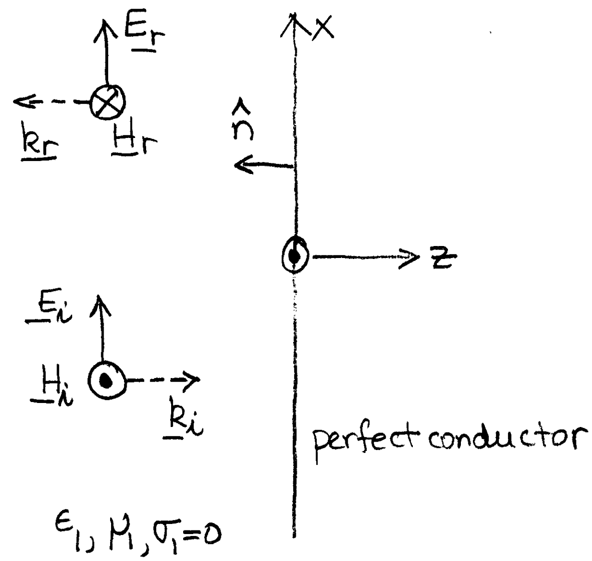
Ch. 3 Reflection, Transmission & Refraction at Planar Interfaces



BOUNDARY CONDITION: E_{tan} vanishes at $z=0$ $\forall x,y$

since $E_2 = 0$

Total fields in medium 1:

$$\therefore \quad \underline{E}_1(z) = \underline{E}_i(z) + \underline{E}_r(z) = \hat{x}\left[E_{io}e^{-j\beta_1 z} + E_{ro}e^{+j\beta_1 z}\right]$$

$$\underline{H}_1(z) = \underline{H}_i(z) + \underline{H}_r(z) = \hat{y}\left[\frac{E_{io}}{\eta_1}e^{-j\beta_1 z} - \frac{E_{ro}}{\eta_c}e^{+j\beta z}\right]$$

sign of impedance

propagation in $-z$ direction

@ $z=0$ $\underline{E}_1(z) = 0$

$$\therefore \quad E_{io} + E_{ro} = 0 \quad \text{or} \quad E_{ro} = -E_{io}$$

$$\therefore \underline{E}_1(z) = \hat{x}\, E_{io}\left\{\underbrace{e^{-j\beta_1 z} - e^{+j\beta_1 z}}_{-2j\sin\beta_1 z}\right\} = -\hat{x}\, E_{io}\, j\, 2\sin(\beta_1 z)$$

$$\underline{H}_1(z) = \underline{H}_i(z) + \underline{H}_r(z) = \hat{y}\left[\frac{E_{io}}{\eta_1} e^{-j\beta_1 z} + \frac{E_{io}}{\eta_1} e^{+j\beta_1 z}\right]$$

$$= \hat{y}\,\frac{2E_{io}}{\eta_1}\cos\beta_1 z$$

@ z=0

since there is no field in region 2

$$\underline{J}_s = \hat{n}\times\underline{H}_i(0) = (-\hat{z}\times\hat{y})\,\frac{2E_{io}}{\eta_1} = \hat{x}\,\frac{2E_o}{\eta_1}\ \frac{A}{m}$$

$$(\underline{S}_{AV})_1 = \frac{1}{2}\mathrm{Re}\left\{\underline{E}_1\times\underline{H}_1^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\hat{x}\left[-2jE_{io}\sin\beta z\right]\times\hat{y}\left[\frac{2E_{io}^*}{\eta_1}\cos\beta z\right]\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\hat{z}\,-j\,\frac{2|E_{io}|^2}{\eta_1}\sin 2\beta_1 z\right\} = 0$$

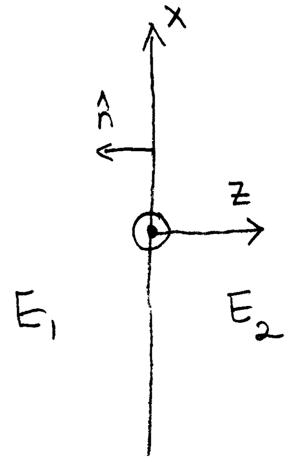
NO POWER FLOW SINCE PERFECT CONDUCTOR REFLECTS ALL INCIDENT ENERGY $\Longrightarrow$ STANDING WAVE

$$w_e(z,t) = \frac{1}{2}\epsilon_1|\mathcal{E}_1(z,t)|^2 = 2\epsilon_1|E_{io}|^2\sin^2\beta_1 z\,\sin^2(\omega t+\zeta)$$

$$w_m(z,t) = \frac{1}{2}\mu_1|\mathcal{H}_1(z,t)|^2 = 2\epsilon_1|E_{io}|^2\cos^2\beta_1 z\,\cos^2(\omega t+\zeta)$$

total stored energy at a point z alternates between fully magnetic $(\omega t+\zeta=0)$ to fully electric $(\omega t+\zeta=\frac{\pi}{2})$

Example 3-2 circularly polarized wave incident on a conductor



$$E_i(z) = c_1(\hat{x} \pm j\hat{y})e^{-j\beta_1 z}$$

Since $E_1(0) = 0$ because $E_2 = 0$

$$E_1(z) = E_i(z) + E_r(z)$$

becomes

$$E_1(0) = c_1(\hat{x} \pm j\hat{y})e^{0} + E_r(0) = 0$$

$$E_r = -c_1(\hat{x} \pm j\hat{y})$$

comparing

$$E_i(z) = c_1(\hat{x} + j\hat{y})e^{-j\beta z} \quad \text{LHCP} \quad \mathcal{E}_i(z,t) = \hat{x}c_1\cos(\omega t - \beta z) + \hat{y}c_1\cos(\omega t - \beta z + \tfrac{\pi}{2})$$

$$= c_1(\hat{x} - j\hat{y})e^{-j\beta z} \quad \text{RHCP} \quad + \hat{x}c_1\cos(\omega t - \beta z) - \hat{y}c_1\underbrace{\cos(\omega t - \beta z + \tfrac{\pi}{2})}_{-\sin(\omega t - \beta_1 z)}$$

sign

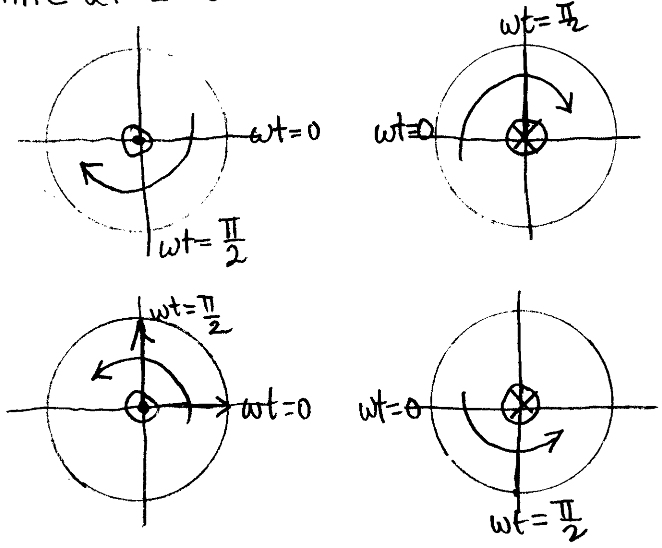
$$E_r(z) = -c_1(\hat{x} + j\hat{y})e^{+j\beta z} \qquad \mathcal{E}_r(z,t) = -\hat{x}c_1\cos(\omega t - \beta z) - \hat{y}c_1\cos(\omega t - \beta z + \tfrac{\pi}{2})$$

$$-c_1(\hat{x} - j\hat{y})e^{+j\beta z} \qquad -\hat{x}c_1\cos(\omega t - \beta z) + \hat{y}c_1\underbrace{\cos(\omega t - \beta z + \tfrac{\pi}{2})}_{-\sin(\omega t - \beta_1 z)}$$

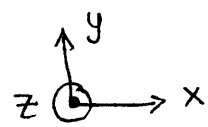
Examine at $z = 0$

LHCP

RHCP

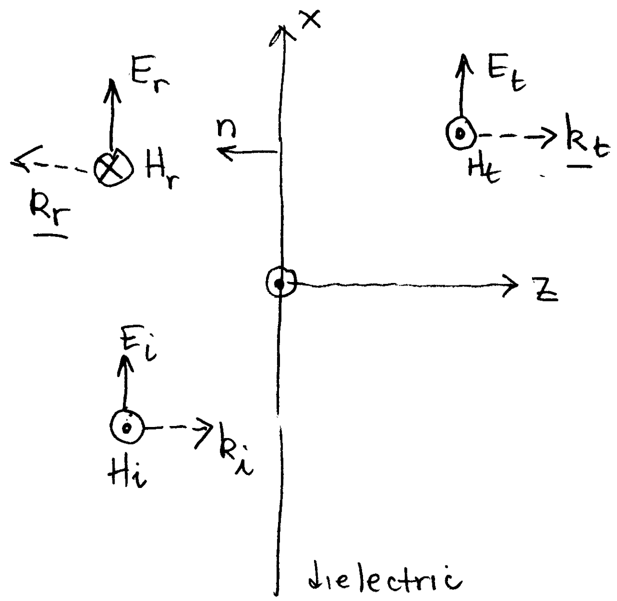


incident reflected



using the right hand rule we note that the polarizations reversed when reflected by a perfect conductor.

3.2 Normal incidence on a lossless dielectric



$$E_i(z) = \hat{x} E_{i0} e^{-j\beta z}$$
$$H_i(z) = \hat{y} \frac{E_{i0}}{\eta_1} e^{-j\beta z}$$

$$E_r(z) = \hat{x} E_{r0} e^{+j\beta z}$$
$$H_r(z) = -\hat{y} \frac{E_{r0}}{\eta_1} e^{+j\beta z}$$

$$E_t(z) = \hat{x} E_{t0} e^{-j\beta z}$$
$$H_r(z) = \hat{y} \frac{E_{t0}}{\eta_2} e^{-j\beta z}$$

We require tan E to be continuous at interface
tan H is also continuous since there is no surface current at interface.

$$\therefore \; E_i(z=0) + E_r(z=0) = E_t(z=0)$$
$$H_i(z=0) + H_r(z=0) = H_r(z=0$$

or

$$E_{i0} + E_{r0} = E_{t0}$$
$$\frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

Solve simultaneously

$$E_{i0} + E_{r0} = E_{t0}$$
$$E_{i0} - E_{r0} = \frac{\eta_1}{\eta_2} E_{t0}$$

Adding $\quad 2E_{i0} = \left(1 + \frac{\eta_1}{\eta_2}\right) E_{t0} \quad$ or $\quad \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_1 + \eta_2} \equiv \mathcal{T}$

$$E_{io} + E_{ro} = E_{to}$$

$$1 + \frac{E_{ro}}{E_{io}} = \frac{E_{to}}{E_{io}}$$

$$\frac{E_{ro}}{E_{io}} = \frac{E_{to}}{E_{io}} - 1 = \frac{2\eta_2}{\eta_2+\eta_1} - 1 = \frac{2\eta_2 - \eta_2 - \eta_1}{\eta_2+\eta_1}$$

$$\frac{E_{ro}}{E_{io}} = \frac{\eta_2 - \eta_1}{\eta_2+\eta_1} \triangleq \Gamma$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2+\eta_1}$$

NOTE: $1 + \Gamma = \mathcal{T}$

$$\mathcal{T} = \frac{2\eta_2}{\eta_1+\eta_2}$$

For optical applications where $\mu_1 = \mu_2 = \mu_0$ the μ's drop out and

$$\Gamma = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad \text{and} \quad \mathcal{T} = \frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}}$$

the optical index of refraction $n = \sqrt{\mu_r \epsilon_r}$ and for optical dielectrics

$$\Gamma = \frac{n_1 - n_2}{n_1 + n_2} \qquad \mathcal{T} = \frac{2n_1}{n_1+n_2}.$$

3.2.2. Propagating & Standing Waves

For a wave incident on a dielectric interface we have both a reflected and a transmitted wave

The total wave in medium #1 can be written as

$$\underline{E}_1(z) = \underline{E}_i(z) + \underline{E}_r(z)$$
$$= \hat{x} E_{io} e^{-j\beta_1 z} + \hat{x}\Gamma E_{io} e^{+j\beta_1 z}$$
$$= \hat{x} E_{io}(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z})$$

This can be rewritten as

$$= \hat{x} E_{io}\left[e^{-j\beta_1 z} + \underbrace{\Gamma e^{-j\beta_1 z} - \Gamma e^{-j\beta_1 z}}_{\text{add \& subtract}} + \Gamma e^{+j\beta_1 z}\right]$$

$$= \hat{x} E_{io}\left[\underbrace{(1+\Gamma)}_{\tau} e^{-j\beta_1 z} + \Gamma\underbrace{(e^{j\beta_1 z} - e^{-j\beta_1 z})}_{2j\sin\beta_1 z}\right]$$

$$\underline{E}_1(z) = \hat{x} E_{io}\left[\tau e^{-j\beta_1 z} + \Gamma\, 2j\sin\beta_1 z\right] \qquad j = e^{j\frac{\pi}{2}}$$

converting from phasor to time domain

$$\underline{\mathcal{E}}_1(z,t) = \hat{x} E_{io}\Big[\underbrace{\tau\cos(\omega t - \beta_1 z)}_{\text{propagating wave}} + \underbrace{2\Gamma\sin\beta_1 z\,\underbrace{\cos(\omega t - \tfrac{\pi}{2})}_{-\sin\omega t}}_{\text{standing wave}}\Big]$$

associated magnetic field comes from $\underline{E}_1(z)$

$$\underline{H}_1(z) = \hat{y}\left[\frac{E_{io}}{\eta_1} e^{-j\beta_1 z} - \Gamma\frac{E_{io}}{\eta_1} e^{+j\beta_1 z}\right]$$

As for the E-field.

$$= \hat{y}\frac{E_{io}}{\eta_1}\left[e^{-j\beta_1 z} + \underbrace{\Gamma e^{-j\beta_1 z} - \Gamma e^{-j\beta_1 z}}_{\text{add \& subtract}} - \Gamma e^{+j\beta_1 z}\right]$$

$$\underline{H}_1(z) = \hat{y}\,\frac{E_{io}}{\eta_1}\left[(1+\Gamma)\,e^{-j\beta_1 z} - \Gamma\left(e^{j\beta_1 z} + e^{-j\beta_1 z}\right)\right]$$

$$\underline{H}_1(z) = \hat{y}\,\frac{E_{io}}{\eta_1}\left[\mathcal{T}\,e^{-j\beta_1 z} - \Gamma\, 2\cos\beta_1 z\right]$$

Converting from phasors to time domain

$$\underline{\mathcal{H}}_1(z,t) = \hat{y}\,\frac{E_{io}}{\eta_1}\Big[\underbrace{\mathcal{T}\cos(\omega t - \beta_1 z)}_{\text{traveling wave}} - \underbrace{2\Gamma\cos\beta_1 z\cos\omega t}_{\text{standing wave}}\Big]$$

The transmitted wave is easier to analyze

$$\underline{E}_2(z) = \underline{E}_t(z) = \hat{x}\,\mathcal{T}E_{io}\,e^{-j\beta_2 z}$$

using impedance we can determine the $\underline{H}$ field.

$$\underline{H}_2(z) = \underline{H}_t(z) = \hat{y}\,\frac{\mathcal{T}}{\eta_2}\,E_{io}\,e^{-j\beta_2 z}$$

converting from phasors to time domain

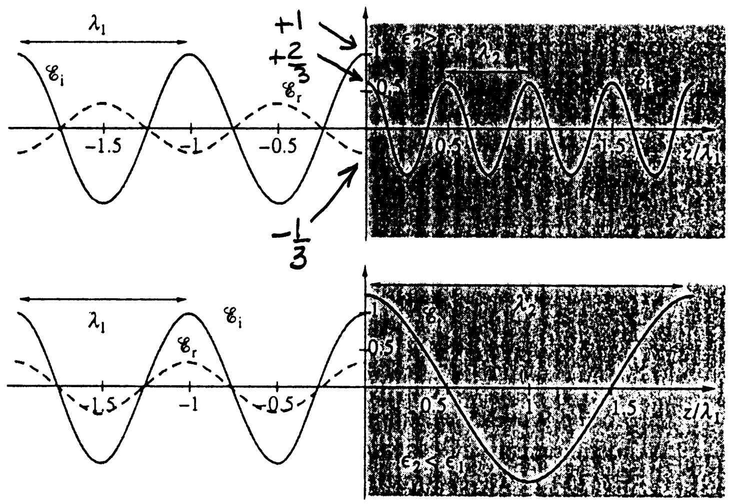
$$\mathcal{E}_2(z,t) = \hat{x}\,\mathcal{T}E_{io}\cos(\omega t - \beta_2 z)$$

$$\mathcal{H}_2(z,t) = \hat{y}\,\frac{\mathcal{T}}{\eta_2}\,E_{io}\cos(\omega t - \beta_2 z)$$

Instantaneous ($t = 0$) waveforms of amplitude

$$\Gamma = \frac{n_1 - n_2}{n_1 + n_2} = \frac{1-2}{1+2} = -\frac{1}{3}$$

$$\tau = \frac{2n_1}{n_1+n_2} = \frac{2(1)}{1+2} = \frac{2}{3}$$



(a)

$\epsilon_2 > \epsilon_1$

$\epsilon_2 = 4\epsilon_1$

$\epsilon_2 < \epsilon_1$

$\epsilon_1 = 4\epsilon_2$

Note (1) The wavelength changes in medium 2 because $\beta = \frac{2\pi}{\lambda} = \omega\sqrt{\mu\epsilon}$. Since ϵ changes λ also changes.

(2) $|\mathcal{E}_t|$ can be larger than $|\mathcal{E}_i|$ since ϵ_2 changes. However, $(S_{AV})_2 = (S_{AV})_1$

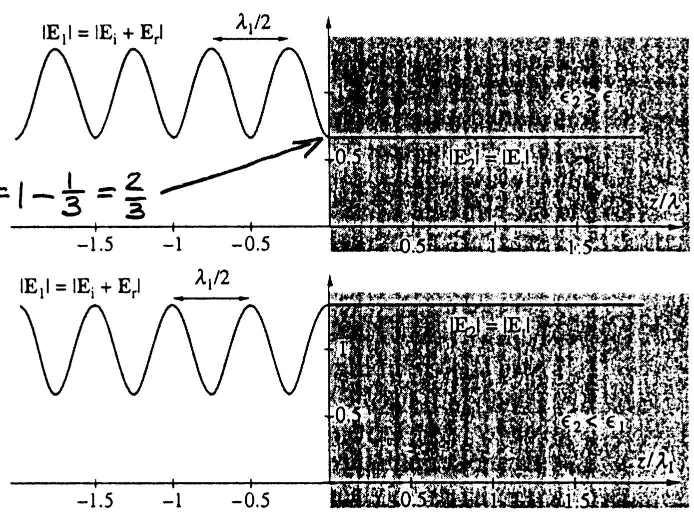
Standing-wave patterns (independent of t)

$1+|\Gamma|$

1

$1-|\Gamma|$

Assuming $E_{i0}=1$ $E_i + E_r = 1 - \frac{1}{3} = \frac{2}{3}$



(b)

$\epsilon_2 = 4\epsilon_1$

$\epsilon_1 = 4\epsilon_2$

Standing waves are very similar to the voltage waves on a transmission line.

3.2.3. EM Power Flow (Poynting vector)

In medium 2

$$(\underline{S}_{AV})_2 = \frac{1}{2}\,\mathcal{Re}\left\{\underline{E}_t(z) \times \underline{H}_t^*(z)\right\}$$

$$= \frac{1}{2}\,\mathcal{Re}\left\{\hat{x}\,\mathcal{T} E_{io} e^{-j\beta_2 z} \times \hat{y}\,\frac{\mathcal{T}}{\eta_2} E_{io} e^{+j\beta_2 z}\right\}$$

$$= \hat{z}\,\frac{E_{io}^2}{2\eta_2}\,\mathcal{T}^2$$

we assumed $\mathcal{T}$ is real.

In medium 1

$$(\underline{S}_{AV})_1 = \frac{1}{2}\,\mathcal{Re}\left\{\underline{E}_1 \times H_1^*(z)\right\}$$

$$= \frac{1}{2}\,\mathcal{Re}\left\{E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z}\right)\hat{x} \times \hat{y}\left(\frac{E_{io}}{\eta_1}\right)\left(e^{+j\beta_1 z} - \Gamma e^{-j\beta_1 z}\right)\right\}$$

$$= \frac{1}{2}\hat{z}\,\frac{E_{io}^2}{\eta_1}\,\mathcal{Re}\left\{e^{-j\beta_1 z}\left(1+\Gamma e^{j2\beta_1 z}\right) e^{j\beta_1 z}\left(1-\Gamma e^{-j2\beta_1 z}\right)\right\}$$

$$= \hat{z}\,\frac{E_{io}^2}{2\eta_1}\,\mathcal{Re}\left\{1 + \Gamma e^{j2\beta_1 z} - \Gamma e^{-j2\beta_1 z} - \Gamma^2\right\}$$

$$= \hat{z}\,\frac{E_{io}^2}{2\eta_1}\,\mathcal{Re}\left\{(1-\Gamma^2) + \Gamma\left(2j\sin(2\beta_1 z)\right)\right\}$$

$$(\underline{S}_{AV})_1 = \hat{z}\,\frac{E_{io}^2}{2\eta_1}\,(1-\Gamma^2)$$

Law of Energy conservation says

$$(\underline{S}_{AV})_1 = (\underline{S}_{AV})_2$$

since no energy stored or dissipated at Interface

$$\hat{z}\,\frac{E_{io}^2}{2\eta_1}(1-\Gamma^2) = \hat{z}\,\frac{E_{io}^2}{2\eta_2}\,\mathcal{T}^2$$

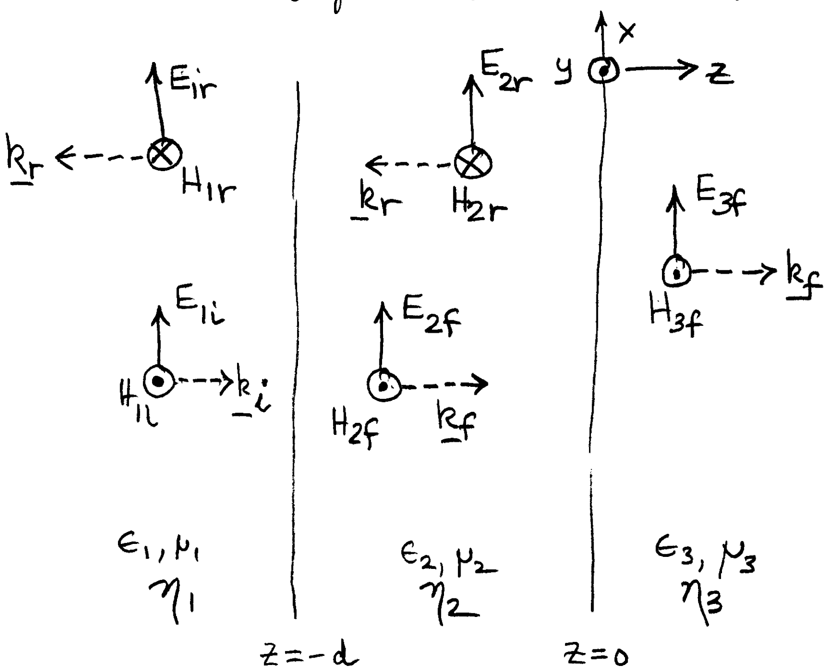
$$\therefore\ 1-\Gamma^2 = \frac{\eta_1}{\eta_2}\,\mathcal{T}^2$$

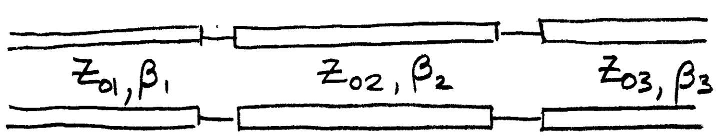
3.3. Multiple Dielectric Interfaces.

(some applications include anti-reflection coatings)

See J. A. Adam, How to design an "invisible" aircraft, IEEE Spectrum, p. 30, April 1988.

You can use a transmission line analogy to derive results but we will do general solution instead.





Transmission line analogy

write total waves in all three media

Medium 1 $\quad z < -d$

$$\underline{E}_1(z) = \underline{E}_{1i} + \underline{E}_{1r} = \hat{x} E_{i0} \underbrace{e^{-j\beta_1(z+d)}}_{\substack{\text{addtl phase} \\ \text{shift since interface} \\ \text{at } z=-d}} + \hat{x} E_{i0} \underbrace{\Gamma_{eff}}_{\substack{\text{effective } \Gamma \\ \text{for multiple} \\ \text{layers}}} e^{+j\beta_1(z+d)}$$

$$\underline{H}_1(z) = \underline{H}_{1i} + \underline{H}_{1r} = \hat{y} \frac{E_{i0}}{\eta_1} e^{-j\beta_1(z+d)} - \hat{y} \frac{E_{i0}}{\eta_1} \Gamma_{eff} e^{+j\beta_1(z+d)}$$

Medium 2 $\quad -d < z < 0$

$$\underline{E}_2(z) = \underline{E}_{2f} + \underline{E}_{2r} = \hat{x} E_{20} e^{-j\beta_2 z} + \hat{x} E_{20} \Gamma_{23} e^{+j\beta_2 z}$$

$$\underline{H}_2(z) = \underline{H}_{2f} + \underline{H}_{2r} = \hat{y} \frac{E_{20}}{\eta_2} e^{-j\beta_2 z} - \hat{y} \frac{E_{20}}{\eta_2} \Gamma_{23} e^{+j\beta_2 z}$$

Medium 3 $\quad z > 0$

$$\underline{E}_3(z) = \underline{E}_{3f} = \hat{x} \mathcal{T}_{eff} E_{i0} e^{-j\beta_3 z}$$

$$\underline{H}_3(z) = \underline{H}_{3f} = \hat{y} \mathcal{T}_{eff} \frac{E_{i0}}{\eta_3} e^{-j\beta_3 z}$$

Now apply boundary conditions at $z=-d$, $z=0$ that tangential $\underline{E}$ & $\underline{H}$ must be continuous. This will give us four equations in four unknowns.

@ $z=-d$ $\qquad E_1(-d) = E_2(-d)$ and $H_1(-d) = H_2(-d)$

@ $z=0$ $\qquad E_2(0) = E_3(0)$ and $H_2(0) = H_3(0)$

Neglecting vectors

$$@z=-d \qquad \underbrace{E_{i0}(1+\Gamma_{eff})}_{E_1(-d)} = \underbrace{E_{20}\left(e^{+j\beta_2 d} + \Gamma_{23}e^{-j\beta_2 d}\right)}_{E_2(-d)} \qquad (1)$$

$$@z=-d \qquad \underbrace{\frac{E_{i0}}{\eta_1}(1-\Gamma_{eff})}_{H_1(-d)} = \underbrace{\frac{E_{20}}{\eta_2}\left(e^{+j\beta_2 d} - \Gamma_{23}e^{-j\beta_2 d}\right)}_{H_2(-d)} \qquad (2)$$

$$@z=0 \qquad \underbrace{E_{20}(1+\Gamma_{23})}_{E_2(0)} = \underbrace{\mathcal{T}_{eff}E_{i0}}_{E_3(0)} \qquad (3)$$

$$@z=0 \qquad \underbrace{\frac{E_{20}}{\eta_2}(1-\Gamma_{23})}_{H_2(0)} = \underbrace{\mathcal{T}_{eff}\frac{E_{i0}}{\eta_3}}_{H_3(0)} \qquad (4)$$

We can solve for Γ_{23} by multiplying (4) by η_3 and subtract from (3)

$$E_{20}(1+\Gamma_{23}) = \mathcal{T}_{eff}E_{i0}$$

$$E_{20}\frac{\eta_3}{\eta_2}(1-\Gamma_{23}) = \mathcal{T}_{eff}E_{i0}$$

$$E_{20} + E_{20}\Gamma_{23} - E_{20}\frac{\eta_3}{\eta_2} + E_{20}\frac{\eta_3}{\eta_2}\Gamma_{23} = 0$$

$$E_{20}\left(1-\frac{\eta_3}{\eta_2}\right) + E_{20}\Gamma_{23}\left(1+\frac{\eta_3}{\eta_2}\right) = 0$$

Solving $\Gamma_{23} = \dfrac{\frac{\eta_3}{\eta_2}-1}{\frac{\eta_3}{\eta_2}+1} = \dfrac{\eta_3-\eta_2}{\eta_3+\eta_2}$

Substituting

$$E_{i0}(1+\Gamma_{eff}) = E_{20}\, e^{j\beta_2 d} + \frac{\eta_3-\eta_2}{\eta_3+\eta_2} E_{20}\, e^{-j\beta_2 d}$$

$$\frac{E_{i0}}{\eta_1}(1-\Gamma_{eff}) = \frac{E_{20}}{\eta_2} e^{j\beta_2 d} - \frac{\eta_3-\eta_2}{\eta_3+\eta_2}\frac{E_{20}}{\eta_2} e^{-j\beta_2 d}$$

dividing

$$\eta_1 \frac{1+\Gamma_{eff}}{\dfrac{1-\Gamma_{eff}}{\eta_1}} = \frac{(\eta_3+\eta_2)e^{j\beta_2 d} + (\eta_3-\eta_2)e^{-j\beta_2 d}}{\dfrac{(\eta_3+\eta_2)e^{j\beta_2 d} - (\eta_3-\eta_2)e^{-j\beta_2 d}}{\eta_2}} \eta_2$$

$$\frac{(1+\Gamma_{eff})}{(1-\Gamma_{eff})} = \frac{\eta_2(\eta_3+\eta_2)e^{j\beta_2 d} + \eta_2(\eta_3-\eta_2)e^{-j\beta_2 d}}{\eta_1(\eta_3+\eta_2)e^{j\beta_2 d} - \eta_1(\eta_3-\eta_2)e^{-j\beta_2 d}}$$

$$\eta_1(\eta_3+\eta_2)e^{j\beta_2 d} - \eta_1(\eta_3-\eta_2)e^{-j\beta_2 d} + \Gamma_{eff}\left[\eta_1(\eta_3+\eta_2)e^{j\beta_2 d} - \eta_1(\eta_3-\eta_2)e^{-j\beta_2 d}\right]$$

$$= \eta_2(\eta_3+\eta_2)e^{j\beta_2 d} + \eta_2(\eta_3-\eta_2)e^{-j\beta_2 d} - \Gamma_{eff}\left[\eta_2(\eta_3+\eta_2)e^{j\beta_2 d} + \eta_2(\eta_3-\eta_2)e^{-j\beta_2 d}\right]$$

rearranging

$$\Gamma_{eff}\left[(\eta_1+\eta_2)(\eta_3+\eta_2)e^{j\beta_2 d} + (\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j\beta_2 d}\right] =$$

$$(\eta_2-\eta_1)(\eta_3+\eta_2)e^{j\beta_2 d} + (\eta_1+\eta_2)(\eta_3-\eta_2)e^{-j\beta_2 d}$$

$$\Gamma_{eff} = \frac{(\eta_2-\eta_1)(\eta_3+\eta_2) + (\eta_2+\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}{(\eta_2+\eta_1)(\eta_3+\eta_2) + (\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}$$

$$\frac{\cancel{E_{i0}}(1+\Gamma_{eff})}{T_{eff}\cancel{E_{i0}}} = \frac{\cancel{E_{20}}\left(e^{j\beta_2 d} + \Gamma_{23}e^{-j\beta_2 d}\right)}{\cancel{E_{20}}(1+\Gamma_{23})}$$

$$\frac{(1+\Gamma_{eff})(1+\Gamma_{23})}{e^{j\beta_2 d} + \Gamma_{23}e^{-j\beta_2 d}} = T_{eff}$$

$$T_{eff} = \frac{\left(1+\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)\left(1+\frac{(\eta_2-\eta_1)(\eta_3+\eta_2)+(\eta_2+\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}{(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}\right)}{e^{j\beta_2 d}+\left(\frac{\eta_3-\eta_2}{\eta_3+\eta_2}\right)e^{-j\beta_2 d}}$$

$$= \frac{\frac{(\eta_3+\cancel{\eta_2}+\eta_3-\cancel{\eta_2})}{\cancel{(\eta_3+\eta_2)}}\left[(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}+(\eta_2-\eta_1)(\eta_3+\eta_2)+(\eta_2+\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}\right]}{\frac{1}{\cancel{(\eta_3+\eta_2)}}\left[(\eta_3+\eta_2)e^{j\beta_2 d}+(\eta_3-\eta_2)e^{-j\beta_2 d}\right]\quad(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}$$

$$= \frac{2\eta_3\left[(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}+(\eta_2+\eta_1)(\eta_3-\eta_2)e^{-j\beta_2 2d}\right]}{\left[(\eta_3+\eta_2)e^{j\beta_2 d}+(\eta_3-\eta_2)e^{-j\beta_2 d}\right]\quad(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}$$

I have tried reducing this algebraically but it is very tedious.

The textbook's result is

$$T_{eff} = \frac{4\eta_2\eta_3 e^{-j\beta_2 d}}{(\eta_2+\eta_1)(\eta_3+\eta_2)+(\eta_2-\eta_1)(\eta_3-\eta_2)e^{-j2\beta_2 d}}.$$

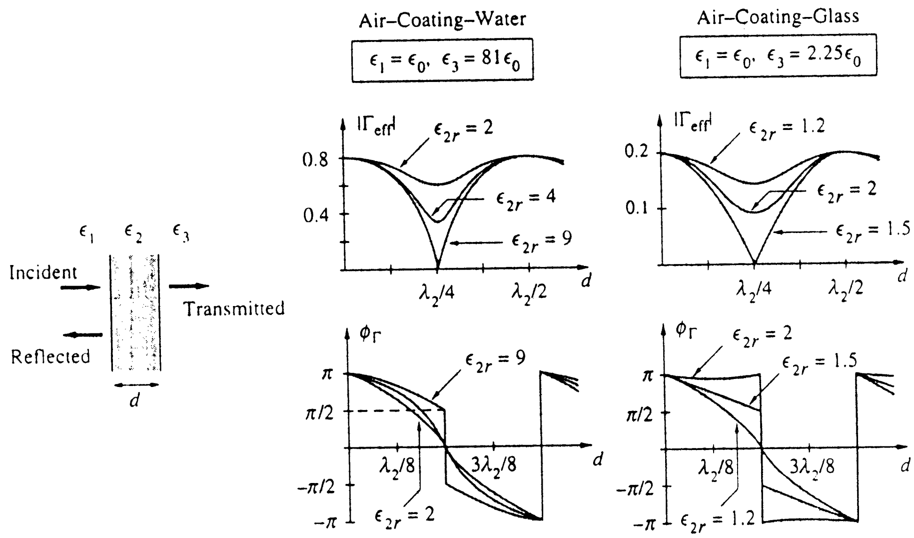
EXTRA CREDIT: DO THIS DERIVATION.

The case of a thin film between two dielectric media is very practically important. In general, $\Gamma_{eff} = \rho_{eff} e^{j\phi_r}$

Not immediately obvious but if you plot Γ_{eff} the minimum value of Γ_{eff} occurs at $d = \frac{\lambda_2}{4}$. Since this is the point of minimum reflection, the transmission is maximized. If $\epsilon_2 = \sqrt{\epsilon_1 \epsilon_2}$ then $\Gamma_{eff} \to 0$ and all energy is transmitted

minimum reflection

minimum reflection



original value repeats at $d = \frac{3\lambda_2}{2}$, etc.

The power relationship is identical to the previous result for a single dielectric–dielectric interface because no power is lost in medium 2.

Again the algebra is tedious with the result

$$1 - |\Gamma_{eff}|^2 = \frac{\eta_1}{\eta_3} |\mathcal{T}_{eff}|^2$$

3.3.2

We shall start using a transmission line approach to avoid this cumbersome algebraically intensive approach to dielectric problems.

Define the wave impedance

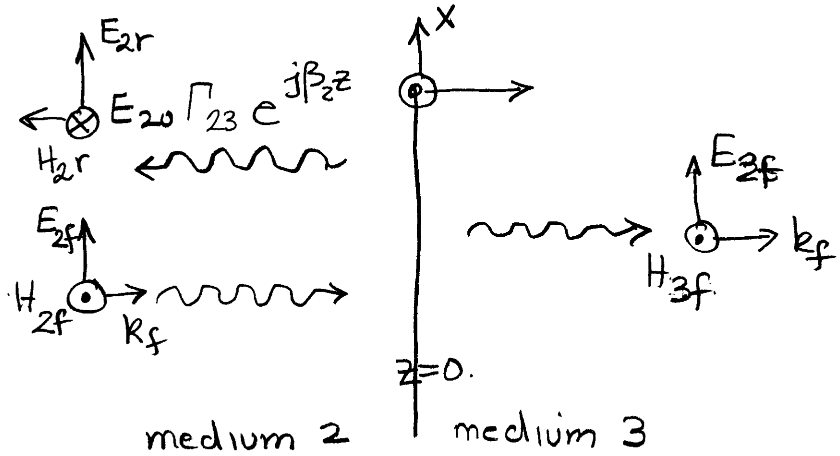
$$Z(z) \equiv \frac{E_x(z)|_{total}}{H_y(z)|_{total}} \quad \Big\} \text{ the total includes the all waves at each point.}$$

In many cases $Z(z)$ is exactly the previously defined impedance of the medium.

In the case of the transmitted field in a three-dielectric situation

$$Z_3(z) = \frac{E_{3x}(z)}{H_{3y}(z)} = \eta_3$$

In the case of traveling waves such as in region 2 of the three-dielectric problem the analysis is more complicated.



$$\Gamma_{23} = \frac{\eta_3 - \eta_2}{\eta_3 + \eta_2}$$

This is simply a two-dielectric problem.

forward wave

reverse wave

$$Z_2(z) = \frac{E_{2x}(z)_{\text{total}}}{H_{2y}(z)_{\text{total}}} = \frac{E_{20}e^{-j\beta_2 z} + E_{20}\Gamma_{23}e^{+j\beta_2 z}}{\dfrac{E_{20}e^{-j\beta_2 z}}{\eta_2} - \dfrac{E_{20}\Gamma_{23}e^{+j\beta_2 z}}{\eta_2}}$$

$$Z_2(z) = \eta_2 \frac{e^{-j\beta_2 z} + \Gamma_{23}e^{j\beta_2 z}}{e^{-j\beta_2 z} - \Gamma_{23}e^{j\beta_2 z}}$$

at $z=0$
$$Z_2(0) = \eta_2 \frac{1+\Gamma_{23}}{1-\Gamma_{23}} = \eta_2 \frac{1 + \dfrac{\eta_3-\eta_2}{\eta_3+\eta_2}}{1 - \dfrac{\eta_3-\eta_2}{\eta_3+\eta_2}} = \frac{\dfrac{2\eta_3}{\eta_3+\eta_2}}{\dfrac{+2\eta_2}{\eta_3+\eta_2}}\eta_2 = \eta_3$$

What is really interesting is what Z looks like at $z=-d$, the interface between medium 1 and medium 2.

There,
$$Z_2(z=-d) = \eta_2 \frac{e^{+j\beta_2 d} + \Gamma_{23}e^{-j\beta_2 d}}{e^{+j\beta_2 d} - \Gamma_{23}e^{-j\beta_2 d}}.$$

$$Z_2(z=-d) = \eta_2 \frac{e^{+j\beta_2 d} + \frac{\eta_3-\eta_2}{\eta_3+\eta_2} e^{-j\beta_2 d}}{e^{j\beta_2 d} - \frac{\eta_3-\eta_2}{\eta_3+\eta_2} e^{-j\beta_2 d}}$$

$$= \eta_2 \frac{(\cos\beta_2 d + j\sin\beta_2 d) + \frac{\eta_3-\eta_2}{\eta_3+\eta_2}(\cos\beta_2 d - j\sin\beta_2 d)}{(\cos\beta_2 d + j\sin\beta_2 d) - \frac{\eta_3-\eta_2}{\eta_3+\eta_2}(\cos\beta_2 d - j\sin\beta_2 d)}$$

$$= \eta_2 \frac{(\eta_3+\eta_2+\eta_3-\eta_2)\cos\beta_2 d + j(\eta_3+\eta_2-\eta_3+\eta_2)\sin\beta_2 d}{(\eta_3+\eta_2-\eta_3+\eta_2)\cos\beta_2 d + j(\eta_3+\eta_2+\eta_3-\eta_2)\sin\beta_2 d}$$

$$= \eta_2 \frac{2\eta_3\cos\beta_2 d + j2\eta_2\sin\beta_2 d}{2\eta_2\cos\beta_2 d + j2\eta_3\sin\beta_2 d}$$

$$= \eta_2 \frac{\eta_3\cos\beta_2 d + j\eta_2\sin\beta_2 d}{\eta_2\cos\beta_2 d + j\eta_3\sin\beta_2 d}$$

$$Z_2(z=-d) = \eta_2 \frac{\eta_3 + j\eta_2\tan\beta_2 d}{\eta_2 + j\eta_3\tan\beta_2 d}.$$

Before we can make effective use of this result we need to determine $Z_1(z)$ in medium 1.

$$Z_1(z) = \frac{E_{1x}(z)\big|_{total}}{H_{1y}(z)\big|_{total}} = \frac{\overbrace{E_{10}e^{-j\beta_1(z+d)}}^{\text{incident}} + \overbrace{\Gamma_{eff}E_{10}e^{+j\beta_1(z+d)}}^{\text{reflected}}}{\frac{E_{10}e^{-j\beta_1(z+d)}}{\eta_1} - \frac{\Gamma_{eff}E_{10}e^{j\beta_1(z+d)}}{\eta_1}}$$

the z+d comes about because we started our coordinate system at the medium 2/ medium 3 interface.

$$Z_1(z) = \eta_1 \frac{e^{-j\beta_1(z+d)} + \Gamma_{eff}\, e^{+j\beta_1(z+d)}}{e^{-j\beta_1(z+d)} - \Gamma_{eff}\, e^{j\beta_1(z+d)}}$$

at $z=-d$ this reduces to

$$Z_1(z=-d) = \eta_1 \frac{1+\Gamma_{eff}}{1-\Gamma_{eff}}$$

Now we can use the previous result for $Z_2(z=-d)$

Since the medium 1/2 interface is a dielectric-dielectric interface the tangential fields on both sides of the interface must be equal. This also means that the wave impedances must be equal on both sides of the interface.

$$\therefore\ Z_1(z=-d) = \eta_1 \frac{1+\Gamma_{eff}}{1-\Gamma_{eff}} = Z_2(z=-d)$$

Solving for Γ_{eff} we get

$$\eta_1 + \eta_1\Gamma_{eff} = Z_2(z=-d) - \Gamma_{eff} Z_2(z=-d)$$

$$(\eta_1 + Z_2(z=-d))\Gamma_{eff} = -\eta_1 + Z_2(z=-d)$$

$$\Gamma_{eff} = \frac{Z_2(z=-d) - \eta_1}{Z_2(z=-d) + \eta_1}$$ where we already know $Z_2(z=-d)$

This formula can be evaluated for some interesting cases.

Case 1: $\eta_1 = \eta_3$, $d = n\frac{\lambda_2}{2}$

$$\beta_2 d = \frac{2\pi}{\cancel{\lambda_2}} \cdot n\frac{\cancel{\lambda_2}}{\cancel{2}} = n\pi \qquad \text{or } \tan\beta_2 d = 0$$

for this case $\quad Z_2(z=-d) = \eta_2 \dfrac{\eta_3 + j\eta_2 \cdot 0}{\eta_2 + j\eta_3 \cdot 0} = \eta_3$

$$\therefore \Gamma_{eff} = \frac{Z_2(z=-d) - \eta_1}{Z_2(z=-d) + \eta_1} = \frac{\eta_3 - \eta_1}{\eta_3 + \eta_1} \to 0 \quad \text{since } \eta_1 = \eta_3$$

Case 2: $\quad \eta_2 = \sqrt{\eta_1 \eta_3}$, $\quad d = (2n+1)\frac{\lambda_2}{4} \quad n = 0, 1, 2, \ldots$

$$\beta_2 d = \frac{2\pi}{\cancel{\lambda_2}} \cdot (2n+1)\frac{\cancel{\lambda_2}}{\cancel{4}_2} = (2n+1)\frac{\pi}{2}$$

$$\text{or } \tan\beta_2 d = \tan(2n+1)\frac{\pi}{2} \to \pm\infty$$

in this case

$$Z_2(-d) = \lim_{\tan\beta d \to \pm\infty} \eta_2 \frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}$$

$$Z_2(z=-d) = \eta_2 \cdot \frac{\eta_2}{\eta_3} = \frac{\eta_2^2}{\eta_3}$$

for $\eta_2 = \sqrt{\eta_1 \eta_3} \qquad Z_2(z=-d) \to \eta_1$

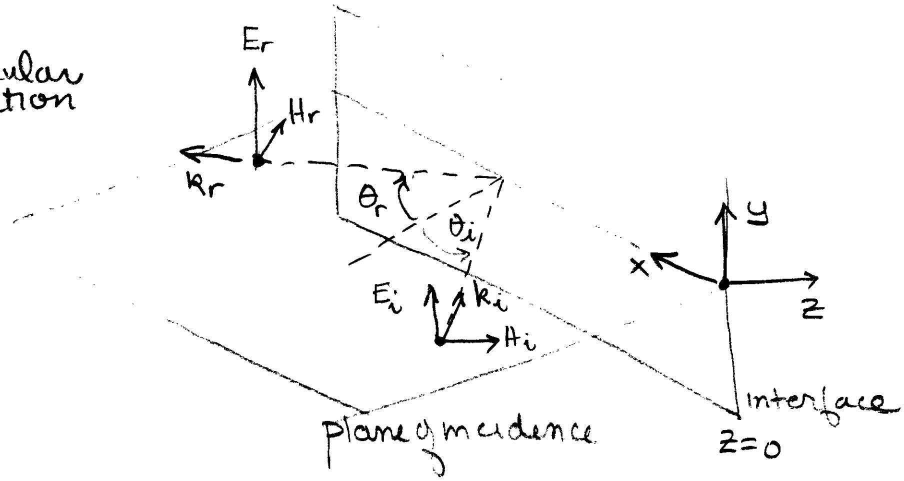
and $\quad \Gamma_{eff} = \dfrac{Z_2(z=-d) - \eta_1}{Z_2(z=-d) + \eta_1} = \dfrac{\eta_1 - \eta_1}{\eta_1 + \eta_1} \to 0$

3.4 Oblique Incidence on Perfect Conductors

parallel – E vector parallel to plane of incidence

perpendicular E vector perpendicular to plane of incidence

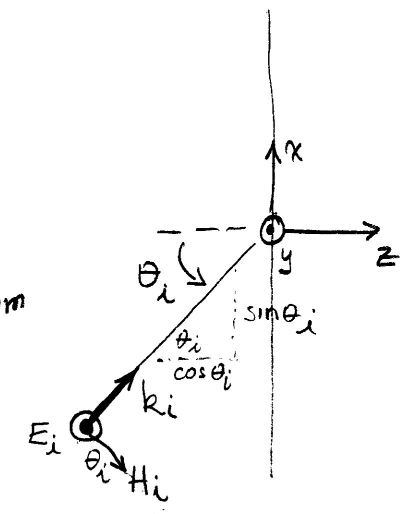
perdendicular polarization



3.4.1 Perpendicular polarization

Incident wave:

$$\underline{E}_i(\underline{r}) = \hat{y} E_{io} e^{-j\beta_1 \hat{\underline{k}}_1 \cdot \underline{r}}$$



recognize that the $\hat{k}$ vector can be written as a sum

$$\hat{k}_i = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i$$

$$\underline{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

so $\underline{E}_i(x,z) = \hat{y} E_{io} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$

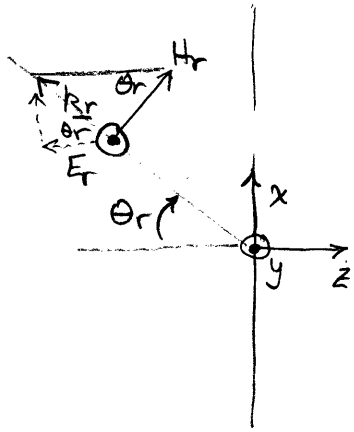
$$H_i(x,z) = \frac{1}{\eta_1}(\hat{\underline{k}}_i \times \underline{E}_i) = \frac{1}{\eta_1}\left[(\hat{x}\sin\theta_i + \hat{z}\cos\theta_i) \times \hat{y} E_{io} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}\right]$$

$$= \frac{E_{io}}{\eta_1}\left[-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i\right] e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

which makes sense in terms of drawing above.

Reflected wave:

$$\underline{E}_r(\underline{r}) = \hat{y}\, E_{ro}\, e^{-j\beta_1 \hat{k}_r \cdot \underline{r}}$$



$$\hat{k}_r = \hat{x}\sin\theta_r - \hat{z}\cos\theta_r$$

$$\underline{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Then

$$\underline{E}_r(x,z) = \hat{y}\, E_{ro}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\underline{H}_r(x,z) = \frac{1}{\eta_1}(\hat{k}_r \times \underline{E}_r) = \frac{1}{\eta_1}\left[(\hat{x}\sin\theta_r - \hat{z}\cos\theta_r) \times \hat{y}\, E_{ro}\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$= \frac{E_{ro}}{\eta_1}\left[\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Boundary condition: tan E = 0 since all E field is tangential and no E field inside conductor

$$\therefore\ E_1(x,0) = E_i(x,0) + E_r(x,0) = 0$$

$$0 = \hat{y}E_{io}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} + \hat{y}\, E_{ro}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

since this must be true independent of x this implies that.

① the exponents are equal $\quad -j\beta_1 x\sin\theta_i = -j\beta_1 x\sin\theta_r \quad$ or $\quad \theta_i = \theta_r$

② if exponents are equal $\quad E_{io} + E_{ro} = 0$

or $E_{ro} = -E_{io}$

This is called Snell's Law of Reflection

This gives rise to the following expressions for fields in medium 1 as.

$$\underline{E}_1(x,z) = \underline{E}_i(x,z) + \underline{E}_r(x,z)$$
$$= \hat{y}\, E_{io}\left(e^{-j\beta_1 z\cos\theta_i} - e^{+j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}$$
$$= -\hat{y}\, j2\, E_{io}\,\sin(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}$$

$$\underline{H}_1(x,z) = \underline{H}_i(x,z) + \underline{H}_r(x,z)$$
$$= -\frac{2E_{io}}{\eta_1}\left[\hat{x}\cos\theta_i\cos(\beta_1 z\cos\theta_i) + \hat{z}j\sin\theta_i\,\sin(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

$$S_{AV} = \frac{1}{2}\mathrm{Re}\left\{\underline{E}_1(x,z)\times\underline{H}_1^*(x,z)\right\}$$
$$= \frac{1}{2}\mathrm{Re}\left\{-\hat{z}j\,\frac{2E_{io}^2}{\eta_1}\cos\theta_i\sin(2\beta_1 z\cos\theta_i) + \hat{x}\,\frac{4E_{io}^2}{\eta_1}\sin\theta_i\sin^2(\beta_1 z\cos\theta_i)\right\}$$
$$= \hat{x}\,\frac{2E_{io}^2}{\eta_1}\sin\theta_i\sin^2(\beta_1 z\cos\theta_i)$$

x-component — real and positive signifying power flow in x-direction along boundary. E_{1y} and H_{1z} are in-phase propagating at $v_{1x} = \frac{\omega}{\beta_{1x}} = \frac{\omega}{\beta_1\sin\theta} = \frac{v_{p1}}{\sin\theta}$. Also $\lambda_{1x} = \frac{2\pi}{\beta_{1x}} = \frac{\lambda_1}{\sin\theta}$

z-component — imaginary, no average power absorbed by conductor
Both E_{1y} and H_{1x} show standing wave patterns because of $\sin(\beta_1 z\cos\theta_i)$ and $\cos(\beta_1 z\cos\theta_i)$ terms.

Some observations

① You expect power flow in $+\hat{x}$ direction because of angle of incident wave

② Wave in $+x$ direction is *NOT* uniform (since amplitude varies across the plane. Note that planes of constant phase are perpendicular to $\underline{k}_i$

③ Total electric field in medium #1 is zero along planes parallel to xy plane (i.e. interface).
This can be seen from expression for $\underline{E}_1(x,z)$

$$\sin(\beta_1 z\cos\theta_i) = 0 \quad \text{for} \quad \beta_1 z \cos\theta_i = m\pi$$

$$\text{or} \quad z = \frac{m\pi}{\beta_1 \cos\theta_i} = \frac{m\pi}{\left(\frac{2\pi}{\lambda_1}\right)\cos\theta_i} = \frac{m\lambda_1}{2\cos\theta_i}$$

[Typical exam or qualifier problem]
You can place a thin conducting sheet at any of these locations without changing fields to the left of the sheet. See p. 160-161 Inan & Inan EM Waves.

④ There is a surface current on conductor.

$$\underline{J}_s(x) = \hat{n}\times\underline{H}_1(x,0) = -\hat{z}\times\left(-\hat{x}\frac{2E_{io}}{\eta_1}\cos\theta_i\, e^{-j\beta_1 x\sin\theta_i}\right)$$

since $\cos(\beta_1 z\cos\theta_i)\to 1$.

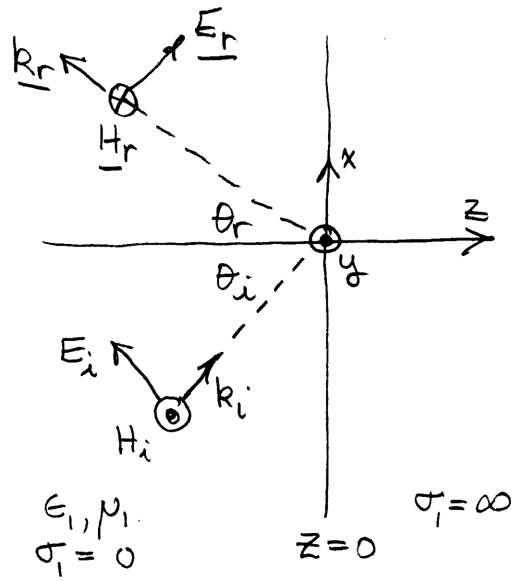
$$= \hat{y}\,\frac{2E_{io}}{\eta_1}\cos\theta_i\, e^{-j\beta_1 x\sin\theta_i}$$

$$\underline{\mathcal{J}}_s(x,t) = \hat{y}\,\frac{2E_{io}}{\eta_1}\cos\theta_i\cos(\omega t-\beta_1 x\sin\theta_i)$$

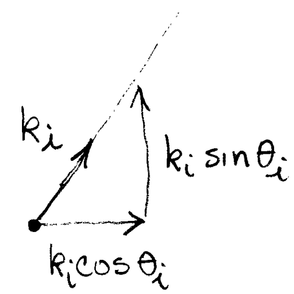
This induced current re-radiates a wave.

Parallel Polarization

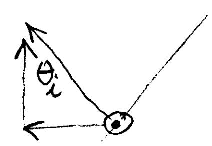
E lies in plane of incidence
(i.e. E is parallel to plane of incidence)



$$k_i = \hat{x}\sin\theta_i + \hat{z}\cos\theta_i$$



$$E_i(\underline{r}) = E_{io}\left[\hat{x}\cos\theta_i - \hat{z}\sin\theta_i\right] e^{-j\beta_1 \hat{k}_1 \cdot \underline{r}}$$



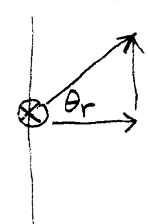
$$E_i(x,z) = E_{io}\left[\hat{x}\cos\theta_i - \hat{z}\sin\theta_i\right] e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Once we have E_i we can use impedance to find H

$$\underline{H_i}(x,z) = \hat{y}\,\frac{E_{io}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

The reflected wave can be described in the same way.

$$\underline{E_r}(x,z) = E_{ro}\underbrace{\left[\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right]}_{\text{total } \underline{E_r}} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$



$$\underline{H_r}(x,z) = -\hat{y}\,\frac{E_{ro}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_t - z\cos\theta_t)}$$

What are the B.C.'s? Same as for perpendicular
E field vanishes in the conductor
which requires $E_{1x}(x,z) = 0$ at $z=0$
for example.

$$E_{1x}(x,0) = E_{ix}(x,0) + E_{rx}(x,0)$$

$$= E_{io}\cos\theta_i e^{-j\beta x\sin\theta_i} + E_{ro}\cos\theta_r e^{-j\beta_i x\sin\theta_r} = 0$$

Let's equate the exponents to get

$$\sin\theta_i = \sin\theta_r$$

$$\text{or } \theta_i = \theta_r$$

with the exponents equal $E_{io} + E_{ro} = 0$

$$\text{or } E_{ro} = -E_{io}$$

with these results we can write the total field in region 1 as

$$\underline{E}_1(x,z) = \underline{E}_i(x,z) + \underline{E}_r(x,z)$$

$$= \left(\hat{x} E_{io}\cos\theta_i e^{-j\beta_1 z\cos\theta_i} + \hat{x} E_{ro}\cos\theta_r e^{-j\beta_1 z\sin\theta_r}\right) e^{-j\beta x\sin\theta_i}$$

pull out the right factors

$$+\left(-\hat{z} E_{io}\sin\theta_i e^{-j\beta_1 z\cos\theta_i} + \hat{z} E_{ro}\sin\theta_r e^{+j\beta_1 z\cos\theta_r}\right) e^{-j\beta_1 x\sin\theta_r}$$

$$\underline{E}_1(x,z) = \hat{x} E_{io}\cos\theta_i \left(e^{-j\beta_1 z\cos\theta_i} - e^{+j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}$$

$$-\hat{z} E_{io}\sin\theta_i \left(e^{-j\beta_1 z\cos\theta_i} + e^{+j\beta_1 z\cos\theta_i}\right) e^{-j\beta_1 x\sin\theta_i}$$

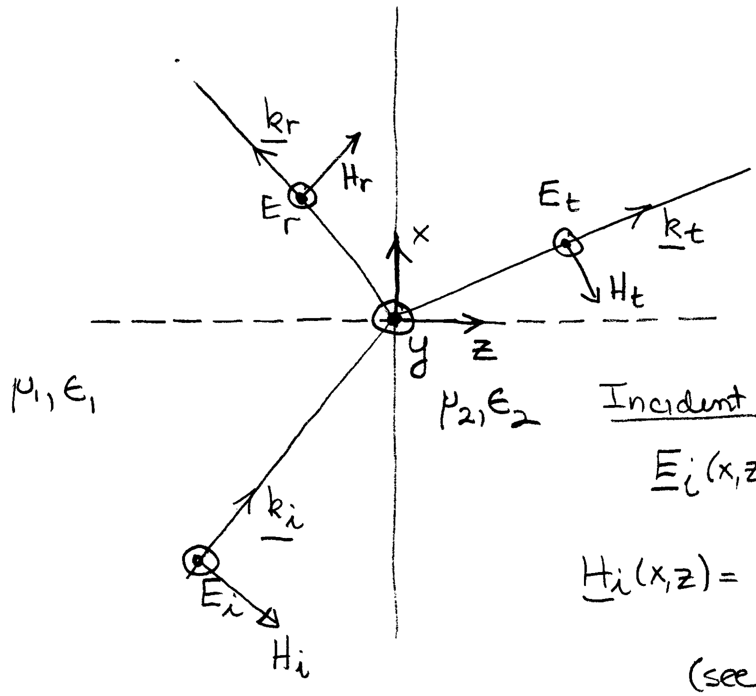
$$= \left[\hat{x}(-2)E_{io}\, j\cos\theta_i \sin(\beta_1 z\cos\theta_i) + \hat{z}\sin\theta_i\cos(\beta_1 z\cos\theta_i)\right] e^{-j\beta_1 x\sin\theta_i}$$

You can then write the magnetic (total) field in medium 1

$$\underline{H}_1(x,z) = \underline{H}_i(x,z) + \underline{H}_r(x,z)$$

$$= \hat{y}\,\frac{2E_{io}}{\eta_1}\cos(\beta_1 z\cos\theta_i)\, e^{-j\beta x\sin\theta_i}$$

3.5 Oblique incidence at a dielectric boundary



This is perpendicular polarization with $E_i \perp$ to the plane of incidence.

We have already written the expressions for the incident and reflected waves before.

Incident

$$\underline{E_i}(x,z) = \hat{y} E_{io} e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)}$$

$$\underline{H_i}(x,z) = \frac{E_{io}}{\eta_1}\left[-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i\right] e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

(see p.21)

Reflected: (p.22)

$$\underline{E_r}(x,z) = \hat{y} E_{ro} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$H_r(x,z) = \frac{E_{ro}}{\eta_1}\left[\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right] e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

Transmitted (this is new but similar to E_i, H_i)

$$E_t(x,z) = \hat{y} E_{to} e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

$$H_t(x,z) = \frac{E_{to}}{\eta_2}\left[-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t\right] e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

Apply B.C.'s! Want E_{tan} continuous across interface everywhere at $z=0$

$$E_i(x,0) + E_r(x,0) = E_t(x,0)$$

$$E_{io} e^{-j\beta_1 x\sin\theta_i} + E_{ro} e^{-j\beta_1 x\sin\theta_r} = E_{to} e^{-j\beta_2 x\sin\theta_t}$$

The only way this will be true for all x is for all three exponents to be equal, i.e.

$$\underbrace{\beta_1 x \sin\theta_i = \beta_1 x \sin\theta_r}_{\text{requires } \theta_i = \theta_r} = \beta_2 x \sin\theta_t$$

$$\underbrace{\beta_1 x \sin\theta_r = \beta_2 x \sin\theta_t}_{\text{requires}}$$

$$\beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

or

$$\frac{\sin\theta_r}{\sin\theta_t} = \frac{\beta_2}{\beta_1} = \frac{\omega\sqrt{\mu_2\epsilon_2}}{\omega\sqrt{\mu_1\epsilon_1}}$$

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}}$$

Snell's Law of Refraction

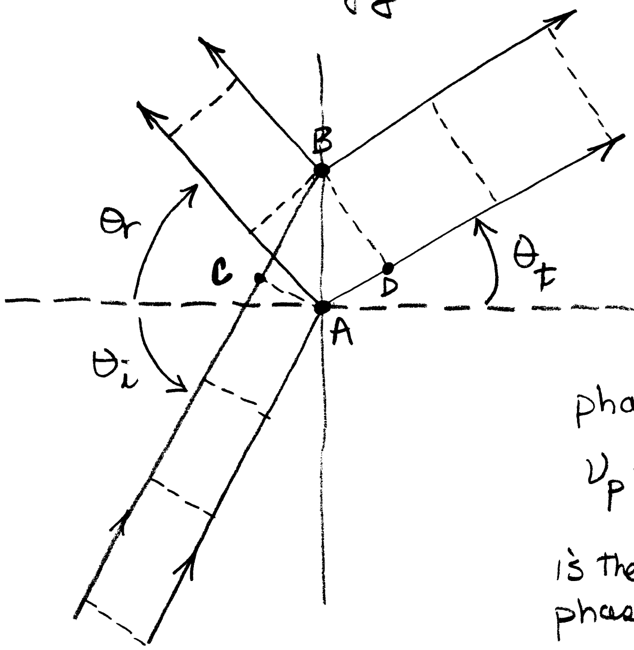
We can now do reflection & transmission coefficients, from magnitudes.

$$E_{io} + E_{ro} = E_{to}$$

$$1 + \frac{E_{ro}}{E_{io}} = \frac{E_{to}}{E_{io}}$$

$$1 + \Gamma_\perp = T_\perp$$

There is a much simpler way to derive reflection coefficient & transmission coefficient using wave concepts and conservation of energy.



phase velocity

$$v_p = \frac{1}{\sqrt{\mu \epsilon}}$$

is the velocity at which the phase fronts propagate.

The clever thing is to note that

$$\frac{CB}{AD} = \frac{v_{p1}}{v_{p2}}$$

Physially the wave front at point B must be in phase with the wave front at point D for the transmitted wave to be planar. This can be converted to Snell's Law by noting

$$CB = AB \sin\theta_i$$
$$AD = AB \sin\theta_t$$

so $$\frac{\sin\theta_i}{\sin\theta_t} = \frac{v_{p1}}{v_{p2}} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}}$$

NOTE: For each wave (incident and transmitted) $\frac{s}{r} = t$

Since points B and D are in phase $\frac{CB}{v_{p1}} = t = \frac{AD}{v_{p2}}$

which gives $\frac{CB}{AD} = \frac{v_{p1}}{v_{p2}}$

The energy of each wave can also be simply written as

Incident wave: power transported in z-direction $\quad |S_{AV}|_i \cos\theta_i = \frac{1}{2\eta_1} E_{io}^2 \cos\theta_i$

Reflected wave $\quad |S_{AV}|_r \cos\theta_r = \frac{1}{2\eta_1} E_{ro}^2 \cos\theta_r$

Transmitted wave: $\quad |S_{AV}|_t \cos\theta_t = \frac{1}{2\eta_2} E_{to}^2 \cos\theta_t$

For conservation of power

$$|S_{AV}|_i \cos\theta_i = |S_{AV}|_r \cos\theta_r + |S_{AV}|_t \cos\theta_t$$

$$\frac{1}{2\eta_1} E_{io}^2 \cos\theta_i = \frac{1}{2\eta_1} E_{ro}^2 \cos\theta_r + \frac{1}{2\eta_2} E_{to}^2 \cos\theta_t$$

$$\frac{\cos\theta_i}{2\eta_1} = \frac{\cos\theta_r}{2\eta_1}\left(\frac{E_{ro}}{E_{io}}\right)^2 + \frac{\cos\theta_t}{2\eta_2}\left(\frac{E_{to}}{E_{io}}\right)^2$$

$$\frac{\cos\theta_i}{2\eta_i} = \frac{\cos\theta_i}{2\eta_1}\Gamma_\perp^2 + \frac{\cos\theta_t}{2\eta_2}T_\perp^2$$

$$\frac{\cos\theta_i}{2\eta_1}\Gamma_\perp^2 = \frac{\cos\theta_i}{2\eta_i} - \frac{\cos\theta_t}{2\eta_2}T_\perp^2$$

$$\Gamma_\perp^2 = 1 - \frac{\eta_1\cos\theta_t}{\eta_2\cos\theta_i}T_\perp^2$$

$$\frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} T^2 = 1 - \Gamma^2$$

but $T = 1 + \Gamma$ so $T^2 = (1+\Gamma)^2$

$$\frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} (1+\Gamma)\cancel{(1+\Gamma)} = \cancel{(1+\Gamma)}(1-\Gamma)$$

$$\frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} = \frac{1-\Gamma}{1+\Gamma}$$

$$\eta_1 \cos\theta_t + \Gamma \eta_1 \cos\theta_t = \eta_2 \cos\theta_i - \Gamma \eta_2 \cos\theta_i$$

$$\Gamma\left(\eta_1 \cos\theta_t + \eta_2 \cos\theta_i\right) = \eta_2 \cos\theta_i - \eta_1 \cos\theta_t$$

$$\therefore \quad \Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_i}$$

Using this result and $\Gamma^2 = 1 - \dfrac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} T^2$.

$$\frac{\eta_1 \cos\theta_t}{\eta_2 \cos\theta_i} T^2 = 1 - \Gamma^2$$

$$T^2 = \frac{\eta_2 \cos\theta_i}{\eta_1 \cos\theta_t}\left(1 - \Gamma^2\right)$$

$$T^2 = \frac{\eta_2 \cos\theta_i}{\eta_1 \cos\theta_t}\left(1 - \left(\frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_i}\right)^2\right)$$

$$= \frac{\eta_2 \cos\theta_i}{\eta_1 \cos\theta_t}\left[\frac{(\eta_1 \cos\theta_t + \eta_2 \cos\theta_i)^2 - (\eta_2 \cos\theta_i - \eta_1 \cos\theta_t)^2}{(\eta_1 \cos\theta_t + \eta_2 \cos\theta_i)^2}\right]$$

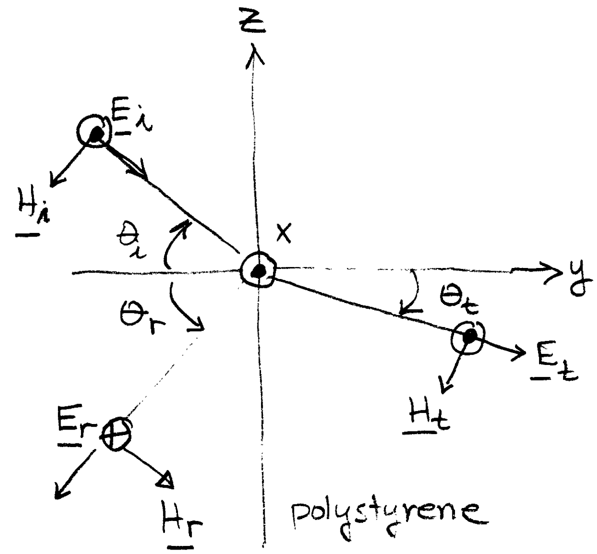
$$= \frac{\eta_2 \cos\theta_i}{\cancel{\eta_1} \cancel{\cos\theta_t}} \; \frac{4 \cancel{\eta_1} \eta_2 \cos\theta_i \cancel{\cos\theta_t}}{(\eta_1 \cos\theta_t + \eta_2 \cos\theta_i)^2}$$

$$T^2 = \frac{4 \eta_2^2 \cos^2\theta_i}{(\eta_1 \cos\theta_t + \eta_1 \cos\theta_i)^2}$$

$$T = \frac{2 \eta_2 \cos\theta_i}{\eta_1 \cos\theta_t + \eta_2 \cos\theta_i}$$

Example 3-10

$$E_i(y,z) = \hat{x} E_o e^{-j\beta(y\cos\theta_i - z\sin\theta_i)}$$



(a) Input power is $1.4 \frac{W}{m^2}$ @ 3 GHz

$\theta_i = 58°$

Calculate amplitude of $\underline{E}$ field as

$$|S_{AV}|^2 = \frac{1}{2}\frac{E_o^2}{\eta_1} = 1.4 \frac{W}{m^2}$$

$$E_o = \sqrt{(1.4)(2)(\eta_1)} = \sqrt{(2)(1.4)(377)}$$

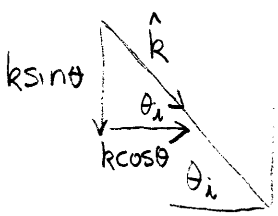
$$E_o \cong 32.5 \frac{V}{m}$$

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c} = \frac{2\pi \times 3 \times 10^9}{3 \times 10^8} = 20\pi$$

(b) The $\underline{H}$ field can be found in several ways

$\underline{\nabla} \times \underline{E} = -j\omega\underline{B}$ always works but is complex

$\underline{H}_i(y,z) = \frac{1}{\eta_1}\hat{k}_i \times \underline{E}_i(y,z)$ works well if you can determine $\hat{k}$ readily



$$\hat{k}_i = \hat{y}\cos\theta_i - \hat{z}\sin\theta_i$$

$$\therefore \underline{H}_i = \frac{1}{\eta_1}(\hat{y}\cos\theta_i - \hat{z}\sin\theta_i) \times \hat{x} E_o e^{-j\beta_1(y\cos\theta_i - z\sin\theta_i)}$$

$$= \frac{E_o}{\eta_1}\left(-\hat{z}\cos\theta_i - \hat{y}\sin\theta_i\right) e^{-j\beta_1(y\cos\theta_i - z\sin\theta_i)}$$

You can put in numerical values to get

$$\underline{H}_i(y,z) = -86.2\left[\hat{y}\,0.848 + \hat{z}\,0.530\right] e^{-j20\pi(0.53y - 0.848z)} \quad \frac{mA}{m}$$

(c) The transmission angle comes from Snell's Law as

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

$$\sin\theta_t = \frac{\sin\theta_i}{\sqrt{\frac{\epsilon_2}{\epsilon_1}}} = \frac{\sin 58^\circ}{\sqrt{\frac{2.56}{1}}} = 0.53$$

you only need relative dielectric constants.

$$\theta_t = 32.0^\circ$$

We have only derived the Fresnel equations for perpendicular polarization and they apply here.

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

since $\mu_1 = \mu_2$ this reduces to $\eta_1 \to 1/\sqrt{\epsilon_{1r}}$, $\eta_2 = 1/\sqrt{\epsilon_{2r}}$

$$\Gamma_\perp = \frac{\sqrt{1}\cos 58^\circ - \sqrt{2.56}\cos 32^\circ}{\sqrt{1}\cos 58^\circ + \sqrt{2.56}\cos 32^\circ} = \frac{.5299 - 1.3568}{.5299 + 1.3568}$$

$$\Gamma_\perp = \frac{-0.8269}{1.8867} = -0.438$$

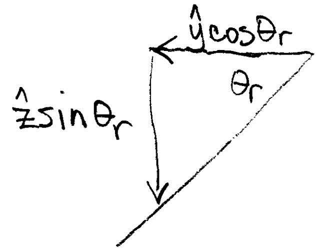
Reflected E field is

$$\underline{E}_r(y,z) = \hat{x}\,\Gamma_\perp E_0 e^{-j\beta_1(y\cos\theta_i - z\sin\theta_i)}$$

$$= \hat{x}\,14.24\, e^{+j20\pi(.53y + .848z)}$$

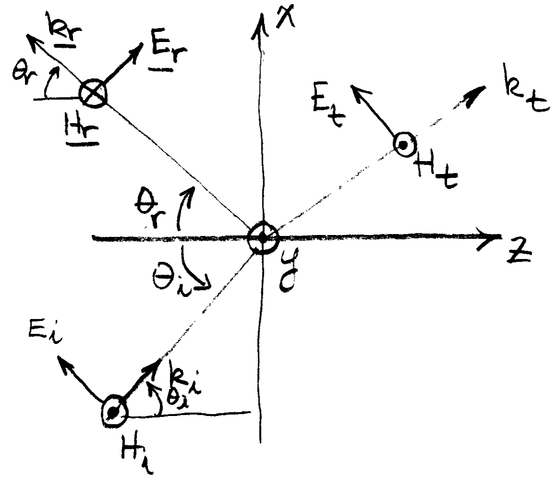
direction of y changed since direction of propagation

$$\underline{H}_r(y,z) = \frac{1}{\eta_1}\hat{k}_r \times \underline{E}_r = \frac{\Gamma_\perp E_0}{\eta_1}\left[\hat{y}\sin\theta_r - \hat{z}\cos\theta_r\right] \times \hat{x}\,14.24\, e^{+j20\pi(.53y+.85z)}$$



$$H_r = -37.8\left[\hat{y}\,.848 - \hat{z}\,.53\right] e^{j20\pi(.53y+.85z)} \ \frac{mA}{m}.$$

3.5.2. Parallel polarization



$$\underline{E}_i(x,z) = E_{io}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{H}_i(x,z) = \hat{y}\,\frac{E_{io}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\underline{E}_r(x,z) = E_{ro}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$H_r(x,z) = -\hat{y}\,\frac{E_{ro}}{\eta_1}\, e^{-j\beta_1(x\sin\theta_t - z\cos\theta_t)}$$

$$E_t(x,z) = E_{to}(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)\, e^{-j\beta_2(x\sin\theta_1 + z\cos\theta_t)}$$

$$H_t(x,z) = \hat{y}\,\frac{E_{to}}{\eta_2}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \qquad \eta_2 = \sqrt{\frac{\mu_2}{\epsilon_2}}$$

use continuity of tangential E (i.e. x)

$$E_{io}\cos\theta_i + E_{ro}\cos\theta_r = E_{to}\cos\theta_t$$

Already showed $\theta_i = \theta_r$ so

$$\frac{(E_{io} + E_{ro})\cos\theta_i}{E_{io}} = \frac{E_{to}\cos\theta_t}{E_{io}}$$

$$\left(1 + \frac{E_{ro}}{E_{to}}\right)\cos\theta_i = \frac{E_{to}}{E_{io}}\cos\theta_t$$

$$\frac{E_{to}}{E_{io}} = \left(1 + \frac{E_{ro}}{E_{to}}\right)\frac{\cos\theta_i}{\cos\theta_t}$$

Use this with conservation of power (polarization independent)

$$\left(\frac{E_{ro}}{E_{io}}\right)^2 = 1 - \frac{\eta_1 E_{to}^2\cos\theta_t}{\eta_2 E_{io}^2\cos\theta_i}$$

to get

$$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{-\eta_1\cos\theta_i + \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

by eliminating E_{to}.

By eliminating E_{ro} you can get

$$\mathcal{T}_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_1}$$

$$1 + \Gamma_{\parallel} = \mathcal{T}_{\parallel}\left(\frac{\cos\theta_t}{\cos\theta_i}\right)$$

One of the most interesting things about parallel polarization is that there is a value of θ_i for which $\Gamma_{\parallel} = 0$.

$$\Gamma_{\parallel} = -\eta_1 \cos\theta_i + \eta_2 \cos\theta_t = 0$$

$$\therefore \cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t$$

$$\cos^2\theta_i = \frac{\eta_2^2}{\eta_1^2}\cos^2\theta_t$$

$$1 - \sin^2\theta_i = \frac{\eta_2^2}{\eta_1^2}\left(1 - \sin^2\theta_t\right)$$

Combine this with Snell's Law

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}}$$

$$\sin^2\theta_t = \frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}\sin^2\theta_i$$

$$1 - \sin^2\theta_i = \frac{\eta_2^2}{\eta_1^2}\left(1 - \frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}\sin^2\theta_i\right)$$

$$1 - \sin^2\theta_i = \frac{\eta_2^2}{\eta_1^2} - \frac{\eta_2^2}{\eta_1^2}\frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}\sin^2\theta_i$$

$$1 - \frac{\eta_2^2}{\eta_1^2} = \sin^2\theta_i - \frac{\eta_2^2}{\eta_1^2}\frac{\epsilon_1 \mu_1}{\epsilon_2 \mu_2}\sin^2\theta_i$$

$$\sin^2\theta_i = \frac{1 - \frac{\mu_2 \epsilon_1}{\mu_1 \epsilon_2}}{1 - \left(\frac{\epsilon_1}{\epsilon_2}\right)^2}$$

For $\mu_1 = \mu_2 = \mu_0$ this reduces to

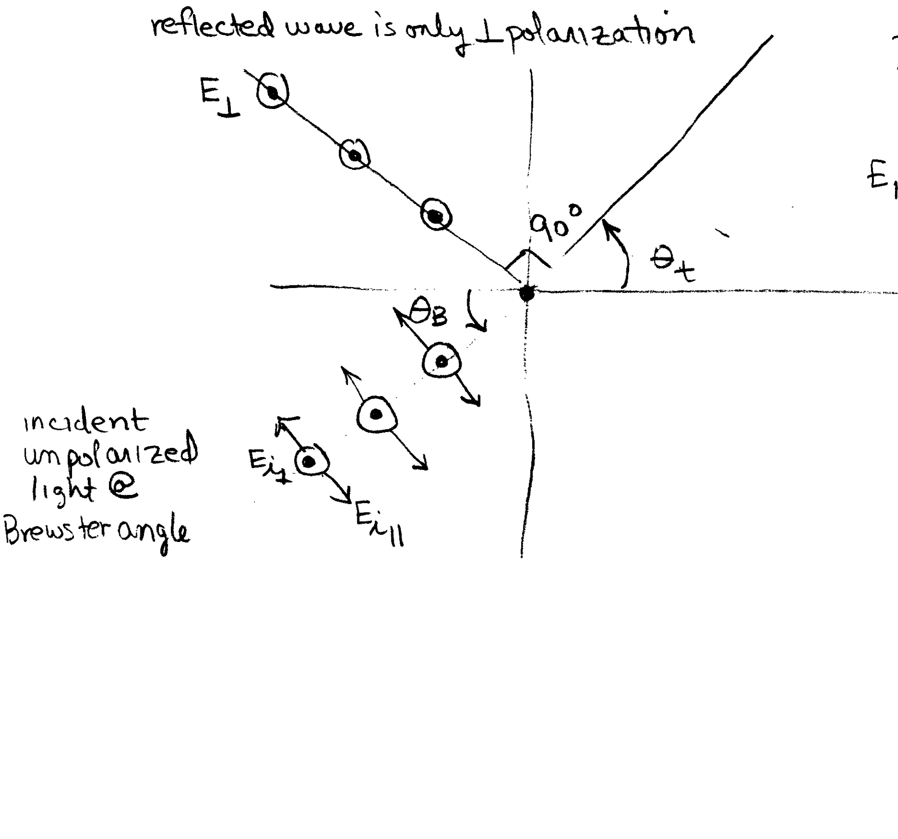
$$\sin^2\theta_i = \frac{1 - \frac{\epsilon_1}{\epsilon_2}}{1 - \left(\frac{\epsilon_1}{\epsilon_2}\right)^2} = \frac{1}{1 + \frac{\epsilon_1}{\epsilon_2}} = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2}$$

We call this angle the Brewster angle

$$\cos^2\theta_i = 1 - \sin^2\theta_i = 1 - \frac{\epsilon_2}{\epsilon_1+\epsilon_2} = \frac{\epsilon_1}{\epsilon_1+\epsilon_2}$$

$$\tan^2\theta_B = \frac{\sin^2\theta_B}{\cos^2\theta_B} = \frac{\epsilon_2}{\epsilon_1}$$

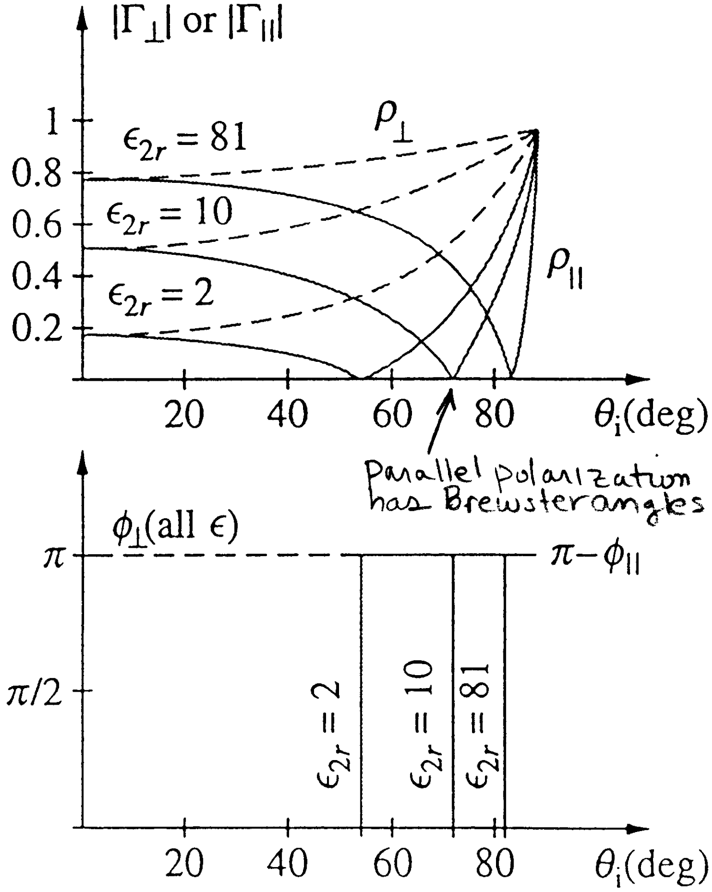
$$\tan\theta_B = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$$

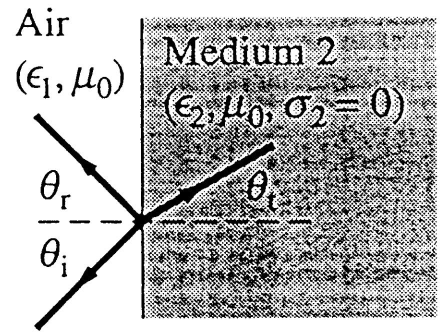


transmitted wave is ∥ polarization only.

$E_\parallel$ get transmitted with no loss

This is a plot of $|\Gamma|$ as a function of θ_i





$\epsilon_r = 2$ paraffin

$\epsilon_r = 10$ flint glass

$\epsilon_r = 81$ distilled water

the sign of the reflection coefficient for $\parallel$ polarization changes sign. For $\theta < \theta_B$ $\Gamma_\parallel < 0$ but for $\theta > \theta_B$ $\Gamma_\parallel > 0$

$\Gamma_\perp$ is always negative.

For $\perp$ polarization and ϵ_r large most of the energy is reflected. This starts to look like a conductor and we can expect $\Gamma_\perp \to -1$ at $0°$

For $\parallel$ polarization for $\theta_i < \theta_B$ transmitted wave is small so reflected e-field must be negative to satisify boundary condition. For $\theta_i > \theta_B$ $E_t > E_i$ so E_r must become positive

For waves which are NOT parallel or perpendicular decompose into $\parallel$ and $\perp$ components.

In general you will get elliptical polarization.

Gets complicated for circular polarization.

For $\theta_i < \theta_B$ reflected will be elliptical rotation changes since $\Gamma_{\parallel}$ & $\Gamma_{\perp} < 0$

For $\theta_i = \theta_B$ $\Gamma_{\parallel} = 0$ so reflection is linearly polarized

For $\theta_i > \theta_B$ $\Gamma_{\perp} < 0$ but $\Gamma_{\parallel} > 0$ rotation will be the same as incident.

Transmitted will be same handedness as incident except as θ_B will it will become linear.

3.6 Total Internal Reflection

Most useful for dielectic waveguides or optical fibers

previously mostly considered cases where $\epsilon_2 > \epsilon_1$

i.e, going into a more dense medium

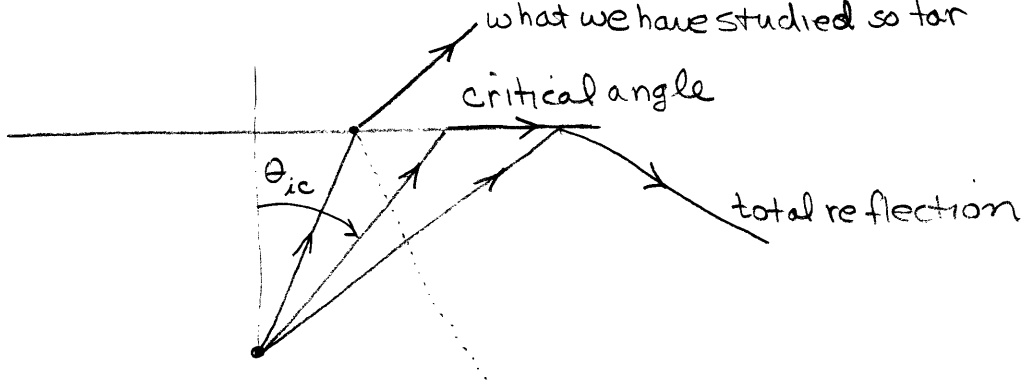
for cases where $\epsilon_1 > \epsilon_2$ we get interesting results.

Snell's Law $\frac{\sin\theta_i}{\sin\theta_t} = \frac{\sqrt{\epsilon_2\mu_2}}{\sqrt{\epsilon_1\mu_1}} \rightarrow \sqrt{\frac{\epsilon_2}{\epsilon_1}}$ for non-magnetic material

or $\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i$

since $\epsilon_1 > \epsilon_2$ $\theta_t > \theta_i$

but $\sin\theta_t$ can never be greater than 1.



For Critical angle θ_{ic} $\theta_t = 90°$ or $\sin\theta_t = 1$

then $\sin\theta_{ic}\sqrt{\frac{\epsilon_1}{\epsilon_2}} = 1$ or $\sin\theta_{ic} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}$

For $\theta_i > \theta_c$ θ_t becomes imaginary

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\underbrace{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i}_{>1}} = \pm j\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}$$

Previous expressions for $\Gamma_\perp$ & $\Gamma_\parallel$ remain valid.

For $\sin\theta_t > 0$ $\Gamma_\perp$ & $\Gamma_{\parallel}$ can be evaluated to give

$$\Gamma_\perp = \frac{\cos\theta_i + j\sqrt{\sin^2\theta_i - \epsilon_{21}}}{\cos\theta_i - j\sqrt{\sin^2\theta_i - \epsilon_{21}}} = 1e^{j\phi_\perp}$$

where $\tan\frac{\phi_\perp}{2} = \frac{\sqrt{\sin^2\theta_i - \epsilon_{21}}}{\cos\theta_i}$

and

$$\Gamma_{\parallel} = -\frac{\epsilon_{21}\cos\theta_i + j\sqrt{\sin^2\theta_i - \epsilon_{21}}}{\epsilon_{21}\cos\theta_i - j\sqrt{\sin^2\theta_i - \epsilon_{21}}}$$

where $\tan\frac{\phi_{\parallel}}{2} = \frac{\sqrt{\sin^2\theta_i - \epsilon_{21}}}{\epsilon_{21}\cos\theta_i}$

and $\epsilon_{21} = \frac{\epsilon_2}{\epsilon_1}$

The interesting thing is that this can be used to convert linear polarization into circular or elliptical

Consider linear polarization at $45°$ so $|E_{\parallel}| = |E_\perp|$

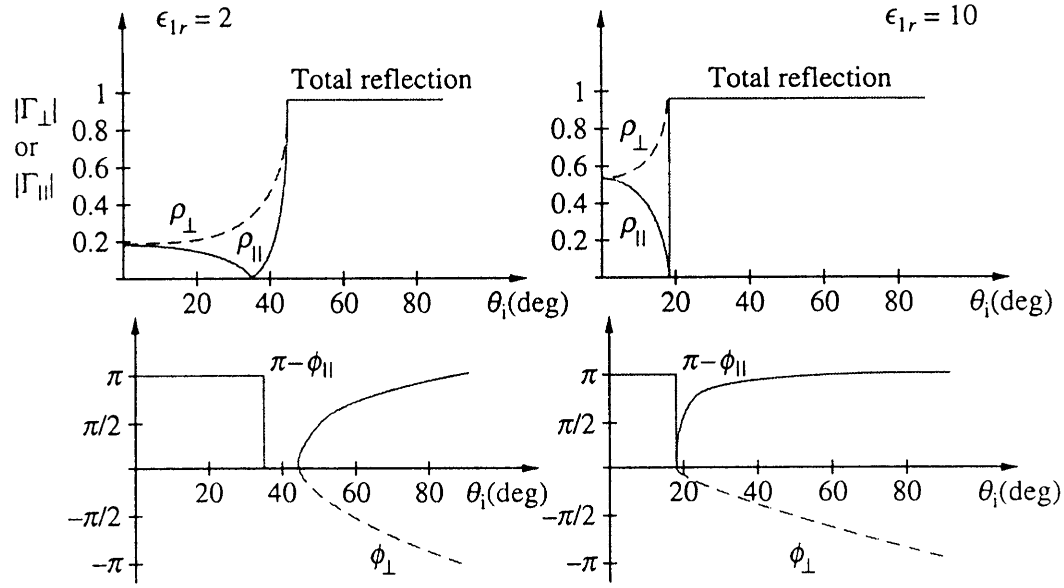
Upon total internal reflection each component undergoes a <u>different</u> phase.

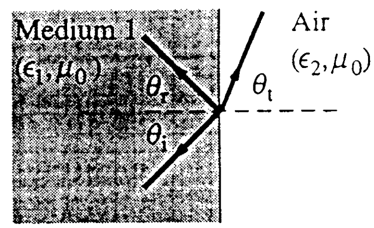
$$\Delta\phi \equiv \phi_\perp - \phi_{\parallel}$$

$$\tan\left(\frac{\Delta\phi}{2}\right) = \frac{\tan\left(\frac{\phi_\perp}{2}\right) - \tan\left(\frac{\phi_{\parallel}}{2}\right)}{1 + \tan\left(\frac{\phi_\perp}{2}\right)\tan\left(\frac{\phi_{\parallel}}{2}\right)} = \frac{\cos\theta_i\sqrt{\sin^2\theta_i - \epsilon_{21}}}{\sin^2\theta_i}$$

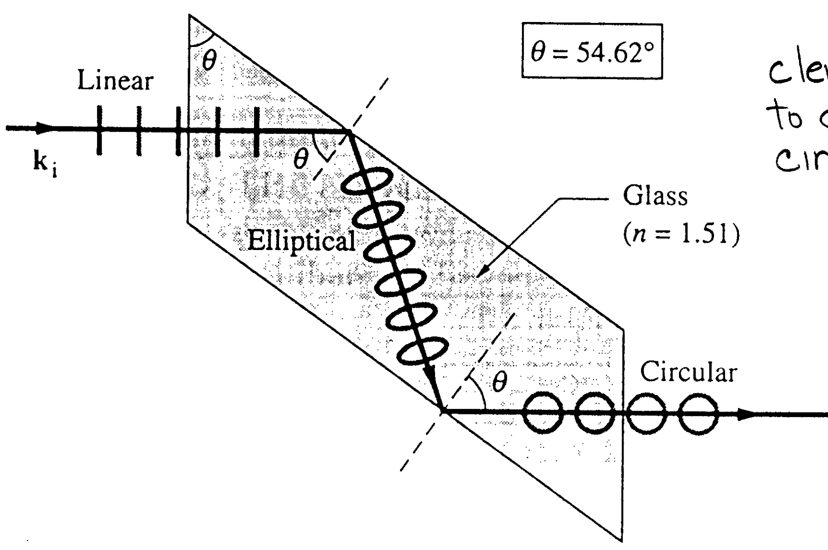
if we can get $\Delta\phi = 90°$ this will have the $\perp$ & $\parallel$ components $90°$ out of phase and will get circular polarization. Other values give elliptical.

paraffin flint glass





In the total reflection region ($\theta_i > \theta_c$) there is a variable phase shift. You can still see the Brewster angle below the critical angle.



clever prism to convert linear to circular polarization

FIGURE 3.35. **Fresnel's rhomb.** Linearly polarized light incident on glass (let n = 1.51) with its electric field at 45° to the plane of incidence acquires a phase difference between its two components ($E_\perp$ and $E_{\|}$) of 45° at each total internal reflection. As a result, the wave exiting the rhomb is circularly polarized.

3.6.2 The Refracted "Wave"

For $\theta_i > \theta_c$ the wave must still satisify the B.C.'s

$$\underline{E}_i + \underline{E}_r = \underline{E}_i + \Gamma \underline{E}_i \neq 0$$

The angle θ_t is complex but field still exists in medium 2.

All previous field expressions & results still valid.

$$\underline{E}_t(x,z) = \hat{y}\, E_{to}\, e^{-j\beta_2 (x \sin\theta_t + z\cos\theta_t)}$$

Using $\cos\theta_t = \pm j\sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1}$ this wave can be written

$$\underline{E}_t(x,z) = \hat{y}\, E_{to}\, e^{-\alpha_t z} e^{-j\beta_t x}$$

pick + sign for attenuation

$$\alpha_t = \pm \beta_2 \sqrt{\frac{\epsilon_1}{\epsilon_2}\sin^2\theta_i - 1} = + \frac{2\pi}{\lambda_1}\sqrt{\sin^2\theta_i - \epsilon_{21}}$$

$$\beta_t = \beta_2 \sqrt{\frac{\epsilon_1}{\epsilon_2}} \sin\theta_i = \underbrace{\omega\sqrt{\mu_0 \epsilon_1}}_{\beta_1} \sin\theta_i$$

For $\theta_i > \theta_c$ there is a wave along the interface

$$\lambda_t = \frac{2\pi}{\beta_t} = \frac{\lambda_1}{\sin\theta_i} > \lambda_1$$

since wave is attenuated in z it is a surface wave

slow wave $\quad v_{pt} = \dfrac{\omega}{\beta_t} < v_{p2} = \dfrac{\omega}{\beta_2}$

phase velocity < velocity of light in medium 2.

total internal reflection

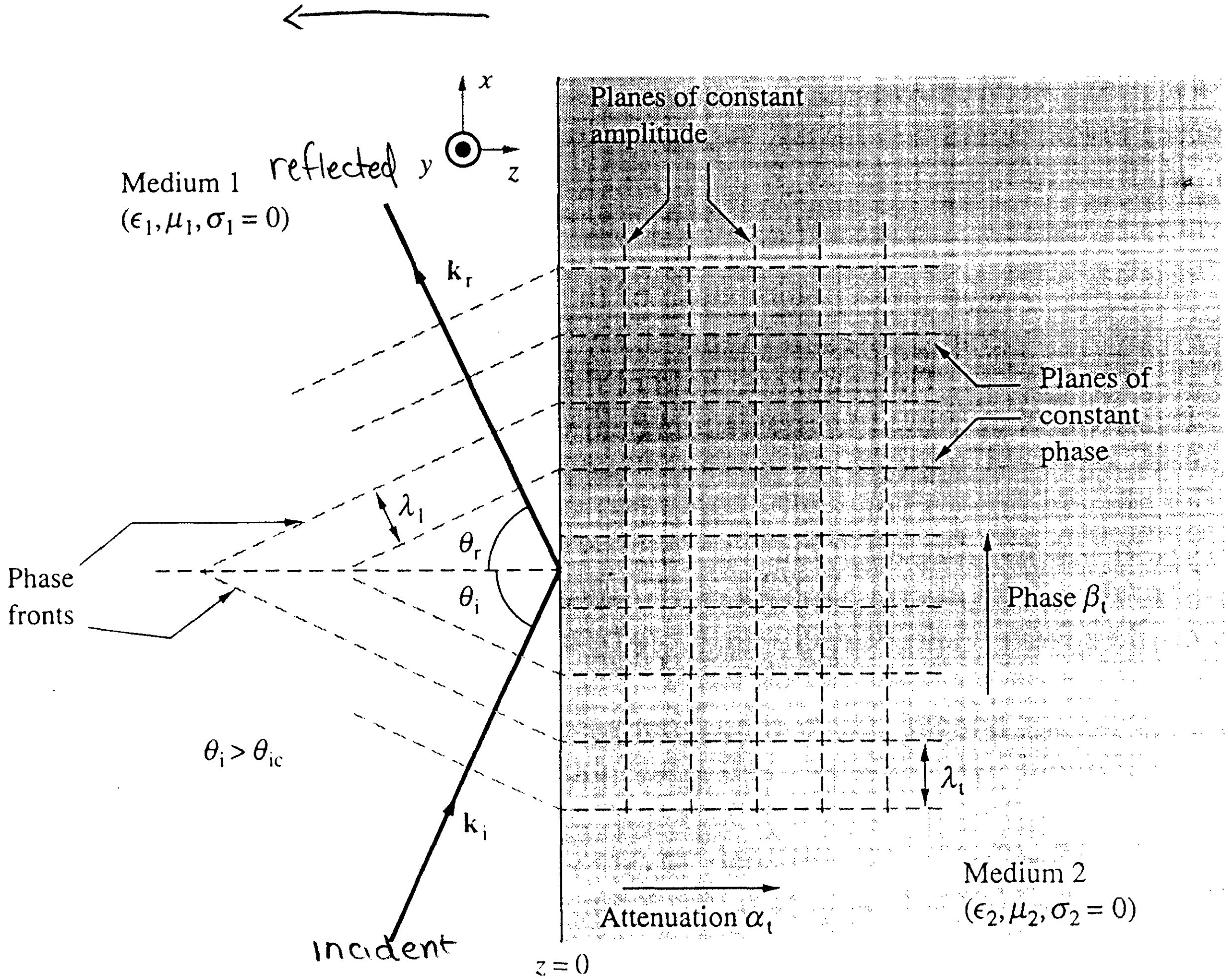


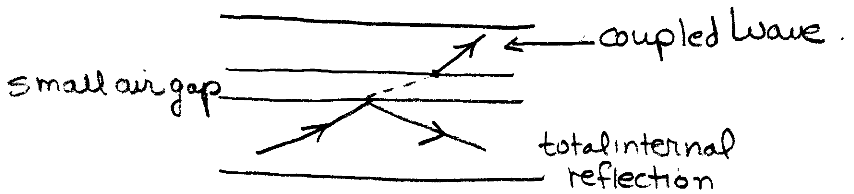
FIGURE 3.36. Total internal reflection: Constant amplitude and phase fronts. Oblique incidence beyond the critical angle $\theta_i > \theta_{ic}$ on a dielectric interface with $\mu_1 = \mu_2 = \mu_0$ and $\epsilon_1 > \epsilon_2$. The planes of constant amplitude are parallel to the interface, whereas the planes of constant phase are normal to it. This picture is valid for both perpendicular and parallel polarization.

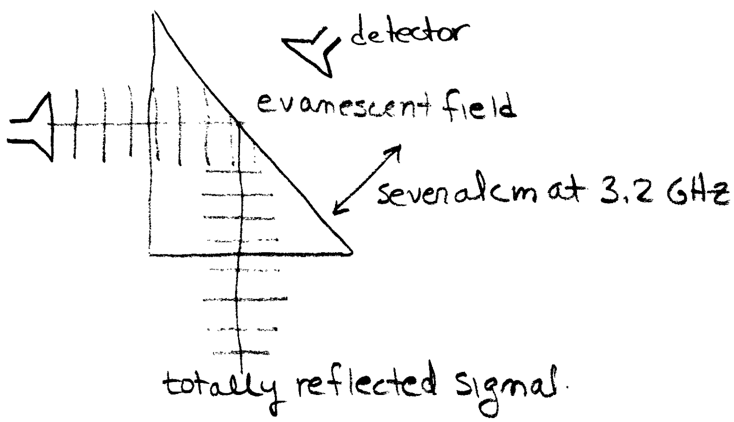
using complex θ_t $\quad \mathcal{T}_\perp = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\cos\theta_i}{\sqrt{1-\epsilon_{21}}}\, e^{j\frac{\phi_\perp}{2}}$

Magnetic field even more complex, and out of phase (90°) with E,
$|S|_z = |E \times H^*|_z$ will be purely imaginary.

Evanescent wave $\equiv \mathrm{Re}\{S\} = 0$, attenuated with distance

That a field exists in medium 2 can be proved by putting a prism against a surface at which total internal reflection is occurring.



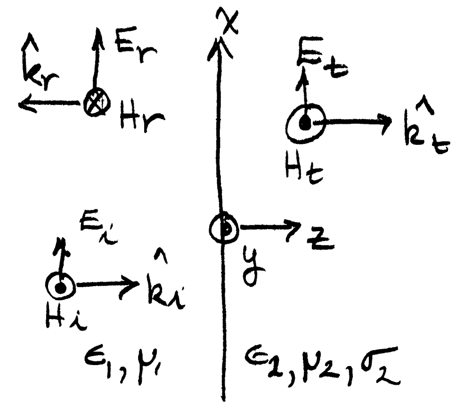


3.7 Normal Incidence on a Lossy Medium

good conductor with conductivity σ
leads to surface impedance $Z_s = \frac{1+j}{\sigma\delta}$

$\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$ skin depth.

Normal expression of fields



$$E_i(z) = \hat{x} E_{io} e^{-j\beta_1 z}$$

$$H_i(z) = \hat{y} \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}$$

$$E_r(z) = \hat{x} E_{ro} e^{+j\beta_1 z}$$

$$H_r(z) = -\hat{y} \frac{E_{ro}}{\eta_1} e^{+j\beta_1 z}$$

$$\left.\begin{aligned} E_t(z) &= \hat{x} E_{to} e^{-\gamma_2 z} \\ H_t(z) &= \hat{y} \frac{E_{to}}{\eta_c} e^{-\gamma_2 z} \end{aligned}\right\} \text{good conductor}$$

For medium 1

$$\beta = \omega\sqrt{\mu_1\epsilon_1}$$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} \cong 377\Omega$$

For medium 2

$$\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\epsilon_2)}$$ See p.41

$$\eta_c = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\epsilon_2}}$$ See [2.22], p.42

This is a re-written form of [2.22]

$$\eta_c = \sqrt{\frac{\mu_2}{\epsilon_2 - j\frac{\sigma_2}{\omega}}} = \sqrt{\frac{\mu_2\omega}{\omega\epsilon_2 - j\sigma_2}} = \sqrt{\frac{j\omega\mu_2}{j\omega\epsilon_2 + \sigma_2}}$$

See p.11-13 of notes
Sect 2.3 of text

Since $\sigma \gg \omega\epsilon$ for a good conductor we can use an effective ϵ

$$\nabla \times \underline{H} = (j\omega\epsilon + \sigma)\,\underline{E} \cong \sigma \underline{E}$$

$$\nabla \times H = j\omega \underbrace{\left(\frac{\sigma}{j\omega}\right)} E$$

this is an "effective" permittivity ϵ_{eff}

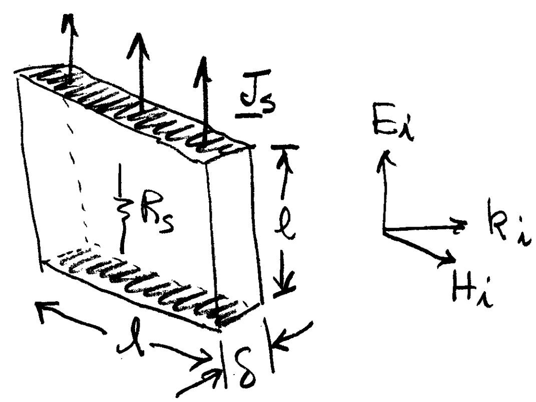
$\therefore$ simply use previous results with $\epsilon \rightarrow \epsilon_{eff}$ in medium 2.

$$\therefore \quad \gamma_2 = \underbrace{j\omega\sqrt{\mu \cdot \epsilon_{eff}}} = j\omega\sqrt{\frac{\mu_2 \sigma_2}{j\omega}} = \sqrt{j\omega\mu_2\sigma_2} = \frac{1+j}{\delta}$$

similar to
$\beta = \omega\sqrt{\mu\epsilon}$

$$\eta_c = Z_s = R_s + jX_s = \sqrt{\frac{\mu_2}{\epsilon_{eff}}} = \sqrt{\frac{\mu_2}{\frac{\sigma_2}{j\omega}}} = \sqrt{\frac{j\omega\mu_2}{\sigma_2}} = \frac{\gamma_2}{\sigma_2} = \frac{1+j}{\sigma\delta}$$

$$R_s = \frac{1}{\sigma\delta}$$



$$R_s = \frac{l}{l\delta\sigma_2} = \frac{1}{\delta\sigma}$$

l ← length of resistor

$l\delta\sigma_2$ ← conductivity/unit length.

$l\delta$ ← area of face.

Since R_s is independent of area it is called a surface resistance.

likewise η_c is the surface impedance.

Apply B.C.'s @ $z=0$

$$\left.\begin{aligned} E_{io} + E_{ro} &= E_{to} \\ \frac{1}{\eta_1}\left(E_{io} - E_{ro}\right) &= \frac{E_{to}}{Z_s} \end{aligned}\right\} \text{ both tangential components equal.}$$

This gives

$$\Gamma = \frac{E_{ro}}{E_{io}} = \frac{Z_s - \eta_1}{Z_s + \eta_1} = \rho e^{j\phi_r}$$

$$\Im = \frac{E_{to}}{E_{io}} = \frac{2Z_s}{Z_s + \eta_1} = \tau e^{j\phi_\tau}$$

For reasonable conductors $\Gamma \to 1$ and $\Im$ is very small.

you can approximate these by

$$\Gamma_1 \cong \frac{\delta - \lambda_1(1-j)}{\delta + \lambda_1(1+j)}$$

$$\Im_1 \cong \frac{2\delta}{\lambda_1(1-j)}$$

where $\lambda_1 \gg \delta$ so that

$$\Gamma_1 \cong 1e^{j\pi}$$

$$\tau \ll 1$$

Back to the original problem of finding the power dissipated in the conductor. In medium 2

$$\underline{J} = \sigma \underline{E}_t = \hat{x}\, \sigma_2 \mathcal{T} E_{io}\, e^{-\gamma_2 z} \quad \frac{A}{m^2}$$

The total current per unit "width" is given by

$$J_s = \sigma_2 \mathcal{T} E_{io} \int_0^\infty e^{-\gamma_2 z} dz = \sigma_2 \mathcal{T} E_{io} \left[\frac{e^{-\gamma_2 z}}{-\gamma_2} \Bigg|_0^\infty \right]$$

$$J_s = \frac{\sigma_2 \mathcal{T} E_{io}}{\gamma_2} \quad \frac{A}{m}$$

The tangential H field is continuous at $z=0$

$$H_1(z=0) = H_t(z=0)$$

but $H_t(z)\ \hat{y}\dfrac{\mathcal{T} E_{io}}{\eta_c} e^{-\gamma_2 z}$

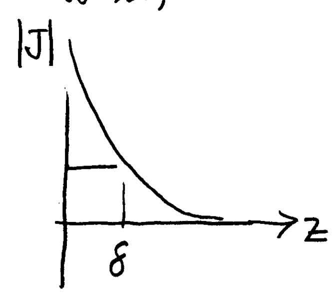
$$H_t(z=0) = \hat{y}\frac{\mathcal{T} E_{io}}{Z_s} \quad \text{where } Z_s = \frac{1+j}{\sigma_2 \delta}$$

$$H_{1y}(0) = \frac{\mathcal{T} E_{io}}{Z_s} \quad \text{or} \quad \mathcal{T} E_{io} = H_{1y}(0) Z_s$$

$$\therefore J_s = \frac{\sigma_2 \mathcal{T} E_{io}}{\gamma_2} = \frac{\sigma_2 H_{1y}(0) Z_s}{\gamma_2} = \frac{\sigma_2 H_{1y}(0)}{\frac{(1+j)}{\delta}} \frac{1+j}{\sigma_2 \delta} = H_{1y}$$

As $\delta \to 0$ for a good conductor $(\sigma \to \infty)$ J_s goes to a true surface current

If σ_2 is finite



Time average power loss comes from Poynting vector at surface.

$$|S_{AV}| = \left|\frac{1}{2}\mathrm{Re}\{\underline{E}\times\underline{H}^*\}\right| = \left|\frac{1}{2}\mathrm{Re}\{E_{1x}H_{1y}^*\}\hat{z}\right|$$

$$= \frac{1}{2}\mathrm{Re}\left\{\mathcal{T}E_{i0}\left(\frac{\mathcal{T}E_{i0}}{Z_s}\right)^*\right\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\mathcal{T}\mathcal{T}^*E_{i0}E_{i0}^*\frac{1}{Z_s^*}\right\} = \frac{1}{2}|\mathcal{T}E_{i0}|^2\mathrm{Re}\left\{\frac{1}{Z_s^*}\right\}$$

$$= \frac{1}{2}|\mathcal{T}E_{i0}|^2\mathrm{Re}\left\{\frac{1}{R_s - jX_s}\right\} = \frac{1}{2}|\mathcal{T}E_{i0}|^2\mathrm{Re}\left\{\frac{R_s + jX_s}{R_s^2 + X_s^2}\right\}$$

$$= \frac{1}{2}|\mathcal{T}E_{i0}|^2\frac{R_s}{R_s^2 + X_s^2}$$

but $R_s + jX_s = \dfrac{1+j}{\delta\sigma_2}$ or $R_s = X_s = \dfrac{1}{\sigma_2\delta}$

So we can write $|S_{AV}|$ as

$$|S_{AV}| = \frac{1}{2}|\mathcal{T}E_{10}|^2\frac{\frac{1}{\sigma_2\delta}}{\frac{1}{(\sigma_2\delta)^2}+\frac{1}{(\sigma_2\delta)^2}} = \underbrace{\frac{1}{4}|\mathcal{T}E_{10}|^2\sigma_2\delta}$$

this power/unit area enters conductor and is entirely dissipated due to $\underline{E}\cdot\underline{J}$ losses.

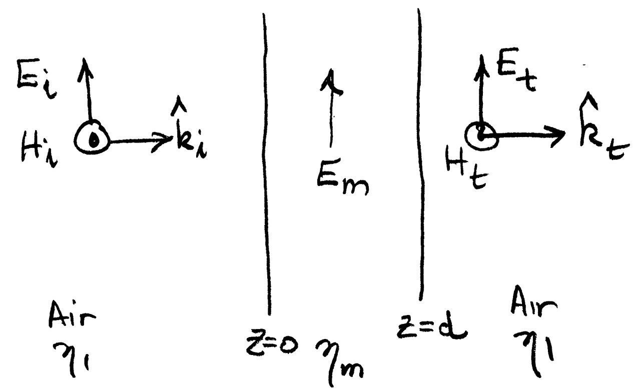
Since $J_s = \dfrac{\sigma_2\mathcal{T}E_{10}}{\gamma_2}$, $\gamma_2 = \dfrac{1+j}{\delta}$, and $R_s = \dfrac{1}{\sigma_2\delta}$

$$|S_{AV}| = \frac{1}{4}\left|\frac{J_s\gamma_2}{\sigma_2}\right|^2\sigma_2\delta = \frac{1}{4}|J_s|^2\left|\frac{1+j}{\sigma_2\delta}\right|^2\sigma_2\delta = \frac{1}{4}|J_s|^2\frac{|1+j|^2}{\frac{1}{R_s^2}}\frac{1}{R_s}$$

$$|S_{AV}| = \frac{1}{2}|J_s|^2R_s$$

so the power dissipates is the surface current density squared times the surface resistance

Example 3-18 RF Shielding



Consider an $\hat{x}$ polarized RF wave normally incident on an aluminum foil of thickness d at $z=0$

(a) Assume multiple reflections can be neglected, determine $|E_t|$

@ $z=0$ $\quad |E_m(z=0)| = \mathcal{T}_1 |E_i(z=0)|$

$$\mathcal{T}_1 = \frac{2\eta_m}{\eta_1+\eta_m} \qquad \eta_1 = 377\Omega$$

$$\eta_m = \sqrt{\frac{j\omega\mu_0}{\sigma_2}} = \sqrt{\frac{\omega\mu_0}{2\sigma_2}}(1+j)$$

@ $z=d$ $\quad |E_m(z=d)| = |E_m(z=0)|e^{-d/\delta} \qquad \delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma_2}}$

$$E_t(z=d) = \mathcal{T}_2 |E_m(z=d)|$$

$$\mathcal{T}_2 = \frac{2\eta_1}{\eta_m+\eta_1}$$

$$\therefore \quad |E_t(z=d)| = \mathcal{T}_1 e^{-d/\delta} \mathcal{T}_2 |E_i(z=0)|$$

(b) This is the calculations for aluminum $d = .025$ mm.

Use $\sigma_2 = 3.54 \times 10^7 \frac{S}{m}$. $f = 100$ mHz. $\epsilon_{al} = \epsilon_0$, $\mu_{al} = \mu_0$

Compute δ, $\quad \delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma_2}} = \frac{1}{\sqrt{\pi(100\times10^6)(4\pi\times10^{-7})(3.54\times10^7)}}$

$$\cong 8.46 \times 10^{-6} \text{ m}$$

$$d = .025\text{mm} = 25\times10^{-6}\text{m} = \frac{25\times10^{-6}}{8.46\times10^{-6}}\delta = 2.96\delta$$

Reflected wave would be attenuated by $e^{-\frac{2(2.96\delta)}{\delta}} \ll 1$.

$$e^{-d/\delta} = e^{-2.96\frac{\delta}{\delta}} = e^{-2.96} = .0518$$

Example 3-18 (cont.)

$$\tau_1 \tau_2 = \left|\frac{2\eta_m}{\eta_1+\eta_m}\right|\left|\frac{2\eta_1}{\eta_1+\eta_m}\right| = \left|\frac{4\eta_1\eta_m}{(\eta_1+\eta_m)^2}\right|$$

calculating

$$\eta_m = \sqrt{\frac{j\omega\mu_0}{\sigma_2}} = \sqrt{\frac{j\,2\pi(100\times10^6)(4\pi\times10^{-7})}{3.54\times10^7}}$$

$$= (3.34\times10^{-3})(1+j)\,\Omega$$

$\eta_1 \gg \eta_m$ so

$$\tau_1\tau_2 \cong \left|\frac{4\eta_1\eta_m}{\eta_1^2}\right| = \frac{4|\eta_m|}{\eta_1} = \frac{4\sqrt{2}(3.34\times10^{-3})}{377} = 5.01\times10^{-5}$$

$$\left|\frac{E_t}{E_i}\right| = \tau_1\tau_2 e^{-d/\delta} = (5.01\times10^{-5})(.0518) = 2.61\times10^{-6}$$

This is the E-field transmission

Since E_t is back into air the power transmission

$$\frac{\frac{1}{2}\frac{|E_t|^2}{\eta_1}}{\frac{1}{2}\frac{|E_i|^2}{\eta_1}} = \frac{|E_t|^2}{|E_i|^2} = 6.73\times10^{-12}$$

VERY LITTLE POWER TRANSMITTED!

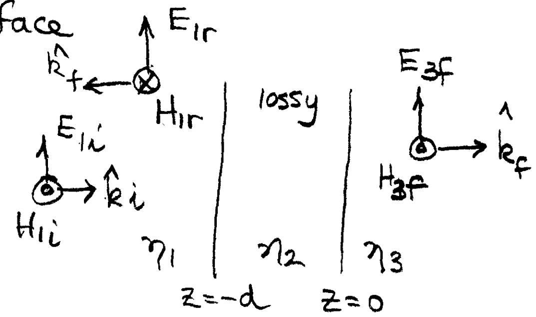
3.7.2 Reflection from multiple lossy interfaces

No equations change.

Use $\gamma_2 = \alpha_2 + j\beta_2$ in place of $j\beta_2$

and use complex impedances

For example, for normal incidence on a multiple dielectric interface



$$\Gamma_{eff} = \rho e^{j\phi_\Gamma} = \frac{(\eta_2-\eta_1)(\eta_3+\eta_2) + (\eta_3+\eta_1)(\eta_3-\eta_2)e^{-2\gamma_2 d}}{(\eta_2+\eta_1)(\eta_3+\eta_2) + (\eta_3-\eta_1)(\eta_3-\eta_2)e^{-2\gamma_2 d}}$$

$$\mathcal{T}_{eff} = \mathcal{T} e^{j\phi_\mathcal{T}} = \frac{4\eta_2\eta_3 e^{-\gamma_2 d}}{(\eta_2+\eta_1)(\eta_3+\eta_2) + (\eta_3-\eta_1)(\eta_3-\eta_2)\underbrace{e^{-2\gamma_2 d}}_{\text{exponents now use } \gamma}}$$

but $1 - |\Gamma_{eff}|^2 \neq \frac{\eta_1}{|\eta_3|} |\mathcal{T}_{eff}|^2$

since medium 2 is lossy.

Example 3.19

Find percentage of incident power absorbed by muscle tissue @ 915 MHz.

$$\Gamma = \rho e^{j\phi} = \frac{\eta_m - \eta_1}{\eta_m + \eta_1}$$

Calculate loss tangent and impedance of muscle.

$$\tan \delta_m = \frac{\sigma_m}{\omega \epsilon_m} = \frac{1.60}{2\pi(915\times10^6)(51)(8.854\times10^{-12})} = \frac{1.60}{2.59} = 0.616$$

$$\eta_m = \frac{\sqrt{\frac{\mu}{\epsilon}}}{\left(1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right)^{\frac{1}{4}}} = \frac{\frac{377}{\sqrt{51}}}{\left[1+(.616)^2\right]^{1/4}} e^{j\frac{1}{2}\mathrm{Tan}^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)}$$

$$\eta_m = \frac{52.7}{1.0837} e^{j\frac{1}{2}(.616)} = 48.62 e^{j17.69^\circ}$$

$$\Gamma = \frac{\eta_m - \eta_1}{\eta_m + \eta_1} = \frac{48.62e^{j17.69} - 377}{48.62e^{j17.69} + 377} = \frac{331.01\, e^{j177.44^\circ}}{423.58\, e^{j2.0^\circ}}$$

$$\Gamma = 0.78 e^{j175.44^\circ}$$

<u>absorbed power</u> $\left[1 - (.78)^2\right] \times 100\% = 38.9\%$

Example 3-20.

If a $1 \frac{mW}{cm^2}$ plane wave in muscle is normally incident on a planar interface between muscle & fat tissues at 915 MHz, find the power density of the wave transmitted into the fat tissue. For fat tissue, take $\sigma_f = 155 \frac{mS}{m}$

$$\eta_m = 48.62 e^{j17.69^\circ}$$

For fat tissue

$$\tan \delta_f = \frac{\sigma_f}{\omega \epsilon_f} = \frac{0.155 \, S/m}{2\pi(915\times10^6)(5.6)(8.854\times10^{-12})}$$

$$\tan \delta_f = 0.54375$$

$$\eta_f = \frac{\sqrt{\frac{\mu}{\epsilon}}\, e^{j\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\epsilon}\right)}}{\left[1+\left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{\frac{1}{4}}} = \frac{\frac{377}{\sqrt{5.6}}\, e^{j\frac{1}{2}(.54375)}}{\left[1+(.54375)^2\right]^{\frac{1}{4}}} = \frac{159.31 e^{j.271875}}{1.0669} = 149.32 e^{j15.6^\circ}$$

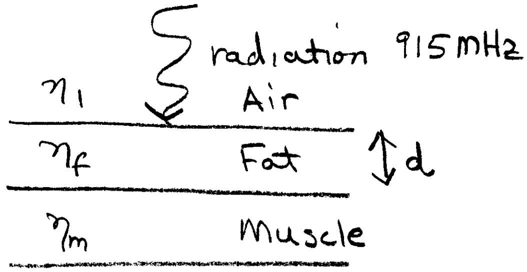
$$\Gamma = \frac{\eta_f - \eta_m}{\eta_f + \eta_m} = \frac{149.32 e^{j15.6^\circ} - 48.62 e^{j17.69^\circ}}{149.32 e^{j15.6^\circ} + 48.62 e^{j17.69^\circ}}$$

$$= \frac{143.82 + j40.16 - 46.32 - j14.77}{143.82 + j40.16 + 46.32 + j14.77}$$

$$\Gamma = \frac{97.5 + j25.39}{190.14 + j54.93} = 0.509 e^{-j.013^\circ}$$

$$\left|(S_{AV})_t\right| = (1-\rho^2)\left|(S_{AV})_i\right| = \left[1-(.509)^2\right]\left(1 \frac{mW}{cm^2}\right) = 0.74 \frac{mW}{cm^2}$$

Example 3.21



(c) For a mature pig we need to calculate the effective reflection & transmission coefficients.

Assume $\sigma_f = 155\,\frac{mS}{m}$

$$\eta_{muscle} = 48.62\, e^{j17.69^\circ} \quad \text{from Example 3-20.}$$

$$\eta_{fat} = 149.32\, e^{j15.6^\circ} \quad \text{from Example 3-21}$$

$\gamma_{fat} = \alpha_f + j\beta_f$ where

$$\begin{bmatrix}\alpha_f \\ \beta_f\end{bmatrix} = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\omega\epsilon}\right)^2} \mp 1\right]^{\frac{1}{2}}$$

$$= \frac{2\pi(915\times10^6)}{3\times10^8}\sqrt{\frac{5.6}{2}}\left[\sqrt{1+(.54375)^2} \mp 1\right]$$

$$= 32.067\left[1.13827 \mp 1\right] = \begin{cases} 4.433 & \text{np/m} \\ 68.568 & \text{rad/m}\end{cases}$$

$$\Gamma_{eff} = \frac{(\eta_f-\eta_1)(\eta_m+\eta_f) + (\eta_f+\eta_1)(\eta_m-\eta_f)e^{-2\gamma_f d}}{(\eta_f+\eta_1)(\eta_m+\eta_f) + (\eta_f-\eta_1)(\eta_m-\eta_f)e^{-2\gamma_f d}}$$

$\cong$

$$T_{eff} = \frac{4\eta_f \eta_m e^{-\gamma_f d}}{(\eta_f+\eta_1)(\eta_m+\eta_f) + (\eta_f-\eta_1)(\eta_m-\eta_f)\, e^{-2\gamma_f d}}$$

I am not going to do these tedious calculations.

However, the power reflected back into the air $\sim 5\%$
power transmitted into muscle $\sim 40\%$

$\therefore$ power absorbed by fat layer must be $\sim 55\%$

The total power absorbed by the pig is approx. 95%.

(a) A young pig has no fat layer and only muscle
We know from Example 3.20 that $\eta_m = 48.62\, e^{j17.69^\circ}$
The reflectance for this air muscle interface is then given by $\Gamma = 0.78 e^{j175^\circ}$ and a young pig only absorbs approx. 40% of the incident power.

$\Rightarrow$ The fat layer acts as an impedance transformer to lower reflections dramatically.