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Transient analysis

Initial Conditions Before we can solve transient problems involving inductors and capacitors we must understand the initial conditions that apply to the differential equations:

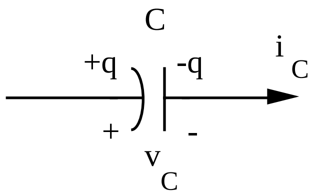
Because no circuit can supply infinite power:
(1) the current through an inductor cannot change instantaneously
(2) the voltage across a capacitor cannot change instantaneously

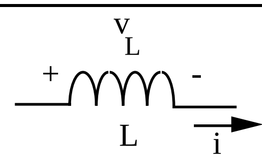
These rules are the boundary conditions which apply to inductors and capacitors and, for purposes of writing time dependent eqpressions for voltage and current, can be written as:

For an inductor
$i(0-)=i(0+)$ where it is assumed that a switch has been opened or closed in the network at $t=0$.

For a capacitor
$v(0-)=v(0+)$ where it is assumed that a switch has been opened or closed in the network at $t=0$.

Basically, this means that for inductors the current is continuous, whereas for capacitors the voltage is continuous.

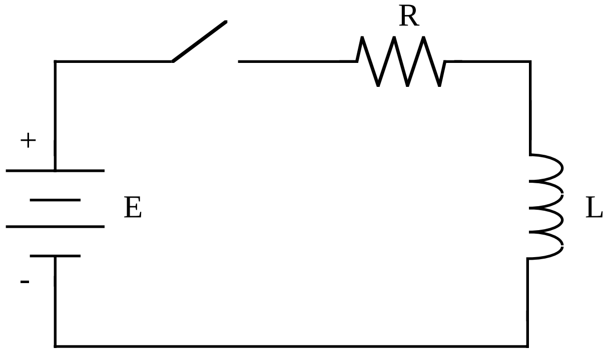
<table>
<tr>
<td>The voltage current relationship for a capacitor cannot be written without using integrals or derivatives.


Using the above definitions we can write "Ohm's Law" for a capacitor as
$i_C = \frac{dq}{dt} = C\frac{dv}{dt}$
or, in integral form, as
$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i_C(t)\, dt$</td>
<td>

For the inductor, "Ohm's Law" is
$v_L = L\frac{di}{dt}$
or, in integral form,
$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v_L(t)\, dt$</td>
</tr>
</table>

A transient is what occurs whenever you open or close a switch in an electrical network. The voltages and currents must quickly re-adjust themselves to the new network. The presence of capacitors and inductors in the network means that that the voltage and current readjustments occur according to the differential equations describing these circuit components.

Consider the simple series circuit shown below:



The question is what is the current in the circuit as a function of time for $t \geq 0$, the switch closes at $t=0$. This requires the use of differential equations. To describe this circuit begin with Kirchoff's voltage law

$$\sum V_n$$

$$E - iR - L\frac{di}{dt} = 0$$

This is a differential equation in i

$$L\frac{di}{dt} + iR = E$$

The homogeneous (or transient) solution comes from solving the equation with the right hand side of the equation set to zero.

$$L\frac{di}{dt} + iR = 0$$

The solution to this equation is an exponential of the form

$$i(t) = Ae^{kt}$$

where A and k are constants to be determined. Substituting this solution into the above differential equation we can find k

$$LAke^{kt} + Ae^{kt}R = 0$$

Dividing through by Ae^{kt}

$$Lk+R=0$$

or

$$k=-R/L$$

The constant A must come from the steady-state solution of the original differential equation. In the steady-state all derivatives are zero since nothing is changing. Therefore, the differential equation for this solution reduces to

$$iR=E$$

or

$$i = \frac{E}{R}$$

The resulting total solution is the sum of the steady-state and transient solutions

$$i(t)=i_{steady\text{-}state} + i_{transient}$$

or

$$i(t) = \frac{E}{R} + Ae^{-\frac{R}{L}t}$$

To find A we must apply the boundary conditions on i, i.e. that

$i(0-)=i(0^+)$. Let $i(0^-)=i_O$. At $t=0^+$

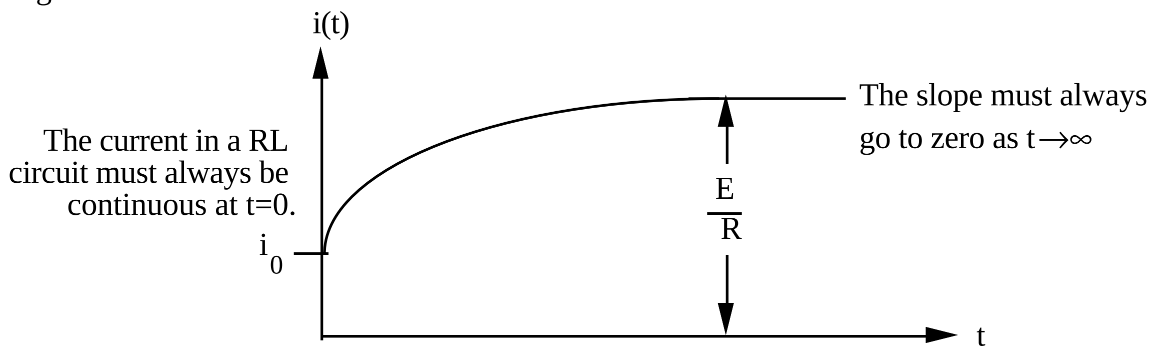
$$i_0 \approx \frac{E}{R} + A$$

$$A = i_0 - \frac{E}{R}$$

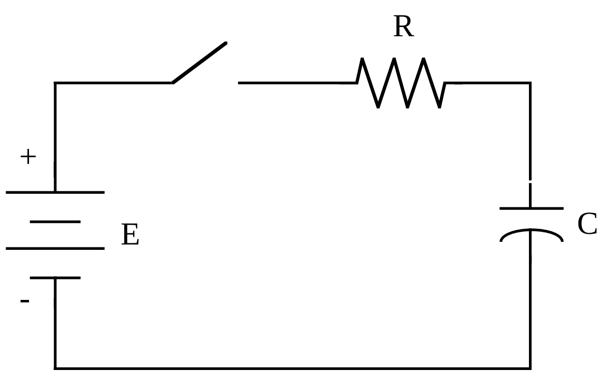
Therefore, the final time dependent expression for i(t) is

$$i(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-\frac{R}{L}t}$$

The factor L/R is called the time constant of the circuit since it determines how rapidly the current changes in the circuit.



For a similar series RC circuit, the objective is to predict the voltage across the capacitor as a function of time given that the capacitor has a voltage v_O across it at time t=0.



Using KVL

$$E - iR - \frac{1}{C}\int_{t_0}^{t} i(t')\ dt' = 0$$

We cannot use i as the variable for capacitors - we must use voltage.

$$v_c(t) = \frac{1}{C}\int_{t_0}^{t} i(t')\ dt'$$

which is the integral form of

$$i = C\frac{dv_c}{dt}$$

The differential equation is then

$$E - R\left(C\frac{dv_c}{dt}\right) - v_c = 0$$

or, rewriting the equation in a more standard form,

$$RC\frac{dv_c}{dt} + v_c = E$$

As before this equation has a homogeneous and a steady-state solution.

For the homogeneous solution:

$$RC\frac{dv_c}{dt} + v_c = 0$$

Letting $v_C(t)=Ae^{kt}$

$RCAke^{kt} + Ae^{kt} = 0$

or, solving for k,

$$k = \frac{-1}{RC}$$

The steady-state solution comes from setting all time derivatives to zero to give

$$v_c = E$$

The total solution is then

$$v_C(t) = v_{steady\text{-}state} + v_{transient}$$

$$v_c(t) = E + Ae^{-\frac{t}{RC}}$$

At t=0, $v_C(t)=v_O$ so

$$v_c(0) = v_0 = E + A$$

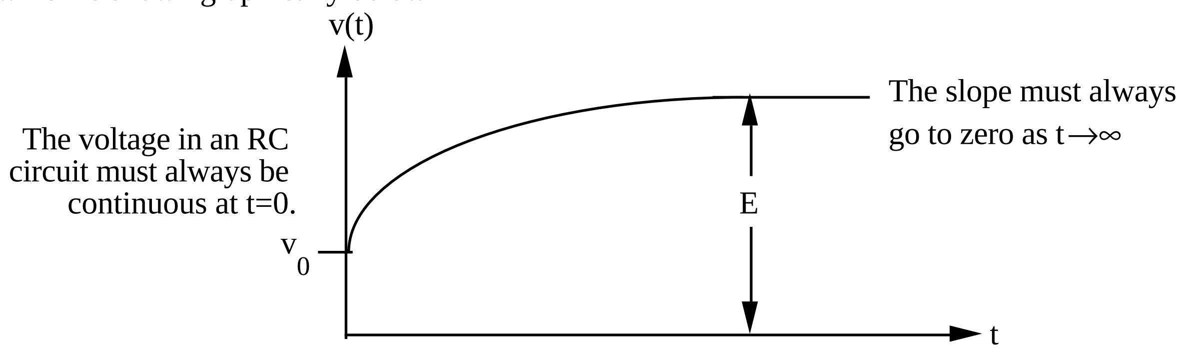
and

$$A = v_O - E$$

The final solution is then

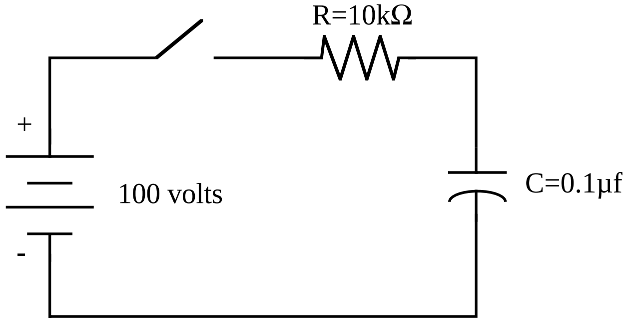
$$v_c(t) = (v_0 - E)e^{-\frac{t}{RC}} + E$$

which is shown graphically below



Example:

Most of the time you do not need any fancy math as shown above to solve the problem. Simply remember the boundary conditions and the general form of the solutions.



Consider the above circuit. The time constant is simply computed as

$$\tau = RC = (10\times10^{-3})(0.1\times10^{-6}) = 10^{-3} \text{ seconds}$$

Assume $v_0=0$ since it was not given. The problem is to determine v across the capacitor at $t=5\times10^{-4}$ seconds. The general form of the solution was shown graphically above. Expressing this general solution mathematically

$$v(t) = E + (v_0 - E)e^{-\frac{t}{RC}}$$

which for the given constants becomes

$$v(t) = 100 + (0-100)e^{-\frac{t}{10^{-3}}}$$

and, at $t=5\times10^{-4}$ seconds, is

$$v(5\times10^{-4}) = 100\left(1 - e^{-\frac{5\times10^{-4}}{10^{-3}}}\right) = 100\left(1 - e^{-\frac{1}{2}}\right)$$

$$v(5\times10^{-4}) \approx 39.35 \text{ volts}$$

Summary of important relationships for solving transient problems.

Component	Initial condition	t=0+	t=∞	τ	Ohm's Law
Capacitor	V(0-)=V(0+)	short	open	RC	$I = C\frac{dV}{dt}$ or $V = \frac{1}{C}\int I\,dt$
Inductor	I(0-)=I(0+)	open	short	L/R	$V = L\frac{dI}{dt}$

The differential equation

$$A\frac{dI}{dt} + BI = C$$

has two solutions. One solution is the d.c. (also called steady state solution) which occurs when all the derivatives go to zero. This solution is simply I=C/B. The other solution is the transient solution (also called the homogeneous solution) and requires that C=0, i.e. solve

$$A\frac{dI}{dt} + BI = 0$$

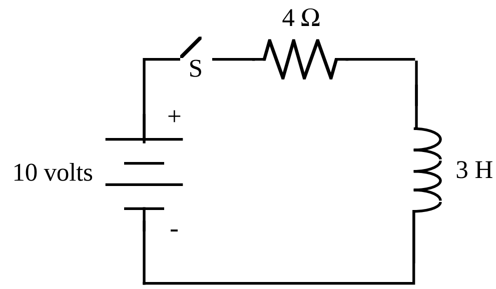
This solution is always of the form $I(t) = Ae^{mt}$ and can be solved for by simply substituting this expression for I(t) into

$$A\frac{dI}{dt} + BI = 0$$

CIRCUITS 31

The switch S closes at T=0. The complete response for i(t) for t>0 is

(a) $2.5 + 6e^{-0.75t}$
(b) $2.5 - 2.5e^{-0.75t}$
(c) $2.5 + 2.5e^{-1.33t}$
(d) 0
(e) $2.5 - 2.5e^{-1.33t}$



We use KVL to write the differential equation for the circuit using the correct expression for the impedance of the inductor.

$$-10 + 4i + 3\frac{di}{dt} = 0$$

Re-writing this in conventional form with the sources on the right hand side of the equation

$$3\frac{di}{dt} + 4i = 10$$

The dc (or homogeneous) solution is obtained by setting the derivatives equal to zero or, in this case

$$4i = 10$$

giving i=2.5 amps.

The transient solution is always an exponential in form. Substituting $i(t)=Ae^{kt}$ into the differential equation and setting the source (the right hand side of the equation) equal to zero we obtain

$$3\frac{di}{dt} + 4i = 0$$

$$3kAe^{kt} + 4Ae^{kt} = 0$$

$$3k+4 = 0$$

$$k=-4/3$$

The total solution is then $i(t) = i_{transient} + i_{homogeneous} = Ae^{-1.33t} + 2.5$

The coefficient A is solved for by using the boundary condition that $i(0^+)=i(0^-)=0$.

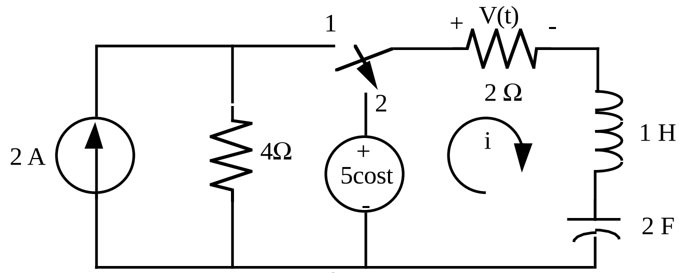
This requires that $i(0^+) = A + 2.5 = 0$, or that A=-2.5.

Then $i(t) = 2.5 - 2.5e^{-1.33t}$ and the correct answer is (e).

Circuits 32

The form of the transient part of V(t) for t>0 is

(a) $(A_1+A_2)e^{-0.5t}$
(b) $A_1\cos(0.5t) + A_2\sin(0.5t)$
(c) $A_1e^{-t}\cos(0.5t) + A_2e^{-t}\sin(0.5t)$
(d) $A_1e^{-1.71t} + A_2e^{-0.29t}$
(e) 0



At t=0 the switch moves from 1 to 2.

Solution:
For t>0 the equation of the circuit is

$$\frac{di}{dt} + 2i + \frac{1}{2}\int_0^t i(\alpha)d\alpha + V_C(0^+) = 5\cos(t)$$

where $V_c(0^+)$, the initial voltage on the capacitor, is zero.
Differentiating the above equation to remove the integral

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + \frac{1}{2}i(t) = -5\sin(t)$$

The left hand side of this equation describes the transient response. For the transient

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + \frac{1}{2}i(t) = 0$$

If we let $i(t)=Ae^{mt}$ we get

$$m^2Ae^{mt} + 2mAe^{mt} + \frac{1}{2}Ae^{mt} = 0$$

which reduces to the characteristic equation

$$m^2 + 2m + 0.5 = 0$$

This equation can be solved using the quadratic formula

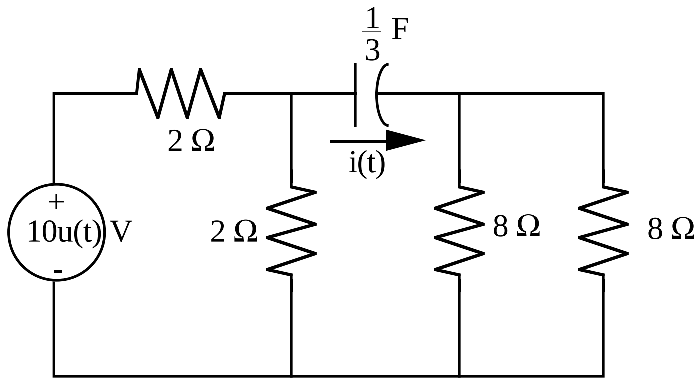
$$m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(0.5)}}{2} = -1.71 \text{ and } -0.29$$

The only answer with these exponents is (d).

Circuits 36

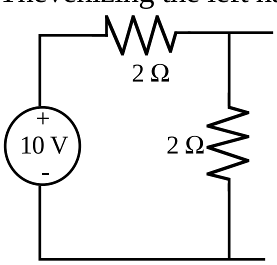
i(t) is

(a) $e^{-0.6t}$
(b) $e^{-1.67t}$
(c) $-e^{-1.67t}$
(d) $-e^{-0.6t}$
(e) $2-e^{-1.67t}$



Solution:
There are many ways to solve this problem but, perhaps, the easiest way is to Thevenize the left hand side of the circuit (the voltage source and the two 2Ω resistors) and replace the right hand side of the circuit (the two 8Ω resistors in parallel) by its equivalent resistance.

Thevenizing the left hand side of the circuit



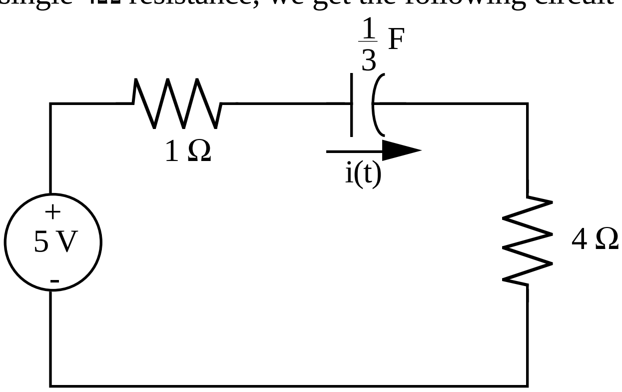
$$V_T = \frac{2}{2+2} 10 = 5 \text{ volts}$$

and

$$R_T = \frac{2 \times 2}{2+2} = 1\ \Omega$$

NOTE: This is a good technique to use to get rid of current sources in problems.

The 8 Ω resistors in parallel can be replaced by a 4 Ω resistor. Redrawing the original circuit and replaceing the left hand side by its Thevenin equivalent and replacing the two 8Ω resistors by a single 4Ω resistance, we get the following circuit



Since $V_C(0^+)=V_C(0^-)=0$

$$i_C(0^+) = \frac{5 \text{ volts}}{1\Omega + 4\Omega} = 1 \text{ amp}$$

The time constant for the circuit can be directly computed as

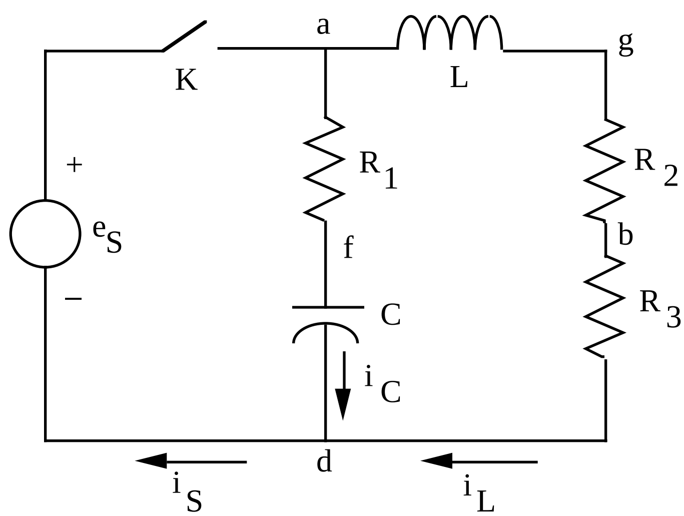
$$\tau = RC = (5\ \Omega)(\tfrac{1}{3} \text{ F}) = 1.67 \text{ seconds}$$

The solutions should be of the form

$$Ae^{-\frac{t}{1.67}} = Ae^{0.6t}$$

The correct answer is (a).

Example afternoon problem (this is actually tougher than most afternoon problems):



You are given that $e_S(t)=E+E_1\sin(500t)+E_2\sin(1000t)$, L=10 millihenries, C=200 microfarads, R_1=10 ohms, R_2=5.0 ohms and R_3=5.0 ohms in the above circuit.

For questions 11–14 assume that switch K is closed at t=0 and answer the questions for the instant immediately after the switch is closed, i.e. for time $t=0^+$.

11. If E=30V, E_1=40V and E_2=20V, the current i_C is most nearly
(A) 0.0 amperes
(B) 1.5 amperes
(C) 2.8 amperes
(D) 3.0 amperes
(E) 6.0 amperes

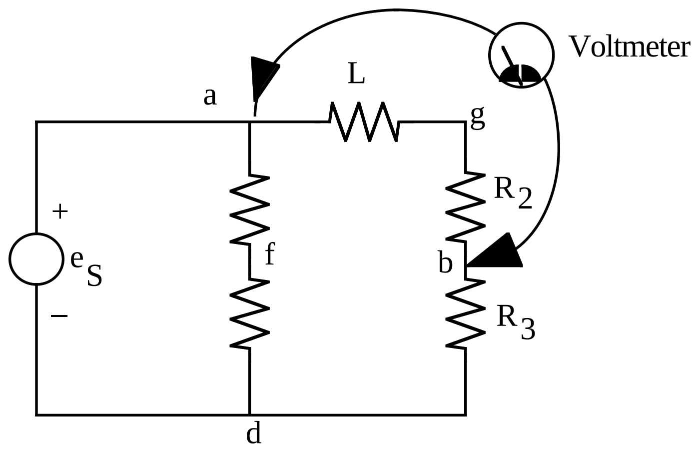
This problem is most easily solved by recalling the initial conditions for capacitors which require that V(0-) = V(0+). The initial voltage on the capacitor is 0 volts so the voltage across the capacitor immediately after the switch is closed must also be 0 volts. The applied voltage $e_S(t=0+) \approx 30$ since $\sin(0+) \approx 0$. At t=0+ e_S appears enitrely across R_1 and the resulting current (which is equal to i_C since R_1 and C are in series) must be given by

$$i_C = \frac{e_S}{R_1} = \frac{30\text{ volts}}{10\Omega} = 3.0\text{ Amperes}$$

The correct answer is (D).

12. If E=30V, E_1=40V and E_2=20V, the magnitude of the voltage between points a and b is most nearly
(A) 5.0 volts
(B) 7.5 volts
(C) 15 volts
(D) 22 volts
(E) 30 volts

The voltage being referred to is across the series combination of the inductor and R_2 as shown in the diagram below.



As in question #11 the voltage $e_S(0^+) \approx 30$ volts, $i_L(0^+)=i_L(0^-)$ since the current through inductors is continuous, and $V_C(0^+)=V_C(0^-)$ since the voltage across capacitors is continuous. Since there is no current flow through L at $t=0^+$ the inductor represents an open circuit. The potential at a is +30 volts; the potential at b is zero since it is connected to ground through R_3 and no current is flowing through R_3. The potential v_{ab} is then 30 volts. The correct answer is (E)

13. If E, E_1 and E_2 are magnitudes such that i_C at $t=0^+$ is 2.0 amperes, the rate of change of the voltage between points f and d is most nearly
(A) 0.0 volts/second
(B) 20 volts/second
(C) 5×10^2 volts/second
(D) 5×10^3 volts/second
(E) 1.0×10^4 volts/second

This question is a lot simpler than it sounds and is a direct application of of the definition of capcitance. By definition,

$$i_C = C\frac{dv}{dt}$$

which includes a direct expression of the rate of change of voltage. Evaluating this expression for $t=0^+$,

$$i_C(0^+) = C\frac{dv}{dt}\bigg|_{0^+}$$

which can be solved for the time rate of change of voltage at $t=0^+$

$$\frac{dv}{dt}\bigg|_{t\,=\,0^+} = \frac{i_C(0^+)}{C} = \frac{2\text{ Amperes}}{200 \times 10^{-6}\text{ Farads}} = 10^4\ \frac{\text{volts}}{\text{second}}$$

The closest answer is (E).

14. If E, E_1 and E_2 are magnitudes such that the voltages between points a and g is 40 volts at $t=0^+$, the rate of change of i_L is most nearly
(A) 0.0 amps/second
(B) 4×10^{-2} amps/second
(C) 4×10^3 amps/second

(D) 2×10^4 amps/second

(E) 4×10^4 amps/second

The solution of this problem is almost identical to that of problem #13 with the exception that the voltage expression must be that for an inductor, i.e.

$V_L = L\,\frac{di}{dt}$

Evaluating this expression at $t=0^+$

$$\left.\frac{di}{dt}\right|_{t\,=\,0^+} = \frac{V_L(0^+)}{L} = \frac{40\text{ volts}}{10\times10^{-3}\text{ Henrys}} = 4000\,\frac{\text{amps}}{\text{second}}$$

The answer is (C).

15. If E=30V, E_1=40V and E_2=20V, the magnitude of the current i_C is most nearly
(A) 0.0 amperes
(B) 1.5 amperes
(C) 2.8 amperes
(D) 3.0 amperes
(E) 6.0 amperes

$e_S(t)=E+E_1\sin(500t)+E_2\sin(1000t)$

$i_C(0^-)=0$ and $i_L(0^-)=0$ since there are no voltage sources for $t<0$. Using our rules for boundary conditions on inductors and capacitors $i_L(0^+)=i_L(0^-)$ but $i_C(0^+)\neq i_C(0^-)$. At t=0+ $e_S(0^+)=30+40\sin(500\times0^+)+20\sin(1000\times0^+) \approx 30$ volts since the sine of a small number is approximately zero. Since $V_C(0^+)=V_C(0^-)$ (remember that the voltage across a capacitor is continuous) the current i_C through R_1 at $t=0^+$ must then be given by i_C = 30volts/10ohms = 3 amperes. The answer is (D).