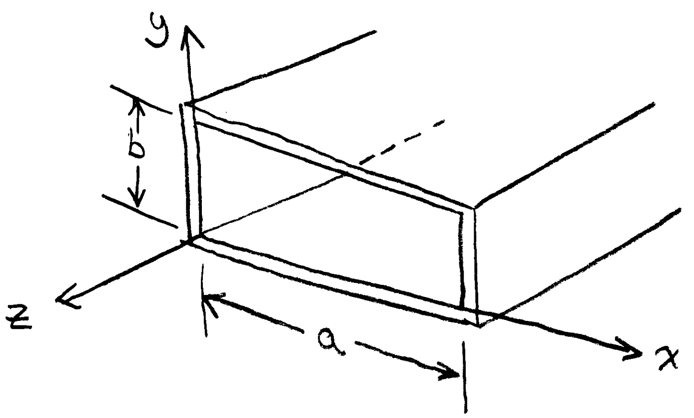
Ch. 5 Cylindrical Waveguides and Cavity Resonators

5.1 Rectangular Waveguides



We assume perfectly conducting waveguide walls which require $E_{tan}=0$ and $H_{norm}=0$

$E_x, E_z = 0$ at $y=0$ and $y=b$

$H_y = 0$ 〃

$E_y, E_z = 0$ at $x=0$ and $x=b$

$H_x = 0$ 〃

We also want the fields to vary in the z-direction as $e^{-\bar{\gamma} z}$

Aside from the boundary conditions this is No different than the parallel plate guide and must satisify the curl equations

$$\nabla \times \underline{H} = j\omega\epsilon \underline{E}$$

$$\nabla \times \underline{E} = -j\omega\mu \underline{H}$$

which we already developed for the parallel plate guide.

Using $\frac{\partial}{\partial z} \rightarrow -\bar{\gamma}$ we have

$$\frac{\partial H_z}{\partial y} + \bar{\gamma} H_y = j\omega\epsilon E_x$$

$$\frac{\partial H_z}{\partial x} + \bar{\gamma} H_x = -j\omega\epsilon E_y \qquad \nabla \times \underline{H} = j\omega\epsilon \underline{E}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z \qquad [5.1]$$

$$\frac{\partial E_z}{\partial y} + \bar{\gamma} E_y = -j\omega\mu H_x$$

$$\frac{\partial E_z}{\partial x} + \bar{\gamma} E_x = j\omega\mu H_y \qquad \nabla \times \underline{E} = -j\omega\mu \underline{H}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

The wave equations for E_z and H_z reduce to

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \bar{\gamma}^2 E_z = -\omega^2\mu\epsilon E_z \qquad [5.2]$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \bar{\gamma}^2 H_z = -\omega^2\mu\epsilon H_z$$

The transverse field components can be written in terms of E_z and H_z.

$$H_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x} + j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y} - j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x} \qquad [5.3]$$

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x} - j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y} + j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

where $h^2 = \bar{\gamma}^2 + \omega^2\mu\epsilon$

5.1.1.

Just as for the parallel plate waveguide the field solutions can be classified as

TE where $E_z = 0$

TM where $H_z = 0$

For waveguides we write the wave equations using a transverse operator ∇_{tr}

which can be written as

$$\nabla_{tr} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$$

and

$$\nabla_{tr}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The wave equations become

$$\nabla_{tr}^2 E_z + (\bar{\gamma}^2 + \omega^2\mu\epsilon) E_z = 0$$

$$\nabla_{tr}^2 H_z + (\bar{\gamma}^2 + \omega^2\mu\epsilon) H_z$$

For TM modes the component equations become

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y}$$

$$\underline{E}_{tr} = \hat{x}E_x + \hat{y}E_y = -\hat{x}\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x} - \hat{y}\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y}$$

$$\underline{E}_{tr} = -\frac{\bar{\gamma}}{h^2}\nabla_{tr}E_z = -\frac{\bar{\gamma}}{\bar{\gamma}^2 + \omega^2\mu\epsilon}\nabla_{tr}E_z$$

Similarily,

$$H_x = j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$\underline{H}_{tr} = \hat{x}H_x + \hat{y}H_y = \frac{j\omega\epsilon}{h^2}\hat{x}\frac{\partial E_z}{\partial y} - \frac{j\omega\epsilon}{h^2}\hat{y}\frac{\partial E_z}{\partial x}$$

$$= \frac{j\omega\epsilon}{h^2}\left(\hat{x}\frac{\partial E_z}{\partial y} - \hat{y}\frac{\partial E_z}{\partial x}\right)$$

$$= \frac{j\omega\epsilon}{h^2}\left(\hat{x}\left(-\frac{h^2E_y}{\bar{\gamma}}\right) - \hat{y}\left(-\frac{h^2E_x}{\bar{\gamma}}\right)\right)$$

$$= -\frac{j\omega\epsilon h^2}{h^2\bar{\gamma}}\left(\hat{x}E_y + \hat{y}E_x\right)$$

$$= \frac{-j\omega\epsilon}{\bar{\gamma}}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ E_x & E_y & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$\underline{H}_{tr} = -\frac{j\omega\epsilon}{\bar{\gamma}}\left(\underline{E}_{tr}\times\hat{z}\right) = \frac{j\omega\epsilon}{\bar{\gamma}}\left(\hat{z}\times\underline{E}_{tr}\right)$$

We can do the component equations for TE waves the same way

$$\underline{H}_{tr} = \hat{x}H_x + \hat{y}H_y = -\frac{\bar{\gamma}}{\bar{\gamma}^2+\omega^2\mu\epsilon}\nabla_{tr}H_z$$

$$\underline{E}_{tr} = \hat{x}E_x + \hat{y}E_y = \frac{j\omega\mu}{\bar{\gamma}}\left(\underline{H}_{tr}\times\hat{z}\right)$$

where the boundary condition is that $\hat{n}\cdot\underline{H}_{tr} = 0$

or $\frac{\partial H_z}{\partial x} = 0$, $\frac{\partial H_z}{\partial y} = 0$

Transverse Magnetic (TM) modes

We use separation of variables similar to that which we used for the parallel plate waveguide

$$E_z(x,y,z) = \underbrace{E_z^{o}(x,y)}\, e^{-\bar{\gamma} z}$$

now a function of two variables
Let $E_z^o(x,y) = f(x)\,g(y)$.

The wave equation becomes

$$\nabla_{tr}^2 E_z + (\bar{\gamma}^2 + \omega^2\mu\epsilon)E_z = 0$$

$$\nabla_{tr}^2 fg + (\bar{\gamma}^2 + \omega^2\mu\epsilon)\, fg = 0$$

The $e^{-\bar{\gamma} z}$ drops out.

$$\nabla_{tr}^2 fg = g\frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2}$$

$$g\frac{d^2f}{dx^2} + f\frac{d^2g}{dy^2} + h^2 fg = 0 \quad \text{where } h^2 = \bar{\gamma}^2 + \omega^2\mu\epsilon$$

divide by fg

$$\frac{1}{f}\frac{d^2f}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} + h^2 = 0$$

Re-arranging

$$\frac{1}{f}\frac{d^2f}{dx^2} + h^2 = \frac{1}{g}\frac{d^2g}{dy^2}$$

Each side must be equal to a constant, call it A^2, which is determined by the boundary conditions

$$\frac{1}{f}\frac{d^2f}{dx^2} + h^2 = A^2 \quad \text{and} \quad \frac{1}{g}\frac{d^2g}{dy^2} = -A^2$$

The two equations have similar solutions

$$f(x) = C_1 \cos(Bx) + C_2 \sin(Bx)$$
$$\text{where} \quad B = \sqrt{h^2 - A^2}$$

and $\quad g(y) = C_3 \cos(Ay) + C_4 \sin(Ay)$

The complete product solution is $E_z^o(x,y) = f(x)\,g(y)$

$$E_z^o(x,y) = C_1C_3 \cos(Bx)\cos(Ay) + C_1C_4 \cos(Bx)\sin(Ay)$$
$$+ C_2C_3 \sin(Bx)\cos(Ay) + C_2C_4 \sin(Bx)\sin(Ay)$$

The B.C.'s are that $E_z^o = 0$ at $x=0$, $x=a$, $y=0$ and $y=b$

$$@x=0 \quad E_z^o(0,y) = C_1C_3 \cancelto{1}{\cos(Bx)}\cos(Ay) + C_1C_4 \cancelto{1}{\cos(Bx)}\sin(Ay)$$
$$+ C_2C_3 \cancelto{0}{\sin(Bx)}\cos(Ay) + C_2C_4 \cancelto{0}{\sin(Bx)}\sin(Ay)$$

$$E_z^o(0,y) = C_1C_3 \cos(Ay) + C_1C_4 \sin(Ay)$$

For $E_z^o(0,y) = 0$ we require $C_1 = 0$

Requiring $C_3 = C_4 = 0$ will result in a trivial solution

$$E_z^o(x,y) = C_2C_3 \sin(Bx)\cos(Ay) + C_2C_4 \sin(Bx)\sin(Ay)$$

Now requiring $E_z^o(x,0) = 0$

$$E_z^o(x,0) = C_2C_3 \sin(Bx)\,\cancelto{1}{\cos(Ay)} + C_2C_4 \sin(Bx)\,\cancelto{0}{\sin(Ay)}$$

$$E_z^o(x,0) = C_2C_3 \sin(Bx)$$

This requires that either C_2 or C_3 equals zero.
We pick $C_3 = 0$ since picking $C_2 = 0$ would be a trivial solution.

This gives the interim solution

$$E_z^o(x,y) = \underbrace{C_2C_4}_{\text{call this } C} \sin(Bx)\sin(Ay)$$

For $E_z^o(a,y) = C\sin(Ba)\sin(Ay) = 0$

requires that $B = \frac{m\pi}{a}$ $\quad m = 1, 2, 3$

For $E_z^o(x,b) = C\sin(Bx)\sin(Ab) = 0$

requires that $A = \frac{n\pi}{b}$ $\quad n = 1, 2, 3.$

The final expression is

$$E_z^o(x,y) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

The final expressions for the fields are

$$E_z(x,y,z) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z} \qquad \bar{\gamma} = j\bar{\beta}_{mn}$$

$$\mathcal{E}_z(x,y,z,t) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos(\omega t - \bar{\beta}_{mn}z)$$

These are propagating waves since $\bar{\gamma} = j\bar{\beta}_{mn}$. We also have the case $\bar{\gamma} = \bar{\alpha}_{mn}$ for evanescent waves, In this case

$$E_z(x,y,z) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-\bar{\alpha}_{mn}z}$$

$$\mathcal{E}_z(x,y,z,t) = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos(\omega t)\,e^{-\bar{\alpha}_{mn}z}$$

The other field components can be calculated using the component equations. For TM modes, $H_z = 0$ so

$$H_x = j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$H_y = -j\frac{\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial x}$$

and $$E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y}$$

so, for propagating rectangular TM_{mn} modes

$$E_z = C\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$E_x = -\frac{j\bar{\beta}_{mn}C}{h^2}\underbrace{\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)}_{\text{derivative}}\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

since $\bar{\gamma} = j\bar{\beta}_{mn}$

Similarly,

$$E_y = -\frac{j\bar{\beta}_{mn}}{h^2}C\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$H_x = \frac{j\omega\epsilon C}{h^2}\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$H_y = -\frac{j\omega\epsilon}{h^2}C\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$

to find $\bar{\gamma} = \sqrt{h^2 - \omega^2 \mu \epsilon}$ we note that

$$\frac{1}{f}\frac{d^2 f}{dx^2} + h^2 = A^2 \qquad \text{where } f(x) = \underbrace{C_2 \sin(Bx)}_{\text{general solution}}$$

$$h^2 = A^2 - \frac{1}{f}\frac{d^2 f}{dx^2} = A^2 - \frac{1}{C_2 \sin(Bx)} C_2(-B^2)\sin(Bx)$$

$$h^2 = A^2 + B^2 = \left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2$$

Then knowing h^2

$$\bar{\gamma} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \epsilon}$$

We see that $\bar{\gamma}$ corresponds to a propagating wave only when $\bar{\gamma}$ is imaginary, i.e., $\omega > \omega_{c_{mn}}$

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2_{c_{mn}} \mu \epsilon = 0$$

$$\omega^2_{c_{mn}} = \frac{1}{\mu\epsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

$$\omega_{c_{mn}} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

for $\omega > \omega_{c_{mn}}$ $\quad \bar{\gamma} = j\bar{\beta}_{mn}$

$$j\bar{\beta}_{mn} = j\sqrt{\omega^2 \mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\bar{\beta}_{mn} = \sqrt{\omega^2 \mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\bar{\beta}_{mn} = \beta\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}$$

where $\beta = \omega\sqrt{\mu\epsilon}$ and $f_{c_{mn}} = \dfrac{\omega_{c_{mn}}}{2\pi}$

You can get a corresponding expression for $\lambda_{c_{m,n}}$

$$\lambda_{c_{mn}} = \frac{v_p}{f_{c_{nm}}} = \frac{1}{\sqrt{\mu\epsilon}\ \frac{\omega_{c_{nm}}}{2\pi}}$$

$$\lambda_{c_{mn}} = \frac{2\pi}{\cancel{\sqrt{\mu\epsilon}}} \frac{1}{\frac{1}{\cancel{\sqrt{\mu\epsilon}}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}}$$

For propagating waves

$$\bar{v}_{p_{mn}} = \frac{\omega}{\bar{\beta}_{mn}} = \frac{\omega}{\beta\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\bar{\lambda}_{mn} = \frac{2\pi}{\bar{\beta}_{mn}} = \frac{2\pi}{\sqrt{\omega^2\mu\epsilon-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}} = \frac{\cancel{\beta}\lambda}{\cancel{\beta}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\bar{\lambda}_{mn} = \frac{\lambda}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

We can also define a wave impedance

$$Z_{TM_{mn}} \equiv \frac{E_x}{H_y} = \frac{E_x^o e^{-j\bar{\beta}z}}{H_y^o e^{-j\bar{\beta}z}} = \frac{E_x^o}{H_y^o}$$

$$Z_{TM_{mn}} = \frac{-\frac{j\bar{\beta}_{mn}}{h^2} C \frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)}{-\frac{j\omega\epsilon}{h^2} C \frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)} = \frac{\bar{\beta}_{mn}}{\omega\epsilon}$$

$$Z_{TM_{mn}} = \frac{\beta\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}{\omega\epsilon} = \frac{\cancel{\omega}\sqrt{\mu\epsilon}\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}{\cancel{\omega}\epsilon}$$

$$Z_{TM_{mn}} = \eta\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}$$

Transverse Electric (TE) modes $\qquad E_z = 0$

$$H_z = C\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

from which we can derive

$$H_x = \frac{j\bar{\beta}_{mn}}{h^2}C\,\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$H_y = \frac{j\bar{\beta}_{mn}}{h^2}C\,\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$E_x = \frac{j\omega\mu}{h^2}C\,\frac{n\pi}{b}\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

$$E_y = -\frac{j\omega\mu}{h^2}C\,\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-j\bar{\beta}_{mn}z}$$

The formulas for $\omega_{c_{mn}}$, $\bar{\beta}_{mn}$, etc. are identical

One different formula is that for impedance

$$Z_{TE_{mn}} = \frac{\eta}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}} = -\frac{E_y}{H_x}$$

A very important mode is the TE_{10} mode $(a>b)$

$$H_z = C\cos\left(\frac{\pi x}{a}\right) e^{-j\bar{\beta}_{10} z}$$

$$H_x = \frac{j\bar{\beta}_{10} a C}{\pi} \sin\left(\frac{\pi x}{a}\right) e^{-j\bar{\beta}_{10} z}$$

$$E_y = -\frac{j\omega\mu a C}{\pi} \sin\left(\frac{\pi x}{a}\right) e^{-j\bar{\beta}_{10} z}$$

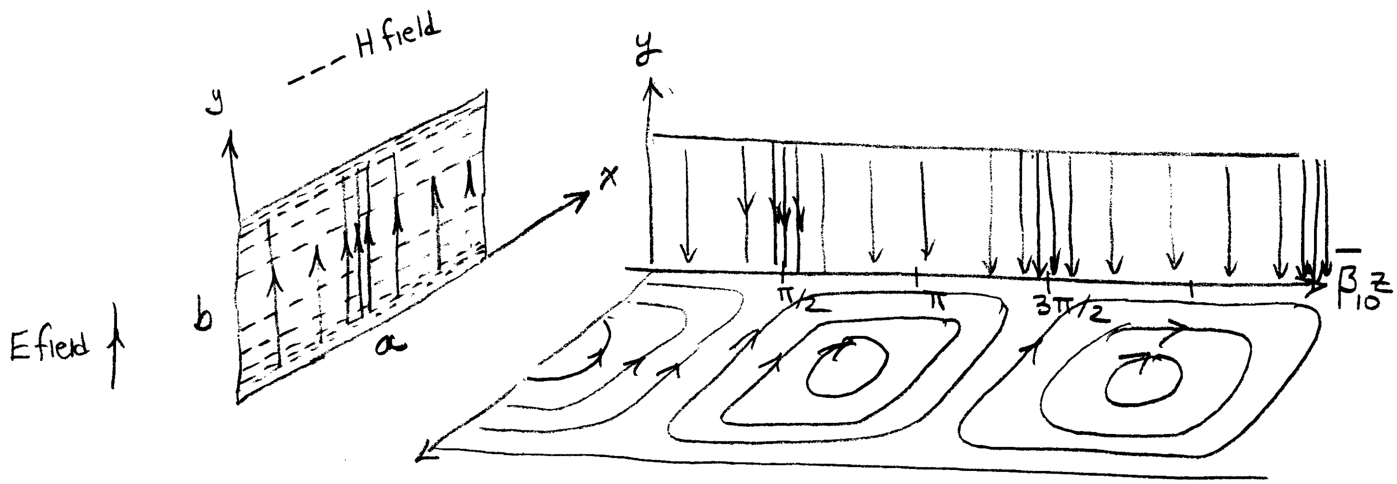
$$E_x = H_y = 0$$

$$\bar{\beta}_{10} = \sqrt{\omega^2\mu\epsilon - \left(\frac{\pi}{a}\right)^2} = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2 - \left(\frac{\pi}{a}\right)^2}$$

$$\bar{\lambda}_{10} = \frac{2\pi}{\bar{\beta}_{10}} = \frac{\lambda}{\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}}$$

$$f_{c_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}}$$

If propagation at a specified f is not possible in the TE_{10} mode it is NOT possible for any mode.



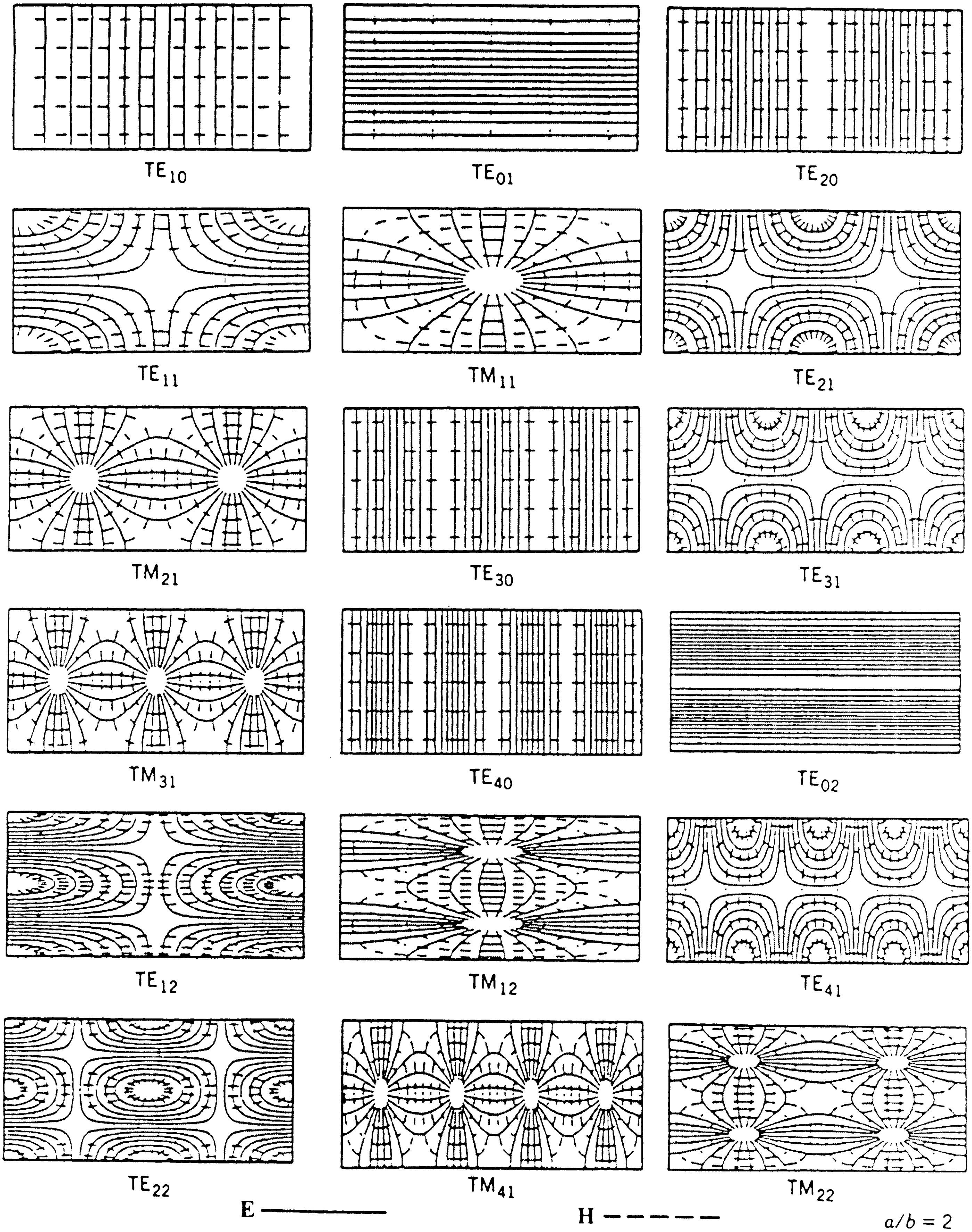
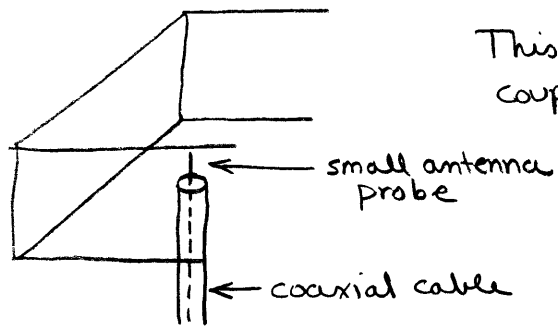


FIGURE 5.3. Field patterns in a rectangular waveguide. Field configurations in the xy plane for selected rectangular waveguide modes. [Taken from C. S. Lee, S. W. Lee, and S. L. Chuang, Plot of modal field distribution in rectangular and circular waveguides, *IEEE Trans. Microwave Theory and Techniques*, 33(3), pp. 271–274, March 1985.]

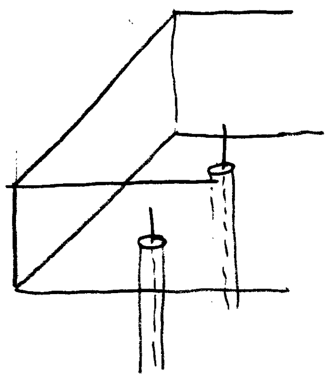
5.1.2

We have not talked about how to couple power into particular modes.

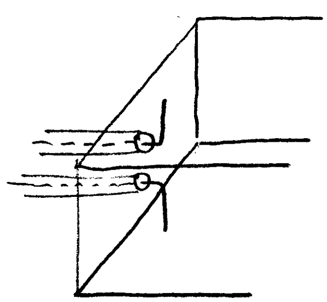
The practice is to use a probe (source) that will produce lines of E and H that are roughly parallel to the lines of E and H for that mode, and that produce the maximum electric field where the field would be maximum for that mode



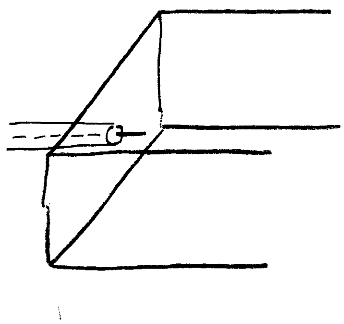
This will excite a TE_{10} mode couples well to E & H fields of mode



To excite a TE_{20} mode use two vertical antenna probles

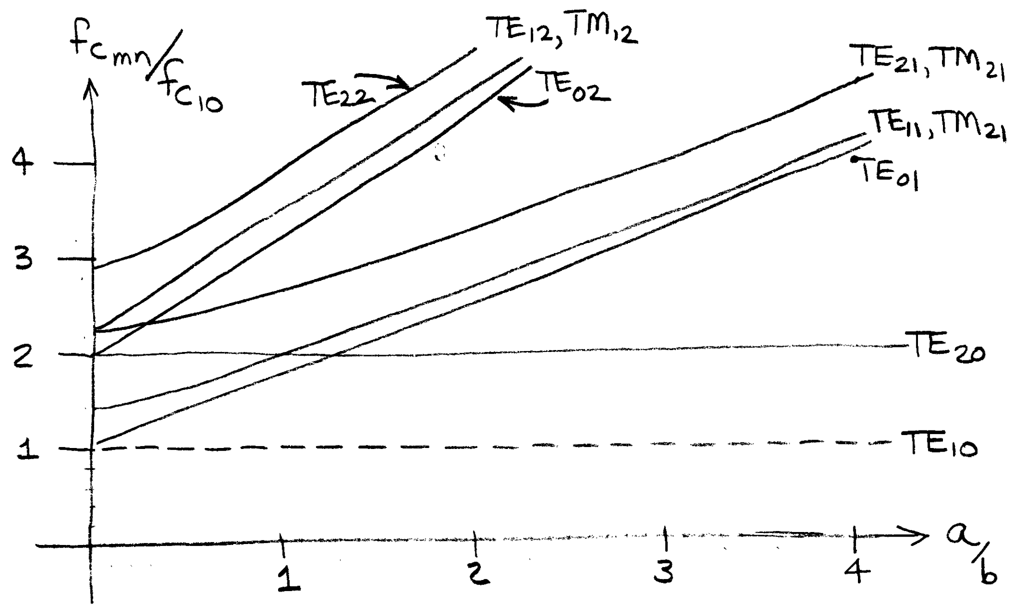


A TE_{11} mode needs parallel excitation of the E field at the wall



To excite the TM_{11} mode we need circular H fields.

In practice waveguide dimensions are chosen to allow only a single mode to propagate.



square guides where $a=b$ are undesirable since modes differ only by rotation

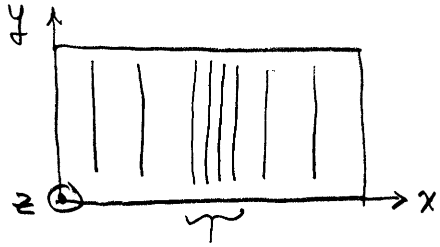
In practice, pick $a \cong 2b$ to separate modes and maximize power transmission.

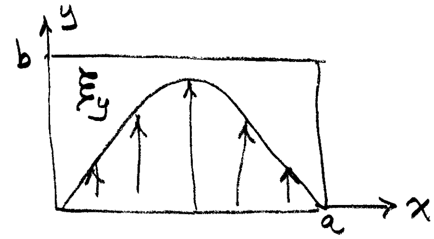
Want single mode guides

- different phase velocities would give different transverse modes & make it difficult to extract energy
- $\frac{\lambda}{2} < a < \lambda$ to ensure transmission of only TE_{10} mode
- often pick $a = 0.7\lambda$ since values near λ may allow next mode to propagate and values near $\frac{\lambda}{2}$ have large variation of v_p and $Z_{TE \text{ or } TM}$ with f.

Power-Handling capacity

For the dominant TE_{10} mode the largest electric field is along the center of the wide wall.





For this TE_{10} mode $E_y = -\frac{j\omega\mu a}{\pi} C \sin\left(\frac{\pi x}{a}\right) e^{-j\bar{\beta}_{10} z}$

The peak value of the electric field is then

$$|E_y|_{max} \equiv E_0 = \frac{\omega\mu a}{\pi} C$$

To avoid dielectric breakdown $E_0 \leq E_{BR}$

The time average power density for the TE_{10} mode can be calculated from the Poynting vector

$$\underline{S}_{AV} = \frac{1}{2} Re\left\{\underline{E}\times\underline{H}^*\right\} = \frac{1}{2} Re\left\{E_y \hat{y} \times H_x^* \hat{x}\right\}$$

$$= \frac{1}{2} Re\left\{-\frac{j\omega\mu a C}{\pi}\sin\left(\frac{\pi x}{a}\right) \cdot \frac{-j\bar{\beta}_{10} a C}{\pi}\sin\left(\frac{\pi x}{a}\right)(-\hat{z})\right\}$$

$$= \frac{1}{2} Re\left\{+\bar{\beta}_{10}\omega\mu C^2 \left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\right\}$$

$$\underline{S}_{AV} = \frac{\bar{\beta}_{10}\omega\mu C^2}{2}\left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)\hat{z}$$

The total average power is gotten by integrating $\underline{S}_{AV}$ over the cross-section

$$P_{AV} = \int_0^b\int_0^a \underline{S}_{AV}\cdot\hat{z}\,dx\,dy = \int_0^b\int_0^a \frac{\bar{\beta}_{10}\omega\mu C^2}{2}\left(\frac{a}{\pi}\right)^2 \sin^2\left(\frac{\pi x}{a}\right)dx\,dy$$

$$= \frac{\bar{\beta}_{10}\omega\mu C^2}{2}\left(\frac{a}{\pi}\right)^2 \frac{ab}{2}$$

We can then relate the power to the peak electricfield strength.

Since $E_0 = \frac{\omega\mu a}{\pi} C \qquad C = \frac{\pi E_0}{\omega\mu a}$ where E_0 is the peak field

$$P_{peak} = 2P_{AV} = 2\frac{\bar{\beta}_{10}\,\omega\mu}{2}\,\frac{\cancel{\pi^2}E_0^2}{\omega^2\mu^2\cancel{a^2}}\left(\frac{\cancel{a}}{\cancel{\pi}}\right)^2\frac{ab}{2}$$

$$P_{peak} = \frac{\bar{\beta}_{10}E_0^2}{\omega\mu}\frac{ab}{2} = \frac{2\pi}{\lambda}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}\,\frac{E_0^2}{\omega\mu}\,\frac{ab}{2}$$

$$= \frac{\cancel{2\pi}}{\lambda\,(\cancel{2\pi}f)\,\mu}\,E_0^2\,\frac{ab}{2}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}$$

$$P_{peak} = \frac{1}{\frac{1}{\sqrt{\mu\epsilon}}\mu}\,E_0^2\,\frac{ab}{2}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2} = \frac{E_0^2}{\eta}\,\frac{ab}{2}\sqrt{1-\left(\frac{\lambda}{2a}\right)^2}$$

For maximumpower maximize a & b. However, you can't increase b too much if you want to keep the TE_{10} mode as the only propagating mode.

Example 5-2

Design an air filled C-band (4-8 GHz) rectangular waveguide such that the center frequency of this band ($f = 6GHz$) is at least 25% higher than the cutoff frequency of the TE_{10} mode and at least 25% lower than the cutoff frequency of the next higher mode, so that the dominant mode of propagation is TE_{10}.

For the TE_{10} mode $\quad f_{c_{10}} = \dfrac{1}{2a\sqrt{\mu\epsilon}} = \dfrac{c}{2a}$

The first criterion can be written as

$$f = 6\,GHz \geq \underbrace{1.25}_{\text{25\% above the cutoff}} f_{c_{10}} = 1.25\left(\frac{c}{2a}\right)$$

$$a \geq \frac{1.25\,c}{2f} = \frac{1.25\,(3\times10^{8})}{2\,(6\times10^{9})} = 0.03125\,m.$$

The next highest mode would be TE_{20} (See Fig 5.5)

$$\omega_{c_{20}} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{2\pi}{a}\right)^2 + (0)^2} = \frac{2\pi}{\sqrt{\mu\epsilon}\,a}$$

$$2\pi f_{c_{20}} = \frac{2\pi\,c}{a}$$

$$f_{c_{20}} = \frac{c}{a}$$

The second design criterion can be written as

$$f = 6GHz \leq 0.75\,f_{c_{20}} = 0.75\,\frac{c}{a}$$

$$a \leq \frac{0.75\,c}{f} = \frac{0.75\,(3\times10^{8})}{(6\times10^{9})} = 0.0375\,m.$$

$$\therefore \quad 3.125\,cm \leq a \leq 3.75\,cm$$

For the TE_{01} mode $f_{c_{01}} = \dfrac{c}{2b}$. For this mode

$$f = 6\,GHz \leq 0.75\,f_{c_{01}} = 0.75\,\frac{c}{2b}$$

$$b \leq 0.75\,\frac{c}{2f} = \frac{0.75\,(3\times10^{8})}{2\,(6\times10^{9})} = .025\,m$$

$$b \leq 2.5\,cm.$$

5.1.3. Attenuation in Rectangular Waveguides

attenuation occurs through three mechanisms

1. losses due to surface currents flowing in the waveguide walls
2. dielectric losses due to a dielectric with $\sigma \neq 0$ or $\epsilon_c = \epsilon' - j\epsilon''$ between the walls
3. evanescent wave attenuation when $f < f_c$

1. Conduction losses

Surface current densities are given by $\underline{J}_s = \hat{n} \times \underline{H}$

Restricting ourselves to the TE_{10} mode

$$J_s^o(x=0,y) = \hat{x} \times \hat{z} H_z^o(x=0,y) = -\hat{y} H_z^o(0,y) = -\hat{y}C$$

$$J_s^o(x=a,y) = -\hat{x} \times \hat{z} H_z^o(x=a,y) = \hat{y} H_z^o(a,y) = -\hat{y}C = J_s^o(x=0,y)$$

$$J_s^o(x,y=0) = \hat{y} \times \underline{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \\ H_x^o(x,0) & 0 & H_z(x,0) \end{vmatrix} = \hat{x} H_z(x,0) - \hat{z} H_x^o(x,0)$$

$$J_s^o(x,y=0) = \hat{x} C \cos\left(\frac{\pi x}{a}\right) - \hat{z} \frac{j\beta_{10} a C}{\pi} \sin\left(\frac{\pi x}{a}\right) = -J_s^o(x,y=b)$$

The attenuation constant associated with power loss was shown to be

$$\alpha_c = \frac{\text{Power loss / unit length}}{2 \times P_{transmitted}} = \frac{P_{LOSS}}{2 P_{AV}}$$

We can calculate $P_{AV}(z)$ for the TE_{10} mode

$$P_{AV}(z) = \int_0^b \int_0^a \frac{1}{2} \mathrm{Re}\{E \times H^*\} \cdot \hat{z} \, dx\,dy$$

$$= \int_0^b \int_0^a -\frac{1}{2} E_y^o (H_x^o)^* \, dx\,dy$$

$$= \int_0^b \int_0^a -\frac{1}{2} \left(\frac{-j\omega\mu a C}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) e^{-j\bar{\beta}_{10} z} \left(\frac{-j\bar{\beta}_{10} a C}{\pi}\right) \sin\left(\frac{\pi x}{a}\right) e^{+j\bar{\beta}_{10} z} \, dx\,dy$$

$$= \int_0^b \int_0^a \frac{1}{2} \left(\frac{\omega\mu a^2 \bar{\beta}_{10} C^2}{\pi^2}\right) \sin^2\left(\frac{\pi x}{a}\right) dx\,dy$$

$$= \frac{\omega\mu a^2 \bar{\beta}_{10} C^2}{2\pi^2} \, b \int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx$$

$$= \frac{\omega\mu \bar{\beta}_{10} C^2}{\pi^2} \, \frac{a^2 b}{2} \, \frac{a}{2}$$

$$P_{AV}(z) = \frac{\omega\mu \bar{\beta}_{10} C^2}{\pi^2} \, ab \left(\frac{a}{2}\right)^2$$

There are four walls so

$$P_{LOSS} = 2\left[P_{LOSS\,1}\right]_{x=0} + 2\left[P_{LOSS_2}\right]_{y=0}$$

$$\left[P_{LOSS2}\right]_{y=0} = \int_0^a \frac{1}{2} \left|J_{sx}^o(y=0)\right|^2 R_s \, dx + \int_0^a \frac{1}{2} \left|J_{sz}^o(y=0)\right|^2 R_s \, dx$$

$$= \int_0^a \frac{1}{2} \left[\left| C \cos\left(\frac{\pi x}{a}\right)\right|^2 + \left| \frac{\bar{\beta}_{10} a C}{\pi} \sin\left(\frac{\pi x}{a}\right)\right|^2 \right] R_s \, dx$$

$$= \frac{R_s C^2}{2} \left[\frac{a}{2} + \frac{\bar{\beta}_{10}^2 a^2}{\pi^2} \frac{a}{2} \right]$$

$$[P_{Loss1}] = \int_0^b \frac{1}{2} |J_{sy}^o(x=0)|^2 R_s\, dy$$

$$= \int_0^b \frac{1}{2} C^2 R_s\, dy = \frac{b}{2} C^2 R_s$$

check math

$$P_{Loss} = 2[P_{Loss2}]_{x=0} + 2[P_{Loss1}]_{y=0}$$

$$P_{Loss} = R_s C^2 \left[a + \frac{\bar{\beta}_{10}^2 a^3}{\pi^2} + b \right] = R_s C^2 \left[b + a \left[1 + \frac{\bar{\beta}_{10}^2 a^2}{\pi^2} \right] \right]$$

$$= R_s C^2 \left[b + a \left[1 + 4f^2 \right. \right]$$

$$= R_s C^2 \left[b + \frac{a}{2} + \frac{4f^2}{C^2} \left[1 - \left(\frac{f_{c10}}{f} \right)^2 \right] a^3 \right]$$

$$= R_s C^2 \left[b + \frac{a}{2} \left(\frac{f}{f_{c10}} \right)^2 \right]$$

$$\alpha_{c_{TE_{10}}} = \frac{P_{Loss}}{2P_{AV}} = \frac{R_s \cancel{C^2} \left[b + \frac{a}{2} + \frac{4f^2}{c^2} \left[1 - \left(\frac{f_{c10}}{f} \right)^2 \right] a^3 \right]}{2\, \frac{\omega \mu \beta_{10} \cancel{C^2}}{\pi^2} ab \left(\frac{a}{2} \right)^2}$$

$$= \frac{R_s \left[1 + \frac{2b}{a} \left(\frac{f_{c10}}{f} \right)^2 \right]}{\eta b \sqrt{1 - \left(\frac{f_{c10}}{f} \right)^2}}$$

losses for TE_{mn} modes are given by (except for m or n = 0)

$$\alpha_{c_{TE_{mn}}} = \frac{\frac{2R_s}{b\eta}}{\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}\left\{\left(1+\frac{b}{a}\right)\left(\frac{f_{c_{mn}}}{f}\right)^2 + \left[1-\left(\frac{f_{c_{mn}}}{f}\right)^2\right]\frac{\frac{b}{a}\left[\frac{b}{a}m^2+n^2\right]}{\frac{b^2}{a^2}m^2+n^2}\right\}$$

losses for TM_{mn} modes are given by

$$\alpha_{c_{TM_{mn}}} = \frac{2R_s}{b\eta\sqrt{1-\left(\frac{f_{c_{mn}}}{f}\right)^2}}\,\frac{m^2\left(\frac{b}{a}\right)^3+n^2}{m^2\left(\frac{b}{a}\right)^2+n^2}$$

Dielectric losses occur when $\epsilon_c = \epsilon' - j\epsilon''$

In this case $\bar{\gamma} = \bar{\alpha}_d + j\bar{\beta}_{mn}$

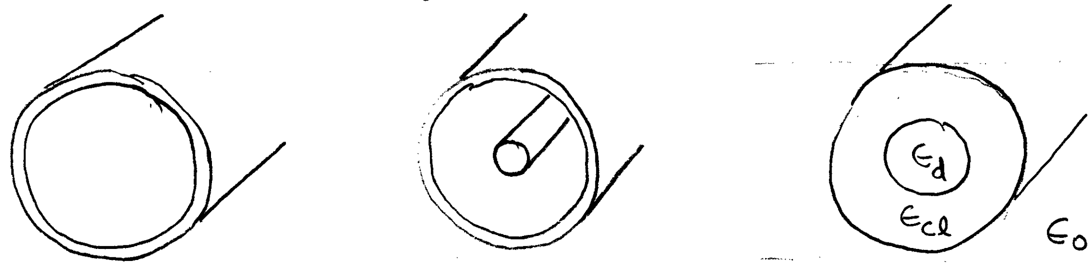
$$\cong \underbrace{\frac{\omega\sqrt{\mu\epsilon'}\,\frac{\epsilon''}{\epsilon'}}{2\sqrt{1-\left(\frac{\omega_{c_{mn}}}{\omega}\right)^2}}}_{\text{attenuation constant}} + j\sqrt{(\omega^2-\omega_{c_{mn}}^2)\mu\epsilon'}$$

For $f < f_{c_{mn}}$

$$\bar{\gamma} = \bar{\alpha}_{mn} = \frac{2\pi}{\lambda}\sqrt{\left(\frac{f_{c_{mn}}}{f}\right)^2-1} = \frac{2\pi}{\lambda_{c_{mn}}}\sqrt{1-\left(\frac{f}{f_{c_{mn}}}\right)^2}$$

5.2 Cylindrical Waveguides with Circular Cross Section

Three different types of cylindrical waveguides



metal tube waveguide hollow or filled with dielectric

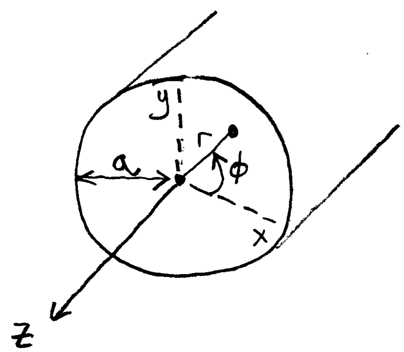
coaxial waveguide center metal conductor, metal shield or braided shield usually dielectric filled.

dielectric waveguide $n_d > n_{cl}$ this is a step index fiber guide

<u>mathematical analysis</u> for TM modes

$$\nabla_{tr}^2 E_z + (\bar{\gamma}^2 + \omega^2 \mu \epsilon) E_z = 0$$

as before for a solution of form $E_z(r, \phi, z) = E_z^0(r, \phi) e^{-\bar{\gamma} z}$



What happens is that ∇_{tr}^2 becomes ∇_{tr}^2 for cylindrical coordinates

$$\nabla_{tr}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

We will also use separation of variables, i.e. $E_z^0(r, \phi) = f(r) g(\phi)$

Substituting $E_z^0(r,\phi) = f(r)\,g(\phi)$ into the wave equation with ∇_{tr}^2 for cylindrical coordinates

$$g(\phi)\frac{\partial^2 f}{\partial r^2} + \frac{g(\phi)}{r}\frac{\partial f}{\partial r} + \frac{f(r)}{r^2}\frac{\partial^2 g}{\partial \phi^2} + \left(\bar{\gamma}^2 + \omega^2\mu\epsilon\right)f(r)g(\phi) = 0$$

dividing by fg and defining $h^2 = \bar{\gamma}^2 + \omega^2\mu\epsilon$

$$\frac{1}{f}\frac{\partial^2 f}{\partial r^2} + \frac{1}{fr}\frac{\partial f}{\partial r} + \frac{1}{g r^2}\frac{\partial^2 g}{\partial \phi^2} + h^2 = 0$$

multiply by r^2

$$\frac{r^2}{f}\frac{d^2 f}{dr^2} + \frac{r}{f}\frac{df}{dr} + \frac{1}{g}\frac{d^2 g}{d\phi^2} + h^2 r^2 = 0$$

collecting terms.

$$\underbrace{\frac{r}{f}\left[r\frac{d^2 f}{dr^2} + \frac{df}{dr}\right] + h^2 r^2}_{\text{only a function of } r} + \underbrace{\frac{1}{g}\frac{d^2 g}{d\phi^2}}_{\text{only a function of } \phi} = 0$$

Recognizing that

$$r\frac{d^2 f}{dr^2} + \frac{df}{dr} = \frac{d}{dr}\left(r\frac{df}{dr}\right)$$

we can write the wave equation as

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + h^2 r^2 = -\frac{1}{g}\frac{d^2 g}{d\phi^2} = n^2 \quad \left(\text{an integer constant}\right)$$

The integer constant requirement comes from the equation for $g(\phi)$ and requiring that it be continuous (i.e. periodic) at 2π

Looking at the ϕ equation

$$-\frac{1}{g}\frac{d^2g}{d\phi^2} = n^2$$

$$\frac{d^2g}{d\phi^2} + n^2 g = 0$$

which has general solutions

$$g(\phi) = c_1 \cos(n\phi) + c_2 \sin(n\phi)$$

Looking at the r equation

$$\frac{r}{f}\frac{d}{dr}\left(r\frac{df}{dr}\right) + h^2 r^2 = n^2$$

we can re-write it as

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)f = 0$$

This is a famous equation called Bessel's equation and has the general solution

$$f(r) = C_3 J_n(hr) + C_4 Y_n(hr)$$

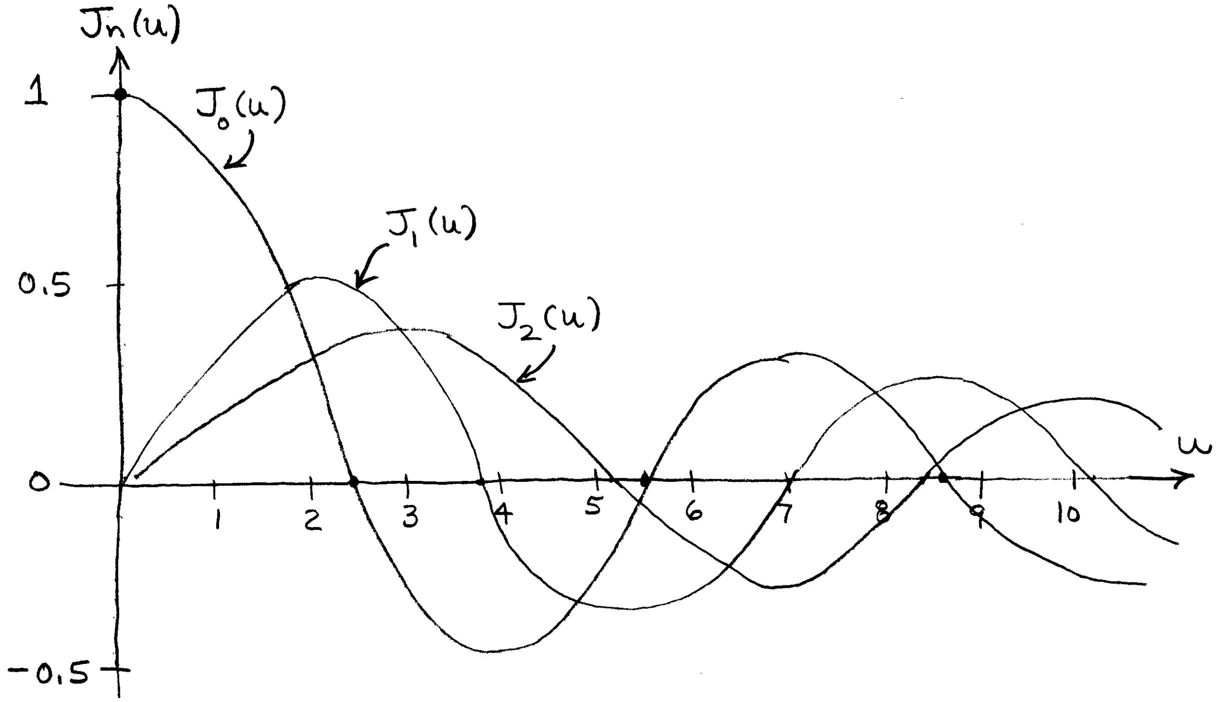
$J_n(\cdot)$ is the n-th order Bessel function of the first kind

$Y_n(\cdot)$ is the n-th order Bessel function of the second kind

The two Bessel functions are tabulated and their properties are well known.

$$Y_n(r) \rightarrow \infty \text{ as } r \rightarrow 0$$

For a cylindrical wave guide we want finite fields at all points (no charge in the guide) so we reject all solutions of the form $Y_n(\cdot)$



These functions behave a lot like a $\frac{\sin x}{x}$. They start at 1 or 0, decrease as u increases, BUT the period changes at r changes.

$J_0(u)$ has zeros at $u = 2.405$
$u = 5.520$
$u = 8.654$
$u = 11.792$

$J_1(u)$ has zeros at $u = 3.832$
$u = 7.016$
$u = 10.173$
$u = 13.323$

$J_2(u)$ has zeros at $u = 5.136$
$u = 8.417$
$u = 11.620$
$u = 14.796$

You will rarely need to use any higher order Bessel functions

The general solution is

$$E_z^o(r,\phi) = C_3 J_n(hr)\left[C_1\cos(n\phi) + C_2\sin(n\phi)\right]$$

Looking at the ϕ terms we expect the solutions to be independent of where we pick the origin, i.e. the $\phi = 0$ point.

So just pick $\cos(n\phi)$ making the solution

$$E_z^o(r,\phi) = C_n J_n(hr)\cos(n\phi)$$

This is almost exactly like the product solutions we got for rectangular waveguides.

For the metal tube waveguides we want

$E_{tan} = 0$ at the metal wall

which requires

$$J_n(ha) = 0$$

where ha is a zero (root) of the n-th order Bessel function of the first kind.

The modes are then given by

$$h_{TM\,n\ell} = \frac{t_{n\ell}}{a}$$

$t_{n\ell}$ ← this is the ℓ-th root of $J_n(\ )$

a ← this is the radius of the guide

n will be the number of circumferential variations

ℓ will be the number of radial variations

The propagation constant

$$\bar{\beta}_{TM_{n\ell}} = \left[\omega^2\mu\epsilon - \left(\frac{t_{n\ell}}{a}\right)^2\right]^{1/2}$$

propagation occurs for $f > f_{c_{TM_{n\ell}}}$

This gives $f_{c_{TM_{n\ell}}} = \dfrac{t_{n\ell}}{2\pi a\sqrt{\mu\epsilon}}$

The lowest cutoff frequency is $t_{01} = 2.405$ which makes TM_{01} the TM mode with the lowest cutoff frequency.

The field components for circular $TM_{n\ell}$ modes are

$$E_z = C_n J_n\left(\frac{t_{n\ell}}{a} r\right)\cos(n\phi)e^{-j\bar{\beta}_{n\ell} z}$$

$$E_r = \frac{-ja\bar{\beta}_{TM_{n\ell}}}{t_{n\ell}} C_n J_n'\left(\frac{t_{n\ell}}{a} r\right)\cos(n\phi)e^{-j\bar{\beta}_{n\ell} z}$$

$$E_\phi = \frac{ja^2 n\bar{\beta}_{TM_{n\ell}}}{t_{n\ell}^2 r} C_n J_n\left(\frac{t_{n\ell}}{a} r\right)\sin(n\phi)e^{-j\bar{\beta}_{n\ell} z}$$

$$H_r = -\frac{j\omega\epsilon n a^2}{t_{n\ell}^2 r} C_n J_n\left(\frac{t_{m\ell}}{a} r\right)\sin(n\phi)e^{-j\bar{\beta}_{n\ell} z}$$

$$H_\phi = -\frac{j\omega\epsilon a}{t_{m\ell}} C_n J_n'\left(\frac{t_{m\ell}}{a} r\right)\cos(n\phi)e^{-j\bar{\beta}_{n\ell} z}$$

$$H_z = 0$$

where $J_n'(\zeta) = \dfrac{dJ_n(\zeta)}{d\zeta}$

The TE_{mn} modes are calculated in a similar manner except that $E_z = 0$.

Let $H_z^0(r,\phi) = C_n J_n(hr)\cos n\phi$

the boundary condition is $E_{tan} = 0$

$\Rightarrow E_\phi^0(r,\phi) = 0$ at $r = a$

or $\frac{\partial H_z^0}{\partial r} = 0$ at $r = a$

The eigenvalues of the solution come from the zeros of $J_n'(u)$ which we call $S_{n\ell}$

$$\bar{\beta}_{TE_{n\ell}} = \sqrt{\omega^2\mu\epsilon - \left(\frac{S_{n\ell}}{a}\right)^2}$$

$$f_{c_{TE_{n\ell}}} = \frac{S_{n\ell}}{2\pi a\sqrt{\mu\epsilon}}$$

roots of $J_n'(\cdot)$

$J_0'(u)$ has zeros at $u = 3.832$
$u = 7.016$
$u = 10.173$
$u = 13.324$

$J_1'(u)$ has zeros at $u = 1.841$
$u = 5.331$
$u = 8.536$
$u = 11.706$

$J_2'(u)$ has zeros at $u = 3.054$
$u = 6.706$
$u = 9.969$
$u = 13.170$

It is very important to notice that $S_{11} = 1.841$ is the lowest zero and gives the TE_{11} mode the lowest cutoff frequency.

The complete $TE_{n\ell}$ solutions are

$$H_z = C_n J_n\left(\frac{s_{n\ell}}{a}r\right)\cos(n\phi)\, e^{-j\bar{\beta}_{n\ell} z}$$

$$H_r = \frac{-ja\bar{\beta}_{TE_{n\ell}}}{s_{n\ell}} C_n J_n'\left(\frac{s_{n\ell}}{a}r\right)\cos n\phi\, e^{-j\bar{\beta}_{n\ell} z}$$

$n=0,1,2,\ldots$

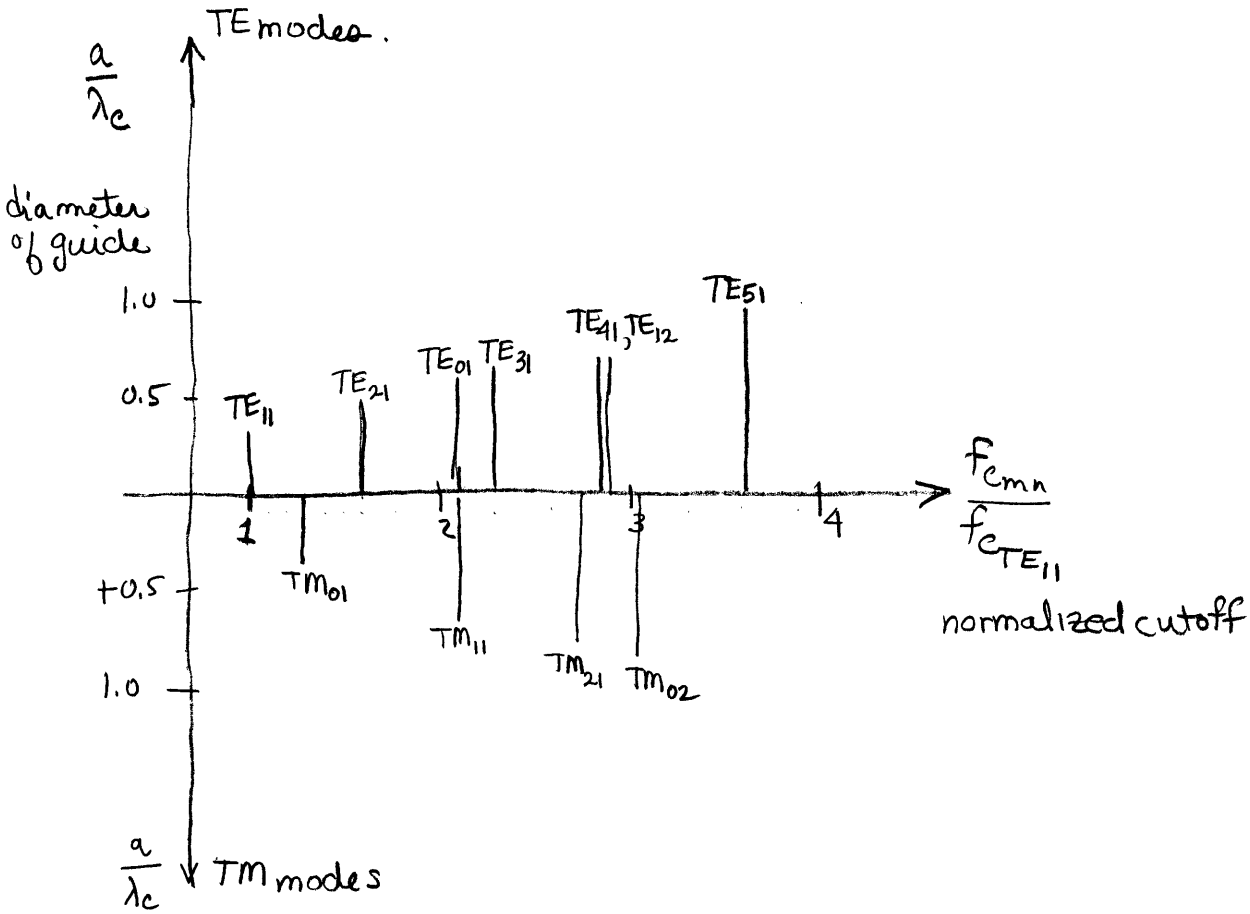
$\ell=1,2,3,\ldots$

$$H_\phi = \frac{jna^2\beta_{TE_{n\ell}}}{s_{n\ell}^2 r} C_n J_n\left(\frac{s_{n\ell}}{a}r\right)\sin n\phi\, e^{-j\bar{\beta}_{n\ell} z}$$

$$E_r = \frac{ja^2\omega\mu n}{s_{n\ell}^2 r} C_n J_n\left(\frac{s_{n\ell}}{a}r\right)\sin n\phi\, e^{-j\bar{\beta}_{n\ell} z}$$

$$E_\phi = \frac{-ja\omega\mu}{s_{n\ell}} C_n J_n'\left(\frac{s_{n\ell}}{a}r\right)\cos n\phi\, e^{-j\bar{\beta}_{n\ell} z}$$

$$E_z = 0$$



Notice that TE_{11} is the overall dominant mode.

If we choose the wavelength λ to lie between

$$\lambda_{c_{TE_{11}}} = \frac{2\pi a}{1.841} = 3.41a$$

and

$$\lambda_{c_{TM_{01}}} = \frac{2\pi a}{2.405} = 2.61a$$

only the TE_{11} mode will propagate,

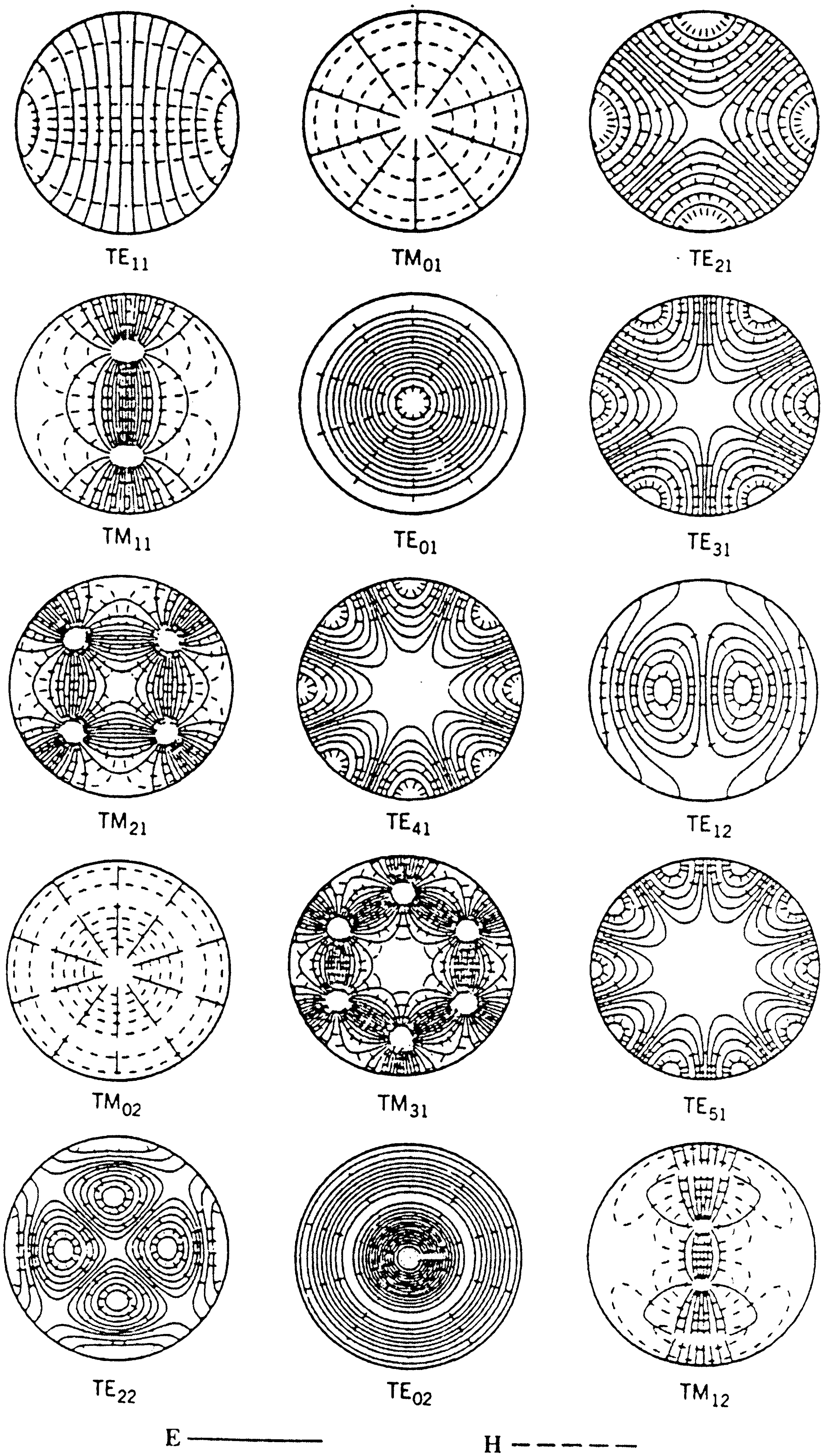


FIGURE 5.12. **Different modes of a circular waveguide.** Field configurations for some circular waveguide modes. [Taken from C. S. Lee, S. W. Lee, and S. L. Chuang, Plot of modal field distribution in rectangular and circular waveguides, *IEEE*

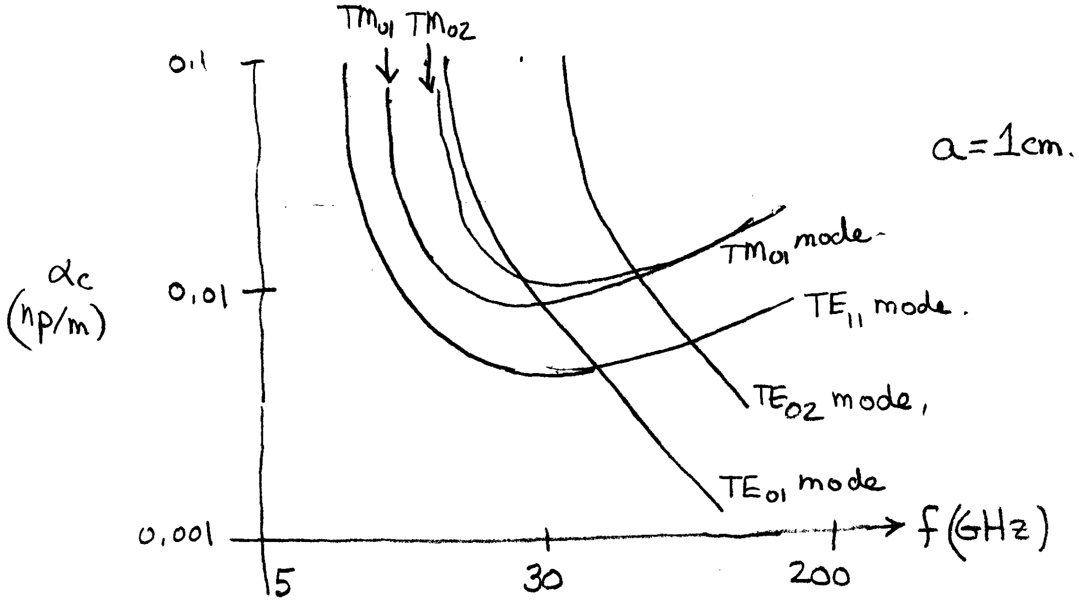
Attenuation in circular waveguides

for $f > f_{c_{TM_{nl}}}$

$$\alpha_{c_{TM_{nl}}} = \frac{R_s}{a\eta}\left[1-\left(\frac{f_{c_{TM_{nl}}}}{f}\right)^2\right]^{-\frac{1}{2}}$$

for $f > f_{c_{TE_{nl}}}$

$$\alpha_{c_{TE_{nl}}} = \frac{R_s}{a\eta}\left[1-\left(\frac{f_{c_{TE_{nl}}}}{f}\right)^2\right]^{-\frac{1}{2}}\left[\left(\frac{f_{c_{TE_{nl}}}}{f}\right)^2 + \frac{n^2}{S_{nl}^2 - n^2}\right]$$



Attenuation is high near cutoff, drops to a minimum near two or three times the cutoff, and increases $\sim \sqrt{f}$ for frequencies well above cutoff.

5.2.2. Coaxial lines

used at frequencies less than 5 GHz
braided outer conductor & inner conductor

tangential E }
normal H } must be zero on conductor surfaces

$$\Rightarrow \begin{matrix} E_\phi = 0 \\ H_r = 0 \end{matrix} \quad \text{at } r = a, b$$

non-zero E_ϕ & H_r can only exist between the conductors if they vary with r.

A TEM solution can only exist with
$\underline{E} = \hat{r} E_r$ and $\underline{H} = \hat{\phi} H_\phi$
to give energy transport in the z-direction.

radial variation of E_ϕ cannot be present since it would create a H_z which cannot exist for TEM

$$-j\omega\mu H_z = [\nabla \times E]_z = \frac{1}{r}\left[\underbrace{\frac{\partial}{\partial r}(rE_\phi)}_{\text{this term would be non-zero}} - \frac{\partial E_r}{\partial \phi}\right].$$

The component equations give

$$-\frac{\partial H_\phi}{\partial z} = j\omega\epsilon E_r$$

$$+j\beta H_\phi^{o}(r) = j\omega\epsilon E_r^{o}(r)$$

and

$$\frac{1}{r} H_\phi + \frac{\partial H_\phi}{\partial r} = 0$$

$$\frac{1}{r} H_\phi^{o}(r) + \frac{\partial H_\phi^{o}(r)}{\partial r} = 0$$

The solution of this equation is $H_\phi^{o}(r) = \frac{H_o}{r}$

$$\therefore \ \underline{H} = \hat{\phi}\, \frac{H_o}{r}\, e^{-j\beta z}$$

$$\cancel{j}\beta H_\phi^o(r) = \cancel{j}\omega\epsilon E_r^o(r)$$

$$E_r^o(r) = \frac{\beta}{\omega\epsilon} H_\phi^o(r) = \frac{\omega\sqrt{\mu\epsilon}}{\omega\epsilon} H_\phi^o(r)$$

$$E_r^o(r) = \sqrt{\frac{\mu}{\epsilon}}\, H_\phi^o(r) = \eta H_\phi^o(r)$$

$$\therefore \ \underline{E} = \hat{r}\,\eta\,\frac{H_o}{r}\, e^{-j\beta z}$$

There is no cutoff frequency since β is that of a plane wave.

The line exhibits loss due to surface currents on the conductors.

$$\underline{J}_s(z) = \hat{n}\times\underline{H} = \begin{cases} \hat{z}\, H_\phi(a,z) & \text{inner conductor} \\ -\hat{z}\, H_\phi(b,z) & \text{outer conductor} \end{cases}$$

$$|\underline{S}_{AV}| = \frac{1}{2}\,\mathrm{Re}\left\{\underline{E}\times\underline{H}^*\right\}\cdot\hat{z} = \frac{1}{2}\, E_r H_\phi^* = \frac{1}{2}\,\eta\,\frac{H_o^2}{r^2}$$

$$P_{AV} = \int_a^b\int_0^{2\pi} |S_{AV}|\, r\, dr\, d\phi = \int_a^b \cancel{2}\pi\, \frac{1}{\cancel{2}}\,\eta\,\frac{H_o^2}{r^2}\, r\, dr$$

$$P_{AV} = \int_a^b \pi\eta H_o^2\, \frac{dr}{r} = \pi\eta H_o^2 \ln\left(\frac{b}{a}\right)$$

The power loss/unit length is calculated as

$$P_{Loss}(z) = \frac{1}{2}\int |J_s|^2 R_s\, ds = \frac{R_s}{2}\int_0^1\int_0^{2\pi} |H_\phi(a,z)|^2\, a\, d\phi$$

$$+ \frac{R_s}{2}\int_0^1\int_0^{2\pi} |H_\phi(b,z)|^2\, b\, d\phi$$

$$P_{LOSS}(z) = \frac{R_s}{2}\int_0^1\int_0^{2\pi}\left|\frac{H_o}{a}\right|^2 a\,d\phi + \frac{R_s}{2}\int_0^1\int_0^{2\pi}\left|\frac{H_o}{b}\right|^2 b\,d\phi$$

$$= \frac{2\pi R_s}{2}\frac{H_o^2}{a^2}a + \frac{2\pi R_s}{2}\frac{H_o^2}{b^2}b$$

$$= \pi R_s H_o^2\left[\frac{1}{a}+\frac{1}{b}\right]$$

The attenuation constant is then

$$\alpha_{c_{TEM}} = \frac{P_{Loss}(z)}{2P_{AV}} = \frac{\cancel{\pi} R_s \cancel{H_o^2}\left[\frac{1}{a}+\frac{1}{b}\right]}{2\cancel{\pi}\eta \cancel{H_o^2}\ln\left(\frac{b}{a}\right)} = \frac{R_s}{2\eta\ln\left(\frac{b}{a}\right)}\left[\frac{1}{a}+\frac{1}{b}\right]$$

and since $R_s = \sqrt{\frac{\omega\mu_o}{2\sigma}}$ $\alpha_{c_{TEM}}$ increases with frequency.

The formula for dielectric losses is the same as that for a dielectric slab waveguide [4.27]

$$\alpha_d \cong \left.\frac{\omega\sqrt{\mu_o\epsilon'}\,\frac{\epsilon''}{\epsilon'}}{2\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}}\right|_{\omega_c=0} = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu_o}{\epsilon'}} \cong \frac{\omega\epsilon''}{2}\eta$$

where we note that $\eta = \sqrt{\frac{\mu_o}{\epsilon'}}$

Just as for the (two conductor) parallel plate waveguide the TEM mode corresponds to voltage and current on a two-conductor transmission line where the voltage and current are given by:

$$V(z) = -\int_a^b \underline{E} \cdot \underline{dl} = -\int_a^b E_r \hat{r} \cdot \hat{r}\, dr = -\int_a^b \frac{H_o \eta e^{-j\beta z}}{r} dr = -\int_a^b \eta H_o e^{-j\beta z} \frac{dr}{r}$$

$$V(z) = -\eta H_o \ln\left(\frac{b}{a}\right) e^{-j\beta z} = -V^+ e^{-j\beta z}$$

$$I(z) = \oint_c \underline{H} \cdot \underline{dl} = \int_0^{2\pi} \frac{H_o}{a} e^{-j\beta z} \hat{\phi} \cdot \hat{\phi}\, a\, d\phi$$ just do on one conductor pick $r=a$

$$I(z) = 2\pi H_o e^{-j\beta z}$$

but from the above equation for $V(z)$

$$H_o = \frac{V^+}{+\eta \ln\left(\frac{b}{a}\right)}$$

$$\therefore \quad I(z) = 2\pi \frac{V^+}{+\eta \ln\left(\frac{b}{a}\right)} e^{-j\beta z} = \frac{V^+}{Z_o} e^{-j\beta z}$$

where $Z_o = \dfrac{\eta \ln\left(\frac{b}{a}\right)}{2\pi} \cong 60 \ln\left(\frac{b}{a}\right)$ since $\eta \cong 120\pi$

The average power is $P_{AV}(z) = \dfrac{1}{2} \dfrac{|V^+|^2}{Z_o}$

You can optimize the coaxial cable in several ways.

Since $E \sim \frac{1}{r}$ the maximum field occurs at $r=a$

For a coaxial geometry

$$E = \frac{V}{r \ln \frac{b}{a}}$$

or

$$E_{max} = \frac{V_{max}}{a \ln(\frac{b}{a})} = \frac{V_{max}}{b} \frac{\zeta}{\ln(\zeta)} \quad \text{where } \zeta = \frac{b}{a}$$

The power transmitted is

$$P_{AV} = \frac{V_{max}^2}{2Z_0} \cong \frac{\left[\frac{E_{max}\, b \ln(\zeta)}{\zeta}\right]^2}{2 \; 60 \ln\zeta} = \frac{E_{max}^2\, b^2}{2 \cdot 60} \frac{\ln^2(\zeta)}{\zeta^2 \ln\zeta}$$

$$P_{AV} = K \frac{\ln\zeta}{\zeta^2}$$

To maximize power transmission

$$\frac{dP_{AV}}{d\zeta} = K\left[\frac{1}{\zeta^3} - \frac{2\ln\zeta}{\zeta^3}\right] = 0$$

$1 - 2\ln\zeta = 0 \quad \ln\zeta = \frac{1}{2}$ or $\frac{b}{a} \cong 1.65$ for max power

This corresponds to a impedance:

$$Z_0 \cong 60 \ln(1.65) = 30\Omega$$

Incidentally, maximum $V(z)$ occurs with 60Ω line
(maximum voltage handling)
minimum conduction losses with 77Ω line

TE & TM modes in a coaxial line

The TEM mode is dominant since it is the lowest cutoff frequency. but TE & TM modes exist for a coaxial waveguide.

The presence of the inner conductor does not allow us to eliminate $Y_n(hr)$ as a possible solution since $r > a$

For TM waves we now have

$$E_z^0(r,\phi) = [C_3 J_n(hr) + C_4 Y_n(hr)]\cos n\phi$$

with the boundary conditions $E_z(r,\phi) = 0$ at $r=a$ and $r=b$

For TE waves we now have

$$H_z^0(r,\phi) = [C_3' J_n(hr) + C_4' Y_n(hr)]\cos n\phi$$

with the boundary conditions $\frac{\partial H_z}{\partial r} = 0$ at $r=a$ and $r=b$

These equations have solutions of the form

$$C_3 J_n(ha) + C_4 Y_n(ha) = 0$$

$$C_3 J_n(hb) + C_4 Y_n(hb) = 0$$

$$\frac{C_3}{C_3}\frac{J_n(ha)}{J_n(hb)} = \frac{-C_4 Y_n(ha)}{-C_4 Y_n(hb)}$$

$$J_n(ha)Y_n(hb) = J_n(hb)Y_n(ha)$$

which is a transcental equation which is best solved numerically.

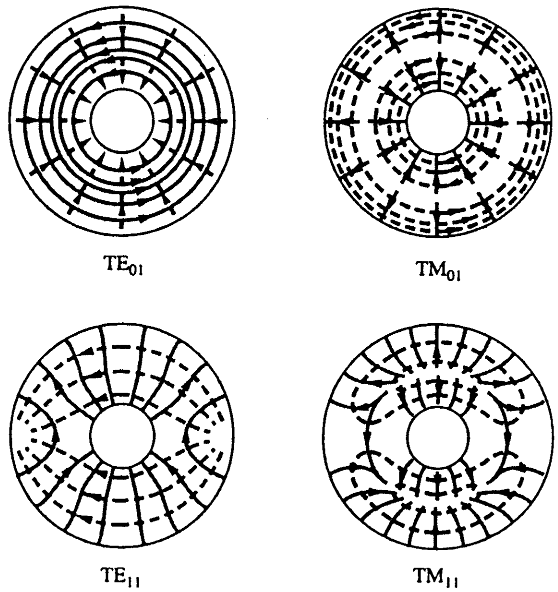


FIGURE 5.15. TE and TM mode fields in a coaxial line. The electric (magnetic) field lines are shown as solid (dashed) lines. [Taken from H. A. Atwater, *Introduction to Microwave Theory*, p. 76, McGraw-Hill, New York, 1962.]

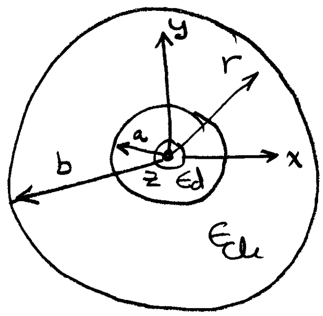
These are a few of the lowest order TE & TM modes.
The lowest non-zero cutoff frequency is for the TE$_{11}$ mode

$$f_{c_{11}} \cong \frac{1}{\pi(a+b)\sqrt{\mu\epsilon}}$$

$$h = \frac{2}{a+b}$$

In practice choose the coax dimensions so only the TEM mode propagates for the frequencies of interest.

Dielectric circular waveguides



We use a cladding as opposed to simple air because

(1) a cladding layer minimizes any effects on the fields in the guide which might cause higher order fields to exist

(2) the use of a cladding ϵ_{cl} gives the designer freedom to design a waveguide so that only one mode propagates

Neither E_z or H_z can be zero so

The fields must satisify the wave equations in both dielectrics

$$\nabla_{tr}^2 E_z + h^2 E_z = 0$$

and

$$\nabla_{tr}^2 H_z + h^2 H_z = 0$$

where $h^2 = h_d^2 = \bar{\gamma}^2 + \omega^2 \mu_d \epsilon_d$ in the dielectric

or $h^2 = -h_{cl}^2 = \bar{\gamma}^2 + \omega^2 \mu_{cl} \epsilon_{cl}$ in the cladding

Assume separable solutions

$$\left.\begin{aligned} E_z(r,\phi,z) &= E_z^0(r,\phi)\, e^{-\bar{\gamma} z} \\ H_z(r,\phi,z) &= H_z^0(r,\phi)\, e^{-\bar{\gamma} z} \end{aligned}\right\} = f(r)\, g(\phi)\, e^{-\bar{\gamma} z}$$

Inside the dielectric core we want $\bar{\gamma} = j\beta$

and $\omega^2 \mu_d \epsilon_d - h_d > 0$

The Bessel functions $J_n(\cdot)$ and $Y_n(\cdot)$ are well defined for this case and give solutions of the form

$$\left.\begin{array}{l} E_z^o(r,\phi) \\ H_z^o(r,\phi) \end{array}\right\} = \left[C_d J_n(h_d r) + C_d' Y_n(h_d r)\right]\left[A_d \cos n\phi + B_d \sin n\phi\right]$$

for $r \le a$.

However in the cladding we want solutions that decay with r

For the dielectric core $h^2 = h_d^2 = \bar{\gamma}^2 + \omega^2 \mu_d \epsilon_d$ and $\bar{\gamma}^2 = h_d^2 - \omega^2 \mu_d \epsilon_d < 0$

and we had solutions of the form

$$C_d J_n(h_d r) + C_d' Y_n(h_d r)$$

For the dielectric slab we required solutions that decayed exponentially, i.e. $\bar{\gamma} = \bar{\alpha} > 0$, outside the slab.

With Bessel functions we have a similar situation in the cladding. Bessel's equation in the core was

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)f = 0 \qquad + \left(T^2 - \frac{\nu^2}{r^2}\right)R$$

and gave the solutions $J_n(hr)$ and $Y_n(hr)$

However, if we allow $h_{cl}^2 < 0$ we get a slightly different equation

$$\frac{d^2f}{dr^2} + \frac{1}{f}\frac{df}{dr} - \left(h^2 + \frac{n^2}{r^2}\right)f = 0$$

For $n=0$ this equation has solutions of the form

$$I_{\pm n}(r) = j^{\pm n} J_{\pm n}(jr)$$

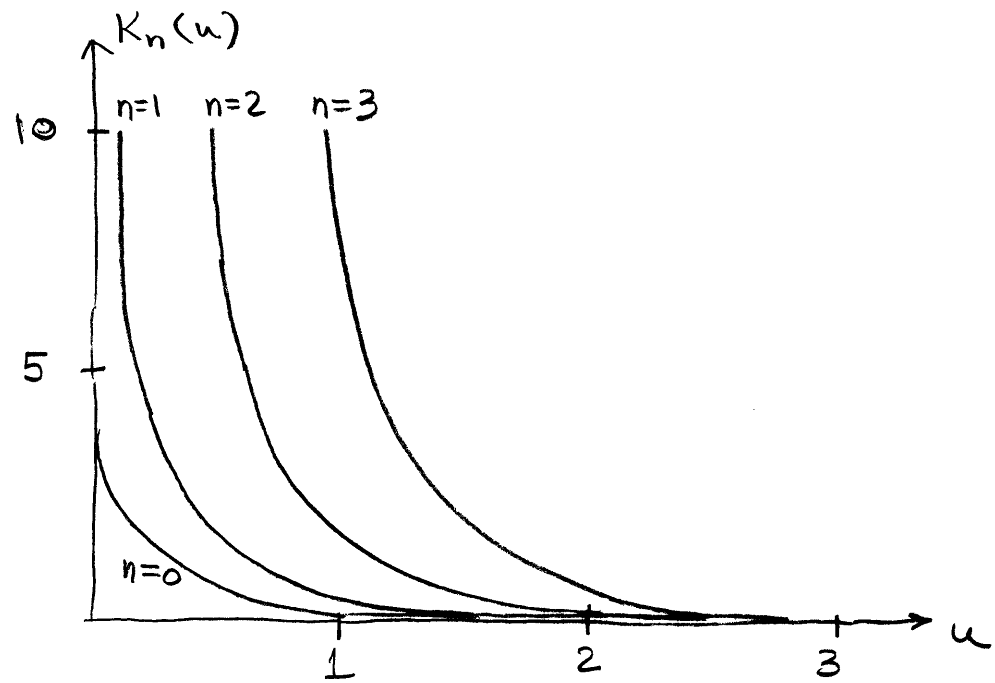
$$K_n(r) = \frac{\pi}{2} j^{n+1} H_n^{(1)}(jr)$$

where $H_n^{(1)}(jr) = \underbrace{J_n(jr) + j\,Y_n(jr)}_{\text{This is a Hankel function}}$

I_n and K_n are called modified Bessel functions and give better solutions in the cladding as they decay much faster than $J_n(hr)$ which only decays as $r^{-1/2}$

<u>See</u>. Ramo, Whinnery, Van Duzer
Fields & Waves in Communications Electronics 2/e
7.14 Bessel Functions

The modified Bessel function K_n decay faster than r^{-1}
Hower, I_n increases and must be eliminated as a possible solution



The complete solution in the cladding is

$$\left.\begin{array}{l} E_z^o(r,\phi) \\ H_z^o(r,\phi) \end{array}\right\} = \left[C_{cl} K_n(h_{cl} r) + C'_{cl} I_n(h_{cl} r)\right]\left[A_{cl}\cos n\phi + B_{cl}\sin n\phi\right]$$

However we set $C'_{cl} = 0$

The complete solution for propagating waves is then

$$\left.\begin{array}{c} E_z(r,\phi,z) \\ \text{or} \\ H_z(r,\phi,z) \end{array}\right\} = \begin{cases} C_d J_n(h_d r)\left[A_d\cos(n\phi) + B_d\sin(n\phi)\right]e^{-j\bar{\beta}z} & r \le a \\ C_{cl} K_n(h_{cl} r)\left[A_{cl}\cos(n\phi) + B_{cl}\sin(n\phi)\right]e^{-j\bar{\beta}z} & r > a \end{cases}$$

where $\omega^2\mu_{cl}\epsilon_{cl} \le \bar{\beta}^2 \le \omega^2\mu_d\epsilon_d$

Since cylindrical waveguides are often used for optical communications we can write this in terms of the optical index of refraction

$$\frac{\omega n_{cl}}{c} \le \bar{\beta} \le \frac{\omega n_d}{c}$$

The propagation constant will lie between that of the core and the cladding.

These solutions must be continuous at $r = a$

One way to do this is to normalize the fields so the radial dependence is 1 at $r = a$, i.e.

$$H_z = \begin{cases} c\,\dfrac{J_n(h_d r)}{J_n(h_d a)}\cos(n\phi)\,e^{-j\bar{\beta}z} & 0 < r \le a \\ c\,\dfrac{K_n(h_{cl} r)}{K_n(h_{cl} a)}\cos(n\phi)\,e^{-j\bar{\beta}z} & r > a \end{cases}$$

Similarily

$$E_z = \begin{cases} c' \dfrac{J_n(h_d r)}{J_n(h_d a)} \sin(n\phi) e^{-j\bar{\beta} z} & 0 < r \le a \\ c' \dfrac{K_n(h_{ce} r)}{K_n(h_{ce} a)} \sin(n\phi) e^{-j\bar{\beta} z} & r > a \end{cases}$$

Note: we arbitrarily picked the origin for ϕ so that we can have $H_z \propto \cos(n\phi)$

This choice forces E_z to be proportional to $\sin(n\phi)$

Once the axial components are specified the transverse field components can be found using Maxwell's equations

Inside the dielectric core

$$\left.\begin{aligned} E_\phi &= \frac{j\omega\mu}{h_d^2} \frac{\partial H_z}{\partial r} - \frac{j\bar{\beta}}{h_d^2 r} \frac{\partial E_z}{\partial \phi} \\ E_r &= \frac{-j\bar{\beta}}{h_d^2} \frac{\partial E_z}{\partial r} - \frac{j\omega\mu_0}{h_d^2 r} \frac{\partial H_z}{\partial \phi} \\ H_\phi &= -\frac{j\bar{\beta}}{h_d^2 r} \frac{\partial H_z}{\partial \phi} - \frac{j\omega\epsilon_d}{h_d^2} \frac{\partial E_z}{\partial r} \\ H_r &= -\frac{j\bar{\beta}}{h_d^2} \frac{\partial H_z}{\partial r} + \frac{j\omega\epsilon_d}{h_d^2 r} \frac{\partial E_z}{\partial \phi} \end{aligned}\right\} r \le a$$

In the cladding

$$E_\phi = \frac{-j\omega\mu}{h_{ce}^2} \frac{\partial H_z}{\partial r} + \frac{j\bar{\beta}}{h_{ce}^2} \frac{\partial E_z}{r\partial \phi}$$

$$H_\phi = \frac{j\bar{\beta}}{h_{ce}^2} \frac{1}{r} \frac{\partial H_z}{\partial \phi} + \frac{j\omega\epsilon_{ce}}{h_{ce}^2} \frac{\partial E_z}{\partial r}$$

plus the other components.

However, we still don't know the value of $\bar{\beta}$

This comes from equation H_ϕ and E_ϕ at $r=a$.

This is pretty complex [See Ramo, Whinnery & Van Duzer, Ch. 14]

Continuity of E_ϕ at $r=a$ gives

$$C\frac{\omega\mu_0}{s} f_n(s) - C'\frac{n\bar{\beta}}{s^2} = -C\frac{\omega\mu_0}{t} g_n(t) + C'\frac{n\bar{\beta}}{t^2}$$

Continuity of H_ϕ at $r=a$ gives

$$C\frac{n\bar{\beta}}{s^2} - C'\frac{\omega\epsilon_z}{s} f_n(s) = -C\frac{n\bar{\beta}}{t^2} + C'\frac{\omega\epsilon_z}{t} g_n(t)$$

where $\quad s = h_d a$

$\qquad\qquad t = h_{ce} a$

$$f_n(h_d a) = \frac{1}{h_d}\left[\frac{d}{dr}\left\{\frac{J_n(h_d r)}{J_n(h_d a)}\right\}\right]_{r=a} = h_d \frac{J_n'(h_d a)}{J_n(h_d a)}$$

$$g_n(h_{ce} a) = \frac{1}{h_{ce}}\left[\frac{d}{dr}\left\{\frac{K_n(h_{ce} r)}{K_n(h_{ce} a)}\right\}\right]_{r=a} = h_{ce}\frac{K_n'(h_{ce} a)}{K_n(h_{ce} a)}$$

Each equation gives a solution for $\bar{\beta}$ in terms of $\frac{C}{C'}$

However, since the C/C' must be the same for each equation we can equate C/C' and get

$$\omega^2\mu_0\left(\frac{f_n(s)}{s} + \frac{g_n(t)}{t}\right)\left(\epsilon_d\frac{f_n(s)}{s} + \epsilon_c\frac{g_n(t)}{t}\right) = (n\bar{\beta})^2\left(\frac{1}{s^2}+\frac{1}{t^2}\right)^2$$

The solutions for $\bar{\beta}$ can be determined numerically.

and result in hybrid modes in which neither E_z or $H_z = 0$.

For $n=0$ these can be broken into TE and TM modes.

For this specialized case

$$\frac{1}{s} f_n(s) = -\frac{1}{t} g_n(t)$$

and $s^2+t^2 = \omega^2 \mu_0 (\epsilon_d - \epsilon_{cl}) a^2 = u^2$

Similarily for TM modes

$$\frac{\epsilon_d}{s} f_n(s) = -\frac{\epsilon_{cl}}{t} g_n(t)$$

Other specialized cases occur when

$$\frac{|\epsilon_d - \epsilon_{cl}|}{|\epsilon_d|} < 1\%$$

or when n gradually varies with r.