EECS412-Electromagnetic Field Theory III

Prof. Frank Merat

Fall 2003 Semester

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Textbook:
Electromagnetic Waves
Umran S. Inan and Aziz S. Inan
Prentice Hall
ISBN 0-201-36179-5
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Web page: http://vorlon.cwru.edu/~flm/eecs412fr03/home.html

TENTATIVE SYLLABUS

TOPIC	READING *	COMMENTS
Review of Fields I Topics (6 lectures)	Any fields textbook	
• Electric Flux and Gauss's Law, electric dipole		
• Poisson's & Laplace's Equations, capacitance		
• Electrostatic boundary conditions		
• Ampere's law, magnetic dipole		
• Inductance		Take home exam
Planes waves in unbounded media (9 lectures)		
• Intro to plane Introduction, Maxwell's Equations	1-14	
• Plane Waves in Lossless Media	15-35	
• Plane Waves in Conducting Media	38-45	
• Conductors/Dielectrics and Skin Effect	45-59	HW#1
• Flow of Electromagnetic Power	59-75	
• Wave Polarization	78-95	
• Arbitrary Uniform Plane Waves	96-104	HW#2
Reflection, Transmission & Refraction at Planar Interfaces (6 lectures)		
• Reflection at Normal Incidence	120-131	
• Normal Incidence on a Dielectric	132-140	
• Multiple Dielectric Interfaces	140-155	HW#3
• Oblique Incidence on a Conductor	155-167	
• Reflection/Refraction at Oblique Incidence	167-188	
• Total Internal Reflection	189-201	HW#4
• Reflection/Refraction from Lossy Media	201-215	

MID-TERM EXAM

Parallel Plate and Dielectric Slab Waveguides (5 lectures)

• Parallel Plate Waveguide	249-286	HW#5
• Dielectric Waveguides	286-307	HW#6
• Wave Velocities	307-319	

Waves and transmission lines (6 lectures)	Supplemental notes	
• Transmission line behavior & circuit models		
• SWR, impedance, multi-port networks		HW#7
• Smith charts & impedance matching		HW#8

Cylindrical Waveguides & Cavity Resonators (6 lectures)

• Rectangular Waveguides	331-353	
• Circular Waveguides	353-378	HW#9
• Cavity Resonators	378-400	

Antennas (5 lectures)	Supplemental notes.	
• Elementary Antennas	476-494	
• Monopoles and Dipoles		HW#10

FINAL EXAM (December 13th, 8:00-11:00 a.m.)

GRADING:

Take home review exam	10%
Mid-term exam	25%
Final exam	25%
Homework	30%
Term paper and presentation	10%

Term Paper and Presentation: Electricity and Magnetism consists of a basic theory (Maxwell's Equations) and a large number of applications. We will consider many of the applications during the course but we cannot cover all topics. You will select a topic beyond the course to explore independently. Below is a suggestive, but not complete set of topics.

- Wave Guides and Cavity Resonators
- Transmission lines
- Green's Functions
- Numerical FielSolutions (2D and 3D)
- Frequency dependence of the dielectric constant (Kramers-Kronig relations).
- Propagation of light in anisotropic crystals
- Wave propagation in nonlinear media

By the end of week 7, you need to submit a proposal and a bibliography for your paper. The proposal will consist of one or two paragraphs that outline your plan of study for the last two weeks of the semester. The content of the proposal should be comparable to a one-week section of the course. Presentations will be a twenty minutes in length during the last week of the term. I will try to schedule some time so that all presentations can be done in the last two days of the semester.

EECS 412 Electromagnetic Fields

Text – Inan & Inan Electromagnetic Waves with supplements.

Maxwell's Equation

$$\nabla \cdot \underline{D} = \rho \qquad \nabla \cdot \underline{B} = 0$$

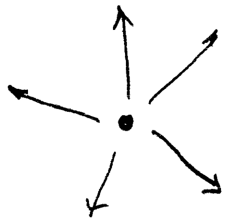
$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \nabla \times \underline{B} = \underline{J} + \frac{\partial D}{\partial t}$$

For static fields

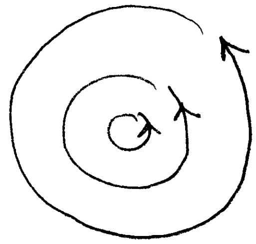
$$\underbrace{\begin{array}{ll} \nabla \cdot \underline{D} = \rho & \nabla \cdot \underline{B} = 0 \\ \nabla \times \underline{E} = 0 & \nabla \times B = \underline{J} \end{array}}$$

These are independent solutions.

General properties of fields given by $\underline{F}(x, y, z, t)$



$\underline{F}(x, y, z, t)$



sources/sinks
irrotational field

closed contours
rotational field

This motivates us to define circulation as the component of $\underline{F}$ which contributes to a closed loop.

$$\text{net circulation} = \oint_c \underline{F} \cdot \underline{dl}$$

By this definition an irrotational field has $\underline{\underline{NO}}$ circulation.

Consider a scalar function $\quad \oint_c d\phi = \phi(P_2) - \phi(P_1)$

Obviously if ϕ is single valued there is $\underline{NO}$ circulation

How do we do this for a vector function?

Using the gradient

$$\oint_c d\underline{\Phi} = \oint_c \underbrace{\frac{d\phi}{dl}\,\underline{dl}}_{\text{scalar}} = \underbrace{\oint \nabla\phi \cdot dl}_{}$$

but this is the circulation of some function

For irrotational fields

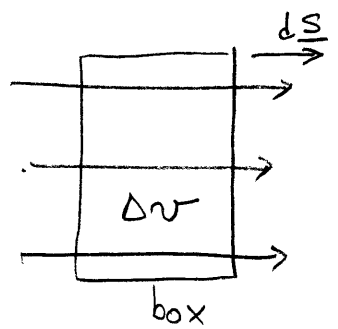
$$F = -\nabla\phi$$

irrotational fields terminate in points (sources / sinks)

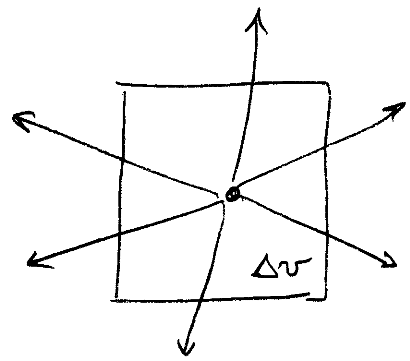
What is a mathematical method of determining if sources / sinks are present?

Ans: divergence $= \underline{\nabla} \cdot \underline{F} = \lim_{\Delta v \to 0} \dfrac{\oint \underline{F} \cdot d\underline{s}}{\Delta v}$

This is simply a way of saying there is a net flux of field lines through a surface.



no sources in box
flux in = flux out
net flux = 0
divergence = 0



source in box
flux in = 0 flux out > 0
net flux > 0
divergence > 0

If there is a source in the volume Δv the integral $\oint \underline{F} \cdot d\underline{s} \neq 0$
If there is no source in the volume Δv $\oint \underline{F} \cdot d\underline{s} = 0$

Note that an irrotational field MUST have non-zero divergence somewhere.

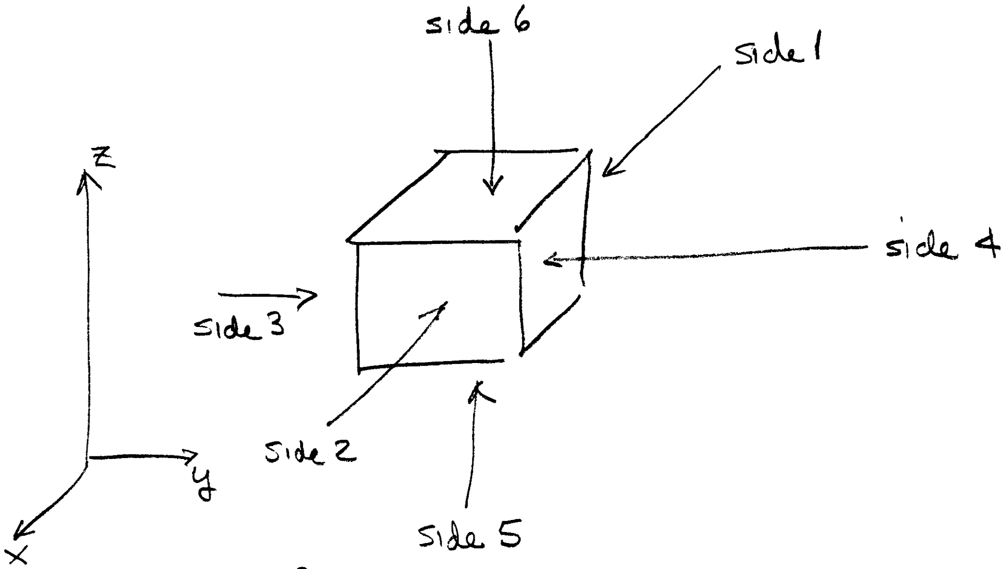
What is flux? $\Psi \equiv \int \underline{F} \cdot d\underline{s}$

$\underline{F}$: something / unit area

$d\underline{s}$: this is unit area

consequently, the flux Ψ is a measure of something

We draw flux plots to indicate the magnitude of that something



Look at how flux changes across a small volume element.

looking at side #1 : normal component is $F_x \hat{x}$

normal vector is $-\hat{x}$ — Normal vector points out.

$$d\underline{S} = \underbrace{-\hat{x}}_{\text{pointing in}}\ \underbrace{dy\,dz}_{\text{surface area}}$$

$$\int_{\text{side 1}} \underline{F} \cdot d\underline{S} = (F_x \hat{x}) \cdot (-\hat{x}\, dy\, dz) = -F_x\, dy\, dz$$

on side #2 — normal component is $\left(F_x + \frac{\partial F_x}{\partial x} dx\right) \hat{x}$

normal vector is $+\hat{x}$

$$d\underline{S} = +\hat{x}\ dy\, dz$$

$$\int_{\text{side 1 + side 2}} \underline{F} \cdot d\underline{S} = \cancel{-F_x\, dy\, dz} + \cancel{F_x\, dy\, dz} + \frac{\partial F_x}{\partial x} dx\, dy\, dz$$

Similarily we get

$$\int_{\text{side 3 + side 4}} \underline{F} \cdot d\underline{S} = \frac{\partial F_y}{\partial y} dx\, dy\, dz$$ in the y direction

$$\int_{\text{side 5 + side 6}} \underline{F} \cdot d\underline{S} = \frac{\partial F_z}{\partial z} dx\, dy\, dz$$ in the z direction

combining all sides

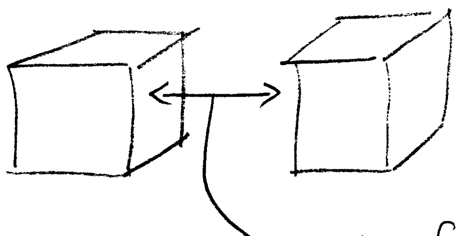
$$\text{div}\,\underline{F} = \lim_{\Delta v \to 0} \frac{\int_{\text{all sides}} F \cdot dS}{\Delta x \Delta y \Delta z}$$

$$\text{div}\,F = \lim_{\Delta v \to 0} \frac{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$\text{div}\,F = \underline{\nabla} \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

divergence Theorem

Consider multiple differential volumes



contributions to $\int \underline{F} \cdot \underline{dS}$ from common surfaces cancel

$$\text{div}\,\underline{F} = \frac{\int_i \underline{F} \cdot \underline{dS}}{\Delta v_i}$$

$$\text{div}\,F\,\Delta v_i = \int_i \underline{F} \cdot \underline{dS}$$

$$\sum_i \text{div}\,F\,\Delta v_i = \sum_i \underbrace{\int_i \underline{F} \cdot \underline{dS}}_{\text{common surface contributions cancel}}$$

$$\int \text{div}\,F\,dv = \int \underline{F} \cdot \underline{dS}$$

$$\int \nabla \cdot \underline{F}\,dv = \int \underline{F} \cdot \underline{dS}$$

For an irrotational field
suppose the sources of the field are given by $\rho(x,y,z,t)$

then $\underline{\nabla} \cdot \underline{F} = \rho$

We already know that $F = -\nabla\phi$ for an irrotational field

Then

$$\nabla \cdot (-\nabla\phi) = \rho$$

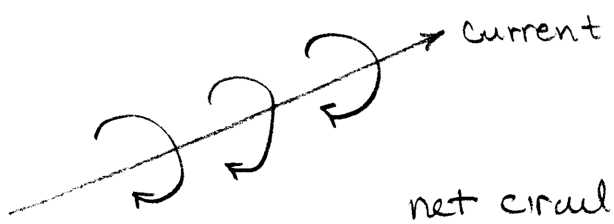
$$\nabla^2\phi = -\rho \qquad \text{Poisson's equation}$$

Fundamental to irrotational fields

$$\nabla^2\phi = -\rho \qquad \text{Poisson's equation}$$

$$\nabla^2\phi = 0 \qquad \text{Laplace's equation}$$

Rotational Fields



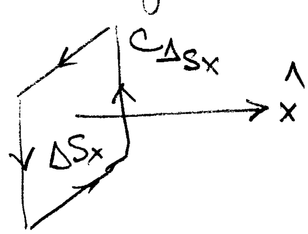
net circulation $= \oint \underline{F} \cdot \underline{dl}$

Just as in divergence, let's define microscopic circulation or curl
(NOTE: curl is a vector quantity)

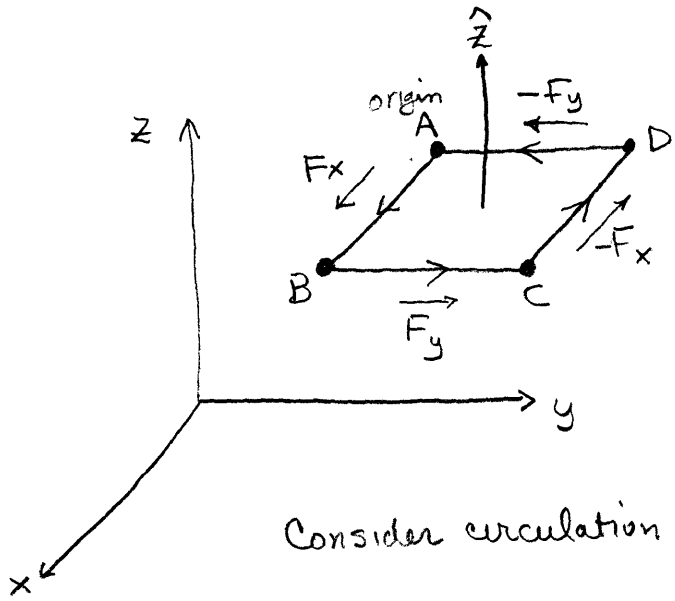
$$\text{curl } \underline{F} \cdot \hat{x} = (\underline{\nabla} \times \underline{F}) \cdot \hat{x} = \lim_{\Delta S_x \to 0} \frac{\oint_{C_{\Delta S_x}} \underline{F} \cdot \underline{dl}}{\Delta S_x}$$

where circulations are defined over a surface.

Contours define surfaces NOT volumes.



normal by right hand rule.



Note right hand rule is used to define surface normal.

Consider circulation for a small loop.

By definition $\oint F \cdot dl = F_x\Big|_{AB} \Delta x + F_y\Big|_{BC} \Delta y - F_x\Big|_{CD} \Delta x - F_y\Big|_{DA} \Delta y$

sign's due to dot products

Now relate sides by Taylor series expansion

@ AB $\quad F_x\Big|_{AB} = F_x$

@ CD $\quad F_x\Big|_{CD} = F_x + \dfrac{\partial F_x}{\partial y} \Delta y$

@ DA $\quad F_y\Big|_{DA} = F_y$

@ BC $\quad F_y\Big|_{BC} = F_y + \dfrac{\partial F_y}{\partial x} \Delta x$

Putting all of these together by multiplying by $\underline{dl}$

$$\oint_c \underline{F} \cdot \underline{dl} = \cancel{F_x \Delta x} + \cancel{F_y \Delta y} + \frac{\partial F_y}{\partial x} \Delta x \Delta y - \cancel{F_x \Delta x} - \frac{\partial F_x}{\partial y} \Delta y \Delta x - \cancel{F_y \Delta y}$$

$$\oint_c \underline{F} \cdot \underline{dl} = \frac{\partial F_y}{\partial x} \Delta x \Delta y - \frac{\partial F_x}{\partial y} \overbrace{\Delta x \Delta y}^{\Delta s}$$

Then $\text{curl } F \cdot \hat{z} = \lim\limits_{\Delta s \to 0} \dfrac{\oint_c F \cdot dl}{\Delta s} = \lim\limits_{\Delta x \Delta y \to 0} \dfrac{\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \cancel{\Delta x \Delta y}}{\cancel{\Delta x \Delta y}}$

We can repeat this in the x and y directions to get

$$\text{curl}\,\underline{F} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

This is very hard to remember so we write it as a cross product

$$\text{curl}\,\underline{F} = \underline{\nabla}\times\underline{F} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix}$$

What are the sources of <u>potential</u> for rotational fields?

We know that if $\underline{F} = -\nabla\phi$ then

$$\underline{\nabla}\times\underline{F} = \underline{\nabla}\times(-\underline{\nabla}\phi) \equiv 0 \qquad \text{(no circulation!)}$$

This means that $\underline{F} = -\nabla\phi$ is only a source of irrotational fields so we need something else for rotational fields.

Try a vector function instead: Let $\underline{F} = \underline{\nabla}\times\underline{A}$

Then $\underline{\nabla}\cdot\underline{F}$ which should be non-zero for a irrotational field is given by

$$\nabla\cdot(\underline{\nabla}\times\underline{A}) \equiv 0 \qquad \text{so } \overset{=\nabla\times A}{\underline{F}} \text{ is purely rotational}$$

However $\underline{\nabla}\times\underline{F} = \underline{\nabla}\times\underline{\nabla}\times\underline{A} \neq 0$

Let $\nabla\times\underline{F} = \underline{J}$ $\qquad$ J is the source of $\underline{F}$ just as before ρ was the source of F

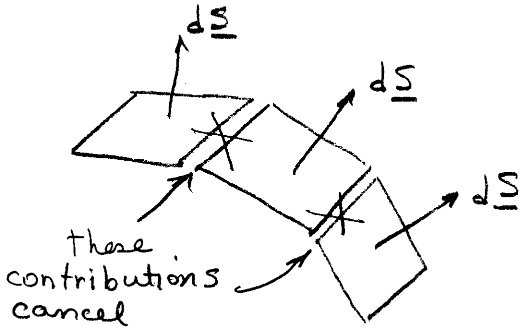
$$\text{Then}\quad \nabla\times\nabla\times\underline{A} = \nabla\cancelto{0}{\nabla\cdot\underline{A}} - \nabla^2\underline{A} = \underline{J}$$

(true for EM fields – prove later)

$$\nabla^2\underline{A} = -\underline{J} \qquad \text{vector Poisson equation}$$

Stoke's Theorem

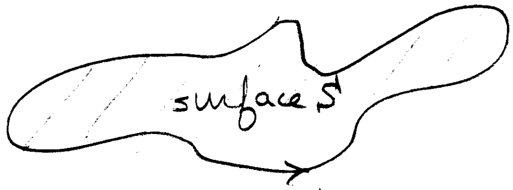
remember $\qquad (\nabla \times F) \cdot d\underline{S} = \oint \underline{F} \cdot d\underline{l}$



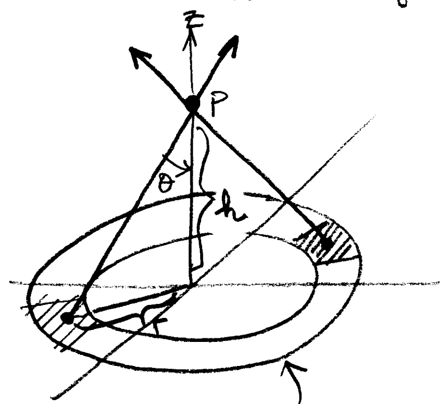
$$\int_S (\nabla \times \underline{F}) \cdot d\underline{S} = \oint_C \underline{F} \cdot d\underline{l}$$

S: surface defined by C

C: closed contour



Example of field from charged plate with charge density ρ_s



charge ρ_s on plate

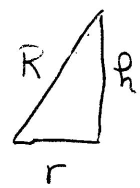
Note: this is not the only way to solve this problem.

The problem is simple if we exploit symmetry. The plate can be broken down into a set of concentric rings centered on the z-axis.

using Coulomb's law and superposition

$$\underline{E} = \int \frac{dq}{4\pi\epsilon_0 R^2} \underline{R}$$

For cylindrical coordinates $dq = \rho_s ds = \rho_s \, r dr d\phi$



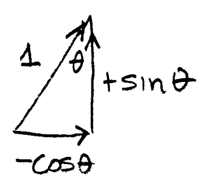
From geometry $R^2 = r^2 + h^2$

$$E = \int_0^{\infty}\int_0^{2\pi} \frac{\rho_s \, r dr d\phi \;\; \underline{R}(r,\phi)}{4\pi\epsilon_0 (r^2+h^2)}$$

explicit function of (r,ϕ)

We can handle $\hat{R}$ in several ways. Basically you want to set up the problem so the radial components of $\hat{R}$ cancel and the z-components add.

$$\hat{R} = -\cos\theta \, \hat{r} + \sin\theta \, \hat{z}$$



$$\therefore E = \underbrace{\int_0^{\infty}\int_0^{\pi} \frac{\rho_s \, r dr d\phi \cos\theta \, (0)}{4\pi\epsilon_0 (r^2+h^2)}}_{\text{this cancels}} + \underbrace{\int_0^{\infty}\int_0^{\pi} \frac{\rho_s \, r dr d\phi \sin\theta (2)}{4\pi\epsilon_0 (r^2+h^2)}}_{\text{the z-components add so } [0,\pi]}$$

Some words about Gauss' Law are in order

Gauss' Law is a mathematical law, not a fields law

It relates the flux through a surface to the sources of flux within the surface.

For any vector field $\underline{U}$

$$\underbrace{\oint_S \underline{U} \cdot \underline{ds}}_{\text{flux through the surface}} = \underbrace{\int_V \underline{\nabla} \cdot \underline{U}\, dv}_{\text{integral of flux sources over volume}}$$

Maxwell's equations give us

$$\underline{\nabla} \cdot \underline{D} = \rho$$

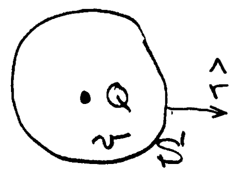
$$\nabla \cdot \epsilon \underline{E} = \rho$$

$$\therefore \quad \oint_S \underline{D} \cdot \underline{ds} = \int_V \underline{\nabla} \cdot \underline{D}\, dv = \int_V \rho\, dv = \underbrace{Q}_{\text{total charge enclosed by } S}$$

This provides another method for solving for the electric field when appropriate symmetry exists.

Electric fields from charge distributions

Electric field from a point charge



1) Field has spherical symmetry, so $\underline{E} = E_r \hat{r}$
2) Use Gauss' Law

$$\oint_S \underline{D} \cdot \underline{ds} = \int_V \rho \, dv$$

$$\oint_S \underline{D} \cdot \underline{ds} = \oint D_r \hat{r} \cdot \hat{r} \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r r^2 \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{4\pi} = 4\pi D_r r^2$$

$$\int_v \rho \, dv = Q$$

$$\therefore \quad D_r 4\pi r^2 = Q \quad \text{or} \quad D_r = \frac{Q}{4\pi r^2}$$

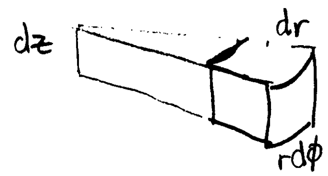
$$E_r = \frac{Q}{4\pi\epsilon_0 r^2}$$

NOTE

cylindrical coordinates

$ds = r \, dr \, d\phi$

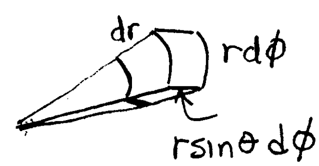
$dv = r \, dr \, d\phi \, dz$



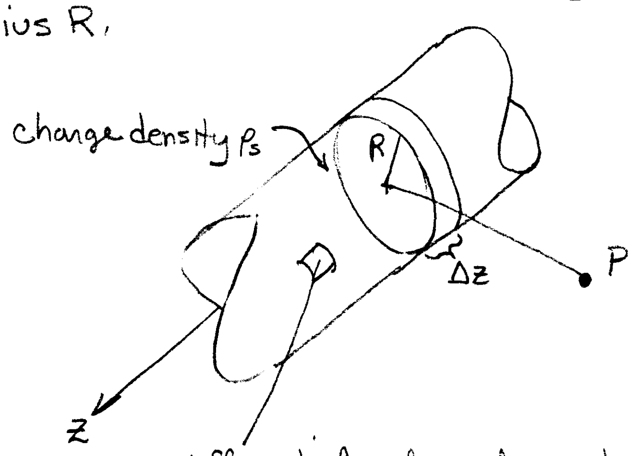
spherical coordinates

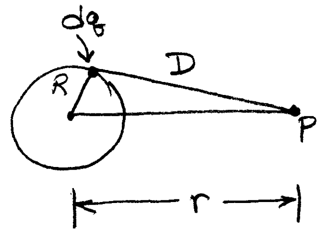
$ds = r^2 \sin\theta \, dr \, d\theta$

$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$

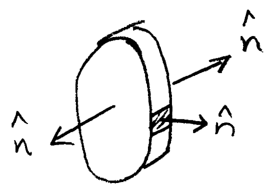


Gauss' Law applied to infinitely long cylindrical shell of charge of radius R,





We take $\underline{E}$ in a small cylindrical slab of material of thickness Δz. For Δz small any fields on the two surface elements cancels since $\hat{n}$ is in opposite directions.



So only the radial part of E survives.

$$\oint \underline{D}\cdot d\underline{S} = \int_0^{2\pi}\int_0^{\Delta z} \epsilon_0 \underline{E}_r \cdot d\underline{S} = \int_0^{2\pi}\int_0^{\Delta z} \epsilon_0 E_r \; r\,d\theta\,dz = \epsilon_0 E_r\, r\, 2\pi\,\Delta z$$

$$= \int \rho\, dv = \rho_s S = \rho_s\, 2\pi R\,\Delta z$$

$$\epsilon_0 E_r\, r \cancel{2\pi}\, \cancel{\Delta z} = \rho_s \cancel{2\pi} R \cancel{\Delta z}$$

$$E_r = \frac{\rho_s R}{\epsilon_0 r}$$

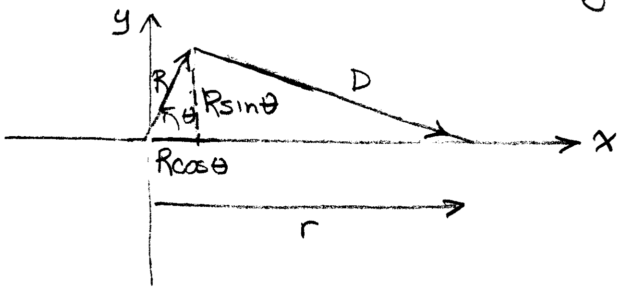
This is a good example because we can also do it by Coulomb's Law

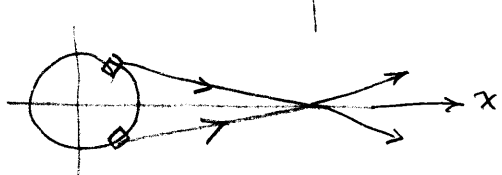
$$E = \int \frac{dq\ \hat{D}}{4\pi\epsilon_0 D^2}$$

which is more mathematically complex to evaluate.

$$D^2 = R^2 + r^2 - 2Rr\cos\theta \quad \text{by law of cosines}$$

$$\hat{D} = (r - R\cos\theta)\,\hat{x} - R\sin\theta\,\hat{y}$$





by symmetry the y components cancel

$$\underline{E} = \iint \frac{\rho_s R\, d\theta\, dz\left[(r - R\cos\theta)\,\hat{x} - R\sin\theta\,\hat{y}\right]}{4\pi\epsilon_0\left[R^2 + r^2 - 2rR\cos\theta\right]}$$

$$\underline{E} = \frac{\rho_s}{4\pi\epsilon_0}\iint \frac{R\, d\theta\, dz\ (r - R\cos\theta)\,\hat{x}}{R^2 + r^2 - 2rR\cos\theta}$$

$$\underline{E} = \frac{\rho_s}{4\pi\epsilon_0}\, 2\int_0^{\Delta z}\int_0^{\pi} \frac{(r - R\cos\theta)\, R\, d\theta\, dz}{R^2 + r^2 - 2rR\cos\theta}\,\hat{x}$$

$$\underline{E} = \frac{2\rho_s R\Delta z}{4\pi\epsilon_0}\int_0^{\pi} \frac{(r - R\cos\theta)\, d\theta}{R^2 + r^2 - 2rR\cos\theta}\,\hat{x} = \frac{\rho_s R}{2\pi\epsilon_0}\,\hat{x}\,\frac{2\pi}{r}\Delta z = \frac{\rho_s R}{\epsilon_0 r}\Delta z\,\hat{x}$$

$$\underline{E} = \frac{E}{\Delta z}\,\hat{x} = \frac{\rho_s R}{\epsilon_0 r}\,\hat{x}$$

Electrostatic potential

There are simpler ways of evaluating fields, especially if we do not know the charge densities.

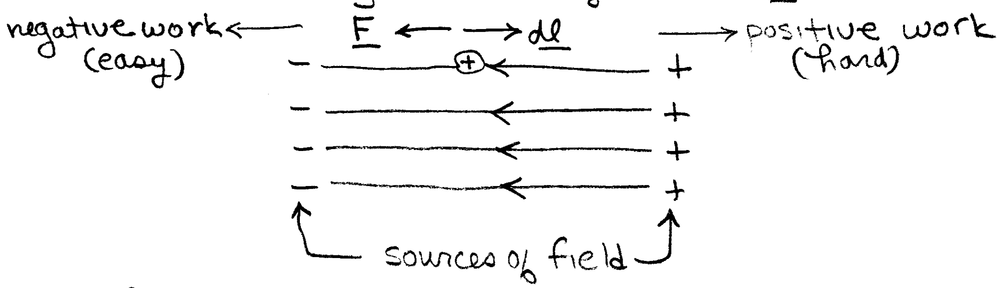
How much work is done in moving a charge to a point in a given electrostatic field.

$$\underset{\text{this is now force}}{\underline{F}} = \Delta q \, \overset{\text{electric field}}{\underline{E}}$$

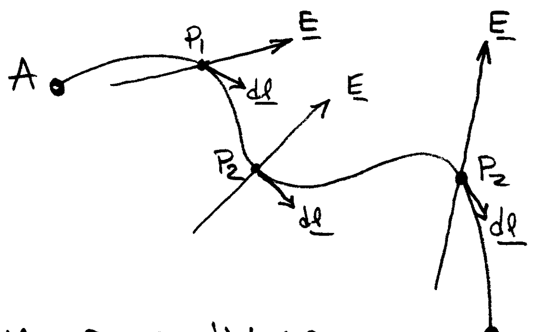
The incremental work dW done in moving Δq along the contour $\underline{dl}$ is given by

$$dW = -\underline{F} \cdot \underline{dl}$$

Consider moving a + charge in an $\underline{E}$ field



If the charge is moved against the field the work done is positive so we need the minus sign to correctly sign the dot product.



at P_1 $\underline{E} \cdot \underline{dl} > 0$ so $dW < 0$
moving in the easy direction
i.e. the field moves the charge this way

at P_2 $\underline{E} \cdot \underline{dl} = 0$ so no work is done

at P_3 $\underline{E} \cdot \underline{dl} < 0$ so $dW > 0$
moving in the hard direction

Just as we did for a electric field we can define work/unitcharge as

$$\frac{dW}{\Delta q} = \frac{-\underline{F}\cdot \underline{d\ell}}{\Delta q} = \frac{-\Delta q\, \underline{E}\cdot \underline{d\ell}}{\Delta q} = -\underline{E}\cdot\underline{d\ell}$$

We define the electrostatic potential $d\Phi \equiv -\underline{E}\cdot\underline{d\ell}$

$$\int_{P_1}^{P_2} d\phi = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{d\ell}$$

$$\phi(P_1) - \phi(P_2) = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{d\ell}$$

If $P_1 = P_2$ this becomes $\oint \underline{E}\cdot\underline{d\ell} = 0$

This is the conservative property of ϕ.

This is also the net circulation and is zero for an irrational

As long as ϕ is single-valued the above expression can be inverted to give

$$E = -\nabla\phi$$

Properties of the scalar potential ϕ:

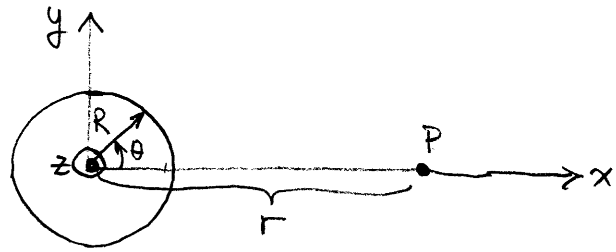
- ϕ is independent of the path since $\phi = -\int_{P_1}^{P_2} \underline{E}\cdot\underline{d\ell}$ along <u>any</u> contour

- ϕ is only relative from the conservative property we can only calculate a potential <u>difference</u>, so we must arbitrarily define some point as reference, Often we pick $\phi(r=\infty)=0$ as an arbitrary reference

- ϕ is <u>NOT</u> unique
  let $\phi' = \phi + k$ (a constant)
  Then $E = -\nabla\phi' = -\nabla\phi$
  and ϕ and ϕ' give the same field

Consider the electric potential from a charged cylinder of radius R.

For a charged cylinder $\underline{E} = \frac{\rho_s R}{\epsilon_0 r}\hat{x}$, ρ_s is uniform inside the cylinder



Since $\underline{E} = -\nabla\phi$ we can simply integrate $\underline{E}$ to obtain ϕ

$$E = -\nabla\phi = -\hat{r}\frac{\partial\phi}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \hat{z}\frac{\partial\phi}{\partial z}$$

notational problem – watch it

In this case $\frac{\partial\phi}{\partial\theta} = \frac{\partial\phi}{\partial z}$ because of symmetry

Since the x-direction was arbitrarily chosen we can let $\hat{x} = \hat{r}$ for this problem

$$E_r = -\frac{\partial\phi}{\partial r} = -\frac{d\phi}{dr} \quad \text{since } \phi = \phi(r)$$

Integrating this we get

$$E_r dr = -d\phi$$

$$\int_{r=a}^{r=b} E_r dr = -\int_{r=a}^{r=b} d\phi = -\phi(r=b) + \phi(r=a)$$

Rearranging

$$\phi(r=a) = \phi(r=b) + \underbrace{\int_{r=a}^{r=b} \frac{\rho_s R}{\epsilon_0 r}\, dr}_{\frac{\rho_s R}{\epsilon_0} \ln r \Big|_{r=a}^{r=b}}$$

$$\phi(r=a) = \phi(r=b) + \frac{\rho_s R}{\epsilon_0}(\ln b - \ln a)$$

$$\phi(r=a) = \phi(r=b) + \frac{\rho_s R}{\epsilon_0}\ln\left(\frac{b}{a}\right)$$

pick a reference point b possibly $\phi(\infty) = 0$

In general, for a collection of N changes we simply sum up the scalar potential from each charge

$$\phi_i(\underline{r_0}) = \sum_{i=1}^{N} \frac{Q_i}{4\pi\epsilon_0 |\underline{r_i} - \underline{r_0}|}$$

This can be readily expanded to a continuous distribution

$$\phi(\underline{r_0}) = \int \frac{\rho(\underline{r}')\,dv'}{4\pi\epsilon_0 |\underline{r}' - \underline{r_0}|}$$

r_0 is the observation point
r_i is the location of the charge

Another way to look at this is to consider the potential from a unit charge as the impulse response associated with a point charge and convolve

$$\phi(\underline{r_0}) = \frac{\delta(\underline{r})}{4\pi\epsilon_0 |r_0|}$$

← locates charge at $r=0$

since $r'=0$

observation point

Then,

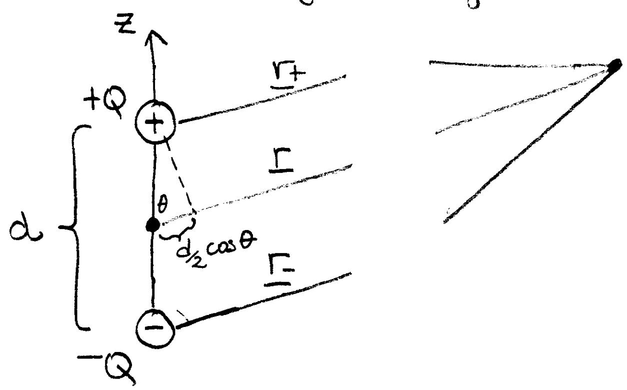
$$\phi(\underline{r_0}) = \int \phi(\underline{r}' - \underline{r_0})\,\rho(\underline{r}')\,dv'$$

which is a Green's Function solution.

Consider the electrostatic potential resulting from multiple point charges

$$\Phi = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|\underline{r}-\underline{r}_k'|}$$

The simplest arrangement of charges is the electric dipole



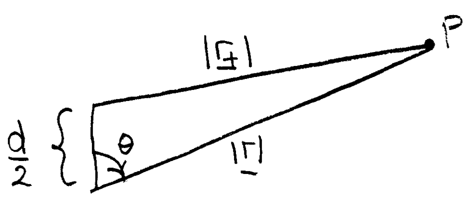
P (the observation point)

Assume that P is far enough away that $\underline{r}_+$, $\underline{r}$ and $\underline{r}_-$ are essentially parallel at P.

Since this is an assembly of charges it is most easily done by summing the potentials at P from the two charges.

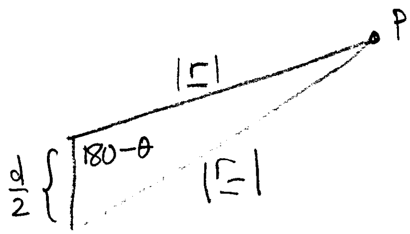
$$\Phi(\underline{r}) = \frac{+Q}{4\pi\epsilon_0|\underline{r}_+|} + \frac{-Q}{4\pi\epsilon_0|\underline{r}_-|} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{|\underline{r}_+|} - \frac{1}{|\underline{r}_-|}\right)$$

The rest of the problem is geometry which is best done by the law of cosines.



$$|\underline{r}_+|^2 = |\underline{r}|^2 + \left(\frac{d}{2}\right)^2 - 2|\underline{r}|\frac{d}{2}\cos\theta$$

$$|\underline{r}_+| = \sqrt{|\underline{r}|^2 + \left(\frac{d}{2}\right)^2 - |\underline{r}|\, d\cos\theta}$$



$$|\underline{r}_-|^2 = |\underline{r}|^2 + \left(\frac{d}{2}\right)^2 - 2|\underline{r}|\frac{d}{2}\cos\theta$$

$$\underline{r}_- = \sqrt{|\underline{r}|^2 + \left(\frac{d}{2}\right)^2 - |\underline{r}|\, d\cos\theta}$$

Let $|\underline{r}| = r$

In most electrostatic problems $r \gg d$, i.e., P is far away

Combining these expressions

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+\left(\frac{d}{2}\right)^2 - rd\cos\theta}} - \frac{1}{\sqrt{r^2+\left(\frac{d}{2}\right)^2 + rd\cos\theta}}\right]$$

Rewrite denominators and expand as a Taylor series

$$\frac{1}{r\sqrt{1+\left(\frac{d}{2r}\right)^2 - \frac{d}{r}\cos\theta}} \cong \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{d}{2r}\right)^2 + \frac{d}{2r}\cos\theta + \ldots\right]$$

neglect this 2nd order term

$$(1+u)^{-\frac{1}{2}} = 1 - \frac{1}{2}u + \ldots$$

Then,

$$\Phi \cong \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right) - \frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right)\right]$$

$$\Phi = \frac{Q}{4\pi\epsilon_0}\left[\frac{d}{r^2}\cos\theta\right]$$

$$\Phi = \frac{Qd\cos\theta}{4\pi\epsilon_0 r^2}$$

$$\underline{E} = -\nabla\Phi$$

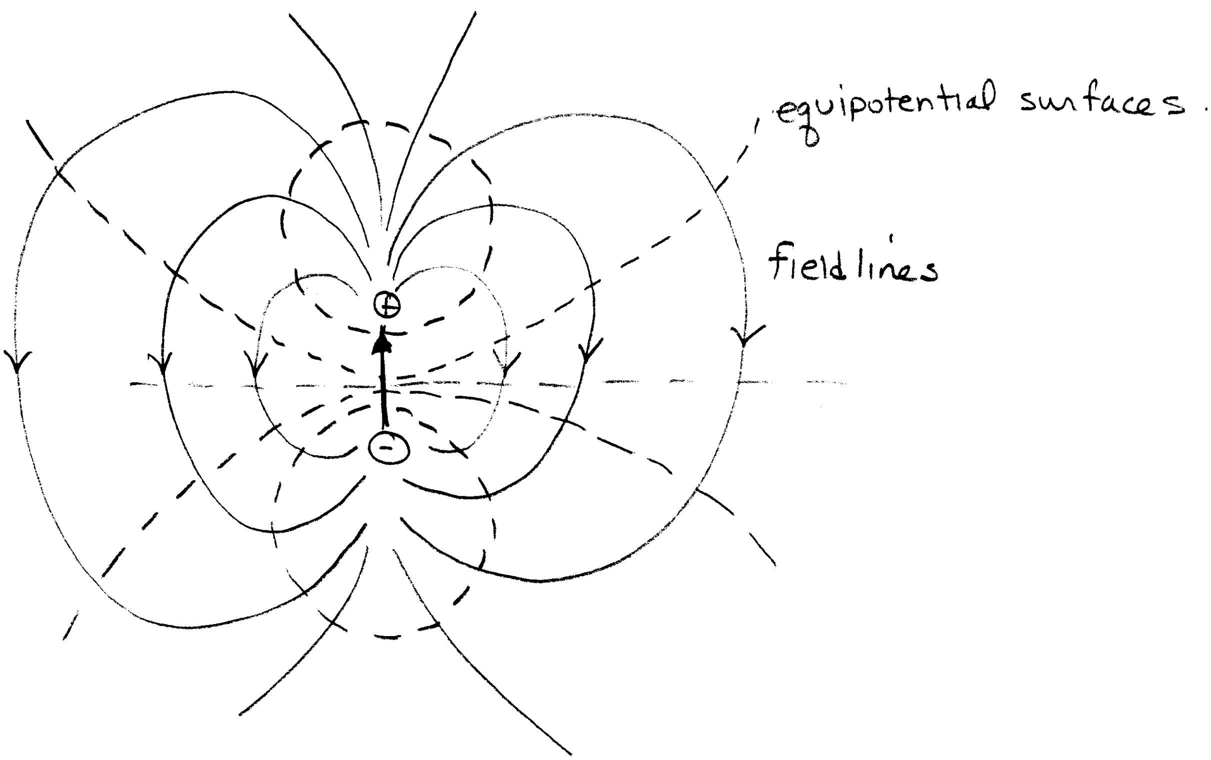
$$\underline{E} = -\left[\hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial\Phi}{\partial\phi}}\right]$$

$$\underline{E} = +\hat{r}\frac{Qd\cos\theta}{2\pi\epsilon_0 r^3} + \hat{\theta}\frac{Qd\sin\theta}{4\pi\epsilon_0 r^3}$$

$$\underline{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right]$$

Qd is the dipole moment

This is a very famous result



For the electrostatic potential

$$\underline{E} = -\nabla\Phi$$

$$\underline{\nabla}\cdot\underline{E} = -\underline{\nabla}\cdot\underline{\nabla}\Phi = -\nabla^2\Phi$$

but $$\underline{\nabla}\cdot\underline{E} = \nabla\cdot\left(\frac{\underline{D}}{\epsilon}\right) = \frac{\underline{\nabla}\cdot\underline{D}}{\epsilon} = \frac{\rho}{\epsilon}$$

Poisson's Equation $\nabla^2\Phi = -\frac{\rho}{\epsilon}$

If the region is charge free $\rho \to 0$

Laplace's Equation $\nabla^2\Phi = 0$

This is probably the most important solution method for electrostatic problems. Boundary conditions for Φ are usually given; and, if not, we have the simple fact that Φ is everywhere continuous (more on this later).

There are many solution techniques such as Gauss' Law, Coulomb's Law, method of images, etc. Solving Laplace's equation directly gives Φ and the resulting electric field in a very straight forward manner.

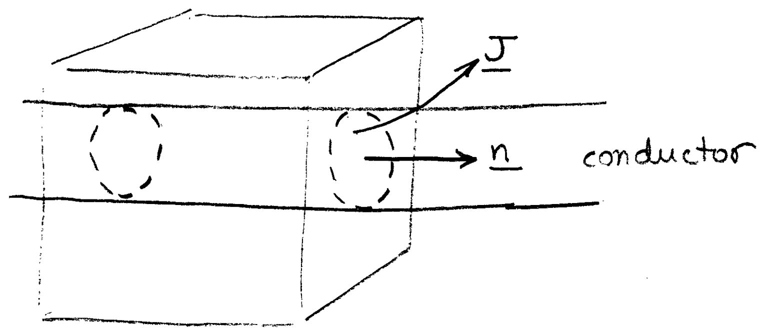
The major clue to using Laplace's equation is that the geometry of the problem and the potential(s) upon metallic (conducting) surfaces. The fundamental method is to find a Φ which satisfies $\nabla^2\Phi = 0$ and the B.C.'s. This is called a boundary value problem.

However, before we study the solution of Laplace's or Poisson's equation we need to study boundary conditions.

For conductors $\underline{J} = \sigma \underline{E}$ (this is Ohm's Law in a fields context)

units of amperes$/m^2$ units of Siemens$/m$

Draw a closed surface S around a conductor.



By conservation of charge the net current thru S gives the rate of change of Q_{TOTAL} within the volume

Mathematically, $$\oint \underline{J} \cdot d\underline{S} = -\frac{dQ_{TOTAL}}{dt} = -\frac{d}{dt}\int \rho \, dv = -\int \frac{\partial \rho}{\partial t} dv$$

sign due to charge leaving because of direction of dS

But, $$\oint \underline{J} \cdot d\underline{S} = \int \underline{\nabla} \cdot \underline{J} \, dv$$

$$\therefore \quad \underline{\nabla} \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot (\sigma \underline{E}) = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \epsilon \underline{E} = -\frac{\epsilon}{\sigma}\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \underline{D} = -\frac{\epsilon}{\sigma}\frac{\partial \rho}{\partial t}$$

$$\therefore \quad \rho = -\frac{\epsilon}{\sigma}\frac{\partial \rho}{\partial t}$$

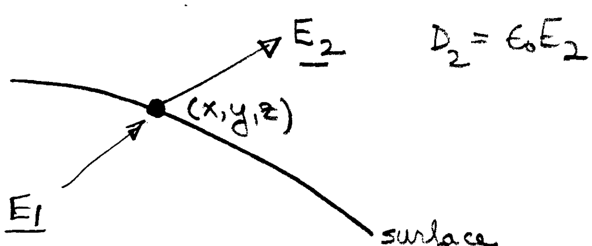
$$\frac{\partial \rho}{\partial t} + \left(\frac{\sigma}{\epsilon}\right)\rho = 0 \quad \Rightarrow \quad \rho = \rho_0 e^{-t/\tau}$$

for conductors $\tau = \frac{\sigma}{\epsilon} \approx 10^{-19}$ sec.

$\therefore$ There are no fields inside conductors

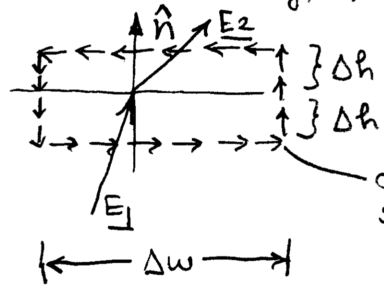
See Section 4.11 Inan & Inan, Electromagnetic Fields

Consider the case of $\underline{E}$ fields on different sides of a surface.



$\underline{D}_1 = \epsilon_0 \underline{E}_1 + \underline{P} = \epsilon \underline{E}_1$

Choose a contour about (x, y, z) as



$\hat{n}$ unit vector from $1 \to 2$

Since $\underline{E}$ is conservative $\oint \underline{E} \cdot d\underline{l} = 0$ (no time dependent B field)

Doing contour integral

$$E_{1n}\cancel{\Delta h} + E_{2n}\cancel{\Delta h} - E_{2t}\,\Delta w - \cancel{E_{2n}}\Delta h - \cancel{E_{1n}}\Delta h + E_{1t}\,\Delta w = 0$$

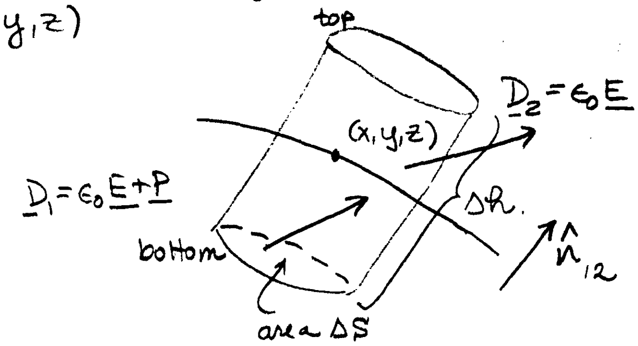
Assume Δh and Δw are small enough that $\underline{E}_1$ and $\underline{E}_2$ are constant on C leaving

$$E_{1t}\,\Delta w - E_{2t}\,\Delta w = 0$$

or $$E_{1t} = E_{2t}$$

Mathematically $\hat{n}_{12} \times (\underline{E}_2 - \underline{E}_1) = 0$

For the normal component of E choose a small volume centered on (x,y,z)

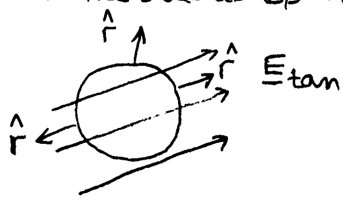


Using Gauss' Law
$$\oint_S \underline{D} \cdot d\underline{s} = \rho_f \, \Delta h \, \Delta S$$
any charge density in volume

$$\int_{top} \underline{D} \cdot d\underline{s} + \int_{sides} \underline{D} \cdot d\underline{s} + \int_{bottom} \underline{D} \cdot d\underline{s} = \rho_f \, \Delta h \, \Delta S$$

Consider the integral over the sides as $\Delta S \to 0$

top view



consider the integral $\underline{D} \cdot d\underline{s}$

$$\underline{D} \cdot d\underline{s} = \epsilon \underline{E}_{tan} \cdot \hat{r}$$

since $\underline{E}_{tan}$ approx. constant over small volume and $\hat{r}$ rotates as it goes around the surface then $\oint \underline{D} \cdot d\underline{s} \to 0$

This leaves
$$\int_{top} \underline{D} \cdot d\underline{s} + \int_{bottom} \underline{D} \cdot d\underline{s} = \rho_f \, \Delta h \, \Delta S$$

Since $\Delta h \, \Delta S \to 0$ we can assume $\underline{D}$ is constant on each surface

$$D_{2n} \cancel{\Delta S} - D_{1n} \cancel{\Delta S} = \rho_f \, \Delta h \, \cancel{\Delta S}$$

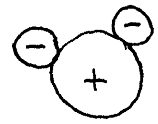
minus sign due to surface normals being in opposite direction

$$D_{2n} - D_{1n} = \rho_f \, \Delta h \to \rho_s$$

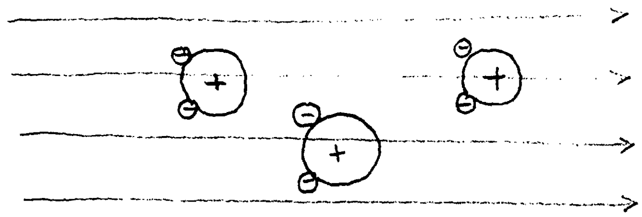
$$\hat{n}_{12} \cdot (\underline{D}_2 - \underline{D}_1) = \rho_s$$

Polarization

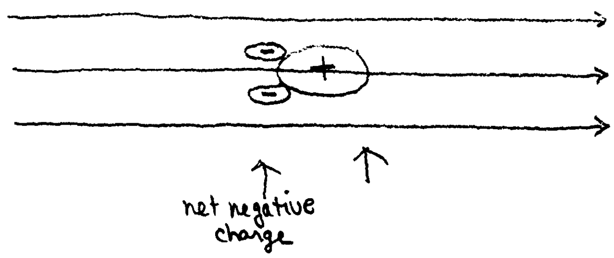
dielectric - insulating material in which charge separation occurs at the microscopic level due to an applied $\underline{E}$ field



consider a water molecule



alignment with applied fields



notice also that field can stretch the atoms

(this creates electric dipoles)

define a macroscopic average

$$\underline{P} \triangleq \lim_{\Delta v \to 0} \frac{\sum_i \underline{p}_i}{\Delta v}$$

dipole moment

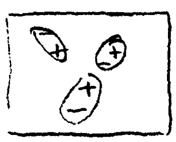
Note: $\underline{p} = Qd\,\hat{z}$

$$= N\underline{p}$$

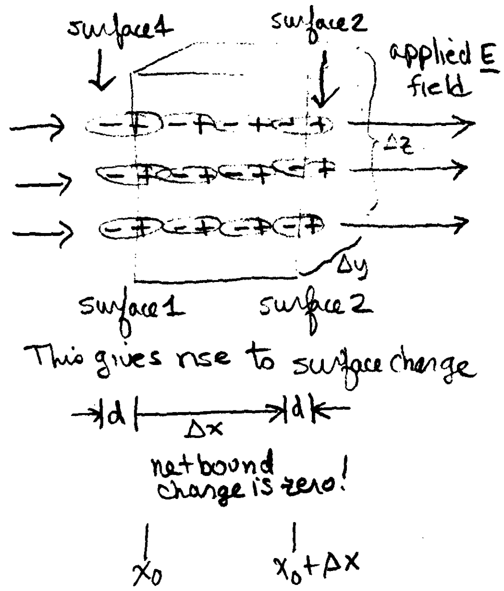
number density of dipoles

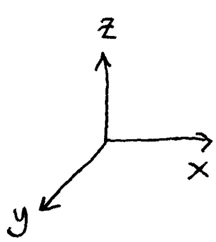
in bulk materials

randomly distributed



align with applied field





The charge on each surface is given by

$$\underbrace{dq_1}_{\text{surface \#1}} = -\underbrace{(Nqd)}_{\substack{\text{dipole}\\\text{density}}}\Big|_{x_0} \underbrace{\Delta y \Delta z}_{\text{volume of surface layer.}}$$ (assuming aligned with field)

Relate this charge to polarization

We have to relate this to bound charge

$$dq_1 = +(Nqd)\Big|_{x_0} \Delta y \Delta z$$ multiply $\underline{P}$ by surface area to get Q at surface 1.

$$P_x(x_0) = Nqd\Big|_{x_0}$$ using $\underline{P} = N\underline{p} = Nq\underline{d}$ polarization is a linear quantity, i.e. dipole moment

$$P_x(x_0+\Delta x) = Nqd\Big|_{x_0+\Delta x}$$ at other surface

The net bound charge is then (this is for volume).

$$dq_{total} = \underbrace{P_x(x_0)\,\Delta y\,\Delta z}_{\substack{\text{note sign is +}\\\text{since this is}\\\text{bound charge}}} - \underbrace{P_x(x_0+\Delta x)\,\Delta y\,\Delta z}_{\substack{\text{this sign is -}\\\text{since it is bound}}}$$

$$= -\frac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x}\underbrace{\Delta x\,\Delta y\,\Delta z}_{\Delta v}$$

The bound volume charge density is then given by

$$\rho_b = \lim_{\Delta v \to 0} \frac{dq_{total}}{\Delta v} = \lim_{\Delta v \to 0} -\frac{P_x(x_0+\Delta x) - P_x(x_0)}{\Delta x}$$

$$= -\frac{dP_x}{dx}\Big|_{x_0}$$

Extending to three dimensions the bound charge is given by

$$\rho_b = -\underline{\nabla}\cdot\underline{P}$$

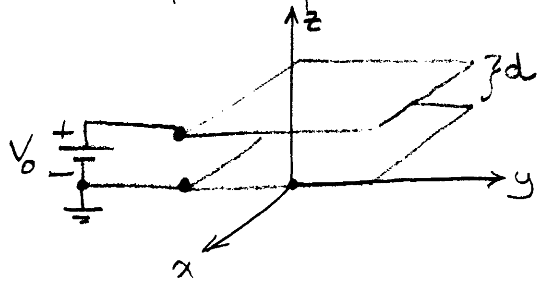
This can be written at the macroscopic level

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$$= \epsilon_0 \underline{E} + \chi_e \epsilon_0 \underline{E}$$

$$= \epsilon \underline{E}$$

Example 4-23: Two parallel plates



Find $\Phi(x,y,z)$ and $\underline{E}(x,y,z)$ between the plates.

In rectangular coordinates Laplace's equation is

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \quad \text{since charge free}$$

From symmetry no dependence on x or y

$$\therefore \quad \nabla^2\Phi = \frac{\partial^2\Phi}{\partial z^2} = 0$$

$$\Rightarrow \Phi(z) = C_1 z + C_2 \quad \text{where } C_1, C_2 \text{ come from the B.C.'s}$$

$$\Phi(0) = 0 \qquad \Phi(+d) = +V_0 \qquad \text{since connected to battery}$$

use potential B.C.'s

$$\Phi(0) = 0 = C_2$$

$$\Phi(+d) = +V_0 = C_1 d \quad \therefore C_1 = \frac{V_0}{d}$$

$$\Phi(z) = \frac{V_0}{d} z$$

$$\underline{E} = -\underline{\nabla}\Phi(z) = -\hat{z}\frac{\partial\Phi}{\partial z} = -\hat{z}\frac{V_0}{d}$$

To find the charge densities on the plates we use the boundary conditions

$$\rho_s = \hat{n}_{12}\cdot(\underline{D_2} - \underline{D_1})$$

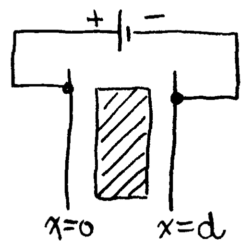
at $z=0$ $\quad \hat{n}_{12} = +\hat{z} \quad \underline{D_1} = 0 \quad D_2 = \epsilon_0 \underline{E} = -\epsilon_0 \frac{V_0}{d}\hat{z}$

$$\rho_s(z=0) = \hat{z}\cdot\hat{z}\left(-\epsilon_0\frac{V_0}{d}\right) = -\frac{\epsilon_0 V_0}{d}$$

at $z=d$ $\quad \hat{n}_{12} = +\hat{z} \quad \underline{D_1} = -\hat{z}\frac{V_0}{d} \quad \underline{D_2} = 0$

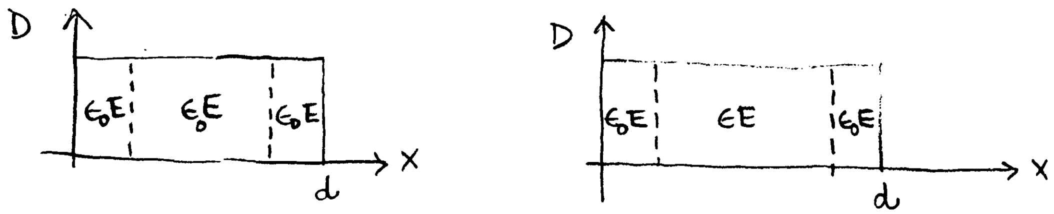
$$\rho_s(z=0) = \hat{z}\cdot\left(0 - \left(-\frac{\epsilon_0 V_0}{d}\hat{z}\right)\right) = +\epsilon_0\frac{V_0}{d}$$

Now lets go back to the parallel plate capacitor with a dielectric, polarizable block between the plates.

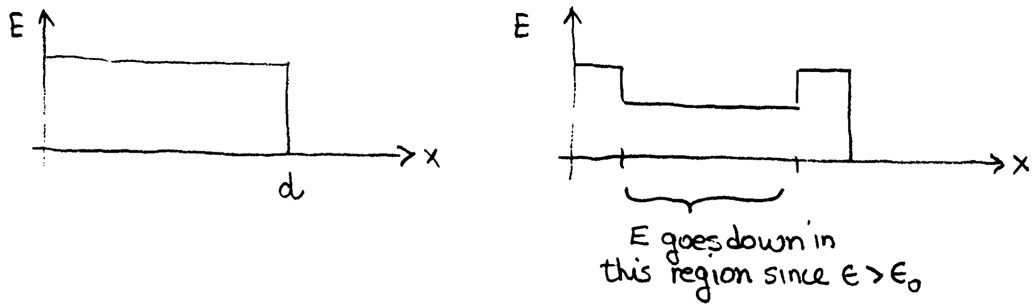


capacitor with no block — with dielectric block

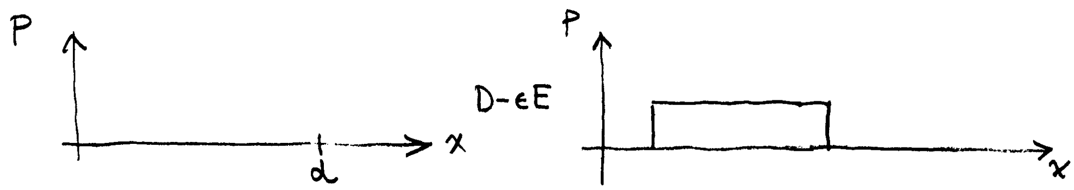
since normal D is continuous



but E is not continuous

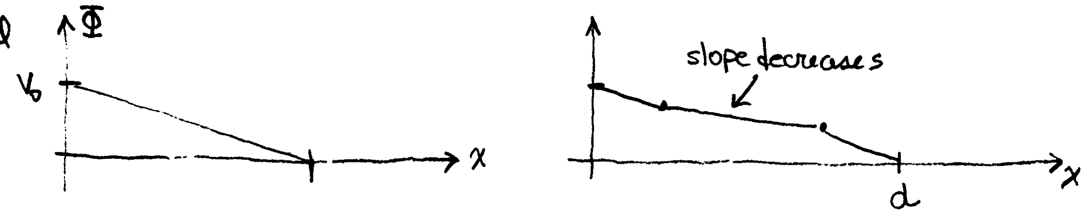


The difference between E and D is P

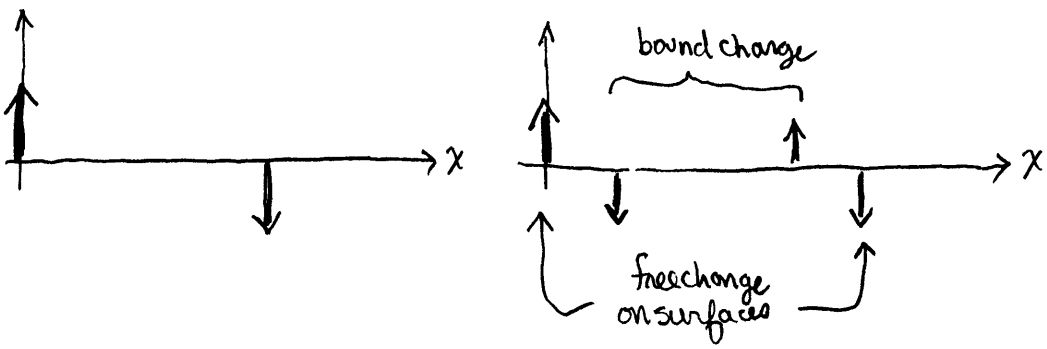


Φ is the integral of E

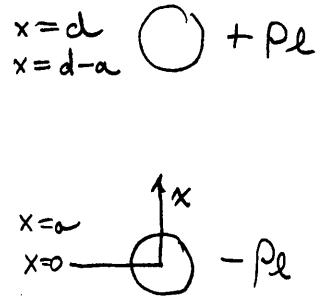
$$E = -\frac{d\Phi}{dx}$$



derivative of E gives surface charge density



Example 4-29. Capacitance of a Two-Wire Line



Very useful problem for telephony, radio transmission lines, and power transmission.

General solution is complicated because "proximity effect" causes charge densities to be larger on facing sides

We will solve for $d \gg a$ to avoid this effect

This is a very difficult problem to solve for the potential directly.

See Inan & Inan, Engineering Electromagnetics, Example 4-11

The potential from a finite length of charge $-\ell$ to $+\ell$

is given by

$$\Phi(\underline{P}) = -\frac{\rho_\ell}{4\pi\epsilon_0} \ln \left[\frac{z-a+\left[r^2+(z-a)^2\right]^{\frac{1}{2}}}{z+a+\left[r^2+(z+a)^2\right]^{\frac{1}{2}}} \right]$$

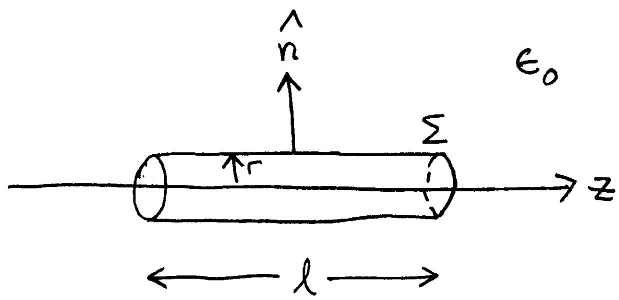
You could compute $\underline{E} = -\underline{\nabla}\Phi$ from this function but it is VERY complex as shown in this example.

Furthermore, we are interested in infinite length lines where $\ell \to \infty$

See Paris & Hurd, Basic Electromagnetic Theory, Example 3-3

As $\ell \to \infty$ the arguement of the $\ln[\;]$ function increases without limit and Φ is undefined. The problem is that our previous expressions for Φ, i.e. $\Phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho}{r}\,dv$, work only for bounded charge distributions.

The best method to find the potential associated with an infinite line charge is the integral form of Gauss' Law.



$$\oint_S \underline{D}\cdot\hat{n}\, ds = \int \rho\, dv = Q_{enclosed}$$

By symmetry $\quad D_r \cdot 2\pi r \cdot l = \rho_l l \quad$ where ρ_l is the line charge density
(ends cancel)

$$D_r = \frac{\rho_l}{2\pi r}$$

$$\underline{D} = \frac{\rho_l}{2\pi r}\hat{r}$$

$$\underline{E} = \frac{\rho_l}{2\pi\epsilon_0 r}\hat{r}$$

Since $\underline{E} = -\underline{\nabla}\Phi$ and the field is circularly symmetric, i.e.

$\frac{\partial\Phi}{\partial\phi} \to 0$ and $\frac{\partial\Phi}{\partial z} = 0$ since it does not matter where we put our origin.

$$\frac{\rho_l}{2\pi\epsilon_0 r}\hat{r} = \underline{E} = -\underline{\nabla}\Phi = -\frac{\partial\Phi}{\partial r}\hat{r}$$

or, $\quad \Phi(r) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r + c \quad$ after integrating

If we pick a reference potential $\Phi(r=r_0)=0$
then $0 = \Phi(r=r_0) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r_0 + c$

and $\Phi(r) = \frac{\rho_l}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right)$

If we use this expression for the E field between the two conductors we can write an expression for the E field between the two conductors as

$$E_x(x,0,0) = \frac{-\rho_\ell}{2\pi\epsilon_0 x} \overset{\text{direction is } -\hat{x}}{\underset{\downarrow}{-}} \frac{+\rho_\ell}{2\pi\epsilon_0 \underbrace{(d-x)}_{\text{positive}}}$$

ignores a

We will need to use the most general definition of capacitance

$$C \triangleq \frac{Q}{\Phi_{12}} = \frac{\oint \underline{D}\cdot \underline{ds}}{-\int_L \underline{E}\cdot \underline{dl}}$$

Gauss Law

defin of potential

Since these are infinite lines Q is readily given as ρ_ℓ, the charge per unit length.

Φ_{12} can be integrated as follows (remember we can use <u>any path</u>)

$$\Phi_{12} = -\frac{1}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{-\rho_\ell}{x} - \frac{\rho_\ell}{d-x}\right]dx$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{1}{x} - \frac{1}{d-x}\right]dx = \frac{\rho_\ell}{2\pi\epsilon_0}\Big[\ln x - \ln(d-x)\Big]_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left(\frac{x}{d-x}\right)\Bigg|_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\left[\ln\left(\frac{d-a}{+a}\right) - \ln\left(\frac{a}{d-a}\right)\right] = \frac{\rho_\ell}{2\pi\epsilon_0}\, 2\ln\left(\frac{d-a}{a}\right)$$

and, for $d \gg a$,

$$\Phi_{12} \cong \frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)$$

We can compute the capacitance <u>per unit length</u> as

$$C \cong \frac{\rho_\ell}{\frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)} = \frac{\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}$$

As an example, a 115 kV transmission line uses two 1.407cm aluminum conductors separated by 3 meters, or

$$C \cong \frac{\pi(8.854\times10^{-12}\ F/m)}{\ln\left(\frac{3}{.01407}\right)} = 5.19\ nF/m$$

<u>Hint for future problems</u>

You can compute C for conductor configurations for which you have derived $\underline{E}$ or Φ. For example, a single cylindrical conductor above a ground plane is that of this twin line with an infinitely large conducting sheet between the two conductors. You could also use the "method of images."

Energy stored in fields

Consider the work done in assemblying a collection of charges.

Let $\Phi(\infty)=0$

Bring Q_1 in from infinity. No work done because no initial fields.

Now bring in Q_2. Work will be done against the field of Q_1.

$$W_{21} = Q_2 V_{21} \quad \text{work done in bringing } Q_2 \text{ in to } P_2$$

Bring in Q_3

$$W_{31} + W_{32} = Q_3 V_{31} + Q_3 V_{32}$$

For N charges the work necessary to assemble these charges is

If there were more than 3 charges

$$W_e = W_{21} + (W_{31} + W_{32}) \quad \{+ \cdots \quad \}$$

Order is not important so let's assemble in reverse order, i.e. Q_3 first

$$W_e = W_{23} + (W_{12} + W_{13})$$

Add these results together to get a general result

$$2W_e = (W_{12} + W_{21}) + (W_{31} + W_{13} + W_{32} + W_{23}) \{+ \cdots\}$$

Re-writing in terms of Q and V

$$2W_e = Q_1 V_{12} + Q_2 V_{21} + Q_3 V_{31} + Q_1 V_{13} + Q_3 V_{32} + Q_2 V_{23} + \cdots$$

Re-arranging

$$2W_e = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32}) + \cdots$$

Generalizing to N charges

$$2W_e = \sum_{i=1}^{N} Q_i V_i \quad \text{where} \quad V_i = \sum_{\substack{j=1 \\ i \neq j}}^{N} V_{ij}$$

sum over all charges

total potential relative to all <u>other</u> charges

For a discrete system the total electrostatic energy is

$$W_e = \frac{1}{2} \sum_{i=1}^{N} Q_i V_i$$

Generalizing to continuous charge distributions

$$W_e = \frac{1}{2} \int \rho V \, dv$$

This can be put into a more conventional form by recognizing that $\underline{\nabla} \cdot \underline{D} = \rho$

$$W_e = \frac{1}{2} \int \underline{\nabla} \cdot \underline{D} \, V \, dv$$

Using a vector identity

$$W_e = \frac{1}{2} \int \left[\underline{\nabla} \cdot (V \underline{D}) - D \cdot \nabla V \right] dv$$

$$= \underbrace{\frac{1}{2} \int \nabla \cdot V \underline{D} \, dv}_{\text{convert to a surface integral at } r = \infty} - \frac{1}{2} \int D \cdot \nabla V \, dv$$

$$\oint_{r=\infty} V \underline{D} \cdot dS = 0 \quad \text{since } V(r=\infty) = 0$$

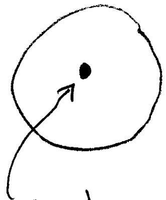
$$\therefore \quad W_e = -\frac{1}{2} \int D \cdot \nabla V \, dv$$

$$= +\frac{1}{2} \int \underline{D} \cdot \underline{E} \, dv$$

$$W_e = +\frac{1}{2} \int \epsilon \, |\underline{E}|^2 \, dv$$

Example: Assemble an atom

electron cloud $-Q$



nucleus of $+Q$

Assume $\rho = \begin{cases} \rho_0 & r < R \\ 0 & r \geq R \end{cases}$

as a simple charge distribution model.

Oak Ridge
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First, let's find the $\underline{E}$ field by Gauss' Law

$$\underline{E} = \begin{cases} \dfrac{r\rho_0}{3\epsilon}\,\hat{r} & 0 \leq r \leq R \\[2ex] \dfrac{R^3\rho_0}{3\epsilon r^2}\,\hat{r} & r > R \end{cases}$$

Then, we integrate to find the energy required to assemble this atom.

$$W_e = \frac{1}{2}\int \epsilon |\underline{E}|^2 dv = \frac{1}{2}\epsilon \left[\underbrace{\int_0^R \int_0^\pi \int_0^{2\pi} \frac{r^2\rho_0^2}{9\epsilon^2} r^2 \sin\theta \, d\theta \, d\phi \, dr}_{\text{inside the electron cloud}} \right.$$

$$\left. + \underbrace{\int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{R^6\rho_0^2}{9\epsilon^2 r^4} r^2 \sin\theta \, d\theta \, d\phi \, dr}_{\text{outside the cloud}} \right]$$

$$W_e = \frac{1}{2}\epsilon \left[\int_0^R \frac{r^2\rho_0^2}{9\epsilon^2} 4\pi r^2 dr + \int_R^\infty \frac{R^6\rho_0^2}{9\epsilon^2 r^4} 4\pi r^2 dr \right]$$

$$W_e = \frac{1}{2}\epsilon \left[\frac{4\pi\rho_0^2}{9\epsilon^2}\frac{R^5}{5} + \frac{4\pi R^6\rho_0^2}{9\epsilon^2}\frac{1}{R} \right] = \frac{1}{2}\epsilon \frac{4\pi\rho_0^2}{9\epsilon^2}\left[\frac{R^5}{5} + R^5 \right]$$

$$W_e = \frac{2\pi\rho^2}{9\epsilon}\left[\frac{6R^5}{5} \right]$$

The charge inside the sphere is $Q = \frac{4}{3}\pi R^3 \rho_0$

Then,

$$W_e = \frac{2\pi\rho^2}{9\epsilon}\frac{6R^5}{5} = \frac{3Q^2}{20\pi\epsilon R}$$

So far we have computed the energy required to assemble the electron cloud.

Now we have to bring in the $+Q$ nucleus using our original definition, i.e.,

$$W_e = QV$$

For a spherical charge distribution

$$\Phi = \begin{cases} \dfrac{-3Q}{8\pi\epsilon R^3}\left(R^2 - \dfrac{r^2}{3}\right) & r < R \\ \dfrac{-Q}{4\pi\epsilon R} & r > R \end{cases}$$

Using the above definition

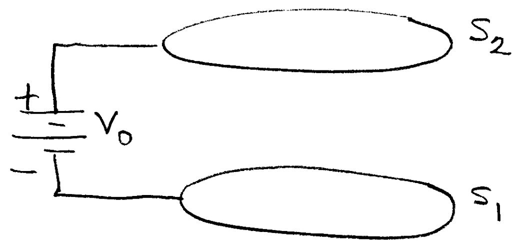
$$W_e = Q\Phi(r=0) = Q\left(\frac{-3Q}{8\pi\epsilon R}\right) = -\frac{3Q^2}{8\pi\epsilon R}$$

Therefore, the total energy required to assemble the atom is

$$W_e(\text{total}) = \underbrace{\frac{3Q^2}{20\pi\epsilon R}}_{\text{electron cloud}} - \underbrace{\frac{3Q^2}{8\pi\epsilon R}}_{\text{nucleus}}$$

$$W_e(\text{total}) = -\frac{9Q^2}{40\pi\epsilon R} \qquad \left(\text{stable since } W_e < 0\right)$$

Very important simple result : capacitor



$$W_e = \frac{1}{2}\int \rho_s V_1 \, dS_1 + \frac{1}{2}\int \rho_s V_2 \, dS_2$$

$$W_e = \frac{1}{2} V_1 \int \rho_s \, dS_1 + \frac{1}{2} V_2 \int \rho_s \, dS_2$$

since S_1, S_2 are equipotential surfaces

$$W_e = \frac{1}{2} V_1 \, (-Q) + \frac{1}{2} V_2 \, (+Q)$$

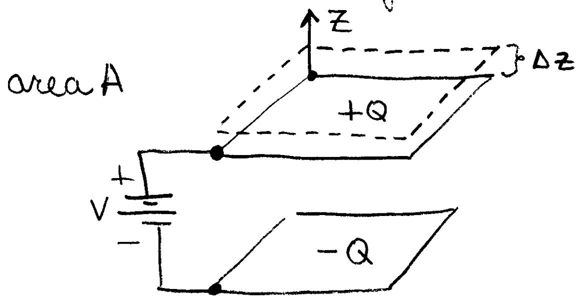
$$W_e = \frac{1}{2} Q \, (V_2 - V_1) = \frac{1}{2} Q V_0$$

but $Q = CV$ for a capacitor

$$W_e = \frac{1}{2} C V^2$$

Principle of virtual work

This is very much related to the earlier discussion of energy of electrostatic systems.



$$W_e = \frac{1}{2} C V^2$$

$$W_e = \frac{1}{2}\left(\frac{\epsilon A}{d}\right) V^2$$

Imagine a small displacement Δz of the upper plate performing work in the z-direction

To move in $+z$ a distance Δz as shown

$$\text{Work} = -F\Delta z$$ where F is the electrostatic force to be overcome

From conservation of energy this mechanical work must go into stored (i.e., potential) energy of the field

$$\Delta W_e = \underbrace{\frac{\epsilon_0}{2} E^2}_{\text{energy density between plates}} \underbrace{A\Delta z}_{\text{volume change associated with } \Delta z}$$

$$\therefore \quad -F\Delta z = \frac{\epsilon_0}{2} E^2 A \Delta z$$

$$\frac{\underline{F}}{A} = -\frac{\epsilon_0 |E|^2}{2} \hat{z}$$

Laplace's Equation solved using separation of variables

$$\nabla^2 \Phi = 0$$

Assume $\Phi = f(u)\, g(v)$

Then $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = g \frac{\partial^2 f}{\partial u^2} + f \frac{\partial^2 g}{\partial v^2} = 0$

Divide by fg to get

$$\frac{1}{f} \frac{\partial^2 f}{\partial u^2} + \frac{1}{g} \frac{\partial^2 g}{\partial v^2} = 0$$

For this to be true the first term must equal the second term independent of variables, i.e.

$$\frac{1}{g} \frac{\partial^2 g}{\partial v^2} = - \frac{1}{f} \frac{\partial^2 f}{\partial u^2} \quad \text{independent of } u \text{ and } v$$

This can be written as

$$\frac{1}{f} \frac{\partial^2 f}{\partial u^2} = k_u^2 \quad \text{where } k_u \text{ is a constant}$$

and $\frac{1}{g} \frac{\partial^2 g}{\partial v^2} = k_v^2$ where $k_u^2 = -k_v^2$

These are differential equations which need to have the boundary conditions specified. You can specify either Φ or $\frac{\partial \Phi}{\partial n}$

Φ Dirichlet boundary condition

$\frac{\partial \Phi}{\partial n}$ Neumann boundary condition

Since $D_{normal} = \epsilon E_{normal} = -\epsilon \frac{\partial \Phi}{\partial n}$

equivalent to specifying charge density.

mixed combination of Dirichlet & Neumann

The solution is dependent upon the actual value of k.

If $k_u^2 > 0$

$$\frac{d^2f}{du^2} - k_u^2 f = 0$$

$$f(u) = c_1 e^{+k_u u} + c_2 e^{-k_u u}$$

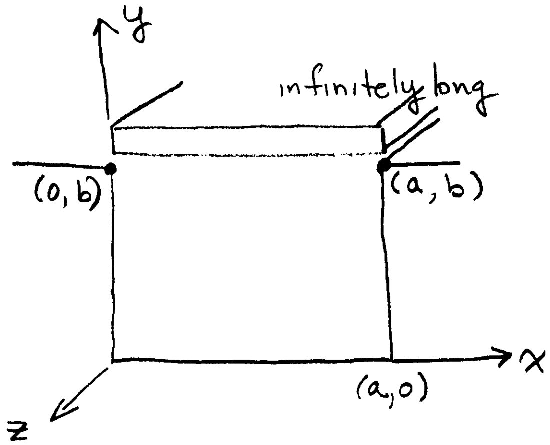
If $k_u^2 < 0$

$$\frac{d^2f}{du^2} + k_u^2 f = 0$$

$$f(u) = c_1 \sin k_u u + c_2 \cos k_u u$$

If $k_u^2 = 0$

$$f(u) = c_1 + c_2 u$$



Use these boundary conditions

$$\Phi(0,y) = 0$$
$$\Phi(x,0) = 0$$
$$\Phi(a,y) = 0$$
$$\Phi(x,b) = V \sin \frac{\pi x}{a}$$

Let $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}$ and further let $\Phi = f(x) g(y)$

As before $\quad g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0$

and dividing $\quad \frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = 0$

Separate to get

$$\frac{1}{f}\frac{d^2 f}{dx^2} = k_x^2 \quad \text{and} \quad \frac{1}{g}\frac{d^2 g}{dy^2} = 0 \quad \text{where } k_x^2 + k_y^2 = 0$$

Which one is positive depends upon the boundary conditions. Since $\Phi(x,b) = V\sin\frac{\pi x}{a}$ we want the x solution to be sinusoidal, or $k_x^2 < 0$.

$$\frac{d^2 f}{dx^2} - k_x^2 f = 0 \Rightarrow f = c_1 \sin k_x x + C_2 \cos k_x x$$

This requires $k_y^2 > 0$ and

$$\frac{d^2 g}{dy^2} + k_y^2 g = 0 \Rightarrow g = c_3 e^{+k_y y} + c_4 e^{-k_y y}$$

$$\Phi(x,y) = (c_1 \sin k_x x + c_2 \cos k_x x)(c_3 e^{+k_y y} + c_4 e^{-k_y y})$$

Since we want $\Phi(x,b) = V \sin \frac{\pi x}{a}$
this allows us to set $c_2 = 0$ and pick $k_x = \frac{\pi}{a}$

Then

$$\Phi(x,y) = (\sin k_x x)(c_3 e^{+k_y y} + c_4 e^{-k_y y})$$

At this point the problem becomes interesting

You cannot use c_3 and c_4 to make $\Phi(0,y) = 0$
and $\Phi(a,y) = 0$ using exponentials.
Solution use the sine function

For $x = 0$ $\sin k_x x = 0$ always

For $x = a$ $\sin k_x a = 0$ when $k_x a = n\pi$ $n = 1, 2, 3$

Using these results we can write $\Phi(x,y)$ as

$$\Phi(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)\left[c_3 e^{-\frac{n\pi}{a} y} + c_4 e^{+\frac{n\pi}{a} y}\right]$$

since $k_y^2 = \left(\frac{n\pi}{a}\right)^2$ Note that my sign is correct since if $k_x^2 < 0$ for sinusoidal solutions, then $k_y^2 > 0$.

At $y = 0$ $\Phi(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{a} x\right)(c_3 + c_4)$

Since this is independent of x $c_3 = -c_4$.

$$\therefore \Phi(x,y) = \sum_{n=1}^{\infty} c \sin\left(\frac{n\pi}{a} x\right)\left[e^{-\frac{n\pi}{a} y} - e^{+\frac{n\pi}{a} y}\right]$$

Now look at $y=b$ where $\Phi(x,b) = V \sin \frac{\pi x}{a}$

$$\Phi(x,b) = \sum_{n=1}^{\infty} c \sin\left(\frac{n\pi x}{a}\right)\left[e^{-\frac{n\pi}{a}b} - e^{+\frac{n\pi}{a}b}\right] = V\sin\left(\frac{\pi x}{a}\right)$$

Now this is only true if $n=1$ where

$$c \cancel{\sin\left(\frac{\pi x}{a}\right)}\left[e^{-\frac{\pi b}{a}} - e^{+\frac{\pi}{a}b}\right] = V\cancel{\sin\left(\frac{\pi x}{a}\right)}$$

$$\therefore \quad c = \frac{V}{e^{-\frac{\pi b}{a}} - e^{+\frac{\pi b}{a}}} = \frac{V}{2\sinh\left(\frac{\pi b}{a}\right)}$$

Final solution,

$$\Phi(x,y) = c \sin\left(\frac{\pi}{a}x\right)\left[e^{-\frac{\pi}{a}y} - e^{+\frac{\pi}{a}y}\right]$$