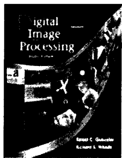

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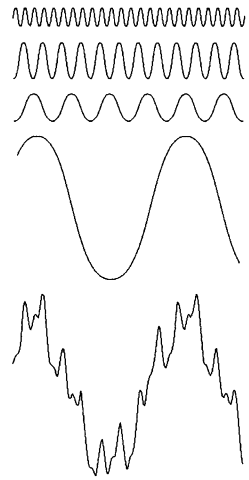


sum of above four functions gives this noisy looking signal.

FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

1-D Fourier transform:

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi ux}\, dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(u)\, e^{j2\pi ux}\, du$$

2-D Fourier transform

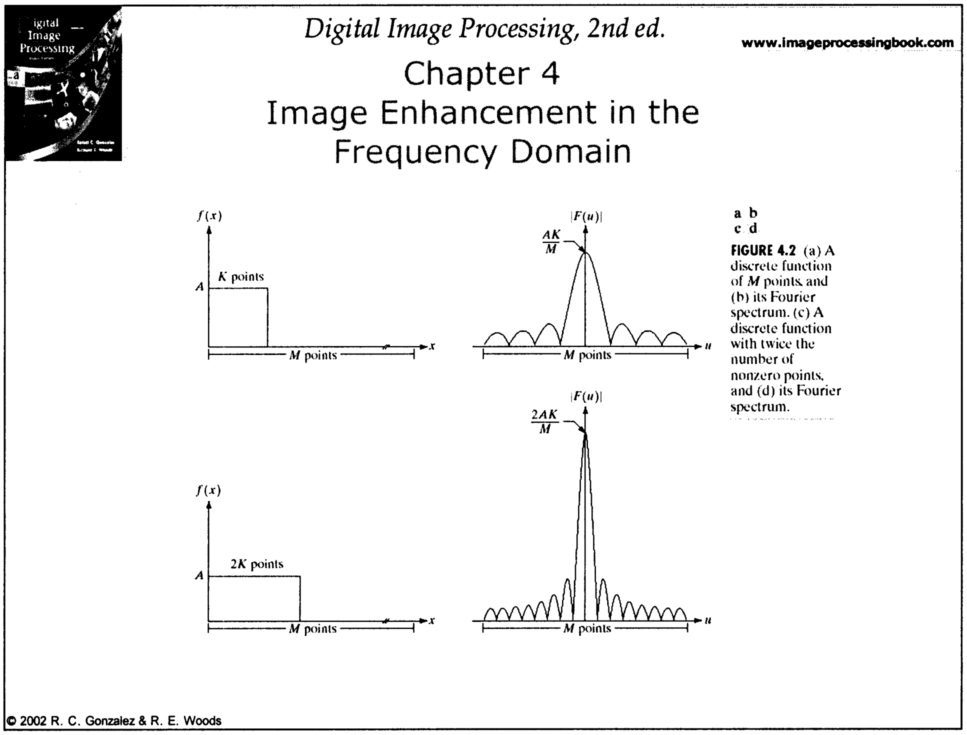
$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)\, e^{+j2\pi(ux+vy)}\, du\, dv$$

Discrete Fourier Transform

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j\frac{2\pi ux}{M}} \qquad u = 0, 1, \ldots, M-1$$

$$f(x) = \sum_{u=0}^{M-1} F(u)\, e^{j\frac{2\pi ux}{M}} \qquad x = 0, 1, \ldots, M-1$$



For above figure $A = 1$

$m = 1024$

$K = 8$

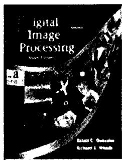
As the width of the function increased by 2 the area under the curve increases by 2 and the peak value increases by 2. Also note that the number of zero's in the transform also increases by 2.

In the future,

$$f(x) \triangleq f(x_0 + x\Delta x)$$

$$F(u) \triangleq F(u\Delta u) \text{ for } u = 0, 1, 2, \ldots, M-1$$

$$\Delta u = \frac{1}{m\Delta x}$$



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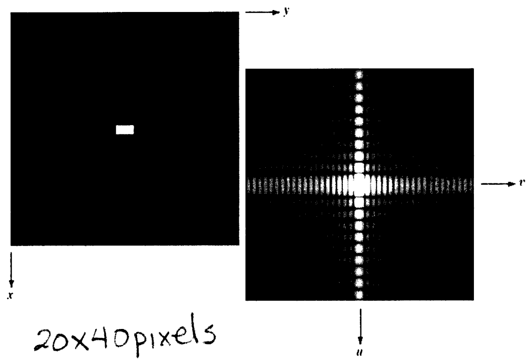


a b
FIGURE 4.3 (a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

20x40pixels

centered
probably $|F(u,v)|$
$0.5\log(1+r)$ transform

Note twice as many zeros in u as compared to v.

2-D DFT

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{+j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Fourier spectrum $\quad |F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$

phase angle $\quad \phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

power spectra $\quad P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$

Multiply input image by $(-1)^{x+y}$ to center transform.

$$\mathfrak{F}\left[f(x,y)(-1)^{x+y}\right] = F\left(u-\frac{M}{2},\, v-\frac{N}{2}\right)$$

i.e. $F(0,0)$ is located at $u=\frac{M}{2}$, $v=\frac{N}{2}$

GWE 4.1

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

where u,v are frequency variables

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} \underbrace{F(u,v)}_{\text{Fourier coefficients}}\, e^{+j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

If $f(x,y)$ is real $F(u,v) = F^*(-u,-v)$

$$\Rightarrow |F(u,v)| = |F(-u,-v)|$$

By substitution

$$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$$

$\Rightarrow$ infinitely periodic in u & v

From the inverse DFT $f(x,y)$ is periodic in x and y.

$$f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,v+N)$$

Some properties of the 2-D Fourier Transform

Translation

$$f(x,y)\, e^{j2\pi\left(\frac{u_0 x}{M} + \frac{v_0 y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)\, e^{-j2\pi\left(\frac{u_0 x}{M} + \frac{v_0 y}{N}\right)}$$

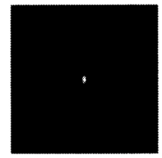
incidentally if we put $u_0 = \frac{M}{2}$ and $v_0 = \frac{N}{2}$ we get $F(u-\frac{M}{2}, v-\frac{N}{2})$ and

$$e^{j2\pi\left(\frac{u_0 x}{M} + \frac{v_0 y}{N}\right)} = e^{j2\pi\left(\frac{x}{2} + \frac{y}{2}\right)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

MATLAB/Image Processing Toolbox

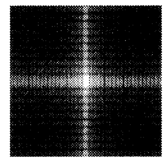
MATLAB Fourier transforms

```
>> f=imread('Figure_Rectangle.jpg');  % load in spatial rectangle
>> F=fft2(f);                          % do 2D FFT
>> S=abs(F);                           % determine magnitude for disp
>> imshow(S, [ ])                      % shows in four corners of displa
>> Fc=fftshift(F);                     % shift FFT to center
>> imshow(abs(Fc), [ ]);               % show magnitude of FFT in cen
```



```
% much tougher to do display transform
>> g=im2unit8(mat2gray(log(1+double(f))));
>> imshow(g)
% double converts the image to double precision floating point
% mat2gray brings the values to the range [0,1]
% im2unit8 brings the values back to the range [0,255]


% general log transform
>> g=im2unit8(mat2gray(c*log(1+double(f))));
```



SEE GWE, Section 4.2 Computing and Visualizing the 2-D DFT in MATLAB
GWE, Section 3.2.2 Logarithmic and Contrast Stretching Transformations

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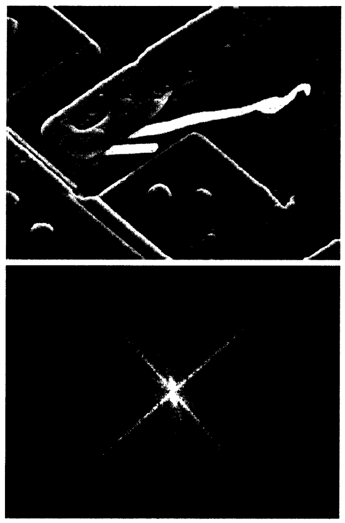

a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

value $F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$ which is the average image intensity

Note that $|F(u,v)| = |F(-u,-v)|$

In this figure the strong axes of $|F(u,v)|$ correspond to the approximate $\pm 45^\circ$ edges in the SEM image.

The angle of the white oxide extrusions wrt horizontal roughly corresponds to the short white off-center line in the transform.

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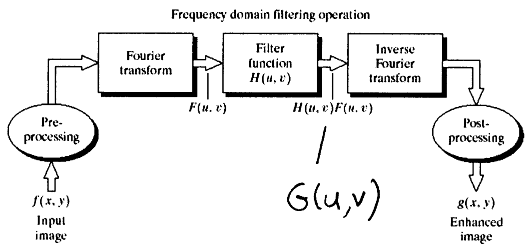

FIGURE 4.5 Basic steps for filtering in the frequency domain.

Preprocessing examples

- $(-1)^{x+y}$ multiplication
- cropping image to even dimension
- gray level scaling
- conversion to floating point

H is known as a zero-phase filter. In general, filters for imaging are real and do not change the phase of $F(u,v)$

Notch filter

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = \left(\frac{M}{2}, \frac{N}{2}\right) \\ 1 & \text{otherwise} \end{cases}$$

Such a filter is called a notch filter because it removes only $F(0,0)$ which is the average value of the image.

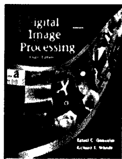

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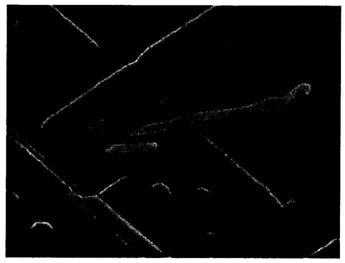
FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.



This is an example of a displayed image after F(0,0) was removed by a notch filter.

Since a display cannot show negative values the image has been shifted so that the most negative value corresponds to zero, i.e. black.

Other kinds of filters:

low pass – attenuates high frequencies while passing low frequencies which come from the large, smooth areas in the image. will show less gray scale variation.

high-pass – attenuates low frequencies while passing high frequencies which come from image detail such as edges and noise

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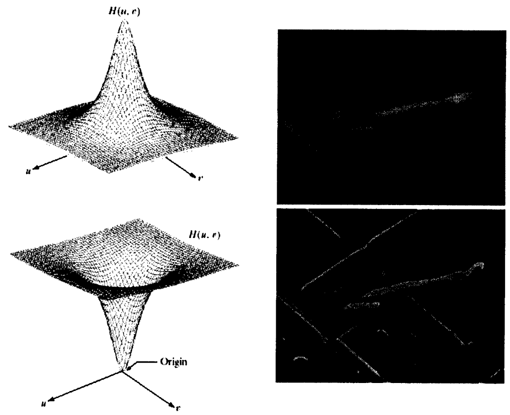

a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

(a) blurred image with high frequency detail missing due to low-pass filter

(b) high-pass filter.
sharp image with prominent edge information
dark because negative information not being shown.

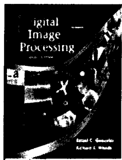

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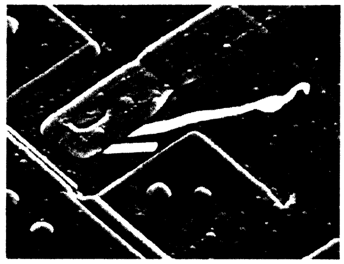
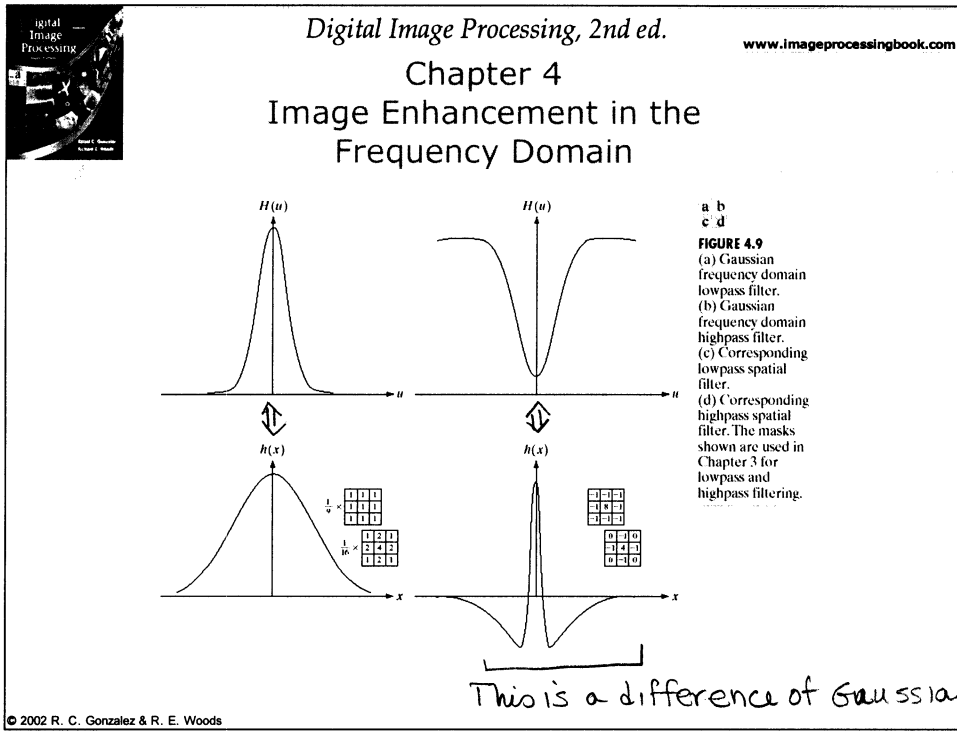
FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



Result of adding a constant to the high-pass filtered image. Adding a constant shifts previously negative values from black into perceivable gray scale.



This is a difference of Gaussians

The Gaussian filter can be shown to be an optimal filter under certain conditions and has the property that both the forward and reverse transform of a Gaussian is a Gaussian.

$$h(x) = \sqrt{2\pi}\,\sigma A e^{-2\pi^2\sigma^2 x^2}$$

$$H(u) = A e^{-\frac{u^2}{2\sigma^2}}$$

both real

construct using Pascal's triangle

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

low-pass

weighted average filter from 3.6.1

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Difference of Gaussians

$$h(x) = \sqrt{2\pi}\,\sigma_1 A e^{-2\pi^2\sigma_1^2 x^2} - \sqrt{2\pi}\,\sigma_2 B e^{-2\pi^2\sigma_2^2 x^2}$$

We usually design a filter in the frequency domain and implement it in the spatial domain.

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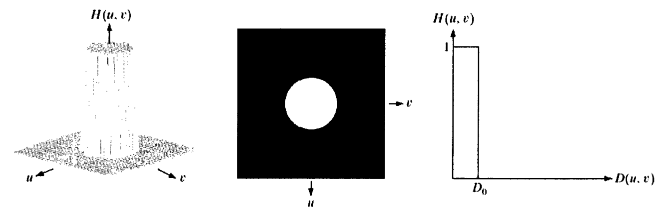


a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Low Pass Filter

Two-dimensional ideal low-pass filter

$$H(u,v) = \begin{cases} 1 & D(u,v) \le D_0 \\ 0 & D(u,v) \ge D_0 \end{cases}$$

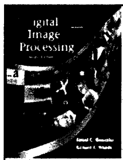
where $D(u,v) = \sqrt{(u-\frac{m}{2})^2 + (v-\frac{N}{2})^2}$ $\quad D_0$ is the cutoff frequency.

Pick cutoff frequencies based upon fraction of image power they remove.

$$P(u,v) = |F(u,v)|^2$$

$$P_T = \sum_{u=0}^{m-1} \sum_{v=0}^{N-1} P(u,v)$$

% of image power $\quad \alpha = 100 \left[\sum_u \sum_v \frac{P(u,v)}{P_T} \right]$



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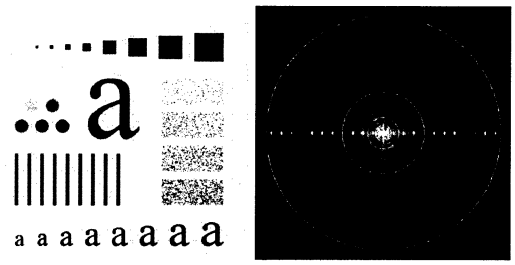


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

There are five circles shown in the Fourier transform of this test image. These correspond to $\alpha = 92.0, 94.6, 96.4, 98$ & 99.5%
The corresponding u, v are 5, 15, 30, 80 & 230

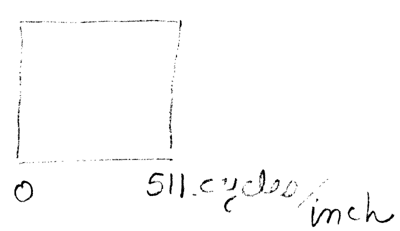
The results of using each filter are shown in the next figure.

$$\Delta u = \frac{1}{M \underline{\Delta x}}$$

↑ spatial interval. Say $\frac{512 \text{ pixels}}{\text{inch}}$ Then $\frac{1}{512} \frac{\text{inches}}{\text{pixel}}$

$= 0.001953 \frac{\text{in}}{\text{pix}}$

$$\Delta u = \frac{1}{512 \times .001953} = 1/\text{inch}$$



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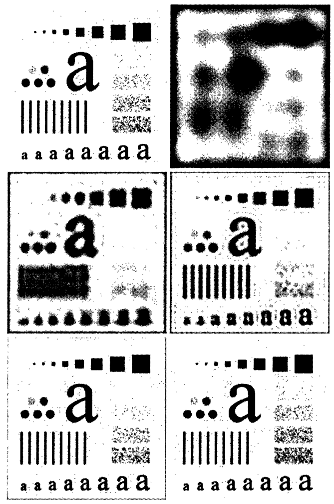
pretty useless except for blurring. all edge information contained in upper 8% of image power.

"ringing" shown in all images.

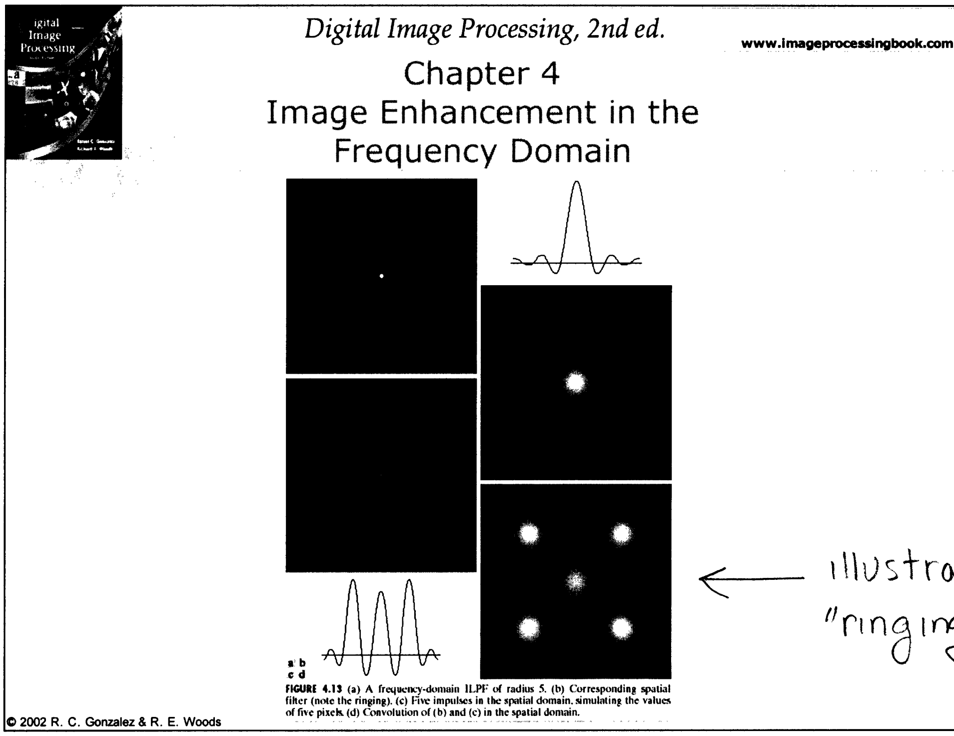


← little edge information in upper 0.5% of power

a b
c d
e f

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

Lowpass filtering



← illustrates "ringing" due to ILPF

(a) ideal low pass filter of radius 5 pixels

(b) corresponding spatial filter is computed

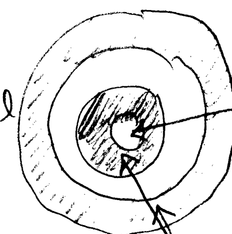
$(-1)^{u+v} H(u,v)$ for centering

inverse DFT

re-centers $(-1)^{x+y} R_e$[inverse DFT]

take real part since it should be real

As pixel radius of (a) increases the radius of the spatial rings decreases.

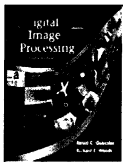


center spot responsible for blurring

rings responsible for "ringing"

(c) 5 pixels in spatial domain "impulses"

(d) convolution of (b) with (c) showing "shifting" property



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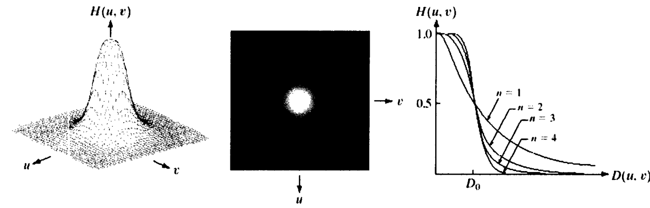


a b c

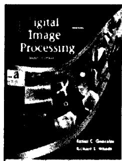
FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Butterworth LPF

Ringing will increase as n increases.

$n=1$ no ringing

$n=2$ bare perceptible



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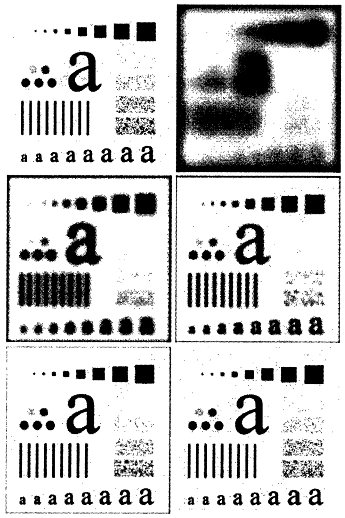

a b
c d
e f
FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

Results of $n=2$ Butterworth low pass filter
Using same radii (D_0) as for ideal low-pass filter.

Blurring linearly decreases as radius increases,
No ringing visible.

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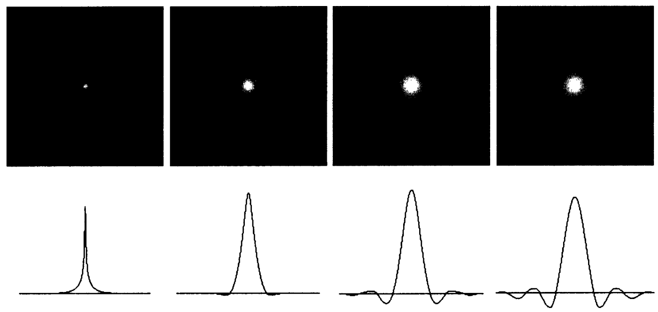

a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Spatial implementations of Butterworth low-pass filters, All filters have $D_0 = 5$.

Note that ringing increases with n.

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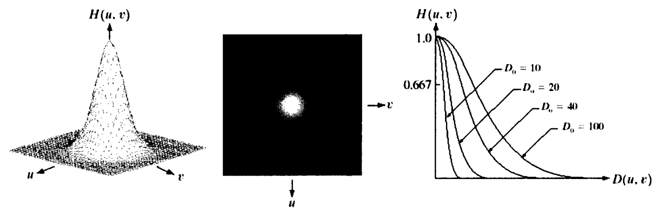


a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Gaussian Low Pass filter

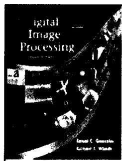
$$H(u,v) = e^{-\frac{D^2(u,v)}{2\sigma^2}}$$

note constant is 1 for consistency with other filter types

pick $\sigma = D_0$

$$H(u,v) = e^{-\frac{D^2(u,v)}{2D_0^2}}$$

Since its always positive we expect no ringing in the resulting image.



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Gaussian Low Pass Filtering

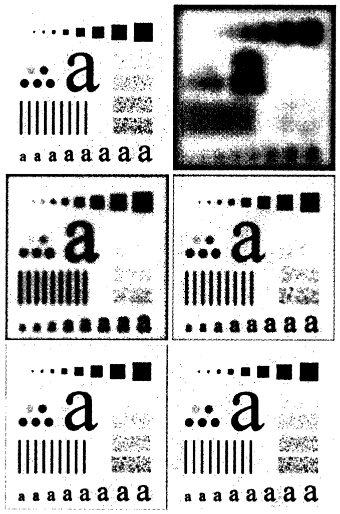
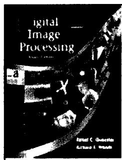


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

Blurring linearly decreases as radius (D_0) increases
No ringing visible

Compare 4.18 (c) Gaussian ← picture looks better so not as much smoothing
to 4.15 (c) Butterworth



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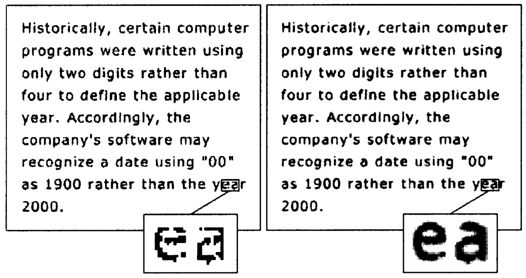
a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

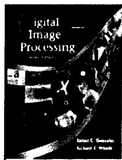


444 x 508 pixels.

broken characters are bad.

fuller and more filled in images using GLPF with $D_0 = 80$

Example of using Gaussian LPF to improve text for OCR.



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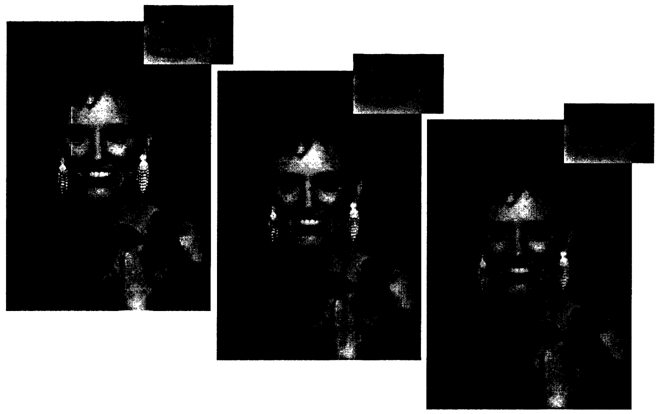
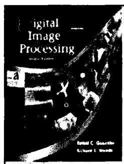


a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Another commercial application of low pass filtering.
Apply a GLPF to eliminate skin lines and blemishes by selectively smoothing images.



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a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

588 × 600 pixel original

GLPF $D_0 = 30$

GLPF $D_0 = 10$

might use this image for registration with a map

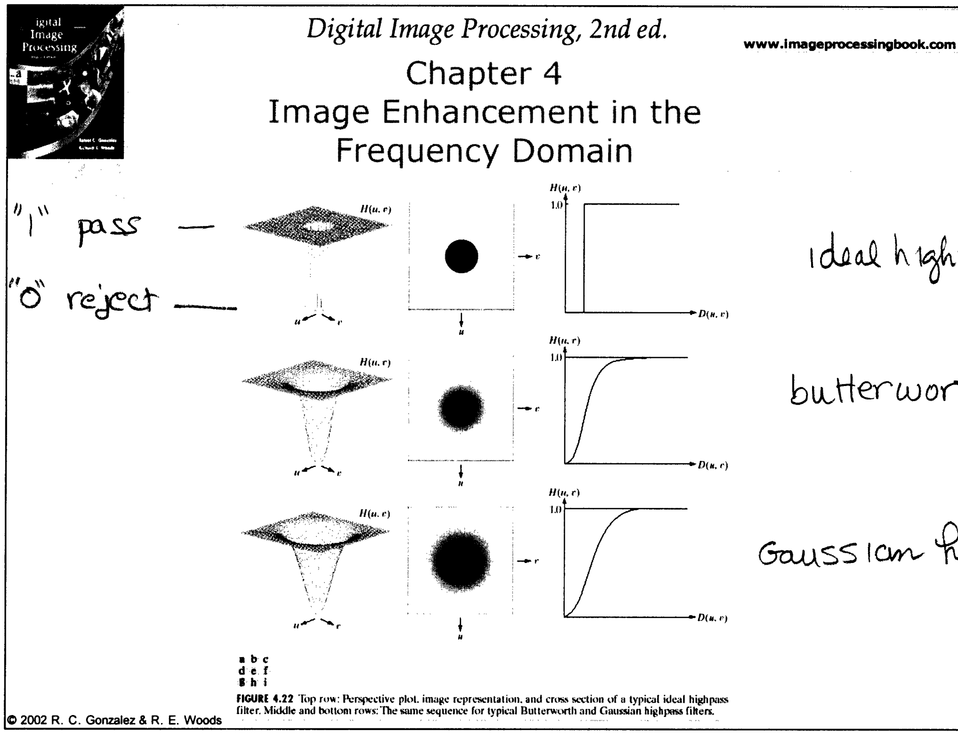


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

In the spatial domain we did unsharp masking as

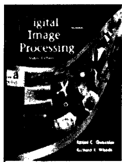
$$\underbrace{f_s(x,y)}_{\text{sharpens detail}} = f(x,y) - \underbrace{\bar{f}(x,y)}_{\text{by subtracting averaged image}}$$

In the frequency domain we can write something roughly comparable

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

Generate spatial masks by

1. multiply $H(u,v)$ by $(-1)^{u+v}$ to center
2. compute inverse DFT
3. multiply real part of inverse DFT by $(-1)^{x+y}$



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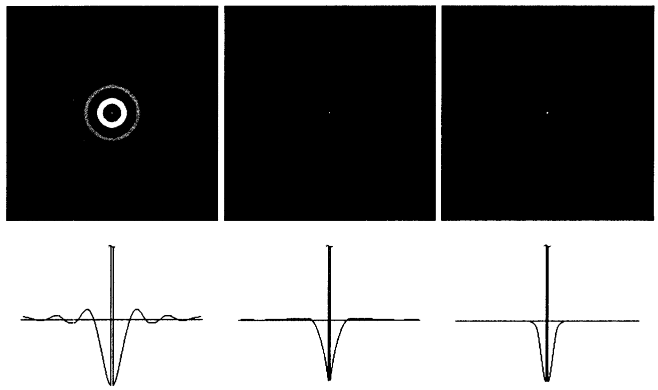

a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Spatial filters generated by

1. multiply $H(u,v)$ by $(-1)^{u+v}$ to center
2. inverse DFT
3. multiply inverse DFT by $(-1)^{x+y}$

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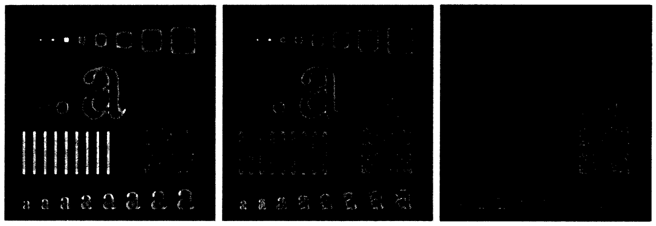
original test image 500 x 500 Pixels



a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with D_0 = 15, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

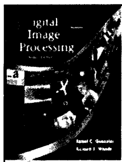
$D_0 = 15$ $\qquad$ $D_0 = 30$ $\qquad$ $D_0 = 80$

Nice edges
Good

2D ideal high-pass filter (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

high-pass filtering emphasizes edge discontinuity information



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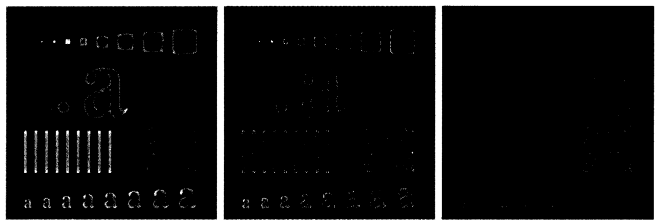

a b c

FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

$D_0 = 15$ $\quad$ $D_0 = 30$ $\quad$ $D_0 = 80$

Much less ringing for small D_0 than IHPF

Butterworth high pass filter (BHPF) of order n

$$H(u,v) = \frac{1}{1 + \left[\frac{D_0}{D(u,v)}\right]^{2n}}$$

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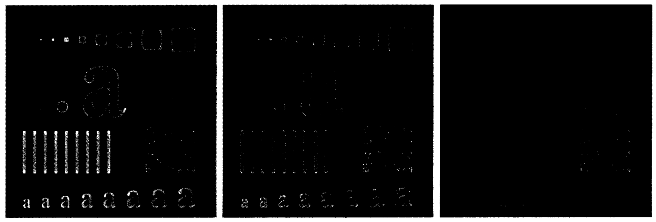


a b c

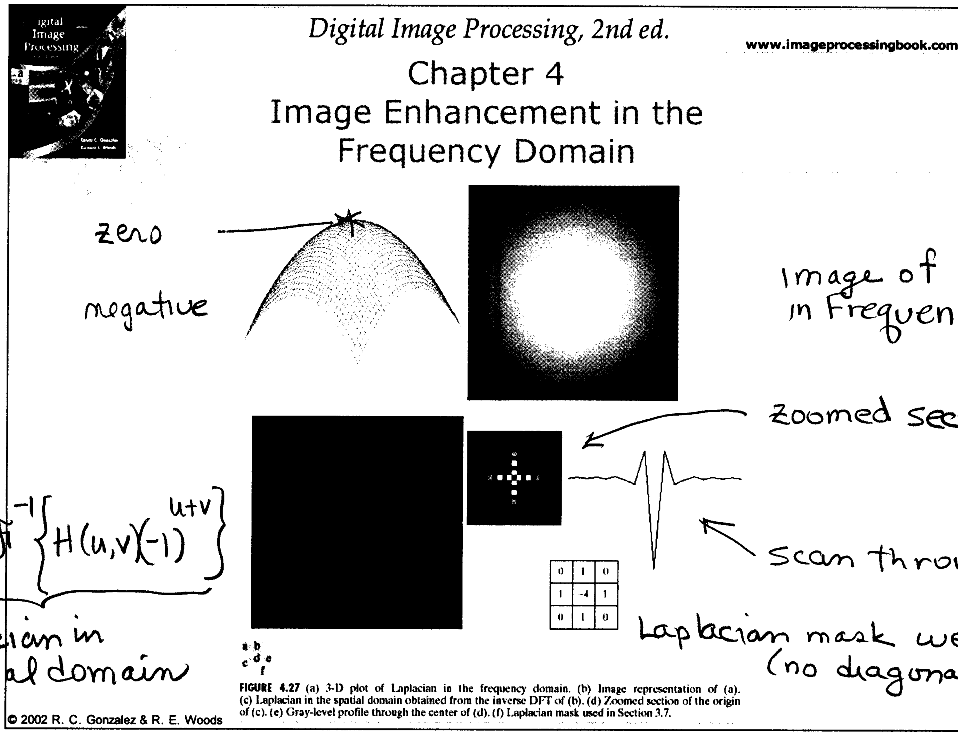
FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

$D_0 = 15$ $D_0 = 30$ $D_0 = 80$

Very clean and smooth results.

Gaussian high-pass filter (GHPF)

$$H(u,v) = 1 - e^{-\frac{D^2(u,v)}{2D_0^2}}$$



zero

negative

Image of Laplacian in Frequency domain

zoomed section at origin

scan through center

$$\underbrace{(-1)^{x+y}\, \mathfrak{F}^{-1}\left\{H(u,v)(-1)^{u+v}\right\}}_{\text{Laplacian in spatial domain}}$$

0	1	0
1	-4	1
0	1	0

Laplacian mask we used (no diagonals)

What does the Laplacian look like in the frequency domain?

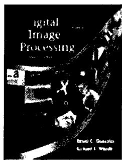
Using $\mathfrak{F}\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u)$

we can write $\mathfrak{F}\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] = (ju)^2 F(u,v) + (jv)^2 F(u,v)$

or $\mathfrak{F}\left[\nabla^2 f(x,y)\right] = -(u^2+v^2)F(u,v)$

We shift this to the center by multiplying $f(x,y)$ by $(-1)^{x+y}$ before we Fourier transform

$$\Rightarrow \quad \underbrace{\nabla^2 f(x,y)}_{\text{multiply by } (-1)^{x+y}} \iff \underbrace{-\left[\left(u-\frac{M}{2}\right)^2 + \left(v-\frac{N}{2}\right)^2\right]}_{\text{Laplacian plotted in (a)}} F(u,v)$$



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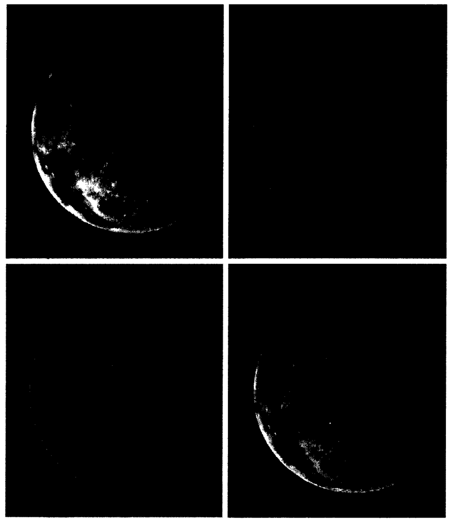


a b
c d

FIGURE 4.28 (a) Image of the North Pole of the moon. (b) Laplacian filtered image. (c) Laplacian image scaled. (d) Image enhanced by using Eq. (4.4-12). (Original image courtesy of NASA.)

scaled image
most positive → 1
most negative → 0

Laplacian
Filtering in the (u,v) domain

$$\nabla^2 f(x,y) = \mathcal{F}^{-1}\left\{ -\left[\left(u-\frac{M}{2}\right)^2 + \left(v-\frac{N}{2}\right)^2\right] F(u,v) \right\}$$

(4.4-12)

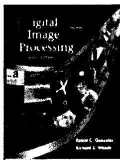
$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

use one frequency domain mask

$$H(u,v) = 1 - \underbrace{\left[\left(u-\frac{M}{2}\right)^2 + \left(v-\frac{N}{2}\right)^2\right]}_{\text{be careful in scaling since these can be} \gg 1}$$

This result is the same one as done before in the spatial domain EXCEPT now done in the frequency domain.



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a b
c d

FIGURE 4.29 Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

Laplacian high-pass filtered image

high-boost image with A=2

high-boost image with A increased to 2.7

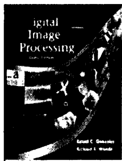
unsharp masking $\quad f_{hp}(x,y) = f(x,y) - f_{lp}(x,y) \qquad (1)$

high boost filtering $\quad f_{hb} = Af(x,y) - f_{lp}(x,y)$

re-writing $\quad f_{hb} = (A-1)f(x,y) + \underbrace{f(x,y) - f_{lp}(x,y)}_{f_{hp}(x,y)}$

In the frequency domain
we can use a composite filter $\quad H_{hb}(u,v) = (A-1)f(x,y)\,H_{hp}(u,v) \quad$ where $A \geq 1$

This is slightly different than previous spatial domain results because this frequency domain representation of the Laplacian does not include diagonal components.



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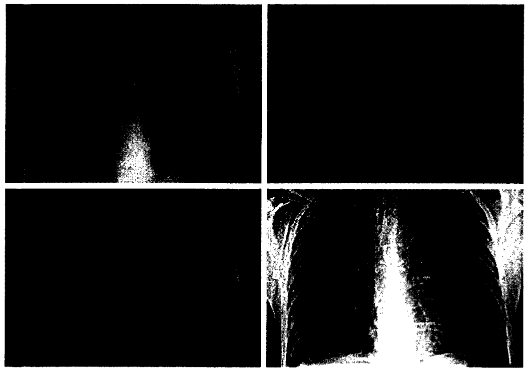


a b
c d

FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

butterworth high pass
$n=2$

high frequency emphasis
$a=0.5$ $b=2$
keeps low freq. tones

histogram equalization of high frequency emphasis image

high boost $H_{hb}(u,v) = (A-1)f(x,y) + H_{hp}(u,v)$

high-frequency emphasis

$$H_{hfe}(u,v) = a + b\,H_{hp}(u,v)$$

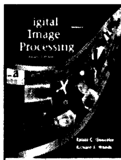
multiply high frequencies by a constant

add an offset so zero frequency term is not eliminated

typically $.25 < a < .5$ $1.5 < b < 2.0$

when $b=1$ this reduces to high-frequency boost

$b>1$ this emphasizes high frequencies



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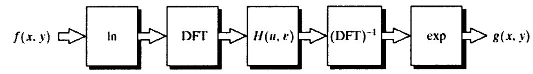


FIGURE 4.31 Homomorphic filtering approach for image enhancement.

simultaneous:

1. dynamic range compression
2. contrast enhancement

$\gamma_H > 1$

$\gamma_L < 1$

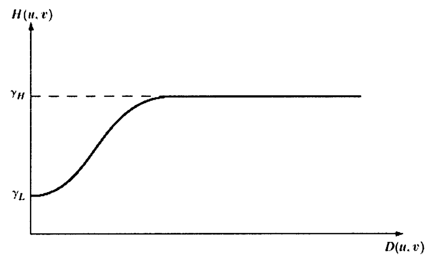
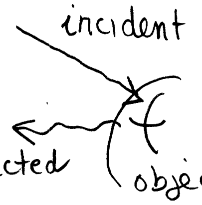


FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u, v)$ is the distance from the origin of the centered transform.

decrease illumination. low frequencies

increase reflectance which varies abruptly for this example



$$f(x,y) = \underbrace{i(x,y)}_{\text{illumination}}\ \underbrace{r(x,y)}_{\text{reflection from object}}$$

define $z(x,y) = \ln(f(x,y)) = \ln i(x,y) + \ln r(x,y)$

(Used log to separate i and r)

Fourier transform $\mathfrak{F}\{z(x,y)\} = \mathfrak{F}\{\ln i(x,y)\} + \mathfrak{F}\{\ln r(x,y)\}$

$$Z(u,v) = F_i(u,v) + F_r(u,v)$$

Now process by a filter $H(u,v)$

$$S(u,v) = H(u,v)Z(u,v) = \underbrace{H(u,v)F_i(u,v)}_{\text{low frequencies}} + \underbrace{H(u,v)F_r(u,v)}_{\text{high frequencies}}$$

inverse

Inverse transforming

$$s(x,y) = \mathfrak{F}^{-1}\{S(u,v)\} = \underbrace{\mathfrak{F}^{-1}\{H(u,v)F_i(u,v)\}}_{i'(x,y)} + \underbrace{\mathfrak{F}^{-1}\{H(u,v)F_r(u,v)\}}_{r'(x,y)}$$

$$s(x,y) = i'(x,y) + r'(x,y)$$

inverse

Use exponential to invert original logarithm

$$g(x,y) = e^{s(x,y)} = e^{i'(x,y)}e^{r'(x,y)} = i_0(x,y)\,r_0(x,y)$$

where $i_0(x,y) = e^{i'(x,y)}$ and $r_0(x,y) = e^{r'(x,y)}$

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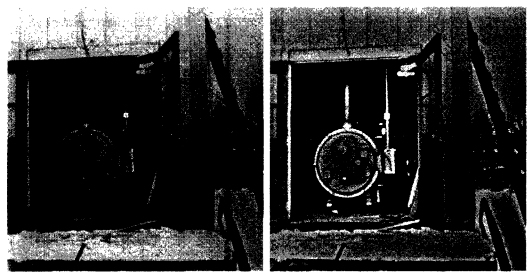


a b
FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

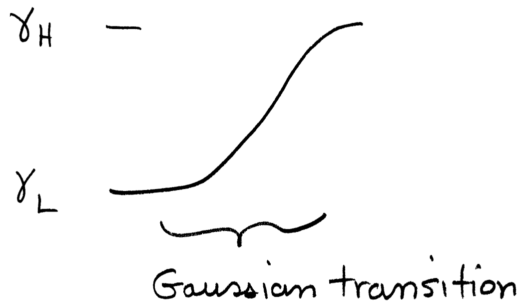
$\gamma_L = 0.5$

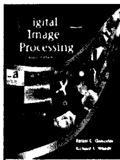
$\gamma_H = 2$

similar to high-frequency emphasis filter

Result of homomorphic processing.

$$H(u,v) = (\gamma_H - \gamma_L)\left[1 - e^{-c\frac{D^2(u,v)}{D_0^2}}\right] + \gamma_L$$





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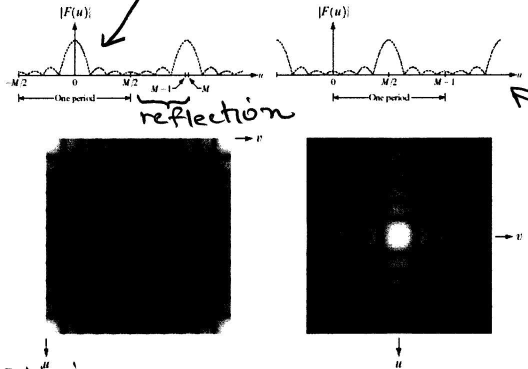
centered on origin due to symmetry



reflection

multiply by $(-1)^x$ before taking transform.

This is the result of fft2

This is what you see after using FFTSHIFT

a b
c d

FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

The DFT is periodic: $F(u,v) = F(u+M, v) = F(u, v+N) = F(u+M, v+N)$

The inverse DFT is periodic

$$f(x,y) = f(x+M, y) = f(x, y+N) = f(x+M, y+N)$$

The DFT is also symmetric

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

(a) In (a) we see $F(u)$ is periodic with a period of length M

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j\frac{2\pi u x}{M}}$$

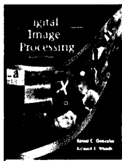
(b) shows the result of premultiplying by $(-1)^x$ which is simply a shift.

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1}(-1)^x f(x)\, e^{-j\frac{2\pi u x}{M}} = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j\frac{\pi}{2}x}\, e^{-j\frac{2\pi u x}{M}}$$

since $-1 = e^{-j\frac{\pi}{2}}$

$$= \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j\frac{2\pi u x}{M} - j\frac{M\pi x}{2M}} = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\, e^{-j\left(2\pi u + \frac{M}{2}\right)\frac{x}{M}}$$

shifted by $+\frac{M}{2}$



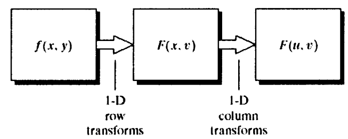


FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

can do either first.

This arguement also works for the inverse DFT.

SEPARABILITY

This is one of the most common 2-D DFT implementations

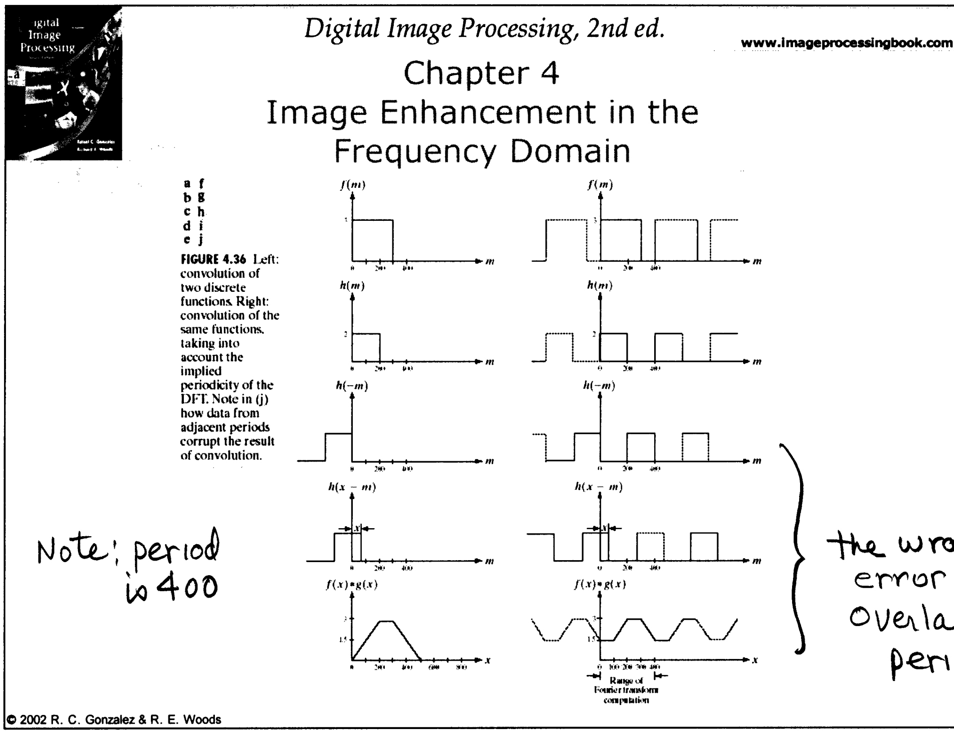
①

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

$$= \frac{1}{M} \sum_{x=0}^{M-1} e^{-j2\pi \frac{ux}{M}} \underbrace{\left[\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi \frac{vy}{N}} \right]}_{\substack{\text{This is the 1-D DFT of } f(x,y) \\ F(x,v)}}$$

$$= \underbrace{\frac{1}{M} \sum_{x=0}^{M-1} F(x,v)\, e^{-j2\pi \frac{ux}{M}}}_{\substack{\text{This is the subsequent} \\ \text{row DFT.} \\ x \text{ going from } 0 \text{ to } M}} \qquad \underbrace{F(x,v)}_{\substack{\text{This is the column DFT} \\ \text{of } f(x,y) \\ y \text{ going from } 0 \text{ to } N}}$$

② The DFT of the DFT yields $\frac{1}{MN} f^*(x,y)$
Since f is real this is not a problem. Multiply by MN to get $f(x,y)$



Note: period is 400

the wraparound error comes from overlap of the periodic functions

analytical convolution

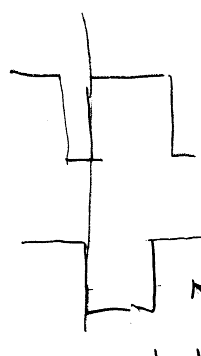
Note that if f and g are 400 points f∗g goes out to 800 points

DFT makes functions periodic so we are really convolving periodic functions

Convolution:

continuous: $$f(x) * h(x) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

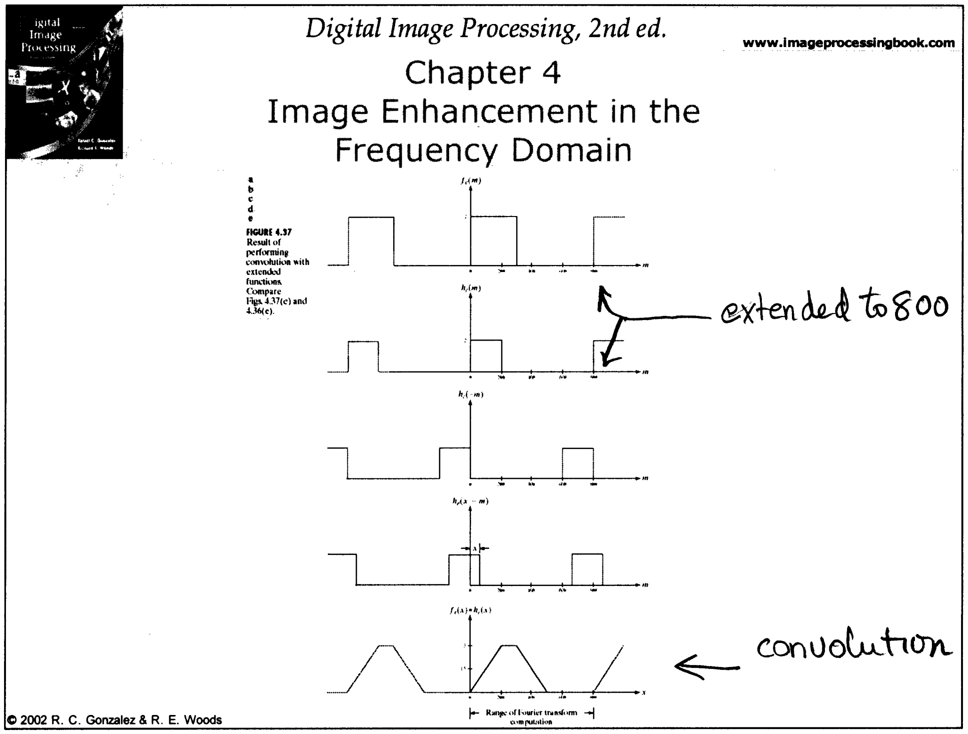
discrete: $$f(x) * h(x) = \frac{1}{M}\sum_{m=0}^{M-1} f(m)\, h(x-m)$$



periodic extension

integral non-zero here

Get rid of this non-zero overlap by simply adding zeros to both functions until the overlap term goes away.



extended to 800

← convolution is now correct!

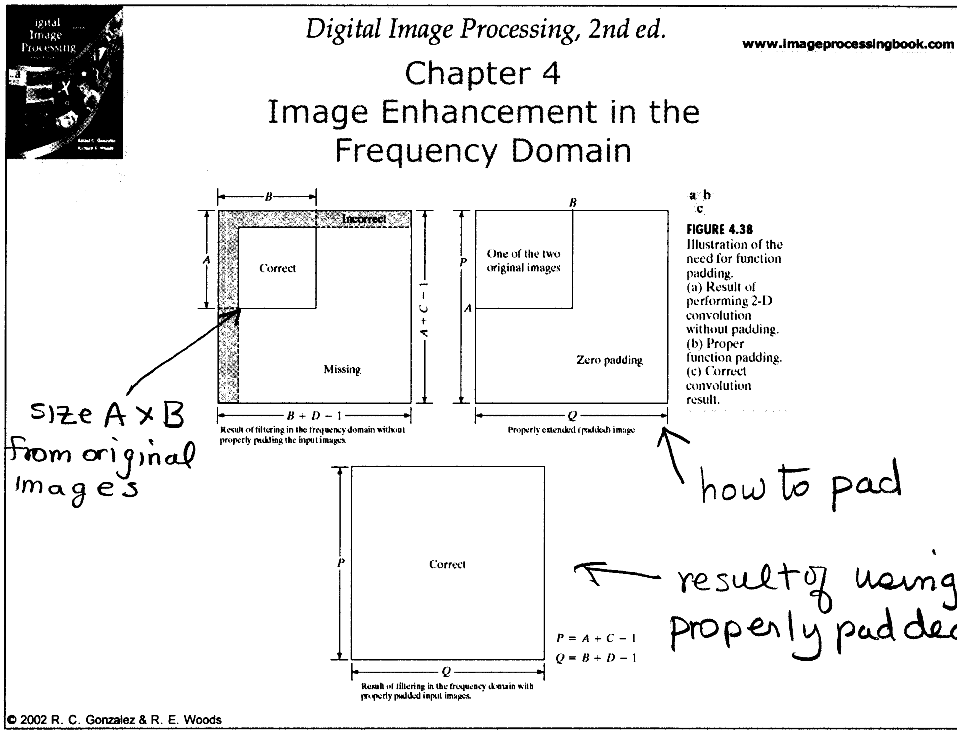
Padded functions $f_e(x) = \begin{cases} f(x) & 0 \le x \le A-1 \\ 0 & A \le x \le P \end{cases}$

$g_e(x) = \begin{cases} g(x) & 0 \le x \le B-1 \\ 0 & B \le x \le P \end{cases}$

both have period P after padding.

P is the new extended (or padded) period

If $P = A+B-1$ there will be no wraparound area



(a) f, h are square and the same size

This is the result of $\mathrm{ifft2}\left[\mathrm{fft2}(f) * \mathrm{fft2}(h)\right]$

There is a band of wrap around error.

(b) f, h are properly padded

$P \geq A + C - 1$

$Q \geq B + D - 1$

for images $A \times B$ and $C \times D$

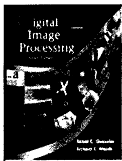


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Image Enhancement in the Frequency Domain

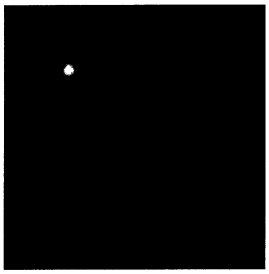


FIGURE 4.39 Padded lowpass filter is the spatial domain (only the real part is shown).

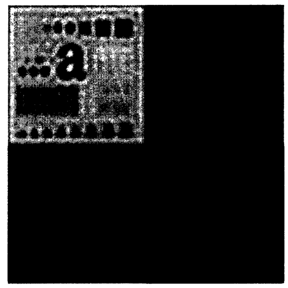
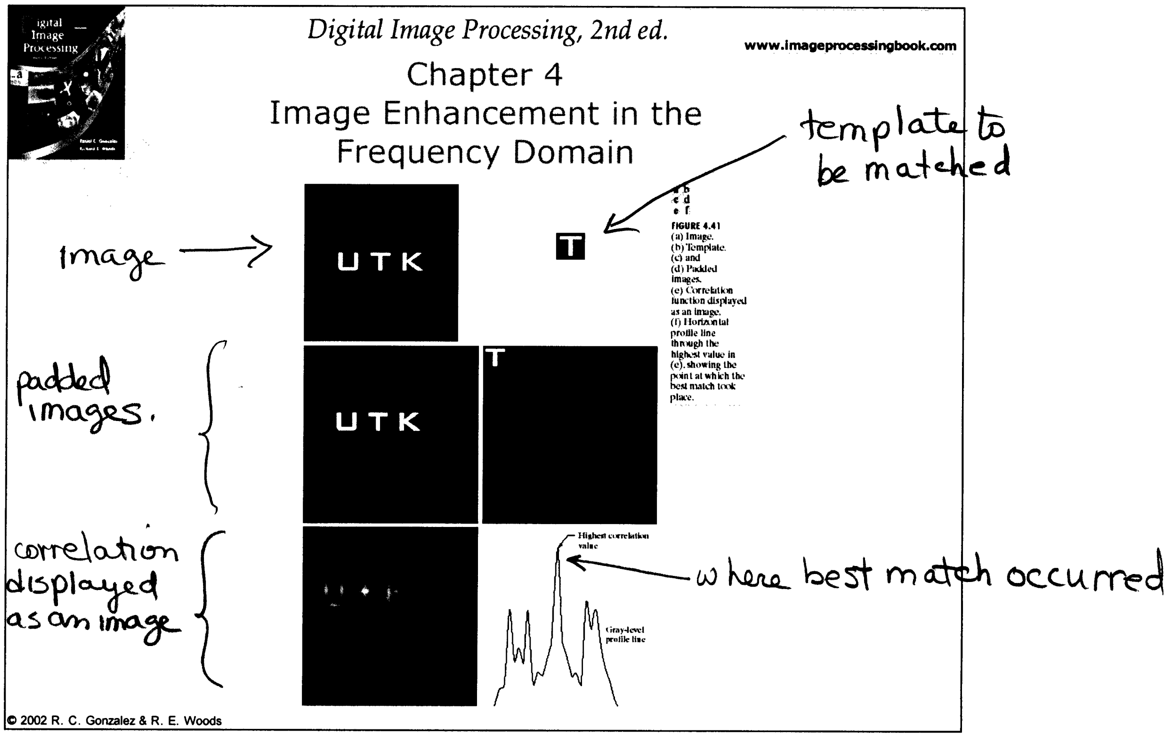


FIGURE 4.40 Result of filtering with padding. The image is usually cropped to its original size since there is little valuable information past the image boundaries.

properly padded low pass filter

result of filtering (using padded filter) a padded test image

Crop back to size of original image.



discrete convolution

$$f(x,y) * h(x,y) \stackrel{\Delta}{=} \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$$

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)\, H(u,v)$$

$$f(x,y)\, h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$

discrete correlation

$$f(x,y) \circ h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \underbrace{f^*(m,n)}_{\text{since images are usually real this doesn't affect us}} \underbrace{h(x+m, y+n)}_{\text{not mirrored about origin}}$$

correlation theorem

$$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v)\, H(u,v)$$

$$f^*(x,y)\, h(x,y) \Leftrightarrow F(u,v) \circ H(u,v)$$