DEFINITIONS

Expansion function – can be a basis for a set of functions

$$f(x) = \sum_k \alpha_k \varphi_k(x)$$

Scaling function – integer translations and binary scalings of a function $\varphi(x)$

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

Wavelet function – spans the difference space W_j

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$$

Refinement equation — (MRA eqn, dilation equation)

$$\underbrace{\varphi(x)}_{\text{expansion functions of } V_j} = \sum_n \underbrace{h_\varphi(n)}_{\text{these are just expansion coefficients}} \sqrt{2}\, \underbrace{\varphi(2x-n)}_{\text{expansion functions of } V_{j+1}}$$

This applies to wavelets giving the wavelet equation

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\,\psi(2x-n)$$

since there are orthogonal space requirements for wavelets $\longrightarrow$ $h_\psi(n) = (-1)^n h_\varphi(1-n)$

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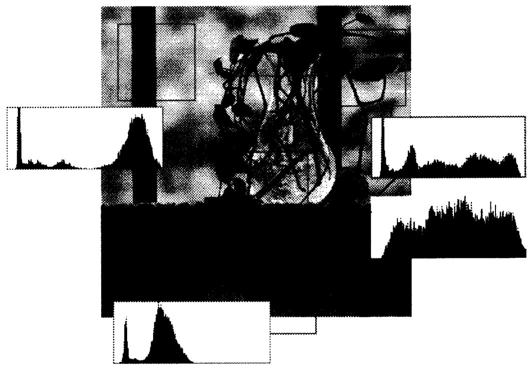
FIGURE 7.1 A natural image and its local histogram variations.



different areas of the image have different histogram distributions resolution decreases by 2^{-2} with each level

down

up

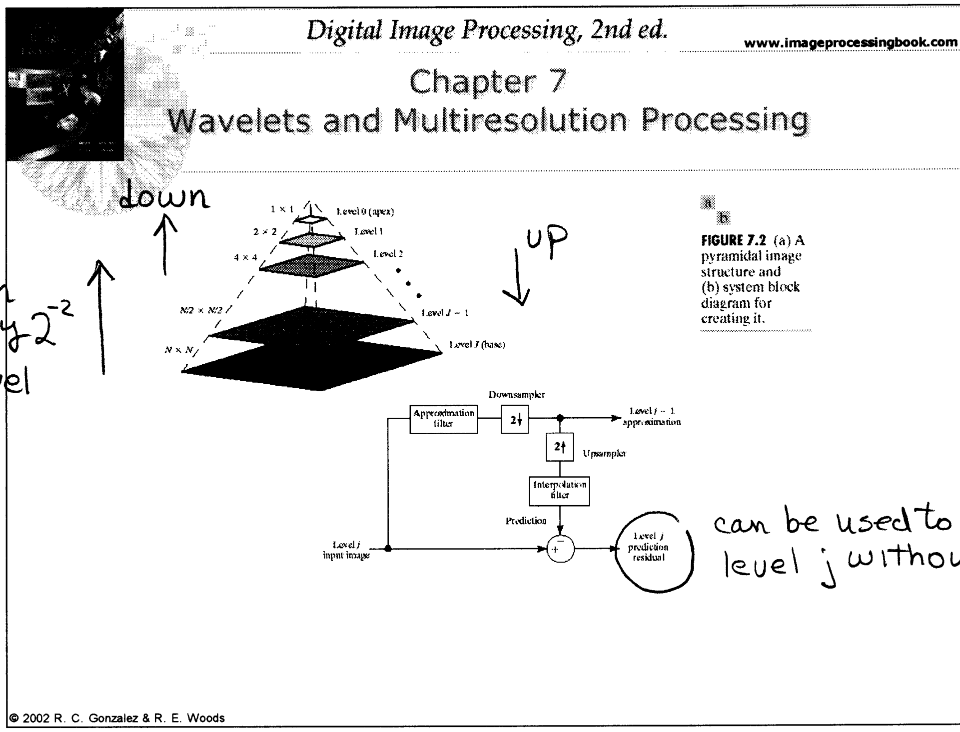


FIGURE 7.2 (a) A pyramidal image structure and (b) system block diagram for creating it.

can be used to reconstruct level j without error.

This is a mathematical representation of what you do as you go up (interpolate) or down (downsample) the image pyramid.

1. The idea is that you compute a reduced resolution image by filtering and down sampling (sub-sampling). Level j-1

   You can use averaging – mean pyramid
   Gaussian LP – Gaussian pyramid
   no filtering – subsampling pyramid

2. To verify how well the downsampling works you upsample and filter the result. This "interpolation" filter reduces the effects of pixel replication (duplication).

3. Compute difference between down sampled/upsampled image and the level j image is called **the level j prediction residual.** You have two pyramids if you keep the residuals.

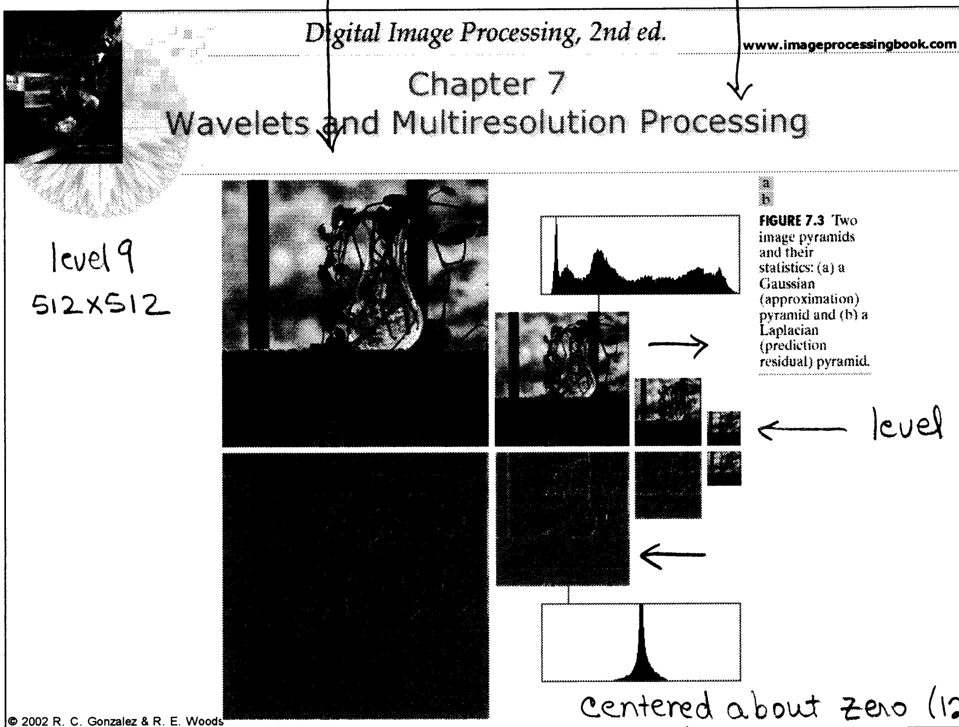
high resolution
good for analyzing
individual objects

low resolution good for
large structures or overall image

Gaussian

level 9
512x512

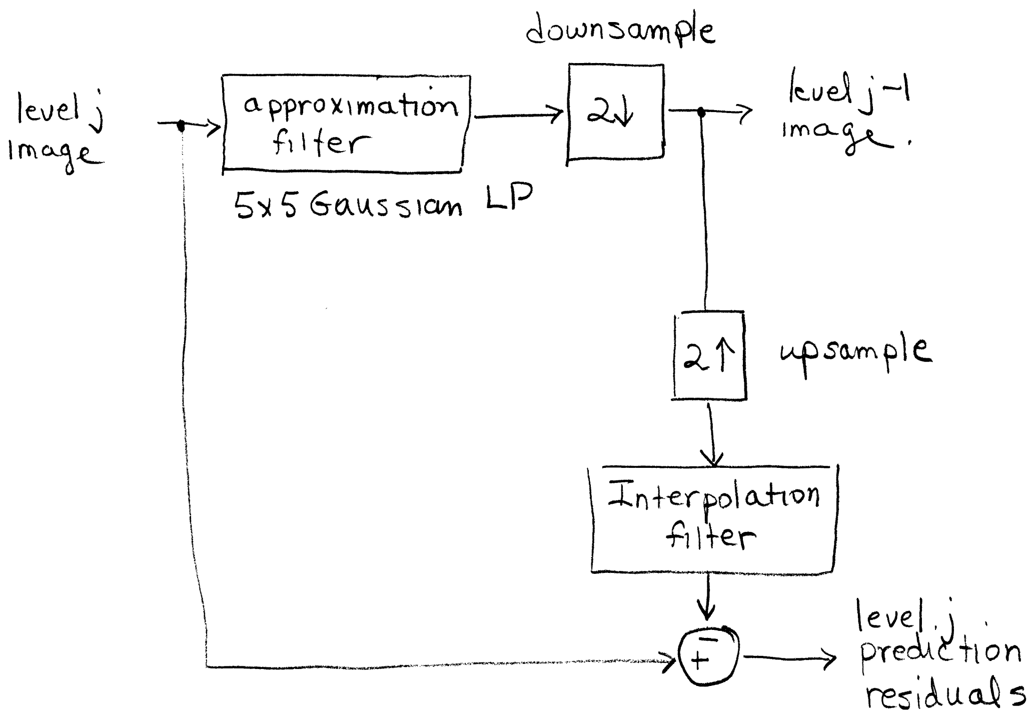
prediction
residuals



level 6 64x64

centered about zero (128)
can be highly compressed

This is an example of a Gaussian pyramid.



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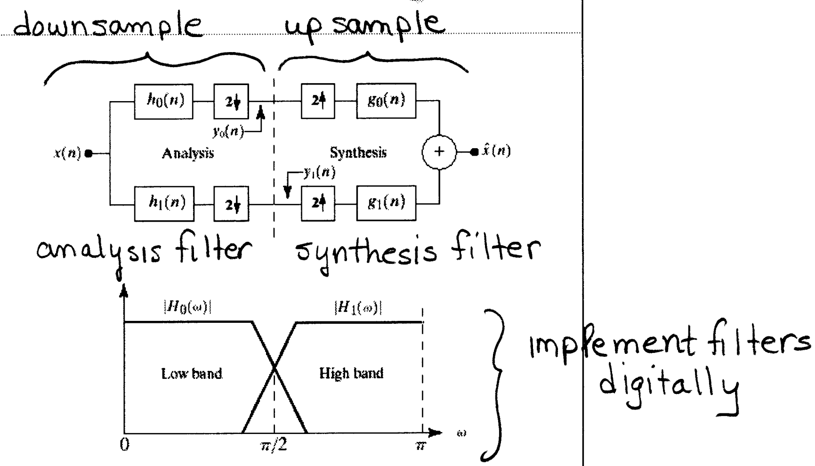

a
b

FIGURE 7.4 (a) A two-band filter bank for one-dimensional subband coding and decoding, and (b) its spectrum splitting properties.

2-band filter bank – each is lower resolution and can be downsampled

This approach was originally developed for speech analysis.

BASIC Z-TRANSFORMS

We want to use z-transforms to implement digital filters. This also is very amenable to up sampling and down sampling.

z-transform $$X(z) = \sum_{-\infty}^{+\infty} x(n) z^{-n}$$

(becomes Fourier when $z = e^{j\omega}$)

Down sampling:

$$x_{down}(n) = x(2n) \Leftrightarrow X_{down}(z) = \frac{1}{2}\left[X(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})\right]$$

check this out

$$x^{up}(n) = \begin{cases} x(n/2) & n = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow X^{up}(z) = X(z^2)$$

in both cases we go up or down by a factor of 2

Combine these to get an estimate of $x(n)$. Call the estimate $\hat{x}(n)$.

downsampling $\quad \frac{1}{2} X(z^{\frac{1}{2}}) + \frac{1}{2} X(-z^{\frac{1}{2}})$

upsample $\quad \frac{1}{2} X(z) + \frac{1}{2} X(-z)$

$$\Updownarrow$$

$$(-1)^n x(n)$$

Analysis of sub-band coding

output of filter $h_0(n)$

$$h_0(n) * x(n) = \sum_k h_0(n-k)\, x(k) \Leftrightarrow H_0(z) X(z)$$

The overall output $\hat{x}(n)$ is given by

$$\hat{X}(z) = \frac{1}{2} G_0(z) \left[\underbrace{H_0(z) X(z) + H_0(-z) X(-z)}_{\text{down sample/upsample}} \right] + \frac{1}{2} G_1(z) \left[\underbrace{H_1(z) X(z) + H_1(-z) X(-z)}_{\text{downsample/upsample}} \right]$$

Rearrange

$$\hat{X}(z) = \frac{1}{2} \left[H_0(z) G_0(z) + H_1(z) G_0(z) \right] X(z) + \underbrace{\frac{1}{2} \left[H_0(-z) G_0(z) + H_1(-z) G_1(z) \right] X(-z)}_{\text{aliasing term from downsampling}}$$

For error-free reconstruction set

$$H_0(z) G_0(z) + H_1(z) G_0(z) = 2$$

$$H_0(-z) G_0(z) + H_1(-z) G_1(z) = 0$$

In matrix form

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \underbrace{\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}}_{\text{analysis modulation matrix } \underline{H}_m(z)} = \begin{bmatrix} 2 & 0 \end{bmatrix}$$

Now transpose to get

$$\underbrace{\begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix}}_{\underline{H}_m^T(z)} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Left multiply by $\left(\underline{H}_m^T(z)\right)^{-1}$

$$\underbrace{\left(\underline{H}_m^T(z)\right)^{-1}\left(\underline{H}_m^T(z)\right)}_{\underline{I}}\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \left(\underline{H}_m^T(z)\right)^{-1}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{1}{\det\left(\underline{H}_m(z)\right)} \underbrace{\begin{bmatrix} H_1(-z) & -H_0(-z) \\ -H_1(z) & H_0(z) \end{bmatrix}^T}_{\text{adjoint matrix}} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\det\left(\underline{H}_m(z)\right)} \begin{bmatrix} H_1(-z) & -H_1(z) \\ -H_0(-z) & H_0(z) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \frac{2}{\det\left(\underline{H}_m(z)\right)} \begin{bmatrix} H_1(-z) \\ -H_0(-z) \end{bmatrix}$$

observations:

1. $G_0(z)$ is a function of $H_1(z)$; $G_1(z)$ is related to $-H_0(-z)$
2. $h_0(z)$ and $g_1(z)$; $h_1(z)$ and $g_0(z)$ are cross-modulated (i.e., $-z$)

   For FIR filters $\det\left(\underline{H}_m(z)\right) = \alpha z^{-(2k+1)}$, a pure delay.

   Ignoring the delay and letting $\alpha = 2$

   $$G_0(z) = H_1(-z) \quad \text{i.e. } g_0(n) = (-1)^n h_1(n)$$

   $$G_1(z) = -H_0(-z) \quad \text{i.e. } g_1(n) = (-1)^{n+1} h_0(n)$$

   $\Rightarrow$ the FIR synthesis filters are cross-modulated copies of the analysis filters.

3. The analysis and synthesis filters are biorthogonal

Write $P(z) = G_0(z)H_0(z) = \dfrac{2}{\det(\underline{H}_m(z))} H_1(-z)H_0(z)$

Then

$$G_1(z)H_1(z) = \frac{-2}{\det(\underline{H}_m(z))} H_0(-z)H_1(z) = P(-z)$$

$$\therefore \quad G_1(z)H_1(z) = P(-z) = G_0(-z)H_0(-z)$$

Going back to the modulation matrix

$$H_0(z)G_0(z) + H_1(z)G_1(z) = 2$$

becomes

$$G_0(z)H_0(z) + G_0(-z)H_0(-z) = 2.$$

Inverse transforming

$$\sum_k g_0(k)h_0(n-k) + (-1)^k \sum g_0(k)h_0(n-k) = 2\delta(n)$$

All odd terms cancel giving

$$\sum_k g_0(k)h_0(\underset{\substack{\uparrow \\ \text{even}}}{2n}-k) = \langle g_0(k), h_0(2n-k)\rangle = \delta(n)$$

Similarly,

$$\langle g_1(k), h_1(2n-k)\rangle = \delta(n)$$

$$\langle g_0(k), h_1(2n-k)\rangle = 0$$

$$\langle g_1(k), h_0(2n-k)\rangle = 0$$

These are biorthogonal filters in general.

$$\langle h_i(2n-k), g_j(k)\rangle = \delta(i-j)\,\delta(n) \qquad i,j = \{0,1\}$$

There are classes of filters which satisify this.

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TABLE 7.1 Perfect reconstruction filter families.

Filter	QMF	CQF	Orthonormal
$H_0(z)$	$H_0^2(z) - H_0^2(-z) = 2$	$H_0(z)H_0(z^{-1}) + H_0^2(-z)H_0(-z^{-1}) = 2$	$G_0(z^{-1})$
$H_1(z)$	$H_0(-z)$	$z^{-1}H_0(-z^{-1})$	$G_1(z^{-1})$
$G_0(z)$	$H_0(z)$	$H_0(z^{-1})$	$G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1}) = 2$
$G_1(z)$	$-H_0(-z)$	$zH_0(-z)$	$-z^{-2K+1}G_0(-z^{-1})$

← $2K$ = # of filter taps (# of coefficients)

quadrature mirror filters

conjugate quadrature filters

orthonormal
- used for fast Wavelet transform
- define perfect reconstruction by

$$\langle g_i(n), g_j(n+2m)\rangle = \delta(i-j)\delta(m)$$

Smith & Barnwell filter
Daubechies filter
Vaidyanathan & Hoang filter

Digital Image Processing, 2nd ed. www.imageprocessingbook.com

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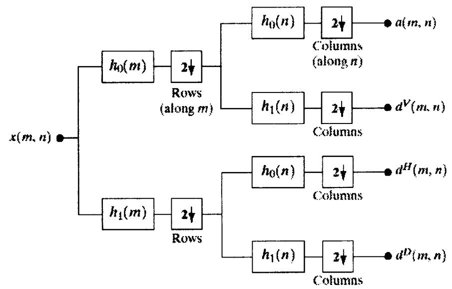

FIGURE 7.5 A two-dimensional, four-band filter bank for subband image coding.

approximation

vertical detail

horizontal detail

diagonal detail

You can apply sub band coding to two-dimensional image data.

Digital Image Processing, 2nd ed. www.imageprocessingbook.com

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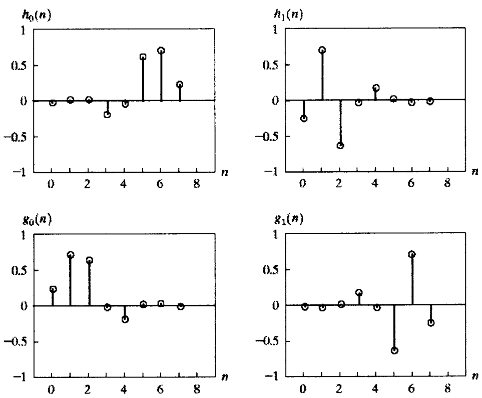
FIGURE 7.6 The impulse responses of four 8-tap Daubechies orthonormal filters.



analysis

synthesis

Impulse responses of four 8-tap orthonormal Daubechies filters.

- note the cross-modulation of analysis & synthesis filters
- the filters are also biorthogonal and orthogonal

7.1.3. The Haar Transform [1910]

oldest and simplest known orthonormal wavelets

Haar transform $\underline{T} = \underline{H}\,\underline{F}\,\underline{H}$

- $\underline{T}$: N×N transform image
- $\underline{F}$: N×N image matrix
- $\underline{H}$: N×N matrix, rows are Haar basis functions

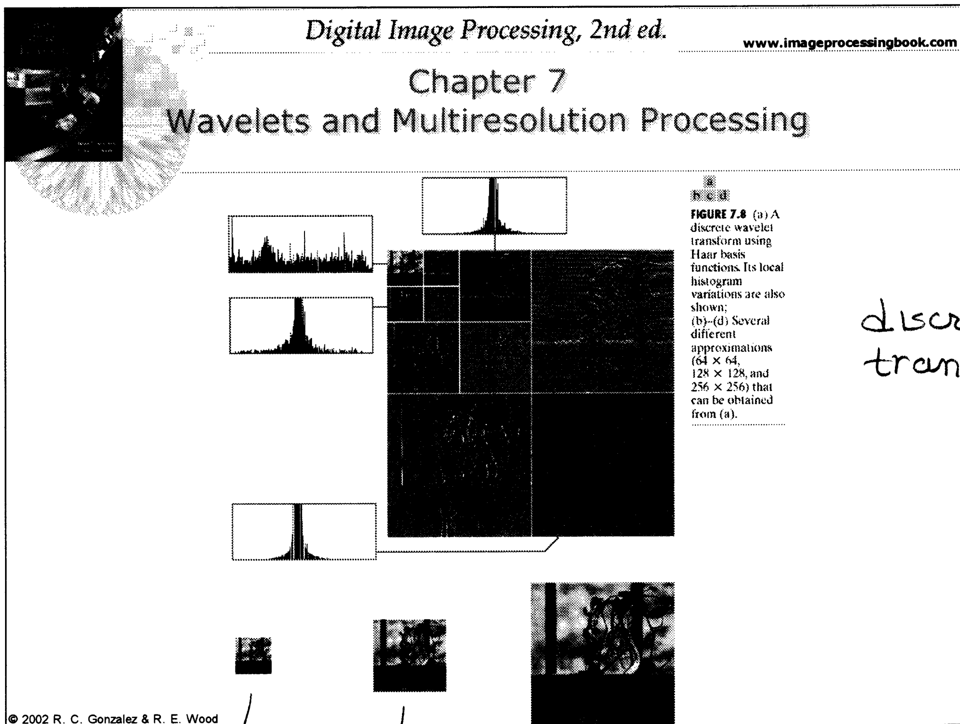
Haar basis functions for wavelets

$$h_0(z) = h_{00}(z) = \frac{1}{\sqrt{N}}$$

$$h_k(z) = h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{(q-1)}{2^p} \le z \le \frac{(q-0.5)}{2^p} \\ -2^{p/2} & \frac{(q-0.5)}{2^p} \le z \le \frac{q}{2^p} \\ 0 & \text{otherwise} \end{cases}$$

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \left.\begin{matrix} h_0(n) \\ h_1(n) \end{matrix}\right\} \text{ for a 2 bank FIR filter}$$

$$H_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$



discrete Haar (wavelet) transform.

64x64 128x128 256x256

Haar transform
1. local statistics are relatively constant
2. many values are close to zero - easy to compress
3. can extract both coarse and fine resolution approximations from the transform

7.2.1 Series Expansions

$$f(x) = \sum_k \underbrace{\alpha_k}_{\text{expansion coefficients}} \underbrace{\varphi_k(x)}_{\substack{\text{expansion functions} \\ \text{(basis functions)}}}$$

$$V = \overline{\operatorname{Span}_k \{\varphi_k(x)\}}$$

function space of the class of functions that can be expressed as an expansion of $\varphi_k(x)$

Also called closed span of $\varphi_k(x)$

$\tilde{\varphi}_k(x)$ is called a dual function

$$\alpha_k = \langle \tilde{\varphi}_k(x), f(x) \rangle = \int \tilde{\varphi}_k^*(x) f(x) dx$$

Case 1: $\langle \varphi_j(x), \varphi_k(x) \rangle = \delta_{jk}$

If the expansion functions form an orthogonal basis set the basis and dual are equivalent and

$$\alpha_k = \langle \varphi_k(x), f(x) \rangle$$

Case 2: If the expansion functions are not orthonormal but orthogonal then the basis and duals are biorthogonal

$$\langle \varphi_j(x), \tilde{\varphi}_k(x) \rangle = \delta_{jk}$$

Case 3: If the expansion set is not a basis for V then it is a spanning set in which there is more than one set of α_k for any $f(x) \in V$,

The expansion functions and their duals form a frame

$$A\|f(x)\|^2 \le \sum_k |\langle \varphi_k(x), f(x) \rangle|^2 \le B\|f(x)\|^2$$

If $A = B$ this is called a tight frame

$$f(x) = \frac{1}{A} \sum_k \langle \varphi_k(x), f(x) \rangle \varphi_k(x)$$

7.2.2. Scaling functions

Consider the set of expansion functions that are integer translations and binary scalings of $\varphi(x)$

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k)$$

k determines position of $\varphi_{j,k}$ along x axis
j determines $\varphi_{j,k}(x)$'s width
$2^{j/2}$ determines height or amplitude
$\varphi(x)$ is called the scaling function

If j is restricted to a specific value then the expansion set $\{\varphi_{j_0,k}(x)\}$ is a subset of $\{\varphi_{j,k}(x)\}$

Increasing j increases the size of $V_j = \overline{\mathrm{Span}_k \{\varphi_{j,k}(x)\}}$

$\varphi_{0,0}(x) = \varphi(x)$

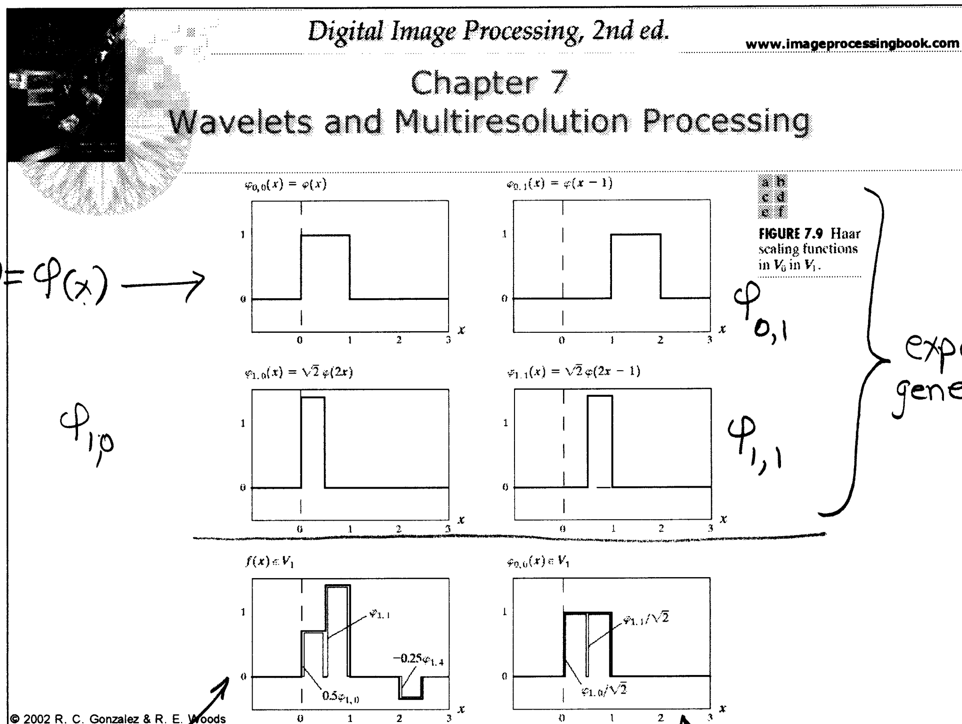


FIGURE 7.9 Haar scaling functions in V_0 in V_1.

$\varphi_{0,1}$

$\varphi_{1,0}$

$\varphi_{1,1}$

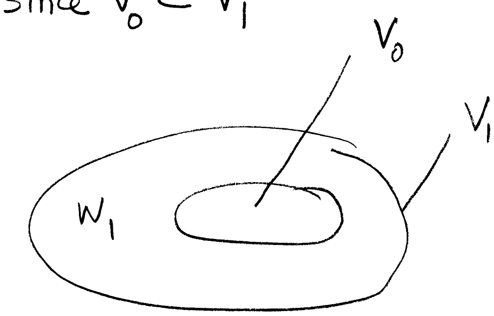
expansion functions generated from $\varphi(x)$

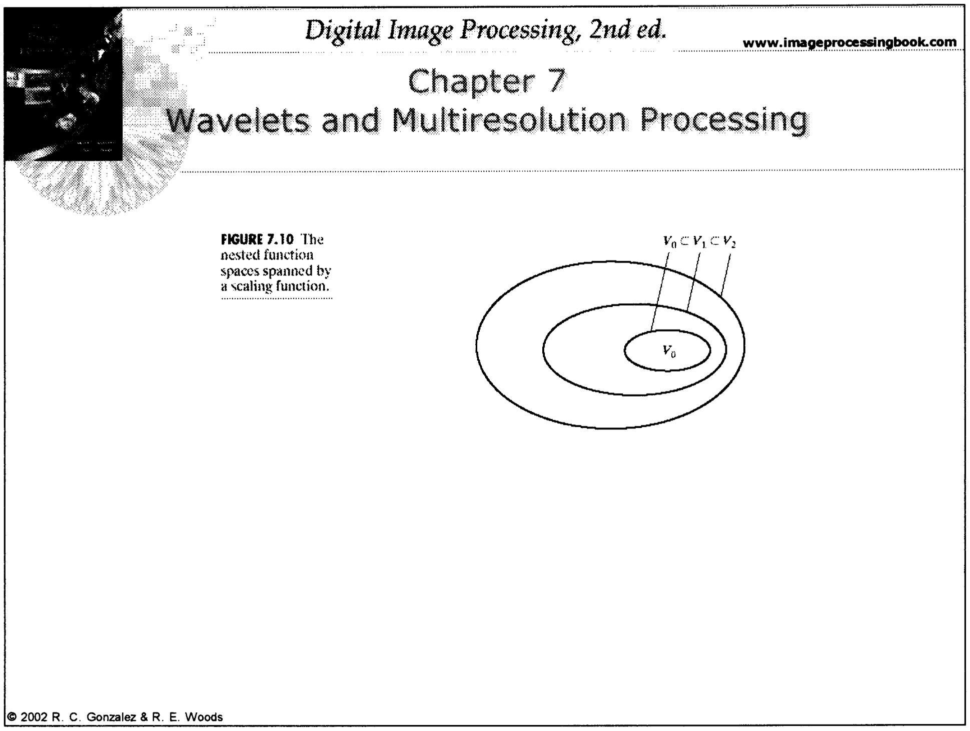
$$f(x) = 0.5\varphi_{1,0}(x) + \varphi_{1,1}(x) - 0.25\varphi_{1,4}(x)$$

decomposition of $\varphi_{0,0}$

$$\varphi_{0,0}(x) = \frac{1}{\sqrt{2}}\varphi_{1,0}(x) + \frac{1}{\sqrt{2}}\varphi_{1,1}$$

since $V_0 \subset V_1$





Multi-Resolution Analysis Requirements (Mallat [1989])

1. The scaling function is orthogonal to its integer translates.

   Example: Haar scaling function has compact support, i.e., zero everywhere outside a finite interval called the support. It is zero on [0,1). Its support is 1.

2. The subspaces spanned by the scaling function at low scales are nested within those spanned at higher scales.

   Figure 7.10

3. The only function that is common to all V_j is $f(x)=0$, the coarsest possible expansion function.

4. Any function can be represented with arbitrary precision as $j \to \infty$.

7.3 Wavelet transforms in one dimension

7.3.1 The wavelet series expansion

$$f(x) = \underbrace{\sum_k c_{j_0}(k)}_{\substack{\text{approximation}\\ \text{(or scaling)}\\ \text{coefficients}}} \underbrace{\varphi_{j_0,k}(x)}_{\substack{\text{scaling}\\ \text{function}}} + \underbrace{\sum_{j=j_0}^{\infty} \underbrace{\sum_k d_j(k)}_{\substack{\text{detail}\\ \text{(or wavelet)}\\ \text{coefficients}}} \underbrace{\psi_{j,k}(x)}_{\text{wavelets}}}_{\substack{\text{adds a finer resolution}\\ \text{function (sum of wavelets)}\\ \text{for each } j \geq j_0}}$$

Note: j_0 is the starting scale

$$c_{j_0}(k) = \langle f(x), \varphi_{j_0,k}(x) \rangle = \int f(x)\, \varphi_{j_0,k}(x)\, dx$$

$$d_j(k) = \langle f(x), \psi_{j,k}(x) \rangle = \int f(x)\, \psi_{j,k}(x)\, dx$$

$$\varphi_{j,k}(x) = \sum_n \alpha_n \varphi_{j+1,n}(x)$$

i.e., expansion functions of V_j can be expressed as a weighted sum of the expansion functions of V_{j+1} (the space which includes V_j)

For the set of integer translations and binary scalings

$$\varphi_{j,k}(x) = \sum_n \underbrace{h_\varphi(n)}_{\text{just change of variable}} 2^{\frac{j+1}{2}} \varphi(2^{j+1}x - n)$$

integer translation

determines amplitude of function

determines width of function

We use the fact that $\varphi(x) = \varphi_{0,0}(x)$ and usually write this equation without subscripts as

$$\varphi(x) = \sum_n h_\varphi(n) \sqrt{2}\, \varphi(2x-n)$$

This is a recursive equation and is often called the refinement equation since it says that the expansion functions can be built from double resolution copies of themselves.

For the Haar function $\varphi(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$.

we get $h_\varphi(0) = h_\varphi(1) = \frac{1}{\sqrt{2}}$

and $\varphi(x) = \frac{1}{\sqrt{2}} \sqrt{2}\, \varphi(2x) + \frac{1}{\sqrt{2}} \sqrt{2}\, \varphi(2x-1) = \varphi(2x) + \varphi(2x-1)$

show's the translation and scaling to get 7.9 (f).

This is the first row of $\underline{H_2}$

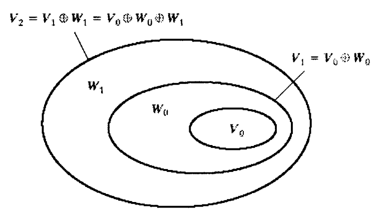


FIGURE 7.11 The relationship between scaling and wavelet function spaces.

Define a wavelet function

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)$$

which spans

$$W_j = \overline{\operatorname{Span}_k \{\psi_{j,k}(x)\}}$$, i.e., The difference space

$$V_{j+1} = V_j \oplus W_j$$

space spanned by the lower resolution functions

union of spaces

space spanned by the higher resolution functions. This is called the wavelet space.

wavelet equation

The wavelet functions which span the difference space can be written as an expansion of double resolution scaling functions.

$$\psi(x) = \sum_n h_\psi(n)\,\sqrt{2}\,\varphi(2x-n)$$

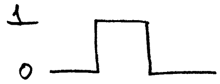
wavelet function coefficients

double resolution scaling function

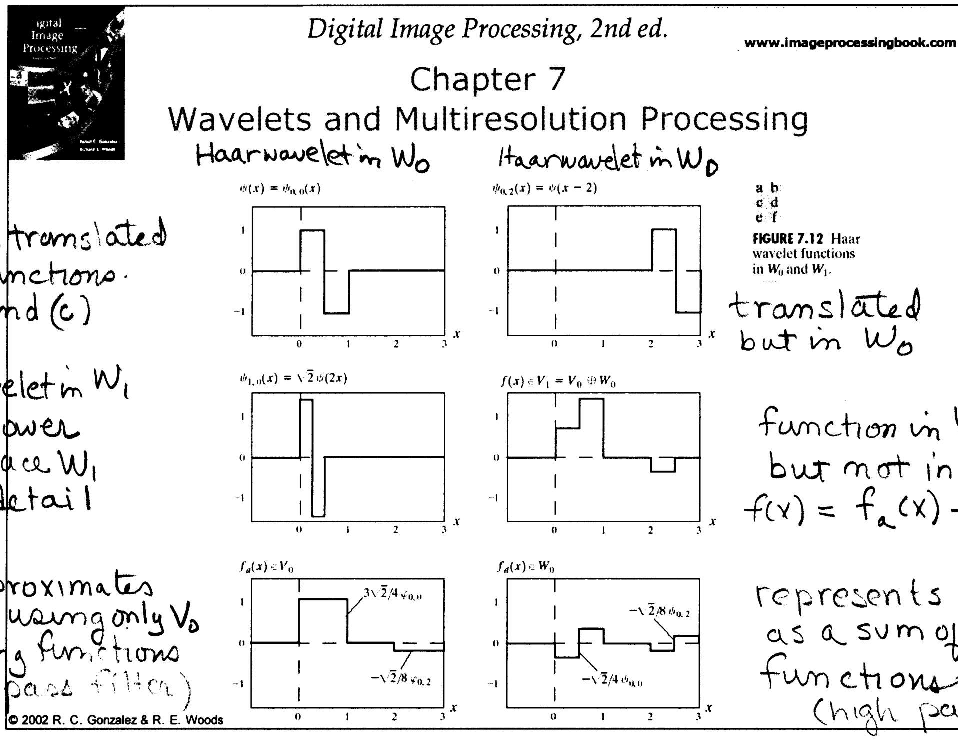
$$\underbrace{h_\psi(n) = (-1)^n\, h_\varphi(1-n)}_{\text{can be shown}}$$

and h_ψ is the wavelet vector

Haar function in V_0



is $\varphi_{0,0}$



scaled & translated wavelet functions in (b) and (c)

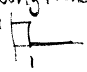
translated but in W_0

Haarwavelet in W_1 narrower for space W_1 finer detail

function in $V_1 = V_0 \oplus W_0$ but not in V_0

$f(x) = f_a(x) + f_d(x)$

basic scaling function $\varphi(x)$



approximates $f(x)$ using only V_0 scaling functions (lowpass filter)

represents $f(x) - f_a(x)$ as a sum of wavelet W_0 functions (high pass filter)

We previously defined the Haar scaling vector as

$$h_\varphi(0) = h_\varphi(1) = \frac{1}{\sqrt{2}}$$

Since $h_\psi(n) = (-1)^n h_\varphi(1-n)$

$$h_\psi(0) = (-1)^0 h_\varphi(1-0) = \frac{1}{\sqrt{2}}$$

$$h_\psi(1) = (-1)^1 h_\varphi(1-1) = -\frac{1}{\sqrt{2}}$$

$$\underline{H}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\underline{H}_2 = \begin{bmatrix} h_\varphi(0) & h_\varphi(1) \\ h_\psi(0) & h_\psi(1) \end{bmatrix}$$

This is the second row of $\underline{H}_2$

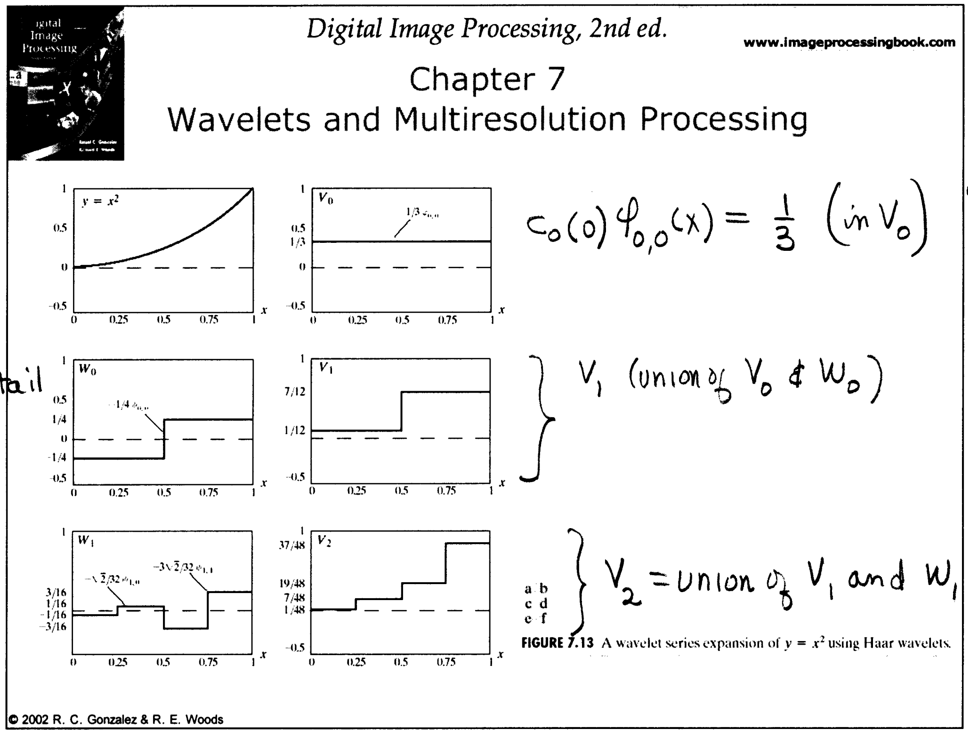
We use the wavelet equation with these values to get

$$\psi(x) = h_\psi(0)\sqrt{2}\,\varphi(2x-0) + h_\psi(1)\sqrt{2}\,\varphi(2x-1)$$

$$= \varphi(2x) - \varphi(2x-1)$$

$\therefore$ The Haar wavelet function as shown in 7.12(a).

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$



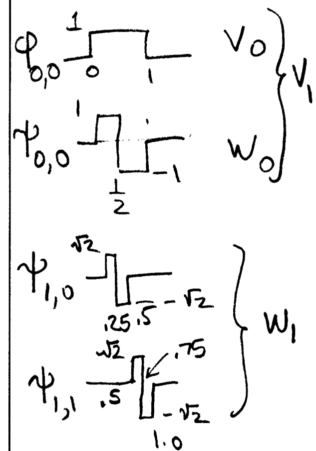
$c_0(0)\,\varphi_{0,0}(x) = \frac{1}{3}$ (in V_0)

next level of detail from wavelet W_0

$\Big\}$ V_1 (union of V_0 & W_0)

detail from subspace coefficients W_1

$\Big\}$ V_2 = union of V_1 and W_1



Use Haar wavelets

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad y = \begin{cases} x^2 & 0 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\psi(x) = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$c_0(0) = \int_0^1 x^2 \varphi_{0,0}(x)\,dx = \int_0^1 x^2 \cdot 1\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$d_0(0) = \int_0^1 x^2 \psi_{0,0}(x)\,dx = \int_0^{0.5} x^2 \cdot 1\,dx + \int_{0.5}^1 x^2 \cdot (-1)\,dx = -\frac{1}{4}$$

next level of resolution

$$d_1(0) = \int_0^1 x^2 \psi_{1,0}(x)\,dx = \int_0^{.25} x^2(\sqrt{2})\,dx + \int_{0.25}^{0.5} x^2(-\sqrt{2})\,dx = -\frac{\sqrt{2}}{32}$$

$$d_1(1) = \int_0^1 x^2 \psi_{1,1}(x)\,dx = \int_{0.5}^{0.75} x^2(\sqrt{2})\,dx + \int_{.75}^{1} x^2(-\sqrt{2})\,dx = -\frac{3\sqrt{2}}{32}$$

$$y = \underbrace{\underbrace{\underbrace{\frac{1}{3}\varphi_{0,0}(x)}_{V_0} + \underbrace{\left[-\frac{1}{4}\psi_{0,0}(x)\right]}_{W_0}}_{V_1 = V_0 \oplus W_0} + \underbrace{\left[-\frac{\sqrt{2}}{32}\psi_{1,0}(x) - \frac{3\sqrt{2}}{32}\psi_{1,1}(x)\right]}_{W_1}}_{V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1} + \cdots$$

7.3.2. The discrete Wavelet transform

$$f(x) = \sum_k c_{j_0}(k)\, \varphi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_k d_j(k)\, \psi_{j,k}(x)$$

If $f(x)$ is a discrete sequence we get the discrete wavelet transform

$$f(x) = \frac{1}{\sqrt{M}} \sum_k W_\varphi(j_0,k)\, \varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}} \sum_{j=j_0}^{\infty} \sum_k W_\psi(j,k)\, \psi_{j,k}(x)$$

$x = 0, 1, 2, \ldots, M-1$

normally $j_0 = 0$ and $M = 2^J$ giving

$j = 0, 1, 2, \ldots, J-1$

$k = 0, 1, 2, \ldots, 2^j - 1$

For Haar wavelets the discretized scaling and wavelet functions are the rows of the Haar matrix

minimum scale is 0

maximum scale is $J-1$

7.3.3 Continuous Wavelet Transform (CWT)

CWT $\quad W_\psi(s,\tau) = \int_{-\infty}^{\infty} f(x)\, \psi_{s,\tau}(x)\, dx$

where $\psi_{s,\tau}(x) = \frac{1}{\sqrt{s}}\, \psi\left(\frac{x-\tau}{s}\right)$

inversely related to scale

s = scale parameter

τ = translation parameter

Inverse CWT

$$f(x) = \frac{1}{C_\psi} \int_0^{\infty} \int_{-\infty}^{+\infty} W_\psi(s,\tau)\, \frac{\psi_{s,\tau}(x)}{s^2}\, d\tau\, ds$$

where $C_\psi = \int_{-\infty}^{+\infty} \frac{|\Psi(u)|^2}{|u|}\, du$

Similarities to the discrete transform

1. continuous parameter τ integer translation k

2. continuous scale parameter s inversely related to scale parameter 2^j
   continuous compressed when $0 < s < 1$ and dilated or expanded when $s > 1$

3. starting scale is $j_0 = -\infty$ so expansion is in terms of only wavelets

4. The CWT can be viewed as an infinite set of transform coefficients $\{W_\psi(s,\tau)\}$ that measure the similarity of $f(x)$ with a set of basis functions $\{\psi_{s,\tau}(x)\}$

7.4 The Fast Wavelet Transform

Start with the multiresolution refinement equation

$$\varphi(x) = \sum_n h_\varphi(n)\sqrt{2}\,\varphi(2x-n)$$

Scale x by 2^j, translate by k, and let $m = 2k+n$

$$\varphi(2^j x - k) = \sum_n h_\varphi(n)\sqrt{2}\,\varphi\left(2(2^j x - k) - n\right)$$

$$\varphi(2^j x - k) = \sum_m h_\varphi(m-2k)\sqrt{2}\,\varphi(2^{j+1}x - m)$$

The corresponding wavelets come from

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\,\varphi(2x-n)$$

which, for this case gives,

$$\psi(2^j x - k) = \sum_m h_\psi(m-2k)\sqrt{2}\,\varphi(2^{j+1}x - m)$$

Substituting into the discrete wavelet transform

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\,\varphi_{j_0,k}(x)$$

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\,\psi_{j,k}(x)$$

shift

resolution

$$f(x) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(x)$$

Substituting gives

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\,2^{j/2}\,\psi(2^j x - k)$$

and using our expression from above

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\,2^{j/2}\left[\sum_m h_\psi(m-2k)\sqrt{2}\,\varphi(2^{j+1}x-m)\right]$$

Interchanging summations and re-arranging

$$W_\psi(j,k) = \sum_m h_\psi(m-2k) \underbrace{\left[\frac{1}{\sqrt{m}} \sum_x f(x)\, 2^{\frac{(j+1)}{2}} \varphi(2^{j+1}x - m) \right]}_{\text{This is the discretized approximation coefficient}}$$

so we have

$$W_\psi(j,k) = \sum_m h_\psi(m-2k)\, W_\varphi(j+1,m)$$

We can start by substituting $\varphi(2^j x - k)$ into the expression for $W_\varphi(j_0,k)$ to get

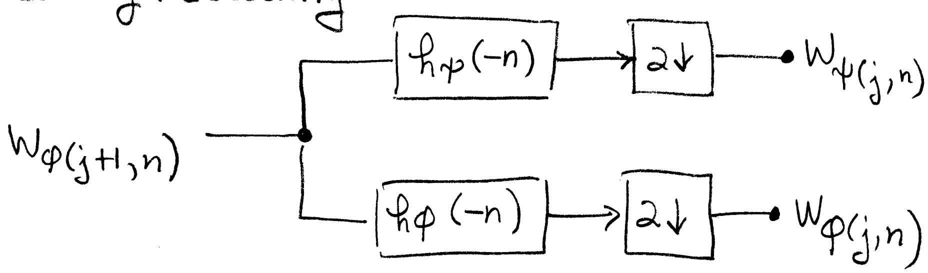
$$W_\varphi(j,k) = \sum_m h_\varphi(m-2k)\, W_\varphi(j+1,m)$$

Observation

the scale j approximation and detail coefficients

convolution of scale j+1 approximation and time reversed scaling and wavelet vectors.
which is subsampled (the k's) since it is 2k

This is identical to the analysis portion of the two-band subband coding & decoding.



$$W_\psi(j,k) = h_\psi(-n) * W_\varphi(j+1,n) \Big|_{n=2k,\, k\geq 0}$$

$$W_\varphi(j,k) = h_\varphi(-n) * W_\varphi(j+1,n) \Big|_{n=2k,\, k\geq 0}$$

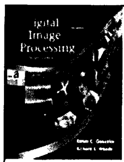


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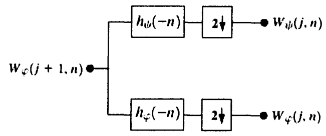
FIGURE 7.15 An FWT analysis bank.



This structure can be "iterated" to compute the DWT for two or more scales.

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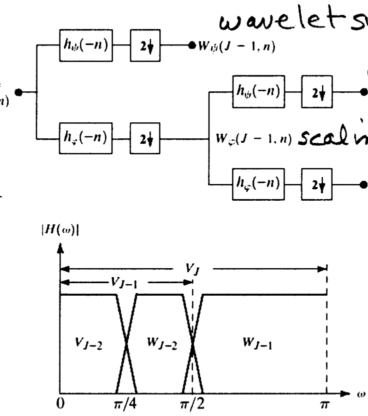

a
b

FIGURE 7.16 (a) A two-stage or two-scale FWT analysis bank and (b) its frequency splitting characteristics.

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start with scale J

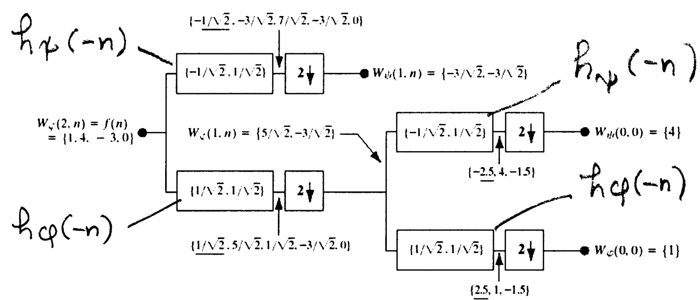


FIGURE 7.17 Computing a two-scale fast wavelet transform of sequence {1, 4, −3, 0} using Haar scaling and wavelet vectors.

scale 2 scale 1 scale 0

Example. $f(n) = \{1, 4, -3, 0\}$

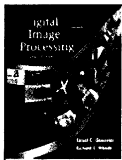
Use the Haar scaling and wavelet vectors.

scaling $h_\varphi(n) = \begin{cases} \frac{1}{\sqrt{2}} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$

wavelet $h_\psi(n) = \begin{cases} \frac{1}{\sqrt{2}} & n = 0 \\ -\frac{1}{\sqrt{2}} & n = 1 \\ 0 & \text{otherwise} \end{cases}$

only uses 0 & 1 coefficients

These are the coefficients to build the FWT filters.



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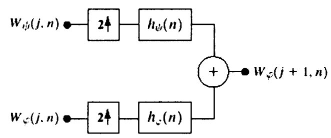

FIGURE 7.18 The FWT^{-1} synthesis filter bank.

This is the corresponding inverse filter.

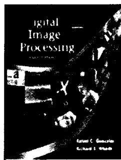

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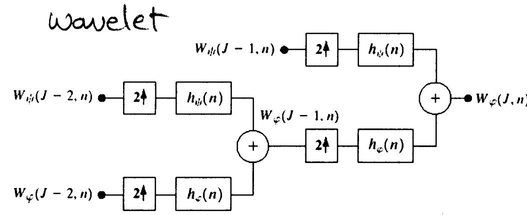
FIGURE 7.19 A two-stage or two-scale FWT^{-1} synthesis bank.

wavelet

wavelet

scaling



The inverse filter can also be scaled.
This is called a synthesis bank.

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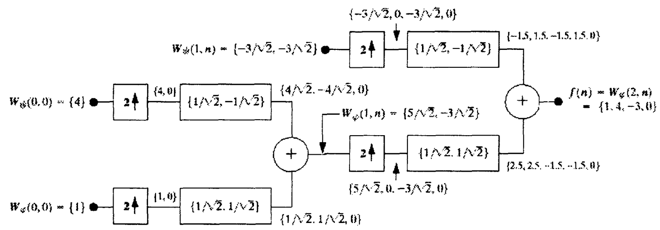

FIGURE 7.20 Computing a two-scale inverse fast wavelet transform of sequence $\{1, 4, -1.5\sqrt{2}, -1.5\sqrt{2}\}$ with Haar scaling and wavelet vectors.

This is the inverse wavelet filter bank for the Haar example.

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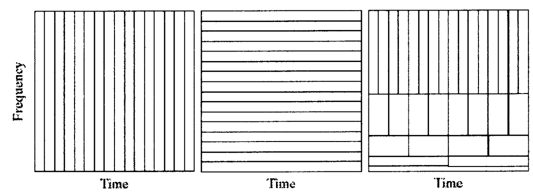


better time resolution

better frequency resolution

a b c

FIGURE 7.21 Time-frequency tilings for (a) sampled data, (b) FFT, and (c) FWT basis functions.

sampled data
all frequencies
at point in time

discrete
frequencies
for each time

each tile
has same area

differences between the FWT and FFT

1. O(m) operation's for FWT. FFT requires O(m log m)
2. requires both scaling function and orthogonality of scaling function and wavelets.
   This restricts functions we can use to those like Haar.
3. FWT links time and frequency uncertainties.
   similar to Heisenberg uncertainty principle

7.5 Wavelet transforms in two dimensions

two-dimensional scaling function $\varphi(x,y)$

$$\varphi(x,y) = \varphi(x)\,\varphi(y)$$

two dimensional wavelets $\psi(x,y)$

$$\psi^H(x,y) = \psi(x)\,\varphi(y)$$
$$\psi^V(x,y) = \varphi(x)\,\psi(y)$$
$$\psi^D(x,y) = \psi(x)\,\psi(y)$$

Define scale and translated basis functions

$$\varphi_{j,m,n}(x,y) = 2^{j/2}\varphi(2^j x - m, 2^j y - n)$$
$$\psi^i_{j,m,n}(x,y) = 2^{j/2}\psi^i(2^j x - m, 2^j y - n)$$

superscript identifies H, V, D

The corresponding discrete wavelet transform

$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$ approximation

$$W^i_\psi(j,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi^i_{j,m,n}(x,y)$$ detail $i=\{H,V,D\}$

j_0 – arbitrary starting scale

Inverse discrete wavelet transform

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_m\sum_n W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$
$$+\frac{1}{\sqrt{MN}}\sum_{i=H,V,D}\sum_{j=j_0}^{\infty}\sum_m\sum_n W^i_\psi(j,m,n)\,\psi^i_{j,m,n}(x,y)$$

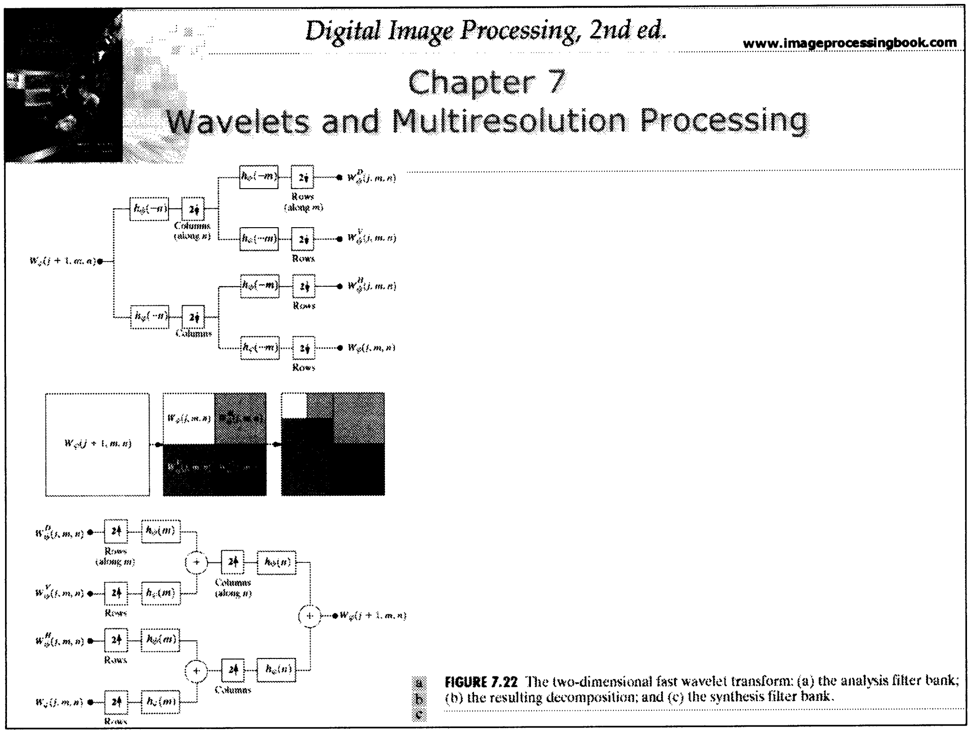


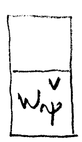
FIGURE 7.22 The two-dimensional fast wavelet transform: (a) the analysis filter bank; (b) the resulting decomposition; and (c) the synthesis filter bank.

Can implement 2-D wavelet transform using digital filters and down samplers.

use $f(x,y)$ as $W_\varphi(J,m,n)$ input

convolve rows with $h_\varphi(-n)$ and $h_\psi(-n)$ and down sample columns

two subimages



low pass vertical

½ resolution since ½ # of columns.

Now filter each sub image columnwise and down sample rows gives four quarter-size images. $W_\varphi, W_\psi^H, W_\psi^V, W_\psi^D$

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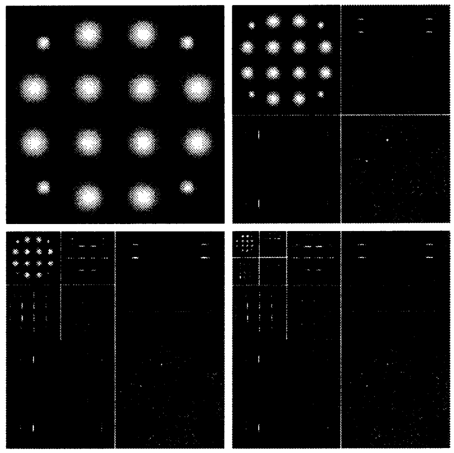
W^H

cosinepulses
128 x 128
level 7



level 6
W^D

level 5

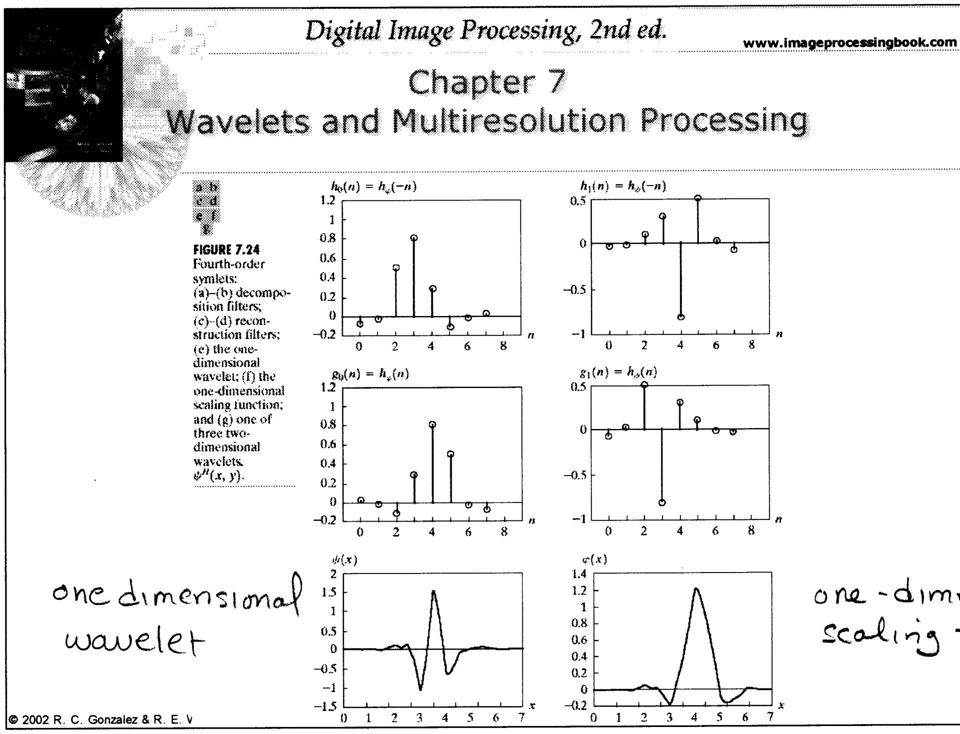
level 4

W^V

a b
c d

FIGURE 7.23 A three-scale FWT.

This example used 4th order symlets rather than Haar functions.



one dimensional wavelet

one-dimensional scaling function

Haar functions are not the only functions that can be used in FWT Filters.

These are Daubechies symlets. They are highly symmetrical and have the highest number of vanishing moments for a given compact support.

$$m(k) = \int x^k \psi(x)\,dx$$

an order n symlet has N vanishing moments

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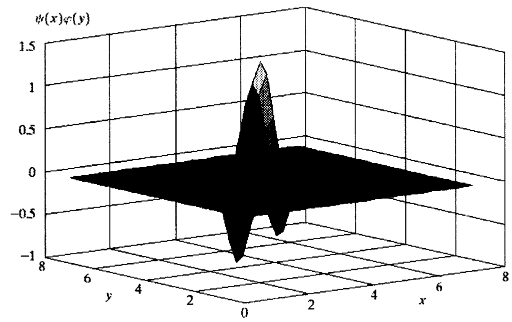
Fig. 7.24 (Con't)



One of three two-dimension wavelets
based on symlets.

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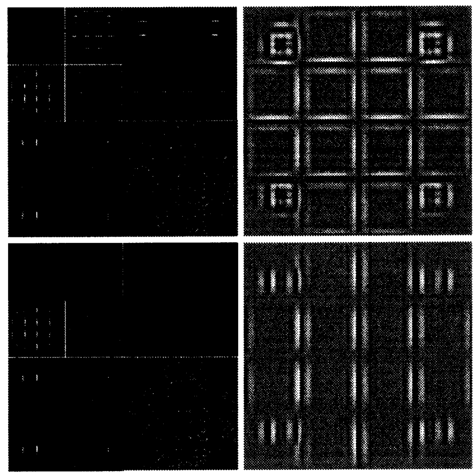
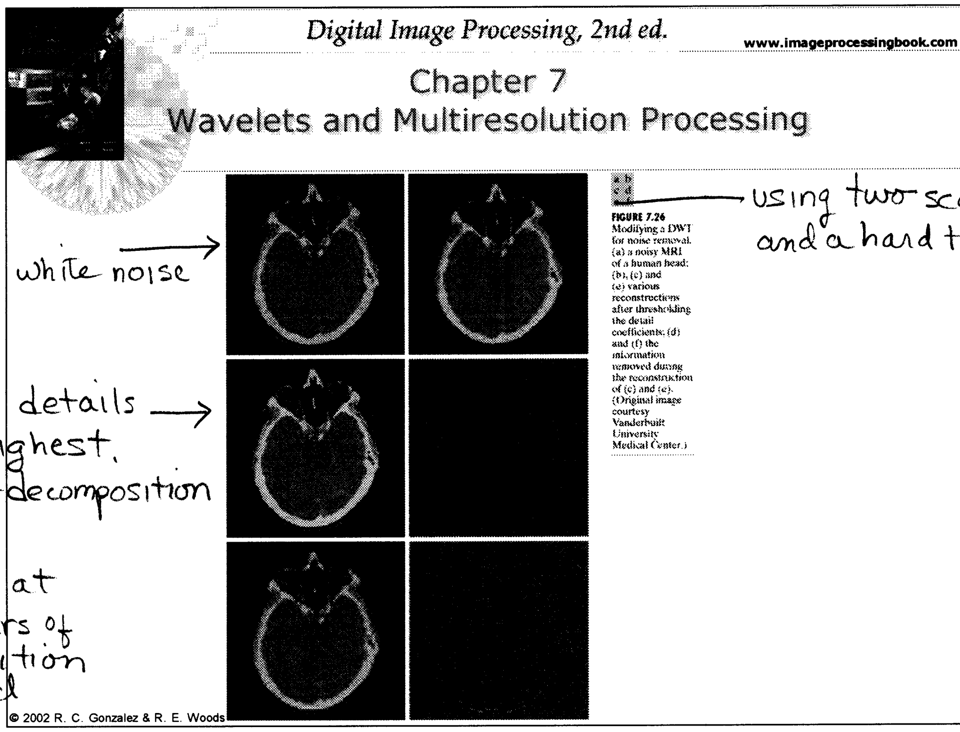
set to zero eliminating lowest scale and reconstruct. emphasizes edges.



a b
c d

FIGURE 7.25 Modifying a DWT for edge detection: (a) and (c) two-scale decompositions with selected coefficients deleted; (b) and (d) the corresponding reconstructions.

zeroing lowest scale and horizontal details can be used to emphasize edges in reconstruction



uniform white noise

zeroed details in only highest order of decomposition

details at both orders of decomposition zeroed

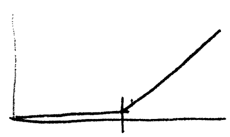
using two scales and a hard threshold

1. Decide upon number of levels or scales.
   Choose wavelet function (Haar, symlet, etc.)

2. Threshold the detail coefficients

   hard threshold — zero if value < threshold

   soft threshold — zero if value < threshold
   Scale other values towards zero at threshold.



for example

3. reconstruct using original approximations plus modified details.

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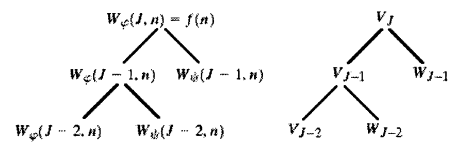

a b

FIGURE 7.27 A coefficient (a) and analysis (b) tree for the two-scale FWT analysis bank of Fig. 7.16.

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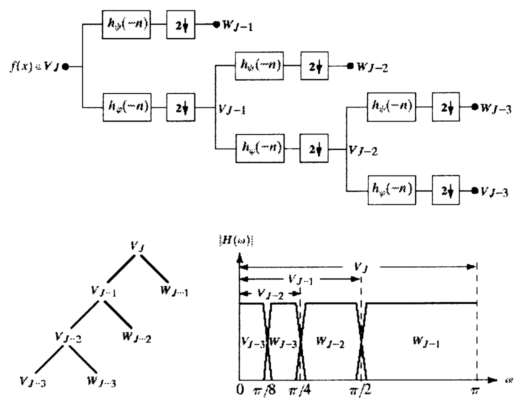

FIGURE 7.28 A three-scale FWT filter bank: (a) block diagram; (b) decomposition space tree; and (c) spectrum splitting characteristics.

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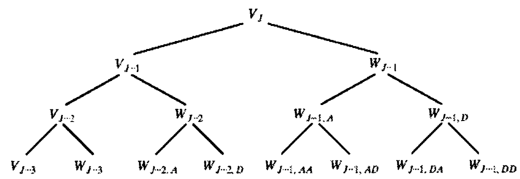

FIGURE 7.29 A three-scale wavelet packet analysis tree.

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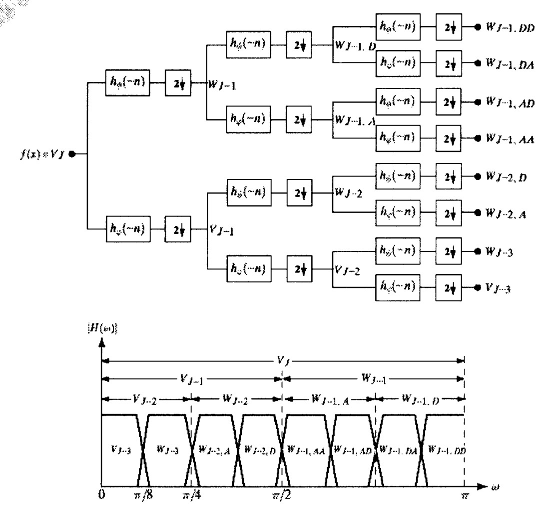

a
b

FIGURE 7.30 The (a) filter bank and (b) spectrum splitting characteristics of a three-scale full wavelet packet analysis tree.

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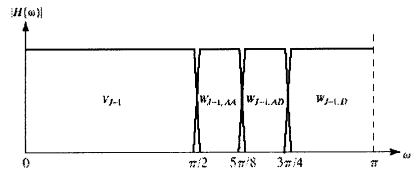

FIGURE 7.31 The spectrum of the decomposition in Eq. (7.6-5).

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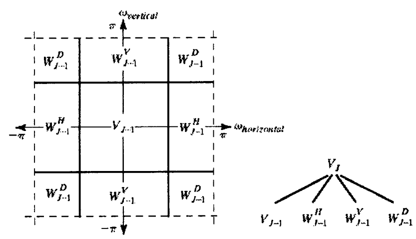
a b

FIGURE 7.32 The first decomposition of a two-dimensional FWT: (a) the spectrum and (b) the subspace analysis tree.



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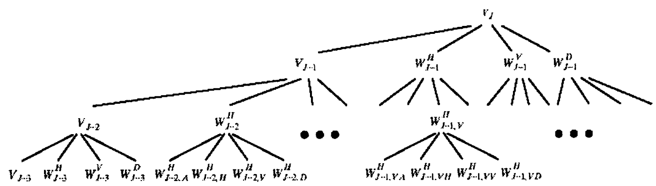

FIGURE 7.33 A three-scale, full wavelet packet decomposition tree. Only a portion of the tree is provided.

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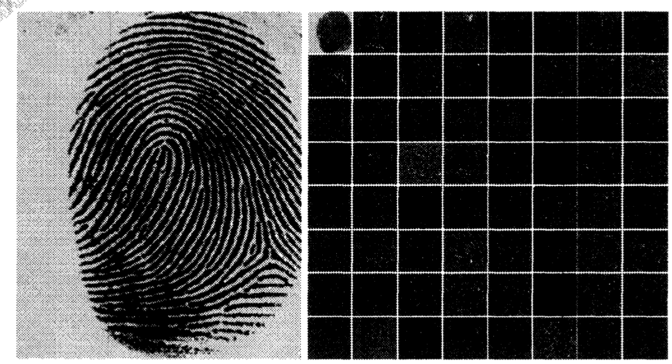

a b

FIGURE 7.34 (a) A scanned fingerprint and (b) its three-scale, full wavelet packet decomposition. (Original image courtesy of the National Institute of Standards and Technology.)

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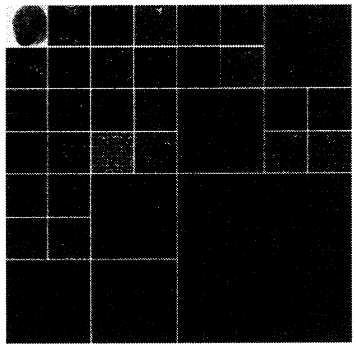

FIGURE 7.35 An optimal wavelet packet decomposition for the fingerprint of Fig. 7.34(a).

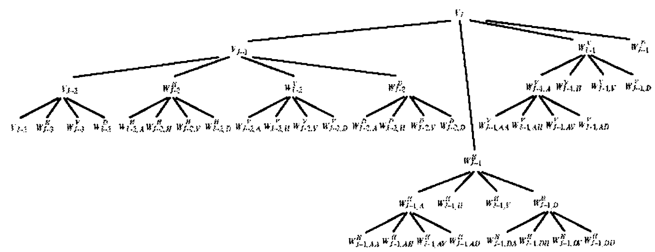

FIGURE 7.36 The optimal wavelet packet analysis tree for the decomposition in Fig. 7.35.

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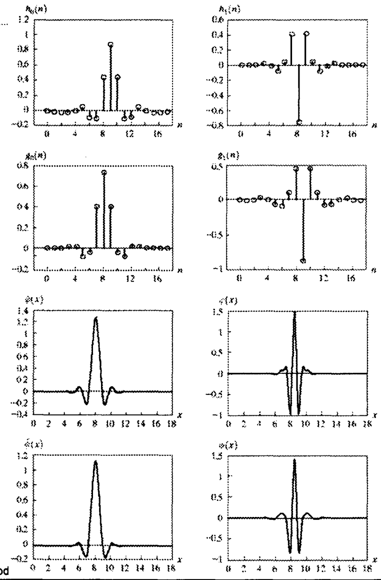

FIGURE 7.37 A member of the Cohen-Daubechies-Feauveau biorthogonal wavelet family: (a) and (b) decomposition filter coefficients. (c) and (d) reconstruction filter coefficients; (e)–(h) dual wavelet and scaling functions.