CHAPTER 3

Steady-State Waves on Transmission Lines

In digital-integrated electronics, computer communication, and many other applications, it is important to understand the response of transmission lines to steplike changes in their inputs, as we studied in detail in Chapter 2. We have seen that waves travel down a transmission line by successively charging the distributed capacitors of the line and establishing current in the distributed inductors. In this context, a wave is a function of both time and space but does not necessarily involve periodic oscillations of a physical quantity (e.g., the height of water in ocean waves). If a disturbance that occurs at a certain point and time causes disturbances at other points in the surrounding region at later times, then wave motion is said to exist. Because of the nonzero travel time along a transmission line, disturbances initiated at one location induce effects that are retarded in time at other locations. The natural response of transmission lines to sudden changes in their inputs typically involves wave motion, with disturbances traveling down the line, producing reflections at terminations, or discontinuities, which in turn propagate back to the source end, and so on. However, except in special cases, the natural response eventually decays after some time interval and is primarily described by the intrinsic properties of the transmission line (characteristic impedance and one-way travel time, or length) and its termination (i.e., the load), rather than the input excitation.

In many engineering applications, however, it is also important to understand the steady-state response of transmission lines to sinusoidal excitations. Electrical power and communication signals are often transmitted as sinusoids or modified sinusoids. Other nonsinusoidal signals, such as pulses utilized in a digitally coded system, may be considered as a superposition (i.e., Fourier series) of sinusoids of different frequencies. In sinusoidal signal applications, the initial onset of a sinusoidal input produces a natural response. However, this initial transient typically decays rapidly in time, while the forced response supported by the sinusoidal excitation continues indefinitely. Once the steady state is reached, voltages and currents on the transmission line vary sinusoidally in time at each point along the line while also traveling down the line. The finite travel time of waves leads to phase differences between the voltages (or currents) at different points along the line. The steady-state solutions for voltage and current waves when reflections are present lead to *standing waves,* which also vary sinusoidally in time at each point on the line but do not travel along the line. The differences between traveling and standing waves will become clear in the following sections.

In discussing the steady-state response of transmission lines to sinusoidal excitation, we take full advantage of powerful tools[1] commonly used for analysis of alternating-current phenomena in lumped electrical circuits, including *phasors* and complex *impedance.* The phasor notation eliminates the need to keep track of the known sinusoidal time dependence of the various quantities and allows us to transform the transmission line equations from partial differential equations to ordinary differential equations. The magnitude of the complex impedance of a device or a load is a measure of the degree to which it opposes the excitation by a sinusoidal voltage source; a load with a higher impedance (in magnitude) at any given frequency requires a higher voltage in order to allow a given amount of current through it *at the given frequency.* The phase of the impedance represents the phase difference between the voltage across it and the current through it. In lumped circuits, the impedance of a load is determined only by its internal dynamics (i.e., its physical makeup as represented by its voltage-current characteristics, e.g., $V = L\,dI/dt$ for an inductor). In transmission line applications, the impedance of a load presented to a source via a transmission line depends on the characteristic impedance and the *electrical length*[2] (physical length per unit of wavelength) of the line connecting them. This additional length dependence makes the performance of transmission line systems dependent on frequency—to a greater degree than is typical in lumped circuit applications.

In this chapter, we exclusively consider excitations that are pure sinusoids. In most applications, transmission line systems have to accommodate signals made up of modified sinusoids, with the energy or information spread over a small bandwidth around a central frequency. The case of excitations involving waveforms that

[1]These tools were first introduced by a famous electrical engineer, C. P. Steinmetz, in his first book, *Theory and Calculation of AC Phenomena,* McGraw-Hill, 1897.

[2]The electrical length of a transmission line is its physical length divided by the wavelength at the frequency of operation. For example, the electrical length of a 1.5-m-long air-filled coaxial line operating at 100 MHz (wavelength $\lambda = 3$ m) is 0.5λ, while the electrical length of the same line operating at 1 MHz (wavelength $\lambda = 300$ m) is 0.005λ.

are other than sinusoidal can usually be handled by suitably decomposing the signal into its Fourier components, each of which can be analyzed as described in this chapter. In the first seven sections we consider lossless (i.e., $R = 0$ and $G = 0$) transmission lines. Lossy transmission lines (i.e., $R \neq 0$ and $G \neq 0$) and resonant transmission line elements are discussed in Sections 3.8 and 3.9, respectively. We shall see that the effects of small but nonzero losses can be accounted for by suitably modifying the lossless analysis. Note also that the loss terms are truly negligible in many applications, so that the lossless cases considered in detail are of practical interest in their own right.

Our topical presentation in this chapter starts in Section 3.1 with a discussion of the solutions of transmission line equations expressed in terms of voltage and current phasors. The important special cases of transmission lines with short- and open-circuited terminations are covered in Section 3.2, followed in Section 3.3 by the analysis of the response of lines terminated in an arbitrary impedance, and in Section 3.4 by a discussion of power and energy relations. The subject of impedance matching, crucially important in practice, is covered in Section 3.5, and the Smith chart, a graphical method useful for both quantitative analysis and qualitative visualization of transmission line behavior, is discussed in Section 3.6. Selected application examples involving the steady-state response of lossless transmission lines are presented in Section 3.7. Lossy and resonant transmission lines are discussed respectively in Sections 3.8 and 3.9.

3.1 WAVE SOLUTIONS USING PHASORS

The fundamental transmission line equations for the lossless case were developed in Chapter 2 (see equations [2.3] and [2.4]):

$$\frac{\partial \mathcal{V}}{\partial z} = -L\frac{\partial \mathcal{I}}{\partial t} \qquad [3.1]$$

$$\frac{\partial \mathcal{I}}{\partial z} = -C\frac{\partial \mathcal{V}}{\partial t} \qquad [3.2]$$

Equations [3.1] and [3.2] are written in terms of the space-time functions describing the instantaneous values of voltage and current at any point z on the line, denoted respectively as $\mathcal{V}(z, t)$ and $\mathcal{I}(z, t)$. When the excitation is sinusoidal, and under steady-state conditions, we can use the phasor concept to reduce the transmission line equations [3.1] and [3.2] to ordinary differential equations (instead of partial differential equations, as they are now) so that we can more easily obtain general solutions. As in circuit analysis, the relations between phasors and actual space-time functions are as follows:

$$\mathcal{V}(z, t) = \Re e\{V(z)e^{j\omega t}\} \qquad [3.3a]$$

$$\mathcal{I}(z, t) = \Re e\{I(z)e^{j\omega t}\} \qquad [3.3b]$$

Here, both phasors $V(z)$ and $I(z)$ are functions of z only and are in general complex.

We can now rewrite[3] equations [3.1] and [3.2] in terms of the phasor quantities by replacing $\partial/\partial t$ with $j\omega$. We have

$$\boxed{\frac{dV(z)}{dz} = -j\omega LI(z)} \qquad [3.4]$$

and

$$\boxed{\frac{dI(z)}{dz} = -j\omega CV(z)} \qquad [3.5]$$

Combining [3.4] and [3.5], we can write a single equation in terms of $V(z)$,

$$\frac{d^2V(z)}{dz^2} = -(\omega^2 LC)V(z) \quad \text{or} \quad \frac{d^2V(z)}{dz^2} = -\beta^2 V(z) \qquad [3.6]$$

where, $\beta = \omega\sqrt{LC}$ is called the *phase constant.* Equation [3.6] is referred to as the complex *wave equation* and is a second-order ordinary differential equation commonly encountered in analysis of physical systems. The general solution of [3.6] is of the form

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \qquad [3.7]$$

where, as we shall see below, $e^{-j\beta z}$ and $e^{j\beta z}$ represent, respectively, wave propagation in the $+z$ and $-z$ directions, and where V^+ and V^- are complex constants to be determined by the boundary conditions. The corresponding expression for the current $I(z)$ can be found by substituting [3.7] in [3.4]. We find

$$I(z) = I^+ e^{-j\beta z} + I^- e^{+j\beta z} = \frac{1}{Z_0}[V^+ e^{-j\beta z} - V^- e^{+j\beta z}] \qquad [3.8]$$

where $Z_0 = V^+/I^+ = -V^-/I^- = \sqrt{L/C}$ is the characteristic impedance of the transmission line.

Using [3.3], and the expressions [3.7] and [3.8] for the voltage and current phasors, we can find the corresponding space-time expressions for the instantaneous voltage and current. We have

$$\begin{aligned} \mathcal{V}(z,t) &= \Re e\{V(z)e^{j\omega t}\} = \Re e\{V^+ e^{-j\beta z}e^{j\omega t} + V^- e^{+j\beta z}e^{j\omega t}\} \\ &= V^+ \cos(\omega t - \beta z) + V^- \cos(\omega t + \beta z) \end{aligned} \qquad [3.9]$$

[3]The actual derivation of [3.4] from [3.1] is as follows:

$$\frac{\partial \mathcal{V}}{\partial z} = -L\frac{\partial \mathcal{I}}{\partial t} \longrightarrow \frac{\partial}{\partial z}\underbrace{[\Re e\{V(z)e^{j\omega t}\}]}_{\mathcal{V}(z,t)} = -L\frac{\partial}{\partial t}\underbrace{[\Re e\{I(z)e^{j\omega t}\}]}_{\mathcal{I}(z,t)}$$

$$\longrightarrow \Re e\left\{e^{j\omega t}\frac{dV(z)}{dz}\right\} = \Re e\{-L(j\omega)e^{j\omega t}I(z)\} \longrightarrow \frac{dV(z)}{dz} = -j\omega LI(z)$$

where we have assumed[4] V^+ and V^- to be real. Similarly, we have

$$\mathcal{I}(z,t) = \frac{1}{Z_0}[V^+\cos(\omega t - \beta z) - V^-\cos(\omega t + \beta z)] \qquad [3.10]$$

The voltage and current solutions consist of a superposition of two waves, one propagating in the $+z$ direction (i.e., toward the load) and the other in the $-z$ direction (i.e., a reflected wave moving away from the load). To see the wave behavior, consider the case of an infinitely long transmission line; in this case no reflected wave is present, and thus $V^- = 0$. The voltage and current for an infinitely long line are

$$V(z) = V^+ e^{-j\beta z}; \qquad \mathcal{V}(z,t) = V^+\cos(\omega t - \beta z) \qquad [3.11]$$

and

$$I(z) = \frac{V^+}{Z_0}e^{-j\beta z}; \qquad \mathcal{I}(z,t) = \frac{V^+}{Z_0}\cos(\omega t - \beta z) \qquad [3.12]$$

Note that, everywhere along the line, the ratio of the voltage to current phasors is Z_0; hence, Z_0 is called the characteristic impedance of the line. Note, however, that this is true not only for an infinitely long line but also for a line of finite length that is terminated at a load impedance $Z_L = Z_0$, as we discuss later.

The solutions [3.11] and [3.12] are in the form of [1.2], which was introduced in Chapter 1 by stating that electromagnetic quantities with such space-time dependencies are often encountered. Here we see that this form of solution is indeed a natural solution of the fundamental transmission line equations.[5] The space-time behavior of the voltage wave given by [3.11] is illustrated in Figure 3.1. We note from Figure 3.1a that the period of the sinusoidal oscillations (as observed at fixed points in space) is $T_p = 2\pi/\omega$. The voltage varies sinusoidally at all points in space, but it reaches its maxima at different times at different positions. Figure 3.1b indicates that the voltage distribution as a function of distance (observed at a fixed instant of time) is also sinusoidal. The distance between the crests of the voltage at a fixed instant of time is the *wavelength* $\lambda = 2\pi/\beta$. As time progresses, the wave propagates to the right ($+z$ direction), as can be seen by observing a particular point on the waveform (e.g., the crest or the minimum) at different instants of time. The speed of this motion is the *phase velocity,* defined as the velocity at which an observer must travel to observe a stationary (i.e., not varying with time) voltage. The voltage observed would be the same as long as the argument of the cosine in [3.11] is the same; thus we have

$$\omega t - \beta z = \text{const.} \quad \longrightarrow \quad v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} \qquad [3.13]$$

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[4]If instead V^+ and V^- were complex, with $V^+ = |V^+|e^{j\phi^+}$ and $V^- = |V^-|e^{j\phi^-}$, we would have

$$\mathcal{V}(z,t) = |V^+|\cos(\omega t - \beta z + \phi^+) + |V^-|\cos(\omega t + \beta z + \phi^-)$$

[5]We will encounter the same type of space-time variation once again in Chapter 8, as the natural solution of Maxwell's equations for time-harmonic (or sinusoidal steady-state) electric and magnetic fields.

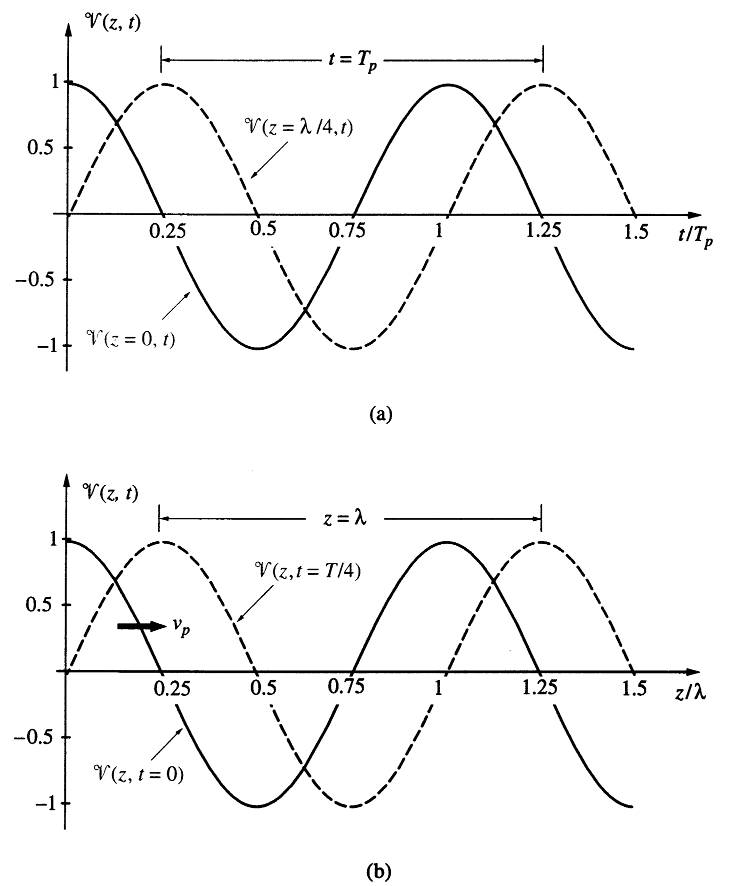


FIGURE 3.1. Wave behavior in space and time. (a) $\mathcal{V}(z, t) = V^+ \cos[2\pi(t/T_p) - 2\pi(z/\lambda)]$ versus t/T_p for $z = 0$ and $z = \lambda/4$. (b) $\mathcal{V}(z, t) = V^+ \cos[2\pi(t/T_p) - 2\pi(z/\lambda)]$ versus z/λ for $t = 0$ and $t = T_p/4$. In both panels we have taken $V^+ = 1$.

where v_p is the phase velocity, which was also introduced in Sections 1.1.3 and 2.2 (see equation [2.7]).

As discussed in Section 2.2, for most of the commonly used two-conductor transmission lines (Figure 2.1), the phase velocity v_p is not a function of the particular geometry of the metallic conductors but is instead solely determined by the electrical and magnetic properties of the surrounding insulating medium. When the surrounding medium is air, the phase velocity is the speed of light in free space, namely, $v_p = c \simeq 3 \times 10^8$ m-s^{-1} = 30 cm-(ns)$^{-1}$. The phase velocity v_p for some other insulating materials was tabulated in Table 2.1. Phase velocities for some additional materials are given in Table 3.1, together with the corresponding values of wavelength at a frequency of 300 MHz. Note that since $\lambda = 2\pi/\beta$, we have $v_p = \omega/\beta = \lambda f$, so that the phase constant β and wavelength λ depend on the electrical

TABLE 3.1. Phase velocity and wavelength in different materials

Material	**Wavelength (m at 300 MHz)**	**Phase velocity speed (cm-(ns)$^{-1}$ at 300 MHz)**
Air	1	30
Silicon	0.29	8.7
Polyethylene	0.67	20.0
Epoxy glass (PC board)	0.45	13.5
GaAs	0.30	9.1
Silicon carbide (SiC)	0.15	4.6
Glycerin	0.14	4.2

and magnetic properties of the material surrounding the transmission line conductors as well as on the frequency of operation.

3.2 VOLTAGE AND CURRENT ON LINES WITH SHORT- OR OPEN-CIRCUIT TERMINATIONS

Most sinusoidal steady-state applications involve transmission lines terminated at a load impedance Z_L. Often, voltages and currents near the load end are of greatest interest, since they determine the degree of matching between the line and the load and the amount of power delivered to (versus that reflected from) the load. A portion of a lossless transmission line terminated in an arbitrary load impedance Z_L is shown in Figure 3.2. We can use this setup to explore the concept of reflected waves on transmission lines, a fundamental feature of distributed circuits in general.

Assume that a forward-propagating ($+z$ direction) wave of the form $V^+e^{-j\beta z}$ produced by a source located at some position z ($z < 0$) is incident on load Z_L located at $z = 0$. Contrary to the case of an infinitely long transmission line, here the ratio of the total voltage $V(z)$ to the total current $I(z)$ at any position z along the line is not equal to Z_0. For example, at the load position ($z = 0$), we must satisfy the

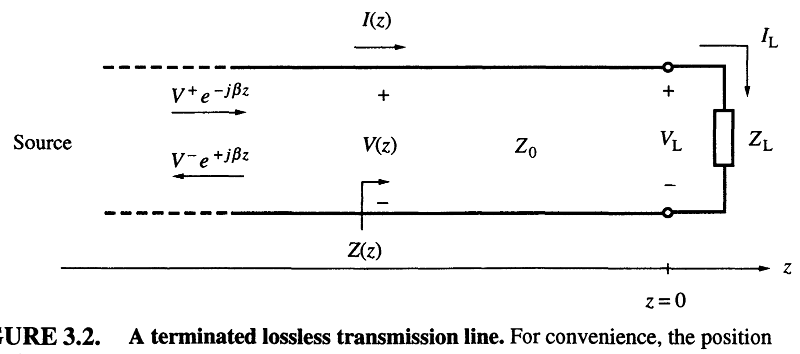


FIGURE 3.2. A terminated lossless transmission line. For convenience, the position of the load is taken to be $z = 0$.

boundary condition $[V(z)/I(z)]_{z=0} = Z_L$, where Z_L is in general not equal to Z_0. Thus, since $V^+/I^+ = Z_0$, and in general $Z_0 \neq Z_L$, a reverse-propagating ($-z$ direction) reflected wave of the form $V^- e^{+j\beta z}$ with the appropriate value for V^- must be present so that the load boundary condition is satisfied. The total voltage and current phasors, $V(z)$ and $I(z)$, at any position on the line consist of the sum of the forward and reverse waves as specified by [3.7] and [3.8], namely

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \quad [3.14]$$

$$I(z) = \frac{1}{Z_0}[V^+ e^{-j\beta z} - V^- e^{+j\beta z}] \quad [3.15]$$

where V^+ and V^- are in general complex constants to be determined by the boundary conditions.

When a transmission line has only a forward-traveling wave with no reflected wave (e.g., in the case of an infinitely long line), the ratio of the total voltage to the current is the characteristic impedance Z_0, as was discussed in Section 3.1. When the line is terminated so that, in general, a reflected wave exists, the ratio of the total line voltage $V(z)$ to the total line current $I(z)$ at any position z—the *line impedance*—is not equal to Z_0. The line impedance is of considerable practical interest; for example, the impedance that the line presents to the source at the source end of the line (called the input impedance of the line, denoted by Z_{in}) is the line impedance evaluated at that position. The source, or the generator, does not know anything about the characteristic impedance of the line or whether a reflected wave exists on the line; it merely sees that when it applies a voltage of V_s to the input terminals of the line, a certain current I_s flows, and thus the source interprets the ratio of V_s/I_s as an impedance of a particular magnitude and phase.

The line impedance as seen by looking toward the load Z_L at any position z along the line (see Figure 3.2) is defined as

$$\boxed{Z(z) \equiv \frac{V(z)}{I(z)} = Z_0 \frac{V^+ e^{-j\beta z} + V^- e^{j\beta z}}{V^+ e^{-j\beta z} - V^- e^{j\beta z}}} \quad [3.16]$$

In general, the line impedance $Z(z)$ is complex and is a function of position z along the line. From electrical circuit analysis, we know that a complex impedance $Z(z)$ can be written as

$$Z(z) = R(z) + jX(z)$$

where the real quantities $R(z)$ and $X(z)$ are, respectively, the resistive and the reactive parts of the line impedance.

The following example considers the case of a *matched load,* defined as a load impedance equal to the characteristic impedance of the line, or $Z_L = Z_0$.

Example 3-1: Matched load. A lossless transmission line is terminated with a load $Z_L = Z_0$, as shown in Figure 3.3a. Find the magnitude of the reflected wave V^- and the line impedance $Z(z)$.

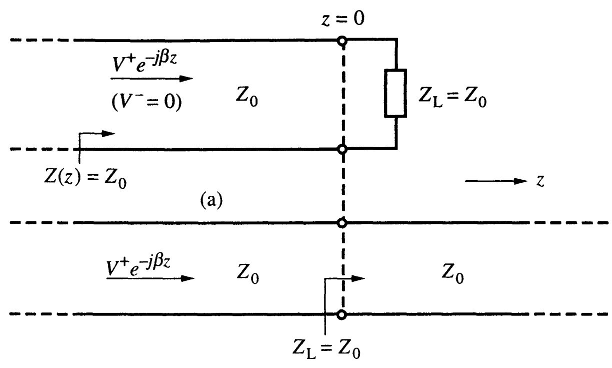


FIGURE 3.3. **Matched load.** (a) Circuit diagram. (b) Z_L is replaced by an infinitely long extension of the same line.

Solution: If $Z_L = Z_0$, the load boundary condition $V(z = 0)/I(z = 0) = Z_L = Z_0$ is satisfied without any reflected wave, so that $V^- = 0$. The line impedance at any position z along the line is $Z(z) = V(z)/I(z) = Z_0 = Z_L$ independent of z. In this case, the load impedance Z_L can be viewed as an infinite extension of the same transmission line, as shown in Figure 3.3b.

Transmission line segments terminated in short or open circuits are commonly used as tuning elements for impedance matching networks (see Sections 3.2.3 and 3.5) and also as resonant circuit elements (see Sections 3.2.3 and 3.9). In the next two subsections, we study these two special cases in detail before considering (in Section 3.3) the more general case of lines terminated in an arbitrary impedance Z_L.

3.2.1 Short-Circuited Line

Figure 3.4 shows a transmission line of length l terminated in a short circuit. Short-circuited termination forces the load voltage V_L to be zero, so we have

$$V_L = [V(z)]_{z=0} = [V^+e^{-j\beta z} + V^-e^{j\beta z}]_{z=0} = V^+ + V^- = 0$$

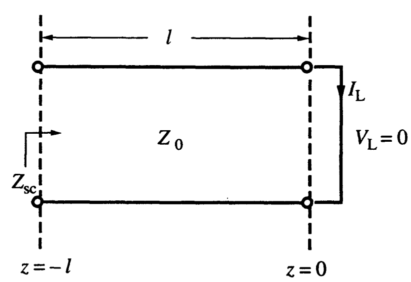


FIGURE 3.4. **Short-circuited line.** The input impedance Z_{sc} of a short-circuited line can be capacitive or inductive, depending on the length l of the line.

leading to

$$V^- = -V^+ \quad \text{or} \quad \frac{V^-}{V^+} = -1$$

Note that the load current flowing through the short circuit can be found from [3.15] using $V^- = -V^+$:

$$I_{\rm L} = [I(z)]_{z=0} = \frac{1}{Z_0}(V^+ - V^-) = \frac{2V^+}{Z_0}$$

Anywhere else along the line we have

$$V(z) = V^+(e^{-j\beta z} - e^{j\beta z}) = -2V^+ j\ \sin(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} + e^{j\beta z}) = \frac{2V^+}{Z_0}\ \cos(\beta z)$$

The instantaneous space-time function for the voltage can be obtained from $V(z)$ using [3.3a]. We have

$$\begin{aligned}\mathcal{V}(z, t) &\equiv \Re e\{V(z)e^{j\omega t}\} = \Re e\{V^+(e^{-j\beta z} - e^{j\beta z})e^{j\omega t}\} \\ &= \Re e\{2|V^+|e^{j\phi^+} \sin(\beta z)e^{-j\pi/2}e^{j\omega t}\} \\ &= 2|V^+|\sin(\beta z)\cos\left(\omega t - \frac{\pi}{2} + \phi^+\right)\end{aligned}$$

where $V^+ = |V^+|e^{j\phi^+}$ is in general a complex constant, to be determined from the boundary condition at the source end of the transmission line. However, note that we can assume $\phi^+ = 0$ (i.e., V^+ is a real constant) without any loss of generality because V^+ is a constant multiplier that appears in front of all voltages and currents everywhere along the line. Accounting for a possible finite phase ϕ^+ simply amounts to shifting the time reference, with no effect on the relationships among the various quantities. Accordingly, we assume $\phi^+ = 0$ throughout the following discussion.

Note that the voltage $\mathcal{V}(z, t)$ along a short-circuited line is a cosinusoid (in time) whose amplitude varies as $2|V^+|\sin(\beta z)$ with position z along the line. It is not a traveling wave, since the peaks (or minima) of the $\cos(\omega t - \pi/2)$ always stay (i.e., stand) at the same positions (i.e., z/λ) along the line as the voltage varies in time (see Figure 3.5b). The voltage $\mathcal{V}(z, t)$ is thus said to represent a pure *standing wave*. Similarly, for current we have

$$\mathcal{I}(z, t) = \frac{2|V^+|}{Z_0}\cos(\beta z)\cos(\omega t)$$

which is also a standing wave (see Figure 3.5c). The absolute amplitudes of both the voltage and current phasors, namely $|V(z)|$ and $Z_0|I(z)|$, are shown as functions of z/λ in Figure 3.5d.

Note that the current and voltage are in time quadrature (i.e., 90° out of phase), such that when $\mathcal{I}(z, t)$ at a given point z is zero, the absolute amplitude of $\mathcal{V}(z, t)$ is

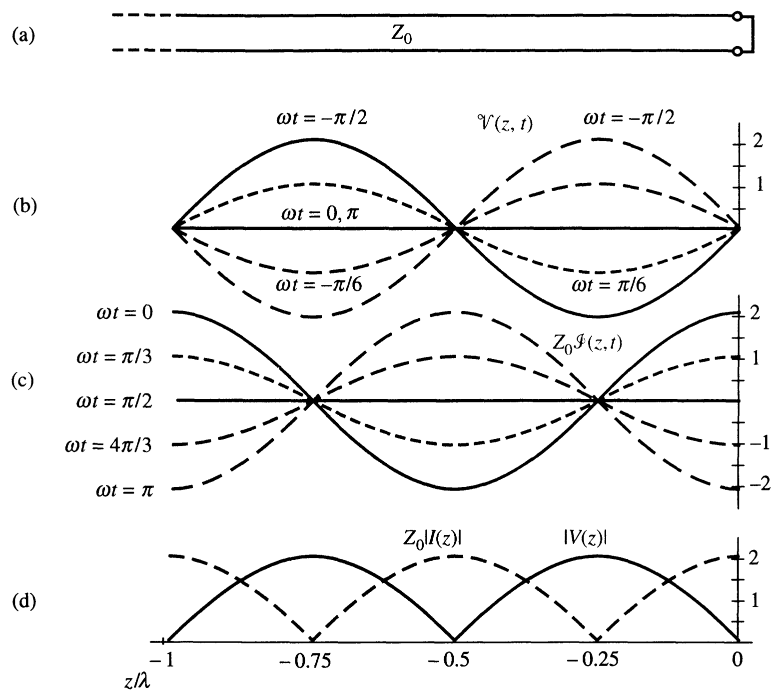


FIGURE 3.5. Voltage and current on a short-circuited line. (a) Schematic of a short-circuited line. (b) Instantaneous voltage $\mathcal{V}(z, t)$ versus z at different times. (c) Corresponding instantaneous current times the characteristic impedance, namely $Z_0\mathcal{I}(z, t)$. (d) Magnitudes of the voltage and current phasors, showing $|V(z)|$ (solid line) and $Z_0|I(z)|$ (dashed line) as functions of z/λ. All of the plots shown are for $V^+ = 1e^{j0°}$.

a maximum, and vice versa. Standing waves stand on the line but do not travel or carry any time-average power to the load;[6] they represent reactive power in a manner

[6]To see this, consider that the instantaneous power carried by the wave is given by

$$\begin{aligned}\mathcal{P}(z, t) &= \mathcal{V}(z, t)\mathcal{I}(z, t) = \left[2|V^+|\sin(\beta z)\cos\left(\omega t - \frac{\pi}{2}\right)\right]\left[\frac{2|V^+|}{Z_0}\cos(\beta z)\cos(\omega t)\right] \\ &= \frac{2|V^+|^2}{Z_0}\sin(2\beta z)\cos\left(\omega t - \frac{\pi}{2}\right)\cos(\omega t) \\ &= \frac{|V^+|^2}{Z_0}\sin(2\beta z)\sin(2\omega t)\end{aligned}$$

where we have used the trigonometric identities of $\cos(\zeta - \pi/2) = \sin\zeta$ and $2\sin\zeta\cos\zeta = \sin(2\zeta)$. Note that the instantaneous power carried by the standing wave oscillates in time at a rate twice that of the voltage and current and that its average over one period (i.e., $T = 2\pi/\omega$) is thus zero, as expected on the basis of the fact that the voltage and current are 90° out of phase.

analogous to the voltage and current, also in time quadrature, of a capacitor or inductor. The power relationships on a transmission line are discussed in more detail in Section 3.4.

The line impedance seen looking toward the short circuit at any position z along the short-circuited line is

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{-2jV^+ \sin(\beta z)}{2V^+ \cos(\beta z)} = -jZ_0 \tan(\beta z)$$

where $z < 0$. The input impedance of a short-circuited line segment of length l can then be found by evaluating the preceding equation at the source end, where $z = -l$:

$$\boxed{Z_{sc} = jZ_0 \tan(\beta l) = jX_{sc}} \qquad [3.17]$$

We note from [3.17] that the input impedance of a short-circuited line of length l is purely reactive. As illustrated in Figure 3.6, the input impedance depends on line length l, or more generally, on the electrical length, which is defined as the ratio of the physical length of the line to the wavelength, i.e., l/λ, where $\lambda = 2\pi/\beta$. The input impedance can be varied by varying the length or the frequency, or both, and can be capacitive (negative X_{sc}) or inductive (positive X_{sc}). It makes physical sense that the input impedance is inductive for very short line lengths ($l < \lambda/4$); a shorted two-wire line of relatively short length resembles a small loop of wire. The fact that any reactive input impedance can be realized by simply varying the length of a short-circuited line (and the open-circuited line as will be discussed in the next subsection) is very useful in tuning- and impedance-matching applications at microwave frequencies, as discussed in Section 3.5.

Example 3-2 illustrates the use of [3.17] to determine the input impedance of a television antenna lead-in wire.

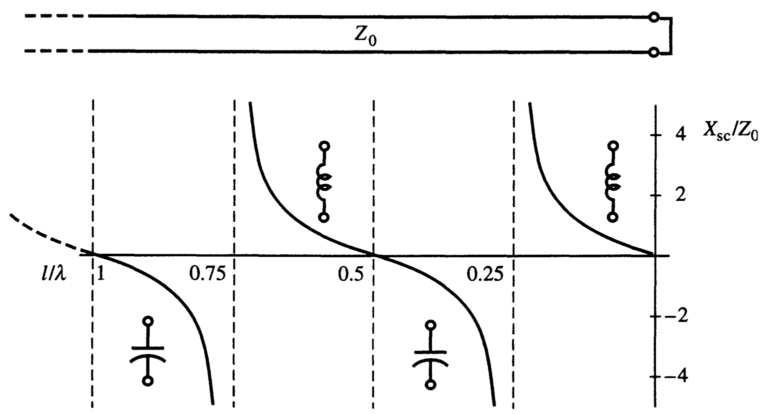


FIGURE 3.6. Input impedance of a short-circuited line. The normalized reactance X_{sc}/Z_0 of a short-circuited line segment of length l is shown as a function of electrical length l/λ.

Example 3-2: Inductance of a short television antenna lead-in wire. Consider a television antenna lead-in wire of length $l = 10$ cm having a characteristic impedance of $Z_0 = 300\Omega$, shorted at one end. Find the input impedance of this line if it is to be used at 300 MHz.

Solution: We can first determine the electrical length of the line. As in Example 2-18, we assume the conductors to be mostly surrounded by air (although they might in fact be held together by some plastic material) so that the phase velocity $v_p \simeq c$. Since the wavelength at 300 MHz is $\lambda = v_p/f \simeq 3 \times 10^8/(300 \times 10^6) = 1$ m, the electrical length is $(l/\lambda) = 0.1$. The input impedance of the shorted line can then be found directly from [3.17]:

$$Z_{sc} = jZ_0 \tan(\beta l) = j(300)\tan\left(\frac{2\pi}{\lambda}l\right) \simeq j218\Omega$$

This is an inductive input impedance. At a frequency of 300 MHz, an inductive reactance of $X_{sc} = \omega L_{sc} = 218\Omega$ corresponds to a lumped inductor having an inductance of $L_{sc} = 218/(2\pi \times 300 \times 10^6) \simeq 0.116\ \mu$H.

According to [3.17] and Figure 3.6, the input impedance of a short-circuited line of length $l = \lambda/4$ is infinite; that is, the line appears as if it is an open circuit. In practice, however, the input impedance of such a line is limited by its distributed conductance. Note that considering the circuit model of the line (Figure 2.5) the shunt conductance per unit length G presents a resistance proportional to $(G)^{-1}$ across the input terminals of a line, even when the input impedance looking toward the load is infinite. Although G was assumed to be zero for our lossless analysis, it nevertheless is a nonzero value and thus limits the input impedance of the line to a finite value. Similarly, equation [3.17] and Figure 3.6 indicate that the input impedance of a short-circuited line of length $l = \lambda/2$ is zero, in other words, that the line appears as if it is a short circuit. In practice, however, the minimum value of input impedance of a short-circuited half-wavelength long line is determined by its series resistance R, which, although small, is nevertheless a nonzero value. The behavior of a short-circuited line of length $l = \lambda/4$ is similar to that of a lumped resonant circuit, as discussed in Section 3.9.

3.2.2 Open-Circuited Line

Figure 3.7 shows an open-circuited transmission line. The analysis of open-circuited lossless transmission lines is very similar to that of short-circuited lines.

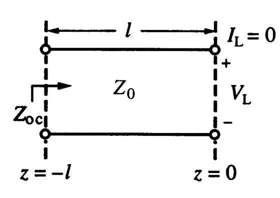


FIGURE 3.7. Open-circuited line. The input impedance Z_{oc} of an open-circuited line can be capacitive or inductive, depending on the length of the line.

Open-circuited termination forces the load current I_L to be zero, so that using [3.15] we have

$$I_L = [I(z)]_{z=0} = \left[\frac{V^+}{Z_0}e^{-j\beta z} - \frac{V^-}{Z_0}e^{j\beta z}\right]_{z=0} = \frac{V^+ - V^-}{Z_0} = 0$$

leading to

$$V^- = V^+ \quad \text{or} \quad \frac{V^-}{V^+} = +1$$

Note that the load voltage appearing across the open circuit can be found from [3.14] using $V^- = V^+$. We have

$$V_L = [V(z)]_{z=0} = [V^+e^{-j\beta z} + V^-e^{j\beta z}]_{z=0} = (V^+ + V^-) = 2V^+$$

Anywhere else along the line we have

$$V(z) = V^+(e^{-j\beta z} + e^{j\beta z}) = 2V^+\cos(\beta z)$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - e^{j\beta z}) = -2\frac{V^+}{Z_0}j\sin(\beta z) = 2\frac{V^+}{Z_0}(e^{-j\pi/2})\sin(\beta z)$$

The instantaneous space-time expressions for the voltage and current[7] are

$$\mathcal{V}(z,t) = \Re e\{V(z)e^{j\omega t}\} = 2|V^+|\cos(\beta z)\cos(\omega t)$$

$$\mathcal{I}(z,t) = \Re e\{I(z)e^{j\omega t}\} = 2\frac{|V^+|}{Z_0}\sin(\beta z)\cos\left(\omega t - \frac{\pi}{2}\right)$$

where we have assumed $V^+ = |V^+|e^{j\phi^+}$, with $\phi^+ = 0$, without any loss of generality.

As for the short-circuited line, the current and voltage on an open-circuited line are in time quadrature (i.e., out of phase by 90°) so that the average power carried is again zero. Both $\mathcal{V}(z,t)$ and $\mathcal{I}(z,t)$ are purely standing waves. Their absolute amplitude patterns are shown in Figure 3.8b, in the same format as in Figure 3.5d.

The line impedance seen looking toward the open circuit at any position z along the line is

$$Z(z) = \frac{V(z)}{I(z)} = jZ_0\cot(\beta z)$$

---

[7]Note for the current that

$$\Re e\{I(z)e^{j\omega t}\} = \Re e\left\{\frac{V^+}{Z_0}[e^{-j\beta z} - e^{j\beta z}]e^{j\omega t}\right\} = \Re e\left\{2\frac{|V^+|}{Z_0}\sin(\beta z)e^{j\phi^+}e^{-j(\pi/2)}e^{j\omega t}\right\}$$

$$= \Re e\left\{2\frac{|V^+|}{Z_0}\sin(\beta z)\left[\cos\left(\omega t - \frac{\pi}{2} + \phi^+\right) + j\sin\left(\omega t - \frac{\pi}{2} + \phi^+\right)\right]\right\}$$

$$= \frac{2|V^+|}{Z_0}\sin(\beta z)\cos\left(\omega t - \frac{\pi}{2} + \phi^+\right)$$

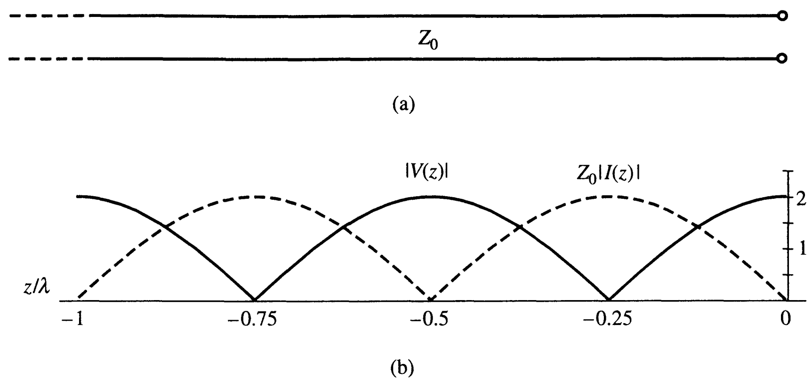


FIGURE 3.8. Voltage and current on an open-circuited line. (a) Schematic of an open-circuited line. (b) Magnitudes of the voltage and current phasors, showing $|V(z)|$ (solid line) and $Z_0|I(z)|$ (dashed line) as functions of z/λ, for $V^+ = 1e^{j0°}$.

where $z < 0$. The input impedance of an open-circuited line of length l (i.e, at $z = -l$) is then given by

$$\boxed{Z_{oc} = Z(z = -l) = -jZ_0 \cot(\beta l) = jX_{oc}} \qquad [3.18]$$

As for a short-circuited line, the input impedance of an open-circuited line of length l is purely reactive. The normalized reactance X_{oc}/Z_0 is plotted in Figure 3.9 as a function of electrical length l/λ. A capacitive or inductive reactance can

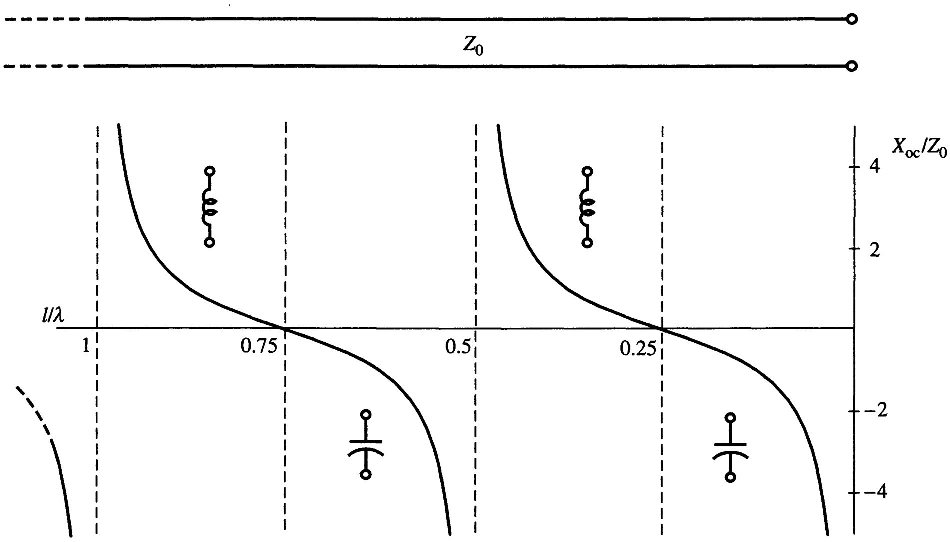


FIGURE 3.9. Input impedance of an open-circuited line. The normalized reactance X_{oc}/Z_0 of an open-circuited line segment of length l is shown as a function of electrical length l/λ.

be obtained simply by adjusting the line length l for a fixed wavelength λ (or frequency ω), or by adjusting the wavelength λ (or frequency ω) for fixed line length l. The fact that for $l < \lambda/4$ the impedance is capacitive makes physical sense, since a relatively short-length open-circuited line consists of two conductors with some separation between them, resembling an ordinary lumped capacitor.

Example 3-3 illustrates the use of [3.18] for a practical transmission line.

Example 3-3: Capacitance of a short television antenna lead-in wire. Consider a television antenna lead-in wire of length $l = 20$ cm having a characteristic impedance of $Z_0 = 300\Omega$ and nothing connected at its end (i.e., open-circuited). Find the input impedance of this line if it is to be used at 300 MHz.

Solution: Once again we assume the phase velocity to be $v_p \simeq c$ and first determine the electrical length of the line. Since the wavelength at 300 MHz is $\lambda = v_p/f \simeq 3 \times 10^8/(300 \times 10^6) = 1$ m, the electrical length is $(l/\lambda) = 0.2$. The input impedance of the open-circuited line can then be found directly from [3.18]:

$$Z_{oc} = -jZ_0 \cot(\beta l) = -j(300)\cot\left(\frac{2\pi}{\lambda}l\right) \simeq -j97.5\Omega$$

This is a capacitive input impedance. At a frequency of 300 MHz, a capacitive reactance of $X_{oc} = -(\omega C_{oc})^{-1} \simeq -97.5\Omega$ corresponds to a lumped capacitor of capacitance $C_{oc} = [(97.5)(2\pi \times 300 \times 10^6)]^{-1} \simeq 5.44$ pF.

According to [3.18] and Figure 3.9, the input impedance of an open-circuited line of length $l = \lambda/4$ is zero; that is, the line appears as if it is short-circuited. In practice, however, the minimum value of input impedance is determined by its series resistance R, which, although small, is nevertheless a nonzero value. Similarly, equation [3.18] and Figure 3.9 both indicate that the input impedance of an open-circuited line of length $l = \lambda/2$ is infinite, that is, that the line appears as if it is an open circuit. In practice, however, the input impedance of such a line is limited by its nonzero distributed conductance G. Open- or short-circuited lines of lengths equal to integer multiples of $\lambda/4$ or $\lambda/2$ are analogous to lumped resonant circuits, and such line segments constitute highly efficient resonators, as discussed in Section 3.9.

3.2.3 Open- and Short-Circuited Lines as Reactive Circuit Elements

An important application of transmission lines involves their use as capacitive or inductive tuning elements in microwave circuits at frequencies between a few gigahertz to a few tens of gigahertz. In this frequency range, lumped inductors and capacitors become exceedingly small and difficult to fabricate. Furthermore, the wavelength is small enough that the physical sizes and separation distances of ordinary circuit components are no longer negligible. On the other hand, transmission line sections of appropriate sizes can be constructed with relative ease. For frequencies higher than ~100 GHz, the physical size of transmission lines is too small, although novel transmission line implementations can operate[8] at frequencies as high as 500 GHz, corresponding to submillimeter wavelengths.

That transmission lines behave as reactive circuit elements is quite evident from Figures 3.6 and 3.9. Consider, for example, the input impedance as a function of frequency of a short-circuited line of length l such that $l = \lambda_0/4$. At a frequency of f_0, for which $\lambda = \lambda_0$, this line presents an infinite impedance (i.e., appears as an open circuit) at its input terminals. For frequencies slightly smaller than f_0, namely $f < f_0$ so that $\lambda > \lambda_0$, the length of the line is slightly shorter than $\lambda/4$, so it presents a very large inductive impedance (Figure 3.6). For frequencies slightly greater than f_0, the electrical length of the line is slightly larger than $\lambda/4$, so it has a very large capacitive impedance (Figure 3.6). Such behavior is similar to that of a lumped circuit consisting of a parallel combination of an inductor and a capacitor.

A similar analysis of a short-circuited line of length l such that $l = \lambda_0/2$ indicates that a half-wavelength line behaves as a lumped circuit consisting of a series combination of an inductor and a capacitor. As can be seen from Figure 3.6, the magnitude of the input impedance of a short-circuited line of length $l = \lambda_0/2$ is very small in the vicinity of its resonant frequency f_0, and the input impedance is inductive for $f > f_0$ and capacitive for $f < f_0$.

Corresponding observations can also be made for open-circuited line segments, for which the input impedance is given as a function of electrical length in Figure 3.9. Lumped circuit counterparts of various transmission line segments are summarized in Figure 3.10.

We see from the preceding discussion that short- or open-circuited transmission line elements act as resonant circuits. In the absence of losses, a short- or open-circuited line segment would store its electrical energy forever, even if the source were removed. When losses are taken into account, transmission line resonators consisting of short- or open-circuited lines of lengths equal to integer multiples of $\lambda/4$ or $\lambda/2$ are highly efficient energy storage devices and exhibit a high degree of frequency selectivity, as discussed in Section 3.9.

The use of a short-circuited transmission line segment as a microwave inductance is illustrated in Example 3-4.

Example 3-4: Transmission line inductor. A short-circuited coaxial line with $v_p = 2.07 \times 10^8$ m-s^{-1} is to be designed to provide a 15-nH inductance for a microwave filter operating at 3 GHz. (a) Find the shortest possible length l if the characteristic impedance of the line is $Z_0 = 50\Omega$, and (b) find the lumped element value of the short-circuited line designed in part (a) at 4 GHz.

[8] Linda P. B. Katehi, Novel transmission lines for the submillimeter-wave region, *Proceedings of the IEEE*, 80(11), p. 1771, November 1992.

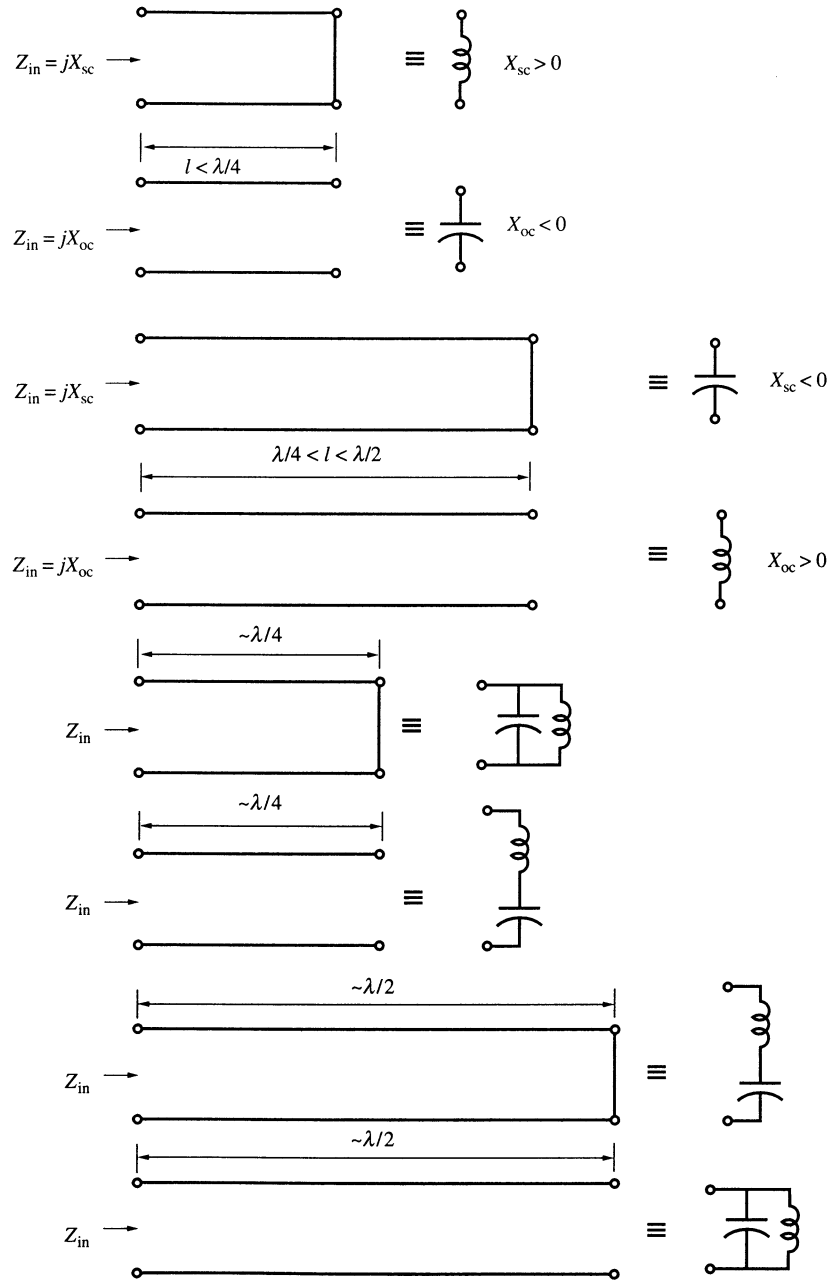


FIGURE 3.10. Lumped circuit models of various open- and short-circuited line segments.

Solution:

(a) Equating the input impedance of a short-circuited line of length l to the impedance of a lumped inductor, we have

$$Z_{sc} = jZ_0 \tan\left(\frac{2\pi}{\lambda}l\right) = j\omega L_{sc}$$

where $\lambda = v_p/f = (2.07 \times 10^{10})/(3 \times 10^9) = 6.9$ cm. For $Z_0 = 50\Omega$, we can write

$$l = \frac{6.9}{2\pi}\tan^{-1}\left(\frac{2\pi \times 3 \times 10^9 \times 15 \times 10^{-9}}{50}\right) \simeq 1.53 \text{ cm}$$

(b) At 4 GHz, $\lambda = 2.07 \times 10^{10}/(4 \times 10^9) = 5.175$ cm. Thus, the input impedance of the short-circuited 50Ω coaxial line of length $\sim$1.53 cm is

$$Z_{sc} \simeq j(50)\tan\left(\frac{2\pi \times 1.53}{5.175}\right) \simeq -j167.4\Omega$$

Therefore, at 4 GHz, the short-circuited 50Ω coaxial line designed in part (a) represents a lumped capacitor of element value given by

$$-\frac{j}{2\pi \times 4 \times 10^9 C} = -j167.4\Omega \quad \rightarrow \quad C_{sc} \simeq 0.238 \text{ pF}$$

The results are summarized in Figure 3.11.

3.3 LINES TERMINATED IN AN ARBITRARY IMPEDANCE

Most sinusoidal steady-state applications involve transmission lines terminated in arbitrary complex load impedances. The load to be driven may be an antenna, the feed-point impedance of which depends in a complicated manner on the antenna characteristics and operating frequency and is in general quite different from the characteristic impedance of the transmission line that connects it to a source. For efficient transmission of the energy from the source to the load, it is often necessary to match the load to the line, using various techniques to be discussed in Section 3.5. In this section, we consider the fundamental behavior of line voltage, current, and impedance for arbitrarily terminated transmission lines. Consider a transmission line of length l terminated in an arbitrary complex load impedance Z_L and excited by a sinusoidal voltage source, the phasor of which is represented by V_0, as shown in Figure 3.12. The line is uniform and lossless (i.e., Z_0 = const., $R = 0$ and $G = 0$), so the voltage and current phasors at any position $z < 0$ along the line are in general given by

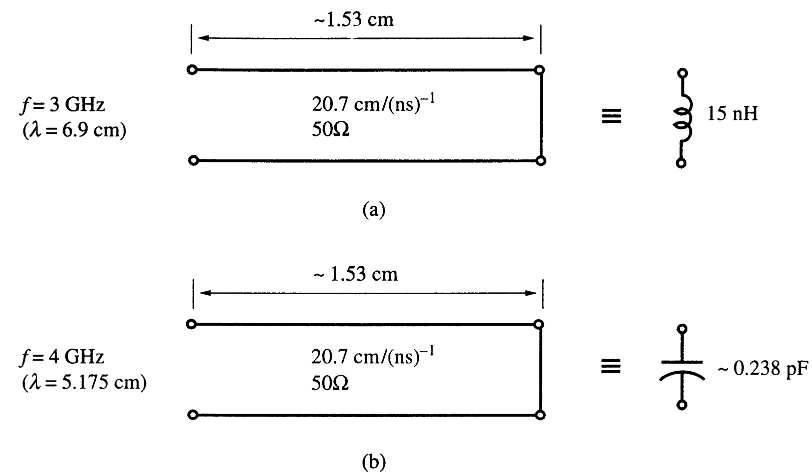


FIGURE 3.11. A short-circuited 50Ω line as an inductor or capacitor. (a) Designed to provide a 15 nH inductance at 3 GHz. (b) Equivalent to a ~ 0.238 pF capacitance at 4 GHz.

$$V(z) = V^+e^{-j\beta z} + V^-e^{+j\beta z} \qquad [3.19]$$

$$I(z) = \frac{1}{Z_0}[V^+e^{-j\beta z} - V^-e^{+j\beta z}] \qquad [3.20]$$

The boundary condition at the load end ($z = 0$) is simply

$$V_L = Z_L I_L \quad \rightarrow \quad V(z)|_{z=0} = Z_L I(z)|_{z=0}$$

or

$$Z_L = Z_0 \frac{V^+ + V^-}{V^+ - V^-}$$

The ratio of the phasors of the reverse and forward waves at the load position ($z = 0$) is the *load voltage reflection coefficient,* defined as $\Gamma_L \equiv V^-/V^+$ such that

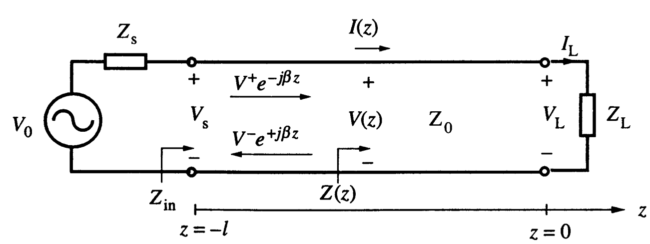


FIGURE 3.12. A terminated line. A lossless transmission line excited by a sinusoidal source terminated in a complex load impedance Z_L.

$$\boxed{Z_L = Z_0\frac{1+\Gamma_L}{1-\Gamma_L} \quad \rightarrow \quad \Gamma_L = \rho e^{j\psi} = \frac{Z_L - Z_0}{Z_L + Z_0}} \tag{3.21}$$

where we explicitly recognize that Γ_L is in general a complex number with a magnitude ρ and phase angle ψ, where $0 \leq \rho \leq 1$. Note that, as shown before, $\Gamma_L = -1$ (i.e., $V^- = -V^+$) for a short-circuited line, $\Gamma_L = +1$ (i.e., $V^- = V^+$) for an open-circuited line, and $\Gamma_L = 0$ (i.e., $V^- = 0$) for a matched load (i.e., $Z_L = Z_0$).

The voltage and current phasors given by equations [3.19] and [3.20] can now be written in terms of Γ_L as

$$V(z) = V^+(e^{-j\beta z} + \Gamma_L e^{j\beta z}) = V^+e^{-j\beta z}[1 + \Gamma(z)] \tag{3.22}$$

$$I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - \Gamma_L e^{j\beta z}) = \frac{V^+}{Z_0}e^{-j\beta z}[1 - \Gamma(z)] \tag{3.23}$$

where

$$\Gamma(z) \equiv \frac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)}$$

is the *voltage reflection coefficient* at any position z along the line, defined as the ratio of the phasors of the reverse and forward propagating waves at that position. The voltage reflection coefficient is a complex number with a constant magnitude ρ (equal to the magnitude of Γ_L) and a phase angle varying with position z. We can view the quantity $\Gamma(z) \equiv \Gamma_L e^{j2\beta z}$ as a generalized reflection coefficient defined not only at the load but also at any point z along the line. Noting that the voltage along the line is given by

$$V(z) = \underbrace{V^+e^{-j\beta z}}_{\text{forward wave}} + \underbrace{\Gamma_L V^+ e^{j\beta z}}_{\text{reflected wave}}$$

we can see that the quantity $\Gamma(z) = \Gamma_L e^{j2\beta z}$ is indeed the ratio of the reflected wave at point z to the forward wave at that same position.

Note that the line impedance seen looking toward the load at any position z along the line can be written in terms of $\Gamma(z)$ as

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \tag{3.24}$$

3.3.1 Voltage and Current Standing-Wave Patterns

To understand the nature of the voltage on the line, it is useful to examine the complete time function $\mathcal{V}(z, t)$. For this, we start our discussion with the case of a real (resistive) load impedance (i.e., $\psi = 0$ or π) and rewrite [3.22] as

$$\begin{aligned} V(z) &= V^+[e^{-j\beta z} \pm \rho e^{j\beta z}] \\ &= V^+[(1 \pm \rho)e^{-j\beta z} \pm \rho(-e^{-j\beta z} + e^{j\beta z})] \end{aligned}$$

which, by using $\mathcal{V}(z, t) = \Re e\{V(z)e^{j\omega t}\}$, gives

$$\mathcal{V}(z,t) = \underbrace{|V^+|(1 \pm \rho)\cos(\omega t - \beta z + \phi^+)}_{\text{propagating wave}} \pm \underbrace{|V^+|(2\rho)\sin(\beta z)\cos\left(\omega t + \phi^+ + \frac{\pi}{2}\right)}_{\text{standing wave}}$$

where we have taken $V^+ = |V^+|e^{j\phi^+}$. In other words, the voltage on the line consists of a standing wave plus a propagating wave.

According to [3.22], the magnitude of the voltage phasor (i.e., $|V(z)|$) alternates between the maximum and minimum values of $V_{\max}$ and $V_{\min}$ given by

$$V_{\max} = |V(z)|_{\max} = |V^+|(1 + |\Gamma_L|) = |V^+|(1 + \rho)$$
$$V_{\min} = |V(z)|_{\min} = |V^+|(1 - |\Gamma_L|) = |V^+|(1 - \rho)$$

Similarly, from Equation [3.23], the magnitude of the current phasor (i.e., $|I(z)|$) alternates between the maximum and minimum values of

$$I_{\max} = |I(z)|_{\max} = \frac{|V^+|}{Z_0}(1 + \rho)$$

$$I_{\min} = |I(z)|_{\min} = \frac{|V^+|}{Z_0}(1 - \rho)$$

where $I_{\max}$ occurs at the same position as $V_{\min}$, and $I_{\min}$ occurs at the same position as $V_{\max}$. For example, Figure 3.13 shows the variations of both voltage and current magnitudes (represented by $|V(z)|$ and $Z_0|I(z)|$) as functions of position with respect to wavelength along the line, for the case of a purely resistive load, with $Z_L = R_L = 2Z_0$. As is apparent from Figure 3.13, the distance between successive voltage maxima (or minima) is $\lambda/2$. Note that for the case shown, with $Z_L = R_L > Z_0$, the reflection coefficient Γ_L is purely real with $\psi = 0$ and $0 \le \rho \le 1$.

The following example illustrates the concepts of reflection coefficient and standing-wave pattern.

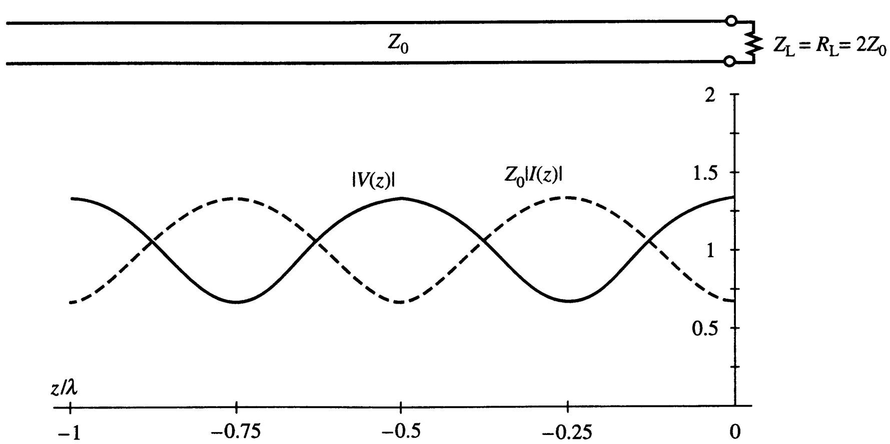


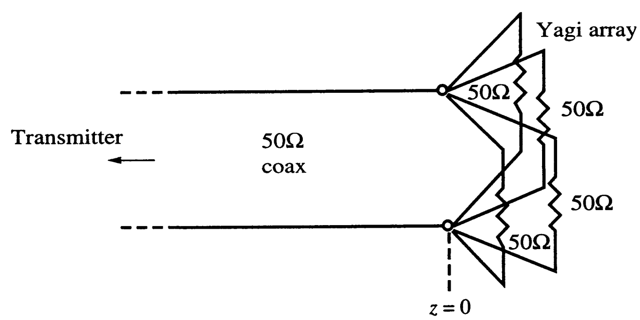
FIGURE 3.13. Standing-wave patterns for $Z_L = R_L = 2Z_0$. Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance z/λ away from the load, for $V^+ = 1$.

Example 3-5: A Yagi antenna array driven by a coaxial line. To increase the geographic coverage area of a broadcast station, four Yagi antennas, each having a feed-point impedance of 50Ω, are stacked in parallel on a single antenna tower and connected to the transmitter by a 50Ω coaxial line, as shown in Figure 3.14a. (a) Calculate the load reflection coefficient Γ_L. (b) Calculate V_{max}, V_{min}, I_{max}, and I_{min} along the line, assuming $V^+ = 1$ V. (c) Sketch $|V(z)|$ and $|I(z)|$ as functions of z/λ, taking the position of the antenna array terminals to be at $z = 0$.

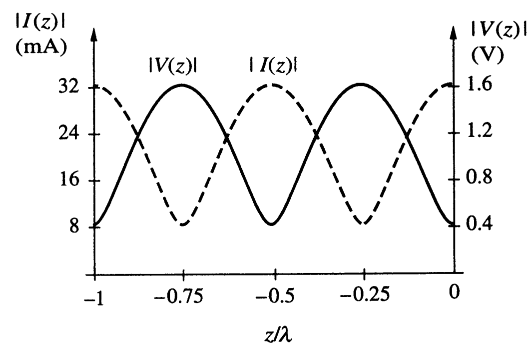
Solution:

(a) The total load impedance seen by the coaxial line is a parallel combination of the four 50Ω impedances, resulting in

$$Z_L = \frac{50}{4} = 12.5\Omega$$



(a)



(b)

FIGURE 3.14. Yagi array driven by a coaxial line. (a) Array of a stack of four Yagi antennas fed by a coaxial line. (b) Voltage and current standing-wave patterns.

The load reflection coefficient is then given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{12.5 - 50}{12.5 + 50} = -0.6 = 0.6e^{j180°}$$

so that $\rho = 0.6$ and $\psi = 180°$.

(b) $V_{max} = |V^+|(1 + \rho) = 1.6$ V $\qquad V_{min} = |V^+|(1 - \rho) = 0.4$ V

$$I_{max} = \frac{|V^+|}{Z_0}(1 + \rho) = 32 \text{ mA} \qquad I_{min} = \frac{|V^+|}{Z_0}(1 - \rho) = 8 \text{ mA}$$

(c) The voltage reflection coefficient at any position z along the line is given by

$$\Gamma(z) = \rho e^{j(\psi + 2\beta z)} = 0.6e^{j\pi(1 + 4z/\lambda)}$$

Using [3.22] and [3.23] with $V^+ = 1$ V, the magnitudes of the line voltage and current (i.e., $|V(z)|$ and $|I(z)|$) are plotted in Figure 3.14b as functions of electrical distance z/λ.

Standing-wave patterns such as those in Figure 3.13 are important in practice because, although the rapid temporal variations of the line voltages and currents are not easily accessible, the locations of the voltage minima and maxima and the ratio of the voltage maxima to minima are often readily measurable. A key parameter that is commonly used to describe the termination of a transmission line is the standing-wave ratio (SWR), or S, defined as

$$\boxed{S = \frac{V_{max}}{V_{min}} = \frac{I_{max}}{I_{min}} = \frac{1 + \rho}{1 - \rho} \quad \longrightarrow \quad \rho = \frac{S - 1}{S + 1}} \qquad [3.25]$$

Note that S varies in the range $1 \leq S \leq \infty$.

The following example illustrates the concepts of reflection coefficient and standing-wave ratio for a UHF antenna.

Example 3-6: UHF blade antenna. A UHF blade antenna installed in the tail-cap of a small aircraft is used for communication over the frequency band 225 MHz to 400 MHz. The following table provides the measured values of the feed-point impedance of the antenna at various frequencies:[9]

f(MHz)	$Z_L(\Omega)$
225	$22.5 - j51$
300	$35 - j16$
400	$45 - j2.5$

[9]R. L. Thomas, *A Practical Introduction to Impedance Matching,* Artech House, Inc., 1976.

A 50Ω coaxial line is used to connect the communication unit to the antenna. Calculate the load reflection coefficient Γ_L and the standing-wave ratio S on the line at (a) 225 MHz, (b) 300 MHz, and (c) 400 MHz.

Solution:

(a) At 225 MHz, the load reflection coefficient is given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{22.5 - j51 - 50}{22.5 - j51 + 50} \simeq \frac{57.9e^{-j118^\circ}}{88.6e^{-j35.1^\circ}} \simeq 0.654e^{-j83.2^\circ}$$

The standing-wave ratio S at 225 MHz can then be obtained as follows:

$$S = \frac{1+\rho}{1-\rho} \simeq \frac{1+0.654}{1-0.654} \simeq 4.78$$

(b) Similarly, at 300 MHz, we have

$$\Gamma_L = \frac{35 - j16 - 50}{35 - j16 + 50} \simeq \frac{21.9e^{-j133^\circ}}{86.5e^{-j10.7^\circ}} \simeq 0.254e^{-j122^\circ}$$

and

$$S \simeq \frac{1+0.254}{1-0.254} \simeq 1.68$$

(c) Similarly, at 400 MHz, we have

$$\Gamma_L = \frac{45 - j2.5 - 50}{45 - j2.5 + 50} \simeq \frac{5.59e^{-j153^\circ}}{95e^{-j1.51^\circ}} \simeq 0.0588e^{-j152^\circ}$$

and

$$S \simeq \frac{1+0.0588}{1-0.0588} \simeq 1.13$$

We see that the reflections on the coaxial line are quite significant near 225 MHz but are much reduced near 400 MHz.

Another quantity that can sometimes be measured in experimental settings is z_{min}, or the distance from the load to the *first* minimum of the voltage standing-wave pattern.[10] From [3.22] we have

$$V(z) = V^+(e^{-j\beta z} + \Gamma_L e^{j\beta z}) = V^+e^{-j\beta z}(1 + \rho e^{j\psi}e^{j2\beta z})$$

Since $|V(z)| = V_{min}$ when $e^{j(\psi+2\beta z)} = -1$, or

$$\psi + 2\beta z_{min} = -(2m+1)\pi \qquad m = 0, 1, 2, 3, \ldots$$

[10]In practice, it may often be difficult to actually measure the first minimum; however, if the location of *any* of the minima can be measured, the location of the first minimum can be deduced by using the fact that successive minima are separated by $\lambda/2$.

where $-\pi \leq \psi < \pi$, and $z_{min} \leq 0$. For any given frequency, measuring the wavelength (by measuring the distance between successive minima) provides a means to determine the phase velocity $v_p = f\lambda$.

At the location of the first minimum we have

$$\psi + 2\beta z_{min} = -\pi \quad \longrightarrow \quad \psi = -\pi - 2\beta z_{min} \qquad [3.26]$$

We see that z_{min} is directly related to the phase ψ of the reflection coefficient Γ_L, whereas S determines its magnitude through [3.25]. Once Γ_L is known, the load can be fully specified (assuming the characteristic impedance Z_0 is known), or Z_0 can be found (if Z_L is known). Thus, the two measurable quantities, S and z_{min}, completely characterize the transmission line terminated in an arbitrary load impedance.

It is often useful to rewrite [3.22] and [3.23] in terms of the load voltage V_L and load current I_L. Using the fact that $V_L = V(z)|_{z=0}$ and $I_L = I(z)|_{z=0}$, and after some manipulation, we have

$$V(z) = V_L \cos(\beta z) - jI_L Z_0 \sin(\beta z) \qquad [3.27]$$

$$I(z) = I_L \cos(\beta z) - j\frac{V_L}{Z_0} \sin(\beta z) \qquad [3.28]$$

The voltages and currents at the source end ($z = -l$) can be found from [3.27] and [3.28] by substituting $z = -l$. Note that for a general complex load impedance Z_L, the load voltage V_L and current I_L are in general complex, so that equations [3.27] and [3.28] do not necessarily constitute a decomposition of $V(z)$ and $I(z)$ into their real and imaginary parts.

In general, the voltage and current standing-wave patterns on a terminated line depend on the nature of the load. Typically what is plotted is $|V(z)|$, as was shown in Figure 3.13 for a purely resistive load $R_L = 2Z_0$. In the general case, with a complex load Z_L, the reflection coefficient Γ_L is complex, with $\psi \neq 0$. From [3.22] we have

$$V(z) = V^+[\cos(\beta z) - j\sin(\beta z) + \rho\cos(\psi + \beta z) + j\rho\sin(\psi + \beta z)]$$

and

$$\boxed{|V(z)| = |V^+|\sqrt{[\cos(\beta z) + \rho\cos(\psi + \beta z)]^2 + [-\sin(\beta z) + \rho\sin(\psi + \beta z)]^2}} \qquad [3.29]$$

which is the quantity plotted in various figures as the voltage standing-wave pattern. For $\psi = 0$ or π (i.e., load is purely resistive) [3.29] reduces to

$$|V(z)| = |V^+|\sqrt{(1 \pm \rho)^2 \cos^2(\beta z) + (-1 \pm \rho)^2 \sin^2(\beta z)} \qquad [3.30]$$

where the lower signs correspond to the case for $\psi = \pi$. Similar expressions can also be written for $|I(z)|$. Voltage and current standing wave patterns for different types of load impedances are shown in Figures 3.15 and 3.16. The interpretation of some of these patterns will become clearer after the discussion of line impedance in the following subsection.

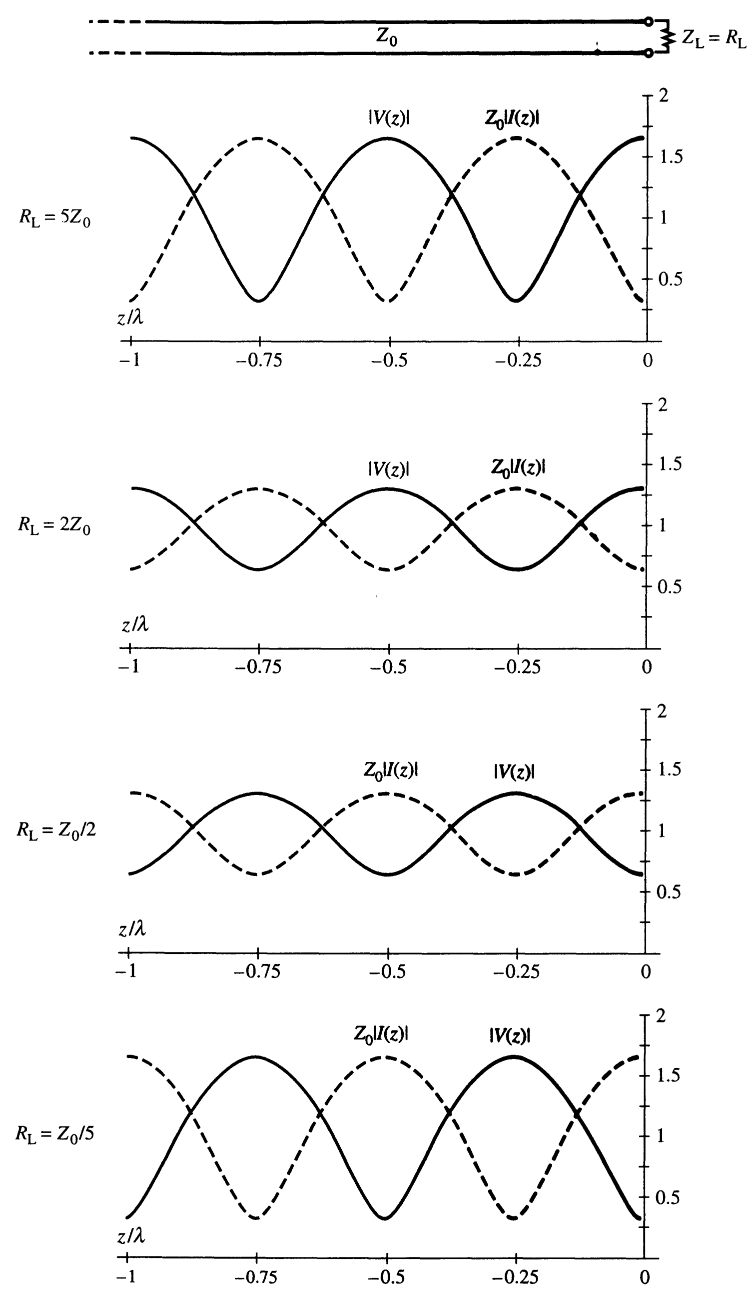


FIGURE 3.15. Voltage and current standing-wave patterns for different purely resistive loads. Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load position z/λ, for $V^+ = 1$ and for $R_L = 5Z_0$, $2Z_0$, $Z_0/2$, and $Z_0/5$.

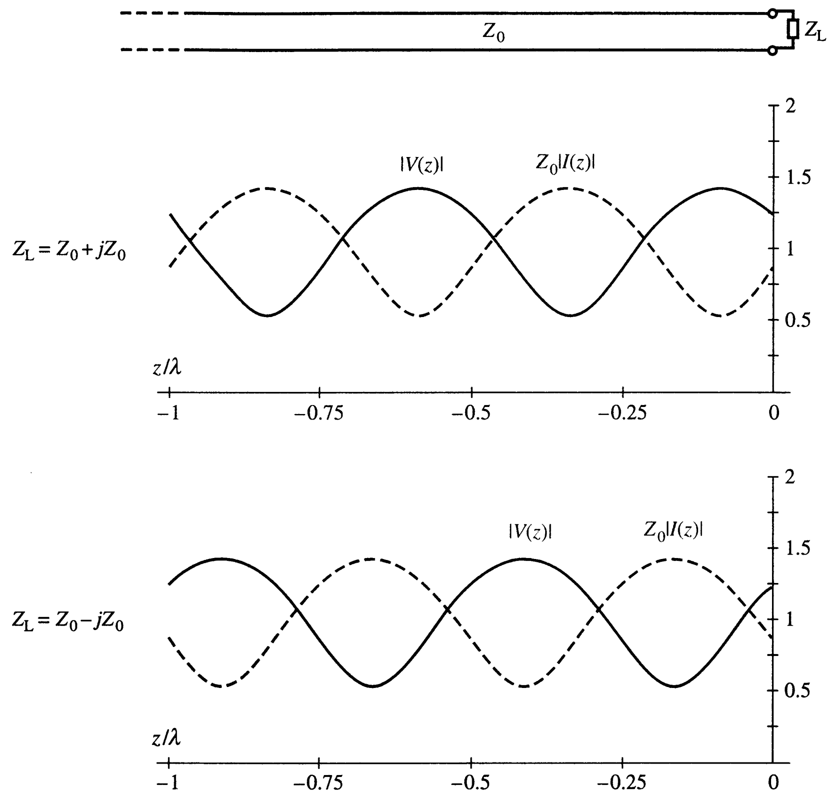


FIGURE 3.16. Voltage and current standing-wave patterns for complex loads. Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load z/λ for $V^+ = 1$ and for $Z_L = Z_0 + jZ_0$ (inductive load) and $Z_L = Z_0 - jZ_0$ (capacitive load).

In general, for purely resistive loads ($Z_L = R_L + j0$), the load position is a point of a voltage maximum or minimum, depending on whether $R_L > Z_0$ or $R_L < Z_0$, respectively. This behavior is apparent from Figure 3.15 and can also be seen by considering [3.22] and [3.23]. For $R_L > Z_0$, $0 < \Gamma_L \le 1$ and $|V(z = 0)| = V_{max} = |V^+|(1 + \rho)$, whereas for $R_L < Z_0$, $-1 \le \Gamma_L < 0$ and $|V(z = 0)| = V_{min} = |V^+|(1 - \rho)$.

The standing-wave patterns in Figure 3.16 for $Z_L = Z_0 \pm jZ_0$ illustrate specific cases of the general behavior for loads with a reactive (capacitive or inductive) component. In general, the sign of the reactance (positive or negative) can be determined by inspection of the voltage standing-wave pattern. For $Z_L = R_L + jX_L$, X_L is negative (i.e., the load is capacitive) when the first minimum is at a distance smaller than one quarter of wavelength from the load (i.e., $-z_{min} < \lambda/4$) and X_L is positive (inductive) when the first minimum is at a distance greater than one quarter of a wavelength from the load (i.e., $\lambda/4 < -z_{min} < \lambda/2$), as illustrated in Figures 3.17a,b.

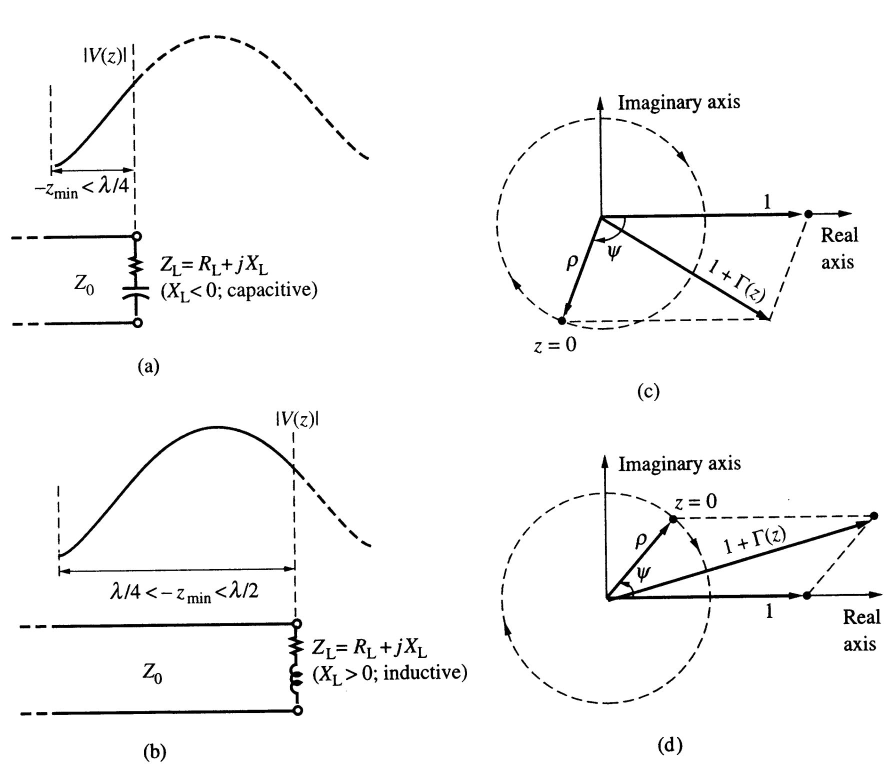


FIGURE 3.17. Variation of the voltage standing-wave pattern in the vicinity of the load for inductive or capacitive loads. In descriptive terms, starting from the load (i.e., $z = 0$), the standing-wave voltage at first increases (decreases) as one moves away from the load (i.e., clockwise in the diagrams shown) for an inductive (capacitive) load. (a) $|V(z)|$ for a capacitive load. (b) $|V(z)|$ for an inductive load. (c) $|V(z)| = |1 + \Gamma(z)|$ as the sum of two complex numbers, 1 and $\Gamma(z)$, for a capacitive load ($-\pi < \psi < 0$). (d) $|V(z)| = |1 + \Gamma(z)|$ as the sum of two complex numbers, 1 and $\Gamma(z)$, for an inductive load ($0 < \psi < \pi$). Note once again that $V^+ = 1$.

The behavior illustrated in Figures 3.17a,b can be understood upon careful examination of [3.21] and [3.22]. For a general complex load impedance, [3.21] can be rewritten as

$$\Gamma_L = \rho e^{j\psi} = \frac{R_L + jX_L - Z_0}{R_L + jX_L + Z_0} = \frac{(R_L - Z_0) + jX_L}{(R_L + Z_0) + jX_L}$$

The phase angle ψ of the reflection coefficient is $0 < \psi < \pi$ if the load impedance is inductive ($X_L > 0$) and $-\pi < \psi < 0$ if the load impedance is capacitive ($X_L < 0$). The magnitude of the voltage along the line can be written from [3.22] as

$$|V(z)| = |V^+||1 + \Gamma(z)| = |V^+||1 + \rho e^{j(\psi + 2\beta z)}|$$

where $z \leq 0$. Noting that $|V^+|$ is a constant, consider the second term and its variation with z (note that z decreases as one moves away from the load (at $z = 0$) along the transmission line). This term is the magnitude of the sum of two numbers, one being the real number 1 and the other being a complex number, $\Gamma(z) = \rho e^{j(\psi+2\beta z)}$, which has a constant magnitude ρ ($0 \le \rho \le 1$, as determined by Z_L and Z_0) and a phase angle $\psi + 2\beta z$ that decreases with decreasing z (corresponding to clockwise rotation of this complex number on a circle with radius ρ centered at the origin on the complex plane). The two different cases of capacitive and inductive load are shown respectively in Figures 3.17c and 3.17d. For an inductive load, we see from Figure 3.17d that as we move away from the load (i.e., starting at $z = 0$ and rotating clockwise), $|V(z)|$ (which is proportional to $|1 + \Gamma(z)|$) first increases, reaches a maximum (at $\psi + 2\beta z = 0$), and then decreases, consistent with the variation of $|V(z)|$ for the inductive load as shown in Figure 3.17b. Similarly, for a capacitive load, we see from Figure 3.17c that as z decreases (starting with $z = 0$), $|V(z)|$ first decreases, reaches a minimum (at $\psi + 2\beta z = -\pi$), and then increases, consistent with Figure 3.17a.

3.3.2 Transmission Line Impedance

An important property of a transmission line is its ability to transform impedances. In Section 3.2, we saw that the input impedance of a short- or open-circuited transmission line segment can be made equal to any arbitrary reactive impedance by simply adjusting its electrical length (l/λ). The input impedance of a transmission line terminated in an arbitrary load impedance Z_L is similarly dependent on the electrical length of the line, or the distance from the load at which the impedance is measured. For the transmission line shown in Figure 3.12, the impedance seen looking toward the load Z_L at any position z along the line ($-l \le z \le 0$) given by [3.24] can be rewritten as

$$\boxed{Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{e^{-j\beta z} + \Gamma_L e^{j\beta z}}{e^{-j\beta z} - \Gamma_L e^{j\beta z}} = Z_0\frac{Z_L - jZ_0\tan(\beta z)}{Z_0 - jZ_L\tan(\beta z)}} \qquad [3.31]$$

using [3.21], [3.22], and [3.23]. Expression [3.31] for the line impedance is often written as

$$Z(z) = Z_0\frac{Z_L\cos(\beta z) - jZ_0\sin(\beta z)}{Z_0\cos(\beta z) - jZ_L\sin(\beta z)}$$

Note that since $\beta = 2\pi/\lambda$, the impedance varies periodically with electrical distance (z/λ) along the line, with the same impedance value attained at intervals in z of $\pm\lambda/2$. At the load, where $z = 0$, we have

$$Z(z = 0) = Z_0\frac{1 + \Gamma_L}{1 - \Gamma_L} = Z_L$$

as expected. In particular, the input impedance seen by the source at the source end, where $z = -l$, is

$$\boxed{Z_{in} = [Z(z)]_{z=-l} = \frac{V_s}{I_s} = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)}} \qquad [3.32]$$

For example, for a short-circuited line ($Z_L = 0$), the input impedance is

$$Z_{in} = jZ_0 \tan(\beta l)$$

and for an open-circuited line ($Z_L = \infty$), it is

$$Z_{in} = -jZ_0 \cot(\beta l)$$

as was shown in Sections 3.2.1 and 3.2.2.

The following example illustrates the dependence of the input impedance of a line on its electrical length.

Example 3-7: Input impedance of a line. Find the input impedance of a 75-cm long transmission line where $Z_0 = 70\Omega$, terminated with a $Z_L = 140\Omega$ load at 50, 100, 150, and 200 MHz. Assume the phase velocity v_p to be equal to the speed of light in free space.

Solution:

(a) At $f = 50$ MHz, we have $\lambda \simeq (3 \times 10^8)/(5 \times 10^7) = 6$ m, so the electrical length of the line is $l/\lambda \simeq 0.75/6 = 0.125$. Noting that $\beta l = 2\pi l/\lambda$, we then have, from [3.32],

$$Z_{in} \simeq 70\frac{140 + j70\tan(2\pi \times 0.125)}{70 + j140\tan(2\pi \times 0.125)} = 70\frac{(140 + j70)(70 - j140)}{(70)^2 + (140)^2}$$
$$= \frac{(2 + j)(70 - j140)}{5} = \frac{280 - j210}{5} = 56 - j42\Omega$$

since $\tan(2\pi \times 0.125) = 1$. Note that the input impedance of the line at 50 MHz is capacitive.

(b) At $f = 100$ MHz, we have $\lambda \simeq 3$ m and $l/\lambda \simeq 0.75/3 = 0.25$. From [3.32] we have

$$Z_{in} \simeq 70\frac{140 + j70\tan(2\pi \times 0.25)}{70 + j140\tan(2\pi \times 0.25)} = \frac{(70)^2}{140} = 35\Omega$$

since $\tan(2\pi \times 0.25) = \infty$. Note that the input impedance of the line at 100 MHz is purely resistive.

(c) At $f = 150$ MHz, $\lambda \simeq 2$ m and $l/\lambda \simeq 0.75/2 = 0.375$. We have from [3.32]

$$Z_{in} \simeq 70\frac{140 + j70\tan(2\pi \times 0.375)}{70 + j140\tan(2\pi \times 0.375)} = 70\frac{(140 - j70)(70 + j140)}{(70)^2 + (140)^2}$$
$$= \frac{(2 - j)(70 + j140)}{5} = 56 + j42\Omega$$

since $\tan(2\pi \times 0.375) = -1$. Note that the input impedance of the line at 150 MHz is inductive.

(d) At $f = 200$ MHz, we have $\lambda \simeq 1.5$ m and $l/\lambda \simeq 0.75/1.5 = 0.5$. We have from [3.32]

$$Z_{\text{in}} \simeq 70\frac{140 + j70\tan(2\pi \times 0.5)}{70 + j140\tan(2\pi \times 0.5)} = \frac{70 \times 140}{70} = 140\Omega$$

since $\tan(2\pi \times 0.5) = 0$. Note that the input impedance of the line at 200 MHz is purely resistive and is exactly equal to the load impedance.

Normalized Line Impedance In transmission line analysis, it is often convenient and common practice to normalize all impedances to the characteristic impedance Z_0 of the transmission line. Denoting the normalized version of any impedance by using a bar at the top, we can rewrite [3.31] to express the normalized line impedance $\bar{Z}(z)$ in terms of the normalized load impedance $\bar{Z}_L$

$$\bar{Z}(z) = \frac{\bar{Z}_L - j\tan(\beta z)}{1 - j\bar{Z}_L\tan(\beta z)} = \frac{\bar{Z}_L\cos(\beta z) - j\sin(\beta z)}{\cos(\beta z) - j\bar{Z}_L\sin(\beta z)} \qquad [3.33]$$

where

$$\bar{Z}_L = \frac{Z_L}{Z_0} = \frac{1 + \Gamma_L}{1 - \Gamma_L} = \frac{1 + \rho e^{j\psi}}{1 - \rho e^{j\psi}}$$

Using [3.25] and [3.26], we can further write

$$\bar{Z}_L = \frac{Z_L}{Z_0} = \frac{1 + jS\tan(\beta z_{\min})}{S + j\tan(\beta z_{\min})} = \frac{\cos(\beta z_{\min}) + jS\sin(\beta z_{\min})}{S\cos(\beta z_{\min}) + j\sin(\beta z_{\min})}$$

which expresses the normalized load impedance $\bar{Z}_L$ in terms of the measurable quantities S and $z_{\min}$.

Sometimes it is useful to express the real and imaginary parts of the load impedance $Z_L = R_L + jX_L$ explicitly in terms of $z_{\min}$ and S:

$$\bar{R}_L = \frac{R_L}{Z_0} = \frac{S}{S^2\cos^2(\beta z_{\min}) + \sin^2(\beta z_{\min})}$$

$$\bar{X}_L = \frac{X_L}{Z_0} = \frac{(S^2 - 1)\cos(\beta z_{\min})\sin(\beta z_{\min})}{S^2\cos^2(\beta z_{\min}) + \sin^2(\beta z_{\min})}$$

The relationship between the polarity of X_L (i.e., inductive versus capacitive) and the distance to the first minimum, as depicted in Figure 3.17, can also be deduced by careful consideration of the preceding equation for $\bar{X}_L$.

Example 3-8 illustrates the determination of an unknown load impedance from measurements of S and $z_{\min}$.

Example 3-8: Unknown load. Determine an unknown load Z_L from S and $z_{\min}$ measurements. The following measurements are carried out on a 100Ω

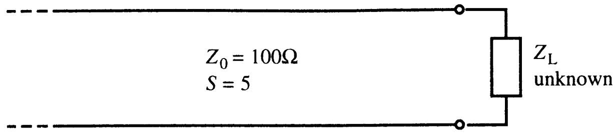


FIGURE 3.18. **Transmission line terminated in an unknown impedance.**

transmission line terminated with an unknown load Z_L, as shown in Figure 3.18. The voltage standing-wave ratio S is 5, the distance between successive voltage minima is 25 cm, and the distance from Z_L to the first voltage minimum is 8 cm. (a) Determine the load reflection coefficient Γ_L. (b) Determine the unknown load impedance Z_L. (c) Determine the location of the first voltage maximum with respect to the load.

Solution:

(a) Using [3.25] and [3.26], we have

$$\rho = \frac{S-1}{S+1} = \frac{5-1}{5+1} \simeq 0.667$$

$$\frac{\lambda}{2} = 25 \text{ cm} \quad \rightarrow \quad \lambda = 50 \text{ cm}$$

$$\psi = -\pi - 2\beta z_{\min} = -\pi + 2\left(\frac{2\pi}{50}\right)(8) = -0.36\pi \text{ rad} \quad \text{or} \; -64.8^\circ$$

$$\Gamma_L = \rho e^{j\psi} \simeq 0.667 e^{-j64.8^\circ}$$

(b) Using the expression derived previously for Z_L in terms of $z_{\min}$ and S, we have

$$(\beta z_{\min}) = \frac{-\pi + 0.36\pi}{2} = -0.32\pi \text{ rad} \quad \rightarrow \quad \tan(\beta z_{\min}) \simeq -1.58$$

and so

$$Z_L = Z_0 \frac{1 + jS\tan(\beta z_{\min})}{S + j\tan(\beta z_{\min})} \simeq 100 \frac{1 - j5(1.58)}{(5 - j1.58)}$$

$$= 100 \frac{(1 - j7.88)(5 + j1.58)}{(5 - j1.58)(5 + j1.58)} \simeq 63.4 - j137.6\Omega$$

(c) The location of the first voltage maximum is $\lambda/4$ away from the location of the voltage minimum. Thus, we have

$$z_{\max} = z_{\min} - \lambda/4 = -8 - 12.5 = -20.5 \text{ cm}$$

Transmission Line Admittance In Sections 3.5 and 3.6, when we discuss impedance matching and the Smith chart, it will be useful at times to work with the line *admittance* rather than the impedance. From [3.31], we can find the expression for line admittance as

$$Y(z) = \frac{1}{Z(z)} = Y_0 \frac{1 - \Gamma_L e^{+j2\beta z}}{1 + \Gamma_L e^{+j2\beta z}} = Y_0 \frac{Y_L - jY_0 \tan(\beta z)}{Y_0 - jY_L \tan(\beta z)} \qquad [3.34]$$

where $Y_0 = (Z_0)^{-1}$. Using [3.33], the normalized line admittance $\bar{Y}(z) = Y(z)/Y_0$ can be written as

$$\bar{Y}(z) = \frac{\bar{Y}_L - j\tan(\beta z)}{1 - j\bar{Y}_L \tan(\beta z)} = \frac{\bar{Y}_L \cos(\beta z) - j\sin(\beta z)}{\cos(\beta z) - j\bar{Y}_L \sin(\beta z)} \qquad [3.35]$$

The load reflection coefficient Γ_L can also be written in terms of admittances as

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L}$$

Line Impedance for Resistive Loads The variation with z of the real and imaginary parts of the normalized line impedance is illustrated in Figure 3.19a for the case of a resistive load with $Z_L = R_L = 2Z_0$. The voltage and current standing-wave

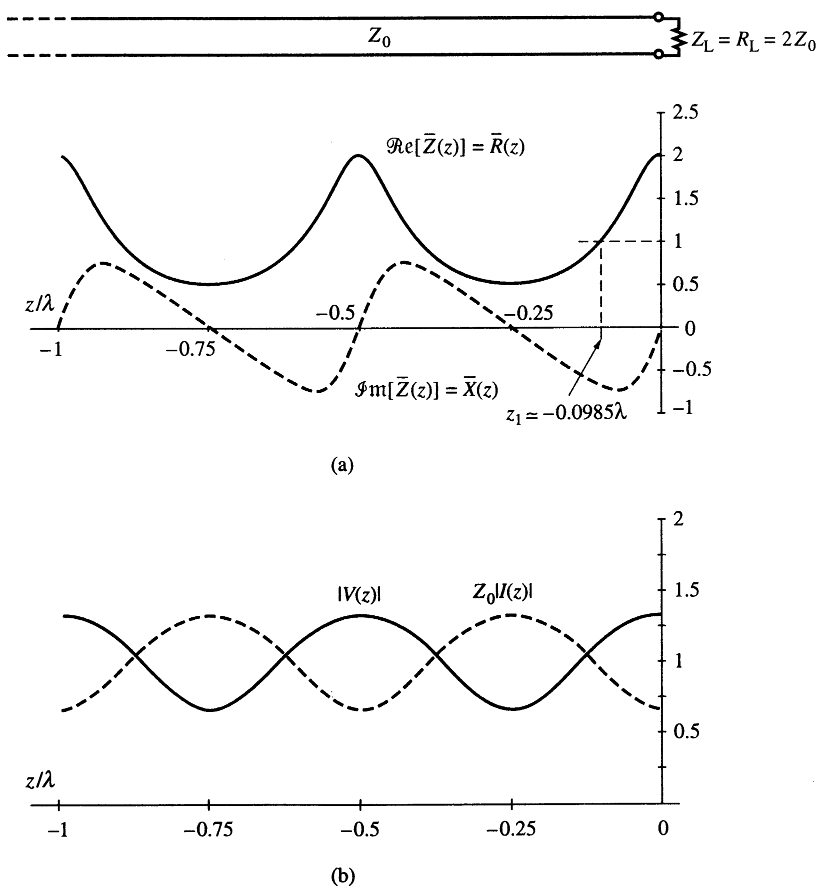


FIGURE 3.19. Impedance along a line terminated with $Z_L = R_L = 2Z_0$. (a) The real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ are shown as functions of z/λ. (b) Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load z/λ, for $V^+ = 1$.

patterns (from Figure 3.13) are also shown for reference in Figure 3.19b. Note from Figure 3.19a that, as viewed from different positions at a distance z from the load, the real part of the normalized line impedance varies between $\bar{R}(z) = 2$ and $\bar{R}(z) = 0.5$, with the distance between successive maxima being $\lambda/2$. The line impedance is purely real at the load and at distances of integer multiples of $\lambda/4$ from the load. These positions also correspond to the positions of voltage maxima and minima along the line. For $-z < \lambda/4$, the line impedance $Z(z)$ is capacitive (i.e., its imaginary part is negative; $X(z) < 0$), reminiscent of the behavior of the open-circuited line[11] (see Figure 3.9). For $\lambda/4 < -z < \lambda/2$, the line impedance $Z(z)$ is inductive ($X(z) > 0$), switching thereafter back and forth between being inductive and capacitive at intervals of $\lambda/4$.

An interesting aspect of the result in Figure 3.19a is the fact that $\Re e\{\bar{Z}(z)\} = 1$ at $z_1 \simeq -0.0985\lambda$.[12] If the imaginary part of the line impedance at that position could somehow be canceled (as we shall see in Section 3.5), the line would appear (from all positions at locations $z < -0.0985\lambda$) as if it were matched (i.e., terminated with an impedance Z_0). For example, such cancellation can in principle be achieved by introducing a purely reactive series impedance that is opposite in sign to the reactive part of $Z(z)$ at that position, as will be discussed in Section 3.5. The following example illustrates the determination of the point at which $\Re e\{\bar{Z}(z)\} = 1$ for a specific complex load impedance.

Example 3-9: An inverted-V antenna. A 50Ω coaxial line filled with teflon ($v_p \simeq 21$ cm-(ns)$^{-1}$) is connected to an inverted-V antenna represented by Z_L, as shown in Figure 3.20. At $f = 29.6$ MHz, the feed-point impedance of the antenna is approximately measured to be $Z_L \simeq 75 + j25\Omega$.[13] Find the two closest positions to the antenna along the line where the real part of the line impedance is equal to the characteristic impedance of the line (i.e., Z_0).

Solution: The line impedance at any position z is given by

$$Z(z) = 50\frac{(75 + j25) - j50\zeta}{50 - j(75 + j25)\zeta} = 50\frac{3 + j(1 - 2\zeta)}{(2 + \zeta) - j3\zeta}$$

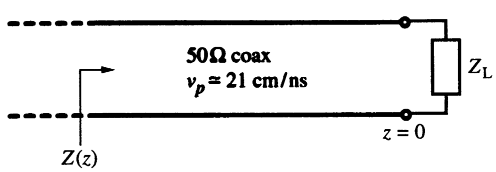


FIGURE 3.20. A coaxial line connected to an antenna.

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[11]Note that this makes sense because in Figure 3.19a the load resistance is larger than the characteristic impedance ($R_L > Z_0$), which is also the case for the open circuit.

[12]The value of z_1 can be read roughly from Figure 3.19a or accurately evaluated from [3.33], by letting $\bar{Z}(z) = 1 + j\bar{X}(z)$.

[13]*The ARRL Antenna Book*, 17th ed., American Radio Relay League, pp. 27-28 and 27-29, 1994–1996.

where $\zeta = \tan(\beta z)$, $\beta = 2\pi/\lambda$, and $\lambda = v_p/f \simeq (2.1 \times 10^8)/(29.6 \times 10^6) \simeq$ 7.09 m. Multiplying both the numerator and the denominator with the complex conjugate of the denominator, we can extract the real part of $Z(z)$ as

$$\Re e\{Z(z)\} = \Re e\left\{50\frac{3 + j(1 - 2\zeta)}{(2 + \zeta) - j3\zeta} \cdot \frac{(2 + \zeta) + j3\zeta}{(2 + \zeta) + j3\zeta}\right\} = 50\frac{3(2 + \zeta) - 3\zeta(1 - 2\zeta)}{(2 + \zeta)^2 + (3\zeta)^2}$$

The value of z for which we have $\Re e\{Z(z)\} = Z_0 = 50\Omega$ can then be found as

$$\Re e\{Z(z)\} = Z_0 \quad \rightarrow \quad 50\frac{3(2 + \zeta) - 3\zeta(1 - 2\zeta)}{(2 + \zeta)^2 + (3\zeta)^2} = 50$$

$$\rightarrow \quad 2\zeta^2 + 2\zeta - 1 = 0 \quad \rightarrow \quad \zeta_1, \zeta_2 \simeq -1.37, 0.366$$

Using

$$\tan(\beta z) = \tan\left(\frac{2\pi z}{\lambda}\right) \simeq \tan\left[\frac{2\pi z}{7.09}\right] = \zeta$$

and noting that $z < 0$, we find

$$z_1 \simeq -0.149\lambda \simeq -1.06 \text{ m} \quad \text{and} \quad z_2 \simeq -0.444\lambda \simeq -3.15 \text{ m}$$

as the locations at which the real part of the line impedance is equal to the characteristic impedance of the line.

Some aspects of the behavior of the real and imaginary parts of the line impedance shown in Figure 3.19 for $\bar{Z}_L = 2$ can be generalized. For example, the line impedance seen at the positions of voltage maxima (minima) is always purely real and has maximum (minimum) magnitude. To see this, consider the voltage along the line at the position of a voltage maximum (i.e., $z = z_{\max}$) given by

$$V(z = z_{\max}) = V^+ e^{-j\beta z_{\max}}[1 + \rho e^{j(\psi + 2\beta z_{\max})}] = V^+ e^{-j\beta z_{\max}}(1 + \rho)$$

with a maximum magnitude of

$$|V(z = z_{\max})|_{\max} = V_{\max} = |V^+|(1 + \rho)$$

occurring at

$$\psi + 2\beta z_{\max} = -m2\pi \qquad m = 0, 1, 2, 3, \ldots$$

where $-\pi \le \psi < \pi$, $z_{\max} \le 0$, and where $m = 0$ does not apply if $-\pi \le \psi < 0$. At the same position, the current is equal to

$$I(z = z_{\max}) = \frac{V^+}{Z_0} e^{-j\beta z_{\max}}(1 - \rho)$$

with a minimum magnitude given by

$$|I(z = z_{\max})|_{\min} = I_{\min} = \frac{|V^+|}{Z_0}(1 - \rho)$$

so that the line impedance $Z(z_{\max}) = V(z_{\max})/I(z_{\max})$ is clearly purely resistive and has a maximum magnitude given by

$$|Z(z_{\max})|_{\max} = R_{\max} = \frac{V_{\max}}{I_{\min}} = Z_0\frac{1+\rho}{1-\rho} = SZ_0 \qquad [3.36]$$

whereas at the voltage minima ($z_{\min} = z_{\max} - \lambda/4$), the line impedance $Z(z_{\min})$ is purely resistive with a minimum magnitude given by

$$|Z(z_{\min})|_{\min} = R_{\min} = \frac{V_{\min}}{I_{\max}} = Z_0\frac{1-\rho}{1+\rho} = \frac{Z_0}{S} \qquad [3.37]$$

Also, for purely resistive terminations ($Z_L = R_L$), the load is at a position of either minimum or maximum for the voltage and therefore, the load impedance R_L is either equal to the minimum ($R_L = Z_0/S$ when $R_L < Z_0$) or maximum ($R_L = SZ_0$ when $R_L > Z_0$) magnitude for the line impedance. Note that we have

$$S = \frac{1+\rho}{1-\rho} = \begin{cases} \dfrac{R_L}{Z_0} & \text{for} \quad R_L > Z_0 \\ \dfrac{Z_0}{R_L} & \text{for} \quad R_L < Z_0 \end{cases}$$

Line Impedance for Complex Load Impedances For general load impedances that are not purely resistive (i.e., $Z_L = R_L + jX_L$), the behavior of the line impedance $Z(z) = R(z) + jX(z)$ is similar to that for purely resistive loads, in that its real part $R(z)$ varies between a maximum value of SZ_0 and a minimum value of Z_0/S, and the imaginary part $X(z)$ alternates sign at intervals of $\lambda/4$. (SZ_0 occurs at the positions of the voltage maxima when the line impedance is purely resistive and therefore is also the maximum magnitude of the line impedance. Z_0/S occurs at the positions of the voltage minima when the line impedance is also purely resistive and therefore is also the minimum magnitude of the line impedance.) However, the maxima and minima of the magnitudes of either the voltage or the line impedance are not at the load position. Figures 3.21 and 3.22 show plots of the real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ as functions of z/λ for selected load impedances.

Example 3-10 illustrates the numerical evaluation of the reflection coefficient, standing-wave ratio, and maximum and minimum line resistances for a complex load termination.

Example 3-10: Reflection coefficient, standing-wave ratio, and maximum and minimum resistances. A radio transmitter is connected to an antenna having a feed-point impedance of $Z_L = 70 + j100\Omega$ with a 50Ω coaxial line, as shown in Figure 3.23. Find (a) the load reflection coefficient, (b) the standing-

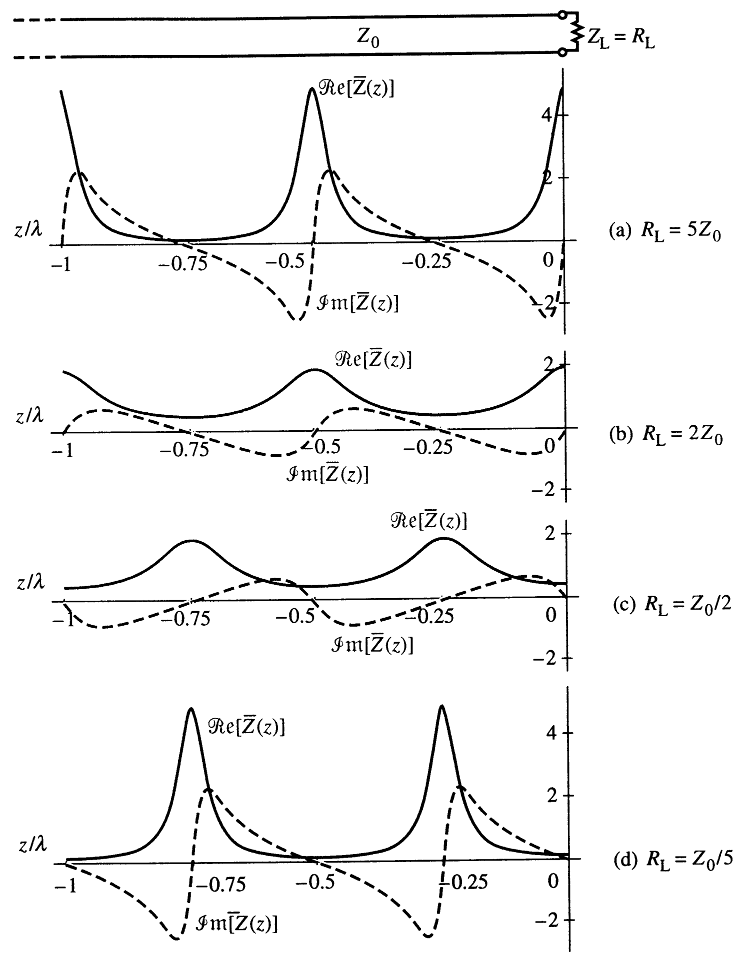

FIGURE 3.21. Line impedance for different purely resistive terminations. The real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ are shown as functions of electrical distance z/λ along the line for Z_L equal to (a) $5Z_0$, (b) $2Z_0$, (c) $Z_0/2$, and (d) $Z_0/5$.

wave ratio, and (c) the two positions closest along the line to the load where the line impedance is purely real, and their corresponding line impedance values.

Solution:

(a) The load reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{70 + j100 - 50}{70 + j100 + 50} = \frac{1 + j5}{6 + j5}$$

$$= \frac{\sqrt{26}e^{j\tan^{-1}(5)}}{\sqrt{61}e^{j\tan^{-1}(5/6)}} \simeq 0.653e^{j38.9^\circ}$$

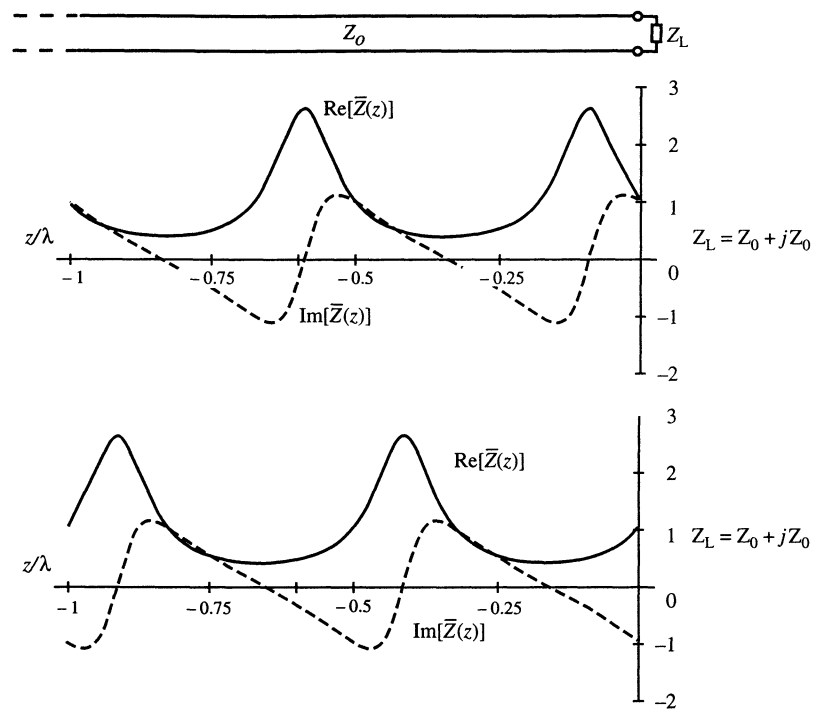

FIGURE 3.22. Line impedance for two different complex load impedances. The real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ are shown as functions of electrical distance z/λ along the line for (a) $Z_L = Z_0 + jZ_0$ and (b) $Z_L = Z_0 - jZ_0$.

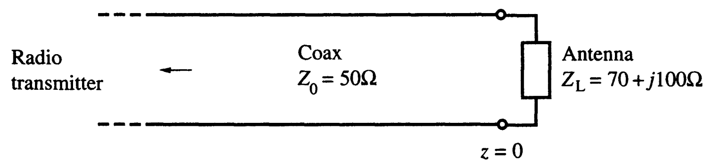

FIGURE 3.23. Transmission line terminated in an antenna. In general, the feed-point impedance of an antenna is complex.

(b) The standing-wave ratio is

$$S = \frac{1+\rho}{1-\rho} \simeq \frac{1+0.653}{1-0.653} \simeq 4.76$$

(c) The maximum voltage position is the position z_{max} at which the line impedance is purely real and the magnitude of the line impedance is a maximum; note that the position of the first voltage maximum is closer to the load than that of the first voltage minimum because the load impedance is inductive. To find the maximum voltage position, use $\psi + 2\beta z_{max} = 0 \quad \rightarrow \quad z_{max} \simeq -0.054\lambda$:

$$R_{max} = SZ_0 \simeq (4.76)(50) = 238\Omega$$

The minimum voltage position is the position $z_{\min}$ at which $|Z(z)|$ is minimum; note that this is the next closest position where the line impedance $Z(z)$ is real. To find the minimum voltage position, use

$$\psi + 2\beta z_{\min} = -\pi \quad \rightarrow \quad z_{\min} \simeq -0.304\lambda$$

Note that as expected, $z_{\min} = z_{\max} - \lambda/4$. We then have

$$R_{\min} = Z_0/S \simeq 50/(4.76) \simeq 10.5\Omega$$

3.3.3 Calculation of V^+

Up to now, we have primarily focused on the line impedance and the variation of the voltage and current along the line without particular attention to the source end of the line. The source that excites the transmission line shown in Figure 3.12 is a voltage source with an open-circuit phasor voltage V_0 and a source impedance Z_s. Using Equation [3.22], the voltage V_s at the source end of the line ($z = -l$) is

$$V_s = V(z = -l) = V^+ e^{j\beta l}(1 + \Gamma_L e^{-j2\beta l})$$

As seen from the source end, the transmission line can be represented by its input impedance, Z_{in}. We can thus also express the source-end voltage phasor V_s in terms of the source parameters V_0 and Z_s by noting the division of voltage between Z_{in} and Z_s, namely,

$$V_s = \frac{Z_{in}}{Z_{in} + Z_s} V_0$$

By equating the two preceding expressions for V_s, we can solve for the constant V^+:

$$V^+ = \frac{Z_{in} V_0}{(Z_{in} + Z_s)e^{j\beta l}(1 + \Gamma_L e^{-j2\beta l})}$$

Note that the knowledge of V^+ and the wavelength $\lambda = 2\pi/\beta$ completely specifies the transmission line voltage and current as given in [3.22] and [3.23], as for any given transmission line with characteristic impedance Z_0 and load Z_L (and hence Γ_L).

Example 3-11 illustrates the calculation of V^+ by considering the source end of a terminated transmission line circuit.

Example 3-11: Coaxial line feeding an antenna. A sinusoidal voltage source of $\mathcal{V}_0(t) = 10\cos(5\pi \times 10^7 t)$ V and $R_s = 20\Omega$ is connected to an antenna with feed-point impedance $Z_L = 100\Omega$ through a 3-m long, lossless coaxial transmission line filled with polyethylene ($v_p = 20$ cm-(ns)$^{-1}$) and with a characteristic impedance of $Z_0 = 50\Omega$, as shown in Figure 3.24a. Find (a) the voltage and current phasors, $V(z)$ and $I(z)$, at any location on the line and (b) the corresponding instantaneous expressions $\mathcal{V}(z, t)$ and $\mathcal{I}(z, t)$.

Solution:

(a) At $f = \omega/(2\pi) = 25$ MHz, the wavelength in a polyethylene-filled coaxial line is

$$\lambda = v_p/f = (20 \text{ cm-(ns)}^{-1})/(25 \text{ MHz}) = 8 \text{ m}$$

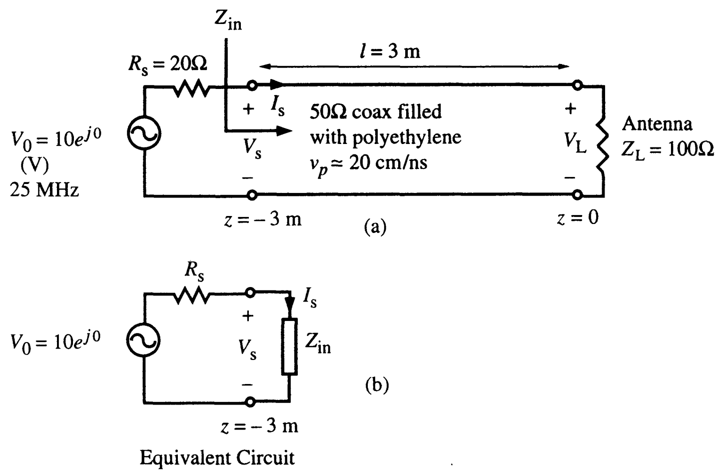

FIGURE 3.24. **Coaxial line feeding an antenna.** (a) Circuit configuration. (b) Thévenin equivalent circuit seen from the source end.

The electrical length of the 3-m line is then $l/\lambda = 3/8 = 0.375$, so we have $\beta l = 2\pi(0.375) = 3\pi/4$ and $\tan(\beta l) = -1$. The input impedance seen at the source end is

$$Z_{in} = Z(z = -l) = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L\tan(\beta l)} = 50\frac{100 + j50(-1)}{50 + j100(-1)}$$

$$= \left(\frac{100 - j50}{1 - j2}\right)\left(\frac{1 + j2}{1 + j2}\right) = (20 - j10)(1 + j2) = 40 + j30\Omega$$

Using the equivalent circuit shown in Figure 3.24b, we have

$$V_s = \frac{Z_{in}}{R_s + Z_{in}}V_0 = \frac{40 + j30}{60 + j30}(10) \simeq \frac{5e^{j36.9°}}{3\sqrt{5}e^{j26.6°}}(10) \simeq 7.45e^{j10.3°}\text{ V}$$

We can also write an expression for V_s by evaluating $V(z)$ at $z = -3$ m as

$$V_s = V(z = -3\text{ m}) = V^+e^{j3\pi/4}(1 + \Gamma_L e^{-j3\pi/2})$$

where Γ_L is the load reflection coefficient given by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - 50}{100 + 50} = \frac{1}{3}$$

Equating the two expressions for V_s, we can determine the complex constant V^+

$$V_s = V^+e^{j3\pi/4}\left(1 + \frac{j}{3}\right) \simeq 7.45e^{j10.3°} \quad \rightarrow \quad V^+ \simeq 7.07e^{-j143°}\text{ V}$$

so that the voltage phasor at any position z from the load is given as

$$V(z) \simeq 7.07e^{-j143°}e^{-j\pi z/4}\left(1 + \tfrac{1}{3}e^{j\pi z/2}\right) \text{ V}$$

and the corresponding current phasor is

$$I(z) \simeq 0.141e^{-j143°}e^{-j\pi z/4}\left(1 - \tfrac{1}{3}e^{j\pi z/2}\right) \text{ A}$$

(b) Using $\mathcal{V}(z, t) = \Re e\{V(z)e^{j\omega t}\}$ and $\mathcal{I}(z, t) = \Re e\{I(z)e^{j\omega t}\}$, we find

$$\mathcal{V}(z, t) \simeq 7.07\cos\left(5\pi 10^7 t - \frac{\pi z}{4} - 143°\right) + 2.36\cos\left(5\pi 10^7 t + \frac{\pi z}{4} - 143°\right) \text{ V}$$

and

$$\mathcal{I}(z, t) \simeq 0.141\cos\left(5\pi 10^7 t - \frac{\pi z}{4} - 143°\right) - 0.0471\cos\left(5\pi 10^7 t + \frac{\pi z}{4} - 143°\right) \text{ A}$$

3.4 POWER FLOW ON A TRANSMISSION LINE

In practice, the primary purpose of most steady-state sinusoidal transmission line applications is to maximize the time-average power delivered to a load. The power and energy flow on a transmission line can be determined from the line voltage and line current; the product of the instantaneous current $\mathcal{I}(z, t)$ and instantaneous voltage $\mathcal{V}(z, t)$ at any point z is by definition the power that flows into the line[14] at that point. In most applications, the quantity of interest is not the rapidly varying instantaneous power but its average over one sinusoidal period T_p, namely, the time-average power, which is given by

$$P_{\text{av}}(z) = \frac{1}{T_p}\int_0^{T_p} \mathcal{V}(z, t)\mathcal{I}(z, t)\, dt$$

where $T_p = 2\pi/\omega$. The time-average power can also be calculated directly from the voltage and current phasors:

$$P_{\text{av}}(z) = \tfrac{1}{2}\Re e\{V(z)[I(z)]^*\}$$

Consider the general expressions for the voltage and current phasors along a lossless uniform transmission line:

$$V(z) = V^+e^{-j\beta z} + \Gamma_L V^+ e^{j\beta z}$$

$$I(z) = \underbrace{\frac{V^+}{Z_0}e^{-j\beta z}}_{\text{forward wave}} - \underbrace{\Gamma_L\frac{V^+}{Z_0}e^{j\beta z}}_{\text{reverse wave}}$$

[14]Note that the product $\mathcal{V}(z, t)\mathcal{I}(z, t)$ represents power flow into the line, rather than out of the line, due to the defined polarity of the current $I(z)$ and voltage $V(z)$ in Figure 3.12.

We denote the time-average power carried by the forward and backward traveling waves as P^+ and P^-, respectively, and we evaluate them directly from the phasors. The time-average power carried by the forward wave is

$$P^+ = \frac{1}{2}\Re e\left\{\frac{(V^+e^{-j\beta z})(V^+e^{-j\beta z})^*}{Z_0}\right\} = \frac{V^+(V^+)^*}{2Z_0} = \frac{|V^+|^2}{2Z_0}$$

Note that although V^+ is in general complex, $V^+(V^+)^* = |V^+|^2$ is a real number.[15] The power carried by the reverse-propagating wave is

$$P^- = \frac{1}{2}\Re e\left\{\frac{(\Gamma_L V^+e^{j\beta z})(-\Gamma_L V^+e^{j\beta z})^*}{Z_0}\right\} = -\Gamma_L\Gamma_L^*\frac{V^+(V^+)^*}{2Z_0} = -\rho^2\frac{|V^+|^2}{2Z_0}$$

The fact that P^- is negative simply indicates that the backward wave carries power in the opposite direction with respect to the defined polarity of $I(z)$ and $V(z)$ in Figure 3.12. The net total power in the forward direction is then given by

$$P_{\text{av}} = P^+ + P^- = \frac{|V^+|^2}{2Z_0} - \rho^2\frac{|V^+|^2}{2Z_0} = \frac{|V^+|^2}{2Z_0}(1-\rho^2) \qquad [3.38]$$

Thus, the net time-average power flow on a transmission line is maximized when the load reflection coefficient Γ_L is zero, which, according to [3.21], occurs when $Z_L = Z_0$. Note that when $\Gamma_L = 0$ and thus $Z_L = Z_0$, the standing-wave ratio $S = 1$, as is evident from [3.25]. Since the transmission line is assumed to be lossless, all of the net power flowing in the $+z$ direction is eventually delivered to the load.

The same result can also be obtained by examining the power dissipation in the load, which is given by[16]

$$P_L = \tfrac{1}{2}\Re e\{V_L I_L^*\} = \tfrac{1}{2}|I_L|^2\Re e\{Z_L\} = \tfrac{1}{2}|I_L|^2 R_L$$

Noting that we have

$$V_L = V(z)|_{z=0} = V^+ + \Gamma_L V^+ = V^+(1+\Gamma_L)$$

$$I_L = I(z)|_{z=0} = \frac{V^+}{Z_0} - \Gamma_L\frac{V^+}{Z_0} = \frac{V^+}{Z_0}(1-\Gamma_L)$$

and substituting in the preceding expression for P_L, we find

---

[15]If $V^+ = A + jB$, then $V^+(V^+)^* = (A+jB)(A-jB) = A^2 + B^2 = |A+jB|^2 = |V^+|^2$.

[16]Note that P_L can also be written in terms of the load voltage V_L and load admittance Y_L as

$$P_L = \tfrac{1}{2}|V_L|^2\Re e\{Y_L\} = \tfrac{1}{2}|V_L|^2 G_L$$

where $Y_L = Z_L^{-1} = G_L + jB_L$.

$$P_L = \frac{1}{2}\Re e\{V_L I_L^*\} = \frac{1}{2}\Re e\left\{\frac{V^+(1+\Gamma_L)[V^+(1-\Gamma_L)]^*}{Z_0}\right\}$$
$$= \frac{1}{2}\Re e\left\{\frac{V^+(V^+)^*[1+(\Gamma_L-\Gamma_L^*)-\Gamma_L\Gamma_L^*]}{Z_0}\right\} = \frac{|V^+|^2}{2Z_0}(1-\rho^2) \quad [3.39]$$

which is identical to the total net forward power P_{av} as derived in [3.38]. We see that $P_{av} = P_L$, as expected, since, for the case of a lossless line as assumed here, *all* of the net power traveling toward the load must be dissipated in the load.

The same result can further be obtained by evaluating $P(z)$ at any point along the line using the total voltage and current phasors (rather than separating them into forward and reverse traveling wave components). In other words,

$$P_{av}(z) = \tfrac{1}{2}\Re e\{V(z)[I(z)]^*\}$$

where $V(z)$ and $I(z)$ are given by [3.22] and [3.23], respectively. (This derivation is left as an exercise for the reader.)

In summary, the total net power propagating in the $+z$ direction is

$$\boxed{P_{av}(z) = \frac{|V^+|^2}{2Z_0}(1-\rho^2)} \quad [3.40]$$

and the following observations concerning power flow on a lossless transmission line can be made:

- For a given V^+, maximum power is delivered to the load when $Z_L = Z_0$, $\Gamma_L = 0$ (i.e., $\rho = 0$), and $S = 1$. Noting that Z_0 is a real number, this condition is realized when the load is purely resistive, that is, when $Z_L = R_L = Z_0$. When $R_L = Z_0$, the load is said to be *matched* to the line and all of the power P^+ is delivered to the load. Detailed discussion of impedance matching is given in Section 3.5.
- To deliver a given amount of power (say, P_L) when the line is not matched (i.e., $S > 1$) requires higher wave power in the incident wave with correspondingly higher voltages ($P^+ = |V^+|^2/(2Z_0)$). The higher voltages are undesirable as they may cause breakdown[17] of the insulation between the two conductors of the line.
- The power efficiency achieved by matching can be assessed by considering the ratio of the power P_L that is dissipated in a given load to the forward wave power P^+ that would be delivered to the load if the line were matched:

$$\frac{P_L}{P^+} = 1 - \rho^2 = \frac{4S}{(1+S)^2}$$

[17]Electrical breakdown of insulating materials will be discussed in Section 4.10.

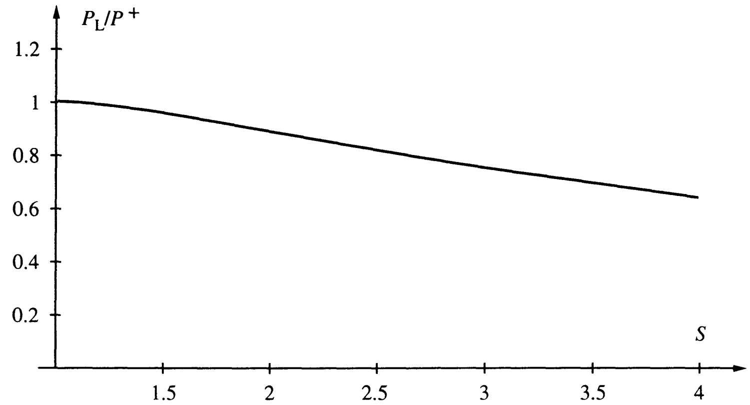

FIGURE 3.25. Power efficiency as a function of standing-wave ratio S.

The variation of P_L/P^+ with S is plotted in Figure 3.25. We see that $P_L/P^+ = 1$ for $S = 1$ and monotonically decreases to zero as S gets larger. Note that for voltage standing-wave ratios $S < 1.5$, which are relatively easy to achieve in practice, more than 90 percent of the power in the forward wave is delivered to the load. In other words, it is not necessary to strive for S very near unity to attain maximum power transfer to the load. Usually the more important issues are ensuring that the value of S is not so large as to make the line performance highly sensitive to frequency (see Section 3.5), and that the design of the line can accommodate large reactive voltages and currents that accompany a large value of S.

The degree of mismatch between the load and the line is sometimes described in terms of *return loss,* which is defined as the decibel value of the ratio of the power carried by the reverse wave to the power carried by the forward wave, given as

$$\text{Return loss} = -20 \log_{10} \rho = 20 \log_{10} \frac{S+1}{S-1}$$

If the load is perfectly matched to the line ($\rho = 0$), the return loss is infinite, which simply indicates that there is no reverse wave. If the load is such that $\rho = 1$ (i.e., a short-circuited or open-circuited line, or a purely reactive load), then the return loss is 0 dB. In practice, a well-matched system has a return loss of 15 dB or more, corresponding to a standing-wave ratio of $\sim$1.43 or less.

Examples 3-12, 3-13, and 3-14 illustrate the calculation of power flow and power delivery to the load for three different transmission line configurations.

Example 3-12: A 125-MHz VHF transmitter-antenna system. A VHF transmitter operating at 125 MHz and developing $V_0 = 100e^{j0°}$ V with a source resistance of $R_s = 50\Omega$ feeds an antenna with a feed-point impedance[18] of

[18]*The ARRL Antenna Compendium,* Vol. 4, p. 56, The American Radio Relay League, 1995–1996.

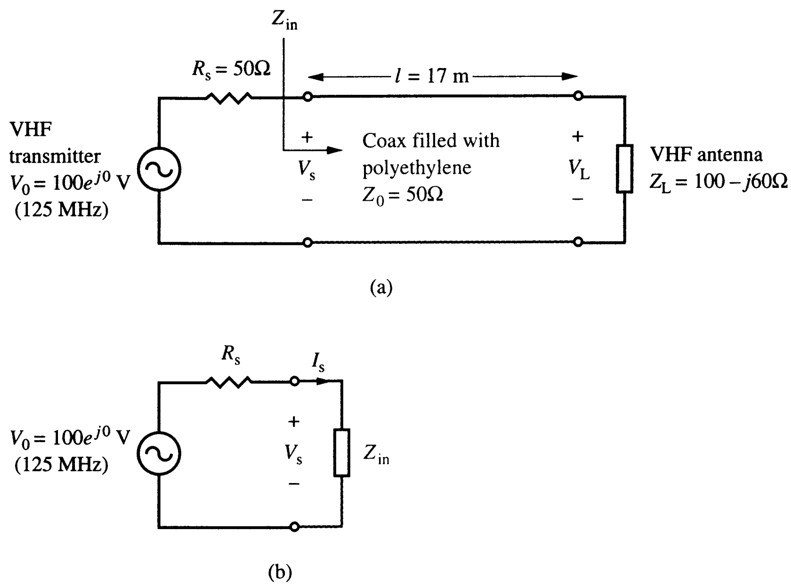

FIGURE 3.26. A 125-MHz VHF transmitter-antenna system. (a) Circuit diagram. (b) Equivalent circuit seen by the source.

$Z_L = 100 - j60$ through a 50Ω, polyethylene-filled coaxial line that is 17 m long. The setup is shown in Figure 3.26a. (a) Find the voltage $V(z)$ on the line. (b) Find the load voltage V_L. (c) Find the time-average power absorbed by the VHF antenna. (d) Find the power absorbed by the source impedance R_s.

Solution:

(a) First we note that for a polyethylene-filled coaxial line, the wavelength at 125 MHz is (using Table 3.1 and assuming v_p at 125 MHz is the same as that at 300 MHz) $\lambda = v_p/f = (2 \times 10^8 \text{ m/s})/(125 \text{ MHz}) = 1.6$ m. The length of the line is then

$$l = 17 \text{ m} = 10.625\lambda = 10.5\lambda + 0.125\lambda$$

Noting that $\tan(\beta l) = \tan[(2\pi/\lambda)(0.125\lambda)] = \tan(\pi/4) = 1$, and the input impedance of the line seen from the source end is then

$$Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} = 50\frac{(100 - j60) + j50\tan(\pi/4)}{50 + j(100 - j60)\tan(\pi/4)}$$

$$= 50\frac{100 - j10}{110 + j100} = \frac{50(100 - j10)(110 - j100)}{(110)^2 + (100)^2} \simeq 22.6 - j25.1\Omega$$

The equivalent circuit at the source end is as shown in Figure 3.26b. The source-end voltage V_s is then

$$V_s = \frac{Z_{in}}{R_s + Z_{in}} V_0 \simeq \frac{22.6 - j25.1}{50 + 22.6 - j25.1} 100e^{j0°}$$
$$\simeq \frac{33.8e^{-j48°}}{76.8e^{-j19.1°}} 100e^{j0°} \simeq 44.0e^{-j28.9°} \text{ V}$$

But we can also evaluate V_s from the expression for the line voltage $V(z)$ as

$$V_s = V(z = -17 \text{ m}) = V^+ e^{-j\beta z}(1 + \Gamma_L e^{j2\beta z})$$
$$\simeq V^+ e^{j5\pi/4}(1 + 0.483e^{-j28.4°} e^{-j\pi/2})$$
$$\simeq V^+ e^{-j3\pi/4}(0.770 - j0.425)$$

where we have used the facts that $e^{j21.25\pi} = e^{j1.25\pi} = e^{-j0.75\pi}$, $e^{-j42.5\pi} = e^{-j0.5\pi}$, and that the load reflection coefficient Γ_L is

$$\Gamma_L = \frac{100 - j60 - 50}{100 - j60 + 50} = \frac{50 - j60}{150 - j60} \simeq 0.483e^{-j28.4°}$$

Equating the two expressions for V_s, we can determine the unknown voltage V^+ as

$$V_s \simeq V^+ e^{-j3\pi/4}(0.770 - j0.425) \simeq 44.0e^{-j28.9°} \quad \rightarrow \quad V^+ = 50e^{j3\pi/4} \text{ V}$$

Thus the expression for the line voltage is

$$V(z) \simeq 50e^{j3\pi/4} e^{-j5\pi z/4}(1 + 0.483e^{-j(28.4° - 5\pi z/2)}) \text{ V}$$

(b) The voltage at the load end of the line is

$$V_L = V(z = 0) \simeq 50e^{j3\pi/4}(1 + 0.483e^{-j28.4°})$$
$$\simeq 50e^{j3\pi/4}(1.43 - j0.230) \simeq 72.2e^{-j126°} \text{ V}$$

(c) Using the value of V_L, the time-average power delivered to the VHF antenna can be calculated as

$$P_L = \frac{1}{2}|I_L|^2 R_L = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L \simeq \frac{(72.2)^2(100)}{2(1.36 \times 10^4)} \simeq 19.2 \text{ W}$$

where $|Z_L|^2 = |100 - j60|^2 = (100)^2 + (60)^2 = 1.36 \times 10^4$. Note that P_L can also be found using the source-end equivalent circuit (Figure 3.26b). Since the line is lossless, all of the time-average power input to the line at the source end must be absorbed by the antenna. In other words,

$$P_L = \frac{1}{2}\left|\frac{V_s}{Z_{in}}\right|^2 R_{in} \simeq \frac{1}{2}\frac{(44.0)^2}{[(22.6)^2 + (25.1)^2]}(22.6) \simeq 19.2 \text{ W}$$

(d) Noting that $R_s = 50\Omega$ and $Z_{in} \simeq 22.6 - j25.1\Omega$, the current at the source end (again considering the lumped equivalent circuit shown in Figure 3.26b) is

$$I_s = \frac{V_0}{R_s + Z_{in}} \simeq \frac{100e^{j0°}}{50 + 22.6 - j25.1} \simeq 1.30e^{j19.1°} \text{ A}$$

Using the value of I_s, the time-average power dissipated in the source resistance R_s is then

$$P_{R_s} = \tfrac{1}{2}|I_s|^2 R_s \simeq \tfrac{1}{2}(1.30)^2(50) \simeq 42.3 \text{ W}$$

Therefore, the total power supplied by the VHF transmitter is $P_{\text{total}} = P_{R_s} + P_L \simeq 61.5$ W.

Example 3-13: Parallel transmission lines. Three lossless transmission lines are connected in parallel, as shown in Figure 3.27. Assuming sinusoidal steady-state excitation with a source to the left of the main line, find the reflection coefficient on the main line and the percentage of the total net forward power that is absorbed by the two loads Z_{L2} and Z_{L3} for the following cases: (a) $Z_{01} = Z_{02} = Z_{03} = Z_{L2} = Z_{L3} = 100\Omega$, (b) $Z_{01} = 50\Omega$, $Z_{02} = Z_{03} = Z_{L2} = Z_{L3} = 100\Omega$, and (c) $Z_{01} = Z_{02} = Z_{03} = 100\Omega$, $Z_{L2} = Z_{L3} = 50 + j50\Omega$.

Solution:

(a) $Z_{01} = Z_{02} = Z_{03} = Z_{L2} = Z_{L3} = 100\Omega$
Since the two parallel branches are both matched, the input impedance seen at the terminals of each branch is independent of its line length and is simply 100Ω. Thus, the line impedance Z_j seen from the main line is the parallel combination of two 100Ω impedances, or $Z_j = 50\Omega$. The reflection coefficient at the junction (as seen from the main line) is then

$$\Gamma_j = \frac{Z_j - Z_{01}}{Z_j + Z_{01}} = \frac{50 - 100}{50 + 100} = -\frac{1}{3}$$

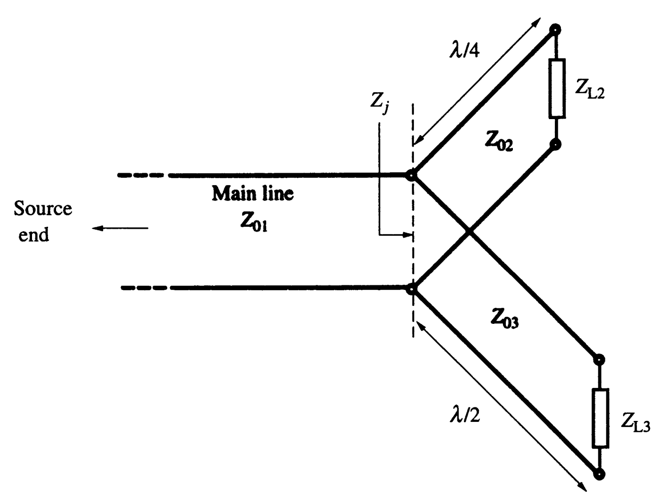

FIGURE 3.27. Parallel transmission lines. A main line with characteristic impedance Z_{01} drives two other lines of lengths λ/4 and λ/2, with characteristic impedances Z_{02} and Z_{03}, respectively.

In other words, the power efficiency, defined as the percentage of total power that is delivered to the loads versus that of the forward wave on the main line, is

$$\frac{P_L}{P^+} = (1 - |\Gamma_j|^2) \times 100 \simeq 88.9\%$$

Since each line presents the same impedance at the junction, each load absorbs half of the total power delivered, or approximately 44.4% of the total power of the incident wave.

(b) $Z_{01} = 50\Omega$, $Z_{02} = Z_{03} = Z_{L2} = Z_{L3} = 100\Omega$
The line impedance at the junction seen from the main line is once again $Z_j = 50\Omega$. However, the reflection coefficient is now zero since

$$\Gamma_j = \frac{Z_j - Z_{01}}{Z_j + Z_{01}} = \frac{50 - 50}{50 + 50} = 0$$

In other words, 100% of the power of the forward wave in this case is delivered, with each load absorbing 50%.

(c) $Z_{01} = Z_{02} = Z_{03} = 100\Omega$, $Z_{L2} = Z_{L3} = 50 + j50\Omega$
We now need to evaluate the input impedances of each of the two parallel sections at the junction as seen from the main line. Note that the length of line 2 is $\lambda/4$, so that using [3.32], its input impedance is

$$Z_{\text{in}_2} = Z_{02}\frac{Z_{L2} + jZ_{02}\tan(\frac{2\pi}{\lambda}\frac{\lambda}{4})}{Z_{02} + jZ_{L2}\tan(\frac{2\pi}{\lambda}\frac{\lambda}{4})}$$
$$= \frac{Z_{02}{}^2}{Z_{L2}} = \frac{(100)^2}{50 + j50} = \frac{200}{1 + j} \times \frac{1 - j}{1 - j} = 100 - j100\Omega$$

whereas line 3 has length $\lambda/2$ and thus presents Z_{L3} at the junction; in other words, $Z_{\text{in}_3} = Z_{L3} = 50 + j50\Omega$. The line impedance Z_j at the junction as seen from the main line is then the parallel combination of Z_{in_2} and Z_{in_3}, namely,

$$Z_j = \frac{(100 - j100)(50 + j50)}{100 - j100 + 50 + j50} = \frac{100(1 - j)(1 + j)}{3 - j}$$
$$= \frac{200}{10}(3 + j) = 60 + j20$$

and the reflection coefficient at the end of the main line is

$$\Gamma_j = \frac{Z_j - Z_{01}}{Z_j + Z_{01}} = \frac{60 + j20 - 100}{60 + j20 + 100} = \frac{-40 + j20}{160 + j20} = \left(\frac{-2 + j}{8 + j}\right)\left(\frac{8 - j}{8 - j}\right)$$
$$= \frac{-15 + j10}{65} = \frac{-3 + j2}{13} \simeq 0.277e^{j146^\circ}$$

The percentage of the incident power delivered to the two loads is thus

$$\frac{P_L}{P^+} = (1 - |\Gamma_j|^2) \times 100 \simeq 92.3\%$$

Since the two impedances Z_{in_2} and Z_{in_3} appear at the junction in parallel, they share the same voltage. In other words, we have

$$P_{\text{L}_2} = \frac{1}{2}\left|\frac{V_j}{Z_{\text{in}_2}}\right|^2 \Re e\{Z_{\text{in}_2}\} \quad \text{and} \quad P_{\text{L}_3} = \frac{1}{2}\left|\frac{V_j}{Z_{\text{in}_3}}\right|^2 \Re e\{Z_{\text{in}_3}\}$$

We can thus calculate the ratio of the powers delivered to the two loads as

$$\frac{P_{\text{L}_2}}{P_{\text{L}_3}} = \frac{|Z_{\text{in}_3}|^2 \Re e\{Z_{\text{in}_2}\}}{|Z_{\text{in}_2}|^2 \Re e\{Z_{\text{in}_3}\}} = \frac{[(50)^2 + (50)^2](100)}{[(100)^2 + (100)^2](50)} = 0.5$$

Therefore we have

$$P_{\text{L}_2} \simeq \tfrac{1}{3} \times 92.3\% \simeq 30.8\% \quad \text{and} \quad P_{\text{L}_3} \simeq \tfrac{2}{3} \times 92.3\% \simeq 61.5\%$$

Example 3-14: Cascaded transmission lines. An antenna of measured feed-point impedance of $72 + j36\Omega$ at 100 MHz is to be driven by a transmitter through two cascaded coaxial lines with the following characteristics:

$$Z_{01} = 120\Omega \quad l_1 = 3.75 \text{ m} \quad \text{air-filled} \quad v_{p1} = c \simeq 30 \text{ cm-(ns)}^{-1}$$
$$Z_{02} = 60\Omega \quad l_2 = 1.75 \text{ m} \quad \text{polyethylene-filled} \quad v_{p2} \simeq 20 \text{ cm-(ns)}^{-1}$$

where we have used Table 3.1 for the phase velocity v_{p2} for the polyethylene-filled coaxial line assuming it to be approximately the same at 100 MHz. (a) Assuming both lines to be lossless and assuming a source voltage of $V_0 = 100e^{j0}$ V and resistance of $R_s = 50\Omega$ for the transmitter, find the time-average power delivered to the load. (b) Repeat part (a) with $l_1 = 4.5$ m. The setup is shown in Figure 3.28a.

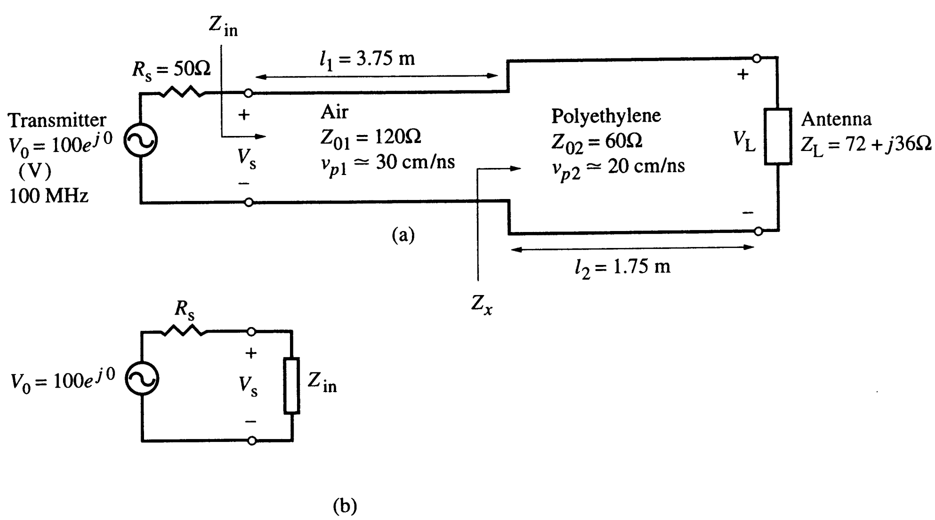

FIGURE 3.28. Cascaded transmission lines. (a) Circuit diagram showing an air-filled line of impedance 120Ω cascaded with a polyethylene-filled line of impedance 60Ω. (b) Equivalent circuit seen by the source.

Solution:

(a) At 100 MHz, the wavelengths for the two coaxial lines are respectively $\lambda_1 = v_{p1}/f \simeq 3 \times 10^8/10^8 = 3$ m, and $\lambda_2 = v_{p2}/f = 2 \times 10^8/10^8 = 2$ m. The lengths of the lines are then

$$l_1 = 3.75 \text{ m} \simeq 1.25\lambda_1 = \lambda_1 + 0.25\lambda_1$$
$$l_2 = 1.75 \text{ m} = 0.875\lambda_2 = 0.5\lambda_2 + 0.375\lambda_2$$

Note that the corresponding phase constants are $\beta_1 = 2\pi/\lambda_1$ and $\beta_2 = 2\pi/\lambda_2$. The impedance Z_x seen looking toward the load at the interface between the two coaxial lines is

$$Z_x = Z_{02}\frac{Z_L + jZ_{02}\tan(\beta_2 l_2)}{Z_{02} + jZ_L\tan(\beta_2 l_2)} = 60\frac{(72 + j36) + j60(-1)}{60 + j(72 + j36)(-1)}$$
$$= 60\frac{72 - j24}{96 + j72} = 60\frac{3 - j}{4 - j3} = 60\frac{(3 - j)(4 + j3)}{4^2 + 3^2} = 36 + j12\Omega$$

since $\tan[(2\pi/\lambda_2)l_2] = \tan(3\pi/4) = -1$. The input impedance is then

$$Z_{in} = Z_{01}\frac{Z_x + jZ_{01}\tan(\beta_0 l_1)}{Z_{01} + jZ_x\tan(\beta_0 l_1)} = \frac{Z_{01}^2}{Z_x} \simeq \frac{(120)^2}{36 + j12}$$
$$= \frac{(120)^2(36 - j12)}{36^2 + 12^2} = 360 - j120\Omega$$

With reference to the equivalent circuit in Figure 3.28b, we have

$$V_s = \frac{Z_{in}}{R_s + Z_{in}}V_0 = \frac{360 - j120}{410 - j120}(100) \simeq 88.8e^{-j2.12°} \text{ V}$$

Thus the power delivered to the antenna is

$$P_L = P_{in} = \frac{1}{2}\left|\frac{V_s}{Z_{in}}\right|^2 \mathcal{R}e\{Z_{in}\} = \frac{1}{2}\frac{(88.8)^2}{(1.44 \times 10^5)}(360) \simeq 9.86 \text{ W}$$

(b) With $l_1 = 4.5$ m $= 1.5\lambda_1$, we have

$$Z_{in} = 36 + j12 \quad \rightarrow \quad V_s = \frac{36 + j12}{86 + j12}(100) \simeq 43.7e^{j10.5°} \text{ V}$$

so that

$$P_L \simeq \frac{1}{2}\frac{(43.7)^2}{(1440)}(36) \simeq 23.9 \text{ W}$$

which is a significant improvement in power delivered, achieved simply by making the first line segment longer. This result indicates that the amount of power delivered to a load sensitively depends on the electrical lengths of the transmission lines.

3.5 IMPEDANCE MATCHING

We have already encountered the concept of impedance matching in previous sections, in connection with standing waves on transmission lines. It was shown that if the characteristic impedance Z_0 of the line is equal to the load impedance Z_L, the reflection coefficient $\Gamma_L = 0$, and the standing-wave ratio is unity. When this situation exists, the characteristic impedance of the line and the load impedance are said to be *matched,* that is, they are equal. In most transmission line applications, it is desirable to match the load impedance to the characteristic impedance of the line in order to reduce reflections and standing waves that jeopardize the power-handling capabilities of the line and also distort the information transmitted. Impedance matching is also desirable in order to drive a given load most efficiently (i.e., to deliver maximum power to the load), although maximum efficiency also requires matching the generator to the line at the source end. In the presence of sensitive components (low-noise amplifiers, etc.), impedance matching improves the signal-to-noise ratio of the system and in other cases generally reduces amplitude and phase errors. In this section, we examine different methods of achieving impedance matching.

3.5.1 Matching Using Lumped Reactive Elements

The simplest way to match a given transmission line to a load is to connect a lumped reactive element in parallel (series) at the point along the line where the real part of the line admittance (impedance) is equal to the line characteristic admittance (impedance).[19] This method is useful only at relatively low frequencies for which lumped elements can be used. The method is depicted in Figure 3.29, which shows a shunt (parallel) lumped reactive element connected to the line at a distance l from the load.

Since the matching element is connected in parallel, it is more convenient to work with line admittance rather than line impedance. The normalized admittance $\bar{Y}(z)$ seen on the line looking toward the load from any position z is given by [3.34]:

$$\bar{Y}(z) = \frac{Y(z)}{Y_0} = \frac{1 - \Gamma_L e^{j2\beta z}}{1 + \Gamma_L e^{j2\beta z}}$$

In Figure 3.29, matching requires that $\bar{Y}_2 = \bar{Y}_1 + \bar{Y}_s = 1$. Thus we first need to choose the position l along the line such that $\bar{Y}(z = -l) = \bar{Y}_1 = 1 - j\bar{B}$, that is, $\Re e\{\bar{Y}_1\} = 1$. Then we choose the lumped shunt element to be purely reactive with an appropriate value such that $\bar{Y}_s = j\bar{B}$, which results in $\bar{Y}_2 = \bar{Y}_1 + \bar{Y}_s = 1$. Substituting $z = -l$, we have

$$\bar{Y}_1 = \bar{Y}(z = -l) = \frac{1 - \Gamma_L e^{-j2\beta l}}{1 + \Gamma_L e^{-j2\beta l}} = 1 - j\bar{B}$$

[19] This possibility was noted earlier in Section 3.2.2 in connection with Figure 3.19, where $\Re e\{\bar{Z}(z_1)\} = 1$ at $z_1 \simeq -0.0985\lambda$. Note that, in general, the real part of the line admittance is equal to unity at some other point $z_2 \neq z_1$.

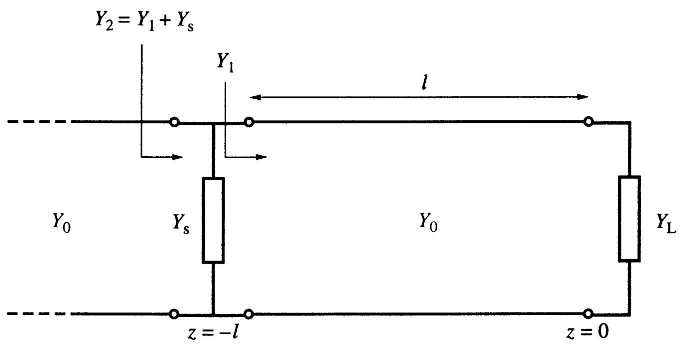

FIGURE 3.29. Matching by a lumped shunt element. The shunt element Y_s is connected at a distance l from the load such that the line admittance $Y_2(z = -l) = Y_1(z = -l) + Y_s = Y_0$.

from which we can solve for the position l of the lumped reactive element, its type (i.e., capacitor or inductor), and its normalized susceptance $-\bar{B}$. Since the load reflection coefficient Γ_L is, in the general case, a complex number given by

$$\Gamma_L = \rho e^{j\psi} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

we have

$$\bar{Y}_1 = \frac{1 - \rho e^{j\psi} e^{-j2\beta l}}{1 + \rho e^{j\psi} e^{-j2\beta l}} = \frac{1 - \rho e^{j\theta}}{1 + \rho e^{j\theta}} \qquad [3.41]$$

where $\theta = \psi - 2\beta l$. By multiplying the numerator and the denominator by the complex conjugate of the denominator,[20] we obtain

$$\bar{Y}_1 = \underbrace{\frac{1 - \rho^2}{1 + 2\rho\cos\theta + \rho^2}}_{\Re e\{\bar{Y}_1\} = 1} - j\underbrace{\frac{2\rho\sin\theta}{1 + 2\rho\cos\theta + \rho^2}}_{-\Im m\{\bar{Y}_1\} = \bar{B}} \qquad [3.42]$$

Since $\Re e\{\bar{Y}_1\} = 1$, we can write

$$\frac{1 - \rho^2}{1 + 2\rho\cos\theta + \rho^2} = 1$$

which yields

$$\theta = \psi - 2\beta l = \cos^{-1}(-\rho) \qquad [3.43]$$

[20] $[1 + \rho e^{j\theta}]^* = 1^* + (\rho e^{j\theta})^* = 1 + \rho e^{-j\theta}$

In other words, the distance l from the load at which $\bar{Y}(z = -l) = 1 - j\bar{B}$ is given by

$$l = \frac{\psi - \theta}{2\beta} = \frac{\psi - \cos^{-1}(-\rho)}{2\beta} = \frac{\lambda}{4\pi}[\psi - \cos^{-1}(-\rho)] \qquad [3.44]$$

Note that, in general, $\theta = \cos^{-1}(-\rho)$ (with $\rho > 0$) has two solutions, one in the range $\pi/2 \leq \theta_1 \leq \pi$ and the other in the range $-\pi \leq \theta_2 \leq -\pi/2$. Also, if [3.44] results in negative values for l, then the corresponding physically meaningful solution can be found by simply adding $\lambda/2$.[21] To find $\bar{B}$, we substitute $\cos\theta = -\rho$ and $\sin\theta = \pm\sqrt{1-\rho^2}$ (where the plus sign corresponds to $\pi/2 \leq \theta_1 \leq \pi$ and the minus sign corresponds to $-\pi \leq \theta_2 \leq -\pi/2$) in the imaginary part of $\bar{Y}_1$ given by [3.42], resulting in

$$\bar{B} = -\mathcal{I}m\{\bar{Y}_1\}|_{\cos\theta=-\rho} = \pm\frac{2\rho\sqrt{1-\rho^2}}{1-2\rho^2+\rho^2} = \pm\frac{2\rho}{\sqrt{1-\rho^2}} \qquad [3.45]$$

where the plus and minus signs correspond to a shunt capacitor ($\bar{B}_1 > 0$) and a shunt inductor ($\bar{B}_2 < 0$), respectively.[22] This susceptance also determines the value of the lumped reactive element $\bar{Y}_s$, which must be connected in parallel to the line in order to cancel out the reactive part of $\bar{Y}_1$. In particular, we should have $\bar{Y}_s = +j\bar{B}$ so that the total admittance $\bar{Y}_2$ seen from the left side of Y_s in Figure 3.29 is

$$\bar{Y}_2 = \bar{Y}_1 + \bar{Y}_s = 1 - j\bar{B} + j\bar{B} = 1$$

When matching with series lumped reactive elements, similar equations can be derived for the distance l away from the load at which $\bar{Z}_1 = \bar{Z}(z = -l) = 1 - j\bar{X}$,

$$l = \frac{\psi - \cos^{-1}\rho}{2\beta} = \frac{\lambda}{4\pi}(\psi - \cos^{-1}\rho) \qquad [3.46]$$

and the normalized reactance of the series lumped element that would provide matching is given by

$$\bar{X} = \pm\frac{2\rho}{\sqrt{1-\rho^2}}$$

Example 3-15 illustrates impedance matching using a single shunt reactive element.

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[21]For negative l, we have $-2\pi < 2\beta l < 0$. By adding $\lambda/2$, we have $2\beta(l+\lambda/2) = 2\beta l + 2(2\pi/\lambda)(\lambda/2) = 2\beta l + 2\pi > 0$, which lies between 0 and 2π.

[22]Note that $\theta = \cos^{-1}(-\rho)$ is an angle that is either in the second quadrant, $\pi/2 \leq \theta_1 \leq \pi$ (when $\sin\theta_1 > 0$, requiring a capacitive element based on the polarity of the imaginary part of $\bar{Y}_1$ as given in [3.42]) or the third quadrant, $-\pi \leq \theta_2 \leq -\pi/2$ (when $\sin\theta_2 < 0$, requiring an inductive element).

Example 3-15: Matching with a single reactive element. An antenna having a feed-point impedance of 110Ω is to be matched to a 50Ω coaxial line with $v_p = 2 \times 10^8$ m/s using a single shunt lumped reactive element, as shown in Figure 3.30. Find the position (nearest to the load) and the appropriate value of the reactive element for operation at 30 MHz using (a) a capacitor, and (b) an inductor.

Solution: The load reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{110 - 50}{110 + 50} = 0.375$$

Using [3.45], the reactive admittance (or susceptance) at the position of the shunt element is

$$\bar{B} = \pm \frac{2 \times 0.375}{\sqrt{1 - (0.375)^2}} \simeq \pm 0.809$$

(a) For $\bar{B}_1 \simeq +0.809$, the shunt element must be a capacitor. The nearest position of the capacitor with respect to the load can be found as

$$l_1 = -\frac{\theta_1}{2\beta} \simeq -\frac{\lambda}{4\pi} \underbrace{\cos^{-1}(-0.375)}_{\pi/2 \le \theta_1 \le \pi} \simeq -\frac{\lambda}{4\pi}(1.955) \simeq -0.156\lambda \simeq 0.344\lambda$$

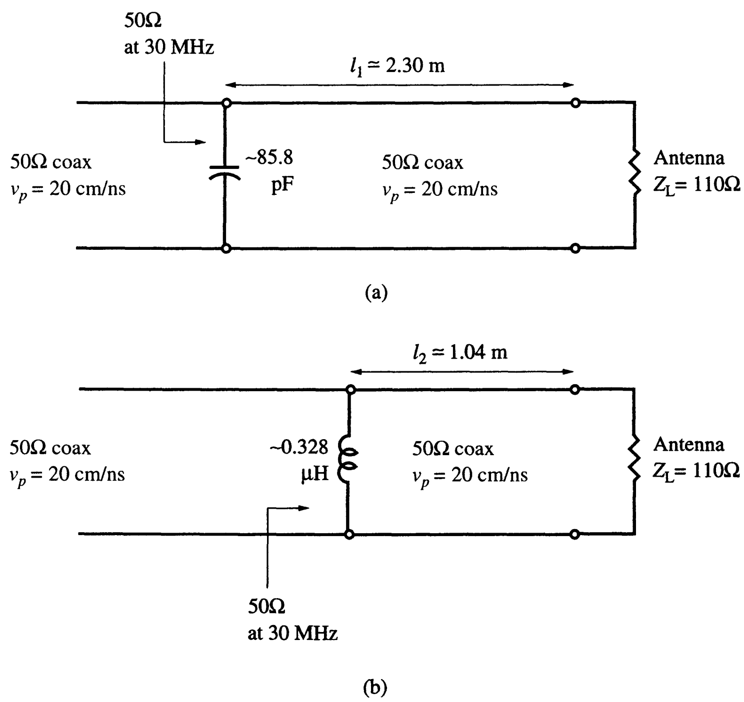

FIGURE 3.30. Matching with a single reactive element. The two solutions determined in Example 3-15. (a) Using a shunt capacitor. (b) Using a shunt inductor.

Since $\lambda = v_p/f = 2 \times 10^8/(30 \times 10^6) \simeq 6.67$ m, the actual position of the shunt capacitor is $l_1 \simeq 0.344 \times 6.67 \simeq 2.30$ m. To determine the capacitance C_s, we use

$$(j\omega C_s)(Z_0) = j\bar{B}_1 \quad \rightarrow \quad [j(2\pi \times 30 \times 10^6 C_s)(50)] \simeq +j0.809$$
$$\rightarrow \quad C_s \simeq 85.8 \text{ pF}$$

(b) For $\bar{B}_2 \simeq -0.809$, the shunt element must be an inductor. Similarly, the nearest position of the inductor is

$$l_2 = -\frac{\theta_2}{2\beta} \simeq -\frac{\lambda}{4\pi} \underbrace{\cos^{-1}(-0.375)}_{-\pi \le \theta_2 \le -\pi/2} \simeq -\frac{\lambda}{4\pi}(-1.955) \simeq 0.156\lambda$$

Using $\lambda \simeq 6.67$ m, the actual position of the shunt inductor is $l_2 \simeq 0.156 \times 6.67 \simeq 1.04$ m. To determine the inductance L_s, we use

$$[-j/(\omega L_s)](Z_0) = j\bar{B}_2 \quad \rightarrow \quad [-j/(2\pi \times 30 \times 10^6 L_s)](50) \simeq -j0.809$$
$$\rightarrow \quad L_s \simeq 0.328 \ \mu\text{H}$$

3.5.2 Matching Using Series or Shunt Stubs

In Section 3.2, we saw that short- or open-circuited transmission lines can be used as reactive circuit elements. At microwave frequencies, it is often impractical or inconvenient to use lumped elements for impedance matching. Instead, we use a common matching technique that uses single open- or short-circuited stubs (i.e., transmission line segments) connected either in series or in parallel, as illustrated in Figure 3.31. In practice, the short-circuited stub is more commonly used for coaxial and waveguide applications because a short-circuited line is less sensitive to external influences (such as capacitive coupling and pick-up) and radiates less than an open-circuited line segment. However, for microstrips and striplines, open-circuited stubs are more common in practice because they are easier to fabricate. For similar practical reasons, the shunt (parallel) stub is more convenient than the series stub; the discontinuity created by breaking the line may disturb the voltage and current in the case of the series stub.

The principle of matching with stubs is identical to that discussed in Section 3.5.1 for matching using shunt lumped reactive elements. The only difference here is that the matching admittance Y_s is introduced by using open- or short-circuited line segments (or stubs) of appropriate length l_s, as shown in Figure 3.32. In the following, we exclusively consider the case of matching with a short-circuited stub, as illustrated in Figure 3.32. The corresponding analysis for open-circuited stubs is similar in all respects and is left as an exercise for the reader.

With the required location l and the normalized admittance $\bar{B}$ of the stub as determined from [3.44] and [3.45], we need only to find the length of the stub l_s necessary to present a normalized admittance of $\bar{Y}_s = +j\bar{B}$ at the junction. For this purpose, we can use expression [3.17] from Section 3.2 for the normalized

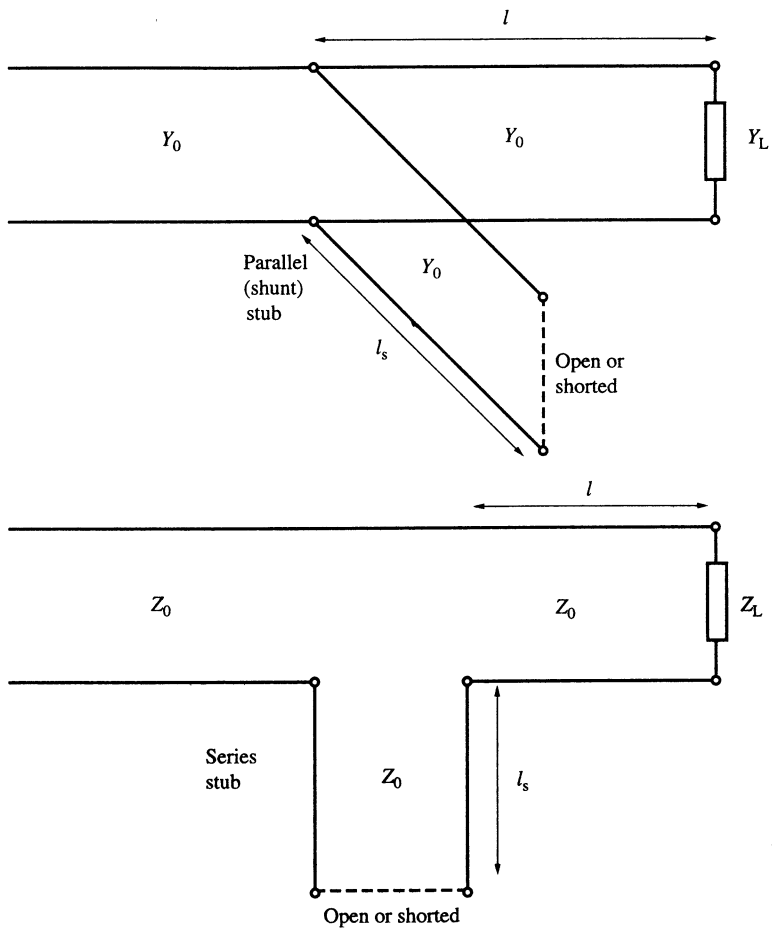

FIGURE 3.31. Matching by shunt or series open- or short-circuited stubs.

input impedance of a short-circuited line of length l_s and set the corresponding normalized admittance equal to $+j\bar{B}$. Recalling that for a short-circuited line $\bar{Z}_{in} = j\tan(\beta l_s)$, we have

$$\bar{Y}_s = \frac{1}{j\tan(\beta l_s)} = +j\bar{B}$$

or

$$\tan(\beta l_s) = -\frac{1}{\bar{B}} \qquad [3.47]$$

The value of $\bar{B}$ determined from [3.45] can be used in [3.47] to find the length l_s of the short-circuited stub. Note that in [3.47], we have assumed the characteristic impedance of the short-circuited stub to be equal to that of the main line.

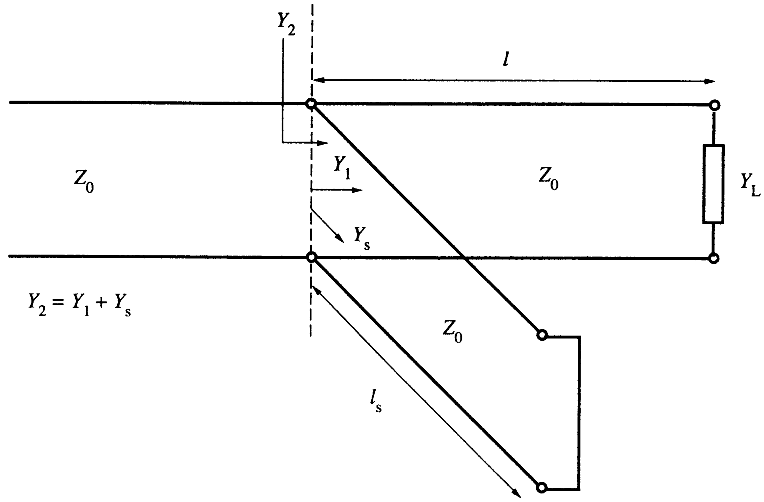

FIGURE 3.32. Matching with a single parallel (shunt) short-circuited stub.

In practice, single-stub matching can be achieved even if the load impedance Z_L is not explicitly known, by relying on measurements of S to determine ρ and measurements of the location of the voltage minimum or maximum to determine ψ. To see this, consider that the stub location l can be measured relative to the position z_{max} of the nearest voltage maximum toward the load end so that $l - \Delta l_{max} = |z_{max}| < \lambda/2$, as shown in Figure 3.33. Using [3.43], we can then write

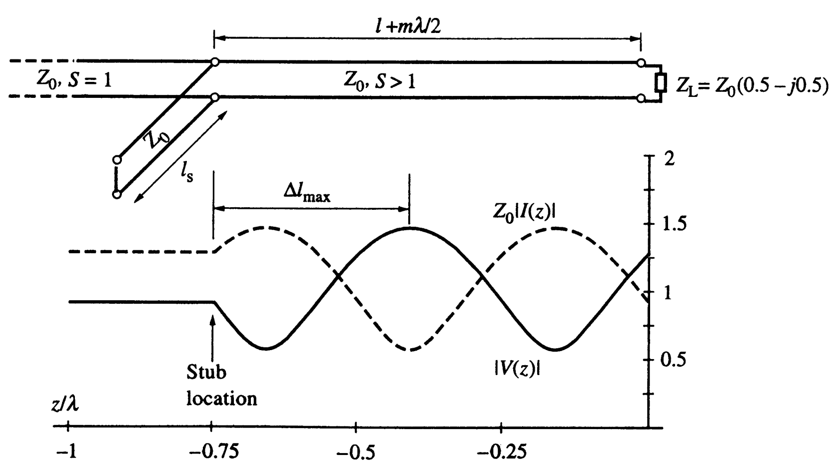

FIGURE 3.33. Voltage standing-wave pattern on a transmission line with single-stub matching. The standing-wave ratio S is unity to the left of the stub. The stub location is at a distance of Δl_{max} from the nearest voltage maximum toward the load end. The particular case shown is for $\bar{Z}_L = 0.5 - j0.5$. Note that, as usual, we have assumed $V^+ = 1$.

$$\theta = \psi - 2\beta l = \psi + 2\beta z_{\max} - 2\beta\Delta l_{\max}$$
$$= -m2\pi - 2\beta\Delta l_{\max} \qquad m = 0, 1, 2, \ldots$$

noting that $m = 0$ does not apply if $-\pi \leq \psi < 0$. Using the preceding expression for θ, we have

$$\Delta l_{\max} = -\frac{1}{2\beta}[\theta + m2\pi] = -\frac{1}{2\beta}[\cos^{-1}(-\rho) + m2\pi], \quad m = 0, 1, 2, \ldots \quad [3.48]$$

Thus, ρ can be directly determined from the measured standing-wave ratio, and [3.48] determines the stub location with respect to the measured location of the voltage maximum.

Figure 3.33 also illustrates the fact that, although the proper choice of the stub location l and its length l_s achieves matching so that the standing-wave ratio on the source side of the stub is unity, a standing wave does exist on the segment of line between the stub and the load.

Impedance matching using a single short-circuited transmission line stub is illustrated in Example 3-16.

Example 3-16: Single-stub matching. Design a single-stub system to match a load consisting of a resistance $R_L = 200\Omega$ in parallel with an inductance $L_L = 200/\pi$ nH to a transmission line with characteristic impedance $Z_0 = 100\Omega$ and operating at 500 MHz. Connect the stub in parallel with the line.

Solution: At 500 MHz, the load admittance is given by

$$Y_L = \frac{1}{R_L} - j\frac{1}{\omega L_L} = \frac{1}{200} - j\frac{1}{[2\pi(500 \times 10^6)(200/\pi)(10^{-9})]} = 0.005 - j0.005 \text{ S}$$

The reflection coefficient at the load is

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = \frac{0.01 - (0.005 - j0.005)}{0.01 + (0.005 - j0.005)} = \frac{1 + j}{3 - j} \simeq 0.447e^{j63.4°}$$

so that $\rho \simeq 0.447$ and $\psi \simeq 63.4°$. Using [3.44], we have

$$l \simeq \frac{1.11 - \cos^{-1}(-0.447)}{2(2\pi/\lambda)} \simeq \left(\frac{1.11 \mp 2.034}{4\pi}\right)\lambda \quad \rightarrow \quad \begin{array}{l} l_1 \simeq -0.073\lambda \\ l_2 = 0.25\lambda \end{array}$$

where the first solution with a negative value of l can be realized by simply adding 0.5λ so that the stub position is between the load and the source. Thus, the stub position for the first solution is $l_1 \simeq -0.073\lambda + 0.5\lambda = 0.426\lambda$. Note that we have used $\theta = \cos^{-1}(-0.447) \simeq \pm 117° \simeq \pm 2.034$ radians. Using [3.45], the normalized susceptance of the input admittance of the short-circuited shunt stub needed is

$$\bar{B} \simeq \pm\frac{2(0.447)}{\sqrt{1 - (0.447)^2}} \simeq \pm 1$$

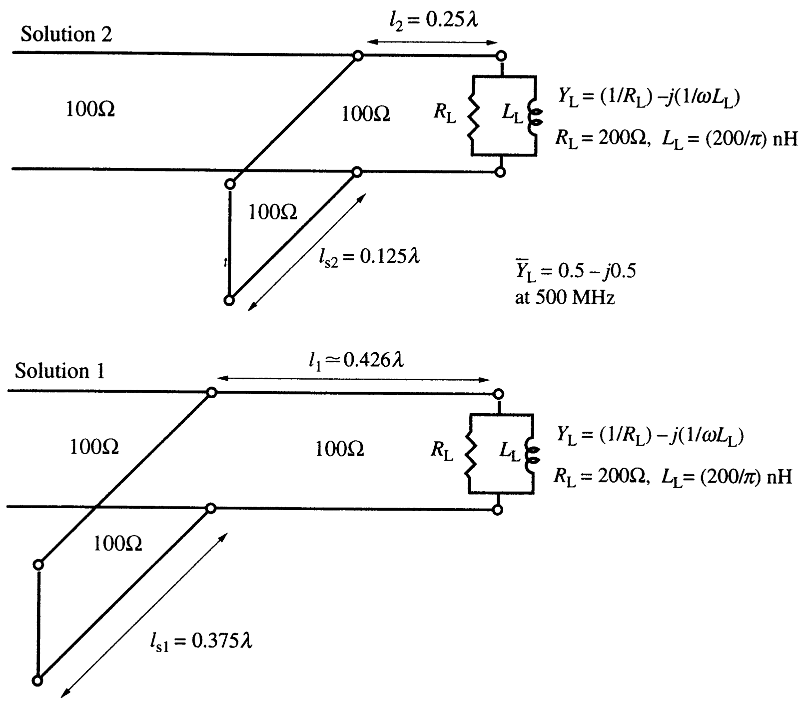

FIGURE 3.34. Two alternative single-stub matching solutions. Solution 2 would in general be preferred since it has shorter segments of line (and a shorter stub) over which the standing-wave ratio differs from unity.

where the plus sign corresponds to the stub position l_1 and the minus sign corresponds to l_2. From [3.47], the length l_{s1} of the short-circuited stub at a distance l_1 from the load is

$$l_{s1} = \frac{\lambda}{2\pi}\tan^{-1}\left(-\frac{1}{\bar{B}}\right) \simeq \frac{\lambda}{2\pi}\tan^{-1}(-1) = -0.125\lambda \quad \rightarrow \quad 0.375\lambda$$

and similarly, the stub length l_{s2} needed at position l_2 is

$$l_{s2} = \frac{\lambda}{2\pi}\tan^{-1}(1) = 0.125\lambda$$

Both of the alternative solutions are shown in Figure 3.34. Since Solution 2 gives a stub position closer to the load and a shorter stub length, it would usually be preferred over Solution 1. In general, standing waves jeopardize power-handling capabilities of a line and also lead to signal distortion. Thus, it is desirable to minimize the lengths of line over which the standing-wave ratio is large. In the case shown in Figure 3.34, more of the line operates under matched ($S = 1$) conditions for Solution 2.

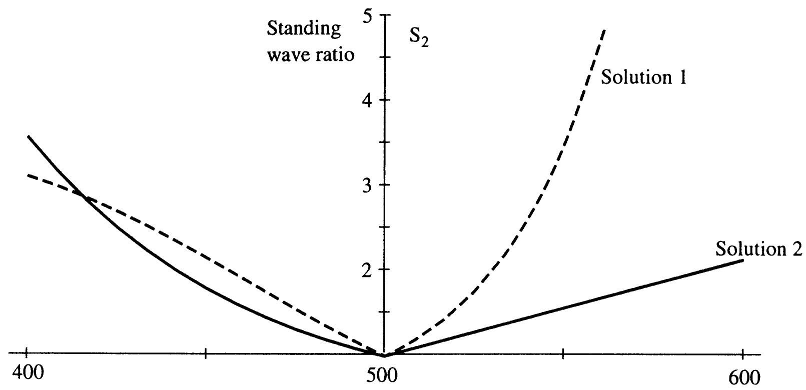

FIGURE 3.35. Frequency sensitivity of single-stub matching. The standing-wave ratio S_2 versus frequency for the two alternative solutions given in Figure 3.34.

The frequency dependence of the various designs can also be important in practice. A comparison of the frequency responses of the two alternative solutions for the previous example is given in Figure 3.35. Note that the load admittance as a function of frequency $f = \omega/(2\pi)$ is given by

$$Y_L(f) = \frac{1}{200} - j\frac{10^9}{400f}$$

Assuming that the phase velocity along the line is equal to the speed of light c, we have $\beta l = 2\pi f l/c$. The line admittance seen just to the right of the short-circuited stub is given as a function of frequency:

$$Y_1(f) = Y_0 \frac{Y_L + jY_0 \tan(2\pi f l/c)}{Y_0 + jY_L \tan(2\pi f l/c)}$$

If we also assume $\beta l_s = 2\pi f l_s/c$, the total line admittance seen from the source side of the short-circuited stub (see Figure 3.32) is

$$Y_2(f) = Y_s + Y_1 = \frac{-jY_0}{\tan(2\pi f l_s/c)} + Y_0 \frac{Y_L + jY_0 \tan(2\pi f l/c)}{Y_0 + jY_L \tan(2\pi f l/c)}$$

The reflection coefficient Γ_2 and the standing-wave ratio S_2 are then given as

$$\Gamma_2(f) = \frac{Y_0 - Y_2(f)}{Y_0 + Y_2(f)}; \quad S_2(f) = \frac{1 + |\Gamma_2(f)|}{1 - |\Gamma_2(f)|}$$

The quantity $S_2(f)$ is plotted in Figure 3.35 as a function of frequency between 400 and 600 MHz. Note that the bandwidths[23] of the two designs are dramatically

[23]Defined as the frequency range over which the standing-wave ratio S_2 is lower than a given amount, for example, $S_2 < 2$. The particular value of S_2 used depends on the application in hand.

different. Which solution to choose depends on the particular application in hand, although in most cases minimizing reflections ($S_2 < 2$, for example) over a wider frequency range is desirable. Note that, for the case shown, Solution 2 provides matching over a substantially broader range of frequencies than does Solution 1.

3.5.3 Quarter-Wave Transformer Matching

A powerful method for matching a given load impedance to a transmission line that is used to drive it is the so-called quarter-wave transformer matching. This method takes advantage of the impedance inverting property of a transmission line of length $l = \lambda/4$, namely, the fact that the input impedance of a line of length $l = \lambda/4$ is given by

$$Z_{\text{in}}\Big|_{l=\lambda/4} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)}\Bigg|_{l=\lambda/4} = \frac{Z_0^2}{Z_L}$$

or in terms of normalized impedances, we have

$$\boxed{\bar{Z}_{\text{in}}\Big|_{l=\lambda/4} = \frac{Z_{\text{in}}}{Z_0}\Big|_{l=\lambda/4} = \frac{Z_0}{Z_L} = \frac{1}{\bar{Z}_L}} \qquad [3.49]$$

hence the term "impedance inverter." Thus, a quarter-wave section transforms impedance in such a way that a kind of inverse of the terminating impedance appears at its input.

Consider a quarter-wavelength transmission line segment of characteristic impedance Z_Q, as shown in Figure 3.36. In the general case of a complex load impedance $Z_L = R_L + jX_L$, we have

$$Z_{\text{in}} = \frac{Z_Q^2}{R_L + jX_L} \quad \longrightarrow \quad Y_{\text{in}} = \frac{R_L}{Z_Q^2} + j\frac{X_L}{Z_Q^2}$$

so that a load consisting of a series resistance (R_L) and an inductive reactance ($X_L > 0$) appears at the input of the quarter-wave section as an admittance consisting of a conductance R_L/Z_Q^2 (or a resistance Z_Q^2/R_L) *in parallel* with a capacitive susceptance $X_L/Z_Q^2 > 0$. Similarly, if the load were capacitive ($X_L < 0$), it would appear as a conductance R_L/Z_Q^2 in parallel with an inductive susceptance $X_L/Z_Q^2 < 0$.

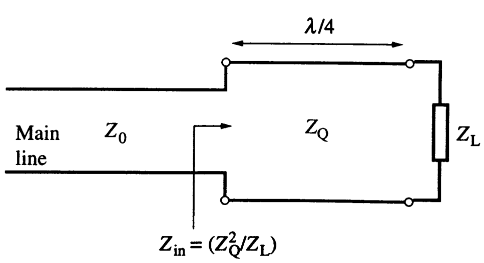

FIGURE 3.36. Quarter-wave transformer. A load Z_L to be driven by a transmission line of characteristic impedance Z_0 connected to the line via a quarter-wavelength-long line of characteristic impedance Z_Q.

Purely Resistive Loads The practical utility of the impedance-inverting property of a quarter-wavelength transmission line becomes apparent when we consider a purely resistive load. Any arbitrary purely resistive load impedance $Z_L = R_L$ is transformed into a purely resistive input impedance of Z_Q^2/R_L. Thus, by appropriately choosing the value of the characteristic impedance Z_Q of the quarter-wavelength line, its input impedance can be made equal to the characteristic impedance Z_0 (a real value for a lossless line) of the main line that is to be used to drive the load. This property of the quarter-wave line can be used to match two transmission lines of different characteristic impedances or to match a load impedance to the characteristic impedance of a transmission line. Note that the matching section must have a characteristic impedance of

$$Z_Q = \sqrt{R_1 R_2} \qquad [3.50]$$

where R_1 and R_2 are the two resistive impedances to be matched. Note that in the case shown in Figure 3.36, $R_1 = Z_0$ and $R_2 = R_L$. Alternatively, R_2 could be the characteristic impedance of another transmission line that may need to be matched to the main line (Z_0) using the quarter-wave section.

The following example illustrates quarter-wave transformer matching of a purely resistive load.

Example 3-17: Quarter-wave transformer for a monopole antenna. Design a quarter-wavelength section to match a thin monopole antenna of length 0.24λ[24] having a purely resistive feed-point impedance of $R_L \simeq 30\Omega$ to a transmission line having a characteristic impedance of $Z_0 = 100\Omega$.

Solution: With reference to Figure 3.37 and according to [3.50], the $\lambda/4$ section must match two impedances—$R_2 = R_L$ and $R_1 = Z_0 = 100\Omega$—and thus

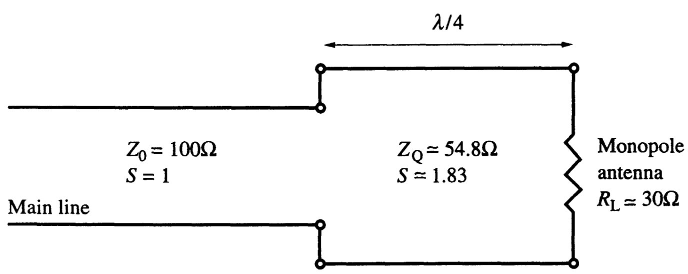

FIGURE 3.37. Quarter-wave transformer. The load is a monopole antenna of length 0.24λ, which has a purely resistive impedance of 30Ω.

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[24]The reactive part of the impedance of such monopole antennas with length just shorter than a quarter wavelength is nearly zero. Monopole or dipole antennas with purely resistive input impedances are referred to as resonant antennas and are used for many applications. (See Section 14.06 of E. Jordan and K. Balmain, *Electromagnetic Waves and Radiating Systems,* Prentice Hall, 1968.)

must have a characteristic impedance Z_Q of

$$Z_Q = \sqrt{R_1 R_2} = \sqrt{Z_0 R_L} = \sqrt{(100)(30)} \simeq 54.8\Omega$$

The standing-wave ratio is unity beyond the quarter-wave section. However, note that $S \simeq 1.83$ within the $\lambda/4$ section.

Complex Load Impedances In using quarter-wave transformers to match a complex load impedance to a lossless transmission line (i.e., where Z_0 is real), it is necessary to insert the quarter-wave segment at the point along the line where the line impedance $\bar{Z}(z)$ is purely resistive. As discussed in previous sections, this point can be the position of either the voltage maximum or minimum. In most cases, it is desirable to choose the point closest to the load in order to minimize the length of the transmission line segment on which $S \neq 1$ because the presence of standing waves jeopardizes power-handling capabilities of the line, tends to reduce signal-to-noise ratio, and may lead to distortion of the signal transmitted. Example 3-18 illustrates quarter-wave matching of a complex load.

Example 3-18: Thin-wire half-wave dipole antenna. A thin-wire half-wave dipole antenna[25] has an input impedance of $Z_L = 73 + j42.50\Omega$. Design a quarter-wave transformer to match this antenna to a transmission line with characteristic impedance $Z_0 = 100\Omega$.

Solution: We start by evaluating the reflection coefficient at the load

$$\Gamma_L = \rho e^{j\psi} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{73 + j42.5 - 100}{73 + j42.5 + 100} \simeq 0.283 e^{j109°}$$

so that $\psi \simeq 109° \simeq 1.896$ radians. Note that the standing-wave ratio is

$$S = \frac{1+\rho}{1-\rho} \simeq \frac{1 + 0.283}{1 - 0.283} \simeq 1.79$$

From previous sections, we know that, for an inductive load, the first voltage maximum is closer to the load than the first voltage minimum (see Figure 3.17). The first voltage maximum is at

$$\psi + 2\beta z_{\max} = 0 \quad \longrightarrow \quad z_{\max} = -\frac{\psi}{2\beta} \simeq -\frac{1.896\lambda}{4\pi} \simeq -0.151\lambda$$

Thus the quarter-wave section should be inserted at $z \simeq -0.151\lambda$, as shown in Figure 3.38.

[25]See, for example, Section 14.06 of E. C. Jordan and K. Balmain, *Electromagnetic Waves and Radiating Systems,* Prentice Hall, 1968.

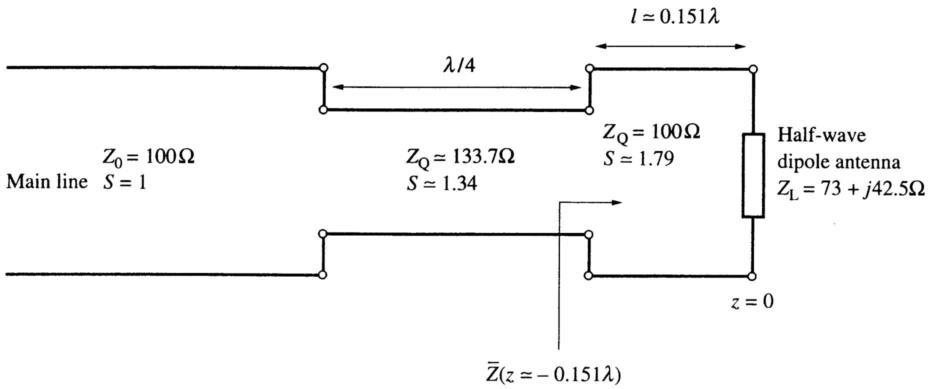


FIGURE 3.38. Quarter-wave matching of the half-wave dipole antenna in Example 3-18.

Noting that the normalized load impedance is $\bar{Z}_L = 0.73 + j0.425$ and that we have $\tan[\beta(0.151\lambda)] \simeq 1.392$, the normalized line impedance $\bar{Z}(z)$ seen toward the load at $z \simeq -0.151\lambda$ is

$$\bar{Z}(z)|_{z\simeq-0.151\lambda} = \left.\frac{\bar{Z}_L - j\tan(\beta z)}{1 - j\bar{Z}_L\tan(\beta z)}\right|_{z\simeq-0.151\lambda} \simeq \frac{0.73 + j0.425 + j1.392}{1 + j(0.73 + j0.425)(1.392)} \simeq 1.79$$

Note that $\bar{Z}(z \simeq -0.151\lambda) \simeq 1.79 = S$, as expected on the basis of the discussion in Section 3.3.2 (i.e., $\bar{R}_{max} = S \simeq 1.79$). The characteristic impedance Z_Q of the quarter-wave section should thus be

$$Z_Q = \sqrt{Z(z \simeq -0.151\lambda)Z_0} \simeq \sqrt{(100)(1.79)(100)} \simeq 133.7\Omega$$

Note that in general, as in this specific example, we have $\bar{Z}(z = z_{max}) = S$, and thus $Z_Q = Z_0\sqrt{S}$. Note also that the standing-wave ratio in the quarter-wave section is $S \simeq 1.34$, as can be calculated by using $Z(z \simeq -0.151\lambda)$ and Z_Q.

Frequency Sensitivity of Quarter-Wave Matching The frequency sensitivity of a quarter-wave transformer is a serious limitation since the design is perfect (i.e., provides $S = 1$) only at the frequency for which the length of the transformer segment is exactly $\lambda/4$. The bandwidth of the transformer can be assessed by plotting the standing-wave ratio S versus frequency, as was done in Section 3.5.2 for the single-stub tuning example and as is shown in the following example.

Example 3-19: Multiple-stage quarter-wave transformers. A resistive load of $R_L = 75\Omega$ is to be matched to a transmission line with characteristic impedance $Z_0 = 300\Omega$. The frequency of operation is $f_0 = 300$ MHz (which

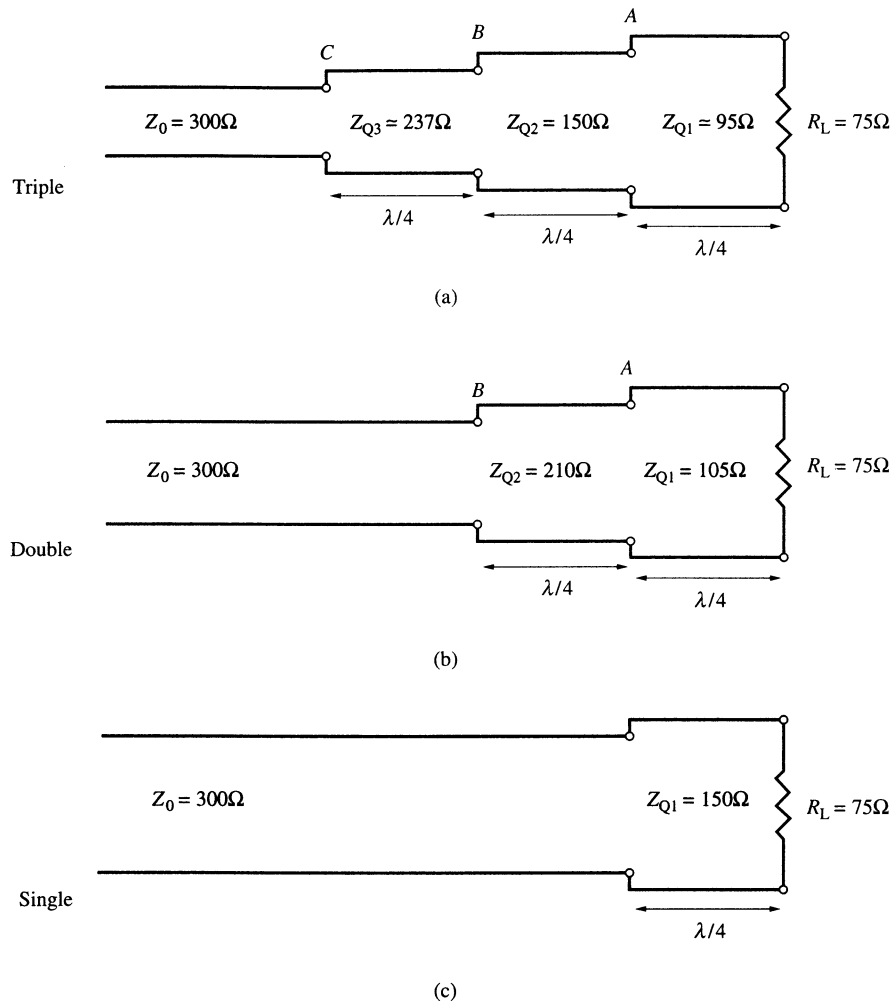


FIGURE 3.39. Multiple-stage quarter-wave matching. Matching with triple, double, and single $\lambda/4$ sections are illustrated respectively in panels (a), (b), and (c).

corresponds to $\lambda_0 = v_p/f_0 \simeq 3 \times 10^8/(300 \times 10^6) = 1$ m, assuming an air-filled coaxial line). Design multiple cascaded quarter-wave matching transformers and at 300 MHz compare their frequency responses between 200 and 400 MHz.

Solution: Three different designs are shown in Figure 3.39. Note that the choice of the impedance Z_{Q1} for the single-transformer case is straightforward. For the double-transformer case, the condition for exact quarter-wave matching is $Z_{Q2} = Z_{Q1}\sqrt{Z_0/R_L}$, allowing for different choices of Z_{Q2} and Z_{Q1} as long as the condition is satisfied. The design shown in Figure 3.39b is one that provides a standing-wave ratio in the first and second quarter-wave sections of

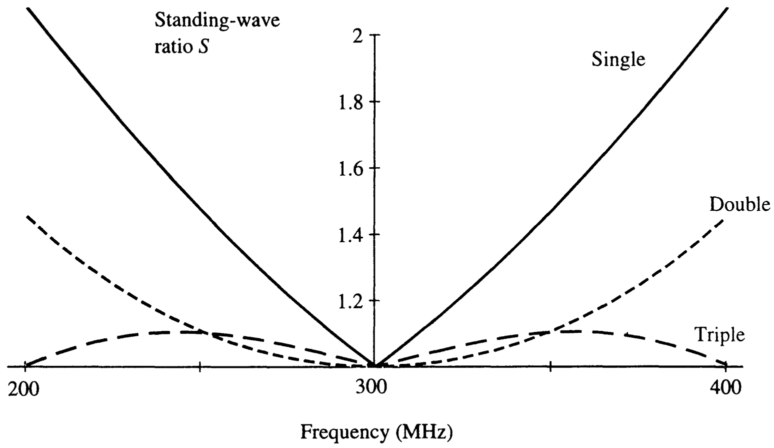


FIGURE 3.40. Quarter-wave transformer bandwidth. Standing-wave ratio S versus frequency for the three different transformers shown in Figure 3.39.

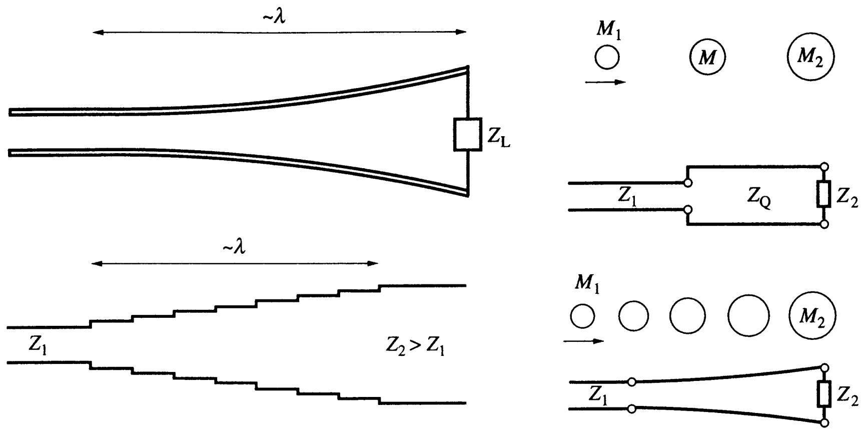
respectively $S \simeq 1.4$ and $S \simeq 1.6$. For the triple transformer case the condition for exact quarter-wave matching can be shown to be $Z_{Q1}Z_{Q3} = Z_{Q2}\sqrt{Z_0 R_L} = 150Z_{Q2}$. Once again, many different combinations of Z_{Q1}, Z_{Q2}, Z_{Q3} satisfy this condition, and other performance criteria (such as minimizing S in the quarter-wave sections) must be used to make particular design choices. One design approach[26] is to require the characteristic impedance of the second quarter-wave segment to be the geometric mean of the two impedances to be matched, namely $Z_{Q2} = \sqrt{(300)(75)} = 150\Omega$. The choices of Z_{Q1} and Z_{Q3} shown in Figure 3.39 is one that provides a relatively low value of $S <\sim 1.26$ in all the transmission line segments.

Figure 3.40 compares the three different designs in terms of their frequency response. As in Figure 3.35, we base this comparison on the behavior of the total standing-wave ratio $S(f)$ as a function of frequency. First we note that since the wavelength at 300 MHz is 1 m (assuming a transmission line with air as the material surrounding the conductors), all of the quarter-wave segments have physical lengths of 0.25 m each. To evaluate $S(f)$, we can start at the load end and transform impedances as we move toward the source end in accordance with

$$Z_i(f) = Z_{0i}\frac{Z_{i-1} + jZ_{0i}\tan[2\pi f(0.25\lambda_0/v_p)]}{Z_{0i} + jZ_{i-1}\tan[2\pi f(0.25\lambda_0/v_p)]}$$

where $\lambda_0 = v_p/f_0 = 1$ m, $v_p = c$, and Z_i and Z_{0i} are, respectively, the input impedance and the characteristic impedance of the ith quarter-wave transformer over which the impedance is being transformed, and Z_{i-1} is the input impedance of the $(i-1)$th quarter-wave transformer seen looking toward the load.

[26]There are various established methods for the design of multisection quarter-wavelength transformers. The approach described here is an ad-hoc one.



(a) (b)

FIGURE 3.41. Tapered impedance transformer and a mechanical analogy.
(a) A gradual taper provides wide bandwidth; many small reflections from a series of small incremental steps add with different phases to produce very small net total reflection.
(b) Mechanical analog of impedance matching to provide better power absorption at a termination.

It is clear from Figure 3.40 that the bandwidth performance improves with the use of multiple segments. The improvement between a single and double quarter-wave section is very significant, with a tolerable standing-wave ratio of $S <\sim 1.2$ being achieved over a much larger range in a double quarter-wave section. The triple transformer provides for $S <\sim 1.1$ over the entire range of 200 to 400 MHz. In practice, little improvement is obtained by cascading more than four sections.[27]

In the limit of adding more and more sections, we would approach an infinitely long, smooth, gradually tapered transmission line with virtually no reflections. This is illustrated in Figure 3.41a. In practice, it is usually sufficient to make the tapered section of length $\sim\lambda$ or more.

A Mechanical Analogy Impedance matching to achieve maximum energy transfer is an essential aspect of not just electrical but also other types of physical systems. One analogy is the transfer of energy in an elastic head-on collision between a mass M_1 moving with a speed v_M and a stationary mass M_2. In the absence of losses, and based on the conservation of momentum, and given the elastic nature of the collision, we know that if $M_2 = M_1$, then all the energy resident in M_1 is transferred to M_2 (i.e., M_1 stops and M_2 moves away at velocity v_M). If the $M_2 = M_1$

[27]S. Guccione, Nomograms for speed design of λ/4 transformers, *Microwaves,* August 1975.

condition is not met, only a fraction of the total energy is transferred to M_2, and M_1 either reflects and moves in the reverse direction (if $M_1 < M_2$) or continues its motion at reduced speed (if $M_1 > M_2$).

The transfer of energy between M_1 and M_2 can be improved by the insertion of a third mass between them, as shown in Figure 3.41b. The energy transfer is optimum if the third mass M is the geometric mean of the other two, that is, if $M = \sqrt{M_1 M_2}$. Further improvement in the energy transfer can be achieved by using several bodies with masses varying monotonically between M_1 and M_2.

3.6 THE SMITH CHART

Many transmission-line problems can be solved easily with graphical procedures, using the so-called *Smith chart.*[28] The Smith chart is also a useful tool for visualizing transmission-line matching and design problems. Many aspects of the voltage, current, and impedance patterns discussed in previous sections can also be interpreted and visualized by similar means using the Smith chart. One might think that graphical techniques are not as useful in this age of powerful computers and calculators, but it is interesting to note that some commonly used pieces of laboratory test equipment have displays that imitate the Smith chart, with the line impedance and standing-wave ratio results presented on such displays. In this section, we describe the Smith chart and provide examples of its use in understanding transmission-line problems.

3.6.1 Mapping of Complex Impedance to Complex Γ

The Smith chart is essentially a conveniently parameterized plot of the normalized line impedance $\bar{Z}(z)$ of a transmission line and the generalized voltage reflection coefficient $\Gamma(z)$ as a function of distance from the load. To understand the utility of the Smith chart, we need to understand the relationship between $\bar{Z}(z)$ and $\Gamma(z)$.

From [3.24], we can write the normalized line impedance as

$$\bar{Z}(z) = \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \qquad [3.51]$$

where we note that $\Gamma(z) = \rho e^{j(\psi + 2\beta z)}$ is the voltage reflection coefficient at any position z along the line. Denoting $\Gamma(z)$ simply as Γ, while keeping in mind that it is a function of z, we can rewrite [3.51] as

$$\boxed{\bar{Z} = \frac{1 + \Gamma}{1 - \Gamma}} \qquad [3.52]$$

[28] P. H. Smith, *Electronics,* January 1939. Also see J. E. Brittain, The Smith chart, *IEEE Spectrum,* 29(8), p. 65, August 1992.

where $\bar{Z} = \bar{R} + j\bar{X}$ and $\Gamma = u + jv$ are both complex numbers, so [3.52] represents a mapping between two complex numbers. Note that if the load is given, then we know Γ_L, and therefore Γ (and thus $\bar{Z} = \bar{R} + j\bar{X}$, from [3.52]), at any position at a distance z from the load. The Smith chart conveniently displays values of $\bar{Z}$ (or $\bar{R}$, $\bar{X}$) on the Γ (or u, v) plane for graphical calculation and visualization. From [3.52] we have

$$\bar{Z} = \bar{R} + j\bar{X} = \frac{1 + (u + jv)}{1 - (u + jv)} = \frac{[1 + (u + jv)][1 - (u - jv)]}{(1 - u)^2 + v^2} \qquad [3.53]$$

Equating real parts in [3.53] and rearranging, we have

$$\boxed{\left(u - \frac{\bar{R}}{1 + \bar{R}}\right)^2 + v^2 = \left(\frac{1}{1 + \bar{R}}\right)^2} \qquad [3.54]$$

which is the equation of a circle in the uv plane centered at $u = \bar{R}/(1 + \bar{R})$, $v = 0$ and having a radius of $1/(1 + \bar{R})$. Examples of such circles are shown in Figure 3.42a. Note that $\bar{R} = 1$ corresponds to a circle centered at $u = \frac{1}{2}$, $v = 0$, having a radius $\frac{1}{2}$, and passing through the origin in the uv plane.

Similarly, by equating the imaginary parts in [3.53] and rearranging, we find

$$\boxed{(u - 1)^2 + \left(v - \frac{1}{\bar{X}}\right)^2 = \frac{1}{\bar{X}^2}} \qquad [3.55]$$

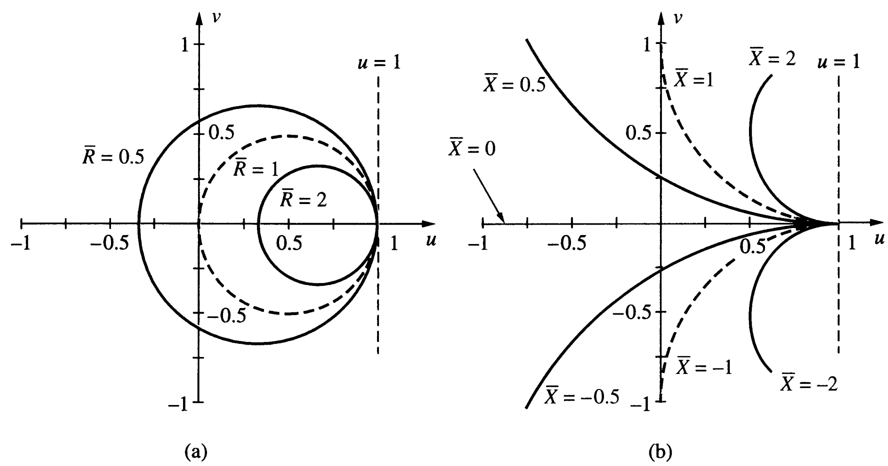


FIGURE 3.42. Contours of constant $\bar{R}$ or $\bar{X}$. (a)The circles in the uv plane are centered at $[\bar{R}(1 + \bar{R})^{-1}, 0]$, with radius $(1 + \bar{R})^{-1}$. Note the $\bar{R} = 1$ circle (dashed lines) passes through the origin (i.e., $u, v = 0$); this circle is centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$. (b) The circles in the uv plane are centered at $(1, \bar{X}^{-1})$, with radius $\bar{X}^{-1}$. Note that for $\bar{X} = \pm 1$ we have two circles (dashed lines) with unity radii and centered at $(1, \pm 1)$.

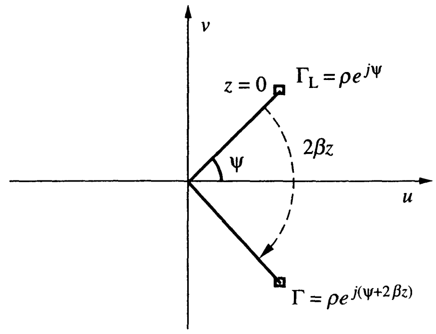


FIGURE 3.43. Complex reflection coefficient Γ. The complex number Γ is shown in the uv plane, together with its variation with z.

which is the equation for a circle in the uv plane centered at $u = 1$, $v = 1/\bar{X}$ and having a radius of $1/\bar{X}$. Examples of such circular segments are shown in Figure 3.42b. Note that $\bar{X} = \pm 1$ corresponds to a circle centered at $u = 1$, $v = \pm 1$; having a radius of 1; and tangent to the v axis at $v = \pm 1$.

The voltage reflection coefficient $\Gamma = u + jv$ is defined on the complex uv plane so that the locus of points of constant $|\Gamma| = |\Gamma_L| = \rho$ are circles centered at the origin. Once ρ is known (the value of which is set by Z_L and Z_0), motion along the line (i.e., variation of z) corresponds to motion around this Γ circle of fixed radius ρ. To see this, consider

$$\Gamma = \Gamma_L e^{j2\beta z} = \rho e^{j\psi} e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)} \qquad [3.56]$$

As illustrated in Figure 3.43, motion away from the load (i.e., decreasing z) corresponds to clockwise rotation of Γ around a circle in the uv plane. Since $\beta = 2\pi/\lambda$, a complete rotation of $2\beta z = -2\pi$ occurs when z decreases by $-\lambda/2$, which is why a complete cycle of line impedance (or admittance) is repeated every $\lambda/2$ length along the line. A circle of constant radius ρ (corresponding to a given load) also corresponds to a fixed-voltage standing-wave ratio S, since $S = (1 + \rho)(1 - \rho)^{-1}$.

Although all Γ values along the line terminated with Z_L lie on the circle of radius ρ, each value of Γ corresponds (through [3.52]) to a different value of $\bar{Z} = \bar{R} + j\bar{X}$, which is the normalized line impedance seen at that position. On a typical Smith chart, shown in Figure 3.44, the contours of constant $\bar{R}$ or $\bar{X}$ are plotted and labeled on the uv plane so that the line impedance at any position along the constant ρ (or S) circle can be easily read from the chart. A summary of various Smith chart contours and key points is provided in Figure 3.45.

As shown in Figures 3.45 and 3.46, the horizontal radius to the left of the chart center (i.e., the negative u axis) is the direction where $\Gamma = u + jv = \rho e^{(\psi + 2\beta z)} = \rho e^{-j\pi}$, or $\psi + 2\beta z = -(2m+1)\pi$ where $m = 0, 1, 2, \ldots$; in this case the magnitude of the line voltage is a minimum. Thus, every crossing of the negative u axis as one moves along the constant ρ circle corresponds to a minimum in the line voltage (and thus a maximum in the line current). The distance from the load to the first voltage minimum, namely $z_{\min}$, can thus be found simply by equating $2\beta z$ to the negative of the angle from the load point (i.e., $\Gamma = \Gamma_L$) to this axis measured in the clockwise direction (i.e., $2\beta z_{\min} = -(\pi + \psi)$, as was discussed in Section 3.3.1). Similarly,

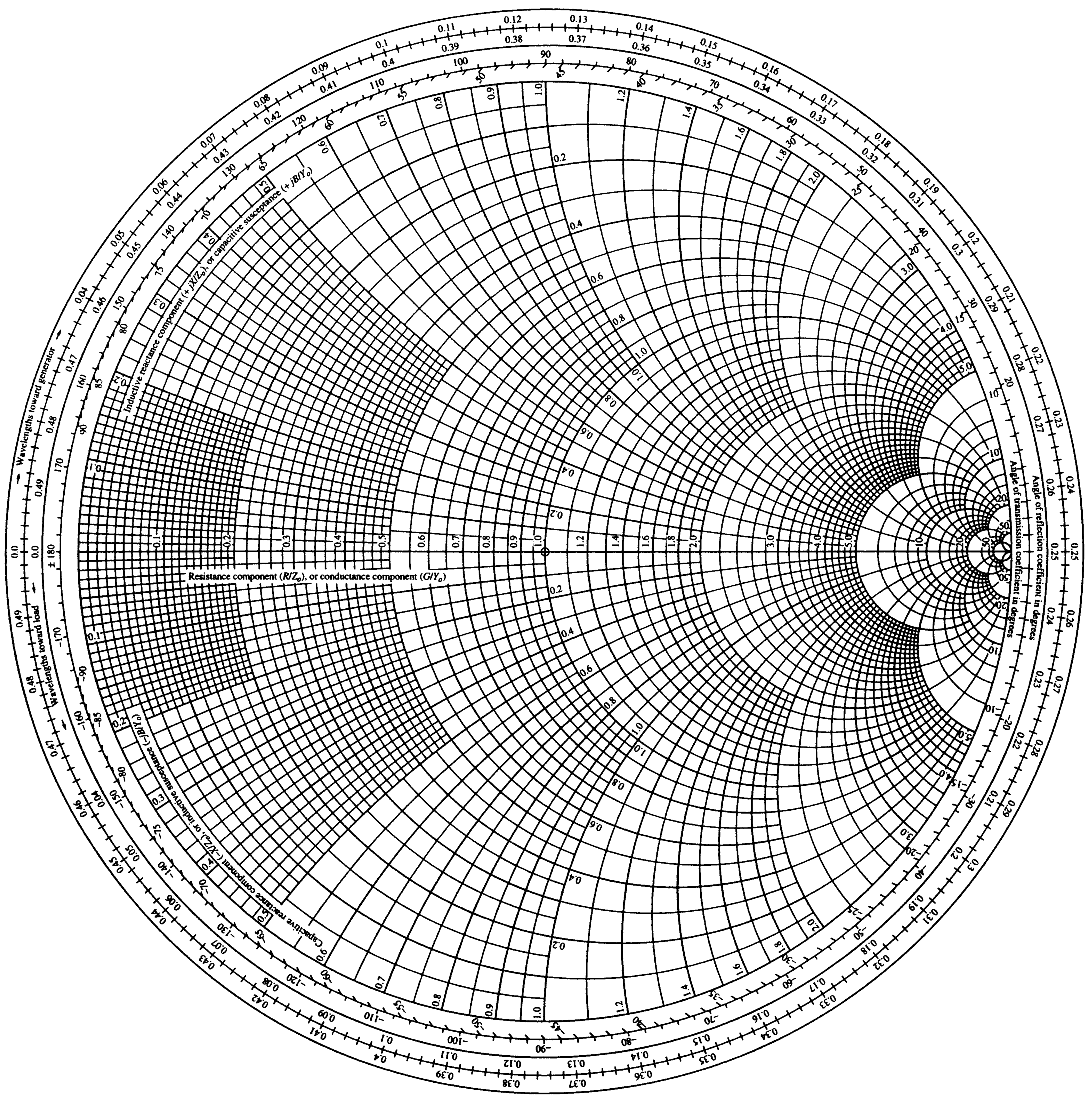


FIGURE 3.44. Smith transmission line chart.

the crossings of the horizontal radius to the right (i.e., the positive u axis) represent voltage maxima. Note that when $\psi + 2\beta z = -m2\pi$ $(m = 0, 1, 2, \ldots)$ where $|V(z)| = V_{\max}$, we have

$$\bar{Z} = \frac{1+\rho}{1-\rho} = \bar{R}_{\max} = S$$

Hence if S is known (instead of ρ), the S circle (which is the same as the ρ circle) can be constructed with its center at the chart center and passing through the same

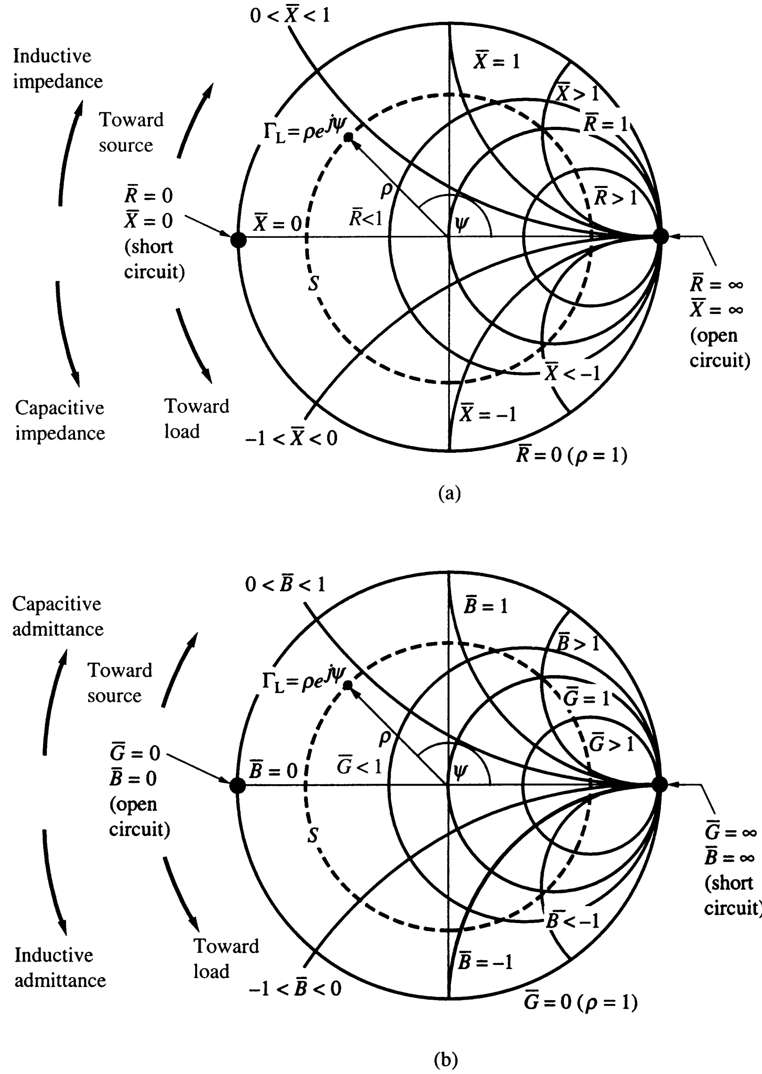


FIGURE 3.45. Summary of various Smith chart contours and locations.
(a) For use as an impedance chart. (b) For use as an admittance chart.

point on the positive u axis as the $\bar{R} = S$ circle. This circle is then the locus of all impedances appearing at various positions along the transmission line, normalized to the characteristic impedance Z_0 of the line.

Once we realize that the upper (lower) half of the impedance Smith chart shown in Figure 3.45a corresponds to inductive (capacitive) reactances, that the negative u axis corresponds to a voltage minimum, and that moving away from the load corresponds to moving clockwise along a constant S (or constant ρ) circle around the chart, the interpretation of voltage standing-wave patterns for inductive versus capacitive

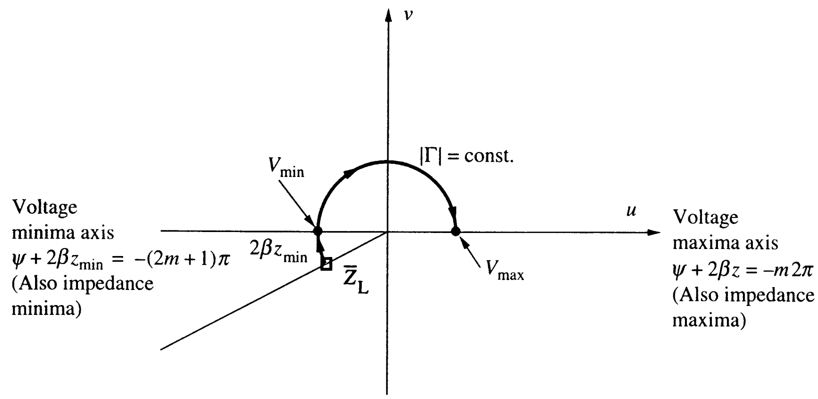


FIGURE 3.46. **Location of voltage minima and maxima on the Smith chart.**

loads (as depicted in Figure 3.17) becomes very clear. When we start anywhere in the upper half of the chart (i.e., inductive load) and move toward the source, we would encounter the voltage maxima (i.e., positive u axis) before the voltage minima so that the voltage magnitude would always first increase as we move away from an inductive load. The reverse would be true for a capacitive load. Many other aspects of the voltage, current, and impedance patterns discussed in previous sections can also be interpreted and visualized similarly using the Smith chart.

In cases for which it is more convenient to work with admittances than impedances, the Smith chart can be effectively used as an admittance chart. For this purpose, we note that since

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = -\frac{Y_L - Y_0}{Y_L + Y_0}$$

we have

$$\bar{Y}(z) = \bar{G} + j\bar{B} = \frac{1 - \Gamma(z)}{1 + \Gamma(z)} \qquad [3.57]$$

instead of [3.52]. In this case, the $\bar{R}$ and $\bar{X}$ circles can be treated, respectively, as $\bar{G}$ and $\bar{B}$ circles. However, note that the upper (lower) half of the chart now corresponds to capacitive (inductive) susceptances, which are represented by positive (negative) values of $\bar{B}$. A summary of various Smith chart contours and key points for its use as an admittance chart is provided in Figure 3.45b.

3.6.2 Examples of the Use of the Smith Chart

We now consider some applications of the Smith chart. The examples selected illustrate the relatively easy determination of line impedance for given resistive, reactive, and complex loads; determination of unknown load impedance based on measurements of standing-wave ratio and location of voltage minimum; single-stub impedance matching; and quarter-wave transformer matching.

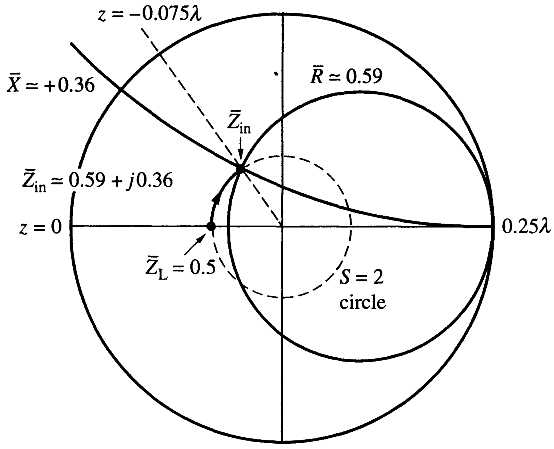


FIGURE 3.47. Graphical solution for Example 3-20.

Example 3-20: Input impedance with pure resistive load. Find the input impedance of a lossless transmission line with the following parameters: $Z_0 = 100\Omega$, $Z_L = 50 + j0\Omega$, line length $l = 86.25$ cm, wavelength $\lambda = 1.5$ m.

Solution: We first note that the electrical length of the line is 0.575λ. Since impedance goes through a full cycle every 0.5λ, the input impedance of this line would be identical to one with length $0.075\lambda = (0.575 - 0.5)\lambda$. The normalized load impedance is $\bar{Z}_L = Z_L/Z_0 = 0.5 + j0$. We enter the Smith chart at the point where the $\bar{R} = 0.5$ circle crosses the horizontal axis (note that the imaginary part of the load impedance is zero). We draw a circle passing through this point and centered at the origin; this is the constant ρ circle. We move along this circle by 0.075λ (from 0 mark to -0.075λ mark) away from the load (i.e., clockwise) and read the impedance as $\bar{Z}_{in}(z = -0.075\lambda) \simeq 0.59 + j0.36$. Since $\bar{Z}$ is the normalized impedance, the actual line impedance is $Z_{in} \simeq 59 + j36\Omega$. The details of the graphical solution are shown in Figure 3.47.

Example 3-21: Input impedance with a pure reactive load. Find the input impedance of a lossless transmission line given the following parameters: $Z_0 = 50\Omega$, $Z_L = 0 - j75\Omega$, line length $l = 1.202\lambda$ (i.e., $\lambda + 0.202\lambda$).

Solution: The normalized load impedance is $\bar{Z}_L = -j1.5$. For a purely reactive load, $R_L = 0$, so that $\rho = 1$ and $S = \infty$. We enter the chart at the point on the outermost circle (which corresponds to $\bar{R} = 0$), which is intersected by the $\bar{X} = -1.5$ circle. The length scale at that point reads $\sim 0.344\lambda$. The angle of Γ_L, or ψ, may be read to be $\sim -67°$. We now move along the outer circle (which in this case is our constant ρ circle) a distance of 0.202λ to the point $\sim 0.046\lambda$. The impedance at that point is $\bar{Z}_{in} \simeq j0.3$, or $Z_{in} \simeq j15\Omega$. The details of the graphical solution are given in Figure 3.48.

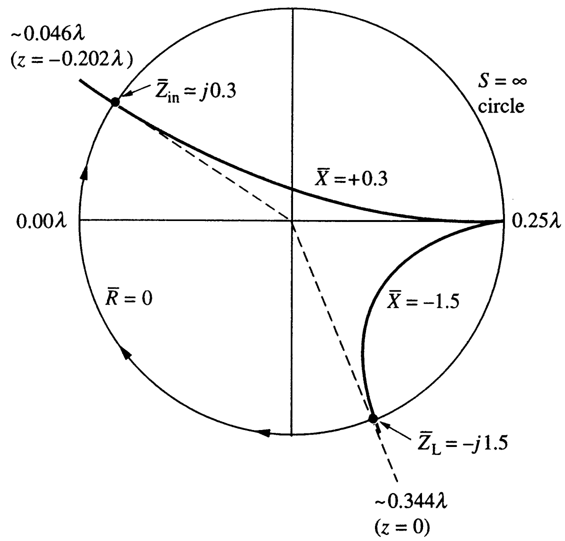


FIGURE 3.48. Graphical solution for Example 3-21.

Example 3-22: Input impedance with a complex load. Find the input impedance of a lossless transmission line given the following parameters: $Z_0 = 100\Omega$, $Z_L = 100 + j100\Omega$, line length $l = 0.676\lambda$ (i.e., $0.5\lambda + 0.176\lambda$).

> **Solution:** The normalized load impedance is $\bar{Z}_L = 1.0 + j1.0$. We find the point on the chart corresponding to $\bar{R} = 1.0$ and $\bar{X} = 1.0$ (i.e., the intersection point of the $\bar{R} = 1.0$ and $\bar{X} = 1.0$ circles) and draw a circle passing through this point and centered at the origin. The intersection of this constant ρ circle with the right horizontal axis is at $\bar{R} \simeq 2.62$, which is also the value of S. To find the input impedance, we simply move along this circle (clockwise from the load position) a distance of 0.176λ and read $\bar{Z}_{in} = 1.0 - j1.0$. The input impedance of the line is then $Z_{in} = 100 - j100\Omega$. The details of the graphical solution are given in Figure 3.49.

Example 3-23: Unknown load impedance. Find the normalized load impedance on a transmission line with the following measured parameters: standing-wave ratio $S = 3.6$ and first voltage minimum $z_{min} = -0.166\lambda$.

> **Solution:** We draw the constant ρ circle corresponding to $S = 3.6$ (i.e., intersecting the positive u axis at $\bar{R} = 3.6$). The point corresponding to z_{min} is that at which this circle crosses the negative u axis. We start at this point of intersection of the constant S circle with the left horizontal axis and move toward the load (i.e., counterclockwise) a distance of 0.166λ to find the normalized load impedance. This gives $\bar{Z}_L \simeq 0.89 - j1.3$. The details of the graphical solution are shown in Figure 3.50.

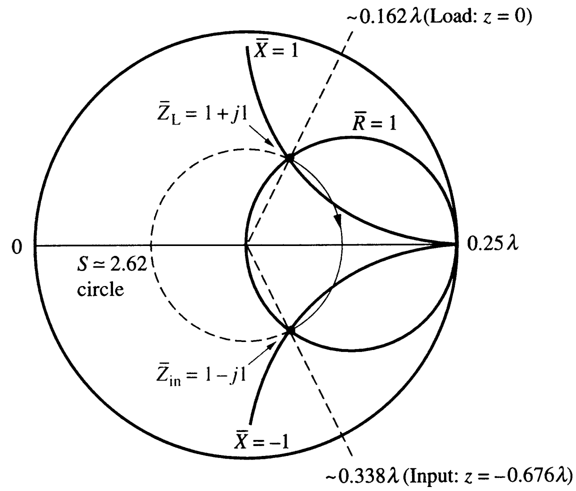


FIGURE 3.49. Graphical solution for Example 3-22.

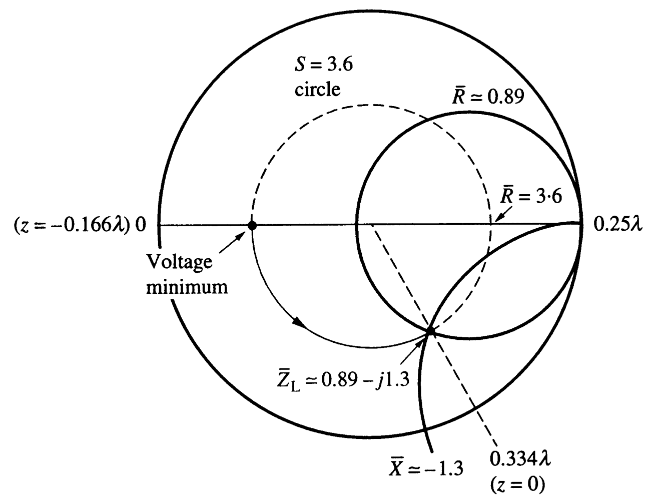


FIGURE 3.50. Graphical solution for Example 3-23.

Example 3-24: Single-stub impedance matching. Given a characteristic impedance $Z_0 = 80\Omega$ and a load impedance $Z_L = 160 - j80\Omega$, match the line to the given load by using a short-circuited shunt stub, as shown in Figure 3.32.

Solution: Refer to Figure 3.51 and to the discussion in Section 3.5 on impedance matching. Note that in view of the shunt connection of the stub, it is more convenient to deal with admittances. For this purpose, we use the Smith chart as an admittance chart. We require

$$\bar{Y}_1 = 1 - j\bar{B} \quad \text{and} \quad \bar{Y}_s = +j\bar{B}$$

where $\bar{Y}_l$ is the admittance seen looking toward the load at the position l where the stub is to be connected, and $\bar{Y}_s$ is the input admittance of the short-circuited stub of length l_s.

The normalized load impedance is $\bar{Z}_L = 2.0 - j1.0$. We enter the Smith chart at the point marked $\bar{Z}_L$ corresponding to the intersection of the resistance $\bar{R} = 2$ circle with the reactance $\bar{X} = -1.0$ circle, noting that negative reactances are in the lower half of the chart. The circle centered at the origin and passing through this point is our constant S (or constant ρ) circle along which the complex reflection coefficient Γ (or the line impedance) varies as we move away from the load. Noting that the Smith chart can be used equally for impedances and admittances, we choose to work with admittances in order to easily handle a parallel connected stub. The normalized load admittance can be found either directly (i.e., $\bar{Y}_L = (\bar{Z}_L)^{-1} = (2 - j)^{-1} = 0.4 + j0.2$) or by moving around the constant S circle by 180°, as shown in Figure 3.51. The normalized load admittance is thus $\bar{Y}_L = 0.4 + j0.2$. Note that when we change an impedance to an admittance on the Smith chart and work from there, all of the $\bar{R}$ and $\bar{X}$ circles can now be used as $\bar{G}$ and $\bar{B}$ circles.

We now move along the constant S circle up to its point of intersection with the conductance $\bar{G} = 1$ circle (P_1). The amount that we need to move determines the stub position at a distance l from the load. For this example, we find $l \simeq 0.126\lambda$. At the intersection point, the line impedance is $\bar{Y}_l \simeq 1.0 + j1.0$, so that for matching we must have $\bar{Y}_s = -j1.0$.

To determine the length of a short-circuited stub that would present an admittance of $-j1.0$, we start from the point on the chart corresponding to a short circuit (i.e., $\bar{Y} = \infty$ on the right horizontal axis, or the positive u axis). We move clockwise until we intersect the circle corresponding to a susceptance $\bar{B} = -1.0$. This determines the length of the stub to be $l_s = 0.125\lambda$.

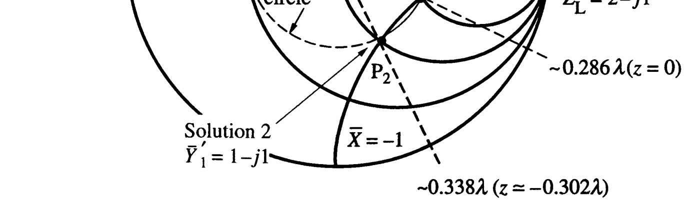


FIGURE 3.51. Graphical solution for Example 3-24.

Note that we might have taken the second intersection of the constant S circle with the $\bar{G} = 1$ circle (P_2), shown in Figure 3.51 as $\bar{Y}_1'$. This would have given $l \simeq 0.302\lambda$ and $\bar{Y}_1' \simeq 1.0 - j1.0$, requiring a stub impedance of $\bar{Y}_s = +j1.0$, which would be presented by a stub of length $l_s = 0.375\lambda$.

Example 3-25: Quarter-wave transformer matching. Given a transmission line with a characteristic impedance $Z_0 = 120\Omega$ and load impedance $Z_L = 72 + j96\Omega$, match the line to the given load using a quarter-wave transformer.

Solution: Refer to Figure 3.52, and the discussions in Section 3.5. We first move along the line a distance of l_1 such that the impedance Z_1 seen looking toward the line is purely resistive. Noting that the normalized load impedance is $\bar{Z}_L = (72 + j96)/120 = 0.6 + j0.8$, we enter the Smith chart at the point where the $\bar{R} = 0.6$ and $\bar{X} = 0.8$ circles intersect. We then move clockwise (away from load) along the constant S circle to its intersection with the horizontal axis, corresponding to the reactive part of the line impedance being zero. As shown in Figure 3.52, we need to move by 0.125λ, which means that the quarter-wave

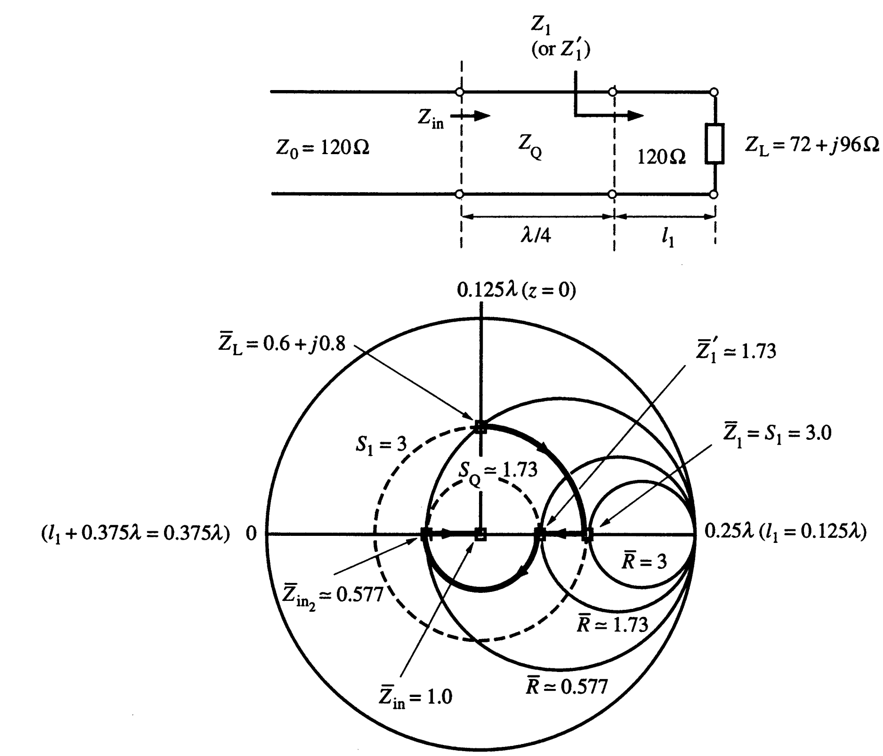


FIGURE 3.52. Quarter-wave transformer matching; graphical solution for Example 3-25. The path that we follow on the chart from the load impedance point to the origin (matching, i.e., $Z_{in} = Z_0$) is indicated by a thick dark line.

transformer can be placed at $l_1 = 0.125\lambda$. At that point, the line impedance normalized to a characteristic impedance of 120Ω is $\bar{Z}_1 = S_1 = 3.0$.

Noting that the quarter-wave transformation will occur on a line with characteristic impedance Z_Q, we now have to normalize the impedance to $Z_Q = \sqrt{Z_0 S_1 Z_0} \simeq 208\Omega$. The line impedance at $z = -0.125\lambda$, normalized to Z_Q, is then $\bar{Z}_1' = \bar{Z}_1(120/Z_Q) \simeq 1.73$. Following along the path on the Smith chart as shown in Figure 3.52, we thus move from $\bar{Z}_1$ to $\bar{Z}_1'$. The transformation along the quarter-wave segment is equivalent to a clockwise (away from load) rotation of 180°, which brings us to $\bar{Z}_{\text{in}_2} = 1/S \simeq 0.577$. Note that this rotation is along the circle of $S_Q \simeq 1.73$, which is the standing-wave pattern within the transformer. We now note that $\bar{Z}_{\text{in}_2} \simeq 0.577$ is an impedance normalized to Z_Q, whereas the characteristic impedance of the line to be matched is 120Ω. Re-normalizing back to 120Ω, we find the input impedance of the line looking into the quarter-wave segment to be $\bar{Z}_{\text{in}} \simeq \bar{Z}_1'(208/120) \simeq 1.0$; this brings us to the center of the chart, which represents a matched line.

3.6.3 Voltage and Current Magnitudes from the Smith Chart

Note that for a lossless transmission line, we have

$$V(z) = V^+ e^{-j\beta z}(1 + \underbrace{\rho e^{j\psi} e^{j2\beta z}}_{\Gamma})$$

so that the magnitude of the line voltage at any position z is given by

$$|V(z)| = |V^+||1 + \Gamma|$$

Since $|V^+|$ is just a scaling constant, the relative value of the line voltage can be obtained from the Smith chart by measuring the amplitude of the complex number $(1 + \Gamma)$. Note that at each position z along the line Γ is a new complex number as determined by a point on the uv plane (i.e., on the Smith chart). The length $|1 + \Gamma|$ can be determined graphically as shown in Figure 3.53. Note that as we move

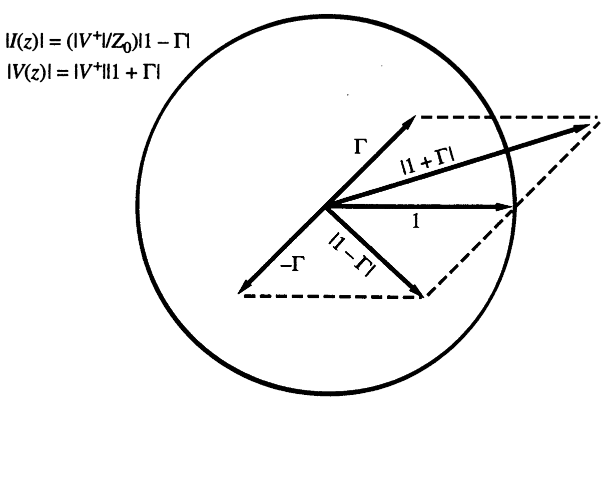


FIGURE 3.53. Line voltage and current from the Smith chart. We have assumed $V^+ = 1$.

along the line away from the load, the generalized reflection coefficient vector Γ rotates clockwise, and the voltage vector $(1 + |\Gamma|)$ rotates clockwise like a crank. By inspection of Figure 3.53, we can see that the maximum and minimum values of the voltage vector (i.e., maximum and minimum values of its length) are, respectively, $(1 + |\Gamma|)$ and $(1 - |\Gamma|)$, so that the standing-wave ratio is $S = (1 + |\Gamma|)/(1 - |\Gamma|) = (1 + \rho)/(1 - \rho)$, as was previously established in Section 3.3.1.

3.7 SELECTED APPLICATION EXAMPLES

In this section, we discuss two selected practical application topics, namely, (a) equivalent circuits for antennas or other loads with complex input impedance, and (b) matching networks.

3.7.1 Lumped Equivalent Circuits for Antennas and Other Loads with Complex Input Impedances

The determination of an unknown impedance from measurements of the standing-wave ratio and location of the voltage minimum, as discussed in Section 3.3, is a practical microwave method for measurement of unknown impedances that are difficult to calculate, such as the feed-point impedance of an antenna. Once the feed-point impedance is determined, an equivalent circuit model of the antenna can be constructed to determine the behavior of the antenna in various transmission line circuits. We illustrate the measurement of the unknown load impedance in Example 3-26 and the use of an equivalent circuit of a dipole antenna in Example 3-27.

Example 3-26: A meteor-damaged spaceship antenna. A spaceship has a microwave transmitter connected to an external antenna via a 50Ω coaxial line that is used to transmit radio waves at 1.5 GHz, as shown in Figure 3.54a. A small meteor strikes the external antenna and causes damage that results in a mismatch between the transmitter and the antenna. One of the crew members, an electrical engineer, decides to use a single short-circuited stub to correct the mismatch. However, the exact length of the coaxial line is unknown since a large portion of it goes through the hull of the spaceship. At first, the engineer measures the first voltage minimum position relative to the point where the coax exits on the inner surface of the hull to be at 8 cm and the next minimum position to be at 18 cm. She also measures the voltage standing-wave ratio on the line to be 4. Subsequently, a second crew member undertakes a spacewalk to short-circuit the external terminals of the damaged antenna, which causes the voltage minimum closest to the exit point of the cable on the hull (which, for a short-circuited antenna, is a deep voltage null) to move to 9.84 cm. Using these measurements, find (a) the feed-point impedance Z_L of the damaged antenna and (b) the appropriate length of the short-circuited 50Ω coaxial stub and the closest position (relative to the exit point on the ship's hull) at which it needs to be connected in parallel with the line to achieve matching.

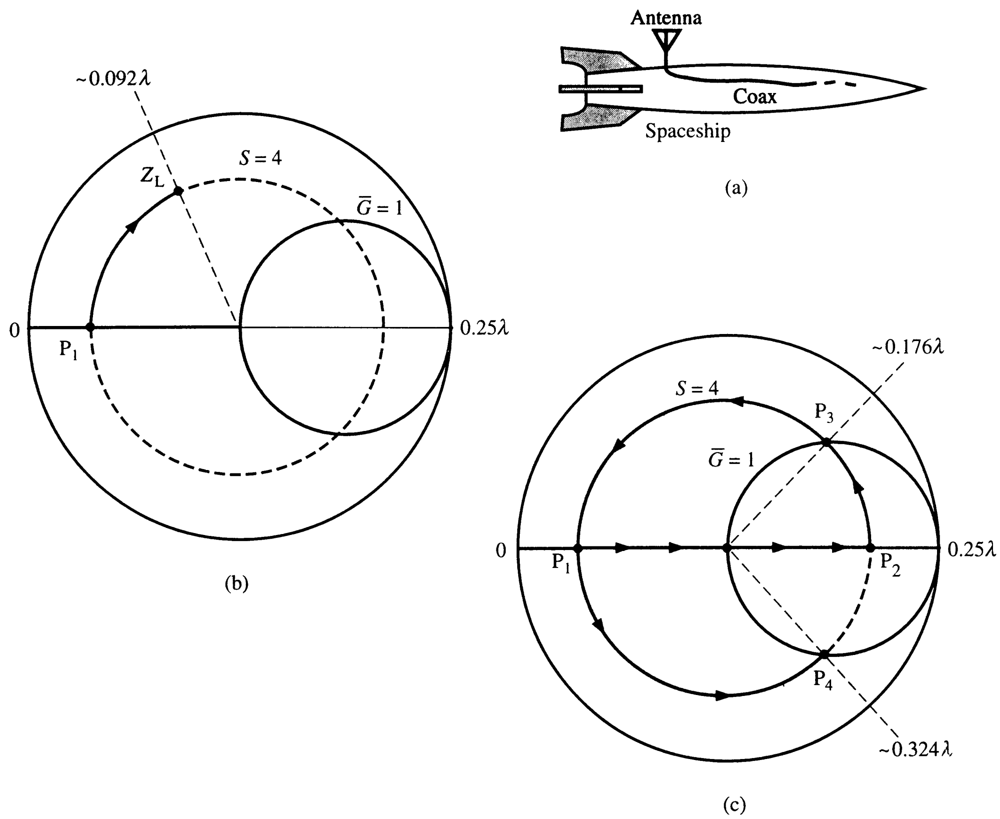


FIGURE 3.54. **A meteor-damaged antenna on a spaceship.** (a) A spaceship uses an externally mounted antenna at the end of a coaxial line. (b) Determination of the unknown terminal impedance of the damaged antenna from S and $z_{\min}$. (c) Determination of the length of a short-circuited stub.

Solution:

(a) Using the Smith chart, we draw the constant voltage standing-wave ratio ($S = 4$) circle, as shown in Figure 3.54b. The normalized line admittance seen looking toward the antenna from the appropriate location of the shunt stub, excluding the stub's admittance, lies on the $S = 4$ circle. Next, we determine the wavelength λ from the distance between successive voltage minima to be $\lambda/2 = 18 - 8 = 10$ cm $\rightarrow \lambda = 20$ cm. When the antenna is shorted all voltage minima along the coax shift toward the antenna by Δl, which can vary in the range $0 \leq \Delta l < \lambda/2$, depending on the feed-point impedance of the antenna Z_L (which lies on the $S = 4$ circle). If Z_L is capacitive, the shift is between 0 and $\lambda/4$, whereas if it is inductive, the shift is between $\lambda/4$ and $\lambda/2$. Since the shift is $18 - 9.84 = 8.16$ cm or 0.408λ in our case, we conclude that Z_L is inductive. Furthermore, if we move on the $S = 4$ circle, starting from point P_1 (i.e., minimum voltage point), a distance of $0.5\lambda - 0.408\lambda = 0.092\lambda$ in the clockwise direction, we reach the point that corresponds to $\bar{Z}_L$, which is read from the Smith chart as $\bar{Z}_L \simeq 0.35 + j0.6$. Thus, the unknown feed-point impedance of the damaged antenna is determined to be $Z_L = Z_0\bar{Z}_L \simeq 17.5 + j30\Omega$, since $Z_0 = 50\Omega$.

(b) We start with the normalized line impedance at 8 cm (point P_1 on the Smith chart) and convert it to line admittance at the same position (i.e., moving to point P_2) so that we can use the Smith chart as an admittance chart, since the matching network consists of a shunt stub (see Figure 3.54c). Next, to find the stub position closest to the exit point along the coaxial line, we move in the counterclockwise direction on the $S = 4$ circle (i.e., going toward the hull of the spaceship) until we find the intersection points with the $\bar{G} = 1$ circle corresponding to points on the inner side of the hull where the normalized line admittance is $\bar{Y}_l = 1 - j\bar{B}$. As seen in Figure 3.54c, there are two such points, marked P_3 and P_4. Since P_4 is a point outside the ship (i.e., the length between P_2 and P_4 is $\sim 0.426\lambda \simeq 8.52 > 8$ cm), P_3 corresponds to the point on the coaxial line closest to the exit point on the ship's hull. Therefore, at point P_3, we read $\bar{Y}_l$ from the chart as $\bar{Y}_l \simeq 1 + j1.5$ and its location relative to the exit point as $l \simeq 8 - 0.074\lambda = 8 - 1.48 = 6.52$ cm. The length of the shorted stub connected at $l \simeq 6.52$ cm can be found from $\bar{Y}_{sc} = -j\cot(\beta l_s) \simeq -j1.5$. From this we have $\beta l_s \simeq 0.588 \rightarrow l_s \simeq 0.0936\lambda \simeq 1.87$ cm.

In Example 3-27, we use a four-element equivalent lumped circuit to represent the feed-point impedance of a dipole antenna, as determined from measurements.[29] The circuit consists of a resistance, an inductance, and two capacitors, as shown in Figure 3.55. The values of these elements depend only on the physical dimensions of the antenna, not on the operation frequency. The empirical equations for these four elements are as follows:

$$C_1 = \frac{6.0337l}{\log(l/a) - 0.7245}\ \text{pF}$$

$$C_2 = l\left\{\frac{0.89075}{[\log(l/a)]^{0.8006} - 0.861} - 0.02541\right\}\ \text{pF}$$

$$L_a = 0.1l\{[1.4813\log(l/a)]^{1.012} - 0.6188\}\ \mu\text{H}$$

$$R_a = 0.41288[\log(l/a)]^2 + 7.40754(l/a)^{-0.02389} - 7.27408\ \text{k}\Omega$$

where l is the total length and a is the radius of the dipole, with both expressed in meters. These equations adequately represent the impedance of the dipole with length up to approximately 0.6 wavelength (i.e., $l \leq 0.6\lambda$).

[29]T. G. Tang, Q. M. Tieng, and M. W. Gunn, Equivalent circuit of a dipole antenna using frequency-independent lumped elements, *IEEE Transactions on Antennas and Propagation*, 41(1), pp. 100–103, 1993.

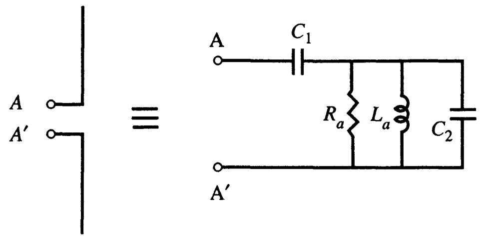


FIGURE 3.55. A dipole antenna and its equivalent lumped circuit.

Example 3-27: Dipole antenna fed by a coaxial line. Consider a dipole antenna of length 1.8 m and radius 2.64 mm connected at the end of an RG-213 (50Ω) coaxial line, as shown in Figure 3.56a. Calculate the standing-wave ratio S on the line at (a) 83.33 MHz and (b) 20.83 MHz.

Solution: For l = 1.8 m and a = 2.64 mm, we can use the empirical equations to calculate the values of the lumped elements in the equivalent circuit as

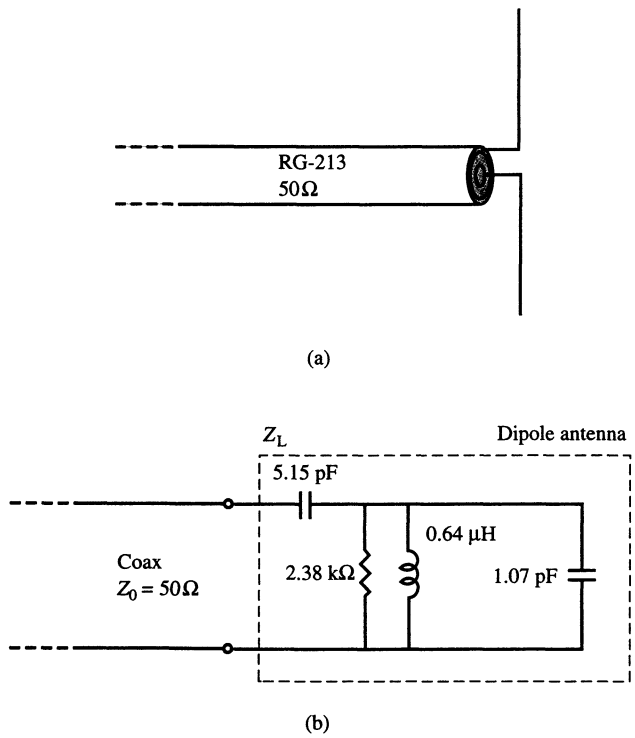


FIGURE 3.56. Dipole antenna fed by a coaxial line. (a) A coaxial line and the antenna wires connected to its inner and outer conductors. (b) Corresponding two-wire equivalent circuit.

$R_a \simeq 2.38$ kΩ, $L_a \simeq 0.6537$ μH, $C_1 \simeq 5.149$ pF, and $C_2 \simeq 1.067$ pF respectively. The corresponding two-wire transmission line circuit is shown in Figure 3.56b.

(a) At $f = 83.33$ MHz, the feed-point impedance of the dipole antenna can be calculated using the expression for the load impedance Z_L

$$Z_L = Z_{C_1} + \frac{1}{Y_{R_a} + Y_{L_a} + Y_{C_2}}$$

where $Z_{C_1} = 1/j\omega C_1 \simeq -j370.9\Omega$, $Y_{R_a} = G = 1/R_a \simeq 4.202 \times 10^{-4}$ S, $Y_{L_a} = 1/j\omega L_a \simeq -j2.906 \times 10^{-3}$ S, and $Y_{C_2} = j\omega C_2 \simeq j5.587 \times 10^{-4}$ S, respectively. Substituting these values in the Z_L expression, we find

$$Z_L \simeq -j370.9 + \frac{1}{4.202 \times 10^{-4} - j2.363 \times 10^{-3}}$$

$$\simeq -j370.9 + \frac{4.202 \times 10^{-4} + j2.363 \times 10^{-3}}{(4.202 \times 10^{-4})^2 + (2.363 \times 10^{-3})^2}$$

$$\simeq -j370.9 + 72.94 + j410.2 \simeq 72.94 + j39.28\Omega$$

The load reflection coefficient Γ_L is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{22.94 + j39.28}{122.9 + j39.28}$$

$$\frac{45.48e^{j59.71°}}{129.1e^{j17.72°}} \simeq 0.3524e^{j42°}$$

and the standing-wave ratio can be found as

$$S = \frac{1+\rho}{1-\rho} \simeq \frac{1 + 0.3524}{1 - 0.3524} \simeq 2.088$$

At 83.33 MHz, the 1.8-m length of the antenna is equal to one-half the wavelength ($\lambda = c/f = 3.6$ m). Such an antenna is an efficient radiator,[30] and we see here that it can be fed by a standard 50Ω coaxial line at a reasonable standing-wave ratio of $S \simeq 2$.

(b) Similarly, at $f = 20.83$ MHz, we have

$$Z_L \simeq -j1484 + \frac{1}{4.202 \times 10^{-4} - j1.169 \times 10^{-2} + j1.396 \times 10^{-4}}$$

$$\simeq -j1484 + \frac{4.202 \times 10^{-4} + j1.155 \times 10^{-2}}{(4.202 \times 10^{-4})^2 + (1.155 \times 10^{-2})^2}$$

$$\simeq 3.146 - j1397\Omega$$

[30]See, for example, Section 14.06 of E. C. Jordan and K. Balmain, *Electromagnetic Waves and Radiating Systems,* Prentice Hall, 1968.

The load-reflection coefficient is then given by

$$\Gamma_L \simeq \frac{3.146 - j1397.4 - 50}{3.146 - j1397.4 + 50} \simeq 0.999839e^{-j4.098°}$$

and the standing-wave ratio is

$$S = \frac{1+\rho}{1-\rho} \simeq \frac{1+0.999839}{1-0.999839} \simeq 12430 \ !$$

The high value of the capacitive reactance of the feed-point impedance at 20.83 MHz ($\lambda \simeq 14.4$ m) is partly due to the fact that the dipole antenna is electrically short ($l/\lambda \simeq 0.125$) and is therefore not an efficient radiator.

3.7.2 Transmission Line Matching Networks

As discussed in Section 3.5, transmission lines are often used in matching networks. In this section, we provide two specific examples involving the use of microstrip lines to realize matching networks for a microwave amplifier (Example 3-28) and for a cellular phone base station (Example 3-29). Microstrip lines, easily fabricated using printed-circuit techniques, are widely used to match impedances in microwave transistor amplifiers. Most microwave transistor amplifiers can be classified as either low-noise amplifiers or power amplifiers. In both cases, the circuit design involves the selection of the appropriate transistor and the optimum design of the matching networks around it to satisfy the design considerations such as power gain, low noise, and bandwidth. Microstrip transmission line segments can be used as open- or short-circuited stubs. In fact, a microstrip line together with a short- or open-circuited shunt stub can transform a 50Ω resistor into any value of impedance.[31] Two such matching networks are illustrated in Example 3-28. In most applications, it is desirable to achieve matching over a broad range of frequencies. As discussed in Section 3.5, most of the simple matching techniques (e.g., quarter-wave transformation, single-stub matching, etc.) do not in general have good frequency response. In Example 3-29, we illustrate a simple and commonly used method called shunt compensation, which can greatly improve the frequency response of a matching network.

Example 3-28: Input matching network of a low-noise microwave amplifier. Design two separate microstrip-line matching networks, as shown in Figure 3.57b,c, for the input stage of a low-noise microwave transistor amplifier to transform a 50Ω load impedance to an input admittance $Y_{in} = G_{in} + jB_{in} =$

[31] See Section 2.5 of G. Gonzalez, *Microwave Transistor Amplifiers, Analysis and Design,* 2nd ed., Prentice Hall, 1997.

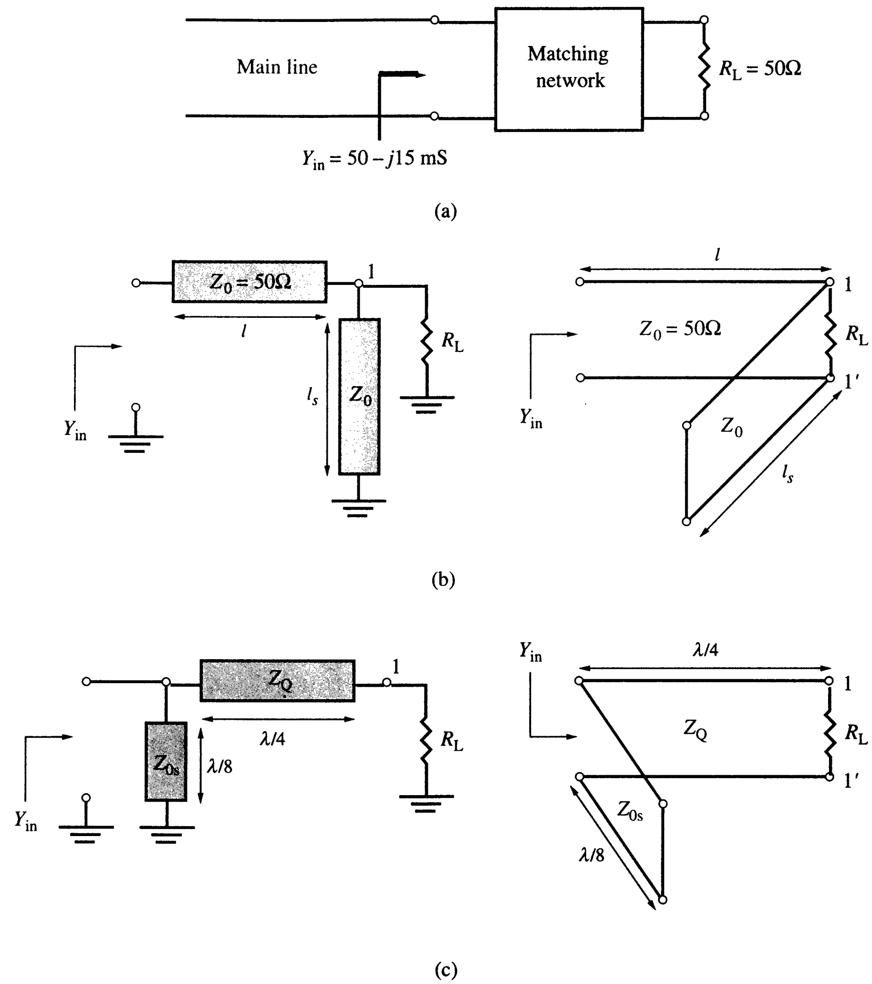


FIGURE 3.57. **Two different matching networks for a microwave amplifier.** (a) The purpose of the matching network is to transform the $R_L = 50\Omega$ load resistance to an input admittance of $Y_{in} = 50 - j15$ mS. (b) Matching network using a shunt short-circuited stub. (c) Matching network using quarter-wave transformation. The right hand panels in (b) and (c) show the two-wire equivalents of the microstrip-line circuits.

$50 - j15$ mS as required to achieve minimum noise performance.[32] (a) The first matching network consists of a short-circuited shunt microstrip stub connected in parallel with R_L followed by a microstrip line of length l, as shown in Figure 3.57b. The characteristic impedance of each of the two microstrip lines is 50Ω. Find the length of each line in terms of wavelength. (b) The second matching network consists

[32]See pp. 316–321 of T. Edwards, *Foundations for Microstrip Circuit Design,* 2nd ed., John Wiley and Sons, 1992.

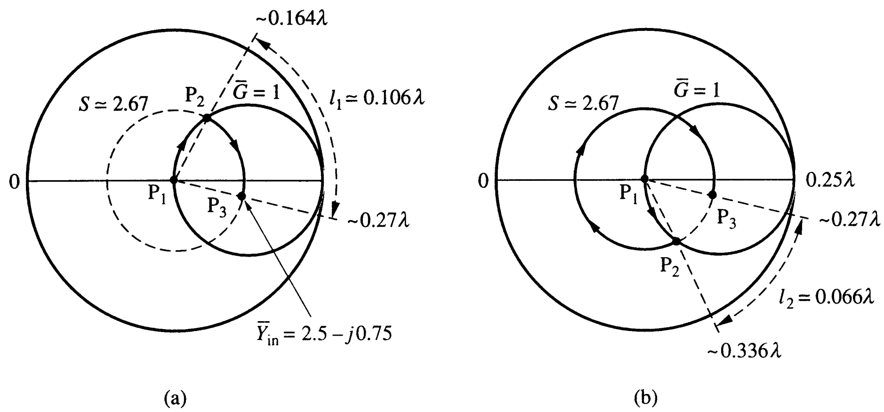


FIGURE 3.58. Smith chart solutions for Example 3-28, part (a). (a) First of two alternative solutions: $l_1 \simeq 0.106\lambda$, $l_{s1} \simeq 0.38\lambda$. (b) Second alternative solution: $l_2 \simeq 0.434\lambda$, $l_{s2} \simeq 0.12\lambda$.

of a quarter-wavelength-long microstrip line of characteristic impedance Z_Q terminated by R_L at one end and having an eighth-wavelength-long short-circuited microstrip stub with characteristic impedance Z_{0s} connected in parallel with it at the other end, as shown in Figure 3.57c. Find Z_Q and Z_{0s}.

Solution:

(a) The matching network shown in Figure 3.57b is somewhat similar to the single-stub matching networks discussed in Section 3.5, except that the short-circuited shunt stub is located at the position of the load. In view of the shunt connections, it is more convenient to use the Smith chart as an admittance chart. Our desired goal is to achieve an input admittance of $Y_{in} = 50 - j15$ mS, which corresponds to a normalized admittance of $\bar{Y}_{in} = Y_{in}/Y_0 = 2.5 - j0.75$, since $Y_0 = (Z_0)^{-1} = (50)^{-1} = 0.02$ S. This normalized admittance point is marked as point P_3 in Figure 3.58a,b. The constant S circle passing through point P_3 is also shown in Figure 3.58a,b; this circle corresponds to $S \simeq 2.76$, as can be determined by reading off the S value from the chart. The standing-wave ratio of $S \simeq 2.76$ can also be calculated by noting that the reflection coefficient on the line with Y_0 terminated in Y_{in} is $\Gamma_{in} = \rho e^{j\psi} = (Y_0 - Y_{in})/(Y_0 + Y_{in})$ and that $S = (1 + \rho)/(1 - \rho)$. However, in the context of the graphical solution using the Smith chart, we do not need to know the numerical value of S explicitly.

The goal of the design is to determine the line length l and the stub length l_s so that we depart from point P_1 and arrive at point P_3 on the Smith chart. We enter the chart at the center point P_1 (i.e., $\bar{Z}_L = Z_L/Z_0 = 1$ or $\bar{Y}_L = Y_L/Y_0 = 1$), as shown in Figure 3.58a. Next, we determine the normalized admittance of the short-circuited shunt stub $\bar{Y}_{sc}$ such that the addition of $\bar{Y}_{sc} = j\bar{B}_{sc}$ to $\bar{Y}_L = 1$ brings us to the point P_2 on the constant S circle, where the normalized admittance is $\bar{Y}_{P_2} = 1 + j\bar{B}_{sc}$. There are two different ways in which this can be done, as illustrated in Figures 3.58a and 3.58b, respectively. In Figure 3.58a, the length of the shunt stub is designed such that it provides a capacitive admittance (i.e., $\bar{B}_{sc} > 0$), so that we move in the clockwise direction on the constant $\bar{G} = 1$ circle, from P_1 to P_2, where $\bar{Y}_{P_2} \simeq 1 + j1.06$. The minimum 50Ω short-circuited stub length that yields a normalized admittance of $\bar{Y}_{sc} = j1.06$ is determined as $\bar{Y}_{sc} = -j\cot(\beta l_s) \simeq j1.06$, so that we have $l_{s1} \simeq 0.38\lambda$. To find the line length l, we move from P_2 to P_3 around the constant S circle in the clockwise direction, yielding a length $l_1 \simeq 0.106\lambda$, as shown in Figure 3.58a. Similarly, in Figure 3.58b, the length of the shunt stub is chosen such that it provides an inductive admittance (i.e., $\bar{B}_{sc} < 0$), so that we move from P_1 to P_2 in the counter clockwise direction along the $\bar{G} = 1$ circle, where $\bar{Y}_{P_2} \simeq 1 - j1.06$. Once again, $\bar{Y}_{P_2}$ lies on the same constant S circle passing through $\bar{Y}_{in}$. The corresponding minimum short-circuited stub length is $l_{s2} \simeq 0.12\lambda$. Furthermore, to move from P_2 to P_3 along the S circle in the clockwise direction requires a minimum line length $l_2 \simeq 0.434\lambda$, as shown in Figure 3.58b.

For both matching circuits, if an open-circuited shunt stub were used instead of the short-circuited one, the only change in the design would have been the length of the open-circuited stub, the minimum value of which can be obtained by adding $\pm\lambda/4$ to the minimum length of the short-circuited stub, depending on whether the minimum length of the short-circuited stub is less than or greater than $\lambda/4$. In addition, although the two designs just discussed both involve a load impedance of 50Ω, this technique can also be applied to an arbitrary complex load impedance, since all of the calculations on the Smith chart were carried out using normalized admittances.

(b) For the second matching network, we can rely on quarter-wave transformer techniques and do not need to use the Smith chart. Noting that the real part (i.e., conductance) of the required Y_{in} is $G_{in} = 50$ mS, the input admittance of the quarter-wave transformer, not including the stub, is given by

$$Y_1 = \frac{Y_Q^2}{G_L} = G_{in} \quad \rightarrow \quad Y_Q \simeq 31.6 \text{ mS}$$

or $Z_Q \simeq 31.6\Omega$. The imaginary part (i.e., susceptance) of the required Y_{in}, that is, $B_{in} = -15$ mS, can be provided by the short-circuited shunt stub of length $\lambda/8$ using

$$Y_{sc} = -jY_{0s}\cot(\beta l_s) = jB_{in} = -j15 \text{ mS}$$

Since $\beta l_s = (2\pi/\lambda)(\lambda/8) = \pi/4$ and $\cot(\beta l_s) = 1$, we have $Y_{0s} = 15$ mS or $Z_{0s} \simeq 66.7\Omega$. Note that both of the characteristic impedances obtained (i.e., $Z_Q \simeq 31.6\Omega$ and $Z_{0s} \simeq 66.7\Omega$) are realizable using microstrips, since microstrip lines with characteristic impedance values ranging between 10Ω and 110Ω are easy to manufacture in practice.[33]

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[33] T. Edwards, *Foundations for Microstrip Circuit Design,* 2nd ed., John Wiley and Sons, 1992.

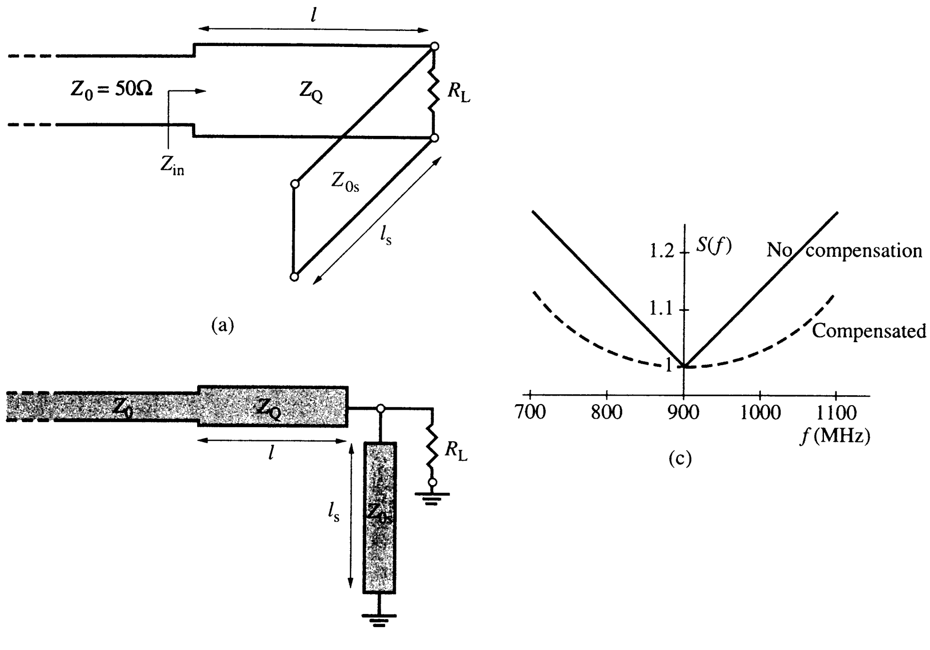


FIGURE 3.59. Quarter-wave matching with shunt compensation. (a) Transmission line circuit diagram. (b) Practical microstrip implementation. (c) Standing-wave ratio versus frequency for a simple quarter-wave transformer (solid line) and a quarter-wave transformer with shunt compensation (dashed line).

Example 3-29: Quarter-wave matching with shunt compensation. A quarter-wave matching network is to be designed for a cellular phone base station power amplifier operating at 900 MHz. The matching network is to match a resistive load of $R_L = 25\Omega$ to a transmission line with characteristic impedance $Z_0 = 50\Omega$, providing a standing-wave ratio $S < 1.1$ over the frequency range 800 to 1000 MHz. (a) Design a quarter-wave transformer to operate at 900 MHz and assess its standing-wave ratio across the specified frequency range. (b) Consider possible improvement of the bandwidth performance by the addition of a short-circuited quarter-wavelength-long (at 900 MHz) stub in shunt with the load, as shown in Figure 3.59a.

Solution:

(a) First consider the design of a quarter-wave transformer without any short-circuited compensation shunt stub. In order to match the 25Ω load to the 50Ω line, the quarter-wave transformer should have a characteristic impedance of $Z_Q = \sqrt{Z_0 R_L} = \sqrt{(50)(25)} \simeq 35.4\Omega$. Assuming an air-filled transmission line, the phase velocity is equal to the speed of light in free space, and the wavelength at the design frequency of 900 MHz is $\lambda_0 = v_p/f_0 = c/f_0 \simeq 3 \times 10^8/(900 \times 10^6) = \frac{1}{3}$ m. Thus, the length of the quarter-wave section should be $l = (\lambda_0/4) = \frac{1}{12}$ m. At the design frequency, the input impedance Z_{in} of the matching network designed is Z_0. At other frequencies, the quarter-wave transformer of length $l = \frac{1}{12}$ m and characteristic impedance Z_Q transforms the load R_L to its input as

$$Z_{\text{in}} = Z_Q \frac{R_L + jZ_Q \tan[2\pi(f/c)(\frac{1}{12})]}{Z_Q + jR_L \tan[2\pi(f/c)(\frac{1}{12})]}$$

The reflection coefficient, as observed at the junction between the main line and the matching network, and the standing-wave ratio on the main line are then

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \rho e^{j\psi} \quad \rightarrow \quad S = \frac{1+\rho}{1-\rho}$$

The frequency dependence of the standing-wave ratio S is shown as a solid line in Figure 3.59c. We see that S varies sensitively with frequency and in fact exceeds the design criteria (i.e., $S > 1.1$) near the edges of the 800–1000 MHz band of interest.

(b) We now consider the use of a short-circuited quarter-wave line in shunt with the load, as shown in Figure 3.59a, to achieve better frequency response. If the length l_s of the short-circuited stub is chosen to be equal to a quarter wavelength at the design frequency of 900 MHz, that is, if $l_s = \frac{1}{12}$ m, the input impedance of the stub as viewed from the load terminals is an open circuit, so that the presence of the stub has no effect on the system performance at 900 MHz. However, at other frequencies, the input impedance of the short-circuited stub is

$$Z_{sc} = jZ_{0s} \tan[2\pi(f/c)(\tfrac{1}{12})]$$

where Z_{0s} is the characteristic impedance of the short-circuited stub. The stub impedance Z_{sc} appears in parallel with the load resistance R_L, so that the load impedance is

$$Z_L = \frac{Z_{sc}R_L}{Z_{sc} + R_L} = \frac{(jZ_{0s}\tan[2\pi(f/c)(\frac{1}{12})])R_L}{jZ_{0s}\tan[2\pi(f/c)(\frac{1}{12})] + R_L}$$

which is transformed to the input of the quarter-wave transformer as

$$Z_{\text{in}}^c = Z_Q \frac{Z_L + jZ_Q \tan[2\pi(f/c)(\frac{1}{12})]}{Z_Q + jZ_L \tan[2\pi(f/c)(\frac{1}{12})]}$$

where the superscript "c" indicates that this input impedance is for the compensated case, and thus it differs from the uncompensated Z_{in} found in part (a). The reflection coefficient and the standing-wave ratio on the main line for the compensated case are then

$$\Gamma^c = \frac{Z_{\text{in}}^c - Z_0}{Z_{\text{in}}^c + Z_0} = \rho^c e^{j\psi^c} \quad \rightarrow \quad S^c = \frac{1+\rho^c}{1-\rho^c}$$

The frequency dependence of the compensated standing-wave ratio S^c is shown as the dashed line in Figure 3.59c, for the case when $Z_{0s} = Z_0 = 50\Omega$. We see that S^c is substantially lower than S over the entire frequency range of interest, so that the compensation has significantly improved the frequency response. The design criteria of $S^c < 1.1$ is easily met over the range 800 to 1000 MHz. In general, both the length of the short-circuited stub and its characteristic impedance Z_{0s} can be optimally chosen to achieve a desired frequency response.

In practice, similar improvement in frequency response can be achieved by using a half-wavelength-long open-circuited stub or by using simple parallel lumped LC networks.

3.8 SINUSOIDAL STEADY-STATE BEHAVIOR OF LOSSY LINES

Our analyses so far have been based on the assumption that there is no power loss in the transmission line itself. A consequence of this assumption was the rather nonphysical result that the line current at a distance of a quarter wavelength from a short circuit is zero, and that the input impedance of a quarter-wavelength short-circuited line is infinite. In reality, every line consumes some power, partly because of the resistive losses (R) in the conductors and partly because of leakage losses (G) through the insulating medium surrounding the conductors. For lines with small losses, the effects of the losses on characteristic impedance, line voltage, line current, and input impedance are usually negligible, so that the lossless analysis is valid. However, in other cases, the losses and the resultant attenuation of signals cannot be ignored. The typical conditions under which losses cannot be neglected are (1) transmission of signals over long distances, (2) high-frequency applications, since resistive losses increase with frequency, and (3) use of quarter-wavelength or half-wavelength-long transmission line segments as circuit elements, when neglecting losses leads to nonphysical results such as zero current and/or infinite input impedance. In the third case, losses become the determining factor on resonant lines when the electrical quantities of interest tend toward zero or infinity. Thus, input impedance of a quarter-wavelength-long open-circuited transmission line is in fact not zero, but is a small nonzero value as determined by the losses. Similarly, the input impedance of a quarter-wavelength-long short-circuited transmission line is not infinite, but a large finite value determined by the losses.

The sinusoidal steady-state behavior of lossy lines can be formulated in a manner quite similar to that of lossless lines. We can start with the most general form of the transmission line equations that were obtained in Section 2.2:

$$\frac{\partial \mathcal{V}(z,t)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right)\mathcal{I}(z,t) \quad [3.58a]$$

$$\frac{\partial \mathcal{I}(z,t)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right)\mathcal{V}(z,t) \quad [3.58b]$$

Under sinusoidal steady-state conditions, it is more convenient to work with the voltage and current equations written in terms of the phasor quantities $V(z)$ and $I(z)$, such that $\mathcal{V}(z,t) = \Re e\{V(z)e^{j\omega t}\}$ and $\mathcal{I}(z,t) = \Re e\{I(z)e^{j\omega t}\}$. We have

$$-\frac{dV(z)}{dz} = (R + j\omega L)I(z) \quad [3.59a]$$

$$-\frac{dI(z)}{dz} = (G + j\omega C)V(z) \quad [3.59b]$$

Taking the derivative of [3.59*a*] and substituting from [3.59*b*], we find

$$\frac{d^2V(z)}{dz^2} = (RG)V(z) + (LG + RC)(j\omega)V(z) + (j\omega)^2(LC)V(z)$$

$$= (R + j\omega L)(G + j\omega C)V(z)$$

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \quad [3.60]$$

where

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \quad [3.61]$$

is the propagation constant. Note that the propagation constant γ is in general a complex number, and its real and imaginary parts, α and β, are known, respectively, as the *attenuation constant* and the *phase constant.*[34] For any given values of R, L, G, and C and the frequency f, the values of α and β can be directly calculated from [3.61].

Equation [3.60] is a second-order differential equation similar to the one we encountered for the lossless case. Its general solution is

$$V(z) = V^+ e^{-\gamma z} + V^- e^{+\gamma z} \quad [3.62]$$

where V^+ and V^- are complex constants to be determined by the boundary conditions.

The current phasor $I(z)$ can be determined by simply substituting [3.62] into [3.59*a*]. Thus, we have

[34]Note that β for the lossy case is not equal to that for the lossless case, which was defined earlier as $\beta = \omega\sqrt{LC}$. In the general lossy case, β is a function of R, L, G, and C and has a more complex dependence on the frequency f.

$$(R + j\omega L)I(z) = -\frac{dV(z)}{dz}$$

$$I(z) = \frac{-1}{R + j\omega L}\frac{dV(z)}{dz} = +\frac{\gamma}{R + j\omega L}(V^+ e^{-\gamma z} - V^- e^{+\gamma z})$$

$$I(z) = \sqrt{\frac{G + j\omega C}{R + j\omega L}}(V^+ e^{-\gamma z} - V^- e^{+\gamma z}) \qquad [3.63]$$

$$= \frac{1}{Z_0}(V^+ e^{-\gamma z} - V^- e^{+\gamma z})$$

where we have defined Z_0 as the *characteristic impedance,* namely,

$$Z_0 \equiv \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_0|e^{j\phi_z}$$

Compared to Z_0 for the lossless case, we see that for the lossy case Z_0 is in general a complex number. Note once again that Z_0 depends on the physical line constants R, L, G, and C (which in turn depend on the physical makeup and dimensions of the line as well as the properties of the surrounding media) but now also on the frequency of operation $\omega = 2\pi f$. For future reference, the general solutions of the transmission line equations for the voltage and current phasors are

$$V(z) = V^+ e^{-\gamma z} + V^- e^{+\gamma z} \qquad [3.64a]$$

$$I(z) = \frac{1}{Z_0}[V^+ e^{-\gamma z} - V^- e^{+\gamma z}] \qquad [3.64b]$$

3.8.1 Infinitely Long or Matched Line

To understand the behavior of the time harmonic solutions for a lossy transmission line, we first consider the case of an infinitely long or a matched-terminated line. By analogy with the lossless case, we can see that the second terms in [3.64*a*] and [3.64*b*], those multiplied by the constant V^-, are zero in these cases, since no reflected wave exists. Accordingly, the voltage and current phasors are

$$V(z) = V^+ e^{-\gamma z}; \quad I(z) = \frac{1}{Z_0}V^+ e^{-\gamma z} \quad \rightarrow \quad \frac{V(z)}{I(z)} = Z_0$$

Note that everywhere on the line the ratio of the voltage to current phasors is the characteristic impedance Z_0, once again underscoring the physical meaning of the characteristic impedance.

It is instructive to write the voltage and current phasors explicitly in terms of the real and imaginary parts of γ. In other words, we have

$$V(z) = V^+ e^{-\alpha z}e^{-j\beta z} \qquad [3.65a]$$

$$I(z) = \frac{V^+}{Z_0}e^{-\alpha z}e^{-j\beta z} \qquad [3.65b]$$

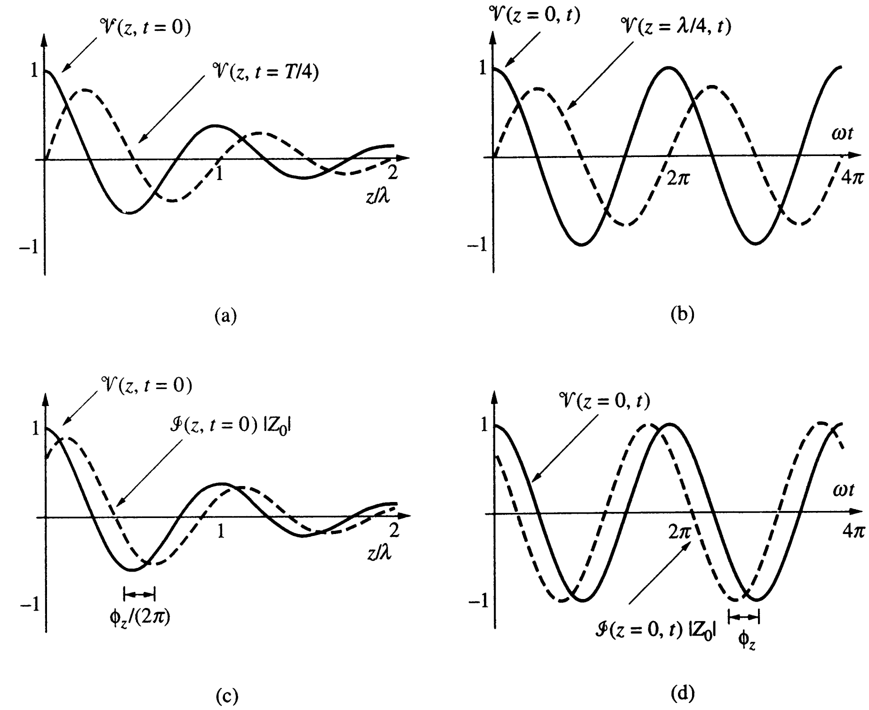


FIGURE 3.60. Voltage and current on a matched lossy line. (a) $\mathcal{V}(z, t) = V^+e^{-\alpha z}\cos(\omega t - \beta z)$ versus z/λ for $t = 0$ and for $t = T_p/4$, where $\beta = 2\pi/\lambda$ and $T_p = 2\pi/\omega$. (b) $\mathcal{V}(z, t)$ vs. ωt for $z = 0$ and $z = \lambda/4$. The attenuation constant was taken to be $\alpha = 1$ neper/λ. A comparison of voltage $\mathcal{V}(z, t)$ and current $\mathcal{I}(z, t)$ (c) at time $t = 0$ as a function of space and (d) at $z = 0$ as a function of time, for an assumed case where the R,L, G, and C values are such that $\phi_z = -\pi/4$. Note that we have assumed $V^+ = 1$.

Using [3.65], we can in turn obtain the space-time voltage and current functions as

$$\mathcal{V}(z, t) = \Re e\{V^+e^{-\alpha z}e^{-j\beta z}\} = V^+e^{-\alpha z}\cos(\omega t - \beta z) \qquad [3.66a]$$

$$\mathcal{I}(z, t) = \Re e\left\{\frac{V^+}{Z_0}e^{-\alpha z}e^{-j\beta z}\right\} = \frac{V^+}{|Z_0|}e^{-\alpha z}\cos(\omega t - \beta z - \phi_z) \qquad [3.66b]$$

where we have assumed V^+ to be real. The solutions for a lossy line are propagating waves with amplitudes exponentially decaying with increased distance. For physically realizable solutions, we must have $\alpha > 0$. Thus, in evaluating the propagation constant γ, the sign of the square root in [3.61] must be taken to be that which gives a value of $\alpha > 0$. To better visualize the behavior of the solutions, we show $\mathcal{V}(z, t)$ in Figure 3.60 as a function of position at fixed times and time at fixed positions. Also shown is the comparison between $\mathcal{V}(z, t)$ and $\mathcal{I}(z, t)$ as a function of space and time, clearly illustrating the phase difference ϕ_z between the two waveforms.

It can be shown from [3.65] that, for a matched or infinitely long line, the magnitude of the ratio of voltages or currents corresponding to two different positions separated by a length l is a constant. In other words, the magnitude of the ratio of the voltage at position $(z + l)$ to the voltage at position z is

$$\left|\frac{V(z)}{V(z+l)}\right| = \frac{e^{-\alpha z}}{e^{-\alpha(z+l)}} = e^{\alpha l} \qquad [3.67]$$

Taking the natural logarithm of both sides, we have

$$\alpha l = \ln\left[\left|\frac{V(z)}{V(z+l)}\right|\right] \qquad [3.68]$$

Note that αl is a dimensionless number, since the units of α are in m^{-1} and l is in meters. However, to underscore the fact that αl expresses the attenuation on the line in terms of the natural (Naperian) logarithm of the magnitude of the ratio of voltages (or currents) at different positions, it is common convention to express αl in units of *nepers* (np). Thus, in conventional usage, the unit of the attenuation constant α is nepers-m^{-1}.

In most engineering applications, a more commonly used unit for attenuation is the *decibel* (dB). The decibel is a unit derived from the *bel,* which in turn was named after Alexander Graham Bell and was used in early work on telephone systems. Specifically, the decibel is defined as

$$\text{Attenuation in decibels} \equiv 20\log_{10}\left[\left|\frac{V(z)}{V(z+l)}\right|\right] \qquad [3.69]$$

It is clear from [3.67] and [3.69] that a relation exists between attenuation expressed in decibels and that expressed in nepers. We have

$$\begin{aligned}\text{Attenuation in dB} &= 20\log_{10} e^{\alpha l} = (\alpha l)20\log_{10} e \simeq 8.686(\alpha l) \\ &\simeq 8.686\ (\text{attenuation in np})\end{aligned} \qquad [3.70]$$

The advantage of a logarithmic unit such as the decibel or neper is that the total loss of several cascaded transmission lines (and other networks connected to them) can simply be found by adding the losses in the individual units. As an example, if sections of a fiber-optic line have attenuations of 10 dB, 20 dB, and 5 dB, then the total attenuation of the signal in its passage through all three of the lines would be $10 + 20 + 5 = 35$ dB. Similarly, total gains of any number of amplifier stages in a system can also be easily calculated with the use of logarithmic units.

Examples 3-30 and 3-31 illustrate the numerical values of the line parameters, respectively, for an open-wire telephone line and a high-speed coplanar strip interconnect.

Example 3-30: Open-wire telephone line. An open-wire telephone line consists of two parallel lines made of copper with diameters ~0.264 mm and spaced ~20 cm apart on the crossarm of the wooden poles. Determine the propagation constant γ, its real and imaginary parts α and β, and the characteristic impedance Z_0. Assume it operates at 1.5 kHz.

Solution: Using the two-wire transmission line formulas given in Table 2.2, we find the transmission line parameters to be $R \simeq 24.4\Omega\text{-(km)}^{-1}$, $L \simeq 2.93$ mH/km, and $C \simeq 3.80$ nF-(km)$^{-1}$. The value of G is assumed to be negligible. We have

$$\begin{aligned}
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
&\simeq \sqrt{(24.4 + j2\pi \times 1.5 \times 10^3 \times 2.93 \times 10^{-3})(0 + j2\pi \times 10^3 \times 3.80 \times 10^{-9})} \\
&\simeq \sqrt{(36.8e^{j48.6^\circ})(3.58 \times 10^{-5}e^{j90^\circ})} \simeq 0.0363e^{j69.3^\circ}\ (\text{km})^{-1} \\
&\simeq 0.0128 + j0.0339\ (\text{km})^{-1} \\
&\rightarrow \alpha \simeq 1.28 \times 10^{-2}\ \text{np-(km)}^{-1}; \quad \beta \simeq 3.39 \times 10^{-2}\ \text{rad-(km)}^{-1}
\end{aligned}$$

The phase velocity and wavelength are

$$v_p = \frac{\omega}{\beta} \simeq \frac{2\pi \times 1.5 \times 10^3}{3.39 \times 10^{-2}} \simeq 2.78 \times 10^5\ \text{km-s}^{-1}; \qquad \lambda = \frac{2\pi}{\beta} \simeq 185\ \text{km}$$

We find that the waves on an open-wire telephone cable propagate at a speed somewhat smaller than the speed of light in free space, namely, $c = 3 \times 10^8$ m-s^{-1}. The characteristic impedance is given by

$$\begin{aligned}
Z_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\
&\simeq \sqrt{\frac{24.4 + j2\pi \times 1.5 \times 10^3 \times 2.93 \times 10^{-3}}{j2\pi \times 1.5 \times 10^3 \times 3.80 \times 10^{-9}}} \simeq 1014.5e^{-j20.7^\circ} \simeq 949 - j359\Omega
\end{aligned}$$

Example 3-31: High-speed GaAs digital circuit coplanar strip interconnects. Transmission line properties of typical high-speed interconnects are experimentally investigated by fabricating and characterizing coplanar strip interconnects on semi-insulating GaAs substrates.[35] Measurements are carried out up to 18 GHz, from which the pertinent per-unit line parameters can be extracted. In one case, the values of the propagation constant γ and characteristic impedance Z_0 at 10 GHz are determined from the measurements to be $\gamma \simeq 1.2$ (np-(cm)$^{-1}$) + j6(rad-(cm)$^{-1}$) and $Z_0 \simeq 105 - j25\Omega$, respectively. Using these values, calculate the per-unit length parameters (R, L, G, and C) of the coplanar strip transmission line at 10 GHz.

[35] K. Kiziloglu, N. Dagli, G. L. Matthaei, and S. I. Long, Experimental analysis of transmission line parameters in high-speed GaAs digital circuit interconnects, *IEEE Transactions on Microwave Theory and Techniques*, 39(8), pp. 1361–1367, August 1991.

Solution: The per-unit length parameters of the transmission line can readily be computed from γ and Z_0 using the relations

$$R + j\omega L = \gamma Z_0$$

$$G + j\omega C = \frac{\gamma}{Z_0}$$

Using the measured values of γ and Z_0 at 10 GHz, we have

$$R + j\omega L = (1.2 + j6)(105 - j25) \simeq 276 + j600$$

from which $R \simeq 276\Omega\text{-(cm)}^{-1}$ and $L \simeq 600/(2\pi \times 10^{10}) \simeq 9.55\ \text{nH-(cm)}^{-1}$, respectively. Similarly, we have

$$G + j\omega C = \frac{1.2 + j6}{105 - j25} \simeq \frac{6.12e^{j78.9^\circ}}{107.9e^{-j13.4^\circ}} \simeq 0.0567e^{j92.1^\circ} \simeq -0.0021 + j0.0567$$

from which $G \simeq -0.0021\ \text{S-(cm)}^{-1}$ and $C \simeq 0.0567/(2\pi \times 10^{10}) \simeq 0.902\ \text{pF-(cm)}^{-1}$, respectively. The negative value of parameter G is nonphysical and is likely a result of measurement error.

The average power delivered into the line at any given point z can be found using the phasor expressions [3.65] for voltage and current:

$$\begin{aligned} P_{\text{av}}(z) &= \frac{1}{2}\mathcal{R}e\{V(z)[I(z)]^*\} \\ &= \frac{1}{2}\mathcal{R}e\left\{ V^+ e^{-\alpha z} e^{-j\beta z} \frac{(V^+)^*}{Z_0^*} e^{-\alpha z} e^{+j\beta z} \right\} \\ &= \frac{|V^+|^2}{2|Z_0|} e^{-2\alpha z} \cos(\phi_z) \end{aligned} \qquad [3.71]$$

We find that the time-average power decreases with distance as $e^{-2\alpha z}$, with an effective attenuation constant that is twice that of the voltage and current. The difference in time-average powers evaluated at any two points z_1 and z_2 is the amount of power dissipated in the segment of the line between z_1 and z_2.

Low-Loss Lines An important practical case is that in which the losses along the line are small but not negligible. If the line is low loss, we can assume that $R \ll \omega L$ and $G \ll \omega C$, which means that the resistive losses and leakage losses in the surrounding medium are both small. In such cases, useful approximate expressions can be derived for the characteristic impedance Z_0 and the propagation constant γ.

We first consider Z_0:

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$= \sqrt{\frac{L}{C}}\sqrt{\frac{1 + R/(j\omega L)}{1 + G/(j\omega C)}} = \sqrt{\frac{L}{C}}\left[\frac{1 + R/(j2\omega L)}{1 + G/(j2\omega C)}\right]$$

where we have used the fact that for $\zeta \ll 1$ $(1+\zeta)^{1/2} = 1+(\zeta/2)+\cdots \simeq 1+(\zeta/2)$. By neglecting the higher-order terms in the numerator and the denominator, and using $(1+\zeta)^{-1} \simeq 1-\zeta$ for $\zeta \ll 1$, we can write the characteristic impedance as

$$Z_0 \simeq \sqrt{\frac{L}{C}}\left(1 + \frac{R}{j2\omega L}\right)\left(1 - \frac{G}{j2\omega C}\right)$$

$$= \sqrt{\frac{L}{C}}\left[\left(1 + \frac{RG}{4\omega^2 LC}\right) + j\frac{1}{2\omega}\left(\frac{G}{C} - \frac{R}{L}\right)\right] \qquad [3.72]$$

In general, the second term in the real part of [3.72] is negligible since it involves the product of two small terms, namely, $R/(\omega L)$ and $G/(\omega C)$. Thus, the important effect of the losses on the transmission line is to introduce a small imaginary component to the characteristic impedance. In many cases, the imaginary part of Z_0 can be neglected, so that the characteristic impedance is, to the first order, equal to that for the lossless line.

A similar simplification can also be obtained for the propagation constant γ, again using $(1+\zeta)^{1/2} = 1 + (\zeta/2) + \cdots \simeq 1 + (\zeta/2)$ for $\zeta \ll 1$. We have

$$\gamma = [(R + j\omega L)(G + j\omega C)]^{1/2}$$

$$= \left[(j\omega L)(j\omega C)\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)\right]^{1/2}$$

$$\simeq j\omega\sqrt{LC}\left[1 - j\frac{1}{2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right]$$

The real and imaginary parts of γ for the low-loss line are thus

$$\alpha \simeq \frac{1}{2}\left[R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right] \qquad [3.73a]$$

$$\beta \simeq \omega\sqrt{LC} \qquad [3.73b]$$

As an example, coaxial lines used at high radio frequencies can be quite accurately represented by the above low-loss formulas of [3.73]. Note that the phase constant β is the same as that in the lossless case, so the phase velocity $v_p = \omega/\beta =$

$1/\sqrt{LC}$, independent of frequency. The loss constant α also does not depend on frequency; it simply accounts for a decrease in the overall signal intensity as the wave propagates along the line. Thus the distortion of an information-carrying signal (consisting of a finite band of frequencies), due to the different speed and attenuation of its frequency components, is minimized for a low-loss line.

Parameter values for a typical low-loss line are illustrated in Example 3-32.

Example 3-32: Low-loss coaxial line. RG17A/U is a low-loss radio frequency coaxial line. The following data for the nominal parameters of this line are available: characteristic impedance $Z_0 = 50\Omega$, line capacitance $C \simeq 96.8$ pF-m^{-1}, and line attenuation ~3 dB/100 m at 100 MHz. Determine the inductance L and resistance R per unit length of this line, assuming that G is negligibly small. Determine the velocity of propagation.

Solution: Using [3.70], we can express the attenuation in np-m^{-1}. We have

$$3 \text{ dB-}(100 \text{ m})^{-1} = 0.03 \text{ dB-m}^{-1} \simeq \frac{0.03}{8.686} \simeq 3.45 \times 10^{-3} \text{ np-m}^{-1} = \alpha$$

Using the low-loss formulas, we have

$$\alpha \simeq \frac{1}{2}\left[\frac{R}{Z_0} + GZ_0\right] = \frac{R}{2Z_0} \simeq 3.45 \times 10^{-3} \text{ np-m}^{-1}$$

which gives us $R \simeq 0.345\Omega$-m^{-1} since $G \simeq 0$ and $Z_0 = 50\Omega$. The inductance can be determined from

$$Z_0 \simeq \sqrt{\frac{L}{C}} = 50\Omega \quad \rightarrow \quad L = Z_0^2 C \simeq (50)^2(96.8 \times 10^{-12}) = 0.242 \ \mu\text{H-m}^{-1}$$

We can check to see that the quantity $|R/(\omega L)| \simeq 2.27 \times 10^{-3}$, or is much smaller than 1, which is apparently why the characteristic impedance for this low-loss line is real. The phase velocity is given by

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{Z_0^2 C^2}} = \frac{1}{Z_0 C} = \frac{1}{50 \times 96.8 \times 10^{-12}} \simeq 2.07 \times 10^8 \text{ m-s}^{-1}$$

3.8.2 Terminated Lossy Lines

An important result of losses is that both the forward wave traveling toward the load and the reflected wave traveling away from the load are attenuated exponentially with distance. As an observer moves away from the load on a terminated lossless line, the standing-wave pattern remains the same. However, on a lossy line, the same observer finds that the attenuation of the reflected wave causes this wave to be less important as he or she moves farther from the load. In addition, since the magnitude of the forward wave becomes larger as the observer moves away from the load, the

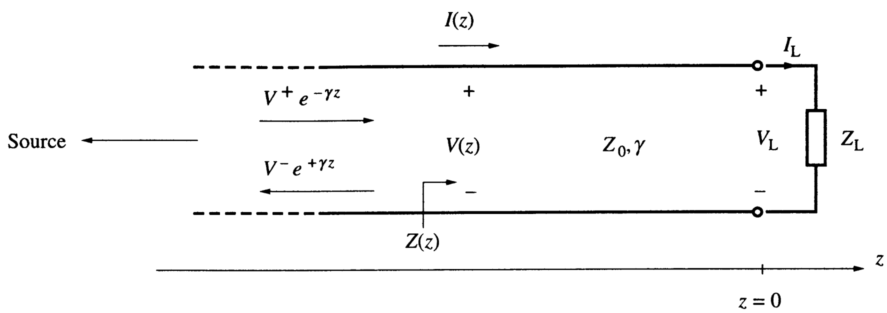


FIGURE 3.61. A terminated lossy transmission line.

relative size of the reflected wave is doubly reduced in moving toward the source. The net result of this effect is that, regardless of its termination, the transmission line begins to appear more and more like an infinite (or matched) line when viewed farther and farther from the load.

We consider a terminated lossy transmission line as shown in Figure 3.61, with $z = 0$ taken to be the position of the load as in the case of lossless lines. In general, the expressions for voltage and current on a terminated lossy transmission line are

$$V(z) = V^+e^{-\gamma z} + V^-e^{\gamma z} = V^+(e^{-\alpha z}e^{-j\beta z} + \Gamma_L e^{\alpha z}e^{j\beta z}) = V^+e^{-\alpha z}e^{-j\beta z}[1 + \Gamma(z)] \quad [3.74a]$$

$$I(z) = \frac{1}{Z_0}V^+e^{-\alpha z}e^{-j\beta z}[1 - \Gamma(z)] \quad [3.74b]$$

where $\Gamma(z)$ is the complex voltage reflection coefficient at any position z along the line defined as

$$\Gamma(z) \equiv \frac{V^-e^{\gamma z}}{V^+e^{-\gamma z}} = \Gamma_L e^{2\alpha z}e^{j2\beta z}$$

and where Γ_L is the complex load reflection coefficient given as

$$\Gamma_L \equiv \frac{V^-}{V^+} = \rho e^{j\psi} = \frac{Z_L - Z_0}{Z_L + Z_0} \quad [3.75]$$

The line impedance $Z(z)$ at any point z on the line is given by the ratio of the voltage and the current:

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{e^{-\alpha z}e^{-j\beta z} + \Gamma_L e^{\alpha z}e^{j\beta z}}{e^{-\alpha z}e^{-j\beta z} - \Gamma_L e^{\alpha z}e^{j\beta z}} \quad [3.76]$$

It is sometimes useful to rewrite the impedance as follows:

$$Z(z) = Z_0\frac{1 + \Gamma_L e^{2\alpha z}e^{j2\beta z}}{1 - \Gamma_L e^{2\alpha z}e^{j2\beta z}} = Z_0\frac{1 + \Gamma(z)}{1 - \Gamma(z)} \quad [3.77]$$

We compare [3.77] for $Z(z)$ to equation [3.31] for the lossless line. We see that Γ_L in [3.31] is replaced with $\Gamma_L e^{2\alpha z}$, so that the magnitude of the reflection coefficient for the lossy case is effectively reduced exponentially between the observation point and the end of the line (i.e., the load position). As viewed from larger and larger distances from the load (i.e., as $z \to -\infty$), the effect of reflections becomes negligible, and the line impedance approaches Z_0, as if the line were an infinitely long or matched line. To understand this effect, consider the general voltage reflection coefficient at any point z, namely

$$\Gamma(z) = \Gamma_L e^{2\alpha z} e^{j2\beta z} = \rho e^{2\alpha z} e^{j(\psi + 2\beta z)} \qquad [3.78]$$

In Section 3.6, we noted that motion along the line away from the load corresponded to clockwise rotation of $\Gamma = u + jv$ in the uv plane (or on the Smith chart), while its magnitude $\Gamma = \rho$ remained constant. On lossy lines, [3.78] indicates that the same type of rotation occurs as determined by the $e^{j2\beta z}$ term, but that, in addition, the magnitude of $\Gamma(z)$, namely $|\Gamma| = \rho e^{2\alpha z}$, decreases as we move away from the load (i.e., as z decreases). Eventually, at some point, $|\Gamma(z)| \to 0$, and looking toward the load from the source side beyond this position, the line is indistinguishable from an infinitely long or matched line.

To examine the behavior of the line voltage, current, and impedance, we first consider a short-circuited line of length l, so that $Z_L = 0$ in Figure 3.61. In this case, the load reflection coefficient is $\Gamma_L = -1$, so the line voltage and current are

$$V(z) = V^+(e^{-\gamma z} - e^{\gamma z}) = -2V^+ \sinh(\gamma z) \qquad [3.79a]$$

$$I(z) = \frac{V^+}{Z_0}(e^{-\gamma z} + e^{\gamma z}) = \frac{2V^+}{Z_0}\cosh(\gamma z) \qquad [3.79b]$$

where we have used the defining expressions for the hyperbolic sine and cosine functions:

$$\sinh\zeta = \frac{e^{\zeta} - e^{-\zeta}}{2} \qquad \cosh\zeta = \frac{e^{\zeta} + e^{-\zeta}}{2}$$

Although the compact form of $V(z)$ in [3.79a] appears very similar to that for the lossless line (with sin replaced by sinh), the evaluation of $\sinh(\gamma z)$ is not trivial,[36] since γ is a complex quantity. Note that when $\alpha \to 0$, equations [3.79] for $V(z)$ and $I(z)$ reduce to their lossless equivalents, since $\sinh(\alpha z) \to 0$ and $\cosh(\alpha z) \to 1$.

Using [3.79a] and [3.79b], we can compactly write the input impedance of a short-circuited line of length l as

$$[Z_{in}]_{sc} = Z_0 \tanh(\gamma l)$$

[36]The hyperbolic sine of the complex number $\gamma = \alpha + j\beta$ can be expressed as

$$\begin{aligned}\sinh(\gamma z) &= \sinh[(\alpha + j\beta)z] \\ &= \sinh(\alpha z)\cosh(j\beta z) + \cosh(\alpha z)\sinh(j\beta z) \\ &= \cos(\beta z)\sinh(\alpha z) + j\cosh(\alpha z)\sin(\beta z)\end{aligned}$$

In practice, the evaluation of $\sinh[(\alpha + j\beta)z]$ would be straightforward using any reasonably sophisticated numerical evaluation tool (e.g., a software package or a scientific calculator); however, it is useful to note for insight the nature of the actual evaluation.

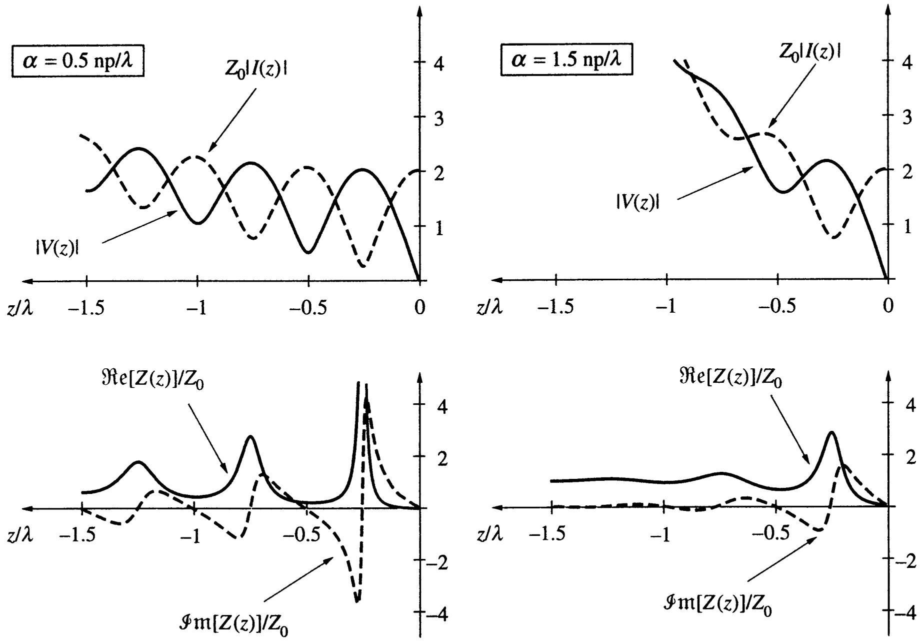


FIGURE 3.62. Voltage and current standing-wave patterns and impedance on a lossy short-circuited line. Results are shown for two different values of the attenuation constant, namely $\alpha = 0.5$ np/λ and 1.5 np/λ, where $\lambda = 2\pi/\beta$. For simplicity, we have assumed the characteristic impedance to be real, i.e., $\phi_z = 0$, and $V^+ = 1$.

Plots of magnitudes of the line voltage and current and the real and imaginary parts of the line impedance for a short-circuited line are provided in Figure 3.62, for two different values of the attenuation constant α, namely $\alpha = 0.5$ np/λ and 1.5 np/λ. For simplicity, we have assumed the phase of the characteristic impedance $\phi_z = 0$ in Figure 3.62. In general this phase angle is small, and leads to a phase difference between the voltage and current, as indicated in [3.66].

The resultant effects of the losses shown in Figure 3.62 become clear upon comparative examination of the lossless equivalents given in Figures 3.5 and 3.6. In the lossless case (Figures 3.5 and 3.6), the line voltage is zero at the load, and every half-wavelength thereafter, while the line current is a maximum at the same positions. The line impedance of the lossless line is zero at the load ($Z_L = 0$), inductive (i.e., $\mathcal{I}m\{Z(z)\} > 0$) in the range $-\lambda/4 < z < 0$, infinite (i.e., an open circuit) at $z = -\lambda/4$, capacitive in the range $-\lambda/2 < z < -\lambda/4$ back to zero at $z = -\lambda/2$, and repeating the same cycle thereafter.

For the lossy case, looking first at the relatively low-loss case of $\alpha = 0.5$ np/λ, we see that although the voltage and current exhibit generally similar cyclic behavior, the maximum and minimum values of both the line voltage and current increase with distance from the load. The line voltage is no longer zero at $z = -\lambda/2$. The differences between the values of the maxima and the minima also become smaller as z becomes increasingly negative, as is clearly evident from the relatively high-loss case of $\alpha = 1.5$ np/λ. At a sufficient distance away from the load, e.g., for $z < -1.5\lambda$ in the case of $\alpha = 1.5$ np/λ, the magnitude of line voltage and current do not vary significantly over a distance of half-wavelength (i.e., the standing-wave ratio is unity), as if the line were infinitely long or matched.

The line impedance for the relatively low-loss ($\alpha = 0.5$ np/λ) case exhibits similar behavior to the lossless case. The impedance is inductive (i.e., $\mathcal{I}m\{Z(z)\} > 0$) in the approximate range $-\lambda/4 < z < 0$, attains a large real value (but not quite an open circuit) at $z = -\lambda/4$, is capacitive in the approximate range $-\lambda/2 < z < -\lambda/4$, but does not quite return to zero at $z = -\lambda/2$. The peak in the real part of the impedance at $z = -3\lambda/4$ is considerably smaller than that at $z = -\lambda/4$. In general, the maxima and minima of the imaginary part of $Z(z)$ both approach zero as z attains larger and larger negative values, while the maxima and minima of the real part of $Z(z)$ both approach Z_0. At sufficient distances from the load, for example, for $z < -1.5\lambda$ in the case of $\alpha = 1.5$ np/λ, the line impedance $Z(z) \simeq Z_0$, just as if the line were infinitely long or matched.

For a lossy line terminated in an open circuit ($Z_L = \infty$), expressions for $V(z)$, $I(z)$, and Z_{in} can be obtained in a manner analogous to the preceding discussion for a short-circuited line. This straightforward procedure is left as an exercise.

The general behavior of the line voltage, current, and impedance for other terminations is quite similar, as illustrated in Figures 3.63 and 3.64 for a resistive load impedance of $Z_L = 5Z_0$. Results are shown for four different values of the

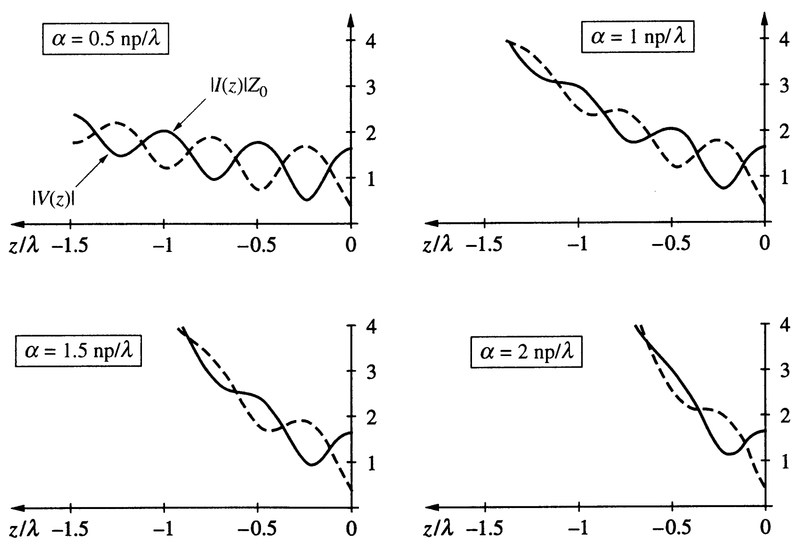


FIGURE 3.63. Voltage and current standing-wave patterns on a terminated lossy transmission line. The magnitudes of the voltage and current phasors (for current, the quantity plotted is $|I(z)|Z_0$) of a lossy line terminated in $Z_L = 5Z_0$ are shown for values of the attenuation constant $\alpha = 0.5, 1, 1.5,$ and 2 np/λ, where $\lambda = 2\pi/\beta$.

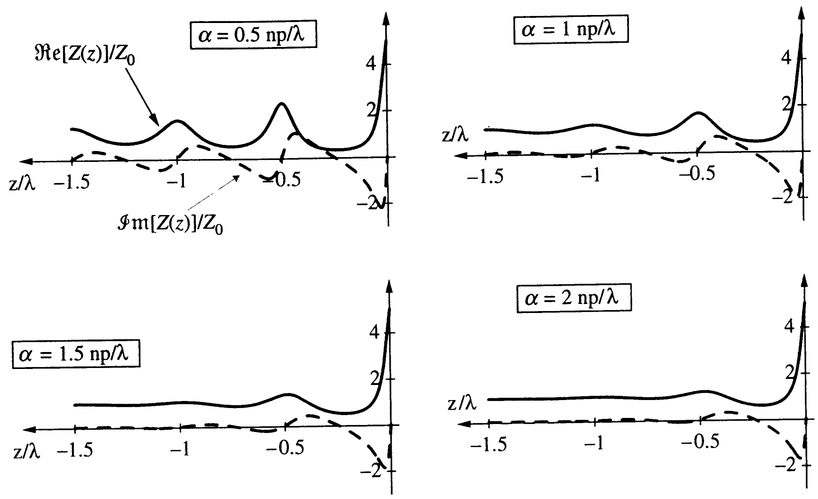


FIGURE 3.64. Line impedance standing on a terminated lossy transmission line. The real and imaginary parts of the line impedance (normalized to Z_0) of a lossy line terminated in $Z_L = 5Z_0$ are shown for values of the attenuation constant $\alpha = 0.5, 1, 1.5,$ and 2 np/λ, where $\lambda = 2\pi/\beta$.

attenuation constant. For simplicity, we have once again assumed the phase of the characteristic impedance $\phi_z = 0$.

The time-average power at any point z along the line can be evaluated using the expressions [3.74] for $V(z)$ and $I(z)$. We have

$$P_{\text{av}}(z) = \frac{1}{2}\Re e\{V(z)[I(z)]^*\}$$

$$= \frac{|V^+|^2e^{-2\alpha z}}{2}\Re e\left\{\frac{[1+\Gamma_L e^{2\alpha z}e^{j2\beta z}][1-\Gamma_L e^{2\alpha z}e^{j2\beta z}]^*}{Z_0^*}\right\}$$

$$= \frac{|V^+|^2e^{-2\alpha z}}{2}\Re e\left\{\frac{1-|\Gamma_L|^2e^{4\alpha z}-\Gamma_L^* e^{2\alpha z}e^{-j2\beta z}+\Gamma_L e^{2\alpha z}e^{+j2\beta z}}{Z_0^*}\right\}$$

Consider a terminated transmission line of length l. The time-average power at its input, namely at $z = -l$, is given by $P_{\text{av}}(z = -l)$, whereas that at the load is given by $P_{\text{av}}(z = 0)$. The difference between these quantities is the average power dissipated in the lossy transmission line. Thus, power lost in the line is given by

$$P_{\text{lost}} = P_{\text{av}}(z = -l) - P_{\text{av}}(z = 0)$$

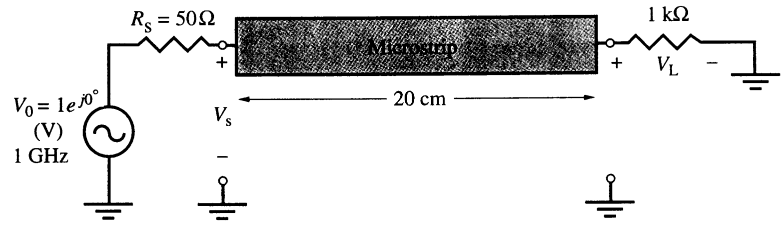
Reflections in lossy lines can lead to substantially increased losses since each time a wave travels down the line it is further attenuated. If the load reflects part of the incident power, more power is dissipated in the lossy line than would have been dissipated if the line were matched (i.e., $\Gamma_L = 0$). If the power dissipated in a lossy line under matched conditions is $P^{\rm m}_{\rm lost}$, it can be shown[37] that the extra power dissipated as a result of reflections is

$$\frac{P_{\rm lost} - P^{\rm m}_{\rm lost}}{P^{\rm m}_{\rm lost}} \simeq |\Gamma_{\rm L}|^2(e^{2\alpha l} - 1)$$

The extra power dissipated due to mismatch can be substantial, especially when $|\Gamma_{\rm L}| > 0.5$ and when the line is long.

Example 3-33 illustrates the concepts of power dissipation in lossy lines in the context of a high-speed microstrip interconnect.

Example 3-33: A high-speed microstrip interconnect. Consider a high-speed microstrip transmission line of length 20 cm used to connect a 1-V amplitude, 1-GHz, 50Ω sinusoidal voltage source to a digital logic gate having an input impedance of 1 kΩ, as shown in Figure 3.65. Based on measurements, the transmission line parameters of this interconnect at 1 GHz are approximately given by R = 5Ω-cm^{-1}, L = 5 nH-cm^{-1}, C = 0.4 pF-cm^{-1}, and G = 0 respectively.



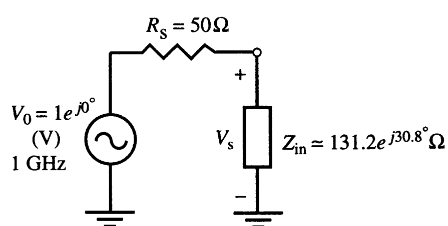


FIGURE 3.65. A lossy high-speed microstrip interconnect. (a) The microstrip transmission line connected to a 1-kΩ load and driven by a 1-GHz source. (b) Thévenin equivalent circuit as seen by the source, where $Z_{\rm in}$ is the input impedance at the source end of the microstrip.

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[37]See Section 6-3 of R. K. Moore, *Traveling Wave Engineering,* McGraw-Hill, New York, 1960.

(a) Find the propagation constant γ and characteristic impedance Z_0 of the line. (b) Find the voltages at the source and the load ends of the line. (c) Find the time-average power delivered to the line by the source and the time-average power delivered to the load. What is the power dissipated along the line?

Solution:

(a) The propagation constant is given by

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$$

$$= \sqrt{(500 + j2\pi \times 10^9 \times 500 \times 10^{-9})(j2\pi \times 10^9 \times 40 \times 10^{-12})}$$

$$\simeq \sqrt{3181e^{j80.96^\circ} \times 0.251e^{j90^\circ}}$$

$$\simeq 28.3e^{j85.5^\circ} \simeq 2.23 + j28.2$$

where $\alpha \simeq 2.23$ np-m^{-1} and $\beta \simeq 28.2$ rad-m^{-1}. The characteristic impedance is given by

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\simeq \sqrt{\frac{500 + j3142}{j0.251}} \simeq \sqrt{\frac{3181e^{j80.96^\circ}}{0.251e^{j90^\circ}}}$$

$$\simeq 112.5e^{-j4.522^\circ}\,\Omega$$

(b) The reflection coefficient at the load end can be found as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \simeq \frac{1000 - 112.5e^{-j4.522^\circ}}{1000 + 112.5e^{-j4.522^\circ}}$$

$$\simeq \frac{887.8 + j8.869}{1112 - j8.869} \simeq 0.798e^{j1.03^\circ}$$

The reflection coefficient at any position along the line is given by

$$\Gamma(z) = \Gamma_L e^{2\gamma z} \simeq 0.798e^{j1.029^\circ}e^{4.458z}e^{j56.37z}$$

$$\simeq 0.798e^{4.458z}e^{j(56.37z+0.018)}$$

The input impedance of the line is given by

$$Z_{in} = Z(z)|_{z=-0.2\text{ m}} = Z_0\frac{1 + \Gamma(-0.2)}{1 - \Gamma(-0.2)}$$

We first find $\Gamma(-0.2)$ as

$$\Gamma(-0.2) \simeq 0.798e^{4.458(-0.2)}e^{j[56.37(-0.2)+0.018]} \simeq 0.327e^{j75.05^\circ}$$

We now use $\Gamma(-0.2)$ to find Z_{in}. We have

$$Z_{\text{in}} \simeq 112.5e^{-j4.522^\circ}\frac{1+0.327e^{j75.05^\circ}}{1-0.327e^{j75.05^\circ}}$$
$$\simeq 112.5e^{-j4.522^\circ}\frac{1.084+j0.316}{0.915-j0.316}$$
$$\simeq (112.5e^{-j4.522^\circ})(1.166e^{j35.31^\circ}) \simeq 131.2e^{j30.79^\circ}\Omega$$

Based on voltage division in the equivalent circuit of Figure 3.65b, the source-end voltage V_s is

$$V_s = \frac{Z_{\text{in}}}{Z_{\text{in}}+Z_s}V_0 \simeq \frac{131.2e^{j30.79^\circ}}{112.7+j67.16+50}(1)$$
$$\simeq \frac{131.2e^{j30.79^\circ}}{176e^{j22.43^\circ}} \simeq 0.745e^{j8.361^\circ}\ \text{V}$$

The voltage at any position z along the line is given by

$$V(z) = V^+e^{-\gamma z}[1+\Gamma(z)]$$

from which V^+ can be written as

$$V^+ = \frac{V(z)}{e^{-\gamma z}[1+\Gamma(z)]}$$

At $z = -0.2$ m, we have

$$V(z=-0.2) = V_s \simeq 0.745e^{j8.361^\circ}\ \text{V}$$

and

$$e^{-\gamma(-0.2)} \simeq e^{0.446}e^{-j37.01^\circ} \quad \text{and} \quad \Gamma(-0.2) \simeq 0.327e^{j75.05^\circ}$$

Using these values, the value of V^+ can be found as

$$V^+ \simeq \frac{0.745e^{j8.361^\circ}}{e^{0.446}e^{-j37.01^\circ}(1+0.327e^{j75.05^\circ})} \simeq 0.423e^{j29.12^\circ}\ \text{V}$$

The voltage at the load end of the line is given as

$$V_L = V(z=0) = V^+[1+\Gamma_L] \simeq (0.423e^{j29.12^\circ})(1+0.798e^{j1.029^\circ}) \simeq 0.760e^{j29.57^\circ}\ \text{V}$$

(c) The time-average power delivered to the line is given by

$$P_s = \frac{1}{2}\left|\frac{V_s}{Z_{\text{in}}}\right|^2 R_{\text{in}} \simeq \frac{1}{2}\left|\frac{0.745}{131.2}\right|^2(112.7)\ \text{W} \simeq 1.82\ \text{mW}$$

Similarly, the time-average power delivered to the load can be found as

$$P_L = \frac{1}{2}\frac{|V_L|^2}{R_L} \simeq \frac{1}{2}\frac{(0.760)^2}{1000}\ \text{W} \simeq 0.289\ \text{mW}$$

Thus, based on conservation of energy, the power dissipated in the lossy line is

$$P_{\text{lost}} = P_s - P_L \simeq 1.82 - 0.289 \simeq 1.53 \text{ mW}$$

3.9 TRANSMISSION LINES AS RESONANT CIRCUIT ELEMENTS

One of the basic elements in a wide variety of dynamical systems is a resonator. In electronic applications, resonant circuits are found in the design of systems that selectively amplify or transmit a single frequency or a narrow band of frequencies. Starting at frequencies of hundreds of MHz, transmission lines and other distributed devices are commonly used as resonant circuit elements in filters, oscillators, tuned amplifiers, phase equalizers, or frequency measuring devices. Since important aspects of the behavior of resonant circuit elements are largely determined by the degree to which the system is lossy, it is appropriate to discuss transmission-line resonators after the general discussion of lossy transmission lines.

Below microwave frequencies (<300 MHz), resonant circuits typically consist of lumped capacitances and inductances. Although microwave integrated circuit elements that behave as capacitances and inductances can be constructed for operation at microwave frequencies (>300 MHz), such elements usually have too high losses to be effective as resonant elements and also are physically too small to handle useful power levels. Accordingly, distributed circuit elements with dimensions comparable to a wavelength are used as resonant elements. Such elements typically consist of sections of transmission line elements (coaxial, two-wire line, parallel-plate line, microstrip, etc.) having lengths of a quarter wavelength or half wavelength.

3.9.1 Lumped Resonant Circuits

Since most concepts underlying lumped resonant circuits carry over to distributed resonators, we first provide a brief review of lumped resonant circuits. This discussion is also useful because transmission line resonators can often be analyzed and represented in terms of lumped equivalent circuits. We consider the series RLC circuit as the simplest example of an electrical resonator, while noting that a nearly identical analysis also applies to the parallel RLC circuit.

The input impedance of the series RLC circuit shown in Figure 3.66a is the ratio of the phasor of the applied voltage V to the phasor of the resultant current I, namely

$$Z_{\text{in}} = \frac{V}{I} = R + j\omega L - \frac{j}{\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

If we view the voltage $\mathcal{V}(t)$ as the input and the current $\mathcal{I}(t)$ as the output, it is clear that the magnitude of Z_{in} determines the magnitude of sinusoidal current fluctuations

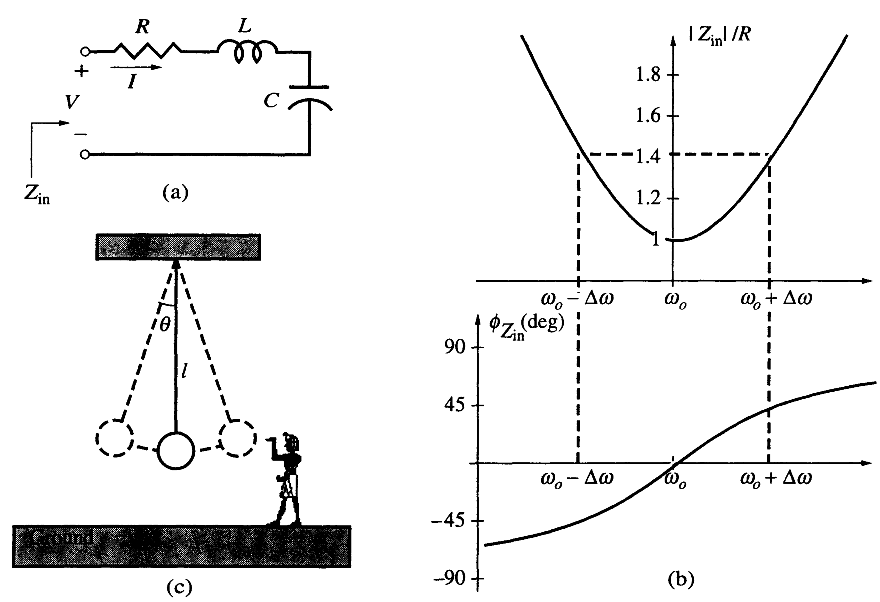

FIGURE 3.66. A series RLC circuit. (a) Circuit diagram. (b) Magnitude and phase of the input impedance as a function of frequency around the resonance frequency $\omega_0 = (LC)^{-1/2}$. (c) An analogous simple pendulum of large physical size. The angular resonant frequency for small oscillations ($\theta \ll \pi/2$) of a simple pendulum is $\omega_0 = \sqrt{g/l}$, where g is the gravitational acceleration and l is the length of the pendulum.

for a given magnitude of applied sinusoidal voltage; in terms of phasor quantities V and I, we have $|I| = |V|/|Z_{in}|$. In resonant circuit applications, the important parameter is the variation of Z_{in} with frequency, which is shown in Figure 3.66b. We note that the magnitude of the impedance $|Z_{in}|$ is a minimum at the *resonant* frequency, $\omega_0 = (LC)^{-1/2}$, for which $Z_{in} = R$. This in turn means that maximum sinusoidal current is established in this circuit when the sinusoidal input voltage is at a frequency $\omega = \omega_0$. Establishing an oscillatory current of the same amplitude at frequencies below or above ω_0 requires larger input voltages.

To better understand the principle of resonance, note that the resonant RLC circuit is analogous to a simple pendulum, which exhibits a natural frequency for small oscillations with an angular frequency of $\omega_0 = \sqrt{g/l}$, where l is the length of the pendulum and g is the gravitational acceleration. Consider a pendulum consisting of a very large and heavy ball hung with a long cable, as shown in Figure 3.66c. If we were to make the pendulum swing back and forth at different frequencies, we would find that it requires a trivially small force to set it into oscillations at ω_0; even for a rather heavy ball, a person could set the pendulum into oscillation (i.e., swinging back and forth repeatedly) with the periodic tap of a finger at time intervals of approximately $2\pi/\omega_0$. However, if we wanted to make the pendulum swing back and forth at a faster or slower rate than its natural frequency of oscillation, we would have to exert an enormous amount of force to carry the large weight of the ball across to make it go faster than its natural frequency or to hold it back in order to make it oscillate slower than its natural frequency.

The magnitude of the oscillatory current that can be established in an RLC circuit at ω_0 is determined by the losses, since $Z_{\text{in}} = R$, and thus $I(\omega_0) = V(\omega_0)/R$. In the absence of losses, that is, if $R = 0$, we can establish an oscillatory current at ω_0 with zero input voltage; in other words, if there were any initial stored energy in the circuit, oscillatory current would flow indefinitely even if we short-circuited the input terminals (i.e., if $V = 0$). In practice, one strives to make R as small as possible, but its value is necessarily nonzero and determines the sharpness, or quality, of resonance.

A useful measure of the sharpness of resonance is the quality factor Q, defined as

$$Q \equiv \omega_0 \frac{\text{time-average energy stored}}{\text{energy lost per second}}$$

with all quantities evaluated at the resonant frequency $\omega = \omega_0$. The energy lost per second (Joules-s^{-1}) is the power loss, given by $P_{\text{loss}} = \frac{1}{2}|I|^2 R$, in watts. The time-average energy stored in the inductance is

$$\bar{W}_{\text{L}} = \frac{1}{T_p}\int_0^{T_p} \frac{1}{2}|\mathcal{I}(t)|^2 L\, dt = \frac{1}{T_p}\int_0^{T_p} \frac{1}{2}[|I|\cos(\omega t + \phi_I)]^2 L\, dt = \frac{1}{4}|I|^2 L$$

where we have recognized that the current phasor is usually a complex number $I = |I|e^{\phi_I}$, so that $\mathcal{I}(t) = \Re e\{Ie^{j\phi_I}e^{j\omega t}\} = |I|\cos(\omega t + \phi_I)$. Similarly, the time-average energy stored in the capacitance is

$$\bar{W}_C = \frac{1}{4}|V_c|^2 C = \frac{1}{4}\frac{|I|^2}{(\omega^2 C)}$$

where V_c is the phasor of the voltage across the series capacitance, the magnitude of which is given by $|V_c| = |I|/(|j\omega C|) = |I|/wC$. Note that at the resonance frequency $\omega = \omega_0$, we have $\bar{W}_{\text{L}} = \bar{W}_C$ since $\omega_0 = (LC)^{-1/2}$. The total stored energy in the RLC circuit at the resonance frequency is $\bar{W} = \bar{W}_{\text{L}} + \bar{W}_C = 2\bar{W}_{\text{L}}$. The quality factor Q is then given by

$$Q = \omega_0 \frac{\bar{W}}{P_{\text{loss}}} = \frac{\omega_0 2(\frac{1}{4}|I|^2 L)}{\frac{1}{2}R|I|^2} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

Let us now examine the behavior of the input impedance as a function of frequency in the vicinity of resonance. At $\omega = \omega_0 + \Delta\omega$, with $\Delta\omega \ll \omega_0$, we have

$$\begin{aligned} Z_{\text{in}} &= R + j(\omega_0 + \Delta\omega)L + \frac{1}{j(\omega_0 + \Delta\omega)C} \\ &= R + j\omega_0 L + j\Delta\omega L + \left[\frac{1}{j\omega_0 C} - \left(\frac{1}{j\omega_0 C}\right)^2 j\Delta\omega C + \cdots\right] \\ &\simeq R + j\Delta\omega L + j\frac{\Delta\omega}{\omega_0^2 C} = R + j\frac{2\Delta\omega}{\omega_0^2 C} = R + j2L\Delta\omega \end{aligned}$$

where we have used $\omega_0^2 = 1/LC$. Since $Q = 1/(\omega_0 RC)$, we can express Z_{in} in terms of Q as

$$Z_{in} \simeq R\left(1 + j2Q\frac{\Delta\omega}{\omega_0}\right)$$

The bandwidth of the series RLC circuit can be determined from the variation with frequency of its input impedance, as shown in Figure 3.66b. The so-called 3-dB bandwidth, defined to be the frequency range over which the magnitude of the impedance is within a factor of $\sqrt{2}$ of that at resonance,[38] is marked in Figure 3.66b by the points at which the real and imaginary parts of the input impedance are equal. Namely,

$$2Q\frac{\Delta\omega}{\omega_0} = 1 \quad \rightarrow \quad \frac{2\Delta\omega}{\omega_0} = \frac{1}{Q}$$

Note that the bandwidth is $2\Delta\omega$, since the behavior around resonance is approximately symmetrical (for $Q \gg 1$) on both sides of ω_0. Thus, we have

$$\text{Bandwidth} = \frac{\omega_0}{Q}$$

The higher the Q of a resonant circuit, the narrower is its bandwidth. All of the preceding concepts apply to different kinds of resonant systems, although they were specifically derived for the series RLC circuit.

3.9.2 Transmission Line Resonators

The simplest examples of distributed resonant circuit elements are short- or open-circuited transmission lines. Note that, in practice, these may be implemented in terms of any of the different two-conductor transmission line configurations shown in Figure 2.1, that is, coaxial, two-wire line, stripline, or others. Consider first the short-circuited half-wavelength-long line shown in Figure 3.67, which we will show

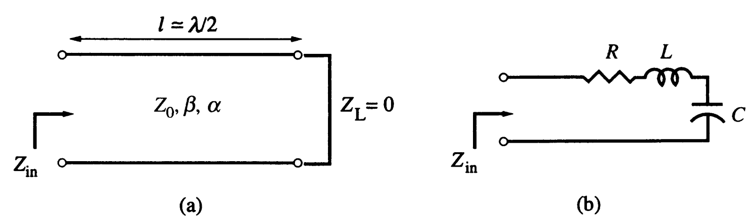

FIGURE 3.67. Short-circuited transmission line resonator. A shorted line shown in (a) exhibits behavior similar to a series RLC circuit (shown in (b)) for $l \simeq \lambda/2$.

[38]Note that, for a given applied voltage V, a reduction in impedance Z_{in} by a factor of $\sqrt{2}$ corresponds to a decrease in the current $I = V/Z_{in}$ by a factor of $\sqrt{2}$, or a decrease in power by a factor of 2, corresponding to 3 dB.

to be equivalent (in terms of the frequency variation of its input impedance) to a series RLC circuit.

The input impedance of such a lossy line of length l is

$$Z_{\text{in}}\big|_{z=-l} = Z_0 \frac{e^{\alpha l} e^{j\beta l} - e^{-\alpha l} e^{-j\beta l}}{e^{\alpha l} e^{j\beta l} + e^{-\alpha l} e^{-j\beta l}}$$

where we have taken into account the fact that $\Gamma_L = -1$ for a short-circuited termination. We consider a line of length $l = \lambda_0/2$ at $\omega = \omega_0$, where $\lambda_0 = 2\pi c/\omega_0$, with c being the speed of light in free space. At any other frequency ω for which $\lambda = 2\pi c/\omega$, we then have

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda_0}{2} = \frac{\pi\omega}{\omega_0} = \frac{\pi(\omega_0 + \Delta\omega)}{\omega_0} = \pi + \frac{\pi\Delta\omega}{\omega_0} \quad \rightarrow \quad e^{\pm j\beta l} = -e^{\pm j\pi\Delta\omega/\omega_0}$$

For a low-loss line as is considered here, we must have $\alpha l \ll 1$, so that the $e^{\pm\alpha l}$ terms in the preceding equation can be approximated by $1 \pm \alpha l$. Making the necessary substitutions, we have

$$\begin{aligned}
Z_{\text{in}} &\simeq Z_0 \frac{(1+\alpha l)(-e^{j\pi\Delta\omega/\omega_0}) - (1-\alpha l)(-e^{-j\pi\Delta\omega/\omega_0})}{(1+\alpha l)(-e^{j\pi\Delta\omega/\omega_0}) + (1-\alpha l)(-e^{-j\pi\Delta\omega/\omega_0})} \\
&= Z_0 \frac{-(\alpha l)\cos(\pi\Delta\omega/\omega_0) - j2\sin(\pi\Delta\omega/\omega_0)}{-j2(\alpha l)\sin(\pi\Delta\omega/\omega_0) - 2\cos(\pi\Delta\omega/\omega_0)} \\
&\simeq Z_0 \frac{-(\alpha l) - j(\pi\Delta\omega/\omega_0)}{-j2(\alpha l)(\pi\Delta\omega/\omega_0) - 2} \\
Z_{\text{in}} &\simeq Z_0\left[\alpha l + j\frac{\pi\Delta\omega}{\omega_0}\right]
\end{aligned}$$

where we have assumed that $\sin(\pi\Delta\omega/\omega_0) \simeq \pi\Delta\omega/\omega_0$ and $\cos(\pi\Delta\omega/\omega_0) \simeq 1$ (since $\Delta\omega \ll \omega_0$).

The preceding expression can now be compared to the input impedance of a series RLC circuit. We have

$$[Z_{\text{in}}]_{\text{RLC}} \simeq R\left[1 + j2Q\frac{\Delta\omega}{\omega_0}\right] = R + j2L\Delta\omega; \qquad [Z_{\text{in}}]_{\lambda/2\ \text{line}} \simeq Z_0\left[\alpha l + j\frac{\pi\Delta\omega}{\omega_0}\right]$$

Thus, the short-circuited line of length $\lambda/2$ can be represented by an equivalent series RLC circuit, with element values

$$R_{\text{eq}} = Z_0\alpha l = Z_0\frac{\alpha\lambda}{2}; \qquad L_{\text{eq}} = \frac{\pi Z_0}{2\omega_0}; \qquad C_{\text{eq}} = \frac{2}{Z_0\pi\omega_0}$$

where $\omega_0 = 1/\sqrt{L_{\text{eq}}C_{\text{eq}}}$. By analogy, we can then deduce the expression for the Q of the short-circuited half-wavelength line to be

$$Q = \frac{1}{\omega_0 R_{\text{eq}} C_{\text{eq}}} = \frac{\pi}{\alpha\lambda_0} = \frac{\beta_0}{2\alpha}$$

where $\beta_0 = 2\pi/\lambda_0$ is the phase constant at $\omega = \omega_0$ and α has to be evaluated at $\omega = \omega_0$. Typical values of Q for short-circuited transmission lines range from several hundred to tens of thousands, much higher than is possible for low-frequency lumped circuits.

Transmission line resonators with high Q values are necessarily low-loss lines so that the simplified expressions [3.73a] and [3.73b] are valid, respectively, for α and β. Thus, the Q of a transmission line resonator can be written in terms of the transmission line parameters as

$$Q \doteq \frac{\beta_0}{2\alpha} \simeq \frac{\omega_0 C Z_0}{G Z_0 + (R/Z_0)}$$

Example 3-34 illustrates the calculation of the Q of an air-filled coaxial line.

Example 3-34: *Q* of an air-filled coaxial line. Determine the Q of an air-filled coaxial line, shorted at one end, made of copper with dimensions $a = 1$ cm and $b = 3$ cm. The operating frequency is 300 MHz.

Solution: We have $\lambda = c/f \simeq 1$ m $= 100$ cm. Thus the resonant line length is $l = \lambda/2 \simeq 50$ cm. Assuming that the losses are low and neglecting shunt losses ($G = 0$), the attenuation constant from [3.73a] is $\alpha \simeq R/(2Z_0)$. From Table 2.2, the series resistance R for a coaxial line made of copper is $R = 4.15 \times 10^{-8}\sqrt{f}(a^{-1} + b^{-1})$, and $Z_0 = 60 \ln(b/a)$. Thus, we have

$$\alpha = \frac{4.15 \times 10^{-8}}{2 \times 60}\sqrt{f}\frac{(a^{-1} + b^{-1})}{\ln[b/a]} \simeq 7.27 \times 10^{-6} \quad \text{np-(cm)}^{-1}$$

and

$$Q = \frac{\pi}{\alpha\lambda} \simeq \frac{\pi}{7.27 \times 10^{-6} \times 100} \simeq 4321$$

Note that we have neglected the losses in the imperfect short circuit. Also, if the space between the conductors of the coaxial line were filled with an insulator other than air, additional high-frequency losses in the insulator would generally tend to reduce Q.

A transmission line resonator that behaves like a series RLC circuit can also be implemented using an open-circuited transmission-line section of length $\lambda/4$. Open-circuited line resonators are easier to implement for microstrip or striplines because short circuits cannot be easily placed on these structures. An analysis of the input impedance similar to that just given shows that for $l = \lambda/4$, we have

$$Z_{\text{in}} \simeq Z_0\left[\alpha l + j\frac{\Delta\omega}{\omega_0}\frac{\pi}{2}\right]$$

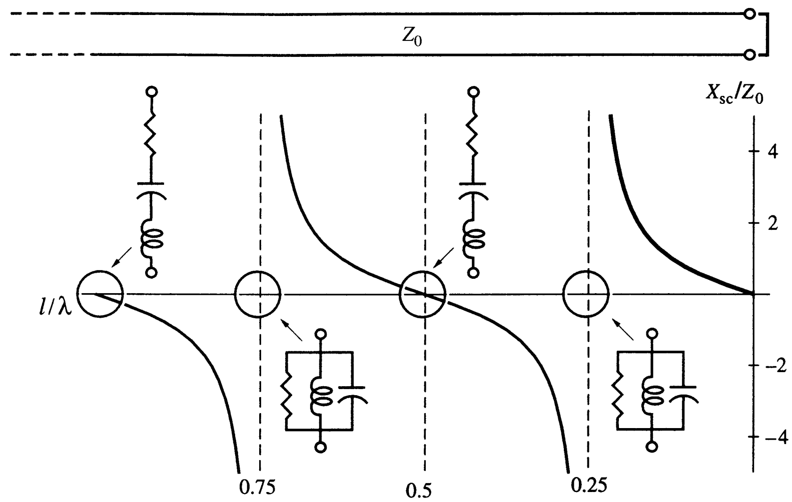

FIGURE 3.68. Resonant behavior of short-circuited transmission line segments. For the purposes of this diagram, the imaginary part of the line impedance for a lossless line of length l is shown, namely $X_{sc} = Z_0 \tan(\beta l)$. Note that in applications of transmission lines as resonators, the losses are generally quite small, so that the behavior of the impedance close to the load is only negligibly different from the lossless case.

so that the equivalent series RLC circuit parameters are

$$R_{eq} = Z_0(\alpha l)^{-1}; \qquad L_{eq} = \frac{\pi Z_0}{4\omega_0}; \qquad C_{eq} = \frac{4}{Z_0 \pi \omega_0}$$

Thus the Q is

$$Q = \frac{1}{\omega_0 R_{eq} C_{eq}} = \frac{\pi}{\alpha \lambda_0} = \frac{\beta_0}{2\alpha}$$

Although we have only considered the series RLC circuit and its transmission line analog, similar results can be obtained for the parallel RLC circuit. A summary of the behaviors of various lengths of shorted transmission lines as parallel or series RLC circuits is given in Figure 3.68. The reader is encouraged to construct an analogous diagram for an open-circuited transmission line.

3.10 SUMMARY

This chapter discussed the following topics:

- **Transmission line equations.** When a transmission line is excited by a sinusoidal source of angular frequency ω at steady state, the variations of the line voltage and current can be analyzed using the phasor form of the transmission line equations, which for a lossless line are

$$\frac{dV(z)}{dz} = -j\omega L I(z)$$

$$\frac{dI(z)}{dz} = -j\omega C V(z)$$

where L and C are the per-unit length distributed parameters of the line, and $V(z)$ and $I(z)$ are, respectively, the voltage and current phasors, which are related to the actual space-time voltage and current expressions as follows:

$$\mathcal{V}(z, t) = \Re e\{V(z)e^{j\omega t}\}; \qquad \mathcal{I}(z, t) = \Re e\{I(z)e^{j\omega t}\}$$

- **Propagating-wave solutions, characteristic impedance, phase velocity, and wavelength.** The solutions of the lossless transmission line equations consist of a superposition of waves traveling in the $+z$ and $-z$ directions. The voltage and current phasors and the corresponding space-time functions have the form

$$V(z) = V^+e^{-j\beta z} + V^-e^{+j\beta z}; \quad \mathcal{V}(z, t) = V^+\cos(\omega t - \beta z) + V^-\cos(\omega t + \beta z)$$

$$I(z) = \frac{V^+}{Z_0}e^{-j\beta z} - \frac{V^-}{Z_0}e^{+j\beta z}; \quad \mathcal{I}(z, t) = \frac{V^+}{Z_0}\cos(\omega t - \beta z) - \frac{V^-}{Z_0}\cos(\omega t + \beta z)$$

The characteristic impedance Z_0 of the line is the ratio of the voltage to the current phasor of the wave propagating in the $+z$ direction (or the negative of the ratio of the voltage to the current phasor of the wave traveling in the $-z$ direction) and, for a lossless line, is given by $Z_0 = \sqrt{L/C}$. The phase velocity and the wavelength for a lossless line are given as

$$v_p = 1/\sqrt{LC}; \qquad \lambda = 2\pi/\beta = v_p/f$$

Note that the phase velocity of a lossless line is independent of frequency.

- **Input impedance of short- and open-circuited lines.** The line impedance of a transmission line seen looking toward the load at any position along the line is defined as

$$Z(z) = \frac{V(z)}{I(z)} = Z_0\frac{V^+e^{-j\beta z} + V^-e^{+j\beta z}}{V^+e^{-j\beta z} - V^-e^{+j\beta z}}$$

The input impedances of short- or open-circuited transmission lines of length l are purely imaginary and are given by

$$Z_{sc} = jZ_0\tan(\beta l) \qquad \text{short-circuited line}$$

$$Z_{oc} = -jZ_0\cot(\beta l) \qquad \text{open-circuited line}$$

Since any arbitrary reactive impedance can be realized by simply adjusting the length l of open- or short-circuited stubs, these stubs are commonly used as reactive circuit elements for impedance matching and other applications.

- **Reflection coefficient.** It is common practice to treat steady-state transmission line problems by considering the wave traveling in the $+z$ direction (toward the load) as the incident wave and the wave traveling in the $-z$

direction (away from the load and toward the source) as the reflected wave. The ratio of the reflected to the incident voltage phasor at any position z along the line is defined as the reflection coefficient, represented by $\Gamma(z)$. The reflection coefficient at the load end of the line (where $z = 0$) is given by

$$\Gamma_L = \frac{V^-}{V^+} = (Z_L - Z_0)/(Z_L + Z_0) = \rho e^{j\psi}$$

The case of $Z_L = Z_0$ is referred to as a matched load, for which there is no reflected wave, since $\Gamma_L = 0$. The reflection coefficient $\Gamma(z)$ at any other location z (where $z < 0$) on a lossless line is given by

$$\Gamma(z) = \frac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)}$$

- **Standing-wave pattern.** The superposition of the incident and reflected waves constitutes a standing-wave pattern that repeats every $\lambda/2$ over the length of the line. The standing-wave ratio S is defined as the ratio of the maximum to minimum voltage (current) magnitude over the line and is given by

$$S = \frac{1 + \rho}{1 - \rho}$$

  where $\rho = |\Gamma_L|$. The standing-wave ratio S has practical significance because it is easily measurable. The value of S varies in the range $1 \leq S \leq \infty$, where $S = 1$ corresponds to $\rho = 0$ (i.e., no reflection case) and $S = \infty$ corresponds to $\rho = 1$ (i.e., the load is either open or short circuit).
- **Transmission line as an impedance transformer.** The line impedance of a lossless transmission line terminated in an arbitrary load impedance, defined as the ratio of the total voltage to current phasor at position z, is in general complex and is a periodic function of z, with period of $\lambda/2$. The line impedance is purely real at locations along the line where the voltage is a maximum or minimum.
- **Power flow.** The net time-average power propagating toward the load on a lossless transmission line is given by

$$P(z) = \frac{|V^+|^2}{2Z_0}(1 - \rho^2)$$

  and is equal to the power P_L delivered to the load. For a given value of $|V^+|$, the power delivered to the load is maximized under matched conditions, or $\rho = 0$. The degree of mismatch between the load and the line can be described in terms of return loss, given as

$$\text{Return loss} = 20 \log_{10} \frac{S + 1}{S - 1}$$

- **Impedance matching.** In most applications it is desirable to match the load impedance to the line in order to reduce reflections and standing waves. In single-stub matching, a short- or open-circuited stub is placed in shunt or series at a location $z = -l$ along the line at which the normalized line admittance or the impedance is given as

$$\bar{Y}_1(z)|_{z=-l} = 1 - j\bar{B}; \qquad \bar{Z}_1(z)|_{z=-l} = 1 - j\bar{X}$$

The matching is then completed by choosing the length l_s of a short- or open-circuited stub so that it presents an admittance or impedance at $z = -l$ of $\bar{Y}_s = j\bar{B}$ or $\bar{Z}_s = j\bar{X}$. In quarter-wave matching, it is first necessary to determine the location l along the line at which the line impedance is purely real, that is, where

$$Z(z)|_{z=-l} = R + j0$$

Matching to a line of impedance Z_0 is then completed by using a quarter-wavelength-long line of characteristic impedance $Z_Q = \sqrt{Z_0 R}$.

- **Smith chart.** The fact that the impedance $Z(z)$ and the reflection coefficient $\Gamma(z)$ on a lossless line are both periodic functions of position z along the line makes it possible to analyze and visualize the behavior of the line using a graphical display of $\Gamma(z)$, S, and $Z(z)$ known as the Smith chart. The Smith chart provides a convenient means of analyzing transmission line problems to determine values of impedance and reflection coefficient (or standing-wave ratio). The Smith chart is also a useful tool for matching network design.
- **Lossy transmission lines.** The solutions for voltage and current propagating in the z direction on a lossy transmission line have the form

$$\mathcal{V}(z, t) = V^+ e^{-\alpha z} \cos(\omega t - \beta z)$$

$$\mathcal{I}(z, t) = \frac{V^+}{|Z_0|} e^{-\alpha z} \cos(\omega t - \beta z - \phi_z)$$

where α and β are the real and imaginary parts of the propagation constant $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)}$, R, L, G, and C are the per-unit distributed parameters of the line, and ω is the angular frequency of the excitation. The characteristic impedance for a lossy line is in general complex and is given by

$$Z_0 = |Z_0| e^{j\phi_z} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

For terminated lines, the general expressions for line voltage and current are

$$V(z) = V^+ e^{-\alpha z} e^{-j\beta z}[1 + \Gamma(z)]$$

$$I(z) = \frac{1}{Z_0} V^+ e^{-\alpha z} e^{-j\beta z}[1 - \Gamma(z)]$$

where $\Gamma(z) = \Gamma_L e^{2\alpha z} e^{j2\beta z}$, with $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$ being the complex load voltage reflection coefficient. The line voltage and current exhibit a standing-wave pattern near the load, but the differences between the maxima and minima become smaller as distance from the load increases. At sufficient distances from the load, the magnitudes of the line voltage and current do not vary significantly with distance, as if the line were matched. The impedance of a lossy transmission line is given by

$$Z(z) = Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)} = Z_0\frac{1+\Gamma_L e^{2\alpha z}e^{j2\beta z}}{1-\Gamma_L e^{2\alpha z}e^{j2\beta z}}$$

The real and imaginary parts of $Z(z)$ exhibit maxima and minima near the load, similar to that of a lossless line. However, at sufficient distances from the load, $Z(z)$ approaches Z_0, as if the line were matched.

- **Transmission line resonators.** Short- or open-circuited transmission lines of lengths that are integer multiples of $\lambda/4$ behave as highly efficient resonators. The Q of a low-loss short-circuited half-wavelength-long line is

$$Q = \frac{\omega_0 C Z_0}{GZ_0 + (R/Z_0)}$$

where C, G, and R are the distributed constants of the line, and $\omega_0 = (LC)^{-1/2}$ is the resonant frequency.

3.11 PROBLEMS

3-1. Transmission line capacitor. An open-circuited 50Ω microstrip transmission line is used in a microwave amplifier circuit to provide a capacitance of 3.2 pF at 2.3 GHz. (a) Find the appropriate electrical length of the line. (b) Find the lumped element values of the open-circuited line designed in part (a) at 2 and 2.6 GHz.

3-2. Resistive load. A 50Ω transmission line is terminated with an antenna having a feed-point impedance of 150Ω. (a) Calculate V_{max}, V_{min}, I_{max}, and I_{min} along the line, assuming $V^+ = 1$ V. (b) Sketch $|V(z)|$ and $|I(z)|$ as functions of z, taking the antenna position to be $z = 0$. Assume $\lambda = 20$ cm.

3-3. Microwave filter. An air-filled coaxial line with $Z_0 = 75\Omega$ is designed to provide an inductive impedance of $j231\Omega$ for a microwave filter to operate at 2.5 GHz. Find the length of the coaxial line if (a) it is short-circuit terminated and (b) it is open-circuit terminated.

3-4. Capacitive termination. A lossless transmission line with $Z_0 = 100\Omega$ is terminated with a capacitive load of $40 - j50\Omega$. (a) Calculate the standing-wave ratio S. (b) Find the position of the first voltage minimum and maximum with respect to the load. (c) Sketch $|V(z)|$ as a function of z/λ. Assume $V^+ = 1$ V.

3-5. Input impedance. A 10-cm-long air transmission line segment with $Z_0 = 100\Omega$ is terminated at $z = 0$ with a resistive load of 200Ω and is operated at 1.5 GHz. Calculate the input impedance of the line if (a) a shunt capacitance of ~ 2.12 pF is connected at a point halfway ($z = -5$ cm) on the line, (b) a series capacitance of ~ 2.12 pF is connected at $z = -5$ cm.

3-6. Resistive load. A transmission line segment with $Z_0 = 50\Omega$ and of length l is terminated at a load resistance of R_L that can be varied . Sketch the input impedance Z_{in} as a function of R_L if (a) $l = \lambda/4$ and (b) $l = \lambda/2$. At what value of R_L do the two curves intersect?

3-7. Inductive termination. An air-filled coaxial line with $Z_0 = 50\Omega$ is terminated with a load of $100 + j50\sqrt{3}\Omega$. If the line is operated at $\lambda = 10$ cm, calculate (a) the standing-wave ratio S on the line, (b) the distance from the load to the first voltage maximum, and (c) the distance from the load to the first current maximum.

3-8. A wireless communication antenna. The following table provides the approximate values at various frequencies of the feed-point impedance of a circularly polarized patch antenna used in the wireless industry for making cellular phone calls in difficult environments, such as sport arenas and office buildings:

f (MHz)	$Z_L(\Omega)$
800	$21.5 - j15.4$
850	$38.5 + j2.24$
900	$43.8 + j9.74$
950	$55.2 - j10.2$
1000	$28.8 - j7.40$

If this antenna is directly fed by a 50Ω transmission line, find and sketch the standing-wave ratio S as a function of frequency.

3-9. Resistive line impedance. A 50Ω coaxial line is terminated with a load impedance of $40 + j80\Omega$ at $z = 0$. Find the minimum electrical length l/λ of the line at which the line impedance (i.e., $Z(z = -l)$) is purely resistive. What is the value of the resistive line impedance?

3-10. Resistive line impedance. A transmission line with $Z_0 = 100\Omega$ is terminated with a load impedance of $120 - j200\Omega$. Find the minimum length l of the line at which the line impedance (i.e., $Z(z = -l)$) is purely resistive. What is the value of the resistive line impedance?

3-11. Resistive load. A lossless line is terminated with a resistive load of 120Ω. If the line presents an impedance of $48 + j36\Omega$ at a position $3\lambda/8$ away from the load, what is the characteristic impedance Z_0 of the line?

3-12. Input impedance. For the lossless transmission-line system shown in Figure 3.69, find Z_{in} for the following load impedances: (a) $Z_L = \infty$ (open circuit), (b) $Z_L = 0$ (short circuit), and (c) $Z_L = Z_0/2$.

3-13. Input impedance. Repeat Problem 3-12 for the circuit shown in Figure 3.70.

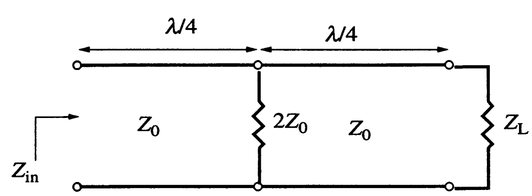

FIGURE 3.69. Input impedance. Problem 3-12.

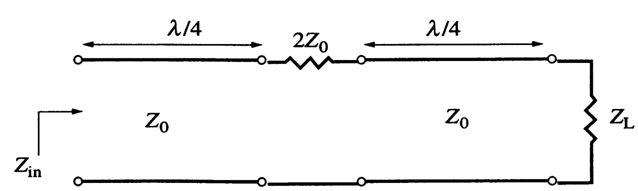

FIGURE 3.70. Input impedance. Problem 3-13.

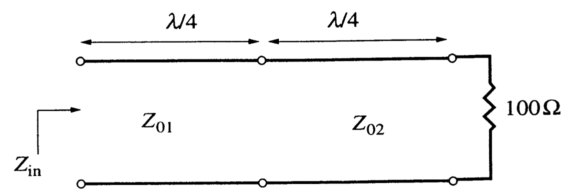

FIGURE 3.71. Input impedance. Problem 3-14.

3-14. Input impedance. For the lossless transmission-line system shown in Figure 3.71, what is the ratio Z_{01}/Z_{02} if $Z_{in} = 225\Omega$?

3-15. Unknown termination. Consider a transmission line with $Z_0 = 50\Omega$ terminated with an unknown load impedance Z_L. (a) Show that

$$Z_L = Z_0 \frac{1 - jS\tan(\beta l_{min})}{S - j\tan(\beta l_{min})}$$

where l_{min} is the length from the load to the first voltage minimum and S is the standing-wave ratio. (b) Measurements on a line with $Z_0 = 50\Omega$ having an unknown termination Z_L show that $S = \sqrt{3}$, $l_{min} = 25$ mm, and that the distance between successive minima is 10 cm. Find the load reflection coefficient Γ_L and the unknown termination Z_L.

3-16. Distance to the first maximum. Derive a formula similar to that in Problem 3-15 in terms of l_{max}, where l_{max} is the distance from the load to the first voltage maximum.

3-17. Power dissipation. For the lossless transmission line system shown in Figure 3.72, with $Z_0 = 100\Omega$, (a) calculate the time-average power dissipated in each load. (b) Switch the values of the load resistors (i.e., $R_{L1} = 200\Omega$, $R_{L2} = 50\Omega$), and repeat part (a).

3-18. Power dissipation. Consider the transmission line system shown in Figure 3.73. (a) Find the time-average power dissipated in the load R_L with the switch S open. (b) Repeat part (a) for the switch S closed. Assume steady state in each case.

3-19. Power dissipation. Repeat Problem 3-18 if the characteristic impedance of the transmission lines on the source side is changed from 50Ω to $25\sqrt{2}\Omega$.

3-20. Two antennas. Two antennas having feed-point impedances of $Z_{L1} = 40 - j30\Omega$ and $Z_{L2} = 100 + j50\Omega$ are fed with a transmission line system, as shown in Figure 3.74.

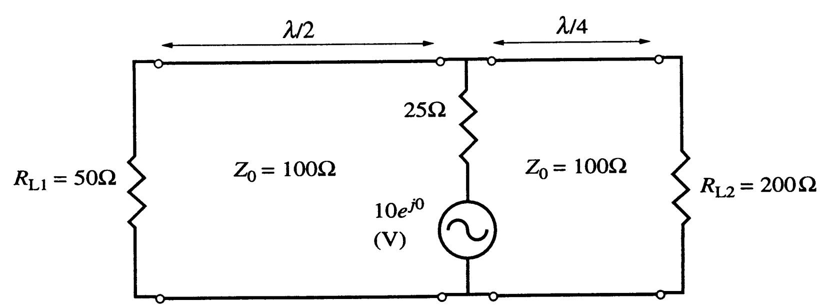

FIGURE 3.72. Power dissipation. Problem 3-17.

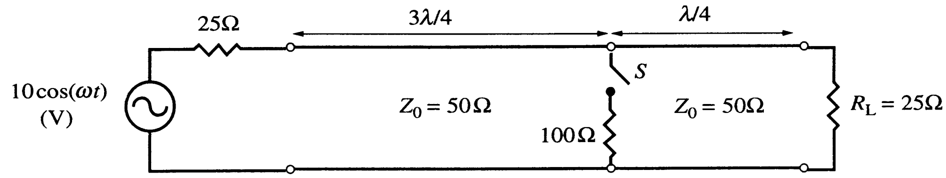

FIGURE 3.73. **Power dissipation.** Problem 3-18.

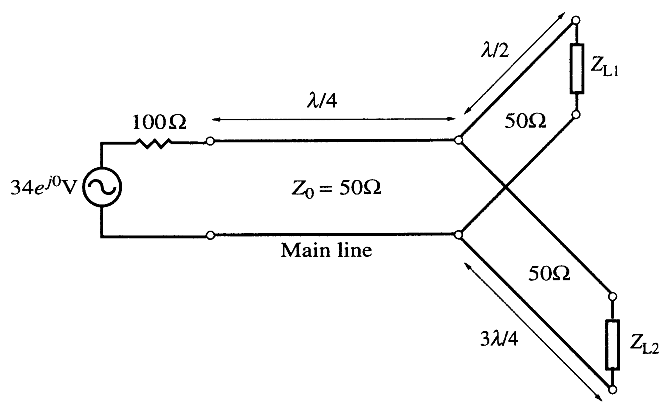

FIGURE 3.74. **Two antennas.** Problem 3-20.

(a) Find S on the main line. (b) Find the time-average power supplied by the sinusoidal source.
(c) Find the time-average power delivered to each antenna. Assume lossless lines.

3-21. Power dissipation. For the transmission line network shown in Figure 3.75, calculate the time-average power dissipated in the load resistor R_L.

3-22. Three identical antennas. Three identical antennas A1, A2, and A3 are fed by a transmission line system, as shown in Figure 3.76. If the feed-point impedance of each antenna is $Z_L = 50 + j50\Omega$, find the time-average power delivered to each antenna.

3-23. Power delivery. For the transmission system shown in Figure 3.77, calculate the percentage of time-average power delivered to R_{L1} and R_{L2} at (a) $f = f_1$, (b) $f = f_2 = 2f_1$, and (c) $f = f_3 = 1.5f_1$.

3-24. Matching with a single lumped element. The transmission line matching networks shown in Figure 3.78 are designed to match a 10Ω load impedance to a 50Ω line. (a) For the network with a shunt element, find the minimum distance l from the load where the unknown shunt element is to be connected such that the input admittance

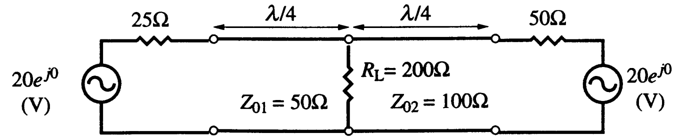

FIGURE 3.75. **Power dissipation.** Problem 3-21.

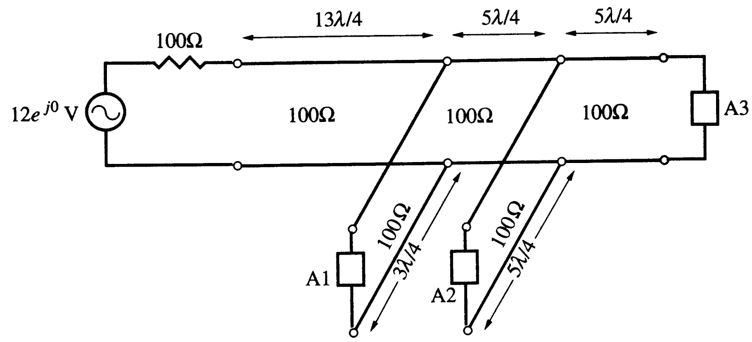

FIGURE 3.76. Three identical antennas. Problem 3-22.

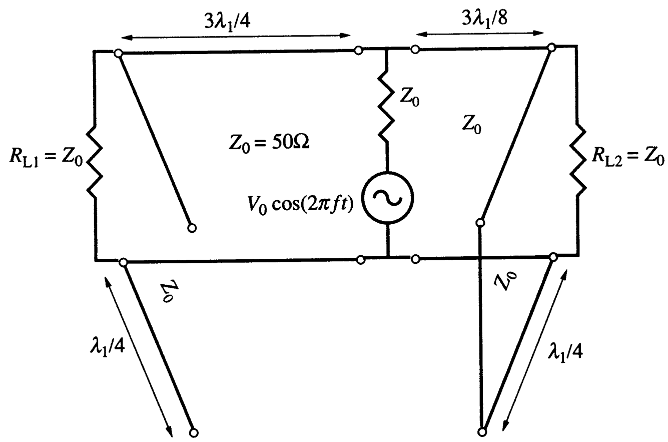

FIGURE 3.77. Power delivery. The normalized line lengths are given at $f = f_1$. Problem 3-23.

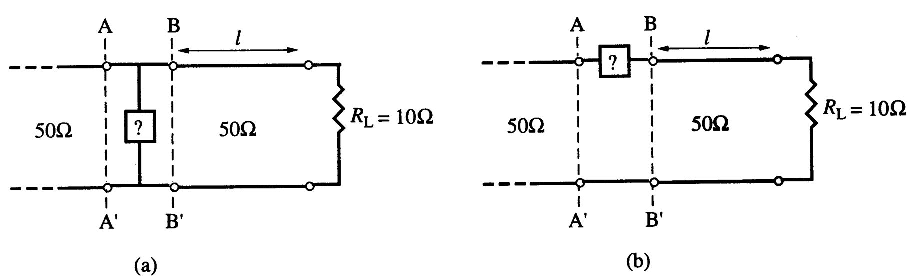

FIGURE 3.78. Matching with a single lumped element. Problem 3-24.

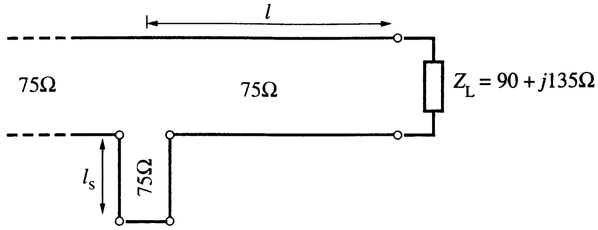

FIGURE 3.79. Matching with series shorted stub. Problem 3-25.

seen at B-B′ has a conductance part equal to 0.02 *S*. (b) Determine the unknown shunt element and its element value such that the input impedance seen at A-A′ is matched to the line (i.e., $Z_{A-A'} = 50\Omega$) at 1 GHz. (c) For the matching network with a lumped series matching element, find the minimum distance l and the unknown element and its value such that a perfect match is achieved at 1 GHz. Assume $v_p = 30$ cm-(ns)$^{-1}$.

3-25. Matching with series stub. A load impedance of $90 + j135\Omega$ is to be matched to a 75Ω lossless transmission line system, as shown in Figure 3.79. If $\lambda = 20$ cm, what minimum length of transmission line l will yield a minimum length l_s for the series stub?

3-26. Open-ended extension. A transmission line with $Z_0 = 50\Omega$ is terminated with a 100Ω load resistance shunted by an open-circuited line having $Z_0 = 50\Omega$ and length 7.4 mm as shown in Figure 3.80. If $\lambda = 10$ cm on both lines, find the length l_s and the position l (measured from the 100Ω load resistance) of the single short-circuited stub to match this load to the line.

3-27. Series stub matching. A series-shorted-stub matching network is designed to match a capacitive load of $R_L = 50\Omega$ and $C_L = 10/(3\pi)$ pF to a 100Ω line at 3 GHz, as shown in Figure 3.81. (a) The stub is positioned at a distance of $\lambda/4$ away from the load. Verify the choice of this position and find the corresponding electrical length of the stub to achieve a perfect match at the design frequency. (b) Calculate the standing-wave ratio S on the main line at 2 GHz. (c) Calculate S on the main line at 4 GHz.

3-28. Quarter-wave transformer. (a) Design a single-section quarter-wave matching transformer to match an $R_L = 20\Omega$ load to a line with $Z_0 = 80\Omega$ operating at 1.5 GHz. (b) Calculate the standing-wave ratio S of the designed circuit at 1.2 and 1.8 GHz.

3-29. Helical antenna. The feed-point impedance of an axial-mode helical antenna with a circumference C on the order of one wavelength is nearly purely resistive and is

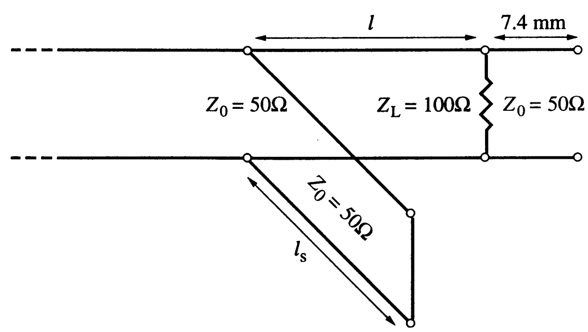

FIGURE 3.80. Open-ended extension. Problem 3-26.

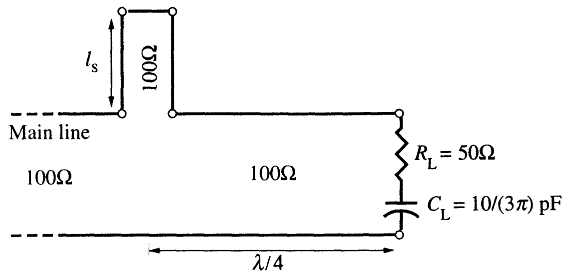

FIGURE 3.81. Series stub matching. Problem 3-27.

approximately given[39] by $R_L \simeq 140(C/\lambda)$, with the restriction that $0.8\lambda \le C \le 1.2\lambda$. Consider a helical antenna designed with a circumference of $C = \lambda_0$ for operation at a frequency f_0 and corresponding wavelength λ_0. The antenna must be matched for use with a 50Ω transmission line at f_0. (a) Design a single-stage quarter-wave transformer to realize the design objective. (b) Using the circuit designed in part (a), calculate the standing-wave ratio S on the 50Ω line at a frequency 15% above the design frequency. (c) Repeat part (b) at a frequency 15% below the design frequency.

3-30. Helical antenna. A helical antenna designed with a feed-point impedance of 125Ω is matched to a 52Ω line by inserting a coaxial transmission line section of characteristic impedance 95Ω and length 0.125λ at a distance of 0.0556λ from the antenna feed point. (See Figure 3.82.) (a) Verify the design by calculating the standing-wave ratio S on the line. (b) Using the same circuit as in part (a), calculate S on the main line at a frequency 20% above the design frequency. Note: Use the approximate expression given in Problem 3-29 to recalculate the feed-point impedance of the helical antenna.

3-31. Quarter-wave matching. Many microwave applications require very low values of S over a broad band of frequencies. The two circuits shown in Figure 3.83 are designed to match a load of $Z_L = R_L = 400\Omega$ to a line with $Z_0 = 50\Omega$, at 900 MHz. The first circuit is an air-filled coaxial quarter-wave transformer, and the second circuit consists of two air-filled coaxial quarter-wave transformers cascaded together. (a) Design both circuits. Assume $Z_{Q1}Z_{Q2} = Z_0Z_L$ for the second circuit. (b) Compare the bandwidth of the two circuits designed by calculating S on each line at frequencies 15% above and below the design frequency.

3-32. Is a match possible? A 75Ω coaxial line is connected directly to an antenna with a feed-point impedance of $Z_L = 156\Omega$. (a) Find the load-reflection coefficient and the standing-wave ratio. (b) An engineer is assigned the task of designing a matching

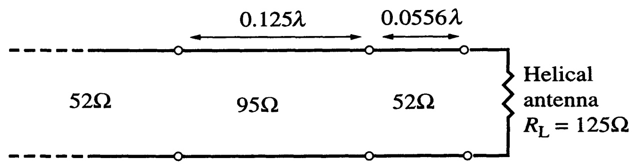

FIGURE 3.82. Helical antenna. Problem 3-30.

[39] See Chapter 7 of J. D. Kraus, *Antennas,* 2nd. ed., McGraw-Hill, 1988.

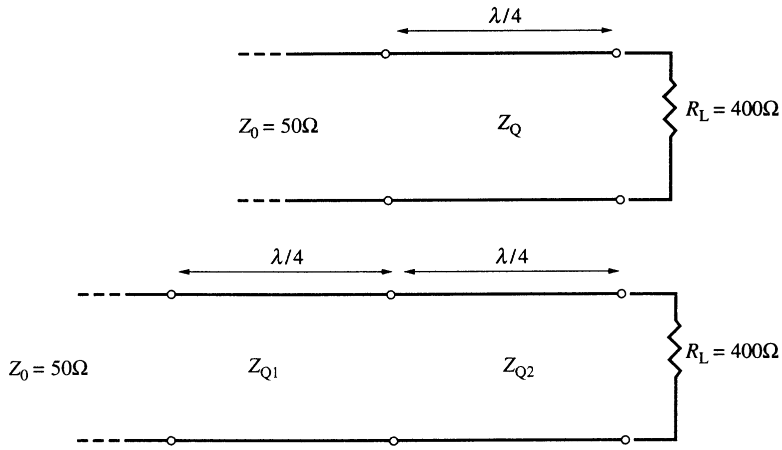

FIGURE 3.83. **Quarter-wave matching.** Problem 3-31.

network to match the feed-point impedance of the antenna to the 75Ω coaxial line. However, all he has available to use for this design is another coaxial line of characteristic impedance 52Ω. Is a match possible?

3-33. L-section matching networks. A simple and practical matching technique is to use the lossless L-section matching network that consists of two reactive elements. (a) Two L-section matching networks marked A1 and A2, each consisting of a lumped inductor and a capacitor, as shown in Figure 3.84, are used to match a load impedance of $Z_L = 60 - j80\Omega$ to a 100Ω line. Determine the L section(s) that make(s) it possible to achieve the design goal, and calculate the appropriate values of the reactive elements at 800 MHz. (b) Repeat part (a) for the two L-section networks marked B1 and B2, consisting of two inductances and two capacitors, respectively.

3-34. Variable capacitor. A shunt stub filter consisting of an air-filled coaxial line terminated in a variable capacitor is designed to eliminate the FM radio frequencies (i.e., 88–108 MHz) on a transmission line with $Z_0 = 100\Omega$, as shown in Figure 3.85. If the

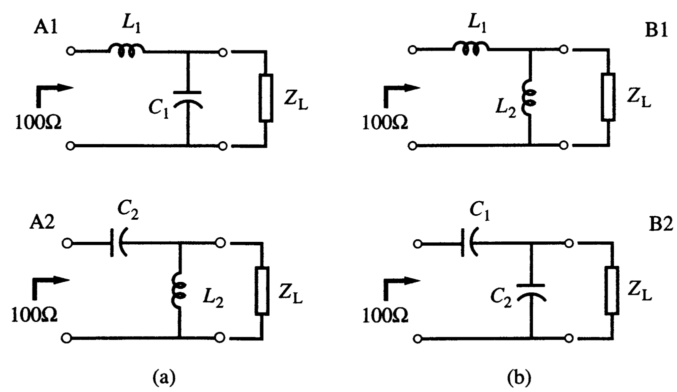

FIGURE 3.84. **L-section matching networks.** Problem 3-33.

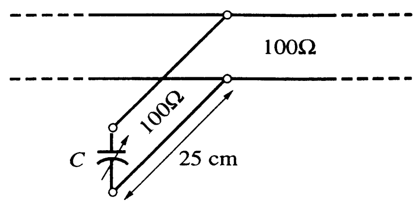

FIGURE 3.85. Variable capacitor. Problem 3-34.

stub length is chosen to be 25 cm, find the range of the variable capacitor needed to eliminate any frequency in the FM band. Assume the characteristic impedance of the stub to be also equal to 100Ω.

3-35. Matching with lumped reactive elements. Two variable reactive elements are positioned on a transmission line to match an antenna having a feed-point impedance of $100 + j100\Omega$ to a $Z_0 = 100\Omega$ air-filled line at 5 GHz, as shown in Figure 3.86. (a) Determine the values of the two reactive elements to achieve matching. (b) If the reactive elements are to be replaced by shorted 50Ω air-filled stubs, determine the corresponding stub lengths.

3-36. Fifth-harmonic filter. The circuit shown in Figure 3.87 has two shunt stubs (one open and one short) that are connected at the same position on a line with $Z_0 = 50\Omega$. The normalized lengths of the two stubs are given at a frequency f_0 (or wavelength λ_0). Assume each stub to have a characteristic impedance of 50Ω. What is the standing-

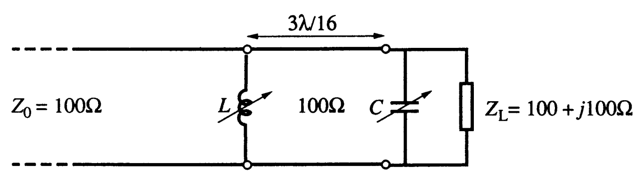

FIGURE 3.86. Matching with lumped reactive elements. Problem 3-35.

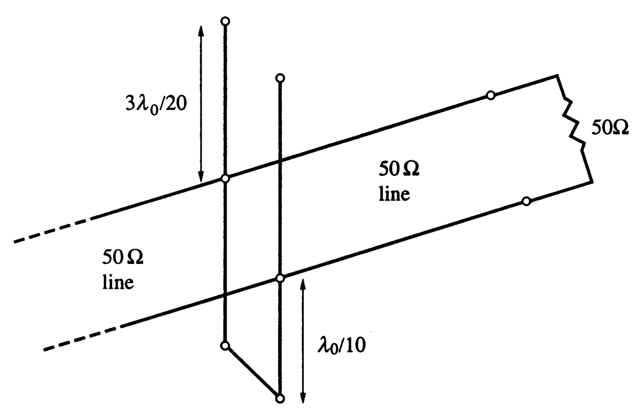

FIGURE 3.87. Fifth-harmonic filter. Problem 3-36.

wave ratio S on the line at f_0? At $3f_0$? At $5f_0$? (Note that this circuit is a fifth-harmonic filter.)

3-37. Standing-wave ratio. For the transmission line shown in Figure 3.88, calculate S on the main line at (a) 800 MHz, (b) 880 MHz, and (c) 960 MHz.

3-38. Quarter-wave matching. (a) For the transmission line system shown in Figure 3.89, determine the value of the characteristic impedance of a quarter-wave transformer (i.e., Z_Q) and its location l with respect to the load needed to achieve matching between Z_L and Z_0. (b) Repeat part (a) for $Z_L = 80 - j60\Omega$.

3-39. Single-stub matching. For the transmission line system shown in Figure 3.90, (a) design a single shorted stub to be as close as possible to the load such that the load is matched to the air line at 3 GHz ($\lambda = 10$ cm). (b) After the matching circuit is built, an engineer experiencing reflections on the main line discovers that the line has an open-circuited extension of 2.5 cm beyond the load position. With the design values of l_s and l found in part (a), what is the actual standing-wave ratio S on the main line caused by the open-circuit extension?

3-40. Quarter-wave transformer. Consider the double quarter-wave transformer system shown in Figure 3.91. (a) Find l, Z_{Q1}, and Z_{Q2} such that the load is matched to the 200Ω line at $\lambda_0 = 12$ cm. Assume $\sqrt{Z_{Q1}Z_{Q2}} = 60\Omega$. (b) Using the values of l, Z_{Q1} found in part (a), find the standing-wave ratio S on the main line at twice the operating frequency.

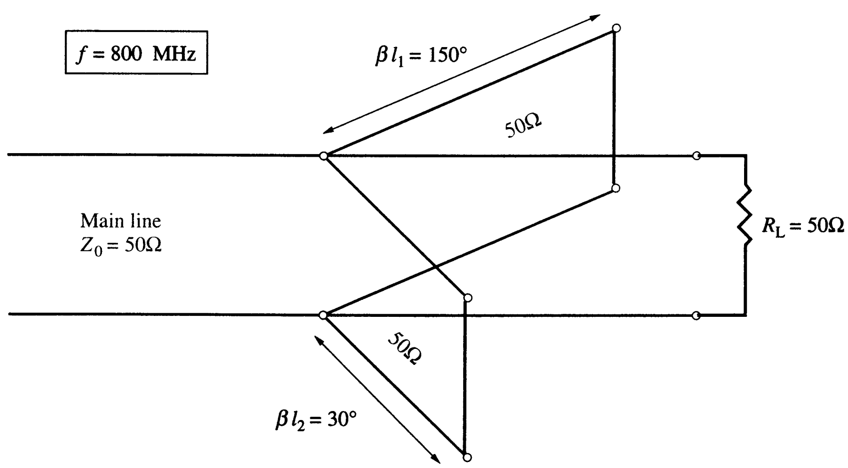

FIGURE 3.88. Standing-wave ratio. Problem 3-37.

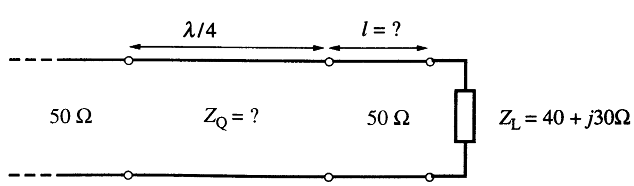

FIGURE 3.89. Quarter-wave matching. Problem 3-38.

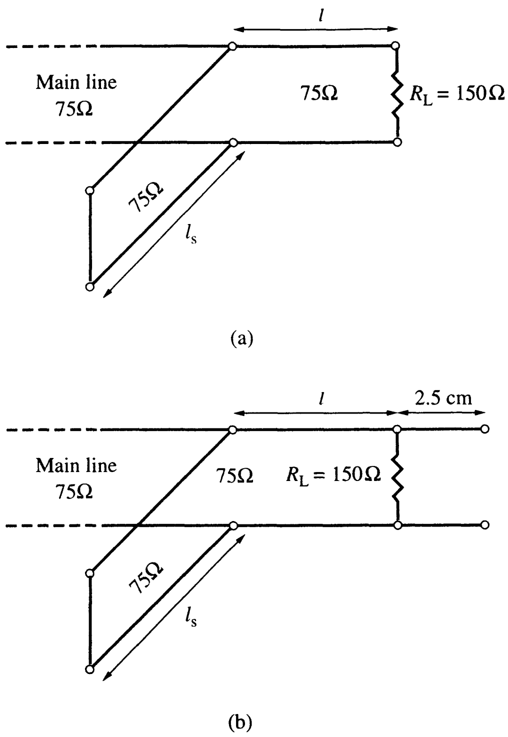

FIGURE 3.90. Single-stub matching. Problem 3-39.

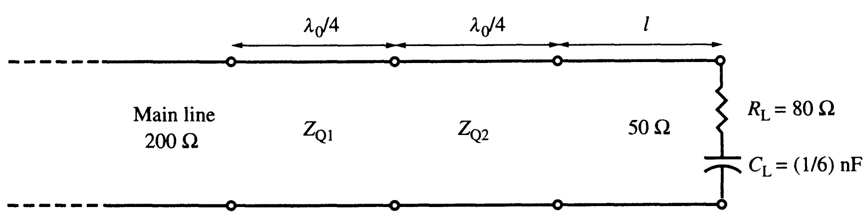

FIGURE 3.91. Quarter-wave transformer. Problem 3-40.

3-41. Unknown feed-point impedance. A 50Ω transmission line is terminated with an antenna that has an unknown feed-point impedance. An engineer runs tests on the line and measures the standing-wave ratio, wavelength, and a voltage minimum location away from the antenna feed point to be, respectively, 3.2, 20 cm, and 74 cm. Use the Smith chart to find the feed-point impedance of the antenna.

3-42. RG218 coaxial line. The RG218 coaxial line is made of copper conductors with polyethylene as the insulator filling. The diameter of the inner conductor and the outer diameter of the insulator are 4.95 mm and 17.27 mm, respectively. The line is to be used at 100 MHz. Find the propagation constant γ and the characteristic impedance Z_0. For polyethylene at 100 MHz, $v_p \simeq 20$ cm-(ns)$^{-1}$, and for a polyethylene-filled coaxial line at 100 MHz, assume $G \simeq 1.58 \times 10^5/[\ln(b/a)]$.

3-43. Two-wire air line. An air-insulated two-wire line made of copper conductors has a characteristic impedance of 500Ω when it is assumed to be lossless. (a) Find the L, C, and R parameters of this line at 144 MHz. Assume a wire diameter of 1.024 mm. (b) Find the propagation constant γ and the characteristic impedance Z_0 with losses included.

3-44. Two-wire matching section. An air-insulated two-wire quarter-wave transmission line section is constructed using a copper wire with diameter 2.54 mm to match a 588Ω load impedance to a 75Ω line at 300 MHz. Assuming the lossless case, find the length and the spacing of the wires of the matching section. (b) Find γ and Z_0 of the matching section with the losses included.

3-45. A parallel-plate line. A certain parallel-plate line is to be made of two copper strips each 5 cm wide and separated by 0.5 cm. The dielectric is air and the frequency of operation is 1 GHz. Find the line parameters L, C, and R, the characteristic impedance Z_0, and the attenuation constant α of the line.

3-46. A lossy high-speed interconnect. The per-unit line parameters of an IC interconnect at 5 GHz are extracted using a high-frequency measurement technique resulting in $R = 143.5\Omega\text{-(cm)}^{-1}$, $L = 10.1\text{ nH-(cm)}^{-1}$, $C = 1.1\text{ pF-(cm)}^{-1}$, and $G = 0.014\text{ S-(cm)}^{-1}$, respectively.[40] Find the propagation constant γ and the characteristic impedance Z_0 of the interconnect at 5 GHz.

3-47. Characterization of a high-speed GaAs interconnect. The propagation constant γ and the characteristic impedance Z_0 at 5 GHz of the GaAs coplanar strip interconnects considered in Example 3-31 are determined from the measurements to be $\gamma \simeq 1.1\text{ np-(cm)}^{-1} + j3\text{ rad-(cm)}^{-1}$ and $Z_0 \simeq 110 - j40\Omega$, respectively. Using these values, calculate the per-unit length parameters (R, L, G, and C) of the coplanar strip line at 5 GHz.

3-48. A lossy high-speed interconnect. Consider a high-speed microstrip transmission line of length 10 cm used to connect a 1-V amplitude, 2-GHz, 50Ω sinusoidal voltage source to an integrated circuit chip having an input impedance of 50Ω. The per-unit parameters of the microstrip line at 2 GHz are measured to be approximately given by $R = 7.5\Omega\text{-(cm)}^{-1}$, $L = 4.6\text{ nH-(cm)}^{-1}$, $C = 0.84\text{ pF-(cm)}^{-1}$, and $G = 0$, respectively. (a) Find the propagation constant γ and the characteristic impedance Z_0 of the line. (b) Find the voltages at the source and the load ends of the line. (c) Find the time-average power delivered to the line by the source and the time-average power delivered to the load. What is the power dissipated along the line?

3-49. Half-wave coaxial line resonator. A $\lambda/2$ resonator is constructed using a piece of copper coaxial line, with an inner conductor of 2-mm diameter and an outer conductor of 8-mm diameter. If the resonant frequency is 3 GHz, find the Q of (a) the air-filled coaxial line and (b) the teflon-filled coaxial line, and compare the results. (c) For an air-filled line, determine the equivalent series RLC circuit parameters, namely R_{eq}, L_{eq}, and C_{eq}. For teflon at 3 GHz, $v_p \simeq 20.7\text{ cm-(ns)}^{-1}$. For a teflon-filled coaxial line at 3 GHz, the per-unit conductance G of a coaxial line is approximately given by $G \simeq 3.3 \times 10^{-4}/[\ln(b/a)]\text{ S-m}^{-1}$.

[40] W. R. Eisenstadt and Y. Eo, S parameter-based IC interconnect transmission line characterization, *IEEE Trans. on Components, Hybrids, and Manufacturing Technology,* 15(4) pp. 483–489, August 1992.