EECS 412 Electromagnetic Fields III
Fall 2003

Homework #1:

Due September 18th

Reference Ramo, Whinnery, Van Duzer, Fields and Waves in Communications Electronics, 3rd Edition, Chapter 1.

Superposition	1.4d
Conservative fields	1.7a
Potential & Gauss' Law	1.8b
Potential (spherical)	1.10d

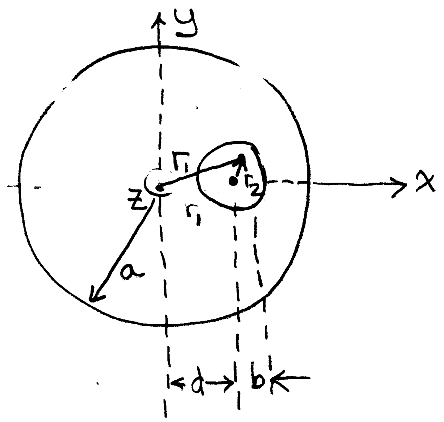
Polarization	1 (attached)
Method of images, electric fields	2 (attached)
Electrostatic forces & energies	3 (attached)
Electrostatic forces and potential	5 (attached)
Lossy capacitor	6 (attached)

Reference Ramo, Whinnery, Van Duzer, Fields and Waves in Communieations Electronics, 3rd Edition, Chapter 7.

Separation of variables, Laplaces equation	4 (attached)

1.4d A sphere of charge of radius a has uniform density ρ_o except for a spherical cavity of zero charge with radius b, centered at $x=d$, $y=0$, $z=0$, where $d<a$ and $b=a-d$. Find the electric field along the x axis from $-\infty < x < \infty$.

Hint: use superposition.



This is a classic problem in superposition. Assume that there exists a second charge density $\rho_2 = -\rho_o$ inside the spherical cavity such that $\rho_o + \rho_2 = \rho_o - \rho_o = 0$ there.

Let b be a point in the uncharged cavity $\underline{r_1}$ from the center of the sphere of radius a, and $\underline{r_2}$ from the center of the uncharged cavity of radius b.

We have the case of the field inside <u>and</u> outside of each volume.

Inside the large sphere Gauss' Law gives

$$\oint_S \underline{D}\cdot\underline{ds} = \int_V \rho\, dv$$

Recognizing symmetry in θ and ϕ

$$\epsilon_o E_r 4\pi r^2 = \frac{4}{3}\pi r^3 \rho_o$$

$$E_r = \frac{r\rho_o}{3\epsilon_o} \qquad r<a$$

For $r>a$ the right hand side becomes $\frac{4}{3}\pi a^3 \rho_o$

$$\epsilon_o E_r 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_o$$

$$E_r = \frac{a^3 \rho_o}{3\epsilon_o r^2} \qquad r \geq a$$

For the small spherical volume we get in a similar manner

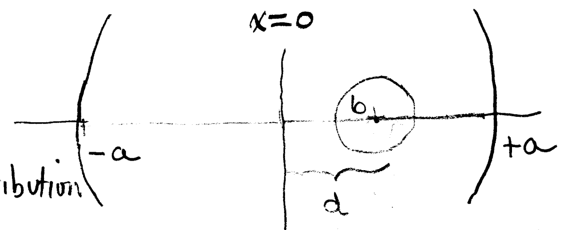
$$E_r' = \frac{-r'\rho_o}{3\epsilon_o} \qquad r'<b \qquad \text{since } \rho_2 = -\rho_o$$

$$E_r' = -\frac{b^3 \rho_o}{3\epsilon_o r'^2} \qquad r' \geq b$$

Since we are only interested in the x-axis let

$|r| = |x|$

$|r'| = |x-d|$



The appropriate E_x fields are then

from $x=0$ distribution; $x=d$ distribution

$x \le -a$
outside both

$$\underbrace{-\frac{a^3\rho_0}{3\epsilon_0 x^2} \qquad + \frac{b^3\rho_0}{3\epsilon_0 (x+d)^2}}_{-\text{ signs since } \underset{-\hat{x}}{E_r} = \underset{+\hat{x}}{-E_x}} \qquad \frac{\rho_0}{3\epsilon}\left[-\frac{a^3}{x^2} + \frac{b^3}{(x+d)^2}\right]$$

$-a \le x \le d-b$
inside large but outside small

$$\underbrace{+\frac{x\rho_0}{3\epsilon_0}}_{\substack{\text{sign depends}\\ \text{on } x}} \qquad \underbrace{+\frac{b^3\rho_0}{3\epsilon_0 (x+d)^2}}_{\text{always in } +x \text{ direction}} \qquad \frac{\rho_0}{3\epsilon_0}\left[x + \frac{b^3}{(x+d)^2}\right]$$

$d-b \le x \le d+b$
inside both spheres

$$\frac{\rho_0 x}{3\epsilon_0} \qquad - \underbrace{\frac{\rho_0(x-d)}{3\epsilon_0}}_{\text{sign depends on } x-d} \qquad = + \frac{\rho_0}{3\epsilon_0} d$$

inside small sphere

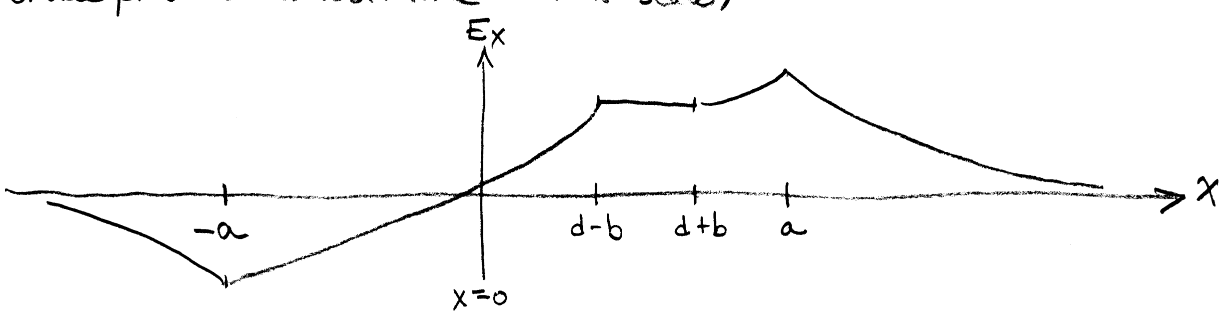
$d+b \le x \le a$
outside small sphere
inside large

$$+\frac{x\rho_0}{3\epsilon_0} \qquad \underbrace{-\frac{b^3\rho_0}{3\epsilon_0 (x-d)^2}}_{\text{field now in } -\hat{x} \text{ direction}} \qquad = \frac{\rho_0}{3\epsilon_0}\left[x - \frac{b^3}{(x-d)^2}\right]$$

$a \le x$
outside both spheres

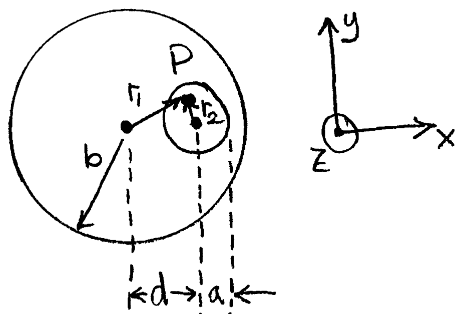
$$\underbrace{+\frac{\rho_0 a^3}{3\epsilon_0 r^2}}_{\text{large sphere}} \quad + \quad \underbrace{-\frac{b^3\rho_0}{3\epsilon_0 (x-d)^2}}_{\text{small sphere}} \qquad = \frac{\rho_0}{3\epsilon}\left[\frac{a^3}{x^2} - \frac{b^3}{(x-d)^2}\right].$$

A crude plot would look like (Not to scale)



This is a similar problem from Fall 2002.

4.24 A spherical region of radius b in free space is uniformly charged with a charge density of $\rho = K$, where K is a constant. The sphere contains an uncharged spherical cavity of radius a. The centers of the two spheres are separated by a distance d such that $d + a < b$. Find the electric field inside the cavity.



This is a classic problem in superposition. Assume that there exists a second charge density $\rho_2 = -K$ inside the spherical cavity such that $\rho_1 + \rho_2 = K - K = 0$ there.

Let P be a point in the uncharged cavity $\underline{r}_1$ from the center of the sphere of radius b, and $\underline{r}_2$ from the center of the uncharged cavity of radius a.

We solved for the field of a spherical cloud of charge on p.4 of the class notes.

For the uncharged cavity the field from charge density ρ_1 is given by

$$\underline{E}_1 = \frac{\rho_1}{3\epsilon_0}\underline{r}_1 = \frac{K}{3\epsilon_0}\underline{r}_1$$

Similarly, the field from charge density ρ_2 is

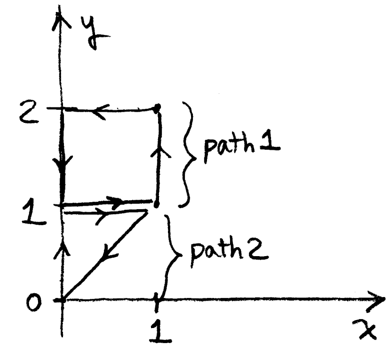
$$\underline{E}_2 = \frac{\rho_2}{3\epsilon_0}\underline{r}_2 = -\frac{K}{3\epsilon_0}\underline{r}_2$$

The electric field is then

$$\underline{E}_{tot} = \underline{E}_1 + \underline{E}_2 = \frac{K}{3\epsilon_0}(\underline{r}_1 - \underline{r}_2)$$

If we choose $\underline{r}_1$ & $\underline{r}_2$ along x axis $\underline{r}_1 - \underline{r}_2 = d\hat{x}$ so $E = \frac{Kd}{3\epsilon_0}\hat{x}$

1.7a Evaluate $\oint \underline{F}\cdot d\underline{l}$ for vectors $\underline{F} = \hat{x}\, zxy + \hat{y}x^2$ and $\underline{F} = \hat{x}y - \hat{y}$ about a rectangular path from $(0,1)$ to $(1,1)$ to $(1,2)$ to $(1,2)$ to $(0,2)$ and back to $(0,1)$. Repeat for a triangular path from $(0,0)$ to $(0,1)$ to $(1,1)$ back to $(0,0)$. Are either or both nonconservative



for $\underline{F} = \hat{x}\,zxy + \hat{y}\,x^2$

$$\oint_{C_1} \underline{F}\cdot d\underline{l} = \int_0^1 (\hat{x}\,zx1 + \hat{y}x^2)\cdot\hat{x}dx + \int_1^2 (\hat{x}\,z1y + \hat{y}(1)^2)\cdot\hat{y}\cdot dy + \int_1^0 (\hat{x}zx2 + \hat{y}x^2)\cdot(-\hat{x}\,dx)$$

$$+ \int_2^1 (\hat{x}z\cdot\phi\cdot y + \hat{y}0^2)\cdot -\hat{y}\,dy$$

$$= \int_0^1 zx\,dx + \int_1^2 dy - \int_1^0 2xz\,dx - \int_2^1 0\,dy = z\frac{x^2}{2}\Big|_0^1 + y\Big|_1^2 + 2z\frac{x^2}{2}\Big|_0^1$$

$$= z\cdot\frac{1}{2} + (2-1) + z(1) = 1 + \frac{3}{2}z$$

$$\oint_{C_2} \underline{F}\cdot d\underline{l} = \int_0^1 (\hat{x}z\cdot 0\cdot y + \hat{y}\cdot 0)\cdot\hat{y}dy + \int_0^1 (\hat{x}zx\cdot 1 + \hat{y}x^2)\cdot\hat{x}dx + \int_1^0\int_1^0 (\hat{x}zxy + \hat{y}x^2)(-\hat{x}dx - \hat{y}dy)$$

$$= 0 + \int_0^1 zx\,dx + \int_1^0\int_1^0 (-zxy\,dx - x^2 dy)$$

$$= z\frac{x^2}{2}\Big|_0^1 + zy\frac{x^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 = \frac{z}{2} + \frac{zy}{2} + \frac{x^3}{3}$$

Field is obviously NOT conservative

for $F = \hat{x}y - \hat{y}$

$$\oint_{C_1} \underline{F}\cdot\underline{dl} = \int_0^1 (\hat{x}\cdot 1 - \hat{y})\cdot\hat{x}dx + \int_1^2 (\hat{x}y - \hat{y})\,\hat{y}\,dy + \int_1^0 (\hat{x}\cdot 2 - \hat{y})(-\hat{x}dx) + \int_2^1 (\hat{x}y - \hat{y})(-\hat{y}dy)$$

$$= \int_0^1 dx + \int_1^2 -dy + \int_1^0 -2dx + \int_2^1 dy$$

$$= 1 - 1 - 2(-1) + (-1) = 1$$

$$\oint_{C_2} \underline{F}\cdot\underline{dl} = \int_0^1 (\hat{x}y - \hat{y})\cdot\hat{y}dy + \int_0^1 (\hat{x}(1) - \hat{y})\cdot\hat{x}dx + \int_1^0\int_1^0 (\hat{x}y - \hat{y})\cdot(-\hat{x}dx - \hat{y}dy)$$

$$= -\int_0^1 dy + \int_0^1 dx - \int_1^0 y\,dx + \int_1^0 dy$$

$$= -(1) + (1) + y - 1 = y - 1$$

Field is not conservative.

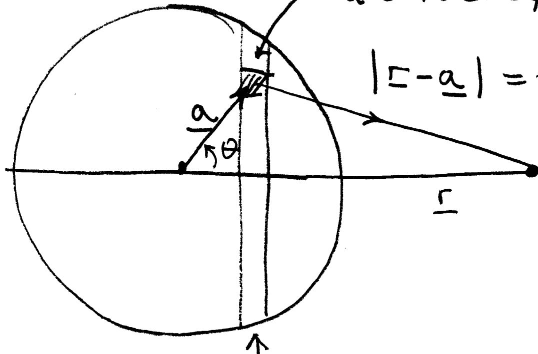
Should be zero independent of anything.

1.8b A charge of surface density ρ_s is spread uniformly over a spherical surface of radius a. Find the potential for $r<a$ and for $r>a$ by integrating contributions from the differential elements of charge. Check the results by making use of Gauss' law and the symmetry of the problem.

This is basically a Green's function solution as described in class.

$a^2 \sin\theta \, d\theta \, d\phi$ is area of surface element.

$$|\underline{r}-\underline{a}| = \sqrt{r^2+a^2-2ra\cos\theta}$$



integrate over rings

See page 10.

$$\Phi(r) = \int_0^{2\pi}\int_0^{\pi} \rho_s \frac{a^2 \sin\theta \, d\theta \, d\phi}{4\pi\epsilon_0 \left[r^2+a^2-2ra\cos\theta\right]^{1/2}}$$

$$= \frac{2\pi a^2}{4\pi\epsilon_0}\int_0^{\pi} \frac{\rho_s \sin\theta \, d\theta}{\left[r^2+a^2-2ra\cos\theta\right]^{1/2}}$$

Here is a clever trick to the integration

$$d\left(r^2+a^2-2ra\cos\theta\right)^{1/2} = \frac{1}{2}\left(r^2+a^2-2ra\cos\theta\right)^{-\frac{1}{2}}(+2ra\sin\theta)\,d\theta$$

$$= \frac{a^2\rho_s}{2\epsilon_0}\,\frac{1}{ra}\int_0^{\pi} d\left(r^2+a^2-2ra\cos\theta\right)^{1/2}$$

$$= \frac{a\rho_s}{2\epsilon_0 r}\sqrt{r^2+a^2-2ra\cos\theta}\,\Big|_0^{\pi}$$

$$= \frac{a\rho_s}{2\epsilon_0 r}\left[\sqrt{r^2+a^2+2ra} - \sqrt{r^2+a^2-2ra}\right]$$

$$= \frac{a\rho_s}{2\epsilon_0 r}\left[(r+a) \mp (r-a)\right]$$

Which signs do we use?

Physically this represents

$-r+a$ outside sphere

$+r-a$ inside sphere.

$$\Phi(r) = \begin{cases} \dfrac{a\rho_s}{2\epsilon_0 r}\cdot 2r = \dfrac{a\rho_s}{\epsilon_0} & r \le a \\[2ex] \dfrac{a\rho_s}{2\epsilon_0 r}\cdot 2a = \dfrac{a^2\rho_s}{\epsilon_0 r} & r > a \end{cases}$$

Check By Gauss' Law $\oint_S \underline{D}\cdot d\underline{s} = \int \rho\, dv$

Inside sphere $\rho = 0$ $\therefore$ $E_r\,(\text{inside}) = 0 = -\dfrac{d\Phi}{dr}$

$\therefore \Phi_{\text{inside}} = \text{constant}.$

Outside sphere $\int \rho\, dv = 4\pi a^2 \rho_s$

$$\oint_S \underline{D}\cdot d\underline{s} = \epsilon_0 E_r\, 4\pi r^2 = 4\pi a^2 \rho_s$$

$$\therefore\quad E_r = \frac{a^2\rho_s}{\epsilon_0 r^2} = -\frac{d\Phi}{dr}$$

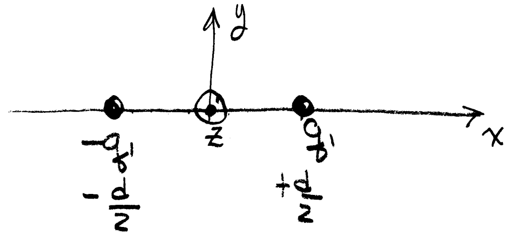
$$d\Phi = -\frac{a^2\rho_s}{\epsilon_0} r^{-2}\, dr$$

$$\Phi = -\frac{a^2\rho_s}{\epsilon_0}\,\frac{r^{-1}}{-1} + \text{const.} = \text{const.} + \frac{a^2\rho_s}{r\epsilon_0}$$

Pick $\Phi = 0$ at $r = \infty \Rightarrow \text{const.} = 0$

$$\Phi(r) = \begin{cases} \dfrac{a^2\rho_s}{r\epsilon_0} & \text{outside} \\[2ex] \dfrac{a\rho_s}{\epsilon_0} & \text{inside} \end{cases}$$

1.10 d For two line charges q_1 and $-q_1$ at $(\frac{d}{2}, 0)$ and $(-\frac{d}{2}, 0)$, respectively, find the potential for any point (x,y) and from this derive the electric field.



q_1 is a charge density

The potential for a line charge was covered in class lecture. See p. 20.

for a line charge at $x=0$ $\Phi(x,y) = \frac{\rho_e}{2\pi\epsilon_0} \ln\left(\frac{r_0}{r}\right)$

For this problem we simply add the potentials

$$\Phi(x,y) = \frac{q}{2\pi\epsilon_0} \ln\left\{\frac{r_0}{[(x-\frac{d}{2})^2 + y^2]^{1/2}}\right\} - \frac{q}{2\pi\epsilon_0} \ln\left\{\frac{r_0}{[(x+\frac{d}{2})^2 + y^2]^{1/2}}\right\}$$

$$\Phi(x,y) = \frac{q}{2\pi\epsilon_0}\left\{ \cancel{\ln r_0} - \frac{1}{2}\ln\left[(x-\tfrac{d}{2})^2 + y^2\right] - \cancel{\ln r_0} + \frac{1}{2}\ln\left[(x+\tfrac{d}{2})^2 + y^2\right]\right\}$$

$$\Phi(x,y) = \frac{q}{4\pi\epsilon_0}\left\{\ln\left[(x+\tfrac{d}{2})^2 + y^2\right] - \ln\left[(x-\tfrac{d}{2})^2 + y^2\right]\right\}$$

$$\Phi(x,y) = \frac{q}{4\pi\epsilon_0} \ln\left[\frac{(x+\frac{d}{2})^2 + y^2}{(x-\frac{d}{2})^2 + y^2}\right]$$

$$\underline{E}(x,y) = -\underline{\nabla}\Phi = -\hat{x}\frac{\partial\Phi}{\partial x} - \hat{y}\frac{\partial\Phi}{\partial y}$$ (no z dependence)

$$\frac{\partial\Phi}{\partial x} = \frac{q}{4\pi\epsilon_0} \frac{1}{\frac{(x+\frac{d}{2})^2 + y^2}{(x-\frac{d}{2})^2 + y^2}} \left[\frac{2(x+\frac{d}{2})}{(x-\frac{d}{2})^2 + y^2} + \left[(x+\tfrac{d}{2})^2 + y^2\right](-1)\left[(x-\tfrac{d}{2})^2 + y^2\right]^{-2} 2(x-\tfrac{d}{2})\right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(x-\frac{d}{2})^2 + y^2}{(x+\frac{d}{2})^2 + y^2} \left[\frac{\left[(x-\frac{d}{2})^2 + y^2\right] 2(x+\frac{d}{2}) - 2(x-\frac{d}{2})\left[(x+\frac{d}{2})^2 + y^2\right]}{\left[(x-\frac{d}{2})^2 + y^2\right]^2}\right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{2d\left[x^2 + \frac{d^2}{4} + y^2\right]}{\left[(x+\frac{d}{2})^2 + y^2\right]\left[(x-\frac{d}{2})^2 + y^2\right]}$$

The derivative for $\hat{y}$ is similar

$$\frac{\partial \Phi}{\partial y} = \frac{q}{4\pi\epsilon_0} \frac{1}{\frac{(x+\frac{d}{2})^2+y^2}{(x-\frac{d}{2})^2+y^2}} \left[\frac{2y}{(x-\frac{d}{2})^2+y^2} + \left[(x+\tfrac{d}{2})^2+y^2\right](-1)\left[(x-\tfrac{d}{2})^2+y^2\right]^{-2} 2y \right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{(x-\frac{d}{2})^2+y^2}{(x+\frac{d}{2})^2+y^2} \left[\frac{2y\left[(x-\frac{d}{2})^2+y^2\right] - 2y\left[(x+\frac{d}{2})^2+y^2\right]}{\left[(x-\frac{d}{2})^2+y^2\right]^2} \right]$$

$$= \frac{q}{4\pi\epsilon_0} \frac{2y\left[\left(x^2 - 2\cdot\frac{dx}{2} + \frac{d^2}{4} + y^2\right) - \left(x^2 + 2\frac{dx}{2} + \frac{d^2}{4} + y^2\right)\right]}{\left[(x+\frac{d}{2})^2+y^2\right]\left[(x-\frac{d}{2})^2+y^2\right]}$$

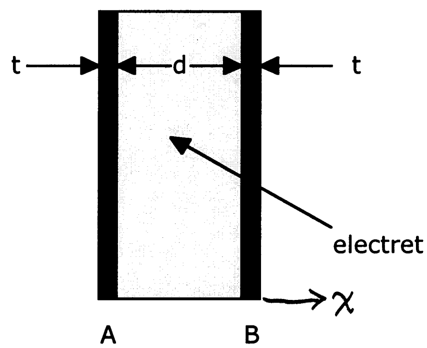
$$= \frac{q}{4\pi\epsilon_0} \frac{2y(-2dx)}{\left[(x+\frac{d}{2})^2+y^2\right]\left[(x-\frac{d}{2})^2+y^2\right]}$$

$$\frac{\partial \Phi}{\partial y} = \frac{q}{4\pi\epsilon_0} \frac{-4dxy}{\left[(x+\frac{d}{2})^2+y^2\right]\left[(x-\frac{d}{2})^2+y^2\right]}$$

Combining

$$\underline{E} = -\frac{qd}{2\pi\epsilon_0} \frac{1}{\left[(x+\frac{d}{2})^2+y^2\right]\left[(x-\frac{d}{2})^2+y^2\right]} \left[\hat{x}\left(x^2 + \frac{d^2}{4} + y^2\right) - \hat{y}\, 2xy \right]$$

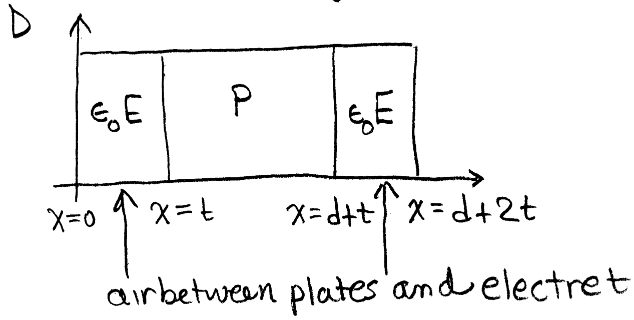
1. An electret (a permanently polarized substance) of polarization P is sandwiched between two metallic plates A and B. Find the distribution of the charges and the electric potentials in the system.

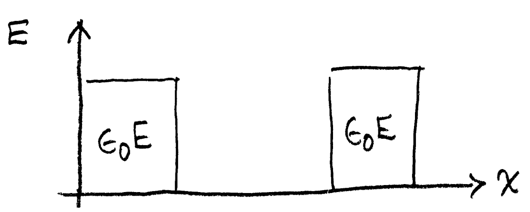


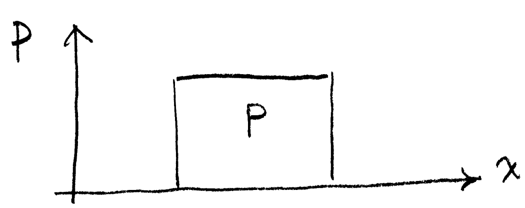
Note: This is not a well written problem since it is not clear if t is the thickness of the plates or an air gap. I assumed it was an air gap. If $t \to 0$ in my solution you get the same as metal plates

This problem is very similar to the capacitor with a polarizable dielectric block between the plates as presented in class. See lecture notes p. 18

Since D is continuous and there is no external voltage source







$$\epsilon_0 E = P \Rightarrow E = \frac{P}{\epsilon_0}$$

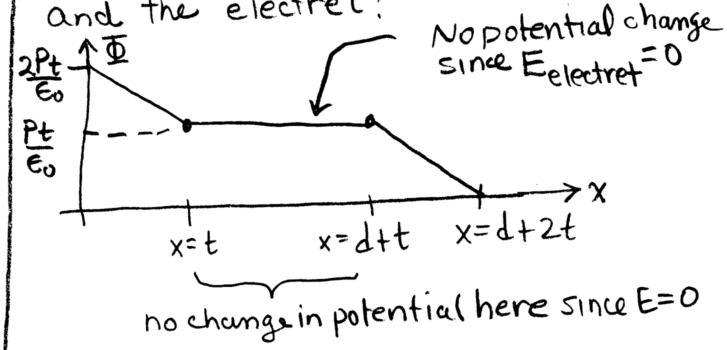
We can integrate E to determine the potential Φ as a function of x.

$$E = -\frac{d\Phi}{dx}$$

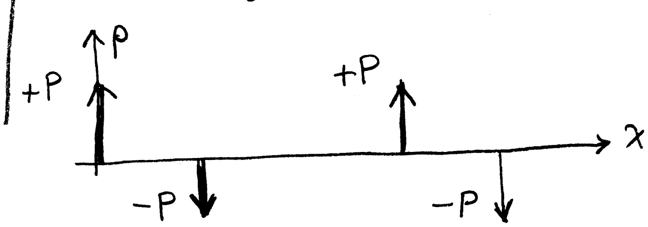
$$\Phi = -\int E\,dx = -\int_0^t \frac{P}{\epsilon_0}\hat{x}\cdot(-\hat{x})dx$$

$$\Phi(t) = \frac{Pt}{\epsilon_0}$$

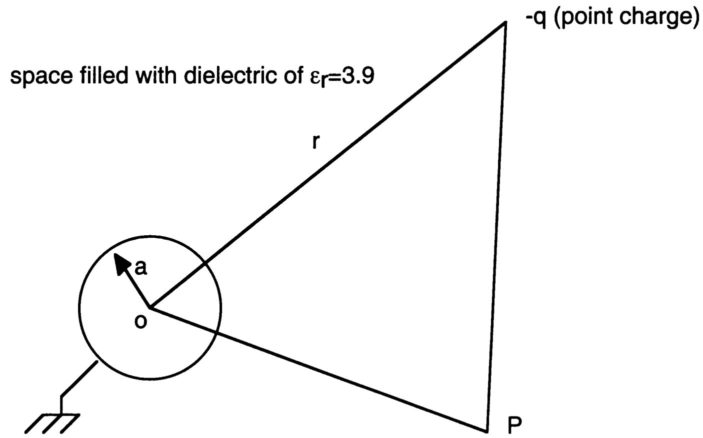
Since E only exists between the plates and the electret:



Since $\rho = \frac{\partial E}{\partial t}$

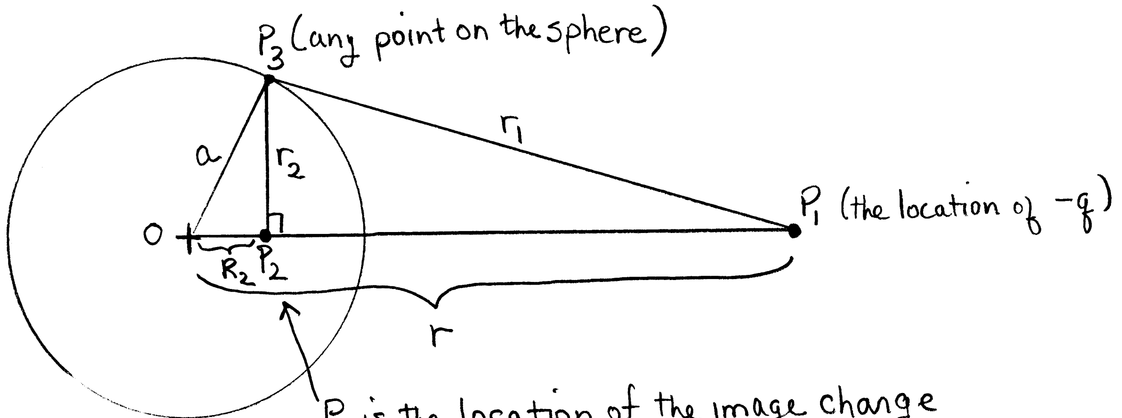


2. Find the electric field at point P in the following structure. The metallic sphere of radius a is grounded and space is filled with a dielectric of ε_r=3.9. Hint: use the method of images.



This is a famous problem called "inversion in a sphere". See Plonsey and Collin, Example 2.8 for a detailed explanation of this problem as well as the case when the sphere is not grounded.

The first part of the solution involves finding the location of the image charge. To do this we consider the following cross-sectional drawing of the problem



P_2 is the location of the image charge on a line from the origin O to the charge at P_1

Since the location of the image charge is imaginary I will also assume that the interior of the sphere is also filled with a dielectric of $\epsilon_r = 3.9$

Since the sphere is grounded the sum of the potentials from the charge and its image must be zero on the surface of the sphere.

$$\underbrace{\frac{-q}{4\pi\epsilon_r\epsilon_0 r_1}}_{\text{potential at sphere from charge } -q} + \underbrace{\frac{q_2}{4\pi\epsilon_r\epsilon_0 r_2}}_{\text{potential at sphere's surface from image charge } q_2} = 0$$

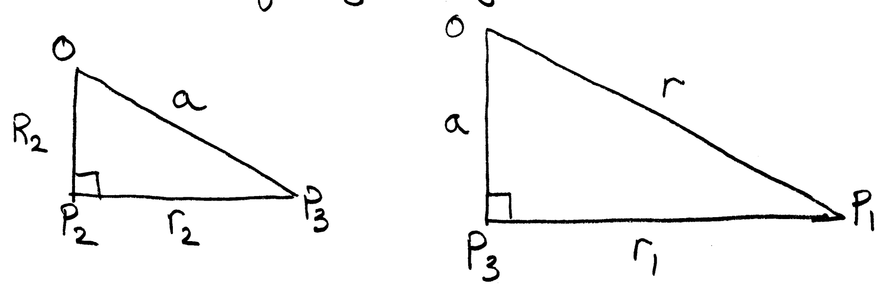
$$-\frac{q}{r_1} + \frac{q_2}{r_2} = 0 \qquad \therefore q_2 = \frac{r_2}{r_1} q \qquad (1)$$

Now we must determine R_2 such that q_2 is always equal to $\frac{r_2}{r_1} q$. This simply requires that $\frac{r_2}{r_1}$ is constant and easily achieved if we require

$$\frac{OP_2}{a} = \frac{a}{OP_1}$$

[Other constraints may be possible]

This is equivalent to requiring triangles OP_2P_3 and OP_3P_1 to be similar.



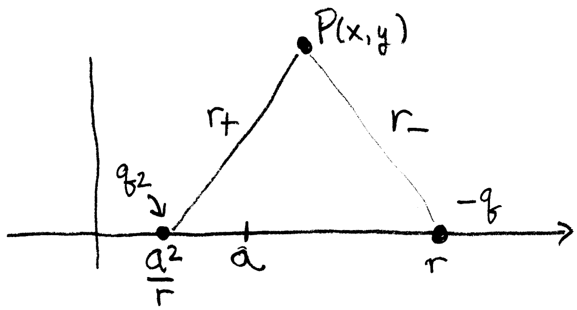
If this is true then $\frac{r_2}{r_1} = \frac{R_2}{a} = \frac{a}{r}$ (2)

Using this result in equation (1) gives $q_2 = \frac{a}{r} q$.

Equation (2) also gives $R_2 = \frac{a^2}{r}$

You can explicitly calculate the $\underline{E}$ field for the given geometry. However, I will use the potential method since we already know the potential.

Using the line containing the charges as my x-axis I can write



define

$$r_+ = \sqrt{\left(x-\frac{a^2}{r}\right)^2 + y^2}$$

$$r_- = \sqrt{(x-r)^2 + y^2}$$

$$\epsilon = 3.9\epsilon_0$$

$$\Phi(P) = \frac{q_2}{4\pi\epsilon r_+} - \frac{q}{4\pi\epsilon r_-}$$

$$\Phi(x,y) = \frac{1}{4\pi\epsilon r}\left[a\left[\left(x-\frac{a^2}{r}\right)^2+y^2\right]^{-\frac{1}{2}} - r\left[(x-r)^2+y^2\right]^{-\frac{1}{2}}\right]$$

$$\underline{E} = -\underline{\nabla}\Phi$$

$$\underline{E} = \frac{1}{4\pi\epsilon r}\left[a\left(-\tfrac{1}{2}\right)\left[\left(x-\frac{a^2}{r}\right)^2+y^2\right]^{-\frac{3}{2}} 2\left(x-\frac{a^2}{r}\right)\hat{x} - r\left(-\tfrac{1}{2}\right)\left[(x-r)^2+y^2\right]^{-\frac{3}{2}} 2(x-r)\hat{x}\right.$$

$$\left. + a\left(-\tfrac{1}{2}\right)\left[\left(x-\frac{a^2}{r}\right)^2+y^2\right]^{-\frac{3}{2}} 2y\hat{y} - r\left(-\tfrac{1}{2}\right)\left[(x-r)^2+y^2\right] 2y\hat{y}\right]$$

$$\underline{E} = \frac{1}{4\pi\epsilon r}\left[\frac{a\left(\frac{a^2}{r}-x\right)\hat{x} - ay\hat{y}}{\left[\left(x-\frac{a^2}{r}\right)+y^2\right]^{3/2}} + \frac{r(x-r)\hat{x} + ry\hat{y}}{\left[(x-r)^2+y^2\right]^{3/2}}\right]$$

You could also simply use the Green's function for the E-field

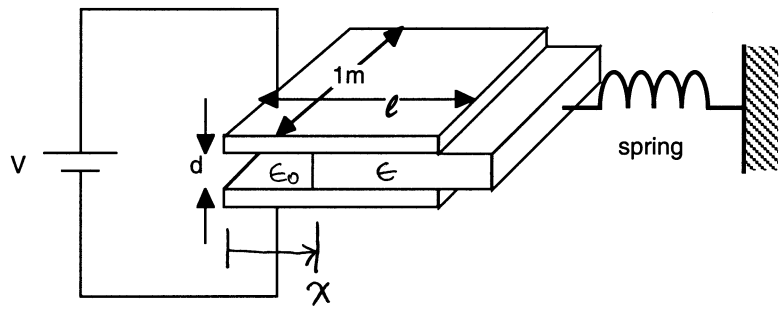
$$\underline{E} = \frac{\delta q}{4\pi\epsilon_0}\frac{\hat{a}_R}{|\underline{R}|^2}$$

where $\underline{R}$ points from the charge to the observation point.

$$\underline{E}(P) = \frac{q}{4\pi\epsilon_r\epsilon_0}\frac{\hat{a}_{P-R}}{|\underline{P}-\underline{R}|^2} - \frac{q_2}{4\pi\epsilon_r\epsilon_0}\frac{\hat{a}_{P-R_2}}{|\underline{P}-R_2|^2}$$

where $\underline{R}_2$ is the position vector of q_2; $\underline{R}$ is the position vector of q; and $\underline{P}$ is the position vector of the observer

3. A slab of dielectric material of electric permittivity e and thickness d meters slides freely between two parallel metallic plates d meters apart. The metallic plates are connected to a battery with an e.m.f. of V volts. Find how far the slab will slide between the plates if the slab is attached to a spring with a stiffness of K newtons/meter. The dimensions are as shown in the figure. Neglect electric fringing in your calculations.



There are multiple ways of solving this problem. I shall follow the method of Plonsey & Collin, Example 3-5

This is a parallel plate capacitor. The capacitance of such a capacitor is given by $C = \frac{\epsilon A}{d}$ where ϵ is the dielectric constant of the region between the plates, A is the area of the plate, and d is the spacing between the plates. The energy stored in such a capacitor is $W_e = \frac{1}{2} C V^2$ where V is the voltage between the plates.

The initial energy of the capacitor (with no dielectric slab between the plates) is given by

$$W_e^{\text{initial}} = \frac{1}{2} \frac{\epsilon_0 A}{d} V^2$$

Now put the dielectric slab between the plates and assume that the spring has zero compression at $x = 0$

We assume $V =$ constant. This means that energy can come from/to the battery as the slab moves in and out. This also means that we would need to include a battery energy term if we used energy balance.

Because of the energy associated with the battery and charge movement it is easier to consider a force balance.

The change in electrical energy due to the slab is

$$\Delta W_e = \underbrace{\frac{1}{2}\,\frac{\epsilon(l-x)+\epsilon_0 x}{d}\,V^2}_{\text{energy of capacitor with slab edge at } x} - \underbrace{\frac{1}{2}\,\frac{\epsilon_0 l}{d}\,V^2}_{\text{initial energy of capacitor slab edge at } x=0}$$

$$F_{slab} = -\frac{dW_e}{dx} = -\left[\frac{-\epsilon+\epsilon_0}{2d}\right]V^2$$

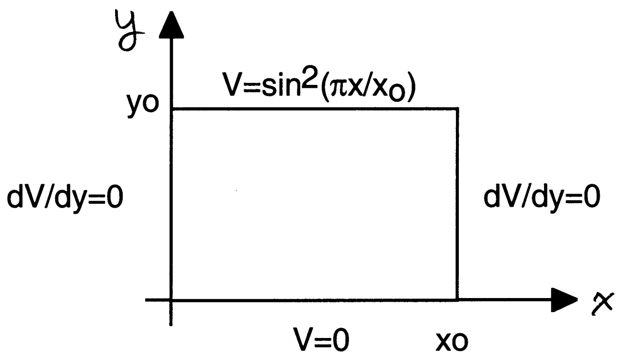
This must be equal at equilibrium to the spring force kx.

$$\therefore \quad F_{slab} = F_{spring}$$

$$\frac{\epsilon-\epsilon_0}{2d}\,V^2 = kx$$

$$x_{equil.} = \frac{\epsilon-\epsilon_0}{2d}\,\frac{V^2}{k}$$

4. A two dimensional box has the potential shown on its boundary. Use a Fourier series expansion to describe the potential inside the box.



The $\frac{dV}{dy}=0$ looks intimidating but isn't. See Eqn (1) below.

This problem is solved using separation of variables in Laplace's equation. $\nabla^2\Phi = 0$.

Let $\Phi(x,y) = f(x)g(y)$ and substitute into $\nabla^2\Phi = 0$ to get

$$g\frac{d^2f}{dx^2} + f\frac{d^2g}{dy^2} = 0$$

Dividing by fg gives

$$\underbrace{\frac{1}{f}\frac{d^2f}{dx^2}}_{k_x^2} + \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{k_y^2} = 0 \qquad \text{where } k_x^2 + k_y^2 = 0$$

Pick $k_x^2 < 0$ giving

$$f = c_1 \sin k_x x + c_2 \cos k_x x$$
$$g = c_3 e^{+k_y y} + c_4 e^{-k_y y}$$

$$\Phi = fg = (c_1 \sin k_x x + c_2 \cos k_x x)(c_3 e^{+k_y y} + c_4 e^{-k_y y})$$

$$\frac{d\Phi}{dy} = (c_1 \sin k_x x + c_2 \cos k_x x)(c_3 k_y e^{+k_y y} - c_4 k_y e^{-k_y y}) \quad (1)$$

For $\frac{d\Phi}{dy}$ to be zero at $x=0$ and $x = x_0$ we simply require $c_2 = 0$ and $k_x x_0 = n\pi$, $n = 1, 2, \ldots$

Then $$\Phi(x,y) = \sum_{n=1}^{\infty} \sin k_x x \left(c_3 e^{k_y y} + c_4 e^{-k_y y} \right)$$

Since $$\Phi(x, y=0) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{x_0}\right)(c_3 + c_4)$$

simply pick $c_3 = -c_4$ to meet this boundary condition.

Then $$\Phi(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{x_0}\right)\left(e^{-\frac{n\pi y}{x_0}} - e^{+\frac{n\pi y}{x_0}}\right)$$

At $y = y_0$

$$\Phi(x, y=y_0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{x_0}\right)\left(e^{-\frac{n\pi y_0}{x_0}} - e^{+n\pi\frac{y_0}{x_0}}\right) = \sin^2\left(\frac{\pi x}{x_0}\right)$$

We find c_m by taking the inner product with $\sin\frac{m\pi x}{x_0}$ and integrating over $[0, x_0]$.

$$\int_0^{x_0} \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{x_0} \sin\frac{m\pi x}{x_0}\left(e^{-\frac{n\pi y_0}{x_0}} - e^{+n\pi\frac{y_0}{x_0}}\right)dx =$$

$$\int_0^{x_0} \sin^2\left(\frac{\pi x}{x_0}\right)\sin\left(\frac{m\pi x}{x_0}\right)dx$$

The integral on the left becomes

$$\int_0^{x_0} \sin\frac{n\pi x}{x_0} \sin\frac{m\pi x}{x_0}\,dx = \int_0^{x_0} \sin^2\frac{m\pi x}{x_0}dx = \left.\frac{1}{2}x - \frac{1}{\frac{4m\pi}{x_0}}\sin 2\frac{m\pi}{x_0}x\right|_0^{x_0}$$

$$= \frac{x_0}{2} \quad \text{for } m=n$$

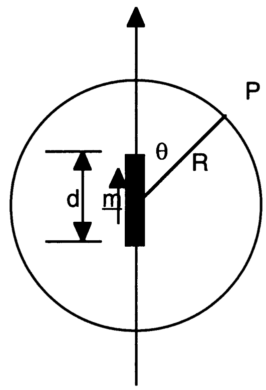
$$= \left[\frac{\sin(m-n)\frac{\pi x}{x_0}}{2(m-n)\frac{\pi}{x_0}} - \frac{\sin(m+n)\frac{\pi x}{x_0}}{2(m+n)\frac{\pi}{x_0}}\right]_0^{x_0}$$

$$= 0 \quad \text{for } m \neq n$$

Then
$$c_m \frac{x_0}{2} = \int_0^{x_0} \sin^2\left(\frac{\pi x}{x_0}\right) \sin\left(\frac{m\pi x}{x_0}\right) dx$$

and
$$\Phi(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{x_0}\right)\left(e^{-\frac{n\pi y_0}{x_0}} - e^{+n\pi\frac{y_0}{x_0}}\right)$$

5. An electric dipole of moment m lies in the x-y plane/ A charge q is placed at the point P a distance R from the dipole. If the charge q is constrained to move in a circle of radius R, find the force on q and from it find the equilibrium position. Assume that d<<R and that the charge has no mass. Use the potential function $\Phi = \frac{m}{4\pi\varepsilon_o}\frac{\cos\theta}{r^2}$



The force on a charge q is given by

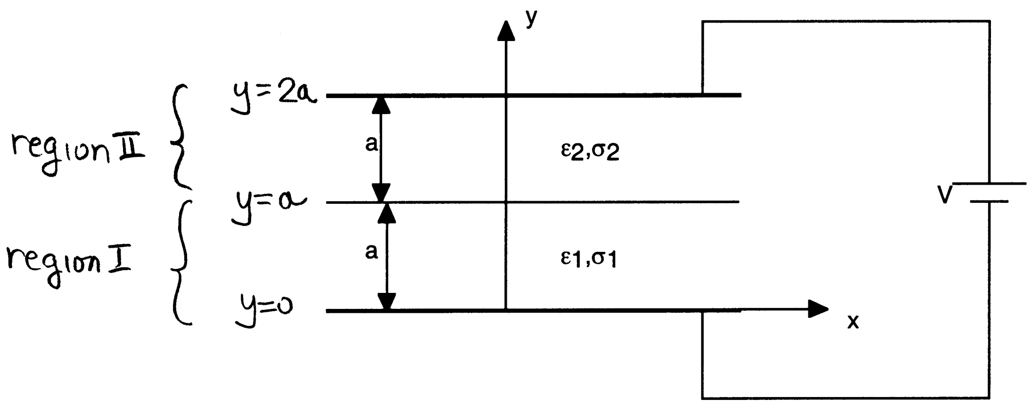
$$\underline{F} = q\underline{E} = -q\underline{\nabla}\Phi = -q\left(\hat{r}\frac{\partial\Phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right)$$

$$\underline{F} = -q\frac{m}{4\pi\epsilon_0}\left[\hat{r}\left(\frac{-2\cos\theta}{r^3}\right) + \hat{\theta}\frac{1}{r}\left(\frac{-\sin\theta}{r^2}\right)\right]$$

$$\underline{F} = \frac{qm}{4\pi\epsilon_0 r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right]$$

The $\hat{r}$ directed force cannot move the charge since it is constrained in that direction. The $\hat{\theta}$ force will act on the charge everywhere except where $\sin\theta = 0$, i.e. $\theta = 0°, 180°$. The equilibrium position is that at the north pole, $\theta = 0°$, of the electric dipole or at its south pole, $\theta = 180°$.

6. Consider the parallel plate capacitor shown below. The region between the plates is filled with two lossy dielectrics having conductivities and permittivities σ_1, ε_1 and σ_2,ε_2 respectively. When the upper plate is at a potential V relative to the lower plate find the displacement flux density and conduction current density that flows between the two plates. You may neglect fringing field effects.



This problem is primarily one of boundary conditions. It is very similar to Plonsey & Collin Example 5.2

The first boundary condition is that the current densities at the boundary between the two dielectrics must be equal, i.e.,

$$J_1 = \sigma_1 E_1 = J_2 = \sigma_2 E_2 \quad @\ y = a \qquad (1)$$

The second boundary condition comes from $E = -\frac{d\Phi}{dy}$ and the potential being continuous.

Integrating gives $\displaystyle\int_0^{2a} d\Phi = -\int_0^{2a} E \cdot dl = -\int_0^{a} E_1 dy - \int_a^{2a} E_2 dy$

$$\cancel{\Phi(2a)}^{V} - \cancel{\Phi(0)}^{0} = -E_1 a - E_2 a.$$

$$\therefore E_1 + E_2 = -\frac{V}{a} \qquad (2)$$

Combine this with equation (1) which gives

$$E_1 = \frac{\sigma_2}{\sigma_1} E_2 \qquad (3)$$

substituting (3) into (2)

$$\left(\frac{\sigma_2}{\sigma_1} E_2\right) + E_2 = -\frac{V}{a}$$

$$E_2\left(1+\frac{\sigma_2}{\sigma_1}\right) = -\frac{V}{a}$$

$$E_2 = -\frac{V/a}{1+\frac{\sigma_2}{\sigma_1}} = -\frac{\sigma_1 V}{a(\sigma_1+\sigma_2)}$$

$$E_1 = -\frac{V}{a} - E_2 = -\frac{V}{a} - \frac{V}{a}\left(\frac{\sigma_1}{\sigma_1+\sigma_2}\right) = -\frac{\sigma_2 V}{a(\sigma_1+\sigma_2)}$$

The Displacement is then

$$\underline{D} = \begin{cases} -\dfrac{\epsilon_1\sigma_2 V}{a(\sigma_1+\sigma_2)}\,\hat{y} & 0 < y < a \\[2ex] -\dfrac{\epsilon_2\sigma_1 V}{a(\sigma_1+\sigma_2)}\,\hat{y} & a < y < 2a \end{cases}$$

$$\underline{J} = \frac{\sigma \underline{D}}{\epsilon} = -\frac{\sigma_1\sigma_2 V}{a(\sigma_1+\sigma_2)}\,\hat{y}$$