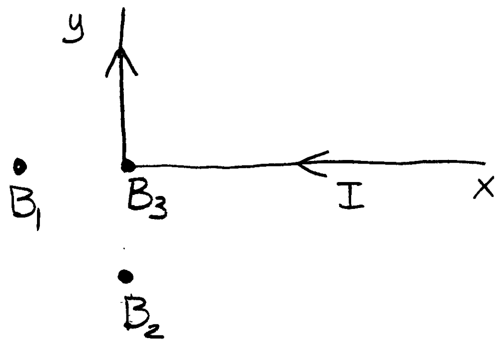
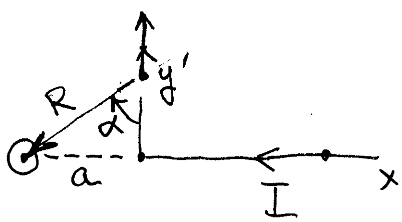
Exam 1, Part 2

6.9 Consider a single wire extending from infinity to the origin along the y-axis and back to infinity along the x-axis and carrying a current I. Find $\underline{B}$ at the following points: (a) $(-a,0,0)$, (b) $(0,-a,0)$ and (c) $(0,0,a)$



This is a problem we can solve by superposition using our previous solution for a finite length current carrying wire. From p. 31a of notes

At the position (a) there will be no contribution from the current along the x-axis. Using the Biot-Savart law



along the x-axis

$$d\underline{H}(\underline{P}) = \frac{1}{4\pi}\frac{I\,\hat{x}\,dx' \times \hat{x}}{(a+x)^2} \to 0$$

along the y-axis

$$d\underline{H}(\underline{P}) = \frac{1}{4\pi}\frac{I\,\hat{y}\,dy' \times \hat{R}}{R^2}$$

where $R^2 = a^2 + y'^2$ $\qquad \hat{y} \times \hat{R} = +\hat{\phi}\,|\hat{y}|\,|\hat{R}|\sin\alpha = \hat{\phi}\sin\alpha$

$$\therefore\ d\underline{H}(P) = \frac{1}{4\pi}\frac{I\,\hat{\phi}\sin\alpha\,dy'}{R^2} = \frac{I\,\hat{\phi}}{4\pi}\underbrace{\frac{a}{\sqrt{a^2+y'^2}}}_{\sin\alpha}\frac{1}{\sqrt{a^2+y'^2}}\,dy'$$

total field at P is

$$\underline{H}_P = \hat{\phi}\,\frac{Ia}{4\pi}\int_{y'=0}^{L}\frac{dy'}{(a^2+y'^2)^{3/2}} = \hat{\phi}\,\frac{Ia}{4\pi}\,\frac{y'}{a^2(a^2+y'^2)^{1/2}}\Bigg|_0^L$$

$$\underline{H}_P = \hat{\phi}\,\frac{Ia}{4\pi}\,\frac{L}{a^2(a^2+L^2)^{1/2}} = \hat{\phi}\,\frac{I}{4\pi a}\,\frac{L}{\sqrt{a^2+L^2}} \rightarrow \hat{\phi}\,\frac{I}{4\pi a}$$

as $L \rightarrow \infty$

since $P = (-a,0,0)$ $\quad \hat{\phi} = +\hat{z}$

$$\therefore \quad \underline{B}_1 = \hat{z}\,\frac{\mu_0 I}{4\pi a}$$

(b) The result for $\underline{B}_2$ is virtually identical except that we replace y' by x' to get

$$B_2 = \hat{z}\,\frac{\mu_0 I}{4\pi a}$$

(c) $\underline{B}_3$ for part (c) looks different but consider that the results for (a) and (b) are radially symmetric (in $\hat{\phi}$)

If we simply rotate the answer for (a) about the y axis to $(0,0,a)$

$$\underline{B}_{3(a)} = \hat{\phi}\,\frac{\mu_0 I}{4\pi a} \rightarrow -\hat{x}\,\frac{\mu_0 I}{4\pi a}$$

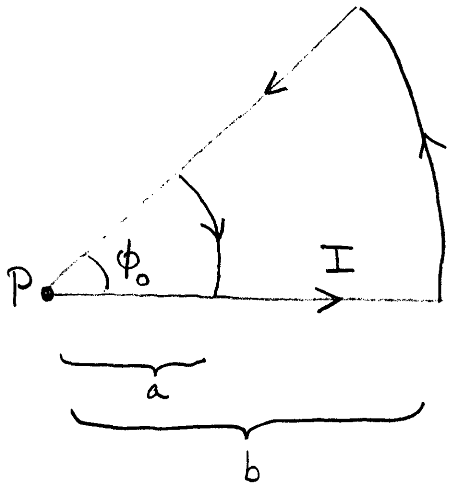
doing the same thing for B_2 we get

$$\underline{B}_{3b} = \hat{\phi}\,\frac{\mu_0 I}{4\pi a} \rightarrow +\hat{y}\,\frac{\mu_0 I}{4\pi a}$$

So, at point $(0,0,a)$

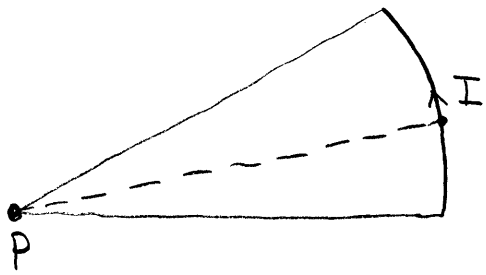
$$\underline{B}_3 = (-\hat{x}+\hat{y})\,\frac{\mu_0 I}{4\pi a}$$

6.13 Consider a loop of wire consisting of two circular and two straight segments carrying a current I, as shown below. Find $\underline{B}$ at the center P of the circular arcs.



This is a Biot-Savart Law problem. The straight segments do not produce any $\underline{B}$ field at P since P is along their axis.

The contribution to the field at P from the arc can be found using the Biot-Savart Law.



$$d\underline{H}(P) = \frac{1}{4\pi}\frac{I\,r\,d\phi\,\hat{\phi}\times\hat{r}}{r^2}$$

$$= \frac{I}{4\pi}\frac{d\phi\,(-\hat{z})}{r} = \hat{z}\,\frac{I}{4\pi r}\,d\phi$$

For the contour at r = b $\quad \underline{B}_b(P) = \int_0^{\phi_o} \hat{z}\frac{\mu_o I}{4\pi}\frac{d\phi}{b} = \frac{\mu_o I \phi_o}{4\pi b}\hat{z}$

The contour at r = a produces essentially the same field except the direction of I changes to −I and the radius is now a to give

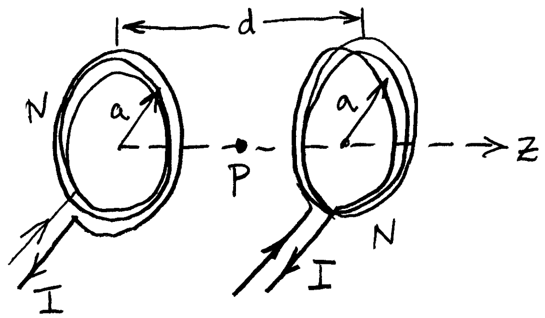
$$\underline{B}_a(P) = \int_0^{\phi_o} \hat{z}\,\frac{\mu_o(-I)}{4\pi}\frac{d\phi}{a} = -\frac{\mu_o I \phi_o}{4\pi a}\hat{z}$$

The total field is given by

$$\underline{B}(P) = \underline{B}_a(P) + \underline{B}_b(P) = \hat{z}\,\frac{\mu_o I \phi_o}{4\pi}\left(\frac{1}{b} - \frac{1}{a}\right)$$

6.16 Two thin circular coaxial coils, each of radius a, having N turns, carring current I, and separated by a distance d, as shown below are referred to as <u>Helm holtz coils</u> for the case when $d=a$. This setup is well known for producing an approximately uniform magnetic field in the vicinity of its center of symmetry.

(a) Find $\underline{B}$ on the axis of symmetry — the z axis — of the Helmholtz coils.



We found the field from a loop of wire to be

$$H_z = \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

[See p. 31 of lecture notes]

For the left hand coil above, the $\underline{B}$ field at P will be given by

$$\underline{B}_L(P) = \hat{z}\, N\mu_0 \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

The corresponding $\underline{B}$ field from the right hand coil will be given by

$$\underline{B}_R(P) = \hat{z}\, N\mu_0 \frac{Ia^2}{2(a^2+(d-z)^2)^{3/2}}$$

$$\underline{B}_{TOT}(P) = \underline{B}_L(P) + \underline{B}_R(P) = \hat{z}\,\frac{\mu_0 N I a^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{(a^2+(d-z)^2)^{3/2}}\right]$$

(b) Show that $\frac{dB_z}{dz} = 0$ at the point P midway between the two coils.

$$\left.\frac{dB_z}{dz}\right|_{z=\frac{d}{2}} = \frac{\mu_0 N I a^2}{2}\left[-\frac{3}{2}(a^2+z^2)^{-\frac{5}{2}}2z - \frac{3}{2}(a^2+(d-z)^2)^{-\frac{5}{2}}(2)(d-z)(-1)\right]_{z=\frac{d}{2}}$$

$$= \frac{\mu_0 N I a^2}{2}\left[-3\left(a^2+\frac{d^2}{4}\right)^{-\frac{5}{2}}\left(\frac{d}{2}\right) - 3\left(a^2+\frac{d^2}{4}\right)^{-\frac{5}{2}}\left(\frac{d}{2}\right)(-1)\right] = 0$$

(c) Show that both $\frac{d^2 B_z}{dz^2}$ and $\frac{d^3 B_z}{dz^3} = 0$ at the midway when $d=a$

Note that d is called the Helmholtz spacing, which corresponds to the coil separation for which the second derivative of B_z vanishes at the center.

For $d=a$

$$\underline{B}_p = \hat{z}\,\frac{\mu_0 N I a^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{(a^2+(a-z)^2)^{3/2}}\right]$$

$$\frac{dB_z}{dz} = \frac{\mu_0 N I a^2}{2}\left[-\frac{3}{2}(a^2+z^2)^{-\frac{5}{2}}(2z) - \frac{3}{2}\left(a^2+(a-z)^2\right)^{-\frac{5}{2}} 2(a-z)(-1)\right]$$

$$= \frac{\mu_0 N I a^2}{2}\left[-3z(a^2+z^2)^{-\frac{5}{2}} + 3(a-z)\left(a^2+(a-z)^2\right)^{-\frac{5}{2}}\right]$$

$$\frac{d^2 B_z}{dz^2} = \frac{\mu_0 N I a^2}{2}\left[-3(a^2+z^2)^{-\frac{5}{2}} - 3z\left(-\frac{5}{\cancel{2}}\right)(a^2+z^2)^{-\frac{7}{2}}(\cancel{2}z) + 3(+1)\left(a^2+(a-z)^2\right)^{-\frac{5}{2}} + 3(a-z)\left(-\frac{5}{\cancel{2}}\right)\left(a^2+(a-z)^2\right)^{-\frac{7}{2}}(\cancel{2})(a-z)(-1)\right]$$

$$= \frac{\mu_0 N I a^2}{2}\left[-3(a^2+z^2)^{-\frac{5}{2}} + 15z^2(a^2+z^2)^{-\frac{7}{2}} + 3\left(a^2+(a-z)^2\right)^{-\frac{5}{2}} + 15(a-z)^2\left(a^2+(a-z)^2\right)^{-\frac{7}{2}}\right]$$

$$\frac{d^3 B_z}{dz^2} = \frac{\mu_0 N I a^2}{2}\left[+\frac{15}{2}(a^2+z^2)^{-\frac{7}{2}}(2z) + 15(2z)(a^2+z^2)^{-\frac{7}{2}} + 15z^2\left(-\frac{7}{\cancel{2}}\right)(a^2+z^2)^{-\frac{9}{2}}(\cancel{2}z) + \frac{15}{2}\left(a^2+(a-z)^2\right)^{-\frac{7}{2}} 2(a-z)(-1) + 15(a-z)\,2(-1)\left(a^2+(a-z)^2\right)^{-\frac{7}{2}} + 15(a-z)^2\left(-\frac{7}{\cancel{2}}\right)\left(a^2+(a-z)^2\right)^{-\frac{9}{2}}(\cancel{2})(a-z)(-1)\right]$$

$$= \frac{\mu_0 N I a^2}{2}\left[15z(a^2+z^2)^{-\frac{7}{2}} + 30z(a^2+z^2)^{-\frac{7}{2}} - 105z^3(a^2+z^2)^{-\frac{9}{2}} - 15(a-z)\left(a^2+(a-z)^2\right)^{-\frac{7}{2}} - 30(a-z)\left(a^2+(a-z)^2\right)^{-\frac{7}{2}} + 105(a-z)^3\left(a^2+(a-z)^2\right)^{-\frac{9}{2}}\right]$$

$$\left.\frac{dB_z}{dz}\right|_{z=\frac{a}{2}} = \frac{\mu_0 N I a^2}{2}\left[-3\frac{a}{2}\left(\frac{5a^2}{4}\right)^{-\frac{5}{2}} + 3\left(\frac{a}{2}\right)\left(\frac{5a^2}{4}\right)^{-\frac{5}{2}}\right] = 0$$

$$\left.\frac{d^2B_z}{dz^2}\right|_{z=\frac{a}{2}} = \frac{\mu_0 N I a^2}{2}\left[-3\left(\frac{5a^2}{4}\right)^{-\frac{5}{2}} + 15\frac{a^2}{4}\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}} + 3\left(\frac{5a^2}{4}\right)^{-\frac{5}{2}} + 15\frac{a^2}{4}\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}}\right] = 0$$

$$\left.\frac{d^3B_z}{dz^3}\right|_{z=\frac{a}{2}} = \frac{\mu_0 N I a^2}{4}\left[15\frac{a}{2}\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}} + 30\left(\frac{a}{2}\right)\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}} - 105\frac{a^3}{8}\left(\frac{5a^2}{4}\right)^{-\frac{9}{2}}\right.$$
$$\left.-15\frac{a}{2}\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}} - 30\left(\frac{a}{2}\right)\left(\frac{5a^2}{4}\right)^{-\frac{7}{2}} + 105\frac{a^3}{8}\left(\frac{5a^2}{4}\right)^{-\frac{9}{2}}\right] = 0$$

Although you were not asked to calculate it the fourth derivative does NOT go to zero. However, having the first three derivatives go to zero at $z=\frac{a}{2}$ for $d=a$ produces a very smooth (linear) field for measurements, etc.

(d) Show that B_z at the midpoint P between the Helmholtz coils is approximately.

For $d=a$

$$\underline{B}_p = \hat{z}\,\frac{\mu_0 N I a^2}{2}\left[\frac{1}{(a^2+z^2)^{3/2}} + \frac{1}{(a^2+(a-z)^2)^{3/2}}\right]$$

For $z=\frac{a}{2}$ this becomes

$$\underline{B}_p = \hat{z}\,\frac{\mu_0 N I a^2}{2}\left[\left(\frac{5a^2}{4}\right)^{-\frac{3}{2}} + \left(\frac{5a^2}{4}\right)^{-\frac{3}{2}}\right] = \hat{z}\,\mu_0 N I a^2\left(\frac{5a^2}{4}\right)^{-\frac{3}{2}}$$

$$= \hat{z}\,N I \mu_0 a^2\,\frac{5^{-\frac{3}{2}}\,a^{-3}}{4^{-\frac{3}{2}}} = \hat{z}\,N I \mu_0\,\frac{\left(\frac{5}{4}\right)^{-\frac{3}{2}}}{a} \cong \hat{z}\,\frac{NI}{a}\,\mu_0\,0.715$$

(e) Find B_z at the center of each loop and compare it with the value at the midpoint between the coils

At the center of the left coil we have (from (a))

$$\underline{B}_L(\text{center}) = \hat{z}\, N\mu_0 \frac{Ia^2}{2(a^2)^{3/2}} = \hat{z}\,\frac{NI}{a}\,\frac{\mu_0}{2}$$

plus the field from the right hand coil

$$B_R(\text{at left center}) = \hat{z}\, N\mu_0 \frac{Ia^2}{2(a^2+a^2)^{3/2}} = \hat{z}\mu_0 NI \frac{a^2}{2\,2^{3/2}a^3}$$

$$= \hat{z}\,\frac{NI}{a}\,\frac{\mu_0}{2\,2^{3/2}}$$

$$\underline{B}(\text{left center}) = \hat{z}\,\frac{NI}{a}\left(\frac{\mu_0}{2} + \frac{\mu_0}{2\,2^{3/2}}\right) \cong \hat{z}\,\frac{NI}{a}(0.676\,\mu_0)$$

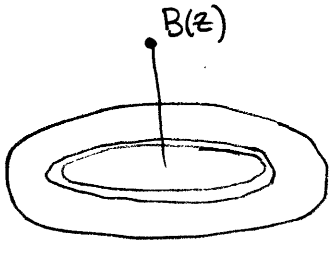
The variation in the field (on-axis) is then only

$$\frac{\Delta B}{B} = \frac{(0.715 - 0.676)}{0.715} \cong 5\%$$

6-22 A circular disk of radius a centered at the origin with its axis along the z-axis carries a surface current flowing in a circular direction around its axis given by

$$\underline{J}_s = \hat{\phi}\, K r \ \frac{A}{m^2}$$

where K is a constant. Find $\underline{B}$ at a point P on the z-axis.



We know the field from a circular loop of current to be

$$B_z(z) = \frac{I a^2}{2(a^2+z^2)^{3/2}}$$ [Lecture Notes p. 31]

So, consider the surface as a series of circular loops of current. Each differential loop is of width dr and has current

$$I = |\underline{J}_s(r)|\, dr = K r\, dr$$

$$\therefore \ \underline{B}(z) = \hat{z}\int_0^a \frac{r^2}{2(r^2+z^2)^{3/2}}(K r\, dr) = \hat{z}\ \frac{K}{2}\int_0^a \frac{r^3\, dr}{(r^2+z^2)^{3/2}}$$

$$= \hat{z}\,\frac{K}{2}\left[\sqrt{r^2+z^2} + \frac{z^2}{\sqrt{r^2+z^2}}\Bigg|_0^a\right]$$ Integral #183 CRC Handbook

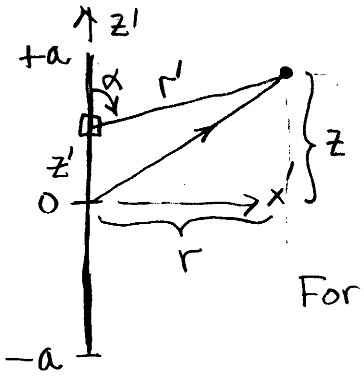
$$= \hat{z}\,\frac{K}{2}\left[\left(\sqrt{a^2+z^2} + \frac{z^2}{\sqrt{a^2+z^2}}\right) - \left(z + \frac{z^2}{z}\right)\right]$$

$$= \hat{z}\,\frac{K}{2}\ \frac{(a^2+z^2)+z^2}{(a^2+z^2)^{1/2}} - 2z$$

$$= \hat{z}\,\frac{K}{2}\left[\frac{a^2+2z^2}{\sqrt{a^2+z^2}} - 2z\right]$$

6-31 Consider the straight, current carrying filamentary conductor of length 2a, as shown below.

(a) Find the magnetic vector potential $\underline{A}$ at an arbitrary point $P(r, \phi, z)$.



$$\underline{A} = \int \frac{\mu J \, dv}{4\pi r'}$$

For this problem's geometry

$$\underline{A} = \hat{z} \frac{\mu_0 I}{4\pi} \int_{z'=-a}^{a} \frac{dz'}{\sqrt{r^2 + (z-z')^2}}$$

let $z'' = z - z'$
$dz'' = -dz'$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \int_{z''=z+a}^{z-a} \frac{-dz''}{\sqrt{r^2 + z''^2}} = \hat{z} \frac{\mu_0 I}{4\pi} \int_{z-a}^{z+a} \frac{dz''}{\sqrt{r^2 + z''^2}}$$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \ln\left(z'' + \sqrt{z''^2 + r^2}\right) \Big|_{z-a}^{z+a}$$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \ln\left((z+a) + \sqrt{(z+a)^2 + r^2}\right) - \ln\left((z-a) + \sqrt{(z-a)^2 + r^2}\right)$$

$$= \hat{z} \frac{\mu_0 I}{4\pi} \ln\left[\frac{(z+a) + \sqrt{r^2 + (z+a)^2}}{(z-a) + \sqrt{r^2 + (z-a)^2}}\right]$$

(b) Find the B-field using $\underline{B} = \underline{\nabla} \times \underline{A}$ to verify the result of Example 6-7.

$$\underline{B} = \underline{\nabla} \times \underline{A} = \nabla \times \left(\hat{z} A_z(r,z)\right) = -\hat{\phi} \frac{\partial A_z}{\partial r}$$

$$= -\hat{\phi} \frac{\mu_0 I}{4\pi} \left[\frac{a + \frac{1}{2}\left((z+a)^2 + r^2\right)^{-1/2} 2(z+a)(1)}{(z+a) + \sqrt{(z+a)^2 + r^2}} - \frac{-a + \frac{1}{2}\left((z-a)^2 + r^2\right)^{-1/2} 2(z-a)1}{(z-a) + \sqrt{(z-a)^2 + r^2}} \right]$$

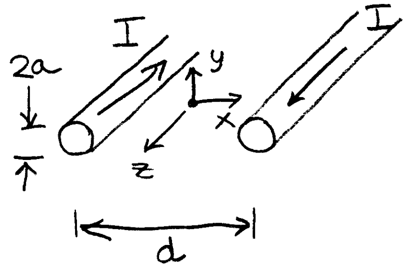
$$= -\hat{\phi} \frac{\mu_0 I}{4\pi} \left[\frac{a + (z+a)\left[(z+a)^2 + r^2\right]^{-1/2}}{(z+a) + \sqrt{(z+a)^2 + r^2}} - \frac{-a + (z-a)\left[(z-a)^2 + r^2\right]^{-\frac{1}{2}}}{(z-a) + \sqrt{(z-a)^2 + r^2}} \right]$$

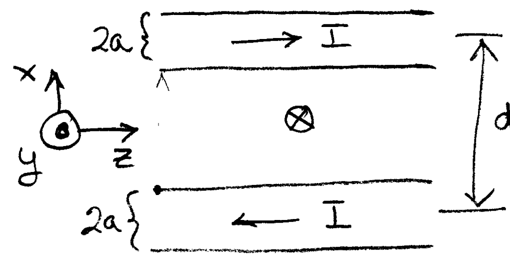
Supposedly this reduces to

$$B = \hat{\phi} \frac{\mu_0 I}{4\pi} \left[\frac{z+a}{\sqrt{r^2 + (z+a)^2}} - \frac{z-a}{\sqrt{r^2 + (z-a)^2}} \right].$$

I tried several approaches but too much algebra was involved to reach any similar form.

6-40 Determine the inductance per unit length of a two-wire transmission line in air as shown below, designed for an amateur radio transmitter, with conductor radius $a = 1\text{mm}$ and spacing $d = 6\text{cm}$.





Superimpose the field from each conductor modeling the fields as those from two infinitely long parallel lines

$$B_y \cong -\frac{\mu_0 I}{2\pi \underbrace{(x+\frac{d}{2})}_{\text{positive}}} + \frac{\mu_0 I}{2\pi \underbrace{(x-\frac{d}{2})}_{\text{negative}}}$$

The flux linked by this circuit of two conductors over a length l is approximated by

$$\Phi = \int_S \underline{B} \cdot d\underline{s} = \frac{\mu_0 I l}{2\pi} \int_{-\frac{d}{2}+a}^{+\frac{d}{2}-a} \left[\frac{1}{x+\frac{d}{2}} - \frac{1}{x-\frac{d}{2}} \right] dx$$

$$= \frac{\mu_0 I l}{2\pi} \left[\ln(x+\tfrac{d}{2}) - \ln(x-\tfrac{d}{2}) \right]_{a-\frac{d}{2}}^{-a+\frac{d}{2}}$$

$$= \frac{\mu_0 I l}{2\pi} \left[\ln(-a+d) - \ln(a) - \ln(-a) + \ln(a-d) \right]$$

$$= \frac{\mu_0 I l}{2\pi} \left[\ln(d-a) + \ln(a-d) - \ln(a) - \ln(-a) \right]$$

$$= \frac{\mu_0 I l}{2\pi} \, 2\left[\ln(d-a) - \ln(a) \right] = \frac{\mu_0 I l}{\pi} \ln\left(\frac{d-a}{a}\right)$$

Since $d \gg a$ $\quad \Psi = \frac{\mu_o I l}{\pi} \ln\left(\frac{d}{a}\right)$

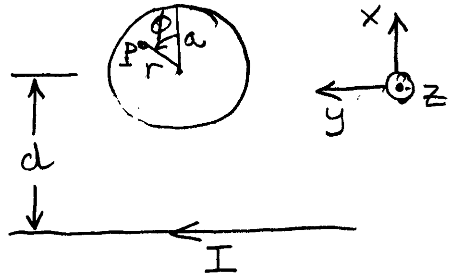
$$\therefore L = \frac{\Psi}{I} = \frac{\mu_o l}{\pi} \ln\left(\frac{d}{a}\right)$$

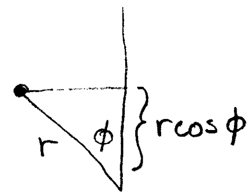
(a) $\frac{L}{l} = \frac{\mu_o}{\pi} \ln\left(\frac{d}{a}\right) = \frac{4\pi \times 10^{-7}}{\pi} \ln\left(\frac{60\,mm}{1\,mm}\right) = 1.64 \times 10^{-6} \frac{H}{m}$.

(b) Repeat part (a) if the conductor spacing is doubled (i.e.) $d = 12cm$.

$$\frac{L}{l} = \frac{\mu_o}{\pi} \ln\left(\frac{d}{a}\right) = \frac{4\pi \times 10^{-7}}{\pi} \ln\left(\frac{120\,mm}{1\,mm}\right) = 1.915 \times 10^{-6} \frac{H}{m}.$$

6-43 Find the mutual inductance between an infinitely long, straight wire and a circular wire loop, as shown below.





consider a point $P(r,\theta)$ inside the loop. The field from the wire is:

$$\underline{B}(P) = \hat{\phi}\,\frac{\mu_0 I}{2\pi(d + r\cos\theta)}$$

The flux linked by the circular loop can be calculated by integrating over the loop

$$\Psi_{12} = \frac{\mu_0 I}{2\pi}\int_0^a\int_0^{2\pi}\frac{r\,dr\,d\phi}{d + r\cos\phi}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^a r\,dr\,\frac{2}{\sqrt{d^2-r^2}}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{d^2-r^2}\,\tan\left(\frac{\phi}{2}\right)}{d+r}\right)\Bigg|_0^{2\pi} \qquad \text{Int. \#341}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^a r\,dr\,\frac{2}{\sqrt{d^2-r^2}}\,\mathrm{Tan}^{-1}\left(\frac{\sqrt{d^2-r^2}\cdot\infty}{d+r}\right)$$

$$= \frac{\mu_0 I}{2\pi}\int_0^a r\,dr\,\frac{2\pi}{\sqrt{d^2-r^2}} = \mu_0 I\int_0^a\frac{r\,dr}{\sqrt{d^2-r^2}} \qquad \text{Int. \# 204}$$

$$= \mu_0 I\left(-\sqrt{d^2-r^2}\,\Big|_0^a\right) = \mu_0 I\left[-\sqrt{d^2-a^2} + \sqrt{d^2}\right]$$

$$\Psi_{12} = \mu_0 I\left[d - \sqrt{d^2-a^2}\right].$$

$$L_{12} = \frac{\Psi_{12}}{I_1} = \frac{\mu_0 I\left[d - \sqrt{d^2-a^2}\right]}{I} = \mu_0\left(d - \sqrt{d^2-a^2}\right)$$