with both polarizations (all three field components are non-zero.

---

2 6.03b

$$E_x = E_1 \cos\omega(t - z/v) = Re\left\{E_1 e^{j\omega t} e^{-jkz}\right\} \qquad k = \omega/v$$

$$E_y = E_2 \cos[\omega(t - z/v) + \psi] = Re\left\{E_2 e^{j\omega t} e^{-jkz + j\psi}\right\}$$

$$\hat{\underline{E}} = a_x E_1 e^{-jkz} + a_y E_2 e^{-jkz + j\psi} = (\underline{a_x} E_1 + \underline{a_y} E_2 e^{j\psi})e^{-jkz}$$

$$= \underline{a_x} E_1 + \underline{a_y} E_2 \cos\psi + a_y j E_2 \sin\psi] e^{-jkz} = (\underline{E_a} + j\underline{E_b})e^{-jkz}$$

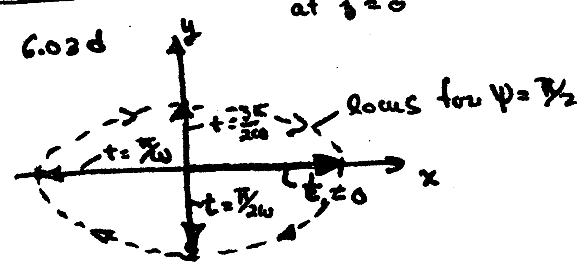
where $\underline{E_a} = \underline{a_x} E_1 + \underline{a_y} E_2 \cos\psi$

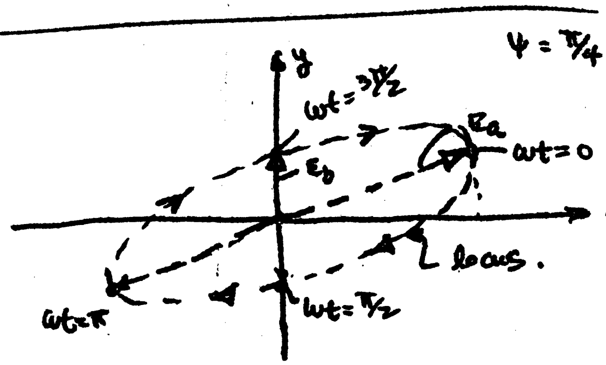
$\underline{E_b} = \underline{a_y} E_2 \sin\psi$.

---

at $z = 0$

3 6.03d





$E_1 = 1$, $E_2 = 1/2$, $\psi = \pi/4$

$\underline{E_a} = a_x + .35 a_y$

$\underline{E_b} = .35\, a_y$

4 9.12b

with $N = 10^{16}$, $\omega_p^2 = \frac{Ne^2}{m\epsilon_0} = \frac{10^{16}\times(1.6\times10^{-19})^2}{9\times10^{-31}\times\frac{1}{36\pi}\times10^{-9}} = 4\pi\times2.56\times10^{18}$

$\omega_p^2 = 3.22\times10^{19}$

$\omega_c = \frac{eB}{m} = \frac{1.6\times10^{-19}}{9\times10^{-31}}\times B$ (B in webers/m²) $= B_{gauss}\times10^{-4}$

$= 1.78\times10^{11}\times10^{-4}\,B_{gauss} = 1.78\times10^{7}\,B_g.$

$\theta = \left(\frac{\beta_{ccw}-\beta_{cw}}{2}\right)z =$

eq. 9.12.6

$\theta/z = \frac{\beta_{ccw}-\beta_{cw}}{2} = \frac{\omega\sqrt{\mu_0\epsilon_0}}{2}\left[\left(1-\frac{\omega_p^2/\omega}{\omega_c+\omega}\right)^{1/2}-\left(1+\frac{\omega_p^2/\omega}{\omega_c-\omega}\right)^{1/2}\right]$

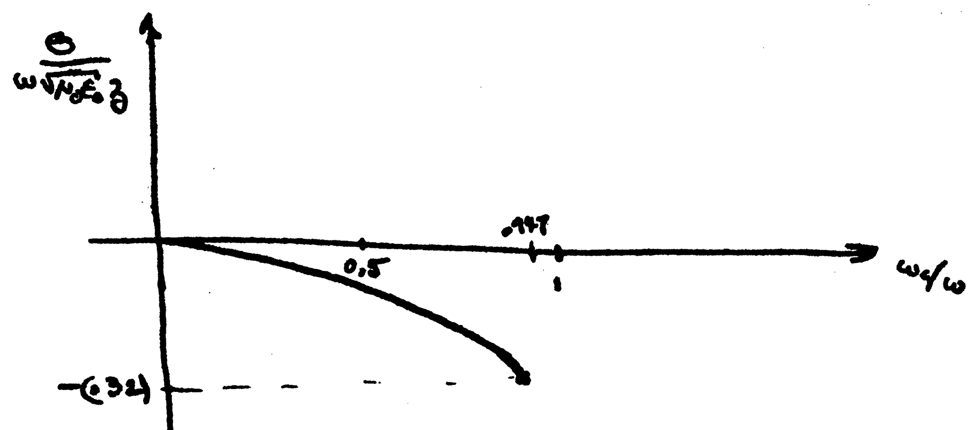
using Eqs. 9.11.4, 9.11.7

$$\theta/z = \frac{\omega\sqrt{\mu_0\epsilon_0}}{2}\left\{\left[1-\frac{\omega_p^2/\omega^2}{\frac{\omega_c}{\omega}+1}\right]^{1/2}-\left[1+\frac{\omega_p^2/\omega^2}{\frac{\omega_c}{\omega}-1}\right]^{1/2}\right\}$$

with $f = 3\times10^{9}$, $\omega = 1.88\times10^{10}$ $\omega_p^2/\omega^2 = \frac{3.22\times10^{19}}{3.55\times10^{20}} = .0907 \ll 1$

$0 < \frac{\omega_c}{\omega} < \frac{1.78\times10^{10}}{1.88\times10^{10}} = .947$, so $(\omega_c/\omega - 1)_{min} = .053$, $\left(\frac{\omega_p^2/\omega^2}{\omega_c/\omega-1}\right)_{max} \approx 1.71$

and $(\omega_c/\omega+1)_{min} = 1.947$ we can neglect $\frac{\omega_p^2/\omega^2}{\omega_c/\omega+1}$ compared with 1 in the first factor



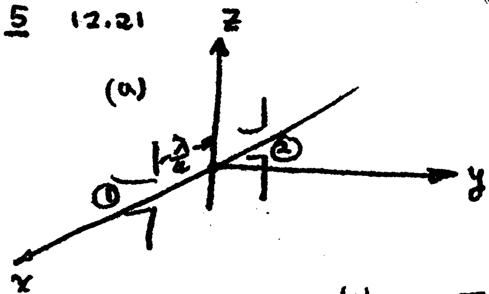
0

5

$k = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$

5 12.21 ⑥

(a)



$$E_{\theta_1} = F(\theta)\, e^{+jk\frac{\lambda}{4}\underline{a_r}\cdot \underline{a_x}} = F(\theta)\, e^{+j\frac{\pi}{2}\cos\phi\sin\theta}$$

$$E_{\theta_2} = F(\theta)\, e^{-jk\frac{\lambda}{4}\underline{a_r}\cdot \underline{a_x}} = F(\theta)\, e^{-j\frac{\pi}{2}\cos\phi\sin\theta}$$

where $F(\theta) = \frac{60 I_m}{r} \frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}$ → the $\lambda/2$ dipole pattern

thus $E_\theta = E_{\theta_1} + E_{\theta_2} = 2F(\theta)\cos(\frac{\pi}{2}\cos\phi\sin\theta) = G(\theta,\phi)$

So the vertical pattern for $\phi = \pi/2$ is as in fig 12.06

the horizontal pattern for $\theta = \pi/2$ is shown in fig a below

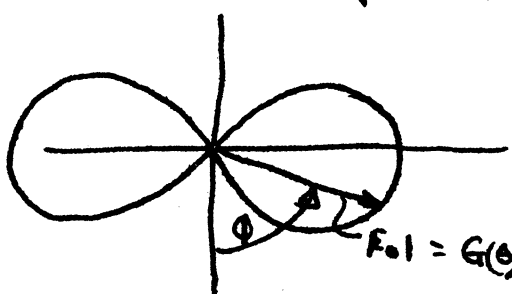


Fig a

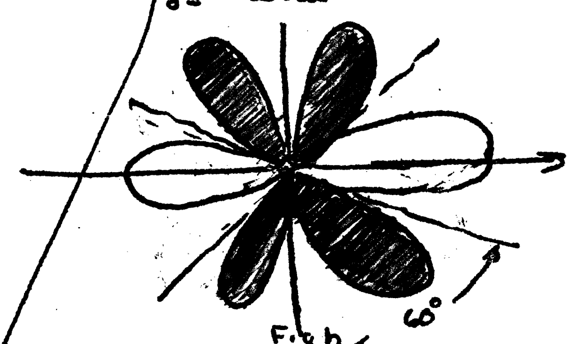
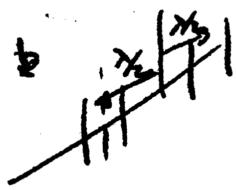


Fig b

b



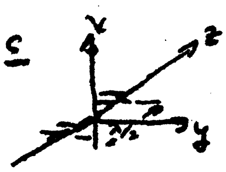
consider this array as composed of two arrays as in part a whose centers are $\lambda/2$ from origin.

define $G(\theta,\phi)$ as above then

$$E_\theta = G(\theta,\phi)\left[e^{+jk\underline{a_r}\cdot \underline{a_x}\lambda/2} + e^{-jk\lambda/2\,\underline{a_r}\cdot \underline{a_x}}\right]$$

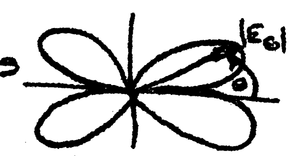
$$= 2\,G(\theta,\phi)\cos(\pi\cos\phi\sin\theta)$$ shown in Fig b for $\theta = \pi/2$

c

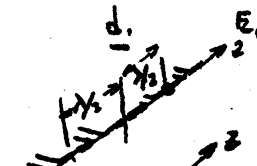


$$E_{\theta_t} = F(\theta)\left[e^{+jk\lambda/4\,\underline{a_z}\cdot\underline{a_r}} + e^{-jk\lambda/4\,\underline{a_z}\cdot\underline{a_r}}\right]$$ $\underline{a_z}\cdot\underline{a_r} = \cos\theta$

$$= 2F(\theta)\cos(\tfrac{\pi}{2}\cos\theta) = 120\frac{I_m}{r}\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}$$ →

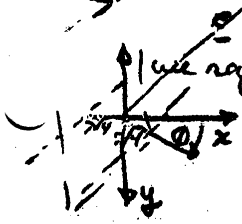


d.



$$E_\theta = 2F(\theta)\cos(\tfrac{\pi}{2}\cos\theta)\left[e^{jk\lambda/2\,\underline{a_z}\cdot\underline{a_r}} + e^{-jk\lambda/2\,\underline{a_z}\cdot\underline{a_r}}\right]$$

$$= 4F(\theta)\cos(\tfrac{\pi}{2}\cos\theta)\cos(\pi\cos\theta)$$ →



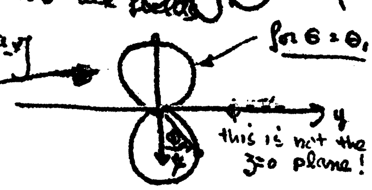
In order to have $E_\theta = 0$ on conducting sheet we require an image set of dipoles located behind sheet $\lambda/4$ now for all fields

sign to set $E_\theta = 0$ on conductor

$$E_\theta = 4F(\theta)\cos(\tfrac{\pi}{2}\cos\theta)\cos(\pi\cos\theta)\left[e^{jk\lambda/4\,\underline{a_r}\cdot\underline{a_x}} - e^{-jk\lambda/4\,\underline{a_r}\cdot\underline{a_x}}\right]$$

$$= j8F(\theta)\cos(\tfrac{\pi}{2}\cos\theta)\cos(\pi\cos\theta)\sin(\tfrac{\pi}{2}\cos\phi\sin\theta)$$

for $\theta = \theta_1$



this is not the $z = 0$ plane!