Single- and Multiport Networks

Ever since single- and multiple port networks were first introduced into the electrical engineering profession through Guillemin and Feldkeller, they have quickly become indispensable tools in restructuring and simplifying complicated circuits as well as in providing fundamental insight into the performance of active and passive electronic devices. Moreover, the importance of network modeling has extended far beyond electrical engineering and has influenced such diverse fields as vibrational analysis in structural and mechanical engineering as well as biomedicine. For example, today's piezoelectric medical transducer elements and their electrical-mechanical conversion mechanisms are most easily modeled as a three-port network.

The ability to reduce most passive and active circuit devices, irrespective of their complicated and often nonlinear behavior, to simple input-output relations has many advantages. Chief among them is the experimental determination of input and output port parameters without the need to know the internal structure of the system. The "black box" methodology has tremendous appeal to engineers whose concern is mostly focused on the overall circuit performance rather than the analysis of individual components. This approach is especially important in RF and MW circuits, where complete field theoretical solutions to Maxwell's equations are either too difficult to derive or the solutions provide more information than is normally needed to develop functional, practical designs involving systems such as filters, resonators, and amplifiers.

In the following sections our objective is to establish the basic network input-output parameter relations such as impedance, admittance, hybrid, and *ABCD*-parameters. We then develop conversions between these sets. Rules of connecting networks are presented to show how more complicated circuits can be constructed by series and parallel cascading of individual network blocks. Finally, the scattering parameters are presented as an important practical way of characterizing RF/MW circuits and devices through the use of power wave relations.

4.1 Basic Definitions

Before embarking on a discussion of electrical networks we have to identify some general definitions pertaining to directions and polarity of voltages and currents. For our purposes we use the convention shown in Figure 4-1. Regardless of whether we deal with a single-port or an N-port network, the port-indexed current is assumed to flow into the respective port and the associated voltage is recorded as indicated.

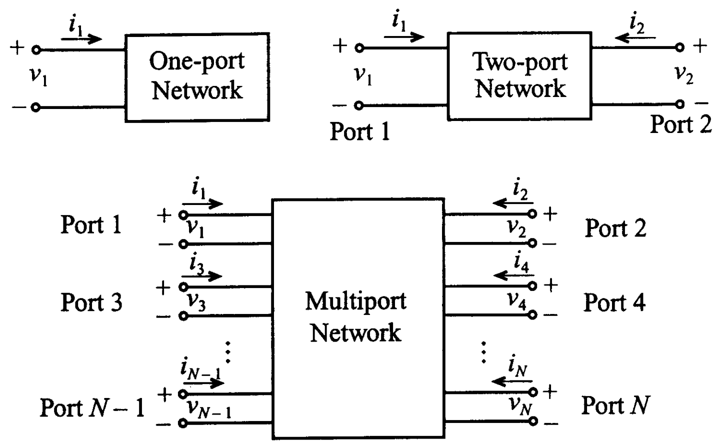

Figure 4-1 Basic voltage and current definitions for single- and multiport network.

In establishing the various parameter conventions we begin with the voltage-current relations through double-indexed impedance coefficients Z_{nm}, where indices n and m range between 1 and N. The voltage at each port $n = 1 \ldots N$ is given by

$$v_1 = Z_{11}i_1 + Z_{12}i_2 + \ldots + Z_{1N}i_N \tag{4.1a}$$

for port 1,

$$v_2 = Z_{21}i_1 + Z_{22}i_2 + \ldots + Z_{2N}i_N \tag{4.1b}$$

for port 2, and

$$v_N = Z_{N1}i_1 + Z_{N2}i_2 + \ldots + Z_{NN}i_N \tag{4.1c}$$

for port N. We see that each port n is affected by its own impedance Z_{nn} as well as by a linear superposition of all other ports. In a more concise notation, (4.1) can be converted into an **impedance** or **Z**-matrix form:

$$\begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{Bmatrix} \quad (4.2)$$

or, in matrix notation,

$$\{\mathbf{V}\} = [\mathbf{Z}]\{\mathbf{I}\} \quad (4.3)$$

where $\{\mathbf{V}\}$ and $\{\mathbf{I}\}$ are vectors of voltages $v_1, v_2, \ldots, v_N$ and currents $i_1, i_2, \ldots, i_N$, respectively, and $[\mathbf{Z}]$ is the impedance matrix.

Each impedance element in (4.2) can be determined via the following protocol:

$$Z_{nm} = \left. \frac{v_n}{i_m} \right|_{i_k = 0 \text{ (for } k \neq m)} \quad (4.4)$$

which means that the voltage v_n is recorded at port n while port m is driven by current i_m and the rest of the ports are maintained under open terminal conditions (i.e. $i_k = 0$ where $k \neq m$).

Instead of voltages as the dependent variable, we can specify currents such that

$$\begin{Bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{Bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix} \quad (4.5)$$

or

$$\{\mathbf{I}\} = [\mathbf{Y}]\{\mathbf{V}\} \quad (4.6)$$

where, similar to (4.4), we define the individual elements of the **admittance** or **Y-matrix** as

$$Y_{nm} = \left. \frac{i_n}{v_m} \right|_{v_k = 0 \text{ (for } k \neq m)} \quad (4.7)$$

Comparing (4.2) and (4.5), it is apparent that impedance and admittance matrices are the inverse of each other:

$$[\mathbf{Z}] = [\mathbf{Y}]^{-1} \quad (4.8)$$

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Example 4-1: Matrix representation of Pi-network

For the pi-network (the name of the network comes from the resemblance with the greek letter Π) shown in Figure 4-2 with generic impedances Z_A, Z_B, and Z_C find the impedance and admittance matrices.

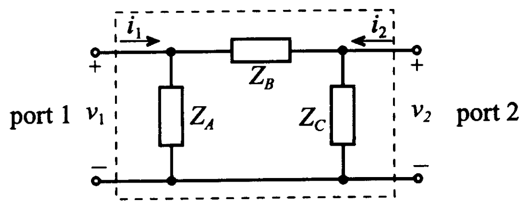

Figure 4-2 Pi-network as a two-port network.

Solution: The impedance elements are found by using (4.4) and the appropriate open- and short-circuit termination conditions.

To find Z_{11} we must compute the ratio of the voltage drop v_1 across port 1 to the current i_1 flowing into this port when the current into port 2 equals zero. The requirement $i_2 = 0$ is equivalent to an open-circuit condition. Thus, the impedance Z_{11} is equal to the parallel combination of impedances Z_A and $Z_B + Z_C$.

$$Z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0} = Z_A \| (Z_B + Z_C) = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C}$$

The value for Z_{12} can be found as the ratio of voltage drop v_1 measured across port 1 to the current i_2. In this case we must ensure that the current i_1 remains zero (i.e., we must treat port 1 as open). Voltage v_1 is equal to the voltage drop across impedance Z_A and can be obtained using the voltage divider rule:

$$v_1 = \frac{Z_A}{Z_B + Z_C} v_{AB}$$

where v_{AB} is a voltage drop across impedances Z_A and Z_B connected in series and computed as $v_{AB} = i_2[Z_C \| (Z_A + Z_B)]$. Thus,

$$Z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0} = \frac{Z_A}{Z_A + Z_B}[Z_C \| (Z_A + Z_B)] = \frac{Z_A Z_C}{Z_A + Z_B + Z_C}$$

Similarly, we can obtain the remaining two coefficients of the impedance matrix:

$$Z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 = 0} = \frac{Z_C}{Z_B + Z_C}[Z_A \| (Z_B + Z_C)] = \frac{Z_A Z_C}{Z_A + Z_B + Z_C}$$

$$Z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1 = 0} = Z_C \| (Z_A + Z_B) = \frac{Z_C(Z_A + Z_B)}{Z_A + Z_B + Z_C}$$

Thus, the impedance matrix for the generic pi-network is written in the form

$$[\mathbf{Z}] = \frac{1}{Z_A + Z_B + Z_C}\begin{bmatrix} Z_A(Z_B + Z_C) & Z_A Z_C \\ Z_A Z_C & Z_C(Z_A + Z_B) \end{bmatrix}$$

The coefficients for the admittance matrix can be derived using (4.7). To find the value for Y_{11} we must find the ratio of current flow into port 1 to the voltage drop across this port when the second port is shortened (i.e., $v_2 = 0$).

$$Y_{11} = \left.\frac{i_1}{v_1}\right|_{v_2 = 0} = \frac{1}{Z_A} + \frac{1}{Z_B}$$

The value for coefficient Y_{12} of the admittance matrix can be obtained by shortening port 1 (i.e., forcing $v_1 = 0$) and measuring the ratio of the current i_1 to the voltage drop across port 2. We note that, when a positive voltage is applied to port 2, the current i_1 will flow away from port 1, resulting in a negative current:

$$Y_{12} = \left.\frac{i_1}{v_2}\right|_{v_1 = 0} = -\frac{1}{Z_B}$$

The rest of the admittance matrix can be derived in the same way, leading to the following final form

$$[\mathbf{Y}] = \begin{bmatrix} \frac{1}{Z_A}+\frac{1}{Z_B} & -\frac{1}{Z_B} \\ -\frac{1}{Z_B} & \frac{1}{Z_B}+\frac{1}{Z_C} \end{bmatrix} = \begin{bmatrix} Y_A+Y_B & -Y_B \\ -Y_B & Y_B+Y_C \end{bmatrix}$$

where $Y_A = Z_A^{-1}$, $Y_B = Z_B^{-1}$, and $Y_C = Z_C^{-1}$.

Direct evaluation shows that the obtained impedance and admittance matrices are indeed inversely related, which supports the validity of (4.8).

The practical determination of the matrix coefficients can be accomplished easily by enforcing open- and short-circuit conditions. However, as the frequency reaches RF limits, parasitic terminal effects can no longer be ignored and a different measurement approach becomes necessary.

Example 4-1 indicates that both impedance and admittance matrices are symmetric. This is generally true for linear, passive networks. *Passive* in this context implies not containing any current or voltage sources. We can state the symmetry as

$$Z_{nm} = Z_{mn} \tag{4.9}$$

which also applies for admittances because of (4.8). In fact, it can be proved that any reciprocal (that is, nonactive, linear) and lossless N-port network is symmetric.

Besides impedance and admittance network descriptions, there are two more useful parameter sets depending on how the voltage and currents are arranged. Restricting our discussion to two-port networks and with reference to Figure 4-1, we define the **chain** or **ABCD-matrix** as

$$\begin{Bmatrix} v_1 \\ i_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} v_2 \\ -i_2 \end{Bmatrix} \tag{4.10}$$

and the **hybrid** or **h-matrix** as

$$\begin{Bmatrix} v_1 \\ i_2 \end{Bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{Bmatrix} i_1 \\ v_2 \end{Bmatrix} \tag{4.11}$$

The determination of the individual matrix coefficients is identical to the method introduced for the impedance and admittance matrices. For instance, to find h_{12} in (4.11), we set i_1 to zero and compute the ratio of v_1 over v_2; that is,

$$h_{12} = \left.\frac{v_1}{v_2}\right|_{i_1 = 0}$$

It is interesting to note that in the hybrid representation parameters h_{21} and h_{12} define the forward current and reverse voltage gain, respectively. The remaining two parameters determine the input impedance (h_{11}) and output admittance (h_{22}) of the network. These properties of the hybrid representation explain why it is most often used for low-frequency transistor models. The following example shows the derivation of the hybrid matrix representation for a **bipolar-junction transistor** (BJT) for low-frequency operation.

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Example 4-2: Low-frequency hybrid network description of a BJT

Describe the common-emitter BJT transistor in terms of its hybrid network parameters for the low-frequency, small-signal transistor model shown in Figure 4-3.

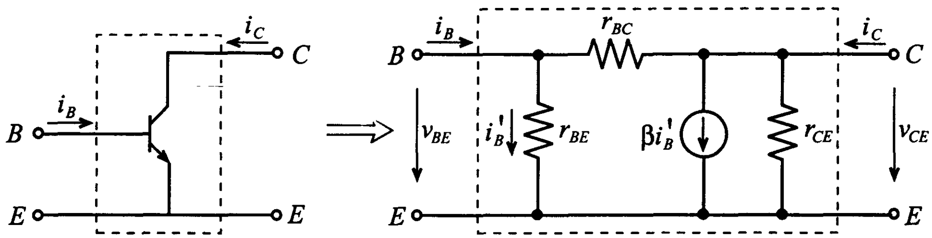

Figure 4-3 Common-emitter low-frequency, small-signal transistor model.

Solution: In the transistor model shown in Figure 4-3 r_{BE}, r_{BC}, and r_{CE} represent base-emitter, base-collector, and collector-emitter internal resistances of the transistor. The current through the current-controlled current source is dependent on the current i_B' flowing through the base-emitter resistance.

To evaluate the h_{11} parameter of the hybrid matrix according to (4.10) we must short-circuit the collector and emitter terminals, thus setting $v_2 = v_{CE} = 0$, and compute the ratio of the base-emitter voltage to the base current. Using the notation established in Figure 4-3, we notice that h_{11} is equal to the parallel combination of r_{BE} and r_{BC}:

$$h_{11} = \left.\frac{v_{BE}}{i_B}\right|_{v_{CE}=0} = \frac{r_{BC}r_{BE}}{r_{BE}+r_{BC}} \quad \text{(input impedance)}$$

Following a similar procedure, the relations for the remaining three parameters of the hybrid representation can be established as follows:

$$h_{12} = \left.\frac{v_{BE}}{v_{CE}}\right|_{i_B=0} = \frac{r_{BE}}{r_{BE}+r_{BC}} \quad \text{(voltage feedback ratio)}$$

$$h_{21} = \left.\frac{i_C}{i_B}\right|_{v_{CE}=0} = \frac{\beta r_{BC}-r_{BE}}{r_{BE}+r_{BC}} \quad \text{(small-signal current gain)}$$

$$h_{22} = \left.\frac{i_C}{v_{CE}}\right|_{i_B=0} = \frac{1}{r_{CE}} + \frac{1+\beta}{r_{BE}+r_{BC}} \quad \text{(output admittance)}$$

In the majority of all practical transistor designs, the current amplification coefficient β is usually much greater than unity and the collector-base resistance is much larger than the base-emitter resistance. Keeping these relations in mind, we can simplify the expressions derived for the h-matrix representation of the transistor:

$$h_{11} = \left.\frac{v_{BE}}{i_B}\right|_{v_{CE}=0} = r_{BE} \quad \text{(input impedance)}$$

$$h_{12} = \left.\frac{v_{BE}}{v_{CE}}\right|_{i_B=0} = 0 \quad \text{(voltage feedback ratio)}$$

$$h_{21} = \left.\frac{i_C}{i_B}\right|_{v_{CE}=0} = \beta \quad \text{(small-signal current gain)}$$

$$h_{22} = \left.\frac{i_C}{v_{CE}}\right|_{i_B = 0} = \frac{1}{r_{CE}} + \frac{\beta}{r_{BC}} \text{ (output admittance)}$$

The hybrid network description is a very popular way to characterize the BJT, and its h-parameter coefficients are widely reported in many data sheets.

Due to the presence of the current source in Example 4-2, the **h**-matrix is no longer symmetric ($h_{12} \neq h_{21}$) and the transistor model is nonreciprocal. In low-frequency electronic circuit design the coefficients of the hybrid matrix representation are often listed as h_{ie} for h_{11}, h_{re} for h_{12}, h_{fe} for h_{21}, and h_{oe} for h_{22}.

Up to this point we considered the problem of deriving the matrix representation based on a known topology and element values of the circuit. However, in practical design tasks it is often required to solve an **inverse** problem and obtain the equivalent circuit for an unknown or incompletely defined device based on a few measurements. This becomes extremely important when the characterization of the device is performed under a particular set of operating conditions, but it becomes necessary to evaluate its performance under completely different circuit conditions. In this case the use of the equivalent circuit representation enables an engineer to predict with reasonable accuracy the response of the device or circuit under changing operating conditions. In the following example we will derive the values of the internal resistances of the BJT from known **h**-matrix parameters.

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Example 4-3: Determination of internal resistances and current gain of a BJT based on *h*-parameter measurements

Use the equivalent circuit representation of the BJT shown in Figure 4-3 and employ the following measured hybrid parameters: $h_{ie} = 5\ \text{k}\Omega$, $h_{re} = 2\times10^{-4}$, $h_{fe} = 250$, $h_{oe} = 20\ \mu\text{S}$ (these parameters correspond to the 2n3904 transistor manufactured by

Motorola). Find the internal resistances r_{BE}, r_{BC}, and r_{CE}, and the current gain β.

Solution: As derived in Example 4-2, the values of the **h**-matrix for the equivalent circuit shown in Figure 4-3 are given by the following four equations:

$$h_{ie} = \frac{r_{BC}r_{BE}}{r_{BE}+r_{BC}} \quad \text{(input impedance)} \tag{4.12}$$

$$h_{re} = \frac{r_{BE}}{r_{BE}+r_{BC}} \quad \text{(voltage feedback ratio)} \tag{4.13}$$

$$h_{fe} = \frac{\beta r_{BC}-r_{BE}}{r_{BE}+r_{BC}} \quad \text{(small-signal current gain)} \tag{4.14}$$

$$h_{oe} = \frac{1}{r_{CE}} + \frac{1+\beta}{r_{BE}+r_{BC}} \quad \text{(output admittance)} \tag{4.15}$$

If we divide (4.12) by (4.13), we determine that the base-collector resistance is equal to the ratio of h_{ie} over h_{re}. Accordingly, for values given in the problem formulation, we obtain: $r_{BC} = h_{ie}/h_{re} = 71\ \text{M}\Omega$. Substituting this value into either equation (4.12) or (4.13), we find $r_{BE} = h_{ie}/(1-h_{re}) = 5\ \text{k}\Omega$. Knowing r_{BC} and r_{BE}, (4.14) allows us to find the current gain coefficient $\beta = (h_{re}-h_{fe})/(h_{re}-1) = 300.02$. Finally, the collector-emitter resistance can be evaluated from (4.15) as

$$r_{CE} = \frac{h_{ie}}{h_{oe}h_{ie}-h_{re}h_{fe}+2h_{re}^2-h_{re}} = 63.35\ \text{k}\Omega$$

We note from the obtained values that r_{BE} is indeed much smaller than r_{BC}.

This example provides a first idea of how the measured h-parameters can be used as a basis to characterize the BJT circuit model. The concept of "inverting" the measurements to determine circuit model parameters will be further analyzed in Chapter 7.

4.2 Interconnecting Networks

4.2.1 Series Connection of Networks

A series connection consisting of two two-port networks is shown in Figure 4-4. The individual networks are shown in impedance matrix representation.

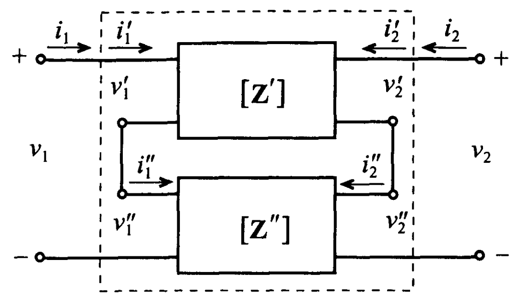


Figure 4-4 Series connection of two two-port networks.

In this case the individual voltages are additive while the currents remain the same. This results in

$$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} v_1' + v_1'' \\ v_2' + v_2'' \end{Bmatrix} = [\mathbf{Z}]\begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix} \tag{4.16}$$

where the new composite network $[\mathbf{Z}]$ takes the form

$$[\mathbf{Z}] = [\mathbf{Z}'] + [\mathbf{Z}''] = \begin{bmatrix} Z_{11}' + Z_{11}'' & Z_{12}' + Z_{12}'' \\ Z_{21}' + Z_{21}'' & Z_{22}' + Z_{22}'' \end{bmatrix} \tag{4.17}$$

Caution has to be exercised in not indiscriminately connecting individual networks, as short circuits may be created. This situation is exemplified in Figure 4-5 (a). The problem can be avoided by including a transformer, as seen in Figure 4-5 (b). The transformer in this case decouples input and output ports of the second network. However, this approach will only work for AC signals since the transformer acts as a high-pass filter and rejects all DC contributions.

When two networks are connected with the output interchanged, as shown in Figure 4-6, the most suitable representation is the hybrid form.

In the network connection that is shown in Figure 4-6, the voltages on the input ports and currents on the output ports are additive (i.e., $v_1 = v_1' + v_1''$ and $i_2 = i_2' + i_2''$), while the voltages on the output ports and currents on input ports are

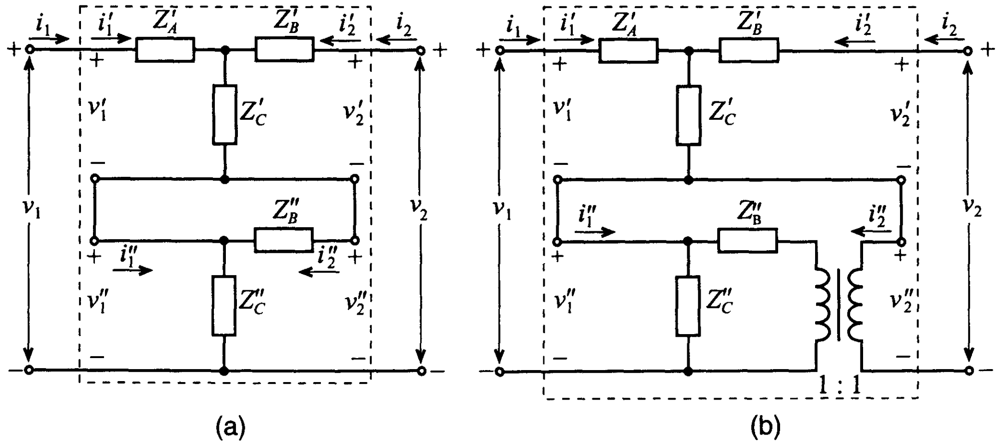


Figure 4-5 (a) Short circuit in series connection. (b) Transformer to avoid short circuit.

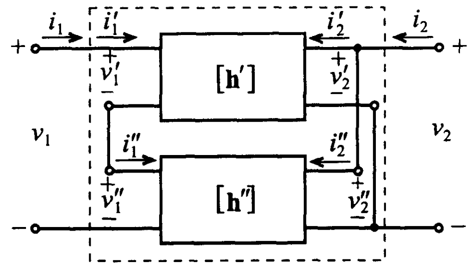


Figure 4-6 Connection of two-port networks suitable for hybrid representation.

the same (i.e., $v_2 = v_2' = v_2''$ and $i_1 = i_1' = i_1''$). From this observation we can conclude that the resulting **h**-matrix for the overall system is equal to the sum of the **h**-matrices of the individual networks:

$$\begin{Bmatrix} v_1 \\ i_2 \end{Bmatrix} = \begin{Bmatrix} v_1' + v_1'' \\ i_2' + i_2'' \end{Bmatrix} = \begin{bmatrix} h_{11}' + h_{11}'' & h_{12}' + h_{12}'' \\ h_{21}' + h_{21}'' & h_{22}' + h_{22}'' \end{bmatrix} \begin{Bmatrix} i_1 \\ v_2 \end{Bmatrix} \tag{4.18}$$

An example of this type of connection is the Darlington transistor pair Q_1 and Q_2 shown in Figure 4-7.

4.2.2 Parallel Connection of Networks

A parallel connection of two dual-port networks is shown in Figure 4-8 for the admittance matrices **Y**′ and **Y**″, where, unlike (4.16), the currents are now additive

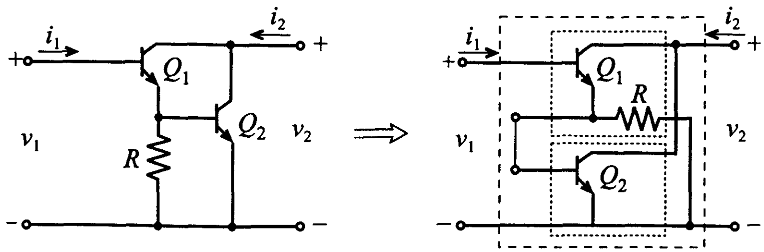


Figure 4-7 Series connection of two hybrid networks.

$$\begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix} = \begin{Bmatrix} i_1' + i_1'' \\ i_2' + i_2'' \end{Bmatrix} = [\mathbf{Y}]\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \tag{4.19}$$

and the new admittance matrix is defined as the sum of the individual admittances

$$[\mathbf{Y}] = [\mathbf{Y}'] + [\mathbf{Y}''] = \begin{bmatrix} Y_{11}' + Y_{11}'' & Y_{12}' + Y_{12}'' \\ Y_{21}' + Y_{21}'' & Y_{22}' + Y_{22}'' \end{bmatrix} \tag{4.20}$$

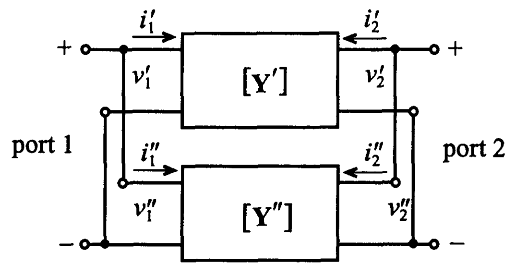


Figure 4-8 Parallel connection of two two-port networks.

4.2.3 Cascading Networks

The *ABCD*-parameter description is most suitable when cascading networks, as depicted in Figure 4-9 for the example of a two-transistor configuration. In this case the current on the output of the first network is equal in value, but opposite in sign, to the input current of the second network (i.e., $i_2' = -i_1''$). The voltage drop v_2' across the output port of the first network is equal to the voltage drop v_1'' across the input port of the second network. Thus, we can write the following relations:

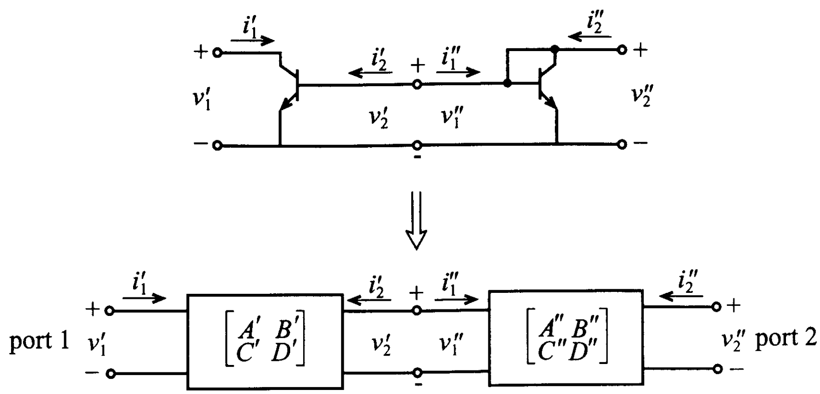


Figure 4-9 Cascading two networks.

$$\begin{Bmatrix} v_1 \\ i_1 \end{Bmatrix} = \begin{Bmatrix} v_1' \\ i_1' \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} v_2' \\ -i_2' \end{Bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{Bmatrix} v_1'' \\ i_1'' \end{Bmatrix}$$
$$= \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \begin{Bmatrix} v_2'' \\ -i_2'' \end{Bmatrix} \tag{4.21}$$

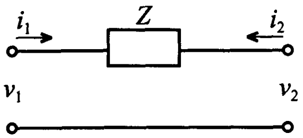
The overall system **ABCD**-matrix is equal to the product of the **ABCD**-matrices of the individual networks.

4.2.4 Summary of *ABCD* Network Representations

As we will see in subsequent chapters, microwave circuits can usually be represented as the result of cascading simpler networks. It is therefore important to develop **ABCD**-matrix representations for simple two-port networks that can be used as building blocks of more complex configurations. In this section several examples are considered for which we will derive *ABCD*-parameters such as transmission line, series impedance, and passive *T*-network. Other very useful circuits, such as parallel impedance, passive pi-network, and transformer, are left as exercises at the end of this chapter (see Problems 4.10–4.12). The results of all the computations are summarized in Table 4-1 at the end of this section.

Example 4-4: ***ABCD*** **network representation of an impedance element**

Compute the *ABCD*-matrix representation for the following network:



Solution: Guided by the definition (4.10), to determine parameter A we have to compute the ratio of the voltage drop across port 1 to the voltage drop across port 2 when the current into this port is equal to zero (i.e., port 2 is disconnected). In this case, it is apparent that for the circuit under consideration, the voltages on both ports are equal to their ratio, which is equal to unity

$$A = \left.\frac{v_1}{v_2}\right|_{i_2 = 0} = 1$$

To obtain the value for B, we need to find the ratio of the voltage drop across port 1 to the current flowing from port 2 when the terminals of port 2 are shortened. From the circuit topology, this ratio is equal to the impedance Z:

$$B = \left.\frac{v_1}{-i_2}\right|_{v_2 = 0} = Z$$

The remaining two parameters are found according to (4.10) of the *ABCD*-representation and can be shown to be

$$C = \left.\frac{i_1}{v_2}\right|_{i_2 = 0} = 0 \text{ and } D = \left.\frac{i_1}{-i_2}\right|_{v_2 = 0} = 1$$

The ***ABCD****-matrix coefficients are determined in a similar manner as the previously discussed* ***Z****-,* ***Y****-, and* ***h****-matrix coefficients.*

The accurate prediction of the coefficients again depends on the ability to enforce open- and short-circuit terminal conditions.

In the following example the $ABCD$-parameters of the passive T-network are determined. In the derivation of the parameters we will rely on the knowledge of $ABCD$ parameters for series and parallel connections of the impedance.

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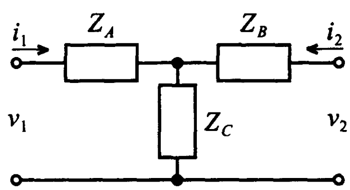
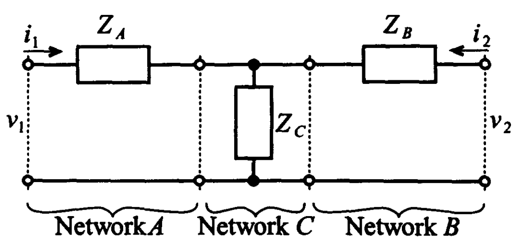
Example 4-5: ABCD matrix computation of a T-network

Compute the **ABCD**-matrix representation for the following T-network:



Solution: This problem can be solved using two different approaches. The first approach involves directly applying the definition of the **ABCD**-matrix coefficients and compute them as done in the previous example. Another approach is to utilize the knowledge of the $ABCD$-parameters for parallel and series connections of a single impedance. If we choose this method, we first have to break the initial circuit into subcircuits as follows:



As discussed previously, the **ABCD**-matrix representation of the entire circuit is equal to the product of the **ABCD**-matrices of the individual subcircuits. Using the results from Example 4-4 and Problem 4.8, we can write

$$[\mathbf{ABCD}] = \begin{bmatrix} 1 & Z_A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Z_C^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & Z_B \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \dfrac{Z_A}{Z_C} & Z_A + Z_B + \dfrac{Z_A Z_B}{Z_C} \\ \dfrac{1}{Z_C} & 1 + \dfrac{Z_B}{Z_C} \end{bmatrix}$$

*Here we see the advantage of using the **ABCD**-matrix representation in that a more complex network can be constructed by multiplication of simpler building blocks.*

As a last example, let us consider the computation of the *ABCD* parameters for a transmission line.

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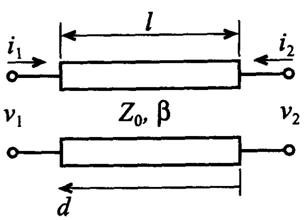
Example 4-6: ABCD-matrix coefficient computation of a transmission line section

Compute the **ABCD**-matrix representation of the following transmission line with characteristic impedance Z_0, propagation constant β, and length l.



Solution: Similar to Example 4-4, we have to apply open- and short-circuit conditions at port 2. For a transmission line these conditions are equivalent to the analysis of open- and short-circuit stub lines. Such lines are simply the open/short-circuit transmission line representations discussed in Sections 2.9.3 and 2.9.2. In these sections we found that for the open-circuit stub the voltage and current are given by the following expressions [see (2.71) and (2.72)]:

$$V(d) = 2V^+\cos(\beta d) \text{ and } I(d) = \frac{2jV^+}{Z_0}\sin(\beta d)$$

where distance d is measured from the open port (i.e., in our case from port 2).

For a short-circuit stub of length l voltages and currents are determined by (2.67) and (2.68):

$$V(d) = 2jV^+\sin(\beta d) \text{ and } I(d) = \frac{2V^+}{Z_0}\cos(\beta d)$$

where distance d is again measured from port 2 to port 1. In addition to these relations, it is important to recall that the current is defined as flowing *toward the load*. Therefore, the current is equal to i_1 at port 1 and equal to $-i_2$ at port 2.

Having determined the relations for voltages and currents, it is now possible to establish equations for the $ABCD$-parameters of the transmission line. Parameter A is defined as the ratio of the voltages at ports 1 and 2 when port 2 is open (i.e., we have to use the formulas for the open-circuit stub):

$$A = \left.\frac{v_1}{v_2}\right|_{i_2 = 0} = \frac{2V^+\cos(\beta l)}{2V^+} = \cos(\beta l)$$

where we employ the fact that $d = 0$ at port 2 and $d = l$ at port 1.

Parameter B is defined as the ratio of the voltage drop across port 1 to the current flowing from port 2 (i.e., toward the load) when port 2 is shorted. For this case we have to use the formulas for voltage and current defined for a short-circuit stub. This yields

$$B = \left.\frac{v_1}{-i_2}\right|_{v_2 = 0} = \frac{2jV^+\sin(\beta l)}{2V^+/Z_0} = jZ_0\sin(\beta l)$$

The remaining two coefficients are obtained in a similar manner:

$$C = \left.\frac{i_1}{v_2}\right|_{i_2=0} = \frac{\frac{2jV^+}{Z_0}\sin(\beta l)}{2V^+} = jY_0\sin(\beta l)$$

$$D = \left.\frac{i_1}{-i_2}\right|_{v_2=0} = \frac{\frac{2V^+}{Z_0}\cos(\beta l)}{\frac{2V^+}{Z_0}} = \cos(\beta l)$$

Thus, a transmission line with characteristic impedance Z_0, propagation constant β, and length l has the following matrix representation:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(\beta l) & jZ_0\sin(\beta l) \\ jY_0\sin(\beta l) & \cos(\beta l) \end{bmatrix}$$

The ABCD transmission line representation has the expected periodic parameter behavior similar to the line input impedance formula derived in Chapter 2.

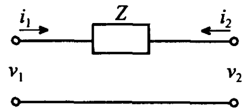
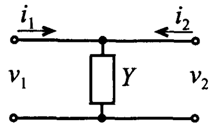
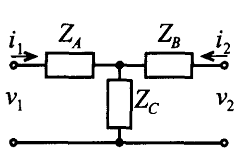
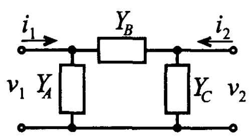
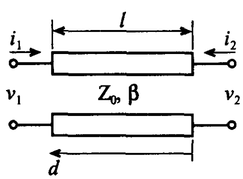
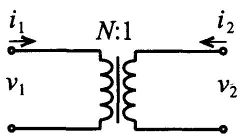
In Table 4-1 six of the most common circuit configurations are summarized in terms of their *ABCD* two-port network representations. From these six basic models, more complicated circuits are readily constructed by suitably combining these elementary networks.

4.3 Network Properties and Applications

4.3.1 Interrelations between Parameter Sets

Depending on the particular circuit configuration, we may be forced to convert between different parameter sets to arrive at a particular input/output description. For instance, the low-frequency transistor parameters are often recorded in **h**-matrix form. However, when cascading the transistor with additional networks, a more useful **ABCD**-matrix form may be appropriate. Thus, converting the **h**-matrix into an **ABCD**-matrix form and vice versa can greatly simplify the analysis.

Table 4-1 *ABCD*-Parameters of Some Useful Two-Port Circuits.

Circuit	*ABCD*-Parameters	
	$A = 1$ $C = 0$	$B = Z$ $D = 1$
	$A = 1$ $C = Y$	$B = 0$ $D = 1$
	$A = 1 + \frac{Z_A}{Z_C}$ $C = \frac{1}{Z_C}$	$B = Z_A + Z_B + \frac{Z_A Z_B}{Z_C}$ $D = 1 + \frac{Z_B}{Z_C}$
	$A = 1 + \frac{Y_B}{Y_C}$ $C = Y_A + Y_B + \frac{Y_A Y_B}{Y_C}$	$B = \frac{1}{Y_C}$ $D = 1 + \frac{Y_A}{Y_C}$
	$A = \cos\beta l$ $C = \frac{j\sin\beta l}{Z_0}$	$B = jZ_0 \sin\beta l$ $D = \cos\beta l$
	$A = N$ $C = 0$	$B = 0$ $D = \frac{1}{N}$

To show how the conversion between the individual parameter sets can be accomplished, let us find an **ABCD**-matrix representation of a given **h**-matrix. From the definition (4.11) we can express parameter A as follows

$$A = \left.\frac{v_1}{v_2}\right|_{i_2 = 0} = \frac{h_{11} i_1 + h_{12} v_2}{v_2} \tag{4.22}$$

In this expression we are able to re-express the current i_1 in (4.11) in terms of the voltage v_2 because $i_2 = 0$. The result is

$$A = \left.\frac{v_1}{v_2}\right|_{i_2 = 0} = \frac{h_{11}\left(-\frac{h_{22}}{h_{21}}v_2\right) + h_{12}v_2}{v_2} = \frac{1}{h_{21}}(h_{22}h_{11} - h_{12}h_{21}) = -\frac{\Delta h}{h_{21}} \quad (4.23)$$

where $\Delta h = h_{11}h_{22} - h_{12}h_{21}$ denotes the determinant of the **h**-matrix. Similarly, for the remaining coefficients we compute

$$B = -\left.\frac{v_1}{i_2}\right|_{v_2 = 0} = -\frac{h_{11}i_1}{i_2} = -\frac{h_{11}\left(\frac{i_2}{h_{21}}\right)}{i_2} = -\frac{h_{11}}{h_{21}} \quad (4.24)$$

$$C = \left.\frac{i_1}{v_2}\right|_{i_2 = 0} = \frac{\frac{-h_{22}}{h_{21}}v_2}{v_2} = -\frac{h_{22}}{h_{21}} \quad (4.25)$$

$$D = -\left.\frac{i_1}{i_2}\right|_{v_2 = 0} = -\frac{\frac{i_2}{h_{21}}}{i_2} = -\frac{1}{h_{21}} \quad (4.26)$$

This concludes the conversion from h-parameters to **ABCD** form. A similar procedure could have been performed from *ABCD*-parameters to **h**-matrix form.

As an additional case, let us investigate the conversion from *ABCD*-parameters to the **Z**-representation. Starting with (4.2) and using (4.11), we can develop the following relations:

$$Z_{11} = \left.\frac{v_1}{i_1}\right|_{i_2 = 0} = \frac{Av_2}{Cv_2} = \frac{A}{C} \quad (4.27)$$

$$Z_{12} = \left.\frac{v_1}{i_2}\right|_{i_1 = 0} = \frac{Av_2 - Bi_2}{\frac{C}{D}v_2} = \frac{Av_2 - \frac{BC}{D}v_2}{\frac{C}{D}v_2} = \frac{AD - BC}{C} = \frac{\Delta ABCD}{C} \quad (4.28)$$

$$Z_{21} = \left.\frac{v_2}{i_1}\right|_{i_2 = 0} = \frac{v_1/A}{Cv_2} = \frac{Av_2/A}{Cv_2} = \frac{1}{C} \quad (4.29)$$

$$Z_{22} = \left.\frac{v_2}{i_2}\right|_{i_1=0} = \frac{v_2}{Cv_2/D} = \frac{D}{C} \tag{4.30}$$

where $\Delta ABCD = AD - BC$ is the determinant of the **ABCD**-matrix.

By relying on the respective defining voltage and current relations, it is relatively straightforward to work out all parameter conversions. For convenience, Table 4-2 summarizes the formulas for the previously defined four network parameter sets (see also Appendix H for a complete list of all conversion formulas).

Table 4-2 Conversion between Different Network Representations

	[Z]	[Y]	[h]	[ABCD]
[Z]	$\begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix}$	$\begin{matrix} \frac{Z_{22}}{\Delta Z} & -\frac{Z_{12}}{\Delta Z} \\ -\frac{Z_{21}}{\Delta Z} & \frac{Z_{11}}{\Delta Z} \end{matrix}$	$\begin{matrix} \frac{\Delta Z}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ -\frac{Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{matrix}$	$\begin{matrix} \frac{Z_{11}}{Z_{21}} & \frac{\Delta Z}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{matrix}$
[Y]	$\begin{matrix} \frac{Y_{22}}{\Delta Y} & -\frac{Y_{12}}{\Delta Y} \\ -\frac{Y_{21}}{\Delta Y} & \frac{Y_{11}}{\Delta Y} \end{matrix}$	$\begin{matrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{matrix}$	$\begin{matrix} \frac{1}{Y_{11}} & -\frac{Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & \frac{\Delta Y}{Y_{11}} \end{matrix}$	$\begin{matrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{\Delta Y}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{matrix}$
[h]	$\begin{matrix} \frac{\Delta h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ -\frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & -\frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta h}{h_{11}} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} -\frac{\Delta h}{h_{21}} & -\frac{h_{11}}{h_{21}} \\ -\frac{h_{22}}{h_{21}} & -\frac{1}{h_{21}} \end{matrix}$
[ABCD]	$\begin{matrix} \frac{A}{C} & \frac{\Delta ABCD}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D}{B} & -\frac{\Delta ABCD}{B} \\ -\frac{1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta ABCD}{D} \\ -\frac{1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$

4.3.2 Analysis of Microwave Amplifier

In this section we consider, by way of an example, the usage of the conversion between different network representations to analyze a relatively complicated circuit. Basis of the analysis is the circuit diagram of a particular microwave amplifier shown in Figure 4-10.

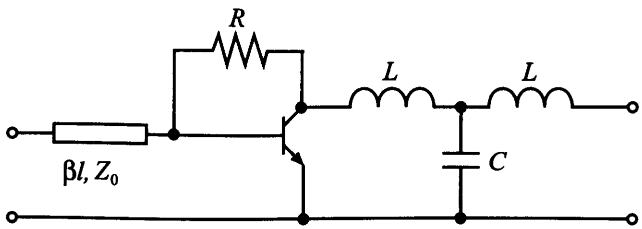


Figure 4-10 Microwave amplifier circuit diagram.

The first step is to break down the circuit into smaller, simpler subnetworks. This can be accomplished in several ways, one of which is shown in Figure 4-11.

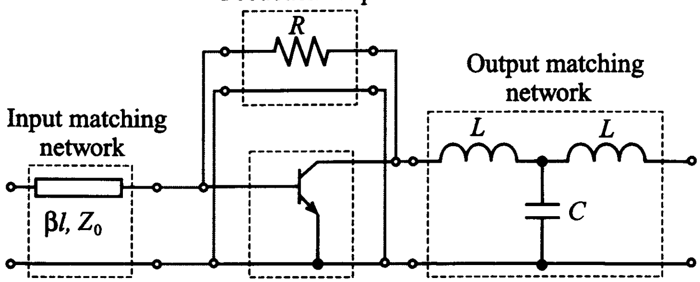


Figure 4-11 Subnetwork representation of the microwave amplifier.

As shown in this figure, the amplifier is divided into a set of four subcircuits. The input matching network consists of a transmission line (for convenience only the upper trace is shown) and is cascaded with a parallel combination of the transistor and a feedback loop. This circuit is then cascaded with an output matching network.

For the transistor we will use a high-frequency hybrid pi-network model (see also Chapter 7), which is shown in Figure 4-12.

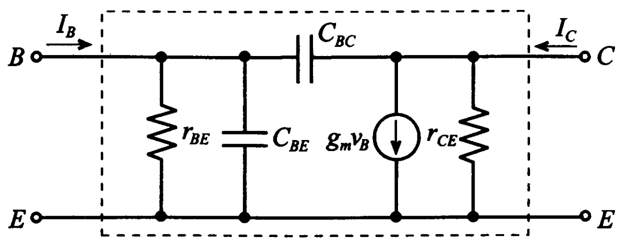


Figure 4-12 High-frequency hybrid transistor model.

The derivation of the h-parameters is left as a problem (Problem 4.13 at the end of this chapter). Here we only list the resulting **h**-matrix for the transistor:

$$h_{11} = h_{ie} = \frac{r_{BE}}{1 + j\omega(C_{BE} + C_{BC})r_{BE}} \tag{4.31a}$$

$$h_{12} = h_{re} = \frac{j\omega C_{BC} r_{BE}}{1 + j\omega(C_{BE} + C_{BC})r_{BE}} \tag{4.31b}$$

$$h_{21} = h_{fe} = \frac{r_{BE}(g_m - j\omega C_{BC})}{1 + j\omega(C_{BE} + C_{BC})r_{BE}} \tag{4.31c}$$

$$h_{22} = h_{oe} = \frac{1}{r_{CE}} + \frac{j\omega C_{BC}(1 + g_m r_{BE} + j\omega C_{BE} r_{BE})}{1 + j\omega(C_{BE} + C_{BC})r_{BE}} \tag{4.31d}$$

To compute the matrix for the parallel combination of the transistor and the feedback loop resistor we have to convert the **h**-matrix into a **Y**-matrix called $[\mathbf{Y}]_{\mathrm{tr}}$ in order to apply the summation rule (4.20). To accomplish this, we can use formulas from Table 4-2 and add the result to the **Y**-matrix of the feedback resistor. The admittance matrix for the feedback resistor can be derived either directly using the definition of the **Y**-matrix or by converting the $ABCD$-parameters derived in Example 4-4 into the **Y**-form. The result of these computations is

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}_R = \begin{bmatrix} R^{-1} & -R^{-1} \\ -R^{-1} & R^{-1} \end{bmatrix} \tag{4.32}$$

After the summation we obtain the admittance matrix for the parallel combination of the transistor and the feedback resistor $[\mathbf{Y}]_{\mathrm{tr}+R}$.

The same result could have been obtained if we had noticed that the feedback resistor is connected in parallel with the capacitor C_{BC} of the transistor. Thus, to obtain the admittance matrix of the parallel combination of the feedback resistor and the transistor, we simply need to replace C_{BC} in the **h**-matrix of the transistor with $C_{BC} + 1/(j\omega R)$ and then convert the resulting matrix into **Y**-representation.

The final step in the analysis is to multiply the **ABCD**-matrices for the input matching network (index: IMN), the transistor with feedback resistor (index: tr + R), and the output matching network (index: OMN)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\mathrm{amp}} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\mathrm{IMN}} \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\mathrm{tr}+R} \begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\mathrm{OMN}} \tag{4.33}$$

where the **ABCD**-matrices for the matching networks are found using the results from Table 4-1:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{IMN}} = \begin{bmatrix} \cos\beta l & jZ_0 \sin\beta l \\ \dfrac{j\sin\beta l}{Z_0} & \cos\beta l \end{bmatrix} \tag{4.34}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}_{\text{OMN}} = \begin{bmatrix} 1-\omega^2 LC & 2j\omega L - j\omega^3 L^2 C \\ j\omega C & 1-\omega^2 LC \end{bmatrix} \tag{4.35}$$

Due to rather lengthy expressions we are not presenting the final result for the *ABCD*-parameters of the entire amplifier. Instead we urge the interested readers to perform these computations by relying on a mathematical spreadsheet program of their choice (MathCad, MATLAB, Mathematica, etc.). One of the results of these computations is shown in Figure 4-13, where the small-signal current gain for the amplifier with short-circuited output (inverse of the *D*-coefficient) is plotted versus frequency for different values of the feedback resistor.

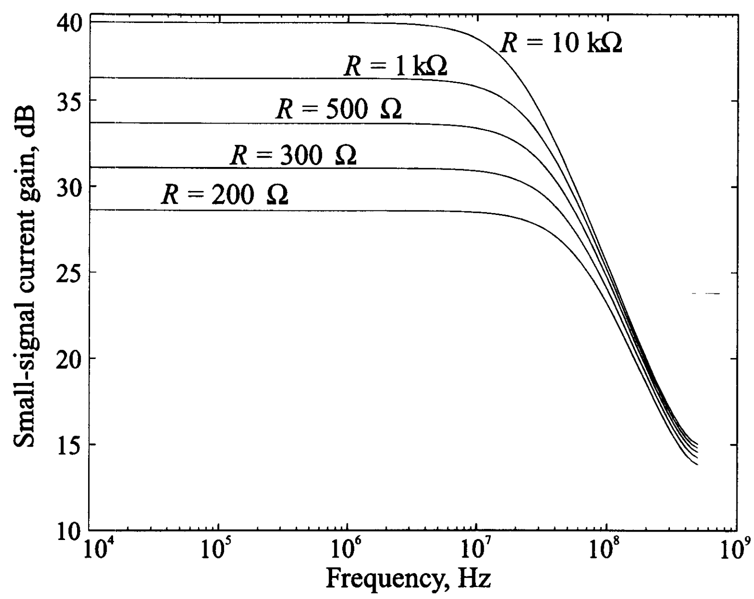


Figure 4-13 Small-signal current gain of the amplifier versus frequency for different values of the feedback resistor.

The computations are based on the circuit in Figure 4-11 with $L = 1\ \text{nH}$, $C = 10\ \text{pF}$, transmission line length of $l = 5\ \text{cm}$, and phase velocity equal to 65% of the speed of light. The transistor is described by the following set of values: $r_{BE} = 520\ \Omega$, $r_{CE} = 80k\ \Omega$, $C_{BE} = 10\ \text{pF}$, $C_{BC} = 1\ \text{pF}$, and $g_m = 0.192\ \text{s}$.

4.4 Scattering Parameters

In almost all databooks and technical literature regarding RF systems, the **scattering** or ***S*-parameter** representation plays a central role. This importance is derived from the fact that practical system characterizations can no longer be accomplished through simple open- or short-circuit measurements, as it is customarily done in low-frequency applications and as discussed at the beginning of this chapter. We should recall what happens when we attempt to create a short circuit with a wire: The wire itself possesses an inductance that can be of substantial magnitude at high frequency. Also, the open circuit leads to capacitive loading at the terminal. In either case, the open/short-circuit conditions needed to determine *Z*-, *Y*-, *h*-, and *ABCD*-parameters can no longer be guaranteed. Moreover, when dealing with wavepropagation phenomena, it is not desirable to introduce a reflection coefficient whose magnitude is unity. For instance, the terminal discontinuity will cause undesirable voltage and/or current wave reflections, leading to oscillations that can result in the destruction of the device. With the *S*-parameters, the RF engineer has a tool to characterize the two-port network description of practically all RF devices without requiring unachievable terminal conditions or causing harm to the **device under test** (DUT).

4.4.1 Definition of Scattering Parameters

Simply put, *S*-parameters are power wave descriptors that permit us to define the input-output relations of a network in terms of incident and reflected power waves. With reference to Figure 4-14 we define an incident *normalized* power wave a_n and a reflected *normalized* power wave b_n as follows:

$$a_n = \frac{1}{2\sqrt{Z_0}}(V_n + Z_0 I_n) \tag{4.36a}$$

$$b_n = \frac{1}{2\sqrt{Z_0}}(V_n - Z_0 I_n) \tag{4.36b}$$

where the index n refers either to port number 1 or 2. The impedance Z_0 is the characteristic impedance of the connecting lines on the input and output side of the network. Under more general conditions the line impedance on the input side can differ from the line impedance on the output side. However, for our initial discussion, we will keep things simple and assume that both impedances are the same.

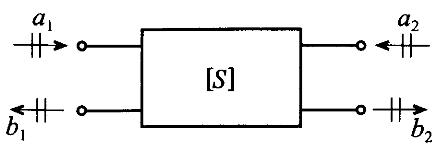


Figure 4-14 Convention used to define *S*-parameters for a two-port network.

Inverting (4.36) leads to the following voltage and current expressions:

$$V_n = \sqrt{Z_0}(a_n + b_n) \tag{4.37a}$$

$$I_n = \frac{1}{\sqrt{Z_0}}(a_n - b_n) \tag{4.37b}$$

The physical meaning of (4.36) becomes clear when we recall the equations for power:

$$P_n = \frac{1}{2}\mathrm{Re}\{V_n I_n^*\} = \frac{1}{2}\left(|a_n|^2 - |b_n|^2\right) \tag{4.38}$$

Isolating forward and backward traveling wave components in (4.37), we immediately see

$$a_n = \frac{V_n^+}{\sqrt{Z_0}} = \sqrt{Z_0}I_n^+ \tag{4.39a}$$

$$b_n = -\frac{V_n^-}{\sqrt{Z_0}} = -\sqrt{Z_0}I_n^- \tag{4.39b}$$

which is consistent with the definitions (4.37) since

$$V_n = V_n^+ + V_n^- = Z_0 I_n^+ - Z_0 I_n^- \tag{4.40}$$

Based on the directional convention shown in Figure 4-14 we are now in a position to define the *S*-parameters:

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{4.41}$$

where the terms are

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2 = 0} \equiv \frac{\text{reflected power wave at port 1}}{\text{incident power wave at port 1}} \tag{4.42a}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2 = 0} \equiv \frac{\text{transmitted power wave at port 2}}{\text{incident power wave at port 1}} \tag{4.42b}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1 = 0} \equiv \frac{\text{reflected power wave at port 2}}{\text{incident power wave at port 2}} \tag{4.42c}$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1 = 0} \equiv \frac{\text{transmitted power wave at port 1}}{\text{incident power wave at port 2}} \tag{4.42d}$$

We observe that the conditions $a_2 = 0$ and $a_1 = 0$ imply that no power waves are returned to the network at either port 2 or port 1. However, these condition can only be ensured when the connecting transmission lines are terminated into their characteristic impedances.

Since the S-parameters are closely related to power relations, we can express the normalized input and output waves in terms of time averaged power. With reference to Section 2.10.2 we note that the average power at port 1 is given by

$$P_1 = \frac{1}{2}\frac{|V_1^+|^2}{Z_0}(1 - |\Gamma_{in}|^2) = \frac{1}{2}\frac{|V_1^+|^2}{Z_0}(1 - |S_{11}|^2) \tag{4.43}$$

where the reflection coefficient at the input side is expressed in terms of S_{11} under matched output according to the following argument:

$$\Gamma_{in} = \frac{V_1^-}{V_1^+} = \left.\frac{b_1}{a_1}\right|_{a_2 = 0} = S_{11} \tag{4.44}$$

This also allows us to redefine the VSWR at port 1 in terms of S_{11} as

$$\text{VSWR} = \frac{1 + |S_{11}|}{1 - |S_{11}|} \tag{4.45}$$

Furthermore, based on (4.39a) we can identify the incident power in (4.43) and express it in terms of a_1:

$$\frac{1}{2}\frac{|V_1^+|^2}{Z_0} = P_{inc} = \frac{|a_1|^2}{2} \tag{4.46}$$

which is the maximal available power from the generator. Using (4.46) and (4.44) in (4.43) finally gives us the total power at port 1 (under matched output condition) expressed as a combination of incident and reflected powers:

$$P_1 = P_{inc} + P_{ref} = \frac{1}{2}(|a_1|^2 - |b_1|^2) = \frac{|a_1|^2}{2}(1 - |\Gamma_{in}|^2) \tag{4.47}$$

If the reflection coefficient, or S_{11}, is zero, all available power from the source is delivered to port 1 of the network. An identical analysis at port 2 yields

$$P_2 = \frac{1}{2}(|a_2|^2 - |b_2|^2) = \frac{|a_2|^2}{2}(1 - |\Gamma_{out}|^2) \tag{4.48}$$

4.4.2 Meaning of S-Parameters

As already mentioned in the previous section, the S-parameters can only be determined under conditions of perfect matching on the input or output side. For instance, in order to record S_{11} and S_{21} we have to ensure that on the output side the line impedance Z_0 is matched for $a_2 = 0$ to be enforced, as shown in Figure 4-15.

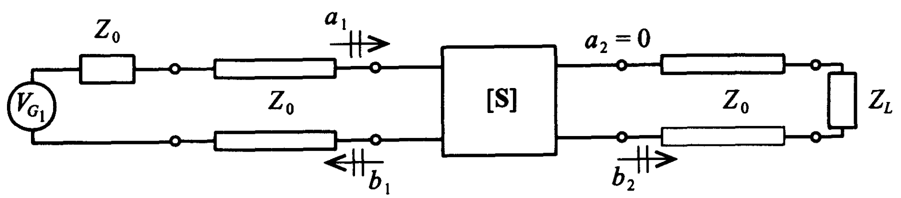


Figure 4-15 Measurement of S_{11} and $S_{21}$by matching the line impedance Z_0 at port 2 through a corresponding load impedance $Z_L = Z_0$.

This configuration allows us to compute S_{11} by finding the input reflection coefficient:

$$S_{11} = \Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \tag{4.49}$$

In addition, taking the logarithm of the magnitude of S_{11} gives us the return loss in dB

$$\text{RL} = -20\log|S_{11}| \tag{4.50}$$

Moreover, with port 2 properly terminated, we find

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \left.\frac{V_2^-/\sqrt{Z_0}}{(V_1+Z_0I_1)/(2\sqrt{Z_0})}\right|_{I_2^+ = V_2^+ = 0} \tag{4.51}$$

Since $a_2 = 0$, we can set to zero the positive traveling voltage and current waves at port 2. Replacing V_1 by the generator voltage V_{G1} minus the voltage drop over the source impedance Z_0, $V_{G1} - Z_0 I_1$ gives

$$S_{21} = \frac{2V_2^-}{V_{G1}} = \frac{2V_2}{V_{G1}} \tag{4.52}$$

Here we observe that the voltage recorded at port 2 is directly related to the generator voltage and thus specifies the **forward voltage gain** of the network. To find the **forward power gain**, we square (4.52) to obtain

$$G_0 = |S_{21}|^2 = \left|\frac{V_2}{V_{G1}/2}\right|^2 \tag{4.53}$$

If we reverse the measurement procedure and attach a generator voltage V_{G2} to port 2 and properly terminate port 1, as shown in Figure 4-16, we can determine the remaining two S-parameters, S_{22} and S_{12}.

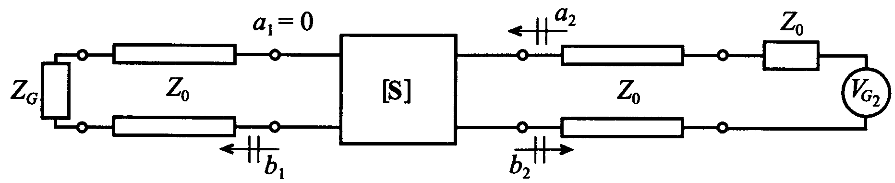


Figure 4-16 Measurement of S_{22} and S_{12} by matching the line impedance Z_0 at port 1 through a corresponding input impedance $Z_G = Z_0$.

To compute S_{22} we need to find the output reflection coefficient Γ_{out} in a similar way as already discussed for S_{11}:

$$S_{22} = \Gamma_{out} = \frac{Z_{out} - Z_0}{Z_{out} + Z_0} \tag{4.54}$$

and for S_{12}

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1 = 0} = \left.\frac{V_1^-/\sqrt{Z_0}}{(V_2 + Z_0 I_2)/(2\sqrt{Z_0})}\right|_{I_1^+ = V_1^+ = 0} \tag{4.55}$$

The term S_{12} can further be manipulated through the substitution of V_2 by $V_{G2} - Z_0 I_2$, leading to the form

$$S_{12} = \frac{2V_1^-}{V_{G2}} = \frac{2V_1}{V_{G2}} \tag{4.56}$$

known as the **reverse voltage gain** and whose square $|S_{12}|^2$ is identified as **reverse power gain**. While determining S_{11} and S_{22} can be directly computed as part of the impedance definitions, S_{12} an S_{21} require the replacement of the defining voltages by the appropriate network parameters. In the following example, the S-parameters are computed for a simple, three element network.

Example 4-7: Determination of a T-network elements

Find the S-parameters and the resistive elements for the 3 dB attenuator network shown in Figure 4-17(a) assuming that the network is placed into a transmission line section with a characteristic line impedance of $Z_0 = 50\ \Omega$.

Solution: An attenuator should be matched to the line impedance and must therefore meet the requirement $S_{11} = S_{22} = 0$. As a result, based on Figure 4-17(b) and consistent with (4.49), we set

$$Z_{in} = R_1 + \frac{R_3(R_2 + 50\ \Omega)}{(R_3 + R_2 + 50\ \Omega)} = 50\ \Omega$$

Because of symmetry, it is immediately clear that $R_1 = R_2$. We now investigate the voltage $V_2 = V_2^-$ at port 2 in terms of $V_1 = V_1^+$. According to the circuit configuration shown in Figure 4-17(c), the following expression is obtained

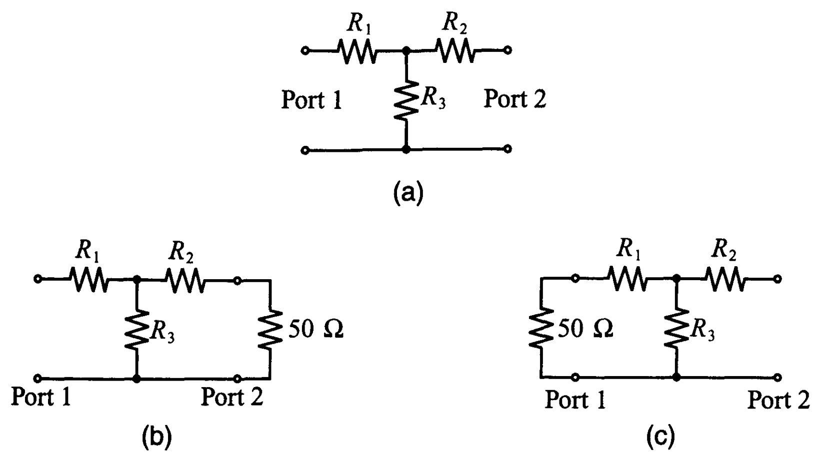


Figure 4-17 S-parameter computation for a T-network. (a) circuit diagram; (b) circuit for S_{11} and S_{21} measurements; (c) circuit for S_{12} and S_{22} measurements.

$$V_2 = \left(\frac{\dfrac{R_3(R_1 + 50\ \Omega)}{R_3 + R_1 + 50\ \Omega}}{\dfrac{R_3(R_1 + 50\ \Omega)}{R_3 + R_1 + 50\ \Omega} + R_1}\right)\left(\frac{50\ \Omega}{50\ \Omega + R_1}\right)V_1$$

For a 3 dB attenuation, we require

$$S_{21} = \frac{2V_2}{V_{G1}} = \frac{V_2}{V_1} = \frac{1}{\sqrt{2}} = 0.707 = S_{12}$$

Setting the ratio of V_2/V_1 to 0.707 in the preceding equation allows us, in combination with the input impedance expression, to determine R_1 and R_3. After simplification it is seen that

$$R_1 = R_2 = \frac{\sqrt{2}-1}{\sqrt{2}+1}Z_0 = 8.58\ \Omega \text{ and } R_3 = 2\sqrt{2}\cdot Z_0 = 141.4\ \Omega$$

The choice of the resistor network ensures that at the input and output ports an impedance of 50 Ω is maintained. This implies that this network can be inserted into a 50 Ω transmission line section without causing undesired reflections, resulting in an insertion loss.

The definitions for the S-parameters require appropriate termination. For instance, if S_{11} is desired, the transmission line connected to port 2 has to be terminated into its characteristic line impedance. This does not necessarily mean that the output impedance Z_{out} of the network has to be matched to the line impedance Z_0. Rather, the line impedance must be matched to ensure that no wave is reflected from the load, as implied by $a_2 = 0$. If this is not the case, we will see in Section 4.4.5 how S_{11} is modified.

4.4.3 Chain Scattering Matrix

To extend the concept of the S-parameter representation to cascaded networks, it is more efficient to rewrite the power wave expressions arranged in terms of input and output ports. This results in the **chain scattering matrix** notation. That is,

$$\begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{Bmatrix} b_2 \\ a_2 \end{Bmatrix} \tag{4.57}$$

It is immediately seen that the cascading of two dual-port networks becomes a simple multiplication. This is apparent in Figure 4-18, where network A (given by matrix $[\mathbf{T}]_A$) is connected to network B (given by matrix $[\mathbf{T}]_B$).

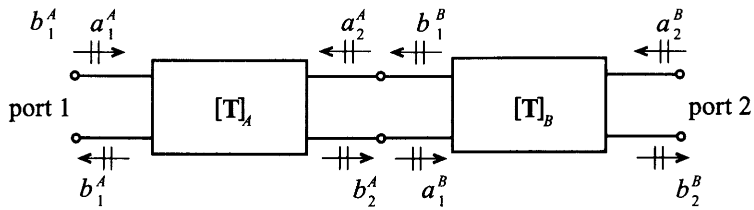


Figure 4-18 Cascading of two networks A and B.

If network A is described by the relation

$$\begin{Bmatrix} a_1^A \\ b_1^A \end{Bmatrix} = \begin{bmatrix} T_{11}^A & T_{12}^A \\ T_{21}^A & T_{22}^A \end{bmatrix} \begin{Bmatrix} b_2^A \\ a_2^A \end{Bmatrix} \tag{4.58a}$$

and network B by

$$\begin{Bmatrix} a_1^B \\ b_1^B \end{Bmatrix} = \begin{bmatrix} T_{11}^B & T_{12}^B \\ T_{21}^B & T_{22}^B \end{bmatrix} \begin{Bmatrix} b_2^B \\ a_2^B \end{Bmatrix} \tag{4.58b}$$

we notice, based on the parameter convention shown in Figure 4-18, that

$$\begin{Bmatrix} b_2^A \\ a_2^A \end{Bmatrix} = \begin{Bmatrix} a_1^B \\ b_1^B \end{Bmatrix} \tag{4.59}$$

Thus, for the combined system, we conclude

$$\begin{Bmatrix} a_1^A \\ b_1^A \end{Bmatrix} = \begin{bmatrix} T_{11}^A & T_{12}^A \\ T_{21}^A & T_{22}^A \end{bmatrix} \begin{bmatrix} T_{11}^B & T_{12}^B \\ T_{21}^B & T_{22}^B \end{bmatrix} \begin{Bmatrix} b_2^B \\ a_2^B \end{Bmatrix} \tag{4.60}$$

which is the desired matrix multiplication. Therefore, the chain scattering matrix plays a similar role as the **ABCD**-matrix discussed earlier.

The conversion from the **S**-matrix to the chain matrix notation follows identical steps as outlined in Section 4.3.1. In particular, to compute T_{11} for instance, we see that

$$T_{11} = \left.\frac{a_1}{b_2}\right|_{a_2 = 0} = \frac{a_1}{S_{21}a_1} = \frac{1}{S_{21}} \tag{4.61}$$

Similarly,

$$T_{12} = -\frac{S_{22}}{S_{21}} \tag{4.62}$$

$$T_{21} = \frac{S_{11}}{S_{21}} \tag{4.63}$$

$$T_{22} = \frac{-(S_{11}S_{22} - S_{12}S_{21})}{S_{21}} = \frac{-\Delta S}{S_{21}} \tag{4.64}$$

Conversely, when the chain scattering parameters are given and we need to convert to S-parameters, we find the following relations:

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2 = 0} = \frac{T_{21}b_2}{T_{11}b_2} = \frac{T_{21}}{T_{11}} \tag{4.65}$$

$$S_{12} = \frac{T_{11}T_{22} - T_{21}T_{12}}{T_{11}} = \frac{\Delta T}{T_{11}} \tag{4.66}$$

$$S_{21} = \frac{1}{T_{11}} \tag{4.67}$$

$$S_{22} = -\frac{T_{12}}{T_{11}} \tag{4.68}$$

Alternatively, a matrix manipulation as discussed in the next section could have been carried out with the same result.

4.4.4 Conversion between *Z*- and *S*-Parameters

We have already seen how certain *S*-parameters can be defined in terms of input and output impedances of a network [i.e., equations (4.49) and (4.54)]. In this section, we go through a formal conversion between the *Z*- and *S*-parameter sets. Once this interrelation is established, we are able to formulate conversion links between all six network parameter sets (*S, Z, Y, ABCD, h, T*).

To find the conversion between the previously defined *S*-parameters and the *Z*-parameters, let us begin with the defining *S*-parameter relation in matrix notation [i.e., (4.41)]

$$\{\mathbf{b}\} = [\mathbf{S}]\{\mathbf{a}\} \tag{4.69}$$

Multiplying by $\sqrt{Z_0}$ gives

$$\sqrt{Z_0}\{\mathbf{b}\} = \{\mathbf{V}^-\} = \sqrt{Z_0}[\mathbf{S}]\{\mathbf{a}\} = [\mathbf{S}]\{\mathbf{V}^+\} \tag{4.70}$$

Adding $\{\mathbf{V}^+\} = \sqrt{Z_0}\{\mathbf{a}\}$ to both sides results in

$$\{\mathbf{V}\} = [\mathbf{S}]\{\mathbf{V}^+\} + \{\mathbf{V}^+\} = ([\mathbf{S}] + [\mathbf{E}])\{\mathbf{V}^+\} \tag{4.71}$$

where $[\mathbf{E}]$ is the identity matrix. To compare this form with the impedance expression $\{\mathbf{V}\} = [\mathbf{Z}]\{\mathbf{I}\}$, we have to express $\{\mathbf{V}^+\}$ in terms of $\{\mathbf{I}\}$. This is accomplished by first subtracting $[\mathbf{S}]\{\mathbf{V}^+\}$ from both sides of $\{\mathbf{V}^+\} = \sqrt{Z_0}\{\mathbf{a}\}$; that is,

$$\{\mathbf{V}^+\} - [\mathbf{S}]\{\mathbf{V}^+\} = \sqrt{Z_0}(\{\mathbf{a}\} - \{\mathbf{b}\}) = Z_0\{\mathbf{I}\} \tag{4.72}$$

Now, by isolating $\{\mathbf{V}^+\}$, it is seen that

$$\{\mathbf{V}^+\} = Z_0([\mathbf{E}] - [\mathbf{S}])^{-1}\{\mathbf{I}\} \tag{4.73}$$

Substituting (4.73) into (4.71) yields the desired result of

$$\{\mathbf{V}\} = ([\mathbf{S}] + [\mathbf{E}])\{\mathbf{V}^+\} = Z_0([\mathbf{S}] + [\mathbf{E}])([\mathbf{E}] - [\mathbf{S}])^{-1}\{\mathbf{I}\} \tag{4.74}$$

or

$$[\mathbf{Z}] = Z_0([\mathbf{S}] + [\mathbf{E}])([\mathbf{E}] - [\mathbf{S}])^{-1} \tag{4.75}$$

Explicit evaluation yields

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = Z_0 \begin{bmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{bmatrix} \begin{bmatrix} 1-S_{11} & -S_{12} \\ -S_{21} & 1-S_{22} \end{bmatrix}^{-1}$$

$$= \frac{Z_0 \begin{bmatrix} 1+S_{11} & S_{12} \\ S_{21} & 1+S_{22} \end{bmatrix}}{(1-S_{11})(1-S_{22}) - S_{21}S_{12}} \begin{bmatrix} 1-S_{22} & S_{12} \\ S_{21} & 1-S_{11} \end{bmatrix} \tag{4.76}$$

Identifying individual terms is now easily carried out. A complete summary of all network coefficient sets is given in Appendix C.

4.4.5 Signal Flow Chart Modeling

The analysis of RF networks and their overall interconnection is greatly facilitated through signal flow charts as commonly used in system and control theory. As originally introduced to seismology and remote sensing, wave propagation can be associated with directed paths and associated nodes connecting these paths. Even complicated networks are easily reduced to input-output relations in which the reflection and transmission coefficients play integral parts. In this section we will briefly summarize key principles needed for a signal flow network analysis.

The main concepts required to construct flow charts are as follows:

1. **Nodes** that are deployed to identify network parameters such as a_1, b_1, a_2, b_2 when dealing with S-parameters
2. **Branches** that are needed when connecting the network parameters
3. Addition and subtraction of branch values in accordance with the directions of the branches

We will now discuss these three items in detail. To this end let us consider a section of a transmission line that is terminated in a load impedance Z_L, as seen in Figure 4-19.

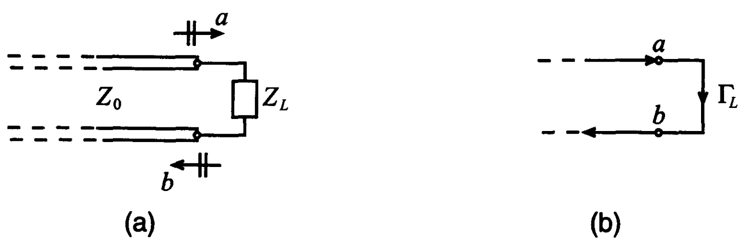


Figure 4-19 Terminated transmission line segment with incident and reflected S-parameter description. (a) Conventional form, and (b) Signal flow form.

Even though we could use voltage values as node identifier, it is the S-parameter representation that finds widespread use. In Figure 4-19(b) the nodes a and b are connected through the load reflection coefficient Γ_L. This makes sense since the reflection coefficient is the ratio b/a, so that it simply states that node b is found as a result of multiplying node a by Γ_L. This is depicted in generic form in Figure 4-20.

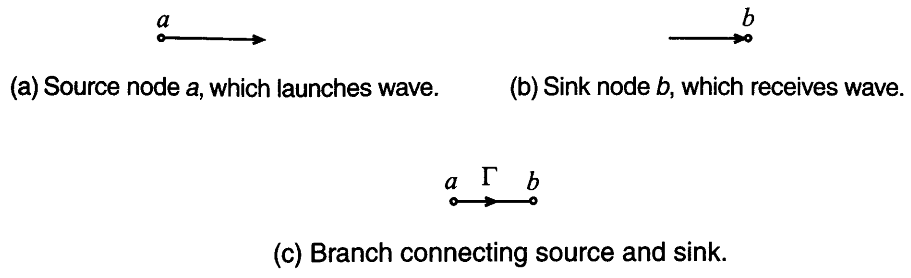


Figure 4-20 Generic source node (a), receiver node (b), and the associated (c) branch connection.

In terms of notation, we can encode the situation shown in Figure 4-20 as

$$b = \Gamma a \tag{4.77}$$

A more complicated situation arises when we need to make the transmission line circuit shown in Figure 4-19 more realistic by including a source term, as seen in Figure 4-21.

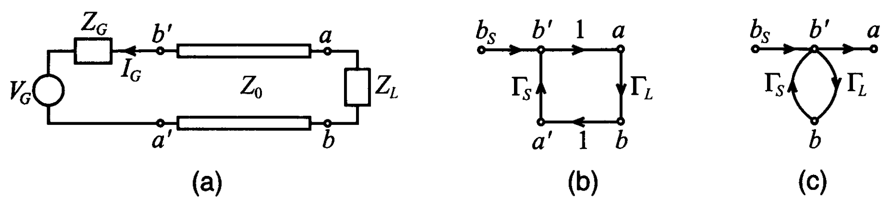


Figure 4-21 Terminated transmission line with source. (a) conventional form, (b) signal flow form, and (c) simplified signal flow form.

Unlike Figure 4-19, the nodes a and b are preceded by two additional nodes that we shall denote a' and b'. The ratio b'/a' defines the source reflection coefficient Γ_S as already discussed in Section 2.11. Here we also see that b' is given by multiplying a' with the source reflection coefficient. By relying on the concept of summation, we define b' as the sum of b_S and $a'\Gamma_S$. Thus, the source b_S is

$$b_S = b' - a'\Gamma_S \tag{4.78}$$

An explicit expression for b_S is obtained by noting that

$$V_S = V_G + I_G Z_G \tag{4.79}$$

based on an outflowing current convention (see Figure 4-21). This can be converted into the form

$$V_S^+ + V_S^- = V_G + Z_G\left(\frac{V_S^+}{Z_0} - \frac{V_S^-}{Z_0}\right) \tag{4.80}$$

Rearranging terms and division by $\sqrt{Z_0}$ gives

$$\frac{\sqrt{Z_0}}{Z_G + Z_0}V_G = \frac{V_S^-}{\sqrt{Z_0}} - \Gamma_S\frac{V_S^+}{\sqrt{Z_0}} \tag{4.81}$$

When comparing (4.81) with (4.78), we immediately see that

$$b_S = \frac{\sqrt{Z_0}}{Z_G + Z_0}V_G \tag{4.82}$$

An important conclusion can be drawn when expressing a' in (4.78) by $\Gamma_L b'$ so that we obtain

$$b' = b_S + \Gamma_L \Gamma_S b' = \frac{b_S}{1 - \Gamma_L \Gamma_S} \tag{4.83}$$

This is a known as a self- or feedback loop (see Figure 4-22), which allows us to represent the nodes b_S and b' by a single branch whose value is given by (4.83).

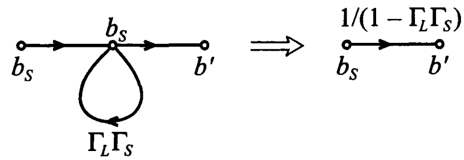
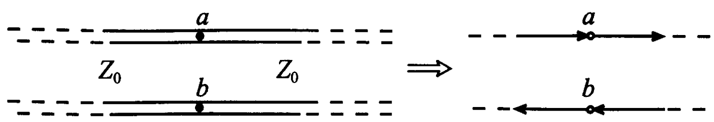
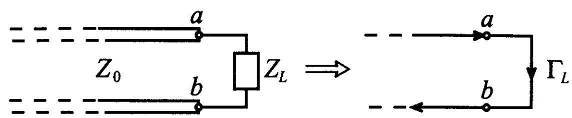
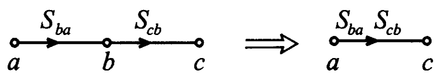
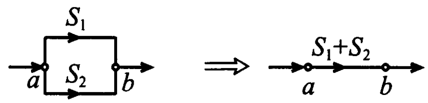
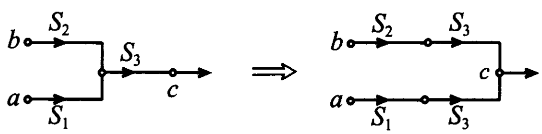
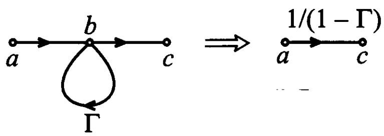


Figure 4-22 A self-loop that collapses to a single branch.

All signal flow chart principles can therefore be reduced to six building blocks, as summarized in Table 4-3.

By way of an example, let us analyze a more complicated RF circuit consisting of a sourced and terminated dual-port network.

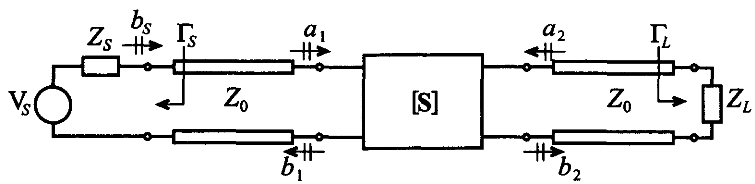
Table 4-3 Signal flow chart building blocks

Description	Graphical Representation
Nodal Assignment	
Branch	
Series Connection	
Parallel Connection	
Splitting of Branches	
Self-loop	

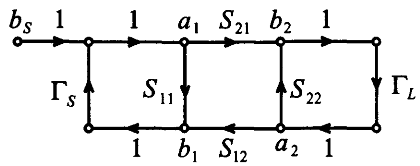
RF&MW→

Example 4-8: Flow chart analysis of a dual-port network

For the network shown in Figure 4-23 find the ratios of b_1/a_1 and a_1/b_S. Assume unity for the multiplication factor of the transmission line segments.



(a) Circuit representation



(b) Signal flow chart

Figure 4-23 Sourced and terminated two-port network.

Solution: The process of setting up the individual ratios is explained best by going through a step-by-step simplification for the ratio a_1/b_S employing the rules summarized in Table 4-3. Figure 4-24 depicts the five steps.

Step 1: Splitting of the rightmost loop between b_2 and a_2, leading to the self-loop $S_{22}\Gamma_L$

Step 2: Decomposition of the self-loop between branches a_1 and b_2, resulting in the multiplication factor $S_{21}/(1-S_{22}\Gamma_L)$, which can be combined with Γ_L and S_{12}

Step 3: Series and parallel connections between a_1 and b_1, leading to the input reflection coefficient

$$\Gamma_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}}{1-S_{22}\Gamma_L}\Gamma_L$$

Step 4: Splitting the loop into a self-loop, resulting in the multiplication factor

$$\left(S_{11} + \frac{S_{12}S_{21}}{1-S_{22}\Gamma_L}\Gamma_L\right)\Gamma_S$$

Step 5: Decomposition of the self-loop at a_1, leading to the expression

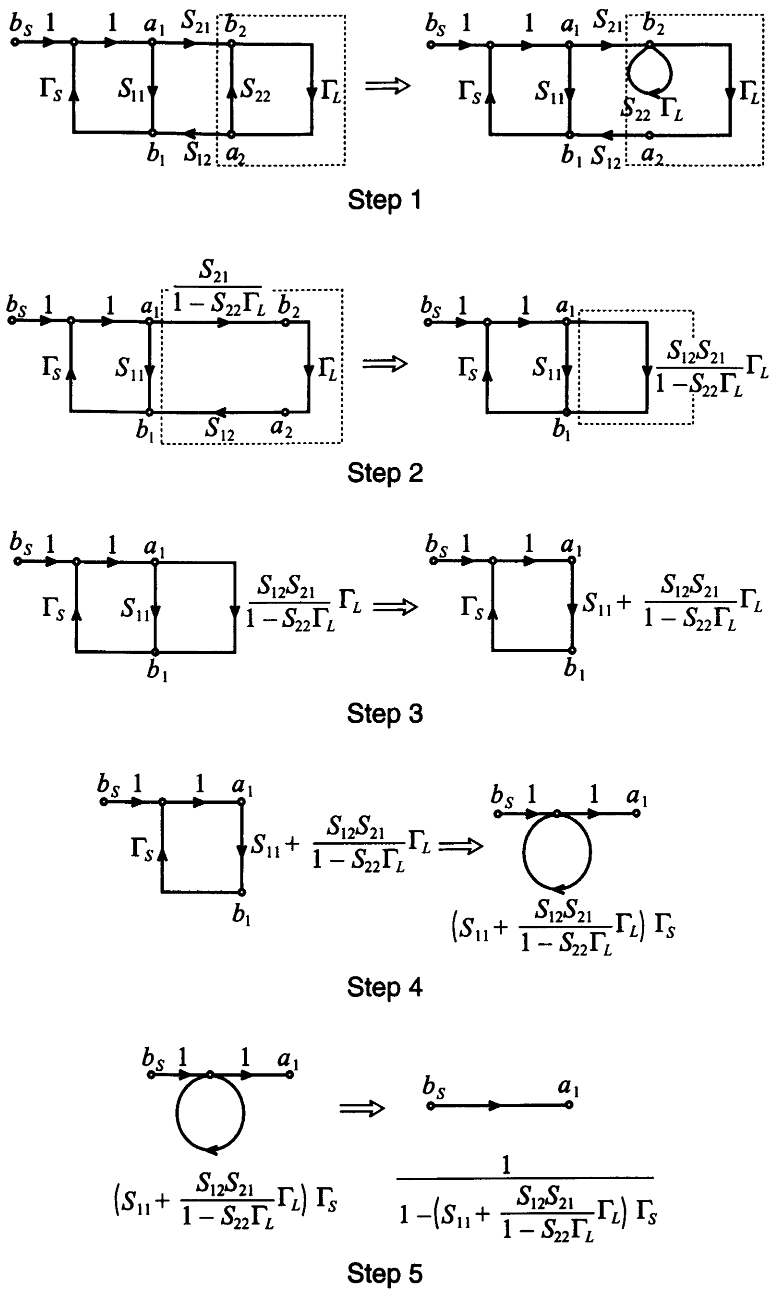


Figure 4-24 Step-by-step simplification to determine the ratio a_1/b_S.

$$a_1 = \frac{1}{1 - \left(S_{11} + \frac{S_{12}S_{21}}{1 - S_{22}\Gamma_L}\Gamma_L\right)\Gamma_S} b_S$$

Rearranging and simplification leads to the final form:

$$\frac{a_1}{b_S} = \frac{1 - S_{22}\Gamma_L}{1 - (S_{11}\Gamma_S + S_{22}\Gamma_L + S_{12}S_{21}\Gamma_S) + S_{11}S_{22}\Gamma_S\Gamma_L}$$

The preceding derivation follows a pattern similar to finding the transfer function of a control system or a signal processor. Even complicated circuits can be reduced efficiently and quickly to establish the nodal dependcies.

The preceding example points out what will happen if the matching condition for recording the S-parameters is not satisfied. As we know, if we compute S_{11} we need to ensure that $a_2 = 0$. However, if $a_2 \neq 0$, as is the case in the preceding example, we see that S_{11} is modified by the additional factor $S_{12}S_{21}\Gamma_L/(1 - S_{22}\Gamma_L)$.

4.4.6 Generalization of S-Parameters

In our discussion thus far it was assumed that the characteristic line impedance at both ports has the same value Z_0. However, this does not have to be the case. Indeed, if we assume that port 1 is connected to line impedance Z_{01} and port 2 to impedance Z_{02}, we have to represent the voltage and current waves at the respective port $(n = 1, 2)$ as

$$V_n = V_n^+ + V_n^- = \sqrt{Z_{0n}}(a_n + b_n) \tag{4.84}$$

and

$$I_n = \frac{V_n^+}{Z_{0n}} - \frac{V_n^-}{Z_{0n}} = \frac{a_n}{\sqrt{Z_{0n}}} - \frac{b_n}{\sqrt{Z_{0n}}} \tag{4.85}$$

where we immediately observe

$$a_n = \frac{V_n^+}{\sqrt{Z_{0n}}}, b_n = \frac{V_n^-}{\sqrt{Z_{0n}}} \tag{4.86}$$

These equations allow the definition of the S-parameters as follows:

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_n = 0(n \neq j)} = \left.\frac{V_i^-/\sqrt{Z_{0i}}}{V_j^+/\sqrt{Z_{0j}}}\right|_{V_n^+ = 0(n \neq j)} \tag{4.87}$$

When compared to the previous S-parameter definitions, we notice that scaling by the appropriate line impedances has to be taken into account. It should also be apparent that although the focus of our derivations was a two-port network, the preceding formulas can be extended to an N-port network where $n = 1, \ldots, N$.

A second consideration is related to the fact that practical measurements involve the determination of the network S-parameters through transmission lines of finite length. In this case we need to investigate a system where the measurement planes are shifted away from the actual network, as depicted in Figure 4-25.

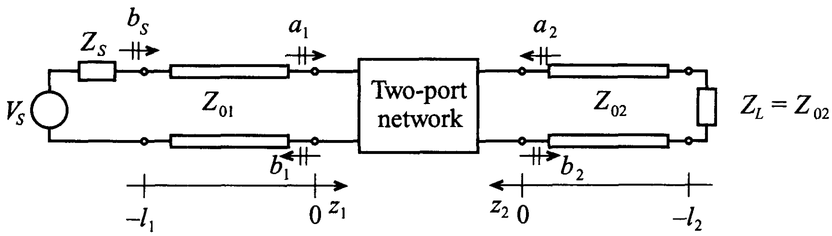


Figure 4-25 Two-port network with finite-length transmission line segments.

An incident voltage wave launched from the power supply will have to travel a distance l_1 in order to reach port 1. Consistent with the notation introduced in Section 2.9, we note that at port 1 the incident voltage is given as

$$V_{in}^+(z_1 = 0) = V_1^+ \tag{4.88}$$

and, at the generator side, as

$$V_{in}^+(z_1 = -l_1) = V_1^+ e^{-j\beta_1(-l_1)} \tag{4.89}$$

The reflected voltage wave at port 1 can be cast in the form

$$V_{in}^-(z_1 = 0) = V_1^- \tag{4.90}$$

and

$$V_{in}^-(z_1 = -l_1) = V_1^- e^{j\beta_1(-l_1)} \tag{4.91}$$

where, as usual, β_1 stands for the lossless propagation constant of line 1. In an identical fashion, the voltage behavior at port 2 can be formulated by simply replacing V_{in} in terms of V_{out} and V_1 in terms of V_2 as well as β_1 in terms of β_2. The preceding equations can be combined in matrix form

$$\begin{Bmatrix} V_{in}^+(-l_1) \\ V_{out}^+(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{j\beta_1 l_1} & 0 \\ 0 & e^{j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_1^+ \\ V_2^+ \end{Bmatrix} \tag{4.92}$$

which links the impinging waves at the network ports to the corresponding voltages shifted by the electric lengths of the attached transmission line segments. For the reflected voltage waves we get the matrix form

$$\begin{Bmatrix} V_{in}^-(-l_1) \\ V_{out}^-(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{-j\beta_1 l_1} & 0 \\ 0 & e^{-j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_1^- \\ V_2^- \end{Bmatrix} \tag{4.93}$$

As the discussion in Section 4.4.1 taught us, the S-parameters are linked to the coefficients a_n and b_n, which in turn can be expressed through voltages (if we assume $Z_{01} = Z_{02}$).

$$\begin{Bmatrix} V_1^- \\ V_2^- \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} V_1^+ \\ V_2^+ \end{Bmatrix} \tag{4.94}$$

It is apparent that if transmission line segments are added, we have to replace the above voltages by the previously derived expressions, leading to the form

$$\begin{Bmatrix} V_{in}^-(-l_1) \\ V_{out}^-(-l_2) \end{Bmatrix} = \begin{bmatrix} e^{-j\beta_1 l_1} & 0 \\ 0 & e^{-j\beta_2 l_2} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} e^{-j\beta_1 l_1} & 0 \\ 0 & e^{-j\beta_2 l_2} \end{bmatrix} \begin{Bmatrix} V_{in}^+(-l_1) \\ V_{out}^+(-l_2) \end{Bmatrix} \tag{4.95}$$

This final reveals that the S-parameters for the shifted network are comprised of three matrices. In terms of the coefficients, we see that

$$[S]^{\text{SHIFT}} = \begin{bmatrix} S_{11}e^{-j2\beta_1 l_1} & S_{12}e^{-j(\beta_1 l_1 + \beta_2 l_2)} \\ S_{21}e^{-j(\beta_1 l_1 + \beta_2 l_2)} & S_{22}e^{-j2\beta_2 l_2} \end{bmatrix} \tag{4.96}$$

The physical meaning of this form is easy to understand. The first matrix coefficient reveals that we have to take into account $2\beta_1 l_1$ or twice the travel time for the incident voltage to reach port 1 and, upon reflection, return. Similarly, for port 2 we see that the phase shift is $2\beta_2 l_2$. Moreover, the cross terms, which are closely related to the forward and reverse gains, require the additive phase shifts associated with transmission line 1 ($\beta_1 l_1$) and transmission line 2 ($\beta_2 l_2$), since the overall input/output configuration now consists of both line segments.

Example 4-9: Input impedance computation of a transmission line based on the use of the signal flow chart

A lossless transmission line system with characteristic line impedance Z_0 and length l is terminated into a load impedance Z_L and attached to a source voltage V_G and source impedance Z_G, as shown in Figure 4-26. (a) Draw the signal flow chart and (b) derive the input impedance formula at port 1 from the signal flow chart representation.

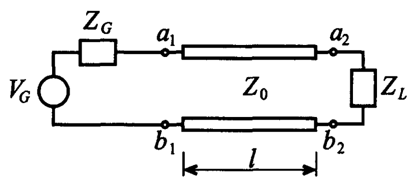


Figure 4-26 Transmission line attached to a voltage source and terminated by a load impedance.

Solution: (a) Consistent with our previously established signal flow chart notation, we can readily convert Figure 4-26 into the form seen in Figure 4-27.

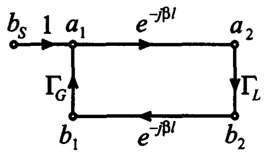


Figure 4-27 Signal flow chart diagram for transmission line system in Figure 4-26.

(b) The input reflection coefficient at port 1 is given by

$$b_1 = \Gamma_L e^{-j2\beta l} a_1$$

which is exactly in the form given in Section 3.1, with $\Gamma_L = \Gamma_0$ and $l = d$. Thus

$$\Gamma_{in}(l) = \Gamma_L e^{-j2\beta l} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

Solving for Z_{in} yields the final result

$$Z_{in} = Z_0 \frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}$$

This example shows how the input impedance of a transmission line can be found quickly and elegantly by using signal flow chart concepts.

4.4.7 Practical Measurements of S-Parameters

Measurement of the S-parameters of a two-port network requires reflection and transmission evaluations of traveling waves at both ports. One of the most popular methods is to use a vector network analyzer. The vector network analyzer is an instrument that can measure voltages in terms of magnitude and phase. Usually network analyzers have one output port, which provides the RF signal either from an internal source or an external signal generator, and three measurement channels, which are denoted as *R, A,* and *B* (see Figure 4-28).

The RF source is typically set to sweep over a specified frequency range. The measurement channel *R* is employed for measuring the incident wave. Channel *R* also serves as a reference port. Channels *A* and *B* usually measure the reflected and transmitted waves. In general, the measurement channels *A* and *B* can be configured to record any two parameters with a single measurement setup. An example of the test arrangement that allows us to measure S_{11} and S_{21} is shown in Figure 4-28.

In this case the value of S_{11} can be obtained by evaluating the ratio A/R, and S_{21} through computing B/R. To measure S_{12} and S_{22} we have to reverse the DUT. In Figure 4-28 the dual-directional coupler allows the separation of the incident and reflected waves at the input port of the DUT. The bias tees are employed to provide necessary biasing conditions, such as a quiescent point for the DUT. Since the most com-

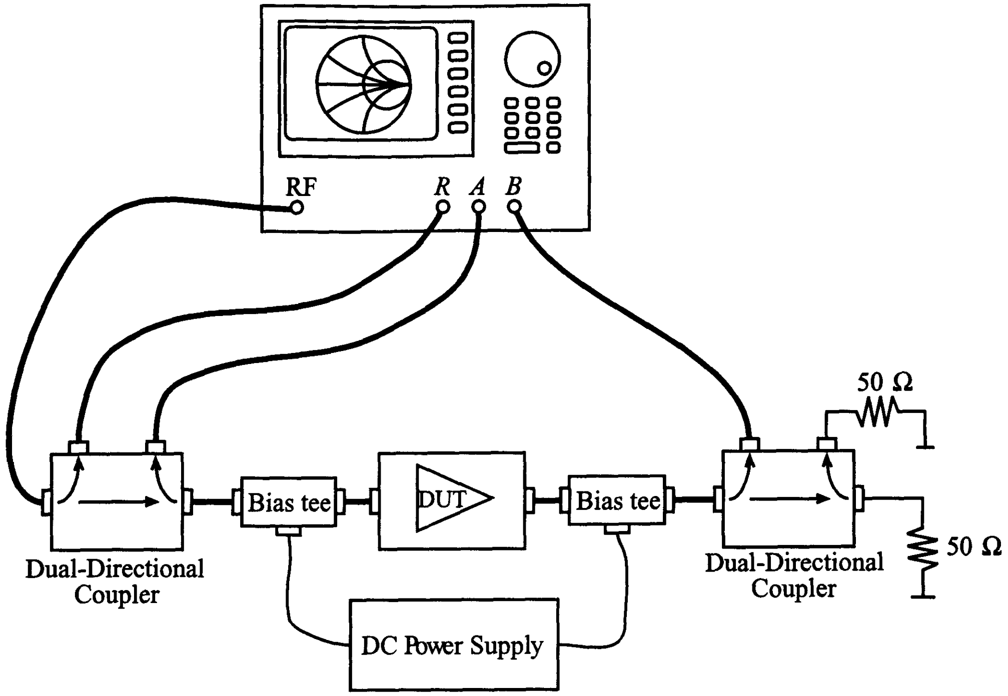


Figure 4-28 Measurement system for S_{11} and S_{21} parameters using a network analyzer.

mon use of network analyzers is the characterization of two-port devices, bias tees, directional couplers, and necessary electronic switches as well as the RF sweep signal generator are all integral parts of most modern analyzers.

As we can see, a practical test arrangement is more complicated when compared with the simple ideal system described in Sections 4.4.4 and 4.4.6, where we assume that the DUT is connected to perfectly matched transmission lines of equal (Section 4.4.4) or unequal (Section 4.4.6) characteristic impedance. In a realistic measurement system we cannot guarantee either matching conditions or ideality of the components. In fact, we have to consider all effects of the external components connected to the input and output ports of the DUT. Furthermore, the primary reference plane for measurements of complex voltages, which are then converted into S-parameters, is usually somewhere inside of the networks analyzer. As a result, it is necessary to take into account not only attenuation and phase shifts due to the external components, but also portions of the internal structure of the network analyzer itself.

In general, the measurement test arrangement can be reduced to the cascade of three networks depicted in Figure 4-29.

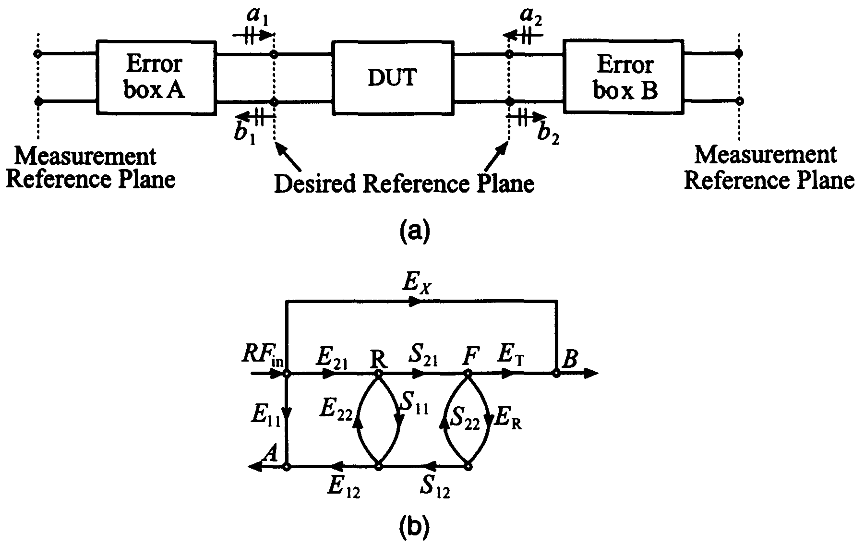


Figure 4-29 (a) Block diagram of the setup for measurement of S-parameters of a two-port network; (b) signal flow chart of the measurement test setup.

In Figure 4-29 the signals R, A, B correspond to the reference port and channels A and B of the network analyzer. RF_{in} is the output line from the signal source. The branch denoted E_X represents possible leakage between the output of the signal source and the channel B.

The network analyzer treats everything between the measurement reference planes as a single device. Therefore, our task is reduced to finding a way to calibrate the network analyzer in such a way that it becomes possible to eliminate the effect of all undesired influences or parasitics. The main goal of a **calibration procedure** is to characterize the error boxes prior to measuring the DUT. This information can then be used by an internal computer to evaluate the error-free S-parameters of the actual DUT.

Assuming that the error box A network is reciprocal, we can state $E_{12} = E_{21}$. Therefore, we have to find six parameters (E_{11}, E_{12}, E_{22}, E_X, E_R, and E_T) to characterize the error boxes.

The simplest calibration method involves three or more known loads (open, short, and matched). The problem with this approach is that such standards are usually imperfect and are likely to introduce additional errors into the measurement procedures. These errors become especially significant at higher frequencies. To avoid the dependency on the accuracy of calibration standards, several methods have been developed (see Eul and Schiek and Engen and Hoer, listed in the Further Reading section at the end of this chapter). In this section we will only consider the so-called **Through-Reflect-Line** (TRL) technique (see Engen and Hoer).

The TRL calibration scheme does not rely on known standard loads. Instead, it is based on the use of three types of connections, which are shown in Figure 4-30.

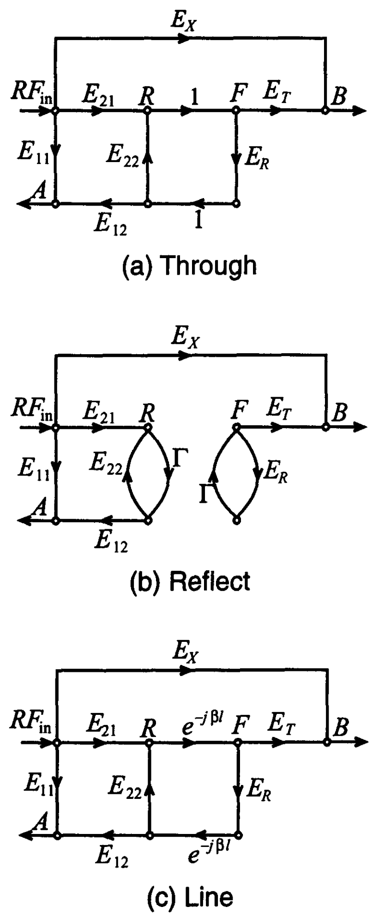


Figure 4-30 Signal flow graphs of TRL method: (a) Through, (b) Reflect, (c) Line configurations.

The *Through* connection is made by directly connecting ports 1 and 2 of the DUT. Next, the *Reflect* connection uses a load with high reflectivity. The reflection coefficient does not have to be known because it will be determined during the calibration process. The only requirement is that the load possesses the same reflection coefficient for both input and output ports. The *Line* connection is made by connecting ports 1 and 2 via a transmission line matched to the impedance of the error boxes. Usually, this impedance is close to 50 Ω. Before we continue with the actual analysis of each particular connection type, let us first consider the system as a general two-port network.

From Figure 4-29(b) it is seen that the signal at node B is a linear combination of the input RF signal and the signal at node F:

$$B = E_X + E_T F \tag{4.97}$$

Applying the self-loop rule, we can write that signal at node F as

$$F = \frac{S_{21}}{1 - E_R S_{22}} R \tag{4.98}$$

To compute the signal at port R, the same method as discussed in Example 4-8 can be used. In this example we first replaced the loop with the signal F through a self-loop and then performed the same transformation for the signal R. The result of these computations is

$$R = \frac{E_{21}}{1 - E_{22}\left(S_{11} + \frac{S_{12}S_{21}E_R}{1 - E_R S_{22}}\right)} \tag{4.99}$$

Substituting (4.99) into (4.98) followed by the substitution of (4.98) into (4.97), we obtain an expression for signal B:

$$B = E_X + E_T \frac{S_{21}}{1 - E_R S_{22}} \frac{E_{21}}{1 - E_{22}\left(S_{11} + \frac{S_{12}S_{21}E_R}{1 - E_R S_{22}}\right)} \tag{4.100}$$

Finally, the value for the signal at node A is obtained by using the summation rule:

$$A = E_{11} + \frac{E_{12}E_{21}}{1 - E_{22}\left(S_{11} + \frac{S_{12}S_{21}E_R}{1 - E_R S_{22}}\right)}\left(S_{11} + S_{12}E_R \frac{S_{21}}{1 - E_R S_{22}}\right) \tag{4.101}$$

If the measurement system does not introduce any errors, then $E_{12} = E_{21} = E_T = 1$ and $E_{11} = E_{22} = E_R = E_X = 0$. Substituting these values into (4.99), (4.100), and (4.101), we find that $R = 1$, $A = S_{11}$, and $B = S_{12}$, which shows the validity of the formulas.

Now we are ready to investigate the TRL connections in more detail. To avoid confusion, let us denote the measured signals R, A, and B for *Through* by subscript T, for *Reflect* by R, and for *Line* by L.

For the *Through* connection we know that $S_{11} = S_{22} = 0$ and $S_{12} = S_{21} = 1$. Setting $E_{12} = E_{21}$ it follows that

$$R_T = \frac{E_{12}}{1 - E_{22}E_R} \tag{4.102a}$$

$$A_T = E_{11} + \frac{E_{12}^2}{1 - E_{22}E_R}E_R \tag{4.102b}$$

$$B_T = E_X + E_T\frac{E_{12}}{1 - E_{22}E_R} \tag{4.102c}$$

For the *Reflect* connection we have $S_{11} = S_{22} = \Gamma$ and $S_{12} = S_{21} = 0$. This results in the equations

$$R_R = \frac{E_{12}}{1 - E_{22}\Gamma} \tag{4.103a}$$

$$A_R = E_{11} + \frac{E_{12}^2\Gamma}{1 - E_{22}\Gamma} \tag{4.103b}$$

$$B_R = E_X \tag{4.103c}$$

Finally, for the *Line* connection we see that $S_{11} = S_{22} = 0$ and $S_{12} = S_{21} = e^{-\gamma l}$, where l is the transmission line length and γ is a complex propagation constant ($\gamma = \alpha + j\beta$) that takes into account attenuation effects. The result is

$$R_L = \frac{E_{12}}{1 - E_{22}E_Re^{-2\gamma l}} \tag{4.104a}$$

$$A_L = E_{11} + \frac{E_{12}^2E_Re^{-2\gamma l}}{1 - E_{22}E_Re^{-2\gamma l}} \tag{4.104b}$$

$$B_L = E_X + E_Te^{-\gamma l}\frac{E_{12}}{1 - E_{22}E_Re^{-2\gamma l}} \tag{4.104c}$$

Equations (4.102a)–(4.104b) allow us to solve for the unknown coefficients of the error boxes E_{11}, E_{12}, E_{22}, E_X, E_R, E_T, the reflection coefficient Γ, and the transmission line parameter $e^{-\gamma l}$. Knowing the error coefficients we are then in a position to process the measured data in order to obtain an error-free S-parameter set of the DUT.

4.5 Summary

Networks play an integral part in analyzing basic low-frequency circuits as well as RF/MW circuits. For instance, the admittance or *Y*-matrix for an *N*-port network can be written in generic form as

$$\begin{Bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{Bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{Bmatrix}$$

where currents and voltages become the defining external port conditions. The evaluation of the matrix coefficients is accomplished through appropriate terminal conditions:

$$Y_{nm} = \left.\frac{i_n}{v_m}\right|_{v_k = 0 \text{ (for } k \neq m)}$$

The concepts of **Z**-, **Y**-, **h**-, and **ABCD**-matrix representations of networks can be directly extended to high-frequency circuits. Unfortunately, we encounter practical difficulties in applying the required open- and short-circuit network conditions needed when defining the respective parameter sets. It is for this reason that the scattering parameters as normalized forward and backward propagating power waves are introduced:

$$a_n = \frac{V_n^+}{\sqrt{Z_0}} = \sqrt{Z_0} I_n^+$$

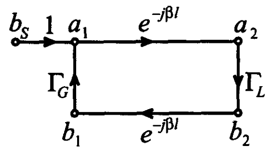
$$b_n = \frac{V_n^-}{\sqrt{Z_0}} = -\sqrt{Z_0} I_n^-$$

For a two-port network this results in the matrix form

$$\begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Unlike open- or short-circuit network conditions, impedance line matching at the respective port is now required to establish the **S**-matrix set. The *S*-parameters can be directly related to the reflection coefficients at the input and output of the two-port network (S_{11}, S_{22}). Furthermore, forward and reverse power gains are readily identified ($|S_{21}|^2$, $|S_{12}|^2$).

The S-parameters are also very useful descriptors when dealing with signal flow diagrams. A signal flow diagram is a circuit representation involving nodes and paths for the sourced and terminated transmission line as follows:



With signal flow diagrams even complicated systems can be examined in terms of specific input output relations in a similar manner as done in control system theory.

Chapter 4 finishes with a brief discussion of the practical recording of the S-parameters for a two-port network (DUT) through the use of a vector network analyzer. To compensate for various error sources associated with the measurement arrangement, the so-called TRL method is presented. Here the *Through*, *Reflect*, and *Line* calibrations are shown to account for the various errors and therefore permit the recording of the actual S-parameters needed to characterize the DUT.

Further Reading

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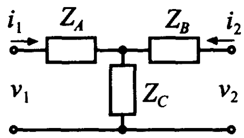
Problems

4.1 From the defining equations (4.3) and (4.6) for the impedance and admittance matrices, show that $[\mathbf{Z}] = [\mathbf{Y}]^{-1}$.

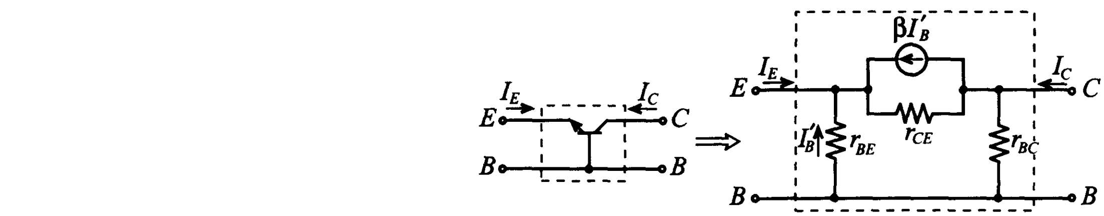
4.2 For the following generic T-network, find the impedance and admittance matrices.



4.3 Show that for a bipolar-junction transistor in a common-base configuration under small-signal low-frequency conditions (whose equivalent circuit is shown below) a hybrid parameter matrix can be established as follows:

$$[\mathbf{h}] = \begin{bmatrix} \dfrac{r_{ce}r_{be}}{r_{be}+(1+\beta)r_{ce}} & \dfrac{r_{be}}{r_{be}+(1+\beta)r_{ce}} \\ -\dfrac{r_{be}+\beta r_{ce}}{r_{be}+(1+\beta)r_{ce}} & \dfrac{1}{r_{bc}}+\dfrac{1}{r_{be}+(1+\beta)r_{ce}} \end{bmatrix}$$

where the individual transistor parameters are denoted in the figure.



4.4 Using the results from Problem 4.3, compute the equivalent circuit parameters for a BJT in common-base configuration if the **h**-matrix is given as

$$[\mathbf{h}] = \begin{bmatrix} 16.6 & 0.262\times10^{-3} \\ -0.99668 & 66.5\times10^{-9} \end{bmatrix}$$

4.5 Employ the conversion table for the different parameter representations of the two-port network and find the **h**-matrix representation for a Darlington pair shown in Figure 4-7 under the assumption that the transistors are specified by the same **h**-matrices derived in Example 4-2.

4.6 Using the definition of the *ABCD* network representation, find the *Y*-parameter description.

4.7 From the results of Problem 4.3 and Example 4.2, establish the conversion equations between the **h**-matrix parameters for the common-base and common-emitter transistor configurations.

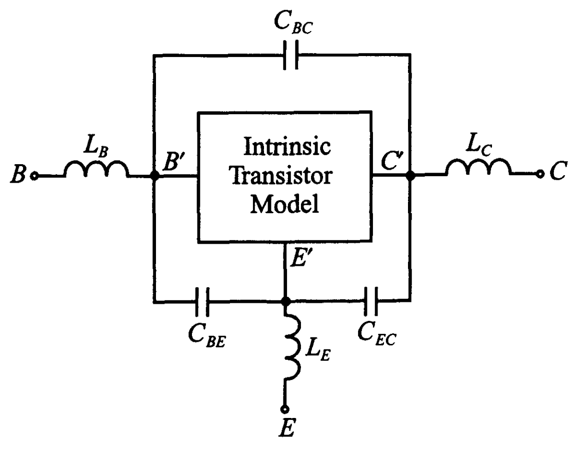
4.8 Unlike the series connection discussed in Example 4-4, derive the *ABCD*-parameters for a two-port network where the impedance Z is connected in parallel.

4.9 Find the *ABCD*-parameters for a generic three-element pi-network, as depicted in Figure 4-2.

4.10 Compute the *ABCD*-parameters for an RF transformer with turn ratio $N = N_1/N_2$, where N_1 is the number of turns a the primary winding and N_2 is the number of turns of the secondary winding.

4.11 Prove that the **h**-matrix parameters for a high-frequency hybrid transistor model shown in Figure 4-12 are given by (4.31).

4.12 In this chapter we have mentioned several **h**-matrix representations of the bipolar-junction transistor for different frequency conditions. In all cases we have neglected the influence of the parasitic components associated with the casing of the transistor. The modification to the equivalent circuit of the transistor that takes into account these parasitics is shown below:



Assuming that the intrinsic transistor model is given by a generic **h**-matrix, derive the modified model that accounts for the casing.

4.13 Compute the return loss for a 25 Ω resistor connected to a 75 Ω lossless transmission line.

4.14 Find the forward gain of the circuit discussed in Example 4-8.

4.15 Given that the input of an amplifier has a VSWR of 2 and the output is given by VSWR = 3, find the magnitudes of the input and output reflection coefficients. What does your result mean in terms of S_{11} and S_{22}?

4.16 Using the same approach as described in Section 4.4.4, show that the S-parameters of the network can be computed from the known Y-parameters using

$$[\mathbf{S}] = ([\mathbf{Y}] + Y_0[\mathbf{E}])^{-1}(Y_0[\mathbf{E}] - [\mathbf{Y}])$$

and the corresponding inverse relation

$$[\mathbf{Y}] = Y_0([\mathbf{E}] - [\mathbf{S}])([\mathbf{S}] + [\mathbf{E}])^{-1}$$

where $Y_0 = 1/Z_0$ is the characteristic line admittance.

4.17 The ideal transformer of Problem 4.10 can also be represented in S-parameter form. Show that the **S**-matrix is given by

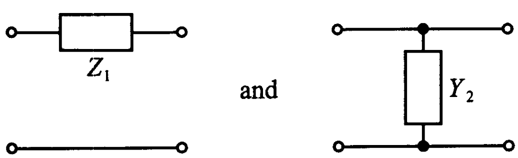
$$[\mathbf{S}] = \left(\frac{1}{1+N^2}\right)\begin{bmatrix}(N^2-1) & (2N)\\(2N) & (1-N^2)\end{bmatrix}$$

where $N = N_1/N_2$.

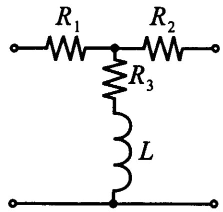
4.18 For the following two circuits, prove that the S-parameters are given as

$$[S] = \begin{bmatrix}\Gamma_1 & 1-\Gamma_1\\1-\Gamma_1 & \Gamma_1\end{bmatrix} \text{ and } [S] = \begin{bmatrix}\Gamma_2 & 1+\Gamma_2\\1+\Gamma_2 & \Gamma_2\end{bmatrix}$$

respectively, where $\Gamma_1 = (1+2Z_0/Z_1)^{-1}$ and $\Gamma_2 = -(1+2Y_0/Y_1)^{-1}$.



4.19 For the following T-network inserted into a transmission line with characteristic impedance of $Z_0 = 50\ \Omega$, the three resistances are $R_1 = R_2 = 8.56\ \Omega$, and $R_3 = 141.8\ \Omega$. Find the S-parameters of this configuration and plot the insertion loss as a function of inductance L for the frequency of $f = 2$ GHz and L changing from 0 to 100 nH.



4.20 In practice, the resistors in the T-network of the previous problem are not frequency independent. At RF frequencies parasitic effects have to be taken into account. Compute the S-parameters at 2 GHz when all resistors have a 0.5 nH parasitic series inductance. Assume L is fixed at 10 nH.

4.21 A BJT is operated in a 50 Ω circuit at 1.5 GHz. For the bias conditions of 4 mA collector current and collector-emitter voltage of 10 V, the manufacturer provides the S-parameters in magnitude and angle as follows:

$S_{11} = 0.6\angle -127°$; $S_{21} = 3.88\angle 87°$; $S_{12} = 0.039\angle 28°$; $S_{22} = 0.76\angle -35°$.

Find (a) the Z-parameter and (b) the h-parameter representation.