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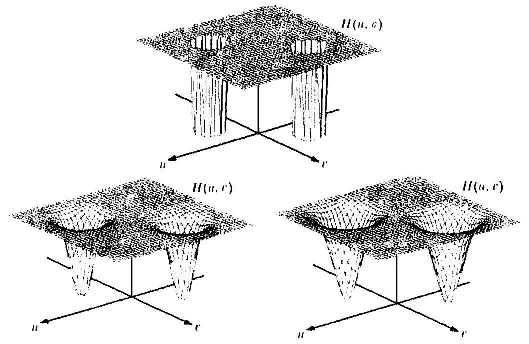
Chapter 5
Image Restoration



a
b c

FIGURE 5.18 Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.

Notch filters:

More complicated formulas:

Ideal:

$$H(u,v) = \begin{cases} 0 & D_1(u,v) \le D_0 \text{ or } D_2(u,v) \le D_0 \\ 1 & \text{otherwise} \end{cases}$$

where $D_1(u,v) = \sqrt{\left(u - \frac{M}{2} - u_0\right)^2 + \left(v - \frac{N}{2} - v_0\right)^2}$

$$D_2(u,v) = \sqrt{\left(u - \frac{M}{2} + u_0\right)^2 + \left(v - \frac{N}{2} + v_0\right)^2}$$

Note: frequency response centered at $\frac{M}{2}, \frac{N}{2}$

Butterworth

$$H(u,v) = \frac{1}{1 + \left[\frac{D_0^2}{D_1(u,v)\, D_2(u,v)}\right]^n}$$

Gaussian

$$H(u,v) = 1 - e^{-\frac{1}{2}\left[\frac{D_1(u,v)\, D_2(u,v)}{D_0^2}\right]}$$

Another class of notch filters (pass) can be constructed as

$$H_{np}(u,v) = 1 - H_{nr}(u,v)$$

where $H_{nr}(u,v)$ are the above notch reject filters.

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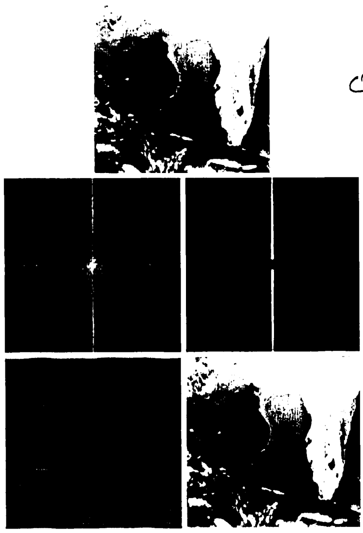

original noise corrupted image (scan lines)

noise source not obvious in frequency domain

construct a simple notch filter and apply this filter to the frequency spectrum

spatial image of noise resulting from applying (c) to (b)

cleaned up image.

a
b c
d e

FIGURE 5.19 (a) Satellite image of Florida and the Gulf of Mexico (note horizontal sensor scan lines). (b) Spectrum of (a). (c) Notch pass filter shown superimposed on (b). (d) Inverse Fourier transform of filtered image, showing noise pattern in the spatial domain. (e) Result of notch reject filtering. (Original image courtesy of NOAA.)

Noise in this case is very regular
and caused by the scanning process.

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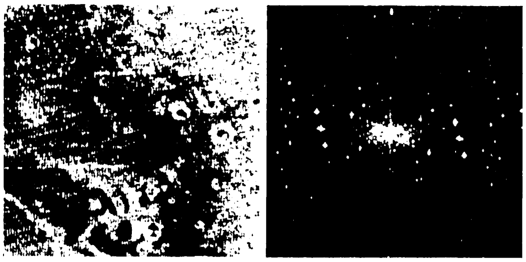
a b

FIGURE 5.20 (a) Image of the Martian terrain taken by *Mariner 6*. (b) Fourier spectrum showing periodic interference. (Courtesy of NASA.)



This is an example of complex periodic noise.
Several noise sources are present.
These are hard to detect and filter.

To handle this we need to develop an optimum method of eliminating the noise, i.e., estimating $f(x,y)$

5.5 Linear, position - invariant degradations

Based upon our model degradation is modeled as an $H(x,y)$
Note H is <u>NOT</u> in the frequency domain.

$$g(x,y) = H[f(x,y)] + \eta(x,y) \qquad (1)$$

$$\Rightarrow g(x,y) = H[f(x,y)] \qquad \text{let } \eta(x,y) = 0$$

<u>Properties</u>

<u>linear</u> if $H[af_1(x,y) + bf_2(x,y)] = aH[f_1(x,y)] + bH[f_2(x,y)]$

This simply says that if H is a linear operator the response to a sum of two inputs is the sum of the two responses

<u>homogeneous</u> if $H[af_1(x,y)] = aH[f_1(x,y)]$

The response to a constant multiple of the input is that same constant multiplied by the response to that input

<u>position (space) invariant</u> if $H[f(x-\alpha, y-\beta)] = g(x-\alpha, y-\beta)$ for any function $g(x,y) = H[f(x,y)]$.

This says that the response is only dependent on the value of the input and <u>NOT</u> its position

The impulse function $A\delta(x-x_0, y-y_0)$ is defined by

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} s(x,y) A\delta(x-x_0, y-y_0) = A s(x_0, y_0) \qquad \text{discrete}$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\alpha,\beta)\, \delta(x-\alpha, y-\beta)\, d\alpha\, d\beta = f(x,y) \qquad \text{continuous}$$

Consider a corrupted image with several interference components

Use a notch filter $H(u,v)$ to isolate the noise in the frequency domain

$$N(u,v) = \underbrace{H(u,v)}_{\text{construct notch filter by observing spectrum } G(u,v) \text{ on a display}} \underbrace{G(u,v)}_{\text{Fourier transform of corrupted image}}$$

In the spatial domain,

$$\eta(x,y) = \mathcal{F}^{-1}\left\{H(u,v)\,G(u,v)\right\}$$ gives the interference noise pattern

write
$$\hat{f}(x,y) = g(x,y) - w(x,y)\,\eta(x,y) \qquad (1)$$

in principal this shoued yield the actual $f(x,y)$
in practice it is an approximation.

$\Rightarrow$ Vary $w(x,y)$ to get the "best" estimate $\hat{f}(x,y)$
since $\eta(x,y)$ is usually not known completely

* One best estimate is to minimize $\sigma_{\hat{f}}^2$ over a specified neighborhood of every point (x,y) in the image.

For a neighborhood $(2a+1) \times (2b+1)$ about (x,y) we can write the variance

$$\sigma_{\hat{f}}^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{a} \sum_{t=-b}^{b} \Big[\hat{f}(x+s,y+t) - \underbrace{\overline{\hat{f}}(x,y)}_{\text{average of } \hat{f} \text{ in the neighborhood of } (x,y)}\Big]^2 \qquad (2)$$

$$\overline{\hat{f}}(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{+a} \sum_{t=-b}^{b} \hat{f}(x+s,y+t)$$

Substituting (1) into (2) gives

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{+a} \sum_{t=-b}^{+b} \Big\{ g(x+s,y+t) - w(x+s,y+t)\eta(x+s,y+t) - \overline{g(x,y)} + \overline{w(x,y)\eta(x,y)} \Big\}^2$$

We assume $w(x,y)$ changes slowly so

$$w(x+s,y+t) \approx w(x,y)$$

and we can write

$$\overline{w(x,y)\eta(x,y)} = w(x,y)\overline{\eta(x,y)}$$

$\therefore$

$$\sigma^2(x,y) = \frac{1}{(2a+1)(2b+1)} \sum_{s=-a}^{+a} \sum_{t=-b}^{+b} \Big\{ g(x+s,y+t) - w(x,y)\eta(x+s,y+t) - \overline{g(x,y)} + w(x,y)\overline{\eta(x+s,y+t)} \Big\}^2$$

minimize by computing $\dfrac{\partial \sigma^2(x,y)}{\partial w(x,y)} = 0$

<u>Extra credit if you prove this result.</u>

$$w(x,y) = \frac{\overline{g(x,y)\eta(x,y)} - \overline{g(x,y)}\,\overline{\eta(x,y)}}{\overline{\eta^2(x,y)} - \left[\overline{\eta(x,y)}\right]^2}$$

Compute $w(x,y)$ for one point in each non-overlapping neighborhood.

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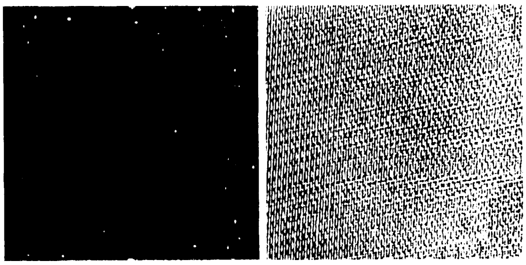

a b

FIGURE 5.22 (a) Fourier spectrum of $N(u, v)$, and (b) corresponding noise interference pattern $\eta(x, y)$. (Courtesy of NASA.)

This is the noise spectrum and the corresponding spatial noise $\eta(x, y)$ obtained by inverse transforming (a).

This was manually estimated.

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Not shifted

(0,0)

FIGURE 5.21 Fourier spectrum (without shifting) of the image shown in Fig. 5.20(a). (Courtesy of NASA.)

Select neighborhood $a = b = 15$ to use for variance calculations

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FIGURE 5.23 Processed image. (Courtesy of NASA.)

This is the image AFTER using

$$\hat{f}(x,y) = g(x,y) - w(x,y)\underbrace{\eta(x,y)}_{\text{noise from previous page}}$$

computed as

$$w(x,y) = \frac{\overline{g(x,y)\eta(x,y)} - \overline{g(x,y)}\ \overline{\eta(x,y)}}{\overline{\eta^2(x,y)} - \left[\overline{\eta(x,y)}\right]^2}$$

$g(x,y)$ – corrupted image

$\eta(x,y)$ – spatial noise, estimated.

averages computed over a $(2a+1) \times (2b+1)$ neighborhood

Using Figure 1 as our reference we can write

$$g(x,y) = H[f(x,y)] = H\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)\,d\alpha\,d\beta\right.$$

for $\eta(x,y)=0$.

Assuming H is linear and using the linearity of integrals we can reverse the order to get

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} H[f(\alpha,\beta)\,\delta(x-\alpha, y-\beta)]\,d\alpha\,d\beta$$

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\,\underbrace{H[\delta(x-\alpha, y-\beta]}_{h(x,\alpha,y,\beta)}\,d\alpha\,d\beta$$

is called the impulse response

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\,h(x,\alpha,y,\beta)]\,d\alpha\,d\beta$$

superposition (Fredholm) integral of the first kind

$\Rightarrow$ If the response to an impulse is known, the response to <u>any</u> input $f(\alpha,\beta)$ can be calculated by this equation.

If H is position invariant then

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\,h(x-\alpha, y-\beta)\,d\alpha\,d\beta$$

the convolution integral

In the presence of noise we have

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\alpha,\beta)\, h(x,\alpha,y,\beta)\, d\alpha\, d\beta + \eta(x,y)$$

or if H is position invariant

$$g(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{f(\alpha,\beta)\, h(x-\alpha,y-\beta)\, d\alpha\, d\beta}_{\text{convolution integral}} + \eta(x,y)$$

Using the book's notation

$$g(x,y) = h(x,y) * f(x,y) + \eta(x,y)$$

$$G(u,v) = H(u,v)\, F(u,v) + N(u,v)$$

because degradations are modeled as convolution
image restoration is often called image deconvolution
the filters used in the restoration process are called deconvolution filters

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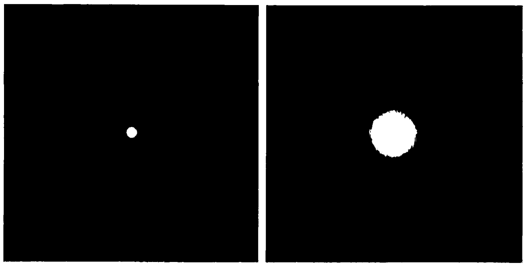

a b
FIGURE 5.24 Degradation estimation by impulse characterization. (a) An impulse of light (shown magnified). (b) Imaged (degraded) impulse.

← observed image

fourier transform of observed image
↓

Estimate the impulse response by imaging a bright spot of light. To minimize noise. Then

$$G(u,v) = H(u,v)F(u,v) + N(u,v) \cong H(u,v)F(u,v)$$

Since the fourier transform of $A\delta(x,y)$ is A we have $H(u,v) \cong \dfrac{G(u,v)}{A}$

Estimating the degradation function H from observation.

$$\underbrace{H_s(u,v)}_{\text{assume position invariance and extend to complete image}} = \frac{G_s(u,v)}{\hat{F}_s(u,v)} \begin{array}{l} \leftarrow \text{observed subimage} \\ \leftarrow \text{reconstructed subimage} \end{array}$$

reconstructed subimage: look for a subimage which has known detail, is relatively noise free, etc.

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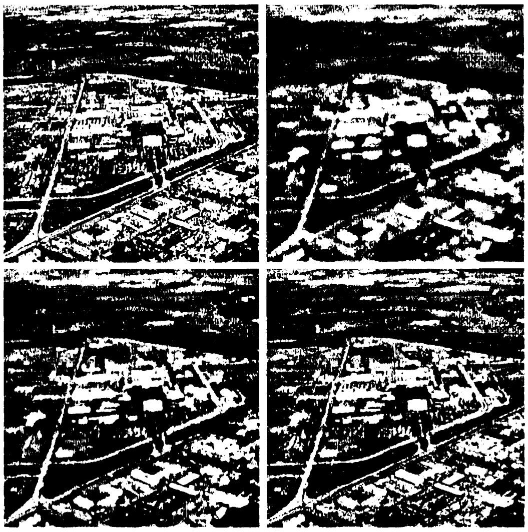

a b
c d

FIGURE 5.25 Illustration of the atmospheric turbulence model. (a) Negligible turbulence. (b) Severe turbulence, $k = 0.0025$. (c) Mild turbulence, $k = 0.001$. (d) Low turbulence, $k = 0.00025$. (Original image courtesy of NASA.)

Image degraded by varying levels of atmospheric turbulence

Estimating the degradation function from a model

Some degradation models have a physical basis

Such a model for atmospheric turbulence is

$$H(u,v) = \underbrace{e^{-k(u^2+v^2)^{5/6}}}_{\text{same form as a Gaussian Low pass filter}}$$

Modeling linear motion as an image degradation

Assume "shutter" opening and closing takes place instantaneously

Let $x_0(t)$, $y_0(t)$ be the time varying x & y motions

For a period T of exposure

$$\underbrace{g(x,y)}_{\text{blurred image}} = \int_0^T \underbrace{f[x-x_0(t),\, y-y_0(t)]}_{\text{moving image}}\, dt$$

Fourier transforming

$$G(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, e^{-j2\pi(ux+vy)}\, dx\,dy$$

substituting $$G(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \int_0^T f[x-x_0(t),\, y-y_0(t)]\, dt\; e^{-j2\pi(ux+vy)}\, dx\,dy$$

reverse order $$G(u,v) = \int_0^T \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f[x-x_0(t),\, y-y_0(t)]\, e^{-j2\pi(ux+vy)}\, dx\,dy\right] dt$$

$$G(u,v) = \int_0^T F(u,v)\, \underbrace{e^{-j2\pi[u x_0(t) + v y_0(t)]}}_{\text{phase shift due to shift of } f(x,y)}\, dt$$

$$G(u,v) = F(u,v) \underbrace{\int_0^T e^{-j2\pi[u x_0(t) + v y_0(t)]}\, dt}_{\text{call this } H(u,v) \text{, the Fourier transform of the degradation}}$$

$$G(u,v) = H(u,v)\, F(u,v)$$

For simple linear motion $x_o(t) = \frac{at}{T}$, $y_o(t) = 0$

we can derive

$$H(u,v) = \int_0^T e^{-j2\pi u x_o(t)} dt$$

$$H(u,v) = \int_0^T e^{-j2\pi \frac{ua t}{T}} dt$$

$$H(u,v) = \frac{T}{\pi u a} \sin(\pi u a)\, e^{-j\pi u a}$$

If we allow $y_o = \frac{bt}{T}$ as well, the degradation function becomes

$$H(u,v) = \frac{T}{\pi(ua+vb)} \sin\left[\pi(ua+vb)\right] e^{-j\pi(ua+vb)}$$

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a b

FIGURE 5.26 (a) Original image. (b) Result of blurring using the function in Eq. (5.6-11) with $a = b = 0.1$ and $T = 1$.

spatial inverse transform of degraded image $G(u,v)$

This is simulated motion degradation

688 × 688 pixel image of Gonzalez & Woods, 1/e

motion is given by

$$\left.\begin{aligned} x_0(t) &= 0.1t \\ y_0(t) &= 0.1t \end{aligned}\right\} \quad \begin{aligned} &a = b = 0.1 \text{ (the velocity)} \\ &T = 1 \text{ (the period of motion)} \end{aligned}$$

$$H(u,v) = \frac{1}{0.1\pi(u+v)} \sin[0.1\pi(u+v)]\, e^{-j0.1\pi(u+v)}$$

5.7 Inverse Filtering

Degraded image is given by $G(u,v) = \underbrace{H(u,v)}_{\text{degradation function}} F(u,v) + \underbrace{N(u,v)}_{\text{noise}}$

Estimate $\tilde{F}(u,v)$ by simply dividing $G(u,v)$ by $H(u,v)$ $\quad \tilde{F}(u,v) = \frac{G(u,v)}{H(u,v)}$

Then $$\tilde{F}(u,v) = \frac{H(u,v)\,F(u,v) + N(u,v)}{H(u,v)}$$

$$\tilde{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

Can never recover $F(u,v)$ exactly

1. $N(u,v)$ is not known since $n(x,y)$ is a random variable
2. If $H(u,v) \to 0$ then $\frac{N(u,v)}{H(u,v)}$ will dominate

   This can be somewhat overcome by restricting the analysis to values near the origin.

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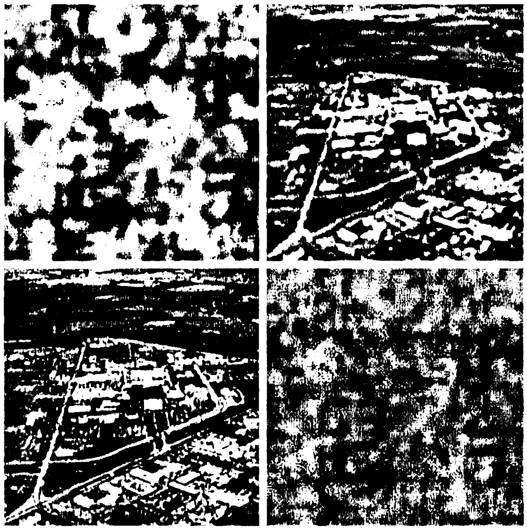
a b
c d

FIGURE 5.27 Restoring Fig. 5.25(b) with Eq. (5.7-1). (a) Result of using the full filter. (b) Result with H cut off outside a radius of 40; (c) outside a radius of 70; and (d) outside a radius of 85.



$D_0 = 40$

$D_0 = 70$

$D_0 = 85$

anything above this resembled (a)

This example shows the problems of small values of $H(u,v)$ in the inversion process.

For $H(u,v) = e^{-k\left[\left(u-\frac{M}{2}\right)^2+\left(v-\frac{N}{2}\right)^2\right]^{5/6}}$

This is never zero but can get small.

Using $\hat{F}(u,v) = \dfrac{G(u,v)}{H(u,v)}$ gives (a) above.

We can improve the result by cutting off values of $\dfrac{G(u,v)}{H(u,v)}$ outside a radius D_0.

The cutoff shown above was done using a Butterworth low-pass filter of order 10.

5.8 Minimum Mean Square Error (Wiener) Filtering

To generate the best estimate $\hat{f}$ of f we minimize

$$e^2 = \underset{\text{expected value}}{E}\left\{ (f-\hat{f})^2 \right\}$$

Assumptions

1. f and n are uncorrelated
2. f and/or n is zero mean
3. gray levels in $\hat{f}$ are a linear function of gray levels in f

Then the best estimate $\hat{F}(u,v)$ is given by

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)\, S_f(u,v)}{S_f(u,v)\,|H(u,v)|^2 + S_\eta(u,v)} \right] G(u,v)$$

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}} \right] G(u,v)$$

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \, \frac{|H(u,v)|^2}{|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}} \right] G(u,v)$$

where $H(u,v)$ = degradation function

$H^*(u,v)$ = complex conjugate of $H(u,v)$

$|H(u,v)|^2 = H^*(u,v)\,H(u,v)$

$S_\eta(u,v) = |N(u,v)|^2$ = power spectrum of noise

$S_f(u,v) = |F(u,v)|^2$ = power spectrum of undegraded image

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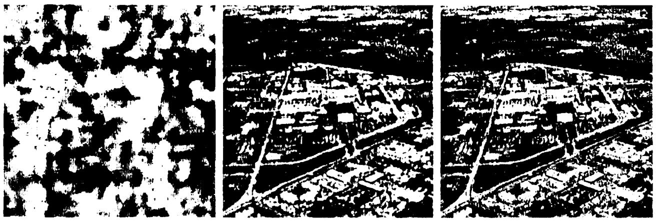

a b c

FIGURE 5.28 Comparison of inverse- and Wiener filtering. (a) Result of full inverse filtering of Fig. 5.25(b). (b) Radially limited inverse filter result. (c) Wiener filter result.

just computing $\dfrac{G(u,v)}{H(u,v)}$

$D_0 = 75$
radially limited $\dfrac{G(u,v)}{H(u,v)}$

Wiener filtering using interactive values of K

In practice $S_f(u,v) = |F(u,v)|^2$ of the undegraded image is not usually known.

So we simply replace $\dfrac{S_\eta(u,v)}{S_f(u,v)}$ by a constant K

$$\hat{F}(u,v) = \frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)| + K} G(u,v)$$