EECS 412 Electromagnetic Fields III
Fall 2003

Homework #2:

Due September 29th

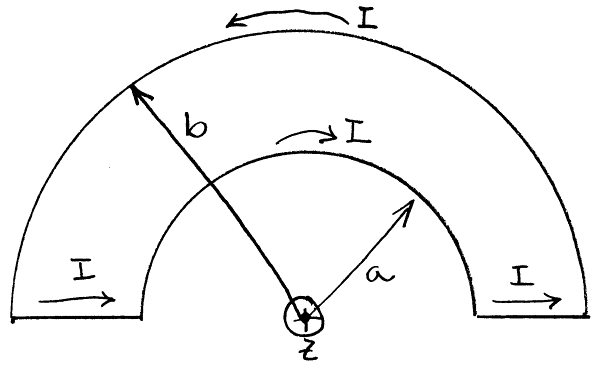
Reference Ramo, Whinnery, Van Duzer, Fields and Waves in Communications Electronics, 3rd Edition, Chapter 2.

Biot-Savart	2.3a
Helmholtz coils	2.3e (not as simple as it looks)
Field from a current distribution	2.4e
Inductance of a coaxial line	2.5

The following will require additional references and are not restricted to subjects covered in class.

Eddy currents	1 (attached)
Induced voltages	2,3 (attached)
Mutual inductance	4 (attached)
Solenoid & magnetic forces	5 (attached)

2.3a A loop of wire is formed by two semicircles, the inner of radius a and the outer of radius b, joined by radial line segments at $\phi = 0$ and $\phi = \pi$ (see Figure below). Find the magnetic field at the origin.



This is a direct application of the Biot-Savart Law

$$dH = \frac{I\, d\underline{l} \times \hat{R}}{4\pi R^2}$$

By inspection the two radial sections do not contribute since $d\underline{l} \times \hat{r} = 0$ for these sections.

For the outer loop $\quad \underline{H}_o = \int_0^{\pi} \frac{+I\, b\, d\phi\, \hat{\phi} \times (-\hat{R})}{4\pi b^2} = \int_0^{\pi} \frac{+I}{4\pi b}\, d\phi\, \hat{z} = +\frac{\pi I}{4\pi b}\, \hat{z}$

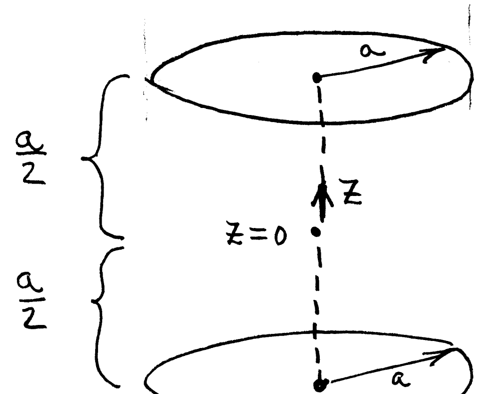
$$\underline{H}_o = +\frac{I}{4b}\,\hat{z}$$

For the inner loop $\quad \underline{H}_i = \int_0^{\pi} \frac{-I a\, d\phi\, \hat{\phi} \times (-\hat{R})}{4\pi a^2} = \int_0^{\pi} \frac{-I}{4\pi a}\, d\phi\, \hat{z} = -\frac{\pi I}{4\pi a}\,\hat{z}$

$$\underline{H}_i = -\frac{I}{4a}\,\hat{z}$$

$$\underline{H}_{total} = \underline{H}_o + \underline{H}_i = \frac{I}{4b}\,\hat{z} - \frac{I}{4a}\,\hat{z} = I\left(\frac{1}{4b} - \frac{1}{4a}\right)\hat{z}$$

2.3e An arrangement that can provide a region of relatively uniform fields consists of a pair of parallel, coaxial loops; the uniform-field region is on the axis midway between the loops. Show that the axial magnetic field, expressed as a Taylor series expansion along the axis about the point midway between the coils, will have a zero first, second, and third derivatives if the loop radii are equal to the spacing d of the loops. This is the so-called Helmholtz configuration.



To solve this problem we can use our previous results for the on-axis field from a current loop, i.e., p. 31 of class notes.

$$H_z = \frac{Ia^2}{2(a^2+z^2)^{3/2}}$$

For the upper loop the $\underline{H}$ field at $z=0$ will be given by.

$$\underline{H}_u(z=0) = \hat{z}\,\frac{Ia^2}{2\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{3/2}}$$

The corresponding $\underline{H}$ field from the lower loop will be given by.

$$H_L(z=0) = \hat{z}\,\frac{Ia^2}{2\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{3/2}}$$

$$H_{TOT,z}(z=0) = H_U(z=0) + H_L(z=0) = \hat{z}\,\frac{Ia^2}{2}\left[\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-3/2} + \left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-3/2}\right]$$

The H_{TOT} as a Taylor series expansion looks like

$$H_{TOT,z}(\Delta z) = H_{TOT,z}(z=0) + \Delta z\,\frac{\partial H_{TOT,z}}{\partial z}\bigg|_{z=0} + \frac{\Delta z^2}{2!}\,\frac{\partial^2 H_{TOT,z}}{\partial z^2}\bigg|_{z=0} + \frac{\Delta z^3}{3!}\,\frac{\partial^3 H_{TOT,z}}{\partial z^3}\bigg|_{z=0} + \ldots$$

The derivatives are calculated as

$$\frac{dH_{TOT,z}}{dz} = \frac{Ia^2}{2}\left[-\frac{3}{2}\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}}(2)\left(\frac{a}{2}+z\right)(1) - \frac{3}{2}\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}}2\left(\frac{a}{2}-z\right)(-1)\right]$$

$$\frac{dH_{TOT,z}}{dz} = \frac{3Ia^2}{2}\left[-\left(\frac{a}{2}+z\right)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} + \left(\frac{a}{2}-z\right)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}}\right]$$

$$\frac{dH_{TOT,z}}{dz} = \frac{3Ia^2}{2}\left[\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}}(-2z)\right]$$

$$\frac{d^2H_{TOT,z}}{dz^2} = \frac{3Ia^2}{2}\Bigg[-(1)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} - \left(\frac{a}{2}+z\right)\left(-\frac{5}{2}\right)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{7}{2}}(2)\left(\frac{a}{2}+z\right)(1)$$
$$+(-1)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} + \left(\frac{a}{2}-z\right)\left(-\frac{5}{2}\right)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{7}{2}}(2)\left(\frac{a}{2}-z\right)(-1)\Bigg]$$

$$\frac{d^2H_{TOT,z}}{dz^2} = \frac{3Ia^2}{2}\Bigg[-\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}+z\right)^2\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{7}{2}}$$
$$-\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}-z\right)^2\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{7}{2}}\Bigg].$$

$$\frac{d^3H_{TOT,z}}{dz^3} = \frac{3Ia^2}{2}\Bigg[-\left(-\frac{5}{2}\right)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{7}{2}}(2)\left(\frac{a}{2}+z\right)(1) + 5(2)\left(\frac{a}{2}+z\right)(1)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{7}{2}}$$
$$+5\left(\frac{a}{2}+z\right)^2\left(-\frac{7}{2}\right)\left(a^2+\left(\frac{a}{2}+z\right)^2\right)^{-\frac{9}{2}}(2)\left(\frac{a}{2}+z\right)(1)$$
$$-\left(-\frac{5}{2}\right)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{7}{2}}(2)\left(\frac{a}{2}-z\right)(-1) + 5(2)\left(\frac{a}{2}-z\right)(-1)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{7}{2}}$$
$$+5\left(\frac{a}{2}-z\right)^2\left(-\frac{7}{2}\right)\left(a^2+\left(\frac{a}{2}-z\right)^2\right)^{-\frac{9}{2}}(2)\left(\frac{a}{2}-z\right)(-1)\Bigg]$$

Now we can evaluate each derivative at $z=0$

$$\left.\frac{dH_{TOT,z}}{dz}\right|_{z=0} = \frac{3Ia^2}{2}\left[\left(a^2+\left(\frac{a}{2}\right)^2\right)^{-\frac{5}{2}}(0)\right] = 0$$

$$\left.\frac{d^2H_{TOT,z}}{dz^2}\right|_{z=0} = \frac{3Ia^2}{2}\left[-\left(a^2+\frac{a^2}{4}\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}\right)^2\left(a^2+\frac{a^2}{4}\right)^{-\frac{7}{2}} - \left(a^2+\frac{a^2}{4}\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}\right)^2\left(a^2+\frac{a^2}{4}\right)^{-\frac{7}{2}}\right]$$

$$= \frac{3Ia^2}{2}\left[-\left(\frac{5a^2}{2^2}\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}\right)^2\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}} - \left(\frac{5a^2}{4}\right)^{-\frac{5}{2}} + 5\left(\frac{a}{2}\right)^2\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}}\right]$$

$$= \frac{3Ia^2}{2}\left[-5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5} + 5\left(\frac{a}{2}\right)^2 5^{-\frac{7}{2}}\left(\frac{a}{2}\right)^{-7} - 5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5} + 5\left(\frac{a}{2}\right)^2 5^{-\frac{7}{2}}\left(\frac{a}{2}\right)^{-7}\right]$$

$$\left.\frac{d^2H_{TOT,z}}{dz^2}\right|_{z=0} = \frac{3Ia^2}{2}\left[-5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5} + 5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5} - 5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5} + 5^{-\frac{5}{2}}\left(\frac{a}{2}\right)^{-5}\right] = 0$$

$$\left.\frac{d^3H_{TOT,z}}{dz^3}\right|_{z=0} = \frac{3Ia^2}{2}\left[5\left(\frac{a}{2}\right)\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}} + 10\left(\frac{a}{2}\right)\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}} - 35\left(\frac{a}{2}\right)^3\left(\frac{5a^2}{2^2}\right)^{-\frac{9}{2}}\right.$$
$$\left. - 5\left(\frac{a}{2}\right)\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}} - 10\left(\frac{a}{2}\right)\left(\frac{5a^2}{2^2}\right)^{-\frac{7}{2}} + 35\left(\frac{a}{2}\right)^3\left(\frac{5a^2}{2^2}\right)^{-\frac{9}{2}}\right]$$

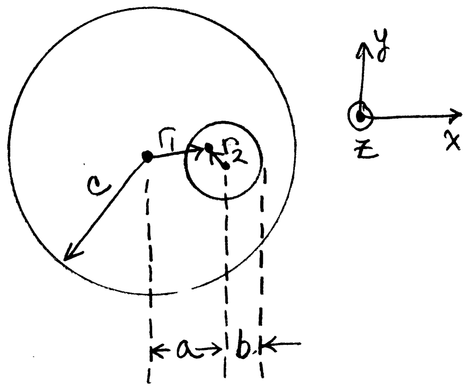
$$\left.\frac{d^3H_{TOT,z}}{dz^3}\right|_{z=0} = 0$$

Note that $\left.\frac{d^4H_{TOT,z}}{dz^4}\right|_{z=0} \neq 0$ although I will not show that here.

See also Problem 6-16 in Exam #1, EECS 412, Fall 2002.

2.4e Consider a round straight wire carrying a uniform current density J throughout, except for a round cylindrical void parallel with the wire axis so that the cross section is constant. Call the radius of the wire c, the radius of the hole b, and the distance of the center of the hole from the center of the wire a. Take $b<a<c$ and $b<c<-a$. Use superposition to find the field $\underline{H}$ as a function of position along a radial line through the center of the hole for all values of radius from the center of the wire.

This problem is the magnetic equivalent of 4.24



Assume a second current density $-J$ inside the cylindrical void such that $J_{TOTAL} = J - J = 0$ there.

We then find the H field from the two current densities in cylindrical coordinates and add the field contributions making appropriate adjustments for their different centers.

Using Ampère's Law for $r > r_0$

$$\oint_C \underline{H} \cdot \underline{dl} = H_\theta \, 2\pi r$$

$$\int_S \underline{J} \cdot \hat{n} \, ds = \underbrace{\int_0^c \int_0^{2\pi} J\hat{z} \cdot \hat{z} \, r dr d\theta}_{J\pi c^2} + \int_c^r \int_0^{2\pi} 0 \cdot \hat{z} \, r dr d\theta$$

$\therefore$ for $r > c$ $\quad H_\theta \, 2\pi r = J\pi c^2 \quad$ or $\quad \underline{H} = \dfrac{Jc^2}{2r} \hat{\theta}$

For $r < c$ the first integral must be evaluated to

$$\int_0^r \int_0^{2\pi} J\hat{z} \cdot \hat{z} \, r dr d\theta = J\pi r^2$$

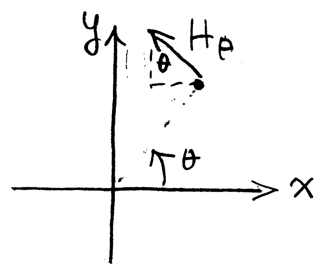
$\therefore$ for $r < c$ $\quad H_\theta \, 2\pi r = J\pi r^2 \quad$ or $\quad \underline{H} = \dfrac{Jr}{2} \hat{\theta}$

Similarily for the small cylindrical void using r' as its radius

$$H'_\theta = \begin{cases} -\dfrac{Jb^2}{2r'} & r' > b \\ -\dfrac{Jr'}{2} & 0 \le r' \le b \end{cases}$$

Note current density is $-J$

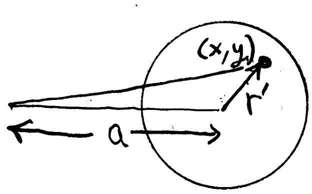
The easiest way to do this is to express H_θ and H'_θ in rectangular coordinates.



$$H_x^+ = -H_\theta \sin\theta = -H_\theta\left(\frac{y}{r}\right)$$

$$H_y^+ = +H_\theta \cos\theta = +H_\theta\left(\frac{x}{r}\right)$$

For the displaced smaller cylinder



$$H_x^- = -H'_\theta \frac{y}{r'}$$

$$H_y^- = +H'_\theta \frac{(x-a)}{r'}$$

where r' is measured from the center of the void.

All we need to do is determine the total field everywhere in the conductor.

Inside the conductor $(r<c)$

$$H_x^+ = -\frac{Jr}{2}\,\frac{y}{r} = -\frac{Jy}{2}$$

$$H_y^+ = +\frac{Jr}{2}\,\frac{x}{r} = +\frac{Jx}{2}$$

The field from the second current distribution is a little more complicated:

$$H_x^- = -H'_\theta \frac{y}{r'} = \begin{cases} -\left(-\dfrac{Jr'}{2}\right)\left(\dfrac{y}{r'}\right) = +\dfrac{Jy}{2} & 0 \le r' \le b \\ -\left(-\dfrac{Jb^2}{2r'}\right)\left(\dfrac{y}{r'}\right) = +\dfrac{Jb^2}{2r'^2}y = \dfrac{Jb^2 y}{2\left[(x-a)^2+y^2\right]} & r' > b \end{cases}$$

$$H_y^- = + H_\theta' \frac{(x-a)}{r'} = \begin{cases} \left(-\frac{Jr'}{2}\right)\left(\frac{x-a}{r'}\right) = -\frac{J(x-a)}{2} & 0 \le r' \le b \\ \left(-\frac{Jb^2}{2r'}\right)\left(\frac{x-a}{r'}\right) = \frac{-Jb^2(x-a)}{2[(x-a)^2+y^2]} & r' > b \end{cases}$$

In the cylindrical void where $0 \le r' \le b$

$$H_y = H_y^+ + H_y^- = +\frac{Jx}{2} - \frac{J(x-a)}{2} = \frac{Ja}{2}$$ uniform in the y-direction

$$H_x = H_x^+ + H_x^- = -\frac{Jy}{2} + \frac{Jy}{2} = 0$$ zero in the x-direction

Outside the void but in the conductor.

$$H_y = H_y^+ + H_y^- = +\frac{Jx}{2} - \frac{Jb^2(x-a)}{2[(x-a)^2+y^2]}$$

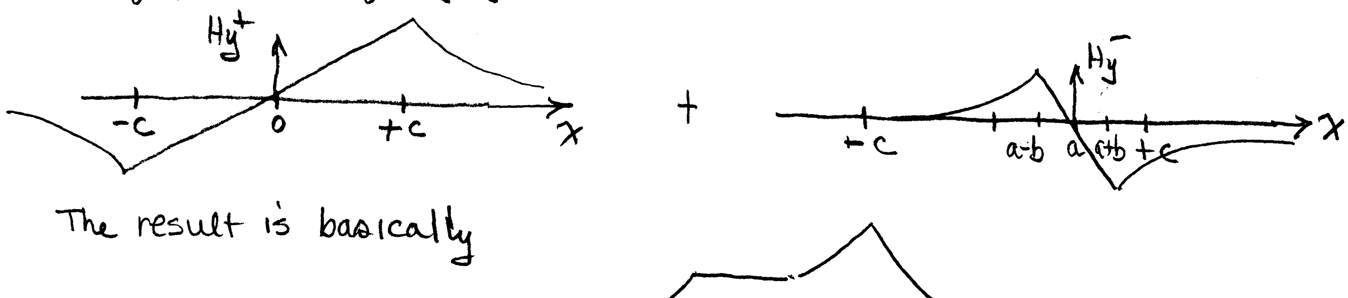
$$H_x = H_x^+ + H_x^- = -\frac{Jy}{2} + \frac{Jb^2 y}{2[(x-a)^2+y^2]}$$

Outside the conductor we simply have the total field from each current distribution

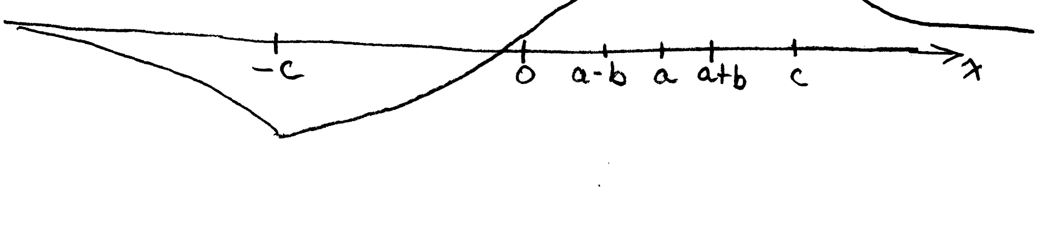
$$H_y = H_y^+ + H_y^- = +\frac{Jc^2 y}{2(x^2+y^2)} - \frac{Jb^2(x-a)}{2[(x-a)^2+y^2]}$$

$$H_x = H_x^+ + H_x^- = -\frac{Jc^2 x}{2(x^2+y^2)} + \frac{Jb^2 y}{2[(x-a)^2+y^2]}$$

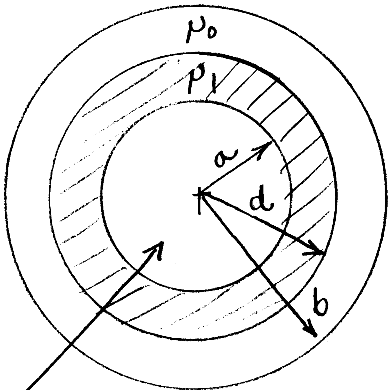
A quick sketch of H_y gives



The result is basically



2.5 A coaxial transmission line with inner conductor of radius a and outer conductor of radius b has a coaxial cylindrical ferrite of permeability μ_1 extending from $r=a$ to $r=d$ (with $d<b$), and air from radius d to b. Find the external inductance per unit length.



Inner conductor with current $\hat{z}I_0$

Using Ampère's Law

$$\oint_C \underline{H}\cdot d\underline{l} = \int_S \underline{J}\cdot\hat{n}\,ds$$

$$\int_0^{2\pi} H_\theta \hat{\theta}\cdot r d\theta \hat{\theta} = I_0 \quad \text{for } r>a$$

$$H_\theta 2\pi r = I_0$$

$$H_\theta = \frac{I_0}{2\pi r}$$

The magnetic flux per unit length

$$\Psi_m = \int_a^d \underline{B}\cdot d\underline{r} + \int_d^b \underline{B}\cdot d\underline{r}$$

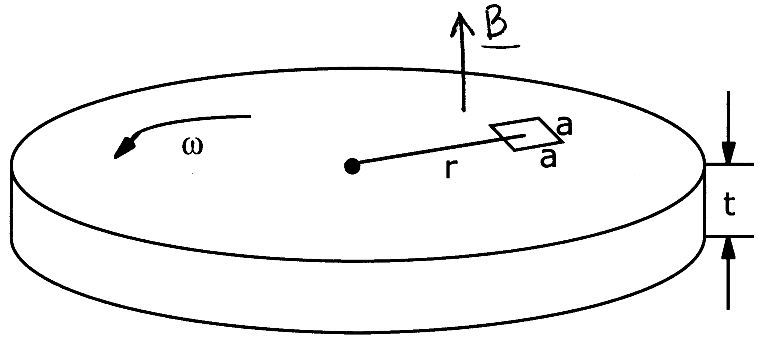
$$\Psi_m = \int_a^d \mu_1 \frac{I_0}{2\pi r}\,dr + \int_d^b \frac{\mu_0 I_0}{2\pi r}\,dr = \frac{I_0}{2\pi}\left[\mu_1 \ln\left(\frac{d}{a}\right) + \mu_0 \ln\left(\frac{b}{d}\right)\right]$$

The inductance per unit length is then

$$\frac{L}{\text{unit length}} = \frac{\Psi_m}{I_0} = \frac{\mu_0}{2\pi}\left[\ln\left(\frac{b}{d}\right) + \frac{\mu_1}{\mu_0}\ln\left(\frac{d}{a}\right)\right].$$

1. An electromagnetic "eddy current" brake consists of a disk of conductivity σ and thickness t rotating about its center with a magnetic field B applied perpendicular to the plane of the disk over a small area a^2.

If the area a^2 is at a distance r from the axis, find an approximate expression for the torque tending to slow down the disk at the instant its angular velocity equals ω.



We may use the Lorentz force law on a single electron to estimate E in the small area.

$$\underline{E} = \frac{\underline{F}}{-e} = \frac{-e\,\underline{v} \times \underline{B}}{-e} = \quad r\omega\,\hat{\phi} \times B_o\hat{z} = +\,r\omega B_o\,\hat{r}$$

this is a standard approach for determining $\underline{E}$

A key is recognizing that $q\underline{v} = I\,d\underline{l}$ (Plonsey & Collin, p. 202)

The braking force is then

$$\underline{F} = q\underline{v} \times \underline{B} = I\,d\underline{l} \times \underline{B}$$ where I is the induced "eddy" current

Recognizing that $\underline{J} = \sigma\underline{E}$ and $\underline{I} = \underline{J}(at)$ — the cross sectional area

$$\underline{F}_{brake} \cong \sigma r\omega B_o\, at\; a\hat{r} \times B_o\hat{z}$$

where we assumed that E is constant across the braking area and that $d\underline{l} = a\hat{r}$

$$\underline{F}_{brake} \cong -\sigma r\omega a^2 t B_o^2\,\hat{\phi}$$

The braking torque is then $\underline{r} \times \underline{F}_{brake} = -\sigma r\omega a^2 t B_o^2\,\hat{z}$

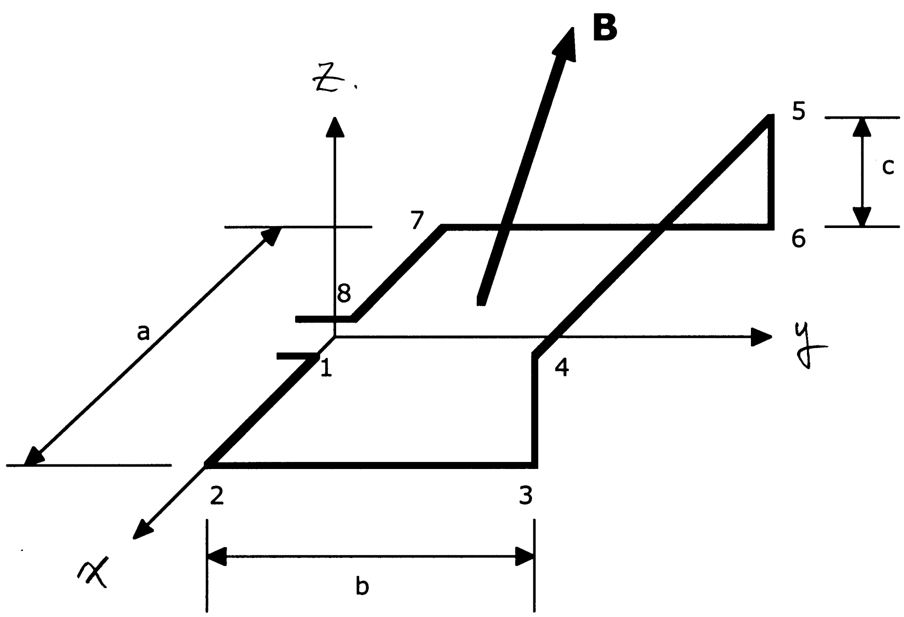
A more detailed discussion of magnetic braking can be found in:

H.D. Wiederick, N. Gauthier, D.A. Campbell and P. Rochon "Magnetic Braking: Simple Theory and Experiment," American Journal of Physics, Vol. 55, No. 6, June 1987.

Mark A. Heald, "Magnetic Braking: Improved Theory" American Journal of Physics, Vol. 56, No. 6, June 1988.

Lee Barnes, John Hardin, Charles A. Gross, Dewain Wasson, "An Eddy Current Braking System" Proceedings IEEE 25th Southeastern Symposium on Systems Theory, March 1993.

2. A conducting wire is bent into the shape shown below. The segments 1-2, 4-5, and 7-8 are parallel to the x-axis. The segments 2-3 and 6-7 are parallel to the y-axis and the segments 3-4 and 5-6 are parallel to the z-axis. This conducting loop is immersed in a magnetic field given by $\overline{B} = B_o(\hat{y} + 2\hat{z})\cos\omega t$. Find the induced voltage that would be measured between terminals 1 and 8.



We use Faraday's Law $V = -\frac{\partial}{\partial t}\int_S \underline{B}\cdot\underline{ds}$

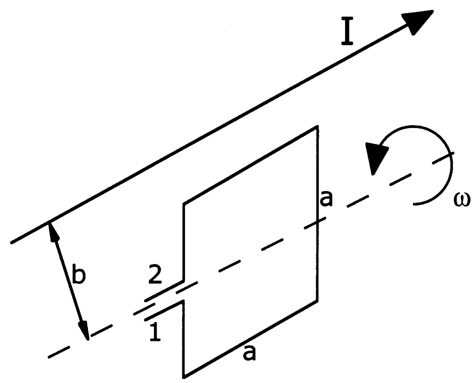
$$V = -\frac{\partial}{\partial t}\left[\int_{S_1}\underline{B}\cdot\underline{ds} + \int_{S_2}\underline{B}\cdot\underline{ds}\right]$$

$$= -\frac{\partial}{\partial t}\left[\int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_0^c (\hat{y}+2\hat{z})\cos\omega t\cdot\hat{y}\,dx\,dz + \int_{-\frac{a}{2}}^{+\frac{a}{2}}\int_0^b (\hat{y}+2\hat{z})\cos\omega t\cdot\hat{z}\,dx\,dy\right]$$

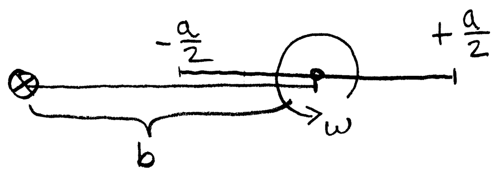
$$= -\frac{\partial}{\partial t}\left[ac\cos\omega t + 2ab\cos\omega t\right] = -\frac{\partial}{\partial t}\left[(ac+2ab)\cos\omega t\right]$$

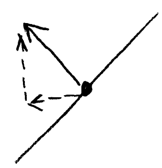
$$V = (ac+2ab)(+\omega\sin\omega t) = (ac+2ab)\omega\sin\omega t$$

3. A very long thin wire has a current I flowing in it. A square single turn coil is rotated about an axis parallel to the wire at an angular rate w. Derive an expression for the EMF between terminals 1 and 2 as a function of time. At t=0 the coil is in the plane defined by the wire and the axis of rotation of the coil.



The field from the long wire is given by $\underline{H} = \hat{\theta}\dfrac{I}{2\pi r}$ (Notes, p. 31a)





At $t=0$ the flux from I through the coil is given by

$$\Psi = \int_0^a \int_{b-\frac{a}{2}}^{b+\frac{a}{2}} \left(\frac{I}{2\pi r}\hat{\theta}\right)\cdot\left(\hat{\theta}\, dr dz\right) = \frac{aI}{2\pi}\int_{b-\frac{a}{2}}^{b+\frac{a}{2}} \frac{dr}{r} = \frac{aI}{2\pi}\left[\ln\left(b+\frac{a}{2}\right) - \ln\left(b-\frac{a}{2}\right)\right]$$

$$\Psi(t=0) = \frac{aI}{2\pi}\ln\left[\frac{2b+a}{2b-a}\right]$$ is the flux through the coil at $t=0$

As the loop rotates the effective dimension in the plane decreases by $\cos\omega t$, i.e. $a \to a\cos\omega t$, substituting this for a gives

$$\Psi(t) = \frac{aI}{2\pi}\ln\left[\frac{2b+a\cos\omega t}{2b-a\cos\omega t}\right]$$

Note: we are assuming that the loop dimensions are small compared to b, i.e., $a \ll b$ so that $\underline{B}$ is uniform over the rotating loop.

Now use Lenz's Law to get

$$V = -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t}\left\{\frac{aI}{2\pi} \ln[2b + a\cos\omega t] - \ln[2b - a\cos\omega t]\right\}$$

$$= -\frac{aI}{2\pi}\left\{\frac{1}{2b+a\cos\omega t}(-a\omega\sin\omega t) - \frac{1}{2b-a\cos\omega t} + a\omega\sin\omega t\right\}$$

$$V = +\frac{aI}{2\pi} a\omega \sin\omega t \left\{\frac{1}{2b+a\cos\omega t} + \frac{1}{2b-a\cos\omega t}\right\}$$

$$V = \frac{Ia^2 \sin\omega t}{\cancel{2}\pi}\left\{\frac{\cancel{2}b - a\cancel{\cos}\omega t + \cancel{2}b + a\cancel{\cos}\omega t}{(2b+a\cos\omega t)(2b-a\cos\omega t)}\right\}$$

$$V = \frac{2Ia^2 b \sin\omega t}{\pi(2b+a\cos\omega t)(2b-a\cos\omega t)}$$

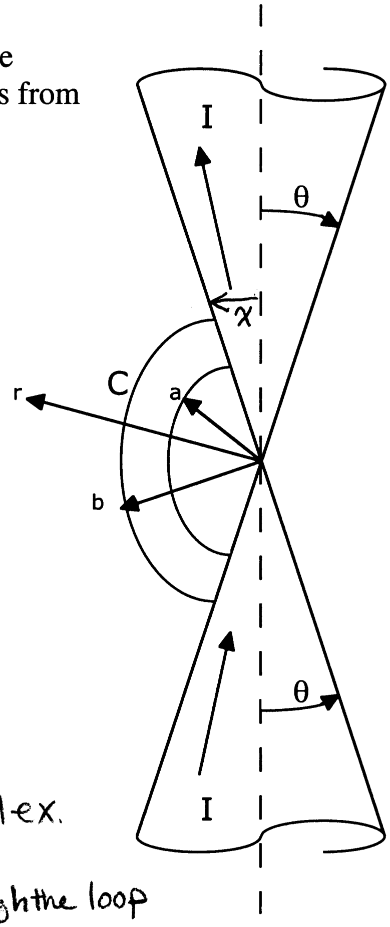
Note: I have tried to set up a more accurate integral which recognizes that $\underline{B}$ is not uniform over the rotating loop; however, my integrals became <u>very</u> complex. I will give extra credit who develops a good way of setting up these integrals.

4. Consider two infinitely long metal cones with a half angle θ_o as shown below. A current I, uniformly distributed over the cone surface, flows in the upward direction on the upper cone and flows towards the origin on the lower cone.

(a) Find the magnetic field **H** surrounding the cones.

(b) Find the mutual inductance between the two cones and the conducting loop labeled C in the figure. The loop extends from r=a to r=b and infinitesimally close to the cone surfaces.

HINT: A useful integral is $\int \frac{d\theta}{\sin\theta} = \ln\left(\tan\frac{\theta}{2}\right)$



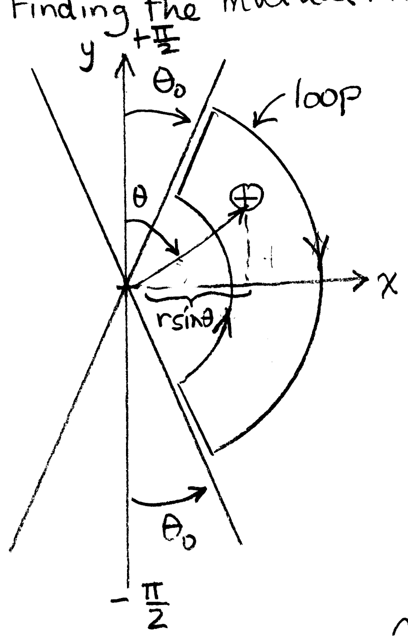
(a) Finding the magnetic field is done using Ampere's Law

$$\oint_C \underline{H}\cdot d\underline{l} = \int_S \underline{J}\cdot\hat{n}\, ds = I$$

Since the current is on the surface H_ϕ only exists outside the cone:

$$H_\phi = \begin{cases} \frac{I}{2\pi x} & x \text{ outside the cone} \\ 0 & \text{inside the cone} \end{cases}$$

(b) Finding the mutual inductance is more complex.



The magnetic flux through the loop is given by

$$\psi_{12} = \iint_{loop} \underline{B}\cdot\hat{n}\, ds = \int_{r=a}^{r=b}\int_{\theta_0}^{\pi-\theta_0} \mu_0 H_\phi\, r\, dr\, d\theta$$

to do this we need to express H_ϕ in terms of r and θ

H_ϕ inside the loop is given by

$$H_\phi = \frac{I}{2\pi r \sin\theta}$$

$$\psi_{12} = \int_{r=a}^{r=b}\int_{\theta_0}^{\pi-\theta_0} \mu_0 \frac{I}{2\pi \cancel{r} \sin\theta}\cancel{r}\, dr\, d\theta$$

H_ϕ is symmetric in θ.

$$\psi_{12} = \frac{\mu_0 I(b-a)}{\pi} \underbrace{2\int_{\theta}^{\pi/2} \frac{d\theta}{\sin\theta}}$$

note that $\int_{\theta_0}^{\pi-\theta_0} = 2\int_{\theta}^{\pi/2}$

$$\psi_{12} = \frac{\mu_0 I(b-a)}{\pi} \ln\left(\tan\left(\frac{\theta}{2}\right)\right)\Big|_{\theta_0}^{\pi/2}$$

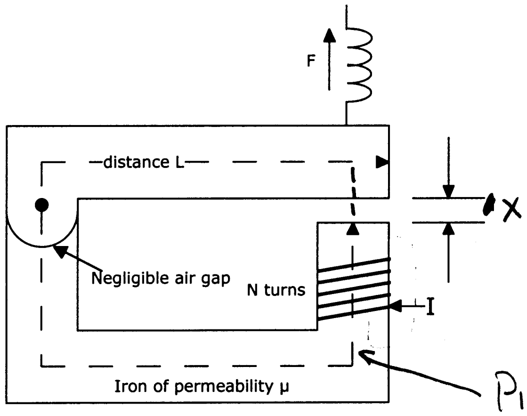
$$= \frac{\mu_0 I(b-a)}{\pi}\left[\ln\left(\cancel{\tan\left(\frac{\pi}{4}\right)}^{1}\right) - \ln\left(\tan\left(\frac{\theta_0}{2}\right)\right)\right]$$

$$\psi_{12} = -\frac{\mu_0 I(b-a)}{\pi} \ln\left[\tan\left(\frac{\theta_0}{2}\right)\right]$$

Note: since $\frac{\theta_0}{2} < 45^\circ$ $\tan\left(\frac{\theta_0}{2}\right) < 1$

$$L_{12} = \frac{\psi_{12}}{I} = -\frac{\mu_0 (b-a)}{\pi} \ln\left[\tan\left(\frac{\theta_0}{2}\right)\right].$$

5. Derive an expression for the required current to cause the armature in the illustrated relay to pull in. All iron paths have a cross-sectional area A. The spring force is F.



The magnetic field source is the solenoidal coil of N turns.

$$\oint_C \underline{H} \cdot d\underline{l} = \int \underline{J} \cdot \underline{n}\, ds$$

Pick The contour shown

Since we know B is continuous we can write for this contour:

$$\frac{B}{\mu} L + \frac{B}{\mu_0} x = NI \quad \text{where } \frac{B}{\mu} \ll \frac{B}{\mu_0}, \ \int \underline{J} \cdot \underline{n}\, dS = NI$$

Then $\frac{B}{\mu_0} x \cong NI$

and $B \cong \mu_0 \frac{NI}{x}$

The energy stored in the air gap is then given by $W_m = \frac{1}{2\mu} \int |B|^2 dv$

$$W_m = \frac{1}{2\mu_0} \frac{\mu_0^2 N^2 I^2}{x^2} \cdot xA = \frac{\mu_0}{2} \frac{N^2 I^2}{x^2}$$

The magnetic force is given by $F_m = -\frac{dW_m}{dx}$

$$F_m = -\frac{\mu_0 N^2 I^2 A}{2} \frac{d}{dx}\left[\frac{x}{x^2}\right] = -\frac{\mu_0}{2} N^2 I^2 A (-1) x^{-2} = + \mu_0 \frac{N^2 I^2 A}{2x^2}$$

For the relay to pull in this must be greater than The spring force F

$$F = \frac{\mu_0}{2} N^2 I^2 A \frac{1}{x^2}$$

$$x^2 = \frac{\mu_0 N^2 I^2 A}{2F} \quad \text{at pull-in} \Rightarrow I = \frac{x}{N} \sqrt{\frac{2F}{\mu_0 A}}$$