CHAPTER 3

The Smith Chart

A transmission line changes its impedance depending on material properties and geometric dimensions. Typical practical realizations include microstrip line, coaxial cable, and parallel-plate line. In addition, both the length and operating frequency of the transmission line significantly influence the input impedance. In the previous chapter we derived the fundamental equation describing the input impedance of a terminated transmission line. We found that this equation incorporates the characteristic line impedance, load impedance, and, through the argument of the tangent function, line length and operating frequency. As we saw in Section 2.9, the input impedance can equivalently be evaluated by using the spatially dependent reflection coefficient. To facilitate the evaluation of the reflection coefficient, P. H. Smith developed a graphical procedure based on conformal mapping principles. This approach permits an easy and intuitive display of the reflection coefficient as well as the line impedance in one single graph. Although this graphical procedure, nowadays known as the Smith Chart, was developed in the 1930s prior to the computer age, it has retained its popularity and today can be found in every data book describing passive and active RF/MW components and systems. Almost all computer-aided design programs utilize the Smith Chart for the analysis of circuit impedances, design of matching networks, and computations of noise figures, gain, and stability circles. Even instruments such as the ubiquitous network analyzer have the option to represent certain measurements in a Smith Chart format.

This chapter reviews the steps necessary to convert the input impedance in its standard complex plane into a suitable complex reflection coefficient representation via a specific conformal transformation originally proposed by Smith. The graphical display of the reflection coefficient in this new complex plane can then be utilized directly to find the input impedance of the transmission line. Moreover, the Smith Chart facilitates evaluation of more complicated circuit configurations, which will be employed in subsequent chapters to build filters and matching networks for active devices.

The following sections present a step-by-step derivation of the Smith Chart followed by several examples of how to use this graphical design tool in computing the impedance of passive circuits.

3.1 From Reflection Coefficient to Load Impedance

In Section 2.9 the reflection coefficient is defined as the ratio of reflected voltage wave to incident voltage wave at a certain fixed spatial location along the transmission line. Of particular interest is the reflection coefficient at the load location $d = 0$. From a physical point of view this coefficient Γ_0 describes the mismatch in impedance between the characteristic line impedance Z_0 and the load impedance Z_L as expressed by (2.52). In moving away from the load in the positive d-direction toward the beginning of the transmission line, we have to multiply Γ_0 by the exponential factor $\exp(-j2\beta d)$, as seen in (2.64), to obtain $\Gamma(d)$. It is this transformation from Γ_0 to $\Gamma(d)$ that constitutes one of the key ingredients in the **Smith Chart** as a graphical design tool.

3.1.1 Reflection Coefficient in Phasor Form

The representation of the reflection coefficient Γ_0 can be cast in the following complex notation.

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_{0r} + j\Gamma_{0i} = |\Gamma_0|e^{j\theta_L} \tag{3.1}$$

where $\theta_L = \tan^{-1}(\Gamma_{0i}/\Gamma_{0r})$. We recall that pure short- and open-circuit conditions in (3.1) correspond to Γ_0 values of -1 and $+1$, located on the real axis in the complex Γ plane.

Example 3-1: Reflection coefficient representations

A transmission line with a characteristic line impedance of $Z_0 = 50\ \Omega$ is terminated into the following load impedances:

(a) $Z_L = 0$ (short circuit)

(b) $Z_L \to \infty$ (open circuit)
(c) $Z_L = 50\ \Omega$
(d) $Z_L = (16.67 - j16.67)\ \Omega$
(e) $Z_L = (50 + j150)\ \Omega$

Find the individual reflection coefficients Γ_0 and display them in the complex Γ-plane.

Solution: Based on (3.1) we compute the following numbers for the reflection coefficients:
(a) $\Gamma_0 = -1$ (short circuit)
(b) $\Gamma_0 = 1$ (open circuit)
(c) $\Gamma_0 = 0$ (matched circuit)
(d) $\Gamma_0 = 0.54\angle 221°$
(e) $\Gamma_0 = 0.83\angle 34°$
The values are displayed in polar form in Figure 3-1.

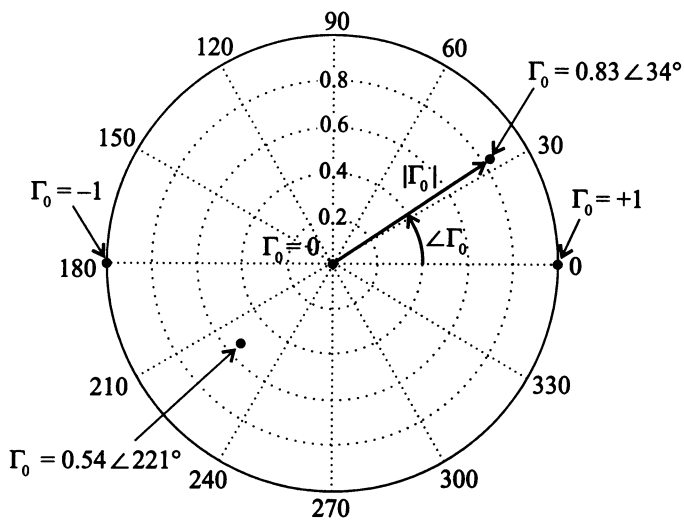

Figure 3-1 Complex Γ-plane and various locations of Γ_0.

The reflection coefficient is represented in phasor form as done when dealing with the conventional voltages and currents in basic circuit theory.

3.1.2 Normalized Impedance Equation

Let us return to our general input impedance expression (2.69), into which we substitute the reflection coefficient

$$\Gamma(d) = |\Gamma_0| e^{j\theta_L} e^{-j2\beta d} = \Gamma_r + j\Gamma_i \tag{3.2}$$

This results in

$$Z_{\text{in}}(d) = Z_0 \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{3.3}$$

In order to generalize the subsequent derivations, we normalize (3.3) with respect to the characteristic line impedance as follows

$$Z_{\text{in}}(d)/Z_0 = z_{\text{in}} = r + jx = \frac{1 + \Gamma(d)}{1 - \Gamma(d)} = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{3.4}$$

The preceding equation represents a mapping from one complex plane, the z_{in}-plane, to a second complex plane, the Γ-plane. Multiplying numerator and denominator of (3.4) by the complex conjugate of the denominator allows us to isolate real and imaginary parts of z_{in} in terms of the reflection coefficient. This means

$$z_{\text{in}} = r + jx = \frac{1 - \Gamma_r^2 - \Gamma_i^2 + 2j\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3.5}$$

can be separated into

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3.6}$$

and

$$x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3.7}$$

Equations (3.6) and (3.7) are explicit transformation rules of finding z_{in} if the reflection coefficient is specified in terms of Γ_r and Γ_i. Therefore, the mapping from the complex Γ-plane into the z_{in}-plane is straightforward, as the following example underscores.

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Example 3-2: Input impedance of a terminated transmission line

A load impedance $Z_L = (30 + j60)\ \Omega$ is connected to a 50 Ω transmission line of 2 cm length and operated at 2 GHz. Use the reflection coefficient concept and find the input impedance Z_{in} under the assumption that the phase velocity is 50% of the speed of light.

Solution: We first determine the load reflection coefficient

$$\Gamma_0 = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{30 + j60 - 50}{30 + j60 + 50} = 0.2 + j0.6 = \sqrt{2/5}e^{j71.56°} \quad (3.8)$$

Next we compute $\Gamma(d = 2\text{cm})$ based on the fact that

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{v_p} = \frac{2\pi f}{0.5c} = 83.77\ \text{m}^{-1}$$

This results in $2\beta d = 191.99°$ and yields for the refection coefficient

$$\Gamma = \Gamma_0 e^{-j2\beta d} = \Gamma_r + j\Gamma_i = -0.32 - j0.55 = \sqrt{2/5}e^{-j120.43°}$$

Having thus determined the reflection coefficient, we can now directly find the corresponding input impedance:

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma}{1 - \Gamma} = R + jX = 14.7 - j26.7\ \Omega$$

We note that the reflection coefficient phasor form at the load, Γ_0, is multiplied with a rotator that incorporates twice the electric line length βd. This mathematical statement thus conveys the idea that voltage/current waves have to travel to the load and return back to the source to define the input impedance.

Example 3.2 could have been solved just as efficiently by using the impedance equation (2.65) developed in Section 2.9.

3.1.3 Parametric Reflection Coefficient Equation

The goal of our investigation is to pursue a different approach toward computing the input impedance. This new approach involves the inversion of (3.6) and (3.7). In other words, we ask ourselves how a point in the z_{in}-domain, expressed through its normalized real, r, and imaginary, x, components, is mapped into the complex Γ-plane, where it then can be expressed in terms of the real, Γ_r, and imaginary, Γ_i, components of the reflection coefficient. Since Γ appears in the numerator and denominator, we have to suspect that straight lines in the impedance plane z_{in} may not be mapped into straight lines in the Γ-plane. All we can say at this point is that the matching of the load impedance to the transmission line impedance $Z_{\text{in}} = Z_0$, or $z_{\text{in}} = 1$, results in a zero reflection coefficient (i.e., $\Gamma_r = \Gamma_i = 0$) located in the center of the Γ-plane.

The inversion of (3.6) is accomplished by going through the following basic algebraic operations:

$$r[(1-\Gamma_r)^2+\Gamma_i^2] = 1-\Gamma_r^2-\Gamma_i^2 \tag{3.9a}$$

$$\Gamma_r^2(r+1)-2r\Gamma_r+\Gamma_i^2(r+1) = 1-r \tag{3.9b}$$

$$\Gamma_r^2-\frac{2r}{r+1}\Gamma_r+\Gamma_i^2 = \frac{1-r}{r+1} \tag{3.9c}$$

At this point the trick consists in recognizing that Γ_r can be written as a complete binomial expression (see also Appendix C)

$$\left(\Gamma_r-\frac{r}{r+1}\right)^2-\frac{r^2}{(r+1)^2}+\Gamma_i^2 = \frac{1-r}{r+1} \tag{3.9d}$$

This finally can be cast in the form

$$\left(\Gamma_r-\frac{r}{r+1}\right)^2+\Gamma_i^2 = \left(\frac{1}{r+1}\right)^2 \tag{3.10}$$

In an identical way as done previously, we proceed to invert (3.7). The result for the normalized reactance is

$$(\Gamma_r-1)^2+\left(\Gamma_i-\frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{3.11}$$

Both (3.10) and (3.11) are parametric equations of circles in the complex Γ-plane that can be written in the generic form $(\Gamma_r-a)^2+(\Gamma_i-b)^2 = c^2$. Here a, b denote shifts along the real and imaginary Γ axes, and c is the radius of the circle.

Figure 3-2 depicts the parametric circle equations of (3.10) for various resistances. For example, if the normalized resistance r is zero, the circle is centered at the origin and possesses a radius of 1, since (3.10) reduces to $\Gamma_r^2 + \Gamma_i^2 = 1$. For $r = 1$ we find $(\Gamma_r - 1/2)^2 + \Gamma_i^2 = (1/2)^2$, which represents a circle of radius 1/2 shifted in the positive Γ_r direction by $1/2$ units. We conclude that as r increases, the radii of the circles are continually reduced and shifted further to the right toward the point 1 on the real axis. In the limit for $r \to \infty$ we see that the shift converges to the point $r/(r+1) \to 1$ and the circle radius approaches $1/(r+1)^2 \to 0$.

It is important to realize that this mapping transforms fixed values of r only and does not involve x. Thus, for a fixed r an infinite range of reactance values x, as indicated by the straight lines in the z-plane, maps onto the same resistance circle. The mapping involving r alone is therefore not a unique point-to-point correspondence.

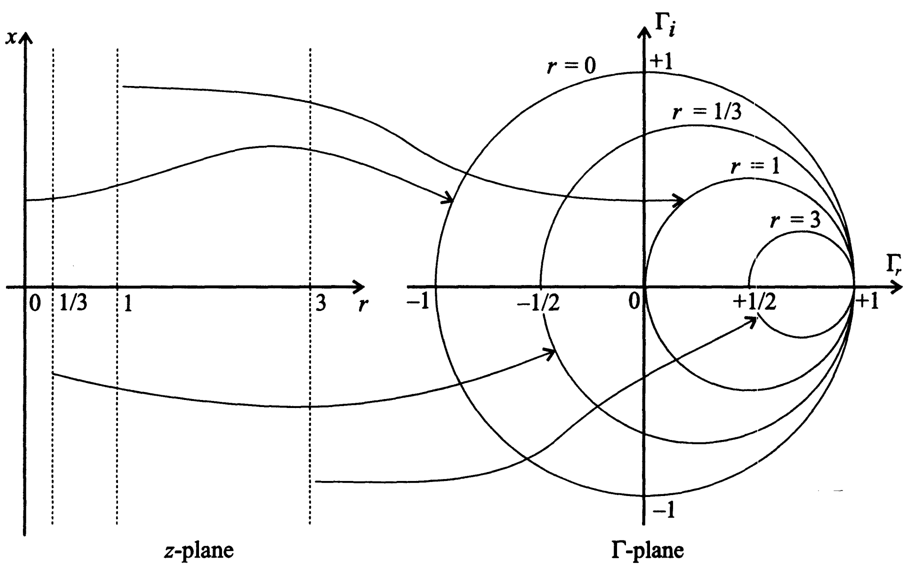

Figure 3-2 Parametric representation of the normalized resistance r in the complex Γ-plane.

A different graphical display results for the circle equation (3.11), which involves the normalized reactance. Here the centers of the circles reside all along a line perpendicular to the $\Gamma_r = 1$ point. For instance, for $x = \infty$ we note that $(\Gamma_r - 1)^2 + \Gamma_i^2 = 0$, which is a circle of zero radius, or a point located at $\Gamma_r = 1$ and $\Gamma_i = 0$. For $x = 1$ we see that the circle equation becomes $(\Gamma_r - 1)^2 + (\Gamma_i - 1)^2 = 1$. As $x \to 0$ the radii and shifts along the positive imaginary axis approach infinity. Interestingly, the shifts can also be along the negative imaginary axis. Here for $x = -1$ we notice that the circle equation becomes $(\Gamma_r - 1)^2 + (\Gamma_i + 1)^2 = 1$ with the center located at $\Gamma_r = 1$ and $\Gamma_i = -1$. We observe that negative x-values refer to capacitive impedances residing in the lower half of the Γ-plane. Figure 3-3 shows the parametric form of the normalized imaginary impedance. For better readability the circles are displayed inside the unit circle only. In contrast to Figure 3-2 we notice that fixed x-values are mapped into circles in the Γ-plane for arbitrary resistance values $0 \le r < \infty$, as indicated by the straight lines in the impedance plane.

The transformations (3.10) and (3.11) taken individually do not constitute unique mappings from the normalized impedance into the reflection coefficient plane. In other words, impedance points mapped into the Γ-plane by either (3.10) or (3.11) cannot uniquely be inverted back into the original impedance points. However, since the transformations complement each other, a unique mapping can be constructed by combining both transformations, as discussed in the next section.

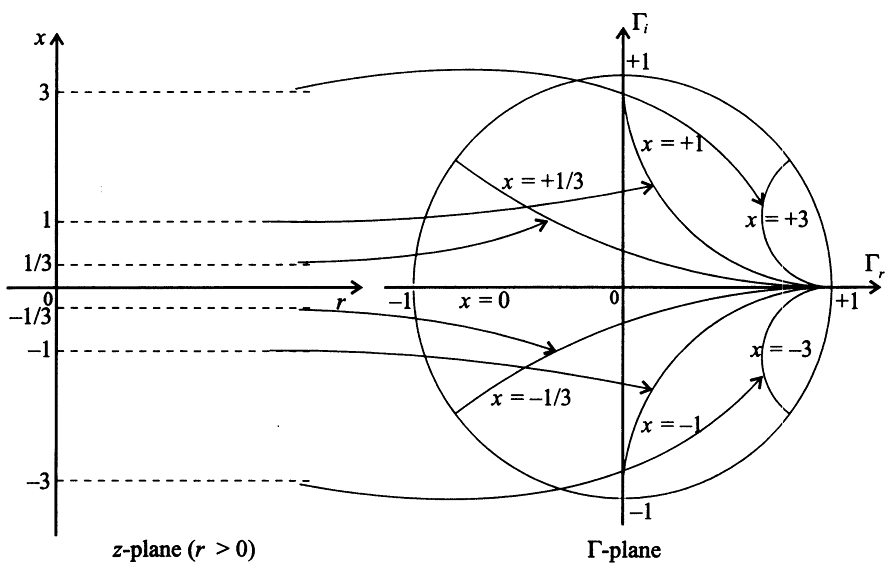

Figure 3-3 Parametric representation of the normalized reactance x in the complex Γ-plane.

3.1.4 Graphical Representation

Combining the parametric representations for normalized resistance and reactance circles (i.e., Figures 3-2 and 3-3) for $|\Gamma| \le 1$ results in the Smith Chart as illustrated in

Figure 3-4. An important observation of the Smith Chart is that there is a *one-to-one mapping* between the normalized impedance plane and the reflection coefficient plane. We notice also that the normalized resistance circles r have a range $0 \le r < \infty$ and the normalized reactance circles x can represent either negative (i.e., capacitive) or positive (i.e., inductive) values in the range $-\infty < x < +\infty$.

It should be pointed out that the reflection coefficient does not have to satisfy $|\Gamma| \le 1$. Negative resistances, encountered for instance as part of the oscillation condition for resonators, lead to the case $|\Gamma| > 1$ and consequently map to points residing outside the unit circle. Graphical displays where the reflection coefficient is greater than 1 are known as **compressed Smith Charts**. These charts, however, play a rather limited role in RF/MW engineering designs and are therefore not further pursued in this text. The interested reader may consult specialized literature (see the Hewlett-Packard application note listed at the end of this chapter).

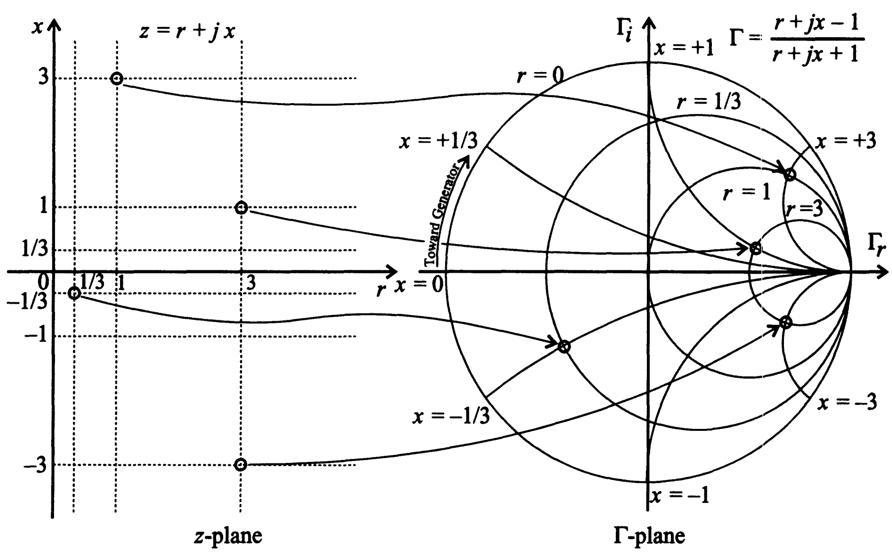

Figure 3-4 Smith Chart representation by combining r and x circles for $|\Gamma| \le 1$.

In Figure 3-4 we must note that the angle of rotation $2\beta d$ introduced by the length of the transmission line is measured from the phasor location of $\Gamma_0 = |\Gamma_0|e^{j\theta_L}$ in clockwise (mathematically negative) direction due to the negative exponent $(-2j\beta d)$ in the reflection coefficient expression (3.2). For the computation of the input impedance of a terminated transmission line, the motion is thus always *away from the load impedance* or *toward the generator*. This rotation is indicated by an arrow on the periphery of the chart. We further observe that a complete revolution around the unit circle requires

$$2\beta d = 2\frac{2\pi}{\lambda}d = 2\pi$$

where $d = \lambda/2$ or $180°$. The quantity βd is sometimes referred to as the **electrical length** of the line.

3.2 Impedance Transformation

3.2.1 Impedance Transformation for General Load

The determination of the impedance response of a high-frequency circuit is often a critical issue for the RF design engineer. Without detailed knowledge of the impedance behavior, RF/MW system performance cannot adequately be predicted. In this section we will elaborate on how the impedance can be determined easily and efficiently with the aid of the previously introduced Smith Chart.

A typical Smith Chart computation involving a load impedance Z_L connected to a transmission line of characteristic line impedance Z_0 and length d proceeds according to the following six steps:

1. Normalize the load impedance Z_L with respect to the line impedance Z_0 to determine z_L.
2. Locate z_L in the Smith Chart.
3. Identify the corresponding load reflection coefficient Γ_0 in the Smith Chart both in terms of its magnitude and phase.
4. Rotate Γ_0 by twice its electrical length βd to obtain $\Gamma_{in}(d)$.
5. Record the normalized input impedance z_{in} at this spatial location d.
6. Convert z_{in} into the actual impedance Z_{in}.

Example 3-3 goes through these steps, which are the standard procedure to arrive at the graphical impedance solution.

Example 3-3: Transmission line input impedance determination with the Smith Chart

Solve Example 3-2 by following the six-step Smith Chart computations given in the preceding list.

Solution: We commence with the load impedance $Z_L = (30 + j60)\ \Omega$ and proceed according to the previously outlined steps:

1. The normalized load impedance is

$$z_L = (30 + j60)/50 = 0.6 + j1.2$$

2. This point can be identified in the Smith Chart as the intersection of the circle of constant resistance $r = 0.6$ with the circle of constant reactance $x = 1.2$, as seen in Figure 3-5.

3. The straight line connecting the origin to point z_L determines the load reflection coefficient Γ_0. The associated angle is recorded with respect to the positive real axis.

4. Keeping in mind that the outside circle on the Smith Chart corresponds to the unity reflection coefficient ($|\Gamma_0| = 1$), we can find its magnitude as the length of the vector connecting the origin to z_L. Rotating this vector by twice the electrical length of the line (i.e., $2 \times \beta d = 2 \times 96° = 192°$) yields the input reflection coefficient Γ_{in}.

5. This point uniquely identifies the associated normalized input impedance $z_{in} = 0.3 - j0.53$.

6. The preceding normalized impedance can be converted back into actual input impedance values by multiplying it by $Z_0 = 50\ \Omega$, resulting in the final solution: $Z_{in} = (15 - j26.5)\Omega$.

We recall that the exact value of the input impedance obtained in Example 3-2 is $(14.7 - j26.7)\ \Omega$. The small discrepancy is understandable because of the approximate processing of the graphical data in the Smith Chart. The entire sequence of steps leading to the determination of the input impedance of the line connected to the load is shown in Figure 3-5.

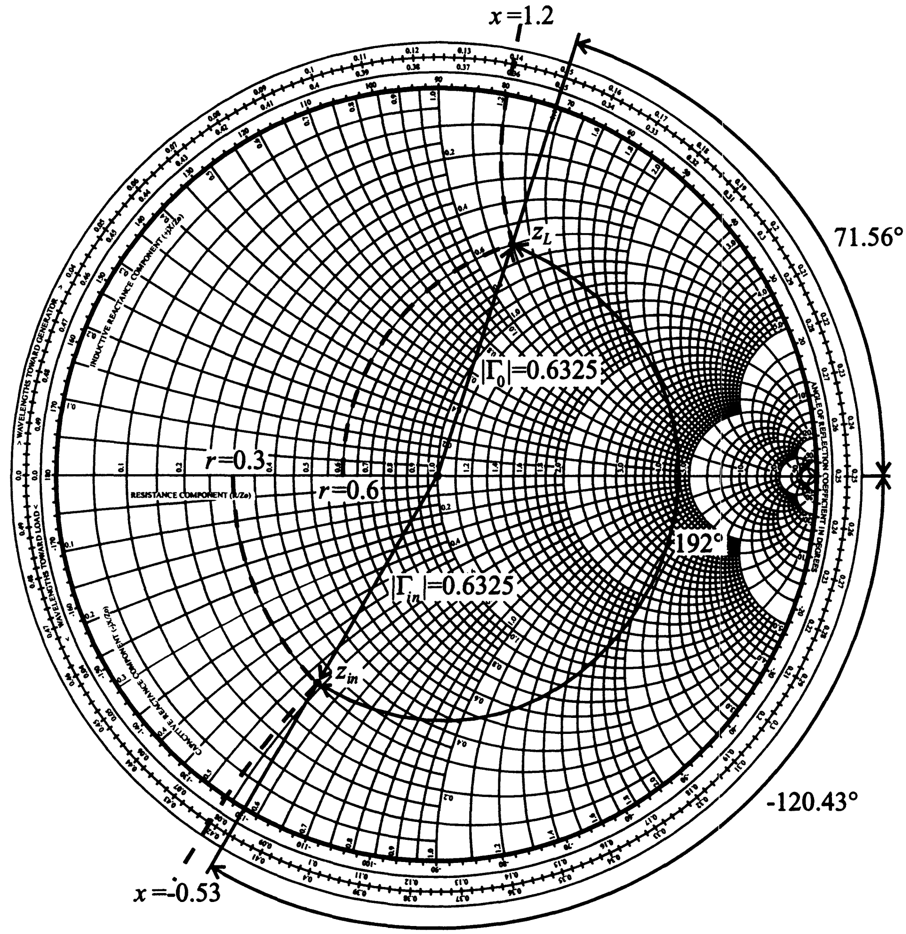

Figure 3-5 Usage of the Smith Chart to determine the input impedance for Example 3-3.

These steps appear at first cumbersome and prone to error if carried out by hand. However, using mathematical spreadsheets and relying on modern computer-based instrumentation, the calculations are routinely done in seconds and with a high degree of accuracy.

3.2.2 Standing Wave Ratio

From the basic definition of the SWR in Section 2.8.3 it follows that for an arbitrary distance d along the transmission line, the standing wave ratio is written

$$\mathrm{SWR}(d) = \frac{1+|\Gamma(d)|}{1-|\Gamma(d)|} \tag{3.12}$$

where $\Gamma(d) = \Gamma_0 \exp(-j2\beta d)$. Equation (3.12) can be inverted to give

$$|\Gamma(d)| = \frac{\mathrm{SWR}-1}{\mathrm{SWR}+1} \tag{3.13}$$

This form of the reflection coefficient permits the representation of the SWR as circles in the Smith Chart with the matched condition $\Gamma(d) = 0$ (or SWR = 1) being the origin.

It is interesting to note that equation (3.12) is very similar in appearance to the expression for determining the impedance from a given reflection coefficient:

$$Z(d) = Z_0 \frac{1+\Gamma(d)}{1-\Gamma(d)} \tag{3.14}$$

This similarity, together with the fact that for $|\Gamma(d)| \le 1$ the SWR is greater or equal to unity, suggests that the actual numerical value for the SWR can be found from the Smith Chart by finding the intersection of the circle of radius $|\Gamma(d)|$ with the right-hand side of the real axis.

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Example 3-4: Reflection coefficient, voltage standing wave ratio, and return loss

Four different load impedances:
(a) $Z_L = 50\ \Omega$, (b) $Z_L = 48.5\ \Omega$, (c) $Z_L = (75 + j25)\ \Omega$, and (d) $Z_L = (10 - j5)\ \Omega$, are sequentially connected to a 50 Ω transmission line. Find the reflection coefficients and the SWR circles, and determine the return loss in dB.

Solution: The normalized load impedances and corresponding reflection coefficients, return loss, and SWR values are computed as follows:

(a) $z_L = 1$, $\Gamma = (z_L - 1)/(z_L + 1) = 0$, $\mathrm{RL_{dB}} = \infty$, $\mathrm{SWR} = 1$

(b) $z_L = 0.97$, $\Gamma = (z_L - 1)/(z_L + 1) = -0.015$, $RL_{dB} = 36.3$, $\mathrm{SWR} = 1.03$

(c) $z_L = 1.5 + j0.5$, $\Gamma = (z_L - 1)/(z_L + 1) = 0.23 + j0.15$, $\mathrm{RL_{dB}} = 11.1$, $\mathrm{SWR} = 1.77$

(d) $z_L = 0.2 - j0.1$, $\Gamma = (z_L - 1)/(z_L + 1) = -0.66 - j0.14$, $\mathrm{RL_{dB}} = 3.5$, $\mathrm{SWR} = 5.05$

To determine the approximate values of the SWR requires us to exploit the similarity with the input impedance, as discussed previously. To this end, we first plot the normalized impedance values in the Smith Chart (see Figure 3-6). Then we draw circles with centers at the origin and radii whose lengths reach the respective impedance points defined in the previous step. From these circles we see that the load refection coefficient for zero load reactance ($x_L = 0$) is

$$\Gamma_0 = \frac{z_L - 1}{z_L + 1} = \frac{r_L - 1}{r_L + 1} = \Gamma_r$$

The SWR can be defined in term of the real load reflection coefficient along the real Γ-axis:

$$\mathrm{SWR} = \frac{1 + |\Gamma_0|}{1 - |\Gamma_0|} = \frac{1 + \Gamma_r}{1 - \Gamma_r}$$

This requires $|\Gamma_0| = \Gamma_r \geq 0$. In other words, for $\Gamma_r \geq 0$ we have to enforce $r_L \geq 1$, meaning that only the intersects of the right-hand-side circles with the real axis define the SWR.

As a graphical design tool, the Smith Chart allows immediate observation of the degree of mismatch between line and load impedances by plotting the radius of the SWR circle.

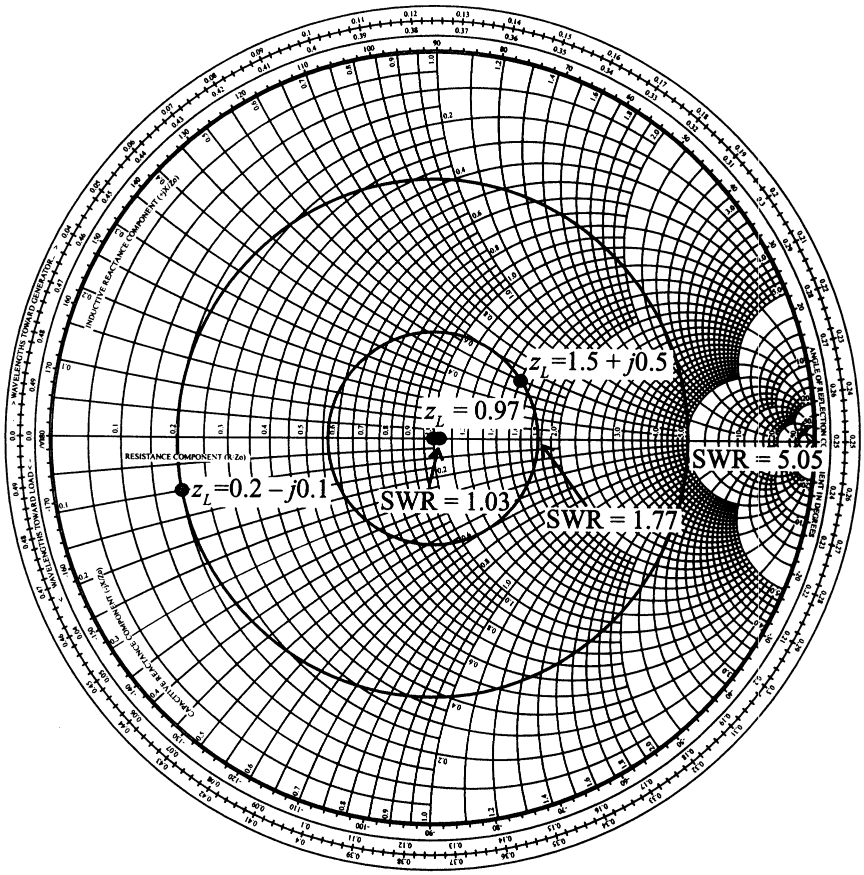

Figure 3-6 SWR circles for various reflection coefficients.

3.2.3 Special Transformation Conditions

The amount of rotation by which the point of the normalized transmission line impedance circles around the Smith Chart is controlled by the length of the line, or alternatively the operating frequency. Consequently, both inductive (upper plane) and capacitive (lower plane) impedances can be generated based on the line length and the termination conditions at a given frequency. These lumped circuit parameter representations, realized through distributed circuit analysis techniques, are of significant practical importance.

The cases of open- and short-circuit line termination are of particular interest in generating inductive and capacitive behavior and are examined in more detail next.

Open Circuit Transformations

To obtain a pure inductive or capacitive impedance behavior, we need to operate along the $r = 0$ circle. The starting point is the right-hand location ($\Gamma_0 = 1$) with rotation toward the generator in a clockwise sense.

A capacitive impedance $-jX_C$ is obtained through the condition

$$\frac{1}{j\omega C}\frac{1}{Z_0} \equiv z_{in} = -j\cot(\beta d_1) \tag{3.15}$$

as direct comparison with (2.70) shows. The line length d_1 is found to be

$$d_1 = \frac{1}{\beta}\left[\cot^{-1}\left(\frac{1}{\omega C Z_0}\right) + n\pi\right] \tag{3.16}$$

where $n\pi$ ($n = 1, 2, \ldots$) is required due to the periodicity of the cotangent function. Alternatively, an inductive impedance jX_L can be realized via the condition

$$j\omega L\frac{1}{Z_0} \equiv z_{in} = -j\cot(\beta d_2) \tag{3.17}$$

The line length d_2 is now found to be

$$d_2 = \frac{1}{\beta}\left[\pi - \cot^{-1}\left(\frac{\omega L}{Z_0}\right) + n\pi\right] \tag{3.18}$$

Both conditions are schematically depicted in Figure 3-7. How to choose a particular open-circuit line length to exhibit capacitive or inductive behavior is discussed in the following example.

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Example 3-5: Representation of passive circuit elements through transmission line section

For an open-ended 50 Ω transmission line operated at 3 GHz and with a phase velocity of 77% of the speed of light, find the line lengths to create a 2 pF capacitor and a 5.3 nH inductor. Perform your computations both by relying on (3.16) and (3.18) and by using the Smith Chart.

Solution: For a given value of phase velocity, the propagation constant is

$$\beta = 2\pi f/v_p = 2\pi f/(0.77c) = 81.6\ \text{m}^{-1}$$

Substituting this value into (3.16) and (3.18), we conclude that for the representation of a 2 pF capacitor we need an open-circuit line or stub with line length $d_1 = 13.27 + n38.5\ \text{mm}$. For the realization of a 5.3 nH inductor, a $d_2 = 32.81 + n38.5\ \text{mm}$ stub is required.

The alternative method for computing the lengths of the required stubs is through the use of the Smith Chart (see Figure 3-7). At a 3-GHz frequency, the reactance of a 2 pF capacitor is $X_C = 1/(\omega C) = 26.5\Omega$. The corresponding normalized imped-

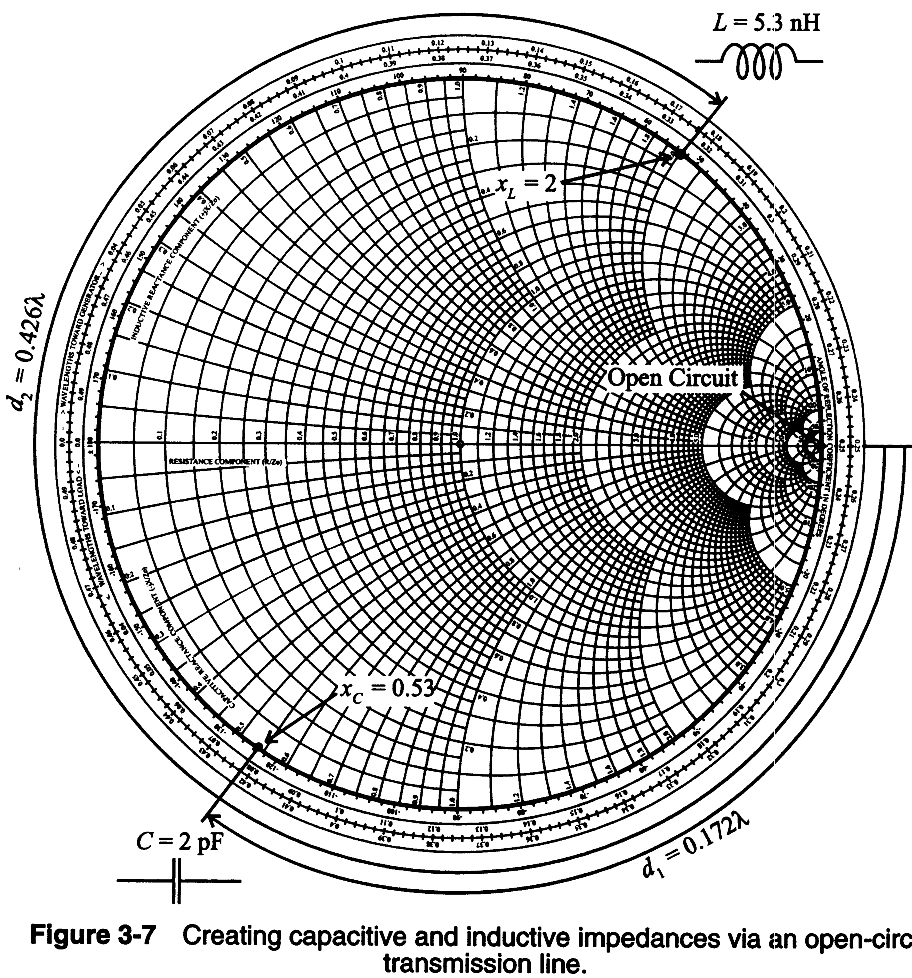


Figure 3-7 Creating capacitive and inductive impedances via an open-circuit transmission line.

ance in this case is $z_C = -jX_C = -j0.53$. From the Smith Chart we can deduce that the required transmission line length has to be approximately 0.172 of one wavelength. We note that for the given phase velocity, the wavelength is $\lambda = v_p/f = 77\text{ mm}$. This results in a line length of $d_1 = 13.24\text{ mm}$ which is very close to the previously computed value of 13.27 mm. Similarly, for the inductance we obtain $z_L = j2$. The line length in this case is 0.426 of one wavelength, which is equal to 32.8 mm.

Circuits are often designed with lumped elements before converting them into transmission line segments, similar to the procedure described in this example.

Short-Circuit Transformations

Here the transformation rules follow similar procedures as outlined previously, except that the starting point in the Smith Chart is now the $\Gamma_0 = -1$ point on the real axis, as indicated in Figure 3-8.

A capacitive impedance $-jX_C$ follows from the condition

$$\frac{1}{j\omega C}\frac{1}{Z_0} \equiv z_{in} = j\tan(\beta d_1) \tag{3.19}$$

where use is made of (2.66). The line length d_1 is found to be

$$d_1 = \frac{1}{\beta}\left[\pi - \tan^{-1}\left(\frac{1}{\omega C Z_0}\right) + n\pi\right] \tag{3.20}$$

Alternatively, an inductive impedance jX_L can be realized via the condition

$$j\omega L\frac{1}{Z_0} \equiv z_{\text{in}} = j\tan(\beta d_2) \tag{3.21}$$

The line length d_2 is now found to be

$$d_2 = \frac{1}{\beta}\left[\tan^{-1}\left(\frac{\omega L}{Z_0}\right) + n\pi\right] \tag{3.22}$$

At high frequencies, it is very difficult to maintain perfect open-circuit conditions because of changing temperatures, humidity, and other parameters of the medium surrounding the open transmission line. For this reason short-circuit conditions are more preferable in practical applications. However, even a short-circuit termination becomes

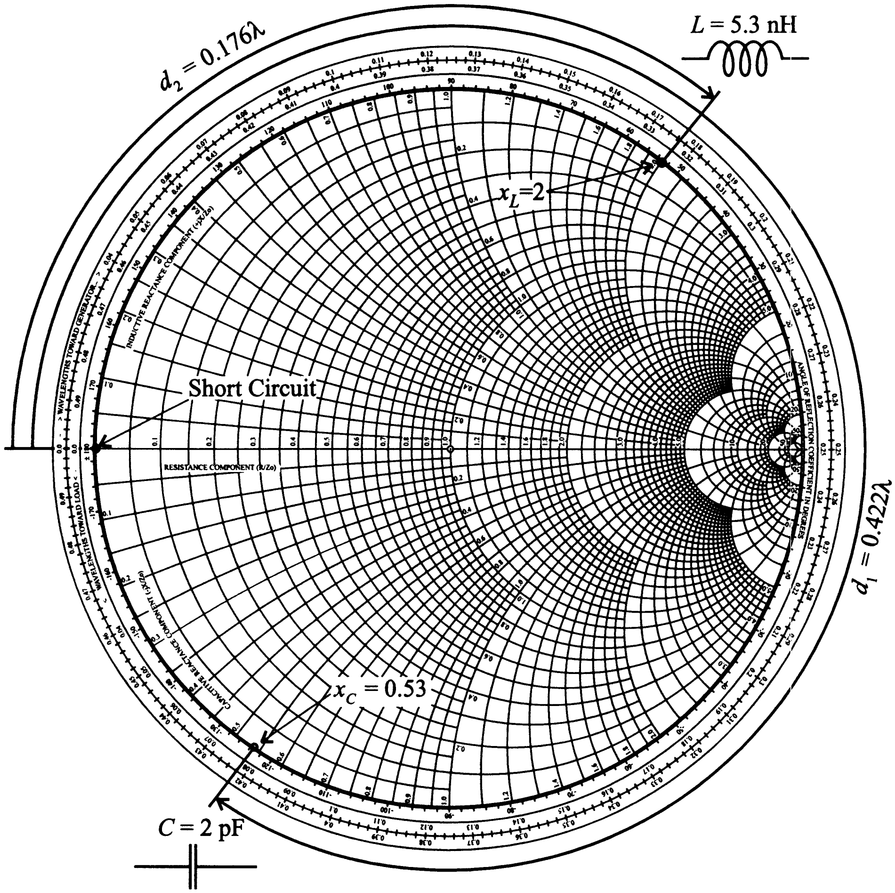


Figure 3-8 Creating capacitive and inductive impedances via a short-circuit transmission line.

problematic at very high frequencies or when through-hole connections in printed circuit boards are involved, since they result in additional parasitic inductances. Moreover, a design engineer may not have a choice if the circuit layout area is to be minimized by requiring the selection of the shortest line segments. For instance, the realization of a capacitor always yields the shortest length for an open-circuit line.

3.2.4 Computer Simulations

There are many **computer aided design** (CAD) programs available to facilitate the RF/MW circuit design and simulation processes. These programs can perform a multitude of tasks, varying from simple impedance calculations to complex circuit optimizations and circuit board layouts. One commercial software package that is used throughout this textbook is called Monolithic and Microwave Integrated Circuit Analysis and Design (**MMICAD**) (Optotek Ltd., Kanata, Ontario, Canada), which is a linear simulator program with optimization tools. Another well-known program with advanced features is EESof's **Libra** package (Hewlett-Packard Corporation, Westlake Village, CA, USA), which is capable of performing linear as well as nonlinear analyses and optimizations.

It is not the purpose of this textbook to review and discuss the various CAD programs presently in industrial and academic use. However, to reproduce the subsequent simulation results, Appendix I provides a brief introduction to the basic features of MATLAB, which was chosen as a tool to carry out most simulations presented in this book.

The main reason for using MATLAB is its wide-spread use as a mathematical spreadsheet which permits easy programming and direct graphical display. This eliminates the need to rely on complex and expensive programs accessible to only a few readers. The benefit of a MATLAB routine will immediately become apparent when the Smith Chart computations have to be performed repetitively for a range of operating frequencies or line lengths as the following discussion underscores.

In this section we revisit Example 3-2, which computed the input reflection coefficient and input impedance of a generic transmission line connected to a load. We now extend this example beyond a single operating frequency and a fixed line length. Our goal is to examine the effect of a frequency sweep in the range from 0.1 GHz to 3 GHz and a change in line length varying from 0.1 cm to 3 cm. The example MATLAB routine, which performs the analysis of the transmission line length changing from 0.1 cm to 3 cm at a fixed operating frequency 2 GHz, is as follows:

```
smith_chart;                % plot smith chart
Set_Z0(50);                 % set characteristic impedance to 50 Ohm
s_Load(30+j*60);            % set load impedance to 30+j60 Ohm
vp=0.5*3e8;                 % compute phase velocity
f=2e9;                      % set frequency to 2 GHz
d=0.0:0.001:0.03;           % set the line length to a range from 0 to
                            % 3 cm in 1 mm increments
betta=2*pi*f/vp;            % compute propagation constant
Gamma=(ZL-Z0)/(ZL+Z0);      % compute load reflection coefficient
rd=abs(Gamma);              % magnitude of the reflection coefficient
alpha=angle(Gamma)-2*betta*d;         % phase of the reflection
                                      % coefficient
plot(rd*cos(alpha),rd*sin(alpha));    % plot the graph
```

In the first line of the MATLAB code (see file fig3_9.m on the accompanying CD) we generate the Smith Chart with the necessary resistance and reactance circles. The next lines define the characteristic line impedance $Z_0 = 50\ \Omega$, load impedance

$Z_L = (30 + j60)\ \Omega$, operation frequency $f = 2\times10^9$ Hz, and phase velocity $v_p = 0.5\times3\times10^8$ m/s. The command line `d=0.0:0.001:0.03` creates an array `d` representing the transmission line length, which is varied from 0 mm to 3 cm in 1-mm increments. After all parameters have been identified, the magnitude and phase of the input reflection coefficients have to be computed. This is accomplished by determining the propagation constant $\beta = 2\pi f/v_p$, load reflection coefficient $\Gamma_0 = (Z_L - Z_0)/(Z_L + Z_0)$ and its magnitude $|\Gamma_0|$, and the total angle of rotation $\alpha = \angle(\Gamma_0) - 2\beta d$. Finally, the display of the impedance as part of the Smith Chart is done through the plot command, which requires both real and imaginary phasor arguments $|\Gamma_0|\cos(\alpha)$ and $|\Gamma_0|\sin(\alpha)$. The final result is shown in Figure 3-9.

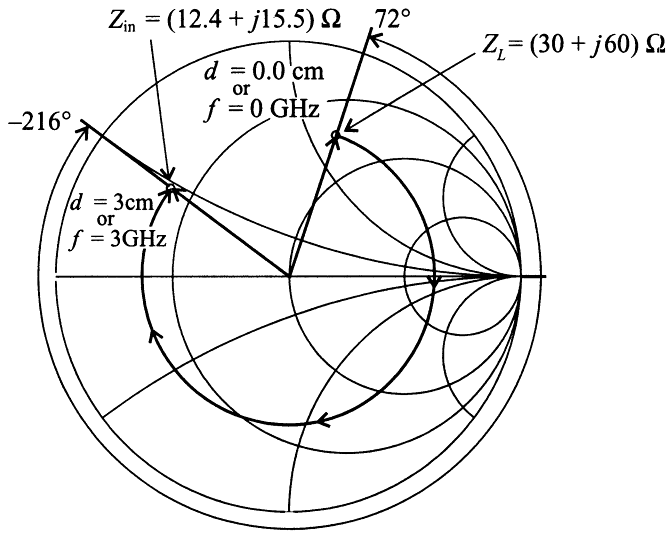


Figure 3-9 Input impedance of a loaded line of 2 cm length for a sweep in operating frequency from 0.0 to 3 GHz. If the operating frequency is fixed at 2 GHz and the line length is varied from 0.0 to 3 cm, the same impedance curve is obtained.

For the case where the length of the line is fixed to be 2 cm and the frequency is swept from values ranging from 0.0 to 3 GHz, the only necessary modification to the above input file is to set `d=0.02`, followed by specifying the frequency range in increments of 100 MHz (i.e., `f=0.0:1e7:3e9`). We should note that in both cases the electrical length (βd) of the line changes from 0° to 144°. Therefore, the impedance graphs produced for both cases are identical.

At the end of the rotation, either by fixing the frequency and varying the length or vice versa, the input impedance is found to be $Z_{in} = (12.4 + j15.5)\ \Omega$. It is reassuring that for a fixed frequency $f = 2$ GHz and a line length range $d = 0 \ldots 2$ cm, we ultimately arrive at the same input impedance of $Z_{in} = (14.7 - j26.7)\ \Omega$ as obtained in Example 3-2.

3.3 Admittance Transformation

3.3.1 Parametric Admittance Equation

From the representation of the normalized input impedance (3.4), it is possible to obtain a normalized admittance equation by simple inversion:

$$y_{in} = \frac{Y_{in}}{Y_0} = \frac{1}{z_{in}} = \frac{1-\Gamma(d)}{1+\Gamma(d)} \tag{3.23}$$

where $Y_0 = 1/Z_0$. To represent (3.23) graphically in the Smith Chart, we have several options. A very intuitive way of displaying admittances in the conventional Smith Chart or **Z-Smith Chart** is to recognize that (3.23) can be found from the standard representation (3.4) via

$$\frac{1-\Gamma(d)}{1+\Gamma(d)} = \frac{1+e^{-j\pi}\Gamma(d)}{1-e^{-j\pi}\Gamma(d)} \tag{3.24}$$

In other words, we take the normalized input impedance representation and multiply the reflection coefficient by $-1 = e^{-j\pi}$, which is equivalent to a $180°$ rotation in the complex Γ-plane.

Example 3-6: Use of the Smith Chart for converting impedance to admittance

Convert the normalized input impedance $z_{in} = 1 + j1 = \sqrt{2}e^{j(\pi/4)}$ into normalized admittance and display it in the Smith Chart.

Solution: The admittance can be found by direct inversion, that is

$$y_{in} = \frac{1}{\sqrt{2}}e^{-j(\pi/4)} = \frac{1}{2} - j\frac{1}{2}$$

In the Smith Chart we simply rotate the reflection coefficient corresponding to z_{in} by $180°$ to obtain the impedance. Its *numerical* value is equal to y_{in} as shown in Figure 3-10. To denormalize y_{in}

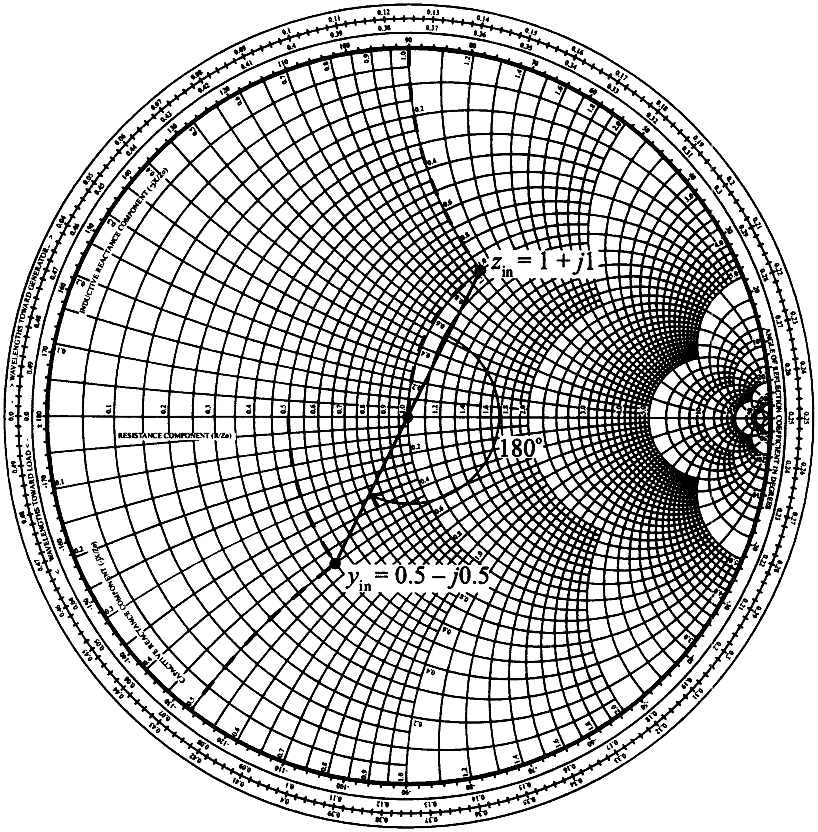


Figure 3-10 Conversion from impedance to admittance by 180° rotation.

we multiply by the inverse of the impedance normalization factor. Thus,

$$Y_{in} = \frac{1}{Z_0} y_{in} = Y_0 y_{in}.$$

Rotations by 180 degrees to convert from the impedance to the admittance representation require only a reflection about the origin in the Γ-plane.

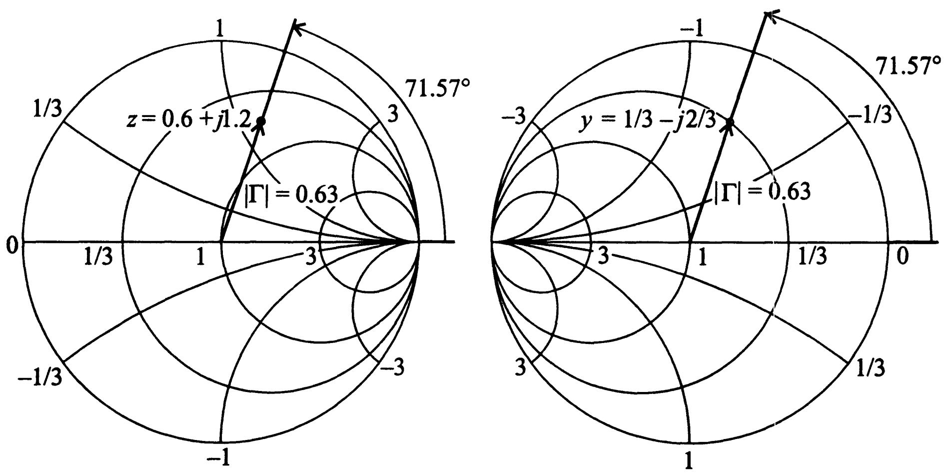
In addition to the preceding operation, there is a widely used additional possibility. Instead of rotating the reflection coefficient by 180° in the Z-Smith Chart, we can rotate the Smith Chart itself. The chart obtained by this transformation is called the **admittance Smith Chart** or the ***Y*-Smith Chart**. The correspondences are such that normalized resistances become normalized conductances and normalized reactances become normalized susceptances. That is,

$$r = \frac{R}{Z_0} \Rightarrow g = \frac{G}{Y_0} = Z_0 G$$

and

$$x = \frac{X}{Z_0} \Rightarrow b = \frac{B}{Y_0} = Z_0 B$$

This reinterpretation is depicted in Figure 3-11 for a particular normalized impedance point $z = 0.6 + j1.2$.



(a) *Z*-Smith Chart (b) *Y*-Smith Chart

Figure 3-11 Reinterpretation of the *Z*-Smith Chart as a *Y*-Smith Chart.

As seen in Figure 3-11, the transformation preserves (a) the direction in which the angle of the reflection coefficient is measured and (b) the direction of rotation (either toward or away from the generator). Attention has to be paid to the proper identification of the extreme points: A short-circuit condition $z_L = 0$ in the *Z*-Smith Chart is $y_L = \infty$ in the *Y*-Smith Chart, and conversely an open-circuit $z_L = \infty$ in the *Z*-Smith Chart is $y_L = 0$ in the *Y*-Smith Chart. Furthermore, negative values of susceptance are plotted now in the upper half of the chart, corresponding to inductive behavior, and positive values in the bottom half, corresponding to capacitive behavior. The real component of the admittance increases from right to left.

To complete our discussion of the *Y*-Smith Chart, we should mention an additional, often employed definition of the admittance chart. Here the admittance is represented in exactly the same manner as the impedance chart without a 180° rotation. In this case the reflection coefficient phase angle is measured from the opposite end of the chart (see the book by Gonzalez listed in Further Reading at the end of this chapter).

3.3.2 Additional Graphical Displays

In many practical design applications it is necessary to switch frequently from impedance to admittance representations and vice versa. To deal with those situations a combined, or so-called ***ZY*-Smith Chart**, can be obtained by overlaying the *Z*- and *Y*-Smith Charts, as shown in Figure 3-12.

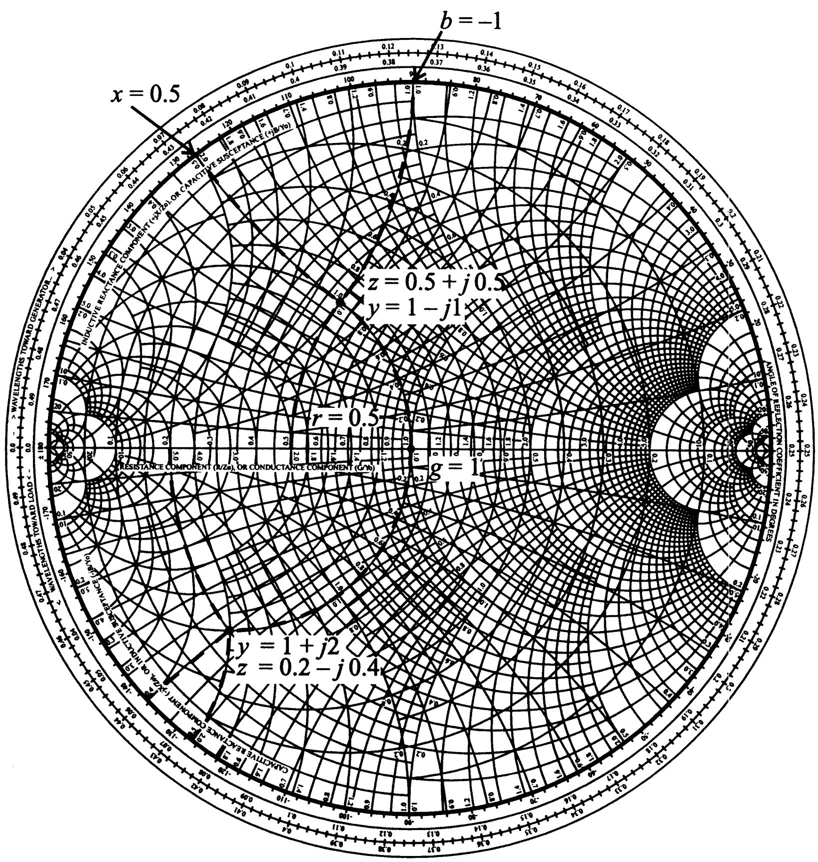


Figure 3-12 The *ZY*-Smith Chart superimposes the *Z*- and *Y*-Smith Charts in one graphical display.

This combined *ZY*-Smith Chart allows direct conversion between impedances and admittances. In other words, a point in this combined chart has two interpretations depending on whether the *Z*-Chart or *Y*-Chart display is chosen.

Example 3-7: Use of the combined *ZY*-Smith Chart

Identify (a) the normalized impedance value $z = 0.5 + j0.5$ and (b) the normalized admittance value $y = 1 + j2$ in the combined *ZY*-Smith Chart and find the corresponding values of normalized admittance and impedance.

Solution: Let us first consider the normalized impedance value $z = 0.5 + j0.5$. In the combined *ZY*-Smith Chart we locate the impedance by using circles of constant resistance $r = 0.5$ and constant reactance $x = 0.5$, as shown in Figure 3-12. The intersection of these two circles determines the specified impedance value $z = 0.5 + j0.5$. To find the corresponding admittance value we simply move along the circles of constant conductance g and susceptance b. The intersection gives us $g = 1$ and $jb = -j1$ (i.e., the admittance for part (a) of this example is $y = 1 - j1$). The solution for the normalized admittance $y = 1 + j2$ is obtained in identical fashion and is also illustrated in Figure 3-12.

The ZY-Smith Chart requires a fair amount of practice due to its "busy" appearance and the fact that inductors and capacitors are counted either in positive or negative units depending on whether an impedance or admittance representation is needed.

3.4 Parallel and Series Connections

In the following sections several basic circuit element configurations are analyzed and their impedance responses are displayed in the Smith Chart as a function of frequency. The aim is to develop insight into how the impedance/admittance behaves over a range of frequencies for different combinations of lumped circuit parameters. A practical understanding of these circuit responses is needed later in the design of matching networks (see Chapter 8) and in the development of equivalent circuit models.

3.4.1 Parallel Connection of *R* and *L* Elements

Recognizing that $g = Z_0/R$ and $b_L = +Z_0/(\omega L)$, we can locate the normalized admittance value in the upper *Y*-Smith Chart plane for a particular, fixed normalized conductance g at a certain angular frequency ω_L:

$$y_{in}(\omega_L) = g - j\frac{Z_0}{\omega_L L} \tag{3.25}$$

As the angular frequency is increased to the upper limit ω_U, we trace out a curve along the constant conductance circle g. Figure 3-13 schematically shows the frequency-dependent admittance behavior for various constant conductance values $g = 0.3$, 0.5, 0.7, and 1 and for frequencies ranging from 500 MHz to 4 GHz. For a fixed inductance value of $L = 10$ nH and a characteristic line impedance $Z_0 = 50\ \Omega$, the susceptance always starts at −1.59 (500 MHz) and ends at −0.20 (4 GHz).

In Figure 3-13 and the following three additional cases, the transmission line characteristic impedance is represented as a lumped impedance of $Z_0 = 50\ \Omega$. This is permissible since our interest is focused on the impedance and admittance behavior of different load configurations. For these cases the characteristic line impedance serves only as a normalization factor.

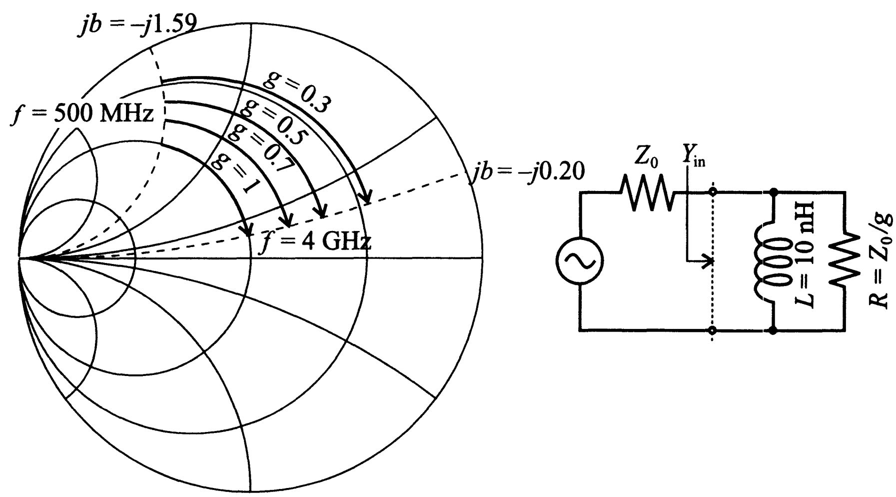


Figure 3-13 Admittance response of parallel *RL* circuit for $\omega_L \le \omega \le \omega_U$ at constant conductances $g = 0.3$, 0.5, 0.7, and 1.

3.4.2 Parallel Connection of *R* and *C* Elements

Here we operate in the lower *Y*-Chart plane because susceptance $b_C = Z_0\omega C$ remains positive. To locate the normalized admittance value for a particular, fixed normalized conductance g and angular frequency ω_L we have

$$y_{in}(\omega_L) = g + jZ_0\omega_L C \tag{3.26}$$

Figure 3-14 depicts the frequency-dependent admittance behavior as a function of various constant conductance values $g = 0.3$, 0.5, 0.7, and 1. The normalized susceptance for $C = 1$ pF and characteristic line impedance $Z_0 = 50\ \Omega$ always starts at 0.16 (500 MHz) and ends at 1.26 (4 GHz).

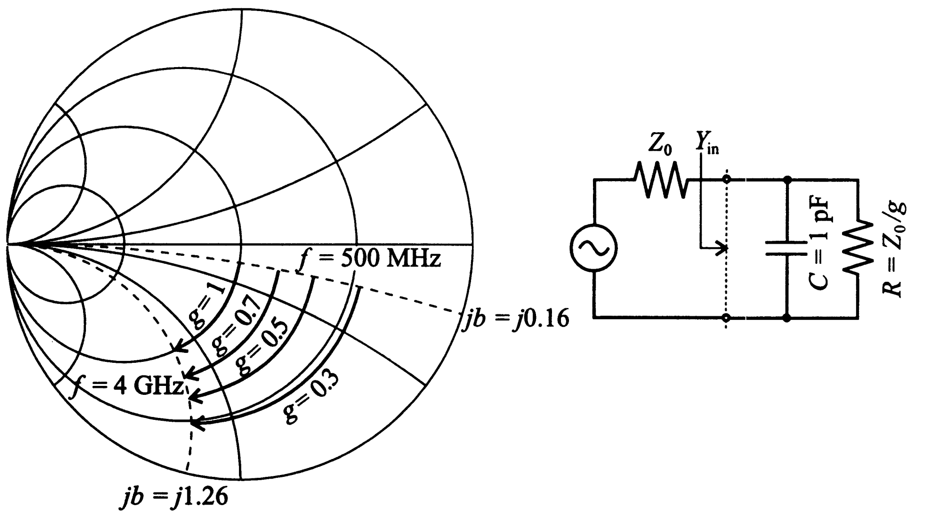


Figure 3-14 Admittance response of parallel *RC* circuit for $\omega_L \le \omega \le \omega_U$ at constant conductances $g = 0.3$, 0.5, 0.7, and 1.

3.4.3 Series Connection of *R* and *L* Elements

When dealing with series connections, we can conveniently choose the Z-Smith Chart for the impedance display. Identifying the normalized reactive component as $x_L = \omega L/Z_0$, it is straightforward to locate the normalized impedance value for a particular, fixed normalized resistance r at a given angular frequency ω_L:

$$z_{in}(\omega_L) = r + j\omega_L L/Z_0 \tag{3.27}$$

In Figure 3-15 the frequency-dependent impedance behavior is shown as a function of various constant resistance values $r = 0.3$, 0.5, 0.7, and 1. For the same inductance of 10 nH and characteristic line impedance of 50 Ω as used in Figure 3-13, we now pick reactance circles associated with 0.63 (500 MHz) and with 5.03 (4 GHz). Because the reactance is positive and since we use the Z-Smith Chart, all impedances have to reside in the upper half plane.

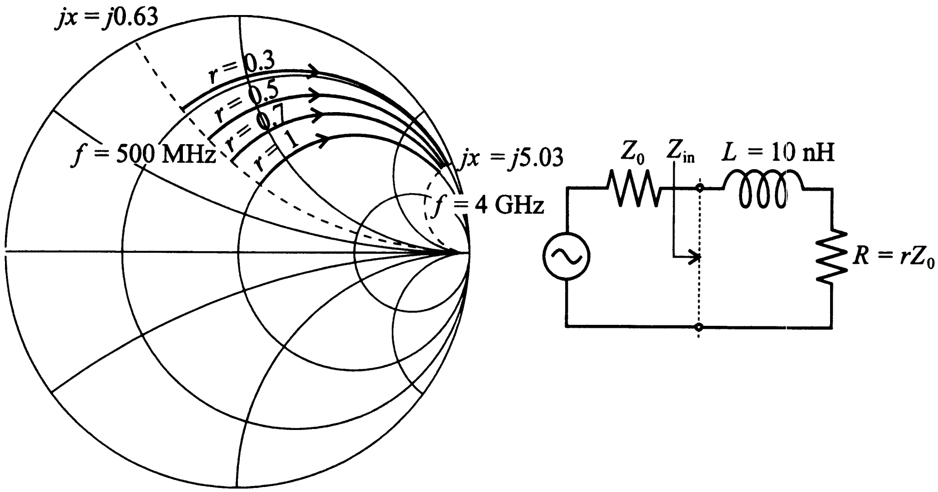


Figure 3-15 Impedance response of series RL circuit for $\omega_L \le \omega \le \omega_U$ and constant resistances $r = 0.3$, 0.5, 0.7, and 1.

3.4.4 Series Connection of *R* and *C* Elements

We again choose the Z-Smith Chart for the impedance display. The normalized reactive component is $x_C = +1/(\omega C Z_0)$, indicating that all curves will reside in the lower half of the Smith Chart. The normalized impedance value for a particular, fixed normalized resistance r at an angular frequency ω_L reads

$$z_{\text{in}}(\omega_L) = r - j\frac{1}{\omega_L C Z_0} \tag{3.28}$$

Figure 3-16 displays the frequency-dependent impedance behavior as a function of various constant resistance values $r = 0.3$, 0.5, 0.7, and 1. The capacitance of 1 pF in series with the variable resistance connected to a characteristic line impedance of 50 Ω now yields circles associated with the reactances of −6.03 (500 MHz) and −0.8 (4 GHz), which intersect with the four resistance circles, uniquely determining upper and lower impedance values.

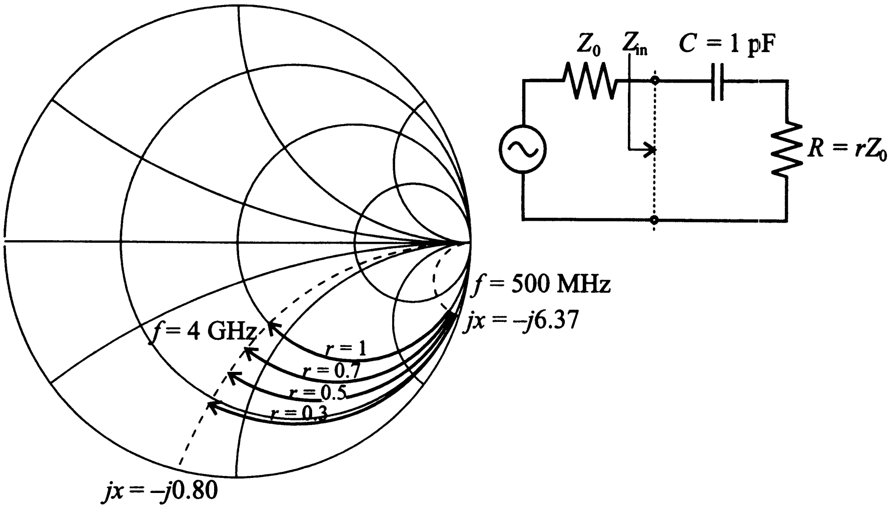


Figure 3-16 Impedance response of series *RC* circuit for $\omega_L \le \omega \le \omega_U$ at constant resistances $r = 0.3, 0.5, 0.7$, and 1.

3.4.5 Example of a *T*-Network

In the previous examples only pure series or shunt configurations have been analyzed. In reality, however, one often encounters combinations of both. To show how easily the *ZY* Chart allows transitions between series and shunt connections, let us investigate by way of an example the behavior of a ***T*-type network** connected to the input of a bipolar transistor. The input port of the transistor is modeled as a parallel *RC* network as depicted in Figure 3-17. As we will see in Chapter 6, R_L approximates the base-emitter resistance and C_L is the base-emitter junction capacitance. The numerical parameter values are listed in Figure 3-17.

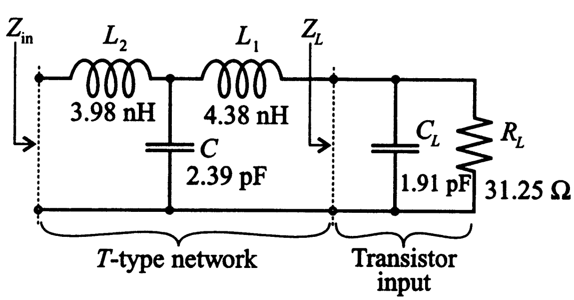


Figure 3-17 *T* network connected to the base-emitter input impedance of a bipolar transistor.

To use the Smith Chart for the computation of the input impedance of this more complicated network, we first analyze this circuit at 2 GHz and then show the entire response of the circuit for a frequency range from 500 MHz to 4 GHz by employing the commercial MMICAD software simulation package.

To obtain the load impedance, or the input impedance of the transistor, we use the *Y*-Smith Chart to identify the conductance point corresponding to the load resistor $R_L = 31.25\ \Omega$. Assuming a 50 Ω characteristic line impedance, we determine the normalized admittance for this case to be $g_A = 1.6$, which corresponds to point *A* in Figure 3-18.

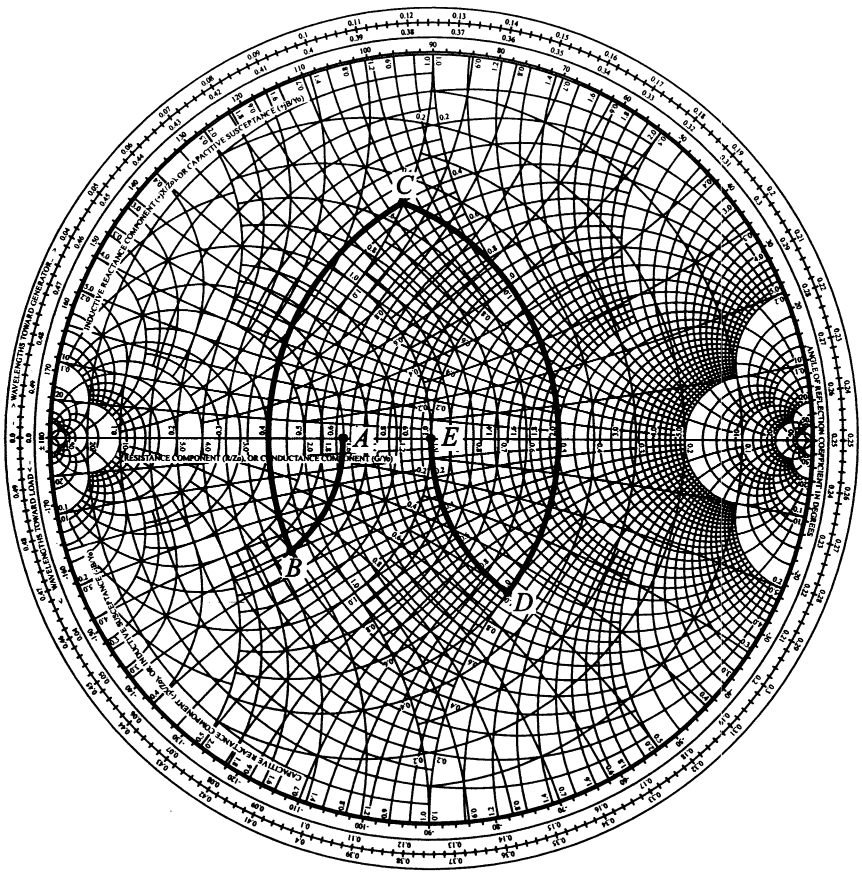


Figure 3-18 Computation of the normalized input impedance of the *T* network shown in Figure 3-17 for a center frequency $f = 2$ GHz.

The next step is to connect the capacitance $C_L = 1.91\,\text{pF}$ in shunt with the resistor R_L. At the angular frequency of $\omega_L = 2\pi 2\times 10^9\ \text{s}^{-1}$, the susceptance of this capacitor becomes $B_{C_L} = \omega_L C_L = 24\ \text{mS}$, which corresponds to a rotation of the original point *A* into the new location *B*. The amount of rotation is determined by the normalized susceptance of the capacitor $b_{C_L} = B_{C_L} Z_0 = 1.2$ and is carried out along the circle of constant conductance in the *Y*-Smith Chart (see Figure 3-18).

Re-evaluating point *B* in the *Z*-Smith Chart, we obtain the normalized impedance of the parallel combination of resistor R_L and capacitor C_L to be $z_B = 0.4 - j0.3$. The series connection of the inductance L_1 results in the new location *C*. This point is obtained through a rotation from $x_B = -0.3$ by an amount $x_{L_1} = \omega_L L_1/Z_0 = 1.1$ to $x_C = 0.8$ along the circle of constant resistance $r = 0.4$ in the *Z*-Smith Chart as discussed in Section 3.4.3.

Converting point *C* into a *Y*-Smith Chart value results in $y_C = 0.5 - j1.0$. The shunt connected capacitance requires the addition of a normalized susceptance $b_C = \omega C Z_0 = 1.5$, which results in the admittance value of $y_D = 0.5 + j0.5$ or point *D* in the Y-Smith Chart. Finally, converting point *D* into the impedance value $z_D = 1 - j1$ in the *Z*-Smith Chart allows us to add the normalized reactance $x_{L_2} = \omega_L L_2/Z_0 = 1$ along the constant $r = 1$ circle. Therefore, we reach $z_{in} = 1$ or point *E* in Figure 3-18. This value happens to match the 50 Ω characteristic transmission line impedance at the given frequency 2 GHz. In other words, $Z_{in} = Z_0 = 50\ \Omega$.

When the frequency changes we need to go through the same steps but will arrive at a different input impedance point z_{in}. It would be extremely tedious to go through the preceding computations for a range of frequencies. This is most efficiently done by the computer.

Relying on the previously mentioned CAD program MMICAD we are able to produce a graphical display of the input impedance in the *Z*-Smith Chart over the entire frequency range in preselected increments of 10 MHz, as shown in Figure 3-19. This figure can also be generated as part of the MATLAB software (see file fig3_18.m on the accompanying CD).

We notice that the impedance trace ranging from 0.5 to 4 GHz is in agreement with our previous calculations at 2 GHz. Also, as the frequency approaches 4 GHz, the capacitor of $C = 2.39$ pF behaves increasingly like a short circuit in series with a single inductor L_2. For this reason, the normalized resistance *r* approaches zero and the reactance grows to large positive values.

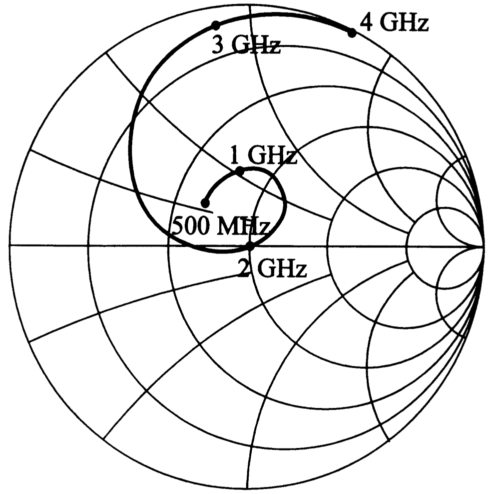


Figure 3-19 CAD simulation of the normalized input impedance Z_{in} for the network depicted in Figure 3-17 over the entire frequency range $500\text{ MHz} \le f \le 4\text{ GHz}$.

3.5 Summary

This chapter has derived the *Smith Chart* as the most widely used RF *graphical design tool* to display the impedance behavior of a transmission line as a function of either line length or frequency. Our approach originated from the representation of the normalized input impedance of a terminated transmission line in the form

$$z_{\text{in}} = r + jx = \frac{1+\Gamma(d)}{1-\Gamma(d)} = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

which can be inverted in terms of the reflection coefficient to yield two circle equations (3.10) and (3.11), which take on the following expressions for the normalized resistance r:

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \left(\frac{1}{r+1}\right)^2$$

and for the normalized reactance x

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

Superimposing the circles described by both equations over the complex polar form of the normalized impedance z-plane on the unit circle yields the Smith Chart. The key feature to remember is that one full rotation is equal to *half a wavelength* because of the exponent $2\beta d$ in the reflection coefficient expression (3.2). In addition to observing the impedance behavior, we can also quantify in the Smith Chart the degree of mismatch expressed by the *standing wave ratio* (SWR) equation (3.12), or

$$\text{SWR}(d) = \frac{1+|\Gamma(d)|}{1-|\Gamma(d)|}$$

which can be directly obtained from the chart.

To facilitate computer-based evaluation of the Smith Chart, a wide range of commercial programs can be utilized. Due to its ease of implementation on a PC and its user-friendly interface, throughout this book we have used the package *MMICAD* developed by Optotek. However, for the relatively incomplicated circuits analyzed in this Chapter, one can also create a custom-tailored Smith Chart and perform simple computations by relying on mathematical spreadsheets such as *Mathematica,* MATLAB, or *MathCad.* To demonstrate the procedure, a number of MATLAB modules have been developed, and the use of these so-called *m.files* as part of a basic Smith Chart computation is demonstrated in Section 3.2.4.

A transition to the admittance, or *Y-Smith Chart*, can be made via (3.23):

$$y_{\text{in}} = \frac{Y_{\text{in}}}{Y_0} = \frac{1}{z_{\text{in}}} = \frac{1-\Gamma(d)}{1+\Gamma(d)}$$

and it is found that the only difference to (3.4) is a sign reversal in front of the reflection coefficient. Consequently, rotating the reflection coefficient in the *Z*-Smith Chart by 180° results in the *Y*-Smith Chart. In practice, this rotation can be avoided by turning the chart itself. Superimposing the rotated chart over the original *Z*-Smith Chart provides a combined *ZY-Smith Chart* display. The benefit of such a display is the easy transition from parallel to series connection in circuit designs. This ease is demonstrated by a *T-network* configuration connected to the input port of a bipolar transistor consisting of a parallel *RC* network. To investigate the impedance behavior as a function of frequency sweep, however, is most easily accomplished through the use of CAD programs.

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Problems

3.1 Consider a load $Z_L = (80 + j40)\ \Omega$ connected to a lossy transmission line with characteristic line impedance of

$$Z_0 = \sqrt{\frac{0.1 + j200}{0.05 - j0.003}}$$

Determine the reflection coefficient and the standing wave ratio (SWR) at the load.

3.2 A coaxial cable of characteristic line impedance $Z_0 = 75\ \Omega$ is terminated by a load impedance of $Z_L = (40 + j35)\ \Omega$. Find the input impedance of the line for each of the following pairs of frequency f and cable length d assuming that the propagation velocity is 77% of the speed of light:
(a) $f = 1$ GHz and $d = 50$ cm
(b) $f = 5$ GHz and $d = 25$ cm
(c) $f = 9$ GHz and $d = 5$ cm

3.3 The attenuation coefficient of a transmission line can be determined by shortening the load side and recording the VSWR at the beginning of the line. We recall that the reflection coefficient for a lossy line takes on the form $\Gamma(d) = \Gamma_0 \exp(-kl) = \Gamma_0 \exp(-\alpha l)\exp(-j\beta l)$. If the line is 100 m in length and the VSWR is 3, find the attenuation coefficient α in Np/m, and dB/m.

3.4 A load impedance of $Z_L = (150 - j50)\ \Omega$ is connected to a 5 cm long transmission line with characteristic line impedance of $Z_0 = 75\ \Omega$. For a wavelength of 6 cm, compute
(a) the input impedance
(b) the operating frequency, if the phase velocity is 77% the speed of light
(c) the SWR

3.5 Identify the following normalized impedances and admittances in the Smith Chart:
(a) $z = 0.1 + j0.7$
(b) $y = 0.3 + j0.5$
(c) $z = 0.2 + j0.1$
(d) $y = 0.1 + j0.2$
Also find the corresponding reflection coefficients and SWRs.

3.6 An unknown load impedance is connected to a 0.3λ long, 50 Ω lossless transmission line. The SWR and phase of the reflection coefficient measured at the input of the line are 2.0 and –20°, respectively. Using the Smith Chart, determine the input and load impedances.

3.7 In Section 3.1.3 the circle equation (3.10) for the normalized resistance r is derived from (3.6). Start with (3.7); that is,

$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

and show that the circle equation

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

can be derived.

3.8 Starting with the equation for normalized admittance

$$y = g + jb = \frac{1-\Gamma}{1+\Gamma}$$

prove that the circle equations for the Y-Smith Chart are given by the following two formulas:
(a) For the constant conductance circle as

$$\left(\Gamma_r + \frac{g}{1+g}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+g}\right)^2$$

(b) For the constant susceptance circle as

$$(\Gamma_r + 1)^2 + (\Gamma_i + 1/b) = (1/b)^2$$

3.9 A lossless transmission line ($Z_0 = 50\ \Omega$) is 10 cm long ($f = 800$ MHz, $v_p = 0.77c$). If the input impedance is $Z_{in} = j60\ \Omega$
(a) Find Z_L (using the Smith Chart)
(b) What length of a short-circuit transmission line would be needed to replace Z_L?

3.10 A transmission line of characteristic impedance $Z_0 = 50\ \Omega$ and length $d = 0.15\lambda$ is terminated into a load impedance of $Z_L = (25 - j30)\ \Omega$. Find Γ_0, $Z_{in}(d)$, and the SWR by using the Z-Smith Chart.

3.11 A short-circuited 50 Ω transmission line section is operated at 1 GHz and possesses a phase velocity of 75% of the speed of light. Use both the analytical and the Smith Chart approach to determine the shortest lengths required to obtain (a) a 5.6 pF capacitor, and (b) a 4.7 nH inductor.

3.12 Determine the shortest length of a 75 Ω open-circuit transmission line that equivalently represents a capacitor of 4.7 pF at 3 GHz. Assume the phase velocity is 66% of the speed of light.

3.13 A circuit is operated at 1.9 GHz and a lossless section of a 50 Ω transmission line is short circuited to construct a reactance of 25 Ω. (a) If the phase velocity is 3/4 of the speed of light, what is the shortest possible length of the line to realize this impedance? (b) If an equivalent capacitive load of 25 Ω is desired, determine the shortest possible length based on the same phase velocity.

3.14 A microstrip line with 50 Ω characteristic line impedance is terminated into a load impedance consisting of a 200 Ω resistor in shunt with a 5 pF capacitor. The line is 10 cm in length and the phase velocity is 50% the speed of light. (a) Find the input impedance in the Smith Chart at 500 MHz, 1 GHz, and 2 GHz, and (b) use the MATLAB routine (see Section 3.2.4) and plot the frequency response from 100 MHz to 3 GHz in the Smith Chart.

3.15 For an FM broadcasting station operated at 100 MHz, the amplifier output impedance of 250 Ω has to be matched to a 75 Ω dipole antenna.
(a) Determine the length and characteristic impedance of a quarter-wave transformer with $v_p = 0.7c$.
(b) Find the spacing D for a two-wire lossless transmission line with AWG

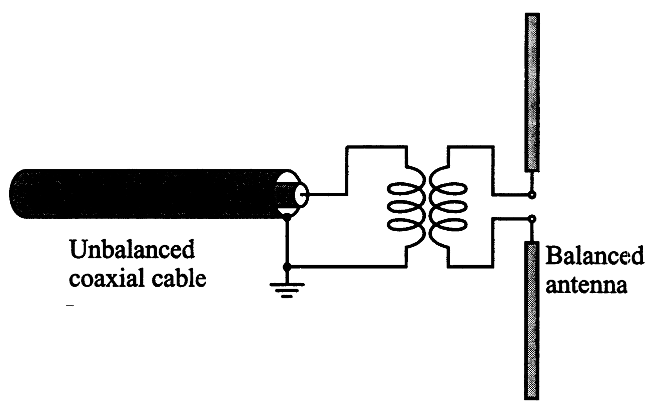
26 wire size and a polysterene dielectric ($\varepsilon_r = 2.55$).

3.16 Consider the case of matching a 73 Ω load to a 50 Ω line by means of a $\lambda/4$ transformer. Assume the matching is achieved for a center frequency of $f_C = 2$ GHz. Plot the SWR for the frequency range $1/3 \le f/f_C \le 3$.

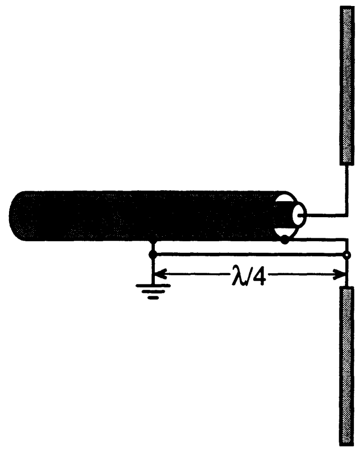
3.17 A line of characteristic impedance of 75 Ω is terminated by a load consisting of a series connection of $R = 30\ \Omega$, $L = 10$ nH, and $C = 2.5$ pF. Find the values of the SWR and minimum line lengths at which a match of the input impedance to the characteristic line is achieved. Consider the following range of frequencies: (a) 100 MHz, (b) 500 MHz, and (c) 2 GHz.

3.18 A 50 Ω lossless coaxial cable ($\varepsilon_r = 2.8$) is connected to a 75 Ω antenna operated at 2 GHz. If the cable length is 25 cm, find the input impedance by using the analytical equation (2.71) and the Z-Smith Chart.

3.19 A **bal**anced to **un**balanced (balun) transformation is often needed to connect a dipole antenna (balanced) to a coaxial cable (unbalanced). The following figure depicts the basic concept.



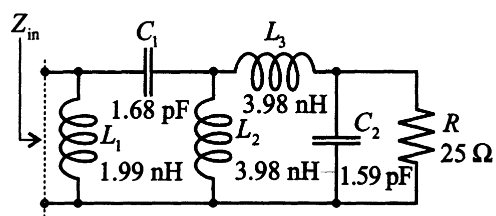
As an alternative of using a transformer, one often uses the following antenna connection.



(a) Explain why one leg of the dipole antenna is connected a distance $\lambda/4$ away from the end of the coax cable.

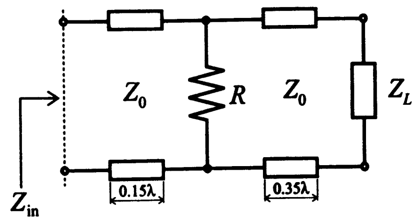
(b) For an FM broadcast band antenna in the frequency range from 88 to 108 MHz, find the average length where the connection has to be made.

3.20 Using the *ZY*-Smith Chart, find the input impedance of the following network at 2 GHz.

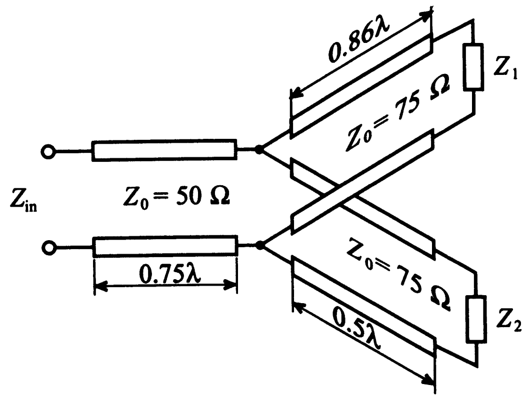


What is the input impedance of this network at 1 GHz?

3.21 A $Z_0 = 50\Omega$ transmission line is 0.5λ in length and terminated into a load of $Z_L = (50 - j30)\Omega$. At 0.35λ away from the load, a resistor of $R = 25\Omega$ is connected in shunt configuration (see figure below). Find the input impedance with the help of the ZY Smith Chart.



3.22 A 50-Ω transmission line of 3/4 wavelength in length is connected to two transmission line sections each of 75 Ω in impedance and length of 0.86 and 0.5 wavelength, respectively, as illustrated in the following figure.

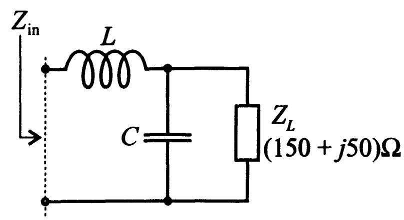


The termination for line 1 is $Z_1 = (30 + j40)\ \Omega$ and $Z_2 = (75 - j80)\ \Omega$ for line 2. Employ the Smith Chart and find the input impedance.

3.23 Repeat the previous problem if all characteristic line impedances are $Z_0 = 50\ \Omega$ and all transmission line sections are $\lambda/4$ in length.

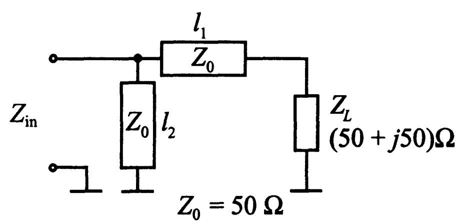
3.24 A dipole antenna of impedance $Z_L = (75 + j20)\ \Omega$ is connected to a 50 Ω lossless transmission whose length is $\lambda/3$. The voltage source $V_G = 25V$ is attached to the transmission line via an unknown resistance R_G. It is determined that an average power of 3W is delivered to the load under load-side matching ($Z_L^{\text{match}} = 50\ \Omega$). Find the generator resistance R_G, and determine the power delivered to the antenna if the generator impedance is matched to the line via a quarter-wave transformer.

3.25 Determine the values of the inductance L and the capacitance C such that they result in a 50 Ω input impedance at 3 GHz operating frequency for the following network.

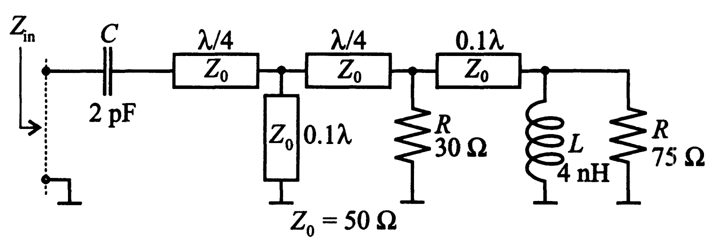


3.26 An open-circuit transmission line (50 Ω) is operated at 500 MHz ($v_p = 0.7c$). Use the *ZY* Smith Chart and find the impedance Z_{in} if the line is 65 cm in length. Find the shortest distance for which the admittance is $Y_{in} = -j0.05S$.

3.27 Find the minimum line length l_1 and the minimum length of the short-circuited stub l_2 in terms of wavelength λ, such that the input impedance of the circuit is equal to 50 Ω.

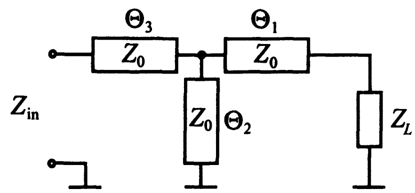


3.28 Find the input impedance in terms of magnitude and phase of the following network at an operating frequency of 950 MHz.



3.29 Repeat your computation and solve Problem 3.28 for a 1.5 GHz operating frequency. Comment on the differences in your results.

3.30 A specific transmission line configuration is as follows:



The characteristic line impedance for all three elements is $Z_0 = 50\ \Omega$. The load impedance has a value of $Z_L = (20 + j40)\ \Omega$, and the electrical lengths of the corresponding line segments are $\Theta_1 = 164.3°$, $\Theta_2 = 57.7°$, and $\Theta_3 = 25.5°$.

(a) Find the input impedance.

(b) Find the input impedance if transmission line segment Θ_2 is open circuit.

(This problem and Problem 3.27 become very important in Chapter 8, when we discuss the problem of matching a particular load impedance to a desired input impedance.)