CHAPTER 2

Transient Response of Transmission Lines

Wave motion is said to occur when a disturbance of a physical quantity at a particular point in space at a particular time is related in a definite manner to what occurs at distant points in later times. *Transient* waves occur in response to sudden, usually brief disturbances at a source point, leading to temporary disturbances at distant points at later times. They are thus different from *steady-state* waves, which are sustained by disturbances involving periodic oscillations at the source point.

Transient waves are of importance in many different contexts. Consider, for example, a line of cars waiting at a red traffic light. When the light turns green, the cars do not all start moving at the same time; instead, the first car starts to move first, followed by the car behind it, and so on, as the act of starting to move travels backwards through the line. This wave travels at a speed that depends on the response properties of the cars and the reaction times of the drivers. As another example, when the end of a stretched rope is suddenly moved sideways, the action of moving sideways travels along the rope as a wave whose speed depends on the tension in the rope and its mass. If the rope is infinitely long, the disturbance simply continues to propagate away from its source. If, on the other hand, the distant end of the rope is held fixed, the wave can be reflected back toward the source.

Other examples of transient waves include the thunderclap, the sound wave emitted from an explosion, and seismic waves launched by an earthquake.

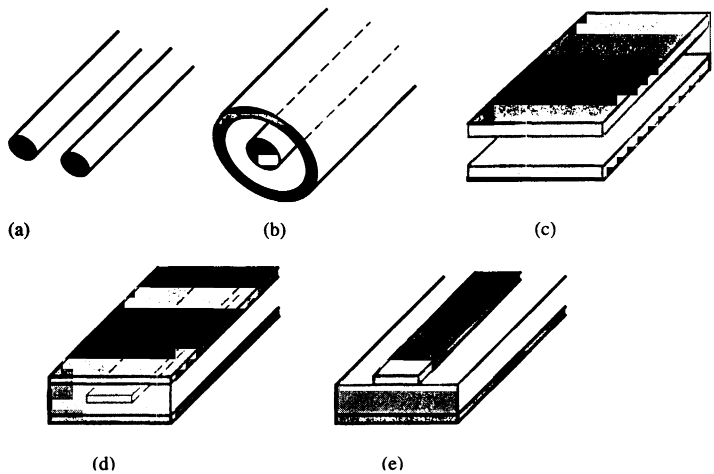

FIGURE 2.1. **Different types of uniform transmission lines:** (a) parallel two-wire; (b) coaxial; (c) parallel-plate; (d) stripline; (e) microstrip.

Transient waves are often used as tools to study the disturbances that create them. The sound wave from a blast can be used for detecting the source of the blast from a long distance; a seismograph measures the strength of a distant earthquake based on tiny transient motions of the earth. Transient waves can also have destructive effects far away from the sources of their initial disturbances. The great Alaskan earthquake of March 27, 1964, produced a giant tsunami, which radiated seaward from Prince William Sound, causing enormous damage when it hit the Hawaiian islands some 5 hours after the disturbance; its remnants produced numerous seiches[1] that sloshed back and forth for more than 24 hours in the various bays, inlets, and harbors along the western coast of North America. Seismic surface waves launched by the same earthquake, propagating by rippling the earth's crust, took 14 minutes to travel from Prince William Sound to the Gulf Coast region of Texas and Louisiana, where they triggered seiches in bays, harbors, rivers, and lakes. Cajun trappers and night fishermen in a Louisiana bayou were surprised to be violently rocked back and forth just after they heard of a large Alaskan earthquake on their radios.

The purpose of this chapter is to study voltage and current transients on electrical transmission lines. A transmission line may consist of two parallel wires (as it will often be illustrated in this book), of coaxial conductors, or of any two conductors separated by an insulating material or vacuum. Some types of two-conductor transmission lines are shown in Figure 2.1.

[1]*Seiche* is a French word that has become the internationally adopted scientific term for transient free (or unforced) surges and oscillations that develop primarily in closed and semiclosed bodies of water. For an excellent, easy-to-read article, see B. J. Korgen, Seiches, *American Scientist,* 83(4), pp. 330–341, July–August, 1995.

Many important electrical engineering applications involve *transients:* temporary variations of current and voltage that propagate down a transmission line. Transients are produced by steplike changes (e.g., sudden on or off) in input voltage or current. Digital signals consist of a sequence of pulses, which represent superposition of successive steplike changes; accordingly, the transient response of transmission lines is of interest in most digital integrated circuit and computer communication applications. Transient transmission line problems arise in many other contexts. The transient response of lines can be used to generate rectangular pulses; the earliest applications of transmission lines involved the use of rectangular pulses for telegraphy. When lightning strikes a power transmission line, a large surge voltage is locally induced and propagates to other parts of the line as a transient.

This chapter is unique; the following chapters are mostly concerned with applications that either involve static quantities, which do not vary in time, or involve steady signals that are sinusoids or modified sinusoids. However, with the rapid advent of digital integrated circuits, digital communications, and computer communication applications, transient responses of transmission lines are becoming increasingly important. Increasing clock speeds make signal integrity[2] analysis a must for the design of high-speed and high-performance boards. Managing signal integrity in today's high-speed printed circuit boards and multichip modules involves features such as interconnect lengths, vias, bends, terminations, and stubs and often requires close attention to transmission line or distributed circuit effects.[3] It is thus fitting that we start our discussion of engineering electromagnetics by studying the transient response of transmission lines. Also, analysis of transients on transmission lines requires relatively little mathematical complexity and brings about an intuitive understanding of concepts such as wave propagation and reflection, which facilitates a better understanding of the following chapters.

Our discussions in this chapter start in Section 2.1 with a heuristic analysis of transmission line behavior, in particular the response to a step (on or off) input, and a discussion of lumped circuit models. Section 2.2 introduces the fundamental circuit equations for a transmission line and their solution for lossless lines in terms of traveling waves. The reflection of the waves at the termination of a transmission line and the step response of lossless lines with open and short circuit terminations are presented in Section 2.3. Section 2.4 covers the step and pulse responses of lossless transmission lines terminated in resistive loads or in other transmission lines, while Section 2.5 treats the cases of reactive loads and loads with nonlinear current-voltage characteristics. Selected practical topics are presented in Section 2.6, followed by a brief discussion of the parameters of some commonly used practical transmission lines in Section 2.7.

[2]The term *signal integrity* refers to the issues of timing and quality of the signal. The timing analysis is performed to ensure that the signal arrives at its destination within a specified time interval and that the signal causes correct logic switching without false switching or transient oscillations, which can result in excessive time delays. See R. Kollipara and V. Tripathi, Printed wiring boards, Chapter 71 in J. C. Whitaker (ed.), *The Electronics Handbook,* CRC Press, 1996, pp. 1069–1083.

[3]See R. Goyal, Managing signal integrity, *IEEE Spectrum,* pp. 54–58, March 1994.

2.1 HEURISTIC DISCUSSION OF TRANSMISSION LINE BEHAVIOR AND CIRCUIT MODELS

Typically, we explain the electrical behavior of a two-conductor transmission line in terms of an equal and opposite current flowing in the two conductors, as measured at any given transverse plane. The flow of this current is accompanied by magnetic fields set up around the conductors (Ampère's law), and when these fields change with time, a voltage (electromotive force) is induced in the conductors (Faraday's law), which affects the current flow.[4] This behavior can be represented by a small inductance associated with each short-length segment of the conductors. Also, any two conductors separated by a distance (such as the short sections of two conductors facing one another) have nonzero capacitance, so that when equal and opposite charges appear on them, there exists a potential drop across them. Hence, each short section of a two-conductor line exhibits some series inductance and parallel capacitance.[5] The values of the inductance and capacitance depend on the physical configuration and material properties of the two-conductor line, including the surface areas, cross-sectional shape, spacing,[6] and layout of the two conductors as well as the electrical and magnetic properties[7] of the substance filling the space between and around the conductors.

2.1.1 Heuristic Discussion of Transmission Line Behavior

We can qualitatively understand the behavior of a two-conductor transmission line by considering a lossless circuit model of the line, consisting of a large number of series inductors and parallel capacitors connected together, representing the short sections Δz of the line, as illustrated in Figure 2.2.

To illustrate the behavior of a lossless transmission line, we now consider the simplest possible transient response: the step response, which occurs upon the sudden application of a constant voltage. At $t = 0$, a battery of voltage V_0 and source resistance R_{s1} is connected to the infinitely long two-conductor transmission line represented by the lossless circuit shown in Figure 2.2, where each pair of

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[4]Detailed discussion of these experimentally based physical laws will be undertaken later; here we simply rely on their qualitative manifestations, drawing on the reader's exposure to these concepts at the freshman physics level.

[5]Neglecting losses for now.

[6]This can be seen at a qualitative level, from the reader's understanding of capacitance and inductance at the freshman physics level. For example, the closer the conductors are to each other, the larger the capacitance is. On the other hand, the inductance of a two-conductor line is smaller if the conductors are closer together, since the magnetic field produced by the current flow is linked by a smaller area, thus inducing less voltage.

[7]At the simplest level, the magnetic properties of a material represent the ability of a material medium to store magnetic energy. Similarly, by electrical properties we refer to the ability of a material to store electric energy or its response to an applied electric field. The microscopic behavior of the materials that determines these properties will be discussed in Chapters 4 and 6.

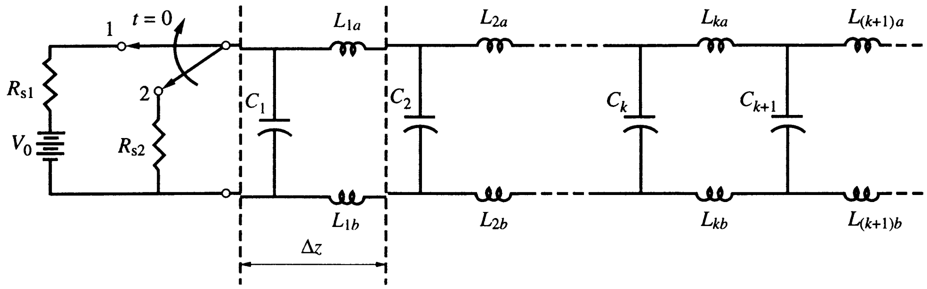

FIGURE 2.2. Circuit model of a two-conductor lossless line.

inductances corresponds to the inductance of a short section of the line of length Δz, and each capacitor corresponds to the capacitance of the same section of length Δz. Initially, the transmission line is completely discharged, so all the capacitances have zero charge (and thus zero voltage) and the inductances have zero current flowing through them. The switch in Figure 2.2 is moved to position 1 at $t = 0$, so that, starting at $t = 0^+$, the source voltage V_0 appears across the source resistance R_{s1} and the terminals of the capacitance C_1, which takes time to charge;[8] and until it charges, there is no voltage across it to drive currents through L_{1a} and L_{1b}. As the voltage across C_1 builds up, the currents in L_{1a} and L_{1b} also take time to increase.[9] When these currents increase enough to cause appreciable flow through C_2, this capacitance now takes some time to charge. As it charges, current starts to flow in inductors L_{2a} and L_{2b}, but this takes time. This same process continues all the way down the line, with the capacitor C_k not starting to charge until the preceding capacitors are charged, just as if it did not know yet that the voltage step had been applied at the input. In this way, the information about the change in the position of the switch travels down the line.

When the switch moves back to position 2 at $t = t_1$, the reverse happens: C_1 has to discharge through R_{s2}, which pulls current (not suddenly) from L_{1a} and L_{1b}, which in turn allows C_2 to discharge, and so on. All of this takes time, so C_k is not affected by the removal of the input signal until the preceding capacitors are discharged first. The rate of charging and discharging depends only on the circuit element values, so the charging and discharging disturbances both continue down the line at the same speed, since $L_{ia} = L_{jb}$ for all i, j and all C_i values are equal, assuming a uniform transmission line.

Note from the above discussion that if the inductance of the line segments is negligible, the line can be approximated as a lumped capacitor (equal to the parallel combination of all of its distributed shunt capacitances); all the points on the line are then at the same potential, and traveling-wave effects are not important. The line inductance becomes important if the line is relatively long or if the rise time of the applied signal (as defined in Figure 1.3) is so fast that the current through the inductor

[8]Voltage across a capacitance cannot change instantaneously.

[9]Current through an inductor cannot change instantaneously.

increases very rapidly, producing appreciable voltage drop ($\mathcal{V} = L\, d\mathcal{I}/dt$) across the inductor even if the value of L is small. By the same token, it is intuitively clear that, even if the line is long (and thus L is large), transmission line effects will be negligible for slow enough rise times, as was discussed in Chapter 1.

2.1.2 Circuit Models of Transmission Lines

It is often more useful to describe transmission line behavior in terms of inductance and capacitance *per unit length,* rather than viewing the line as an infinite number of discrete inductances and capacitances, as implied in Figure 2.2. We must also note that, in general, the conductors of a transmission line exhibit both inductance and resistance and that there can be leakage losses through the material surrounding the conductors. The inductance per unit length (L) of the line, in units of henrys per meter, depends on the physical configuration of the conductors (e.g., the separation between conductors and their cross-sectional shape and dimensions) and on the magnetic properties of the material surrounding the conductors. The series resistance per unit length R, in units of ohms per meter, depends on the cross-sectional shape, dimensions, and electrical conductivity of the conductors[10] and the frequency of operation. Between the conductors there is a capacitance (C), expressed as farads per meter; there is also a leakage conductance (G) of the material surrounding the conductors, in units of siemens per meter. The capacitance depends on the shape, surface area, and proximity of the conductors as well as the electrical properties of the insulating material surrounding the conductors. The conductance depends on the shape and dimensions of the outside surface of the conductors[11] and on the degree to which the insulating material is lossy. A ***uniform*** transmission line consists of two conductors of uniform cross section and spacing throughout their length, surrounded by a material that is also uniform throughout the length of the line. An equivalent circuit of a uniform transmission line can be drawn in terms of the per-unit-length parameters, which are the same throughout the line. One such circuit is shown in Figure 2.3, where each short section of length Δz of the line is modeled as a lumped circuit whose element values are given in terms of the per-unit parameters of the line. The electrical behavior of a uniform transmission line can be studied in terms of such a circuit model if the length of the line (Δz) represented by a single L-R-C-G section is very small compared to, for example, the wavelength of electromagnetic waves in the surrounding material at the frequency of operation. Four different circuit models are shown in Figure 2.4.

Expressions for L, R, C, and G for some of the commonly used uniform transmission lines shown in Figure 2.1 are provided in Section 2.7. The values of these quantities depend on the geometric shapes and the cross-sectional dimensions

[10]The resistance simply represents the ohmic losses due to the current flowing through the conductors; hence, it depends on the cross-sectional area and the conductivity (see Chapter 5) of the conductors.

[11]The leakage current flows from one conductor to the other, through the surrounding material, and in the direction transverse to the main current flowing along the conductors of the line; hence, it depends on the outer surface area of the conductors.

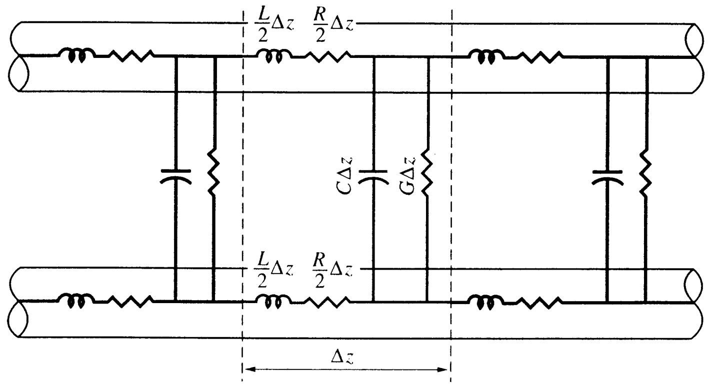

FIGURE 2.3. Distributed circuit of a uniform transmission line.

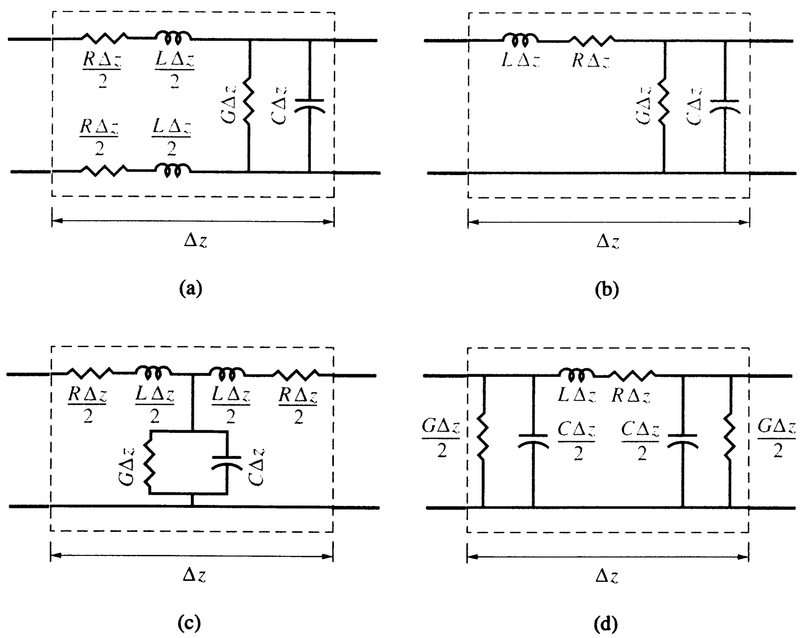

FIGURE 2.4. Lumped circuit models for a short segment of a uniform transmission line.

of the line, the electrical conductivity of the metallic conductors used, the electrical and magnetic properties of the surrounding medium, and the frequency of operation. The expressions for L, R, C, and G for the various lines can be obtained by means of electromagnetic field analysis of the particular geometries involved. For some cases (such as the parallel wire, coaxial line, and parallel plate lines shown in Figure 2.1), compact analytical expressions for R, L, C, and G can be found. For other, more complicated structures (e.g., the stripline and microstrip lines in Figure 2.1), calculation of the R, L, C, and G parameters usually requires numerical computation. Methods for the determination of transmission line parameters are discussed in Chapters 4 through 6 as we introduce the governing electromagnetic equations, so that we can formally derive the expressions for the transmission line parameters. For our distributed circuit analyses of transmission lines in this and the following chapter, it suffices to know that the values of L, R, C, and G are directly calculable for any uniform transmission line configuration. We can thus proceed by using their values as specified by the expressions in Section 2.7, as given in handbooks, or as measured in specific cases.

2.2 TRANSMISSION LINE EQUATIONS AND WAVE SOLUTIONS

In this section we develop the fundamental equations that govern wave propagation along general two-conductor transmission lines. Various lumped circuit models of a single short segment of a transmission line are shown in Figure 2.4. In the limit of $\Delta z \to 0$, any one of the circuit models of Figure 2.4 can be used to derive the fundamental transmission line equations. In the following, we use the simplest of these models (Figure 2.4b), shown in further detail in Figure 2.5.

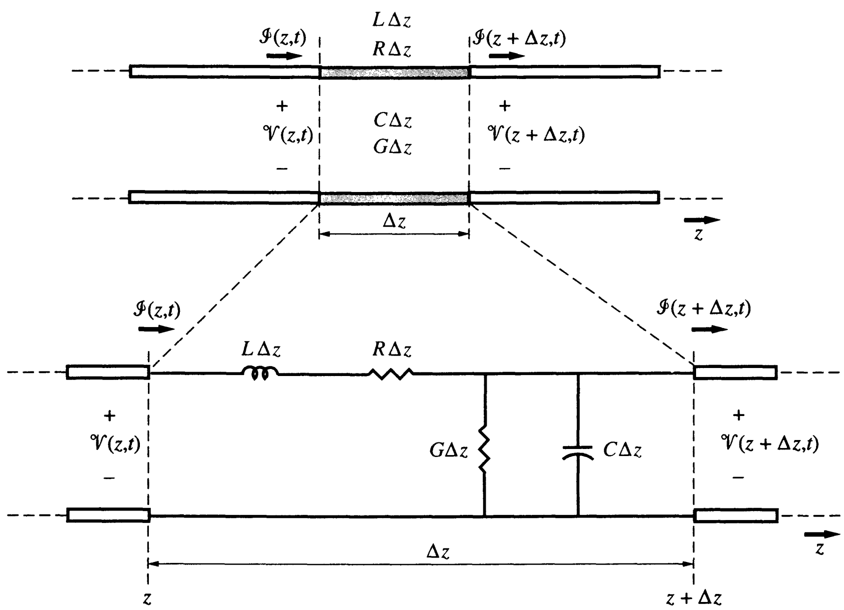

FIGURE 2.5. Equivalent circuit of a short length of Δz of a two-conductor transmission line.

2.2.1 Transmission Line Equations

The section of line of length Δz in Figure 2.5 is assumed to be located at a distance z from a selected point of reference along the transmission line. We consider the total voltage and current at the input and output terminals of this line section: that is, at points z and $(z + \Delta z)$. The input and output voltages and currents are denoted as $\mathcal{V}(z, t)$, $\mathcal{I}(z, t)$ and $\mathcal{V}(z + \Delta z, t)$, $\mathcal{I}(z + \Delta z, t)$, respectively. Using Kirchhoff's voltage law, we can see that the difference in voltage between the input and output terminals is due to the voltage drop across the series elements $R\Delta z$ and $L\Delta z$, so we have

$$\mathcal{V}(z + \Delta z, t) - \mathcal{V}(z, t) = -R\Delta z\ \mathcal{I}(z, t) - L\Delta z \frac{\partial \mathcal{I}(z, t)}{\partial t}$$

Note that we shall consider Δz to be as small as needed so that the lumped circuit model of the segment can accurately represent the actual distributed line. In the limit when $\Delta z \to 0$, we have

$$\lim_{\Delta z \to 0} \frac{\mathcal{V}(z + \Delta z, t) - \mathcal{V}(z, t)}{\Delta z} = \frac{\partial \mathcal{V}(z, t)}{\partial z} = -R\mathcal{I}(z, t) - L\frac{\partial \mathcal{I}(z, t)}{\partial t}$$

or

$$\boxed{\frac{\partial \mathcal{V}(z, t)}{\partial z} = -\left(R + L\frac{\partial}{\partial t}\right)\mathcal{I}(z, t)} \qquad [2.1]$$

Similarly, using Kirchhoff's current law, the difference between the current at the input and output terminals is equal to the total current through the parallel elements $G\Delta z$ and $C\Delta z$, so we have

$$\mathcal{I}(z + \Delta z, t) - \mathcal{I}(z, t) = -G\Delta z\ \mathcal{V}(z + \Delta z, t) - C\Delta z \frac{\partial \mathcal{V}(z + \Delta z, t)}{\partial t}$$

Upon dividing by Δz and expanding $\mathcal{V}(z + \Delta z, t)$ in a Taylor series,[12] and taking $\Delta z \to 0$, we have:

$$\lim_{\Delta z \to 0} \left\{ \frac{\mathcal{I}(z + \Delta z, t) - \mathcal{I}(z, t)}{\Delta z} \right\} = -G\mathcal{V}(z, t) - C\frac{\partial \mathcal{V}(z, t)}{\partial t} - \lim_{\Delta z \to 0} \{\Delta z(\cdots)\}$$

or

$$\boxed{\frac{\partial \mathcal{I}(z, t)}{\partial z} = -\left(G + C\frac{\partial}{\partial t}\right)\mathcal{V}(z, t)} \qquad [2.2]$$

Equations [2.1] and [2.2] are known as the *transmission line equations* or *telegrapher's equations*. We shall see in Chapter 8 that uniform plane electromagnetic wave propagation is based on very similar equations, written in terms of the

[12] $\mathcal{V}(z + \Delta z, t) = \mathcal{V}(z, t) + [\partial \mathcal{V}(z, t)/\partial z]\Delta z + \cdots$

components of electric and magnetic fields instead of voltages and currents. Most other types of wave phenomena are governed by similar equations; for acoustic waves in fluids, for example, one replaces voltage with pressure and current with velocity.

2.2.2 Traveling-Wave Solutions for Lossless Lines

Solutions of [2.1] and [2.2] are in general quite difficult and require numerical treatments for the general transient case, when all of the transmission line parameters are nonzero. However, in a wide range of transmission line applications the series and parallel loss terms (R and G) can be neglected, in which case analytical solutions of [2.1] and [2.2] become possible. In fact, practical applications in which transmission lines can be treated as lossless lines are at least as common as those in which losses are important. Accordingly, our transmission line analyses in this chapter deal exclusively with lossless transmission lines. A brief discussion of transients on lossy lines is provided in Section 2.6.3, and steady-state response of lossy lines is discussed in Section 3.8.

We now apply [2.1] and [2.2] to the analysis of the transient response of lossless transmission lines. For a lossless line we have $R = 0$ and $G = 0$, so that [2.1] and [2.2] reduce to

$$\frac{\partial \mathcal{V}}{\partial z} = -L\frac{\partial \mathcal{I}}{\partial t} \qquad [2.3]$$

$$\frac{\partial \mathcal{I}}{\partial z} = -C\frac{\partial \mathcal{V}}{\partial t} \qquad [2.4]$$

By combining [2.3] and [2.4] we obtain the *wave equations* for voltage and current,

$$\boxed{\frac{\partial^2 \mathcal{V}}{\partial z^2} = LC\frac{\partial^2 \mathcal{V}}{\partial t^2}} \qquad [2.5]$$

or

$$\boxed{\frac{\partial^2 \mathcal{I}}{\partial z^2} = LC\frac{\partial^2 \mathcal{I}}{\partial t^2}} \qquad [2.6]$$

Either one of [2.5] or [2.6] can be solved for the voltage or current. We follow the usual practice and consider the solution of the voltage equation [2.5], which can be rewritten as

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 \mathcal{V}}{\partial t^2}; \qquad v_p = \frac{1}{\sqrt{LC}} \qquad [2.7]$$

Note that once $\mathcal{V}(z, t)$ is determined, we can use [2.3] or [2.4] to find $\mathcal{I}(z, t)$. The quantity v_p represents the speed of propagation of a disturbance, as will be evident in the following discussion. For reasons that will become clear in Chapter 3, v_p is also referred to as the *phase velocity;* hence the subscript p.

Any function[13] $f(\cdot)$ of the variable $\xi = (t - z/v_p)$ is a solution of [2.7]. To see that

$$\mathcal{V}(z, t) = f\left(t - \frac{z}{v_p}\right) = f(\xi)$$

is a solution of [2.7], we can express the time and space derivatives of $\mathcal{V}(z, t)$ in terms of the derivatives of $f(\xi)$ with respect to ξ:

$$\frac{\partial \mathcal{V}}{\partial t} = \frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial f}{\partial \xi} \qquad \text{and} \qquad \frac{\partial^2 \mathcal{V}}{\partial t^2} = \frac{\partial^2 f}{\partial \xi^2}\frac{\partial \xi}{\partial t} = \frac{\partial^2 f}{\partial \xi^2}$$

since $\partial \xi / \partial t = 1$. Similarly, noting that $\partial \xi / \partial z = -(1/v_p)$, we have

$$\frac{\partial \mathcal{V}}{\partial z} = -\frac{1}{v_p}\frac{\partial f}{\partial \xi} \qquad \text{and} \qquad \frac{\partial^2 \mathcal{V}}{\partial z^2} = \frac{1}{v_p^2}\frac{\partial^2 f}{\partial \xi^2}$$

Substituting in [2.7] we find that the wave equation is indeed satisfied by any function $f(\cdot)$ of the variable $\xi = (t - z/v_p)$.

That an arbitrary function $f(t - z/v_p)$ represents a wave traveling in the $+z$ direction is illustrated in Figure 2.6 for $v_p = 1$ m-s^{-1}. By comparing $f(t - z/v_p)$ at two different times $t = 2$ and $t = 3$ s, we note that the function maintains its shape and moves in the $+z$ direction as time t advances, as seen in Figure 2.6a. Similarly, by comparing $f(t - z/v_p)$ at two different positions $z = 0$ and $z = 1$ m, we note that the function maintains its shape but appears at $z = 1$ m exactly 1 s after its appearance at $z = 0$, as seen in Figure 2.6b. Figure 2.6c shows a three-dimensional display of $f(t - z/v_p)$ as a function of time at different points z_1, z_2, and z_3. To determine the speed with which the function moves in the $+z$ direction, we can simply keep track of any point on the function by setting its argument to a constant. In other words, we have

$$t - \frac{z}{v_p} = \text{const} \quad \rightarrow \quad \frac{dz}{dt} = v_p$$

The speed of propagation of waves on a transmission line is one of its most important characteristics. It is evident from [2.7] that v_p depends on the line inductance L and C. In the case of most (all except the microstrip line) of the commonly used two-conductor transmission lines shown in Figure 2.1, the phase velocity v_p in the absence of losses is not a function of the particular geometry of the metallic conductors

[13] An important function of $(t - z/v_p)$ that is encountered often and that we shall study in later chapters is the sinusoidal traveling-wave function, $A\cos[\omega(t - z/v_p)]$. Depending on the location of the observation point z along the z axis, this function replicates the sinusoidal variation $A\cos(\omega t)$ observed at $z = 0$, except delayed by (z/v_p) seconds at the new z. Thus, (z/v_p) represents a time shift, or delay, which is a characteristic of the class of wave functions of the variable $(t - z/v_p)$.

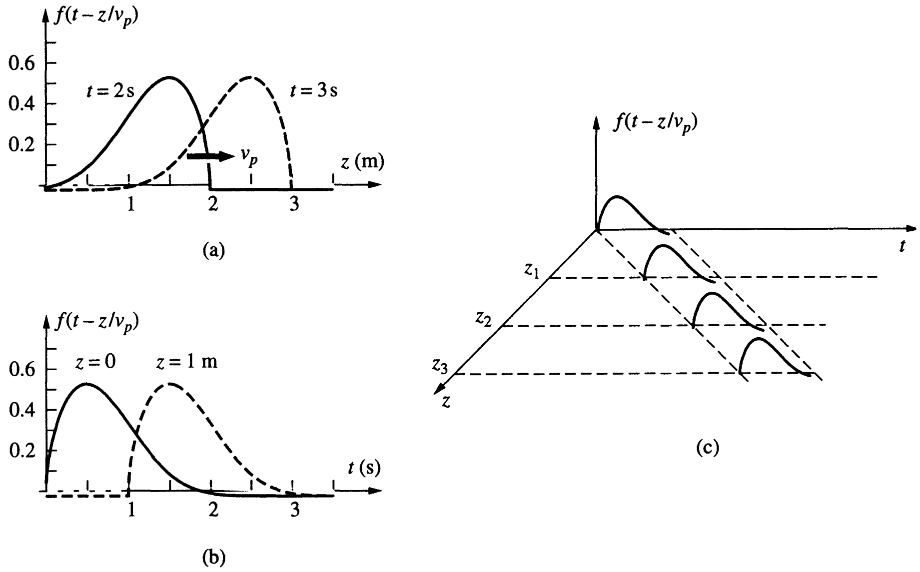

FIGURE 2.6. **Variation in space and time of an arbitrary function $f(t - z/v_p)$.** The phase velocity is taken to be equal to unity, i.e., $v_p = 1$ m/s. (a) $f(t - z/v_p)$ versus z at two different times. (b) $f(t - z/v_p)$ versus t at two different locations. (c) Three-dimensional display of $f(t - z/v_p)$ as a function of time at different points z_1, z_2, and z_3.

but is solely determined by the electrical and magnetic properties of the surrounding medium.[14] When the medium surrounding the metallic conductors is air, the phase velocity is equal to the speed of light in free space, namely $v_p = c$. The propagation speeds for other insulating materials are tabulated in Table 2.1.

It is clear from the above analysis that any function of the argument $(t + z/v_p)$ is an equally valid solution of [2.7]. Thus, the general solution for the voltage $\mathcal{V}(z, t)$ is

$$\mathcal{V}(z, t) = f^{+}\left(t - \frac{z}{v_p}\right) + f^{-}\left(t + \frac{z}{v_p}\right) \qquad [2.8]$$

where $f^{+}(t - z/v_p)$ and $f^{-}(t + z/v_p)$, respectively, represent waves traveling in the $+z$ and $-z$ directions. Note that $f^{+}(\cdot)$ and $f^{-}(\cdot)$ can in general be completely different functions.

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[14]This result will become evident in Chapters 4 and 6 as we determine the capacitance and inductance of selected transmission lines from first principles. That $v_p = (LC)^{-1/2}$ does not depend on the geometrical arrangement of the conductors can also be seen by considering the inductance and capacitance expressions given in Table 2.2, Section 2.7. For transmission lines that do not exhibit symmetry in the cross-sectional plane, such as the microstrip line of Figure 2.1e, the phase velocity depends in a complicated manner on the properties of the surrounding material, the shape and dimensions of the conductors, and the operating frequency. See Section 8.6 of S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics,* 3rd ed., John Wiley & Sons, New York, 1994.

TABLE 2.1. Propagation speeds in some materials

Material	**Propagation speed at ~20°C (cm/ns at 3 GHz)**
Air	30
Glass	(3 to 15)*
Mica (ruby)	12.9
Porcelain	(10 to 13)*
Fused quartz (SiO_2)	15.4
Alumina (Al_2O_3)	10.1
Polystyrene	18.8
Polyethylene	20.0
Teflon	20.7
Vaseline	20.4
Amber (fossil resin)	18.6
Wood (balsa)	27.2
Water (distilled)	3.43
Ice (pure distilled)	16.8**
Soil (sandy, dry)	18.8

*Approximate range valid for most types of this material.
**At −12°C.

To find the general solution for the current $\mathcal{I}(z, t)$ associated with the voltage $\mathcal{V}(z, t)$, we can substitute [2.8] in [2.3] and [2.4], integrate with respect to time, and then take the derivative with respect to z to find

$$\mathcal{I}(z, t) = \frac{1}{Z_0}\left[f^+\left(t - \frac{z}{v_p}\right) - f^-\left(t + \frac{z}{v_p}\right)\right]; \qquad Z_0 \equiv \sqrt{\frac{L}{C}} \qquad [2.9]$$

where Z_0 is known as the *characteristic impedance* of the transmission line.[15] The characteristic impedance is the ratio of voltage to current for a single wave propagating in the $+z$ direction, as is evident from [2.8] and [2.9]. Note that the current associated with the wave traveling in the $-z$ direction (i.e., toward the left) has a negative sign—as expected, since the direction of positive current as defined in Figure 2.5 is to the right. In other words, since the polarity of voltage and the direction of current are defined so that the voltage and current have the same signs for forward (to the right)-traveling waves, the voltage and current for waves traveling to the left have opposite signs.

[15]To find $\mathcal{I}(z, t)$, we can also note that the wave equation [2.6] for the current is identical to equation [2.5] for voltage, so its solution should have the same general form. Thus, the general solution for the current should be

$$\mathcal{I}(z, t) = g^+\left(t - \frac{z}{v_p}\right) + g^-\left(t + \frac{z}{v_p}\right)$$

Substituting this expression for $\mathcal{I}(z, t)$ and [2.8] into [2.3] or [2.4] yields $g^+ = Z_0^{-1}f^+$ and $g^- = -Z_0^{-1}f^-$.

The characteristic impedance of a line is one of the most important parameters in the equations describing its behavior as a distributed circuit. For lossless lines, as considered here, Z_0 is a real number having units of ohms. Since Z_0 for a lossless line depends only on L and C, and since these quantities can be calculated from the geometric shape and physical dimensions of the line and the properties of the surrounding material, Z_0 can be expressed in terms of these physical dimensions for a given type of line. Formulas for Z_0 for some common lines are provided in Section 2.7. Characteristic impedances for other types of transmission lines are given elsewhere. [16]

The following example illustrates the meaning of the characteristic impedance of a line. An infinitely long and initially uncharged line (i.e., all capacitors and inductors in the distributed circuit have zero initial conditions) is considered, so there is no need for the $f(t + z/v_p)$ term in either [2.8] or [2.9], which would be produced only as a result of the reflections of the voltage disturbance at the end of the line.

Example 2-1: Step response of an infinitely long lossless line. As a simple example of the excitation of a transmission line by a source, consider an infinitely long lossless transmission line characterized by L and C and connected to an ideal step voltage source of amplitude V_0 and source resistance R_s, as shown in Figure 2.7a. Find the voltage, the current, and power propagating down the transmission line.

Solution: Before $t = 0$, the voltage and current on the line are identically zero, since the line is assumed to be initially uncharged. At $t = 0$, the step voltage source changes from 0 to V_0, launching a voltage $\mathcal{V}^+(z, t)$ propagating toward the right. In the absence of a reflected wave (infinitely long line), the accompanying current $\mathcal{I}^+(z, t) = (Z_0)^{-1}\mathcal{V}^+(z, t)$, as can be seen from equations [2.8] and [2.9]. In other words, the characteristic impedance $Z_0 = \sqrt{L/C}$ is the resistance that the transmission line *initially* presents to the source, as shown in the equivalent circuit of Figure 2.7b. Accordingly, the initial voltage and current established at the source end ($z = 0$) of the line are

$$\mathcal{V}_s(t) = \mathcal{V}^+(z = 0, t) = \frac{V_0 Z_0}{Z_0 + R_s}$$

and

$$\mathcal{I}_s(t) = \mathcal{I}^+(z = 0, t) = \frac{V_0}{Z_0 + R_s}$$

The propagation of the voltage $\mathcal{V}^+(z, t)$ and the current $\mathcal{I}^+(z, t)$ down the line are illustrated in Figures 2.7c and 2.7d at $t = l/v_p$ as a function of z.

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[16] *Reference Data for Engineers,* 8th ed., Sams Prentice Hall Computer Publishing, Carmel, Indiana, 1993.

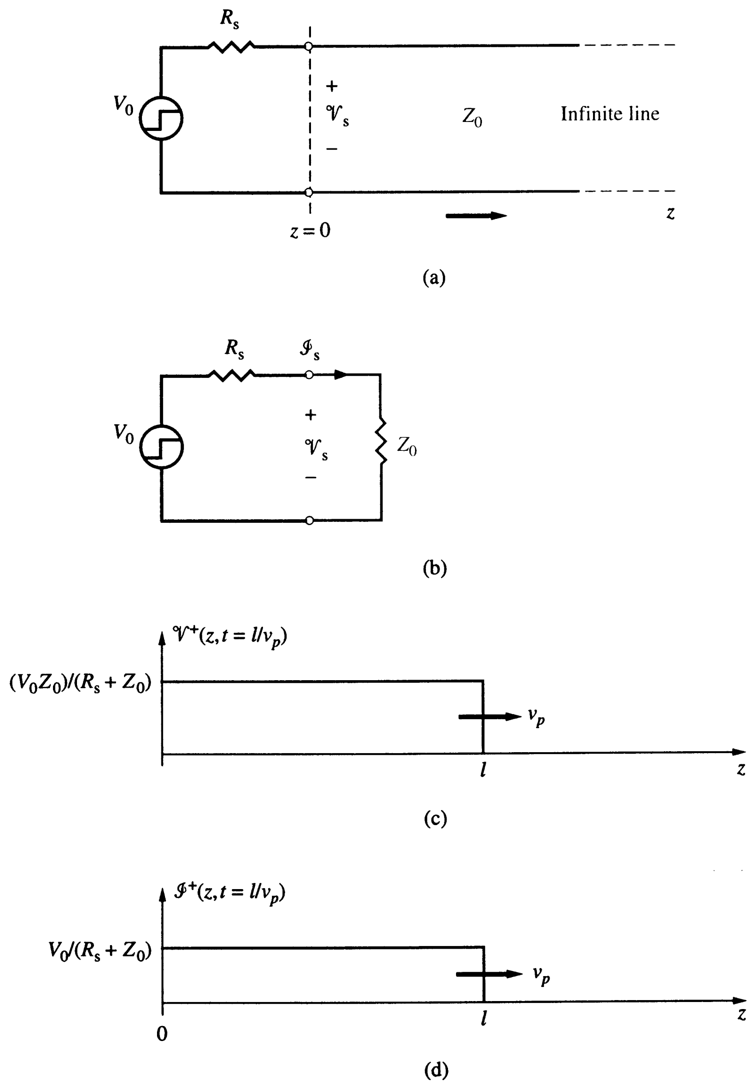

FIGURE 2.7. Step excitation of a lossless line. (a) Step voltage applied to an infinite lossless line. (b) Initial equivalent circuit seen from the source. (c) The voltage disturbance $\mathcal{V}^+(z,t)$. (d) The current disturbance $\mathcal{I}^+(z,t)$.

Note that the flow of a current $\mathcal{I}_s$ outward from a source producing a voltage V_0 represents a total power of $P_0 = \mathcal{I}_s V_0$ supplied by the source. Part of this power, given by $\mathcal{I}_s^2 R_s$, is dissipated in the source resistance. The remainder, given by

$$P^+_{\text{line}} = \mathcal{I}^+(0,t)\mathcal{V}^+(0,t) = \mathcal{I}_s\mathcal{V}_s = \frac{V_0^2 Z_0}{(Z_0 + R_s)^2}$$

is supplied to the line. Because the line is lossless, there is no power dissipation on the line. Therefore, the power P^+_{line} goes into charging the capacitances and the inductances[17] of the line, as discussed in connection with Figure 2.2. Note that as P^+_{line} travels down the line the amounts of energy stored respectively in the capacitance and inductance of a fully charged portion of the line of length dl are given by $\frac{1}{2}(C\,dl)\mathcal{V}_s^2$ and $\frac{1}{2}(L\,dl)\mathcal{I}_s^2$.

2.3 REFLECTION AT DISCONTINUITIES

In most transmission line applications, lines are connected to resistive loads, other lines (with different characteristic impedances), reactive loads, combinations of resistive and reactive loads, or loads with nonlinear current-voltage characteristics. Such discontinuities impose boundary conditions, which cause reflection of the incident voltages and currents from the discontinuities, while new voltages and currents are launched in the opposite direction. In this section we consider the reflection process and also provide examples of step responses of transmission lines terminated with short- and open-circuited terminations.

Consider a transmission line terminated in a load resistance R_L located at $z = l$, as shown in Figure 2.8, on which a voltage of $\mathcal{V}_1^+(z, t)$ is initially ($t = 0$) launched by the source at $z = 0$. Note that for an ideal step voltage source of amplitude V_0 and a source resistance R_s, as shown in Figure 2.8, the amplitude of $\mathcal{V}_1^+(z, t)$ is given by

$$\mathcal{V}_1^+(0, 0) = \frac{Z_0 V_0}{R_s + Z_0}$$

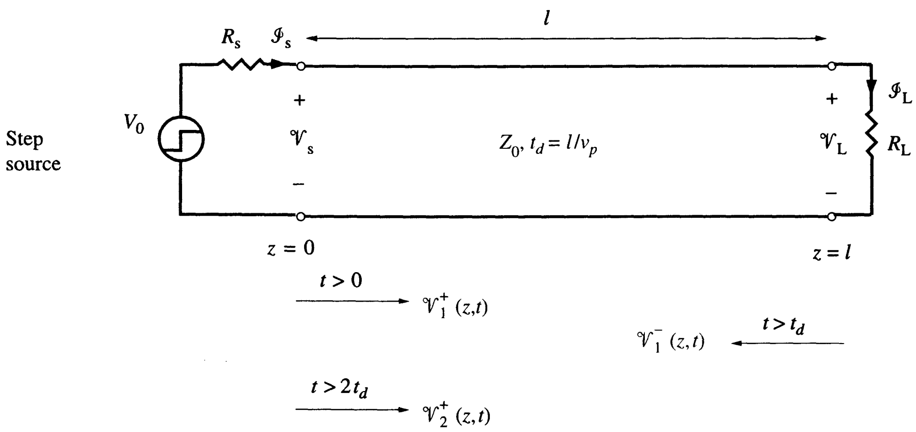

FIGURE 2.8. A terminated transmission line. The load R_L is located at $z = l$, while the source end is at $z = 0$.

[17]"Charging" an inductance can be thought of as establishing a current in it.

In general, a new reflected voltage $\mathcal{V}_1^-(z,t) = \mathcal{V}_1^-(l,t)$ is generated when the disturbance $\mathcal{V}_1^+(z,t)$ reaches the load position at time $t = t_d$, where t_d is the one-way travel time on the line, or $t_d = l/v_p$. In order to determine the amplitude of the reflected wave, we write the total voltage and current at the load position (i.e., $z = l$) at $t = t_d^+$ (i.e., immediately after the arrival of the incident wave) as

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(l,t) + \mathcal{V}_1^-(l,t) \qquad [2.10a]$$

$$\mathcal{I}_L(t) = \frac{1}{Z_0}\mathcal{V}_1^+(l,t) - \frac{1}{Z_0}\mathcal{V}_1^-(l,t) \qquad [2.10b]$$

Using [2.10a] and [2.10b] and the *boundary condition* $\mathcal{V}_L(t) = \mathcal{I}_L(t)R_L$ imposed by the purely resistive termination R_L, we can write

$$\mathcal{I}_L(t) = \frac{\mathcal{V}_L(t)}{R_L} \quad \rightarrow \quad \frac{\mathcal{V}_1^+(l,t)}{Z_0} - \frac{\mathcal{V}_1^-(l,t)}{Z_0} = \frac{\mathcal{V}_1^+(l,t) + \mathcal{V}_1^-(l,t)}{R_L} \qquad [2.11]$$

From [2.11] we can find the ratio of the reflected voltage $\mathcal{V}_1^-(l,t)$ to the incident one $\mathcal{V}_1^+(l,t)$. This ratio is defined as the *load voltage reflection coefficient,* Γ_L,

$$\boxed{\Gamma_L \equiv \frac{\mathcal{V}_1^-(l,t)}{\mathcal{V}_1^+(l,t)} = \frac{R_L - Z_0}{R_L + Z_0}} \qquad [2.12]$$

and it follows that

$$\frac{R_L}{Z_0} = \frac{1+\Gamma_L}{1-\Gamma_L} \qquad [2.13]$$

The reflection coefficient is one of the most important parameters in transmission line analysis. Accordingly, the simple expression [2.12] for Γ_L should be memorized. For lines terminated in resistive loads, Γ_L can have values in the range $-1 \le \Gamma_L \le +1$, where the extreme values of -1 and $+1$ occur when the load is, respectively, a short circuit (i.e., $R_L = 0$) and an open circuit (i.e., $R_L = \infty$). The special case of $\Gamma_L = 0$ occurs when $R_L = Z_0$; meaning that the load resistance is the same as the characteristic impedance,[18] that there is no reflected voltage, and that all the power carried by the incident voltage is absorbed by the load. It is important to note that expression [2.12] for the reflection coefficient was arrived at in a completely general fashion. In other words, whenever a voltage $\mathcal{V}_k^+(z,t)$ is incident on a load R_L from a transmission line with characteristic impedance Z_0, the amplitude of the reflected voltage $\mathcal{V}_k^-(l,t)$ is $\Gamma_L\mathcal{V}_k^+(l,t)$, with Γ_L given by [2.12].

The generality of [2.12] can also be used to determine the reflection of the new voltage $\mathcal{V}_1^-(z,t)$ when it arrives at the source end of the line. Having originated at the load end at $t = t_d$, the reflected voltage $\mathcal{V}_1^-(z,t)$ arrives at the source end (terminated with the source resistance R_s) at $t = 2t_d$. At that point, it can be viewed as a new voltage disturbance propagating on a line with characteristic

[18]This condition is referred to as a *matched* load and is highly preferred in most applications.

impedance Z_0 that is incident on a resistance of R_s. Thus, its arrival at the source end results in the generation of a new reflected voltage traveling toward the load. We denote this new voltage $\mathcal{V}_2^+(z, t)$, where the subscript distinguishes it from the original voltage $\mathcal{V}_1^+(z, t)$ and the superscript underscores the fact that it is propagating in the $+z$ direction. The amplitude of the new reflected voltage $\mathcal{V}_2^+(z, t)$ is determined by the *source reflection coefficient* Γ_s, which applies at the source end of the line and is defined as

$$\Gamma_s \equiv \frac{\mathcal{V}_2^+(0, t)}{\mathcal{V}_1^-(0, t)} = \frac{R_s - Z_0}{R_s + Z_0} \qquad [2.14]$$

Thus we have $\mathcal{V}_2^+(z, t) = \mathcal{V}_2^+(0, t) = \Gamma_s \mathcal{V}_1^-(0, t)$.

Note that the voltage $\mathcal{V}_1^+(z, t)$ was created at $t = 0$ and still continues to exist. Thus the source-end voltage and current at $t = 2t_d^+$ are

$$\begin{aligned} \mathcal{V}_s(t) &= \mathcal{V}_1^+(0, t) + \mathcal{V}_1^-(0, t) + \mathcal{V}_2^+(0, t) \\ &= \mathcal{V}_1^+(0, t)(1 + \Gamma_L + \Gamma_s\Gamma_L) \end{aligned} \qquad [2.15a]$$

$$\begin{aligned} \mathcal{I}_s(t) &= \frac{1}{Z_0}\mathcal{V}_1^+(0, t) - \frac{1}{Z_0}\mathcal{V}_1^-(0, t) + \frac{1}{Z_0}\mathcal{V}_2^+(0, t) \\ &= \frac{1}{Z_0}\mathcal{V}_1^+(0, t)(1 - \Gamma_L + \Gamma_s\Gamma_L) \end{aligned} \qquad [2.15b]$$

The newly generated voltage $\mathcal{V}_2^+(z, t)$ will now arrive at the load end at $t = 3t_d$ and lead to the creation of a new reflected voltage $\mathcal{V}_2^-(z, t)$, and this process will continue indefinitely. In practice, the step-by-step calculation of the successively generated voltages becomes tedious, especially for arbitrary resistive terminations. In such cases, the graphical construction of a *bounce diagram* is very helpful. We introduce the bounce diagram in the following subsection.

2.3.1 Bounce Diagrams

A bounce diagram, illustrated in Figure 2.9, also called a reflection diagram or lattice diagram, is a distance-time plot used to illustrate successive reflections along a transmission line. The distance along the line is shown on the horizontal axis, and time is shown on the vertical axis. The bounce diagram is a plot of the time elapsed versus distance z from the source end, showing the voltages traveling in the $+z$ and $-z$ directions as straight lines sloping[19] downward from left to right and right to left, respectively. Each sloping line corresponds to an individual traveling voltage and is labeled with its amplitude. The amplitude of each reflected voltage is obtained by multiplying the amplitude of the preceding voltage by the reflection coefficient at the position where the reflection occurs.

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[19]The slope of the lines on the bounce diagram (i.e., dt/dz) can be thought of as corresponding to $(\pm v_p)^{-1}$.

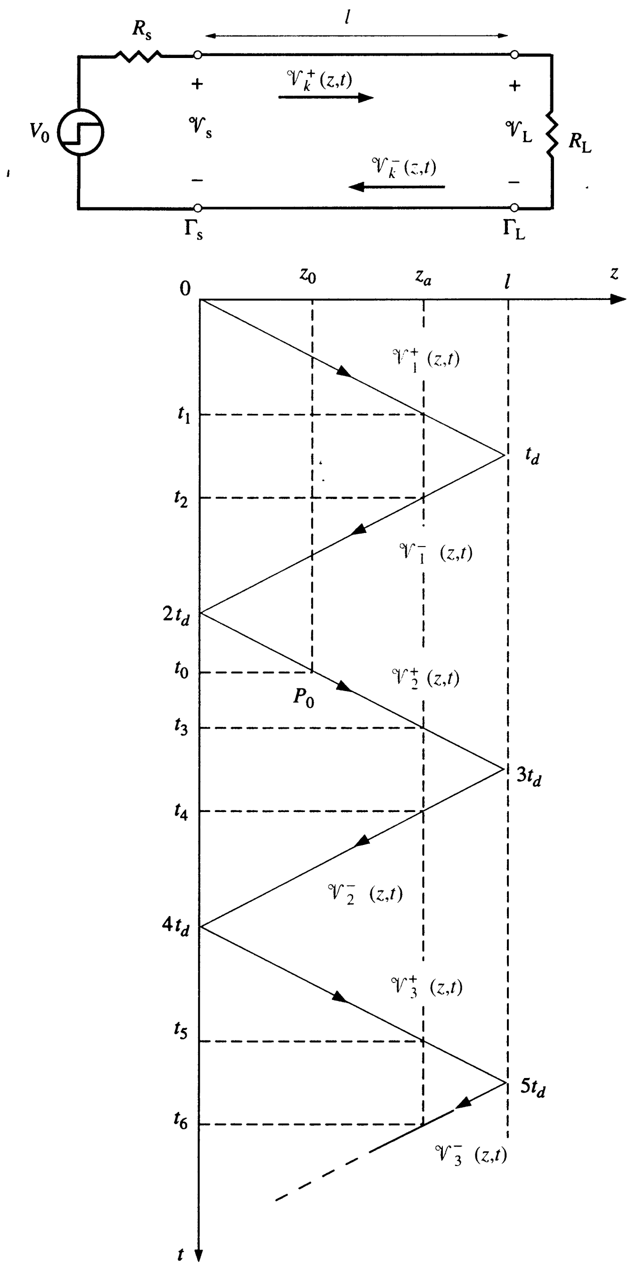

FIGURE 2.9. Bounce diagram.

The time sequence of events starting with the first application of the step voltage at the source end can be easily visualized from the bounce diagram. The application of the step voltage launches $\mathcal{V}_1^+(z, t)$ toward the load. This voltage arrives at the load end at $t = t_d$, and its arrival leads to the generation of $\mathcal{V}_1^-(z, t) = \Gamma_L \mathcal{V}_1^+(z, t)$ propagating toward the source. This new voltage $\mathcal{V}_1^-(z, t)$ arrives at the source end and leads to the generation of $\mathcal{V}_2^+(z, t)$, and this process continues back and forth indefinitely.

Once constructed, a bounce diagram can be used conveniently to determine the voltage distribution along the transmission line at any given time, as well as the variation of voltage with time at any given position. Suppose we wish to find the voltage distribution $\mathcal{V}(z, t_0)$ along the line at $t = t_0$, chosen arbitrarily to be $2t_d < t_0 < 3t_d$. To determine $\mathcal{V}(z, t_0)$, we mark t_0 on the time axis of the bounce diagram and draw a horizontal line from t_0, intersecting the sloping line marked $\mathcal{V}_2^+(z, t)$ at point P_0, as shown in Figure 2.9. Note that all sloping lines below the point P_0 are irrelevant for $\mathcal{V}(z, t_0)$, since they correspond to later times. If we now draw a vertical dashed line through P_0, we find that it intersects the z axis at z_0. At time $t = t_0$, all points along the line have received voltages $\mathcal{V}_1^+(z, t)$ and $\mathcal{V}_1^-(z, t)$. However, only points to the left of z_0 have yet received the voltage $\mathcal{V}_2^+(z, t)$, so a discontinuity exists in the voltage distribution at $z = z_0$. In other words, we have

$$\mathcal{V}(z, t_0) = \begin{cases} \mathcal{V}_1^+(z, t)(1 + \Gamma_L + \Gamma_s\Gamma_L) & z < z_0 \\ \mathcal{V}_1^+(z, t)(1 + \Gamma_L) & z > z_0 \end{cases}$$

Alternatively, we may wish to determine the variation of voltage as a function of time at a fixed position, say z_a. To determine $\mathcal{V}(z_a, t)$, we simply look at the intersection points with the sloping lines of the vertical line passing through z_a, as shown in Figure 2.9. Horizontal lines drawn from these intersection points, crossing the time axis at $t_1, t_2, t_3, t_4, \ldots$, are the time instants at which each of the new voltages $\mathcal{V}_1^+(z, t)$, $\mathcal{V}_1^-(z, t)$, $\mathcal{V}_2^+(z, t)$, $\mathcal{V}_2^-(z, t)$, $\ldots$, arrive at $z = z_a$ and abruptly change the total voltage at that point. Thus, the time variation of the voltage at $z = z_a$, namely $\mathcal{V}(z_a, t)$ is given as

$$\mathcal{V}(z_a, t) = \begin{cases} 0 & 0 < t < t_1 \\ \mathcal{V}_1^+(z_a, t) & t_1 < t < t_2 \\ \mathcal{V}_1^+(z_a, t)(1 + \Gamma_L) & t_2 < t < t_3 \\ \mathcal{V}_1^+(z_a, t)(1 + \Gamma_L + \Gamma_s\Gamma_L) & t_3 < t < t_4 \\ \mathcal{V}_1^+(z_a, t)(1 + \Gamma_L + \Gamma_s\Gamma_L + \Gamma_L\Gamma_s\Gamma_L) & t_4 < t < t_5 \\ \cdots & \cdots \end{cases}$$

where $\mathcal{V}_1^+(z_a, t) = \mathcal{V}_1^+(z, t)$.

A bounce diagram can also be constructed to keep track of the component current waves. However, it is also possible and less cumbersome to derive the component line current, $\mathcal{I}_k^{\pm}(z, t)$, simply from the corresponding voltage $\mathcal{V}_k^{\pm}(z, t)$. In this connection, all we need to remember is that the current associated with any voltage disturbance $\mathcal{V}_k^+(z, t)$ propagating in the $+z$ direction is simply $\mathcal{I}_k^+(z, t) = (Z_0^{-1})\mathcal{V}_k^+(z, t)$, whereas that associated with a voltage disturbance $\mathcal{V}_k^-(z,t)$ propagating in the $-z$ direction is $\mathcal{I}_k^-(z,t) = -(Z_0^{-1})\mathcal{V}_k^-(z,t)$.

The uses of the bounce diagram in specific cases are illustrated in Examples 2-2 through 2-9.

2.3.2 The Reflection Process

Before we proceed with specific examples, we now provide a heuristic discussion of the reflection process for the case of a transmission line terminated in an open circuit.[20] This discussion involves the same considerations as the heuristic discussion in connection with Figure 2.2 of the propagation of disturbances along a transmission line in terms of successive charging of capacitors and inductors. When the voltage disturbance reaches the open-circuited end of the line, its orderly progress of successively charging the distributed circuit elements is interrupted. Consider the approach of a disturbance to the end of an open-circuited transmission line, shown in Figure 2.10a. Figure 2.10b shows the voltage and current progressing together; L_y carries current but L_z does not, and C_y is charged to the source voltage V_0 but C_z

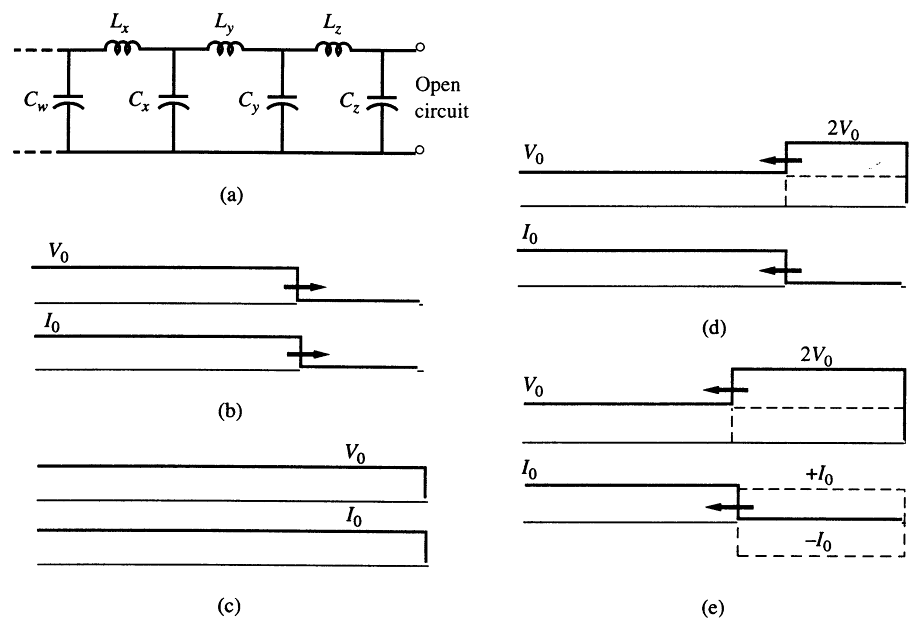

FIGURE 2.10. The reflection process. The orderly progress of the disturbances of current and voltage, propagating initially from left to right, is interrupted when they reach the open end of the line, leading to the reflection of the disturbance.

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[20]The qualitative discussion presented herein is based on a similar discussion in Chapter 14 of H. H. Skilling, *Electric Transmission Lines*, McGraw-Hill, 1951.

is not. The voltage on C_y, however, causes current to flow through L_z, and thus through C_z, charging it to a voltage V_0 (Figure 2.10c).

At the time C_z is charged, all of the inductances (including L_z) carry the full current I_0. The progress of the disturbance along the line cannot continue any more, since there is no inductance beyond C_z to serve as an outlet for current as C_z is charged. As a result, C_z becomes overcharged, since the current through L_z cannot stop until the stored magnetic energy is exhausted. Thus, current continues to flow through C_z until it is charged[21] to twice its normal value ($2V_0$), at which time the current through L_z drops to zero (i.e., L_z now acts like an open circuit) (Figure 2.10d).

When L_z stops carrying current, the current carried by L_y is now driven solely into C_y, doubling its voltage and forcing the current in L_y to stop. At the same time, the voltages at the two ends of L_z are both $2V_0$, so that the current through L_z remains zero and the doubly charged capacitor C_z stays at $2V_0$. We now have the condition depicted in Figure 2.10e, where C_y and C_z are both at $2V_0$ and L_y and L_z both have zero current. This procedure now continues along the line from right to left as the voltage on the line is doubled and the current drops to zero.

The above described phenomenon can be viewed as a reflection, since the original disturbance, traveling from left to right, appears to be reflected from the end of the line and to begin to progress from right to left. In Figure 2.10e, the reflected voltage disturbance of amplitude V_0 travels toward the left, and adds to the previously existing line voltage, making the total voltage $2V_0$. It is accompanied by the reflected current of amplitude $-I_0$, which is added on top of the existing line current I_0, making the total current on the line zero. Although the front of the current disturbance is progressing toward the left, it should be noted that this does not imply any reversal of current flow. The current originally flowing from the source toward the end of the line (i.e., $\mathcal{I}_1^+(z, t)$) continues to do so even after reflection. This current flows from left to right to charge the capacitances when the disturbance progresses toward the end of the line, and it continues to flow from left to right as the reflected disturbance returns, doubling the charge on the line as the voltage is raised to $2V_0$.

All of these physical effects of charging of capacitors and current flows through the inductors are simply accounted for by the general solutions for voltage and current as given respectively by [2.8] and [2.9] and by the application of the boundary condition at the termination—namely, that there must always be zero current at the end of an open-circuited line. The purpose of this heuristic discussion is simply to provide a qualitative understanding of the reflection process in terms of the equivalent circuit of the line.

[21] It is not obvious why the capacitor would charge to precisely twice its normal value. The circuit model of Figure 2.10a, consisting of discrete elements, is not adequate for the determination of the precise value of the reflected voltage, which is unambiguously determined by the governing wave equations [2.5], [2.6] and their solutions [2.10] as applied to an open-circuited termination, as shown in the next section. Nevertheless, consider the fact that the amount that the capacitor voltage is charged to is determined by $\int I\,dt$; thus, with no other outlet for the inductor current, twice the normal current goes through the capacitance, charging it to twice its normal value.

2.3.3 Open- and Short-Circuited Transmission Lines

We now consider examples of step responses of transmission lines with the simplest type of terminations, namely an open or a short circuit. The circumstances treated in Examples 2-2 and 2-3 are commonly encountered in practice, especially in computer-communication problems; for example, when the voltage at one end of an interconnect switches to the HIGH state due to a change in the status of a logic gate. The resultant response is then similar to the short-circuited line case (Example 2-2) if the interconnect is a short-circuited matching stub or drives (i.e., is terminated in) a subsequent gate (or another interconnect) with low input impedance ($R_L \ll Z_0$). Alternatively, and more commonly in practice, the interconnect would be driving a gate with a very high input impedance ($R_L \gg Z_0$), corresponding to an open-circuited termination (Example 2-3).

Example 2-2: Step voltage applied to a short-circuited lossless line. Consider the transmission line of length l terminated with a short circuit at the end (i.e., $R_L = 0$), as shown in Figure 2.11a. Sketch the voltages $\mathcal{V}_s$ and $\mathcal{V}_{l/2}$ as a function of time.

Solution: Initially, the applied voltage is divided between the source resistance R_s and the line impedance Z_0 in the same manner as for the infinite line in Example 2-1, and at $t = 0^+$, a voltage $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = (V_0 Z_0)/(Z_0 + R_s)$ is launched at the source end of the line. The corresponding

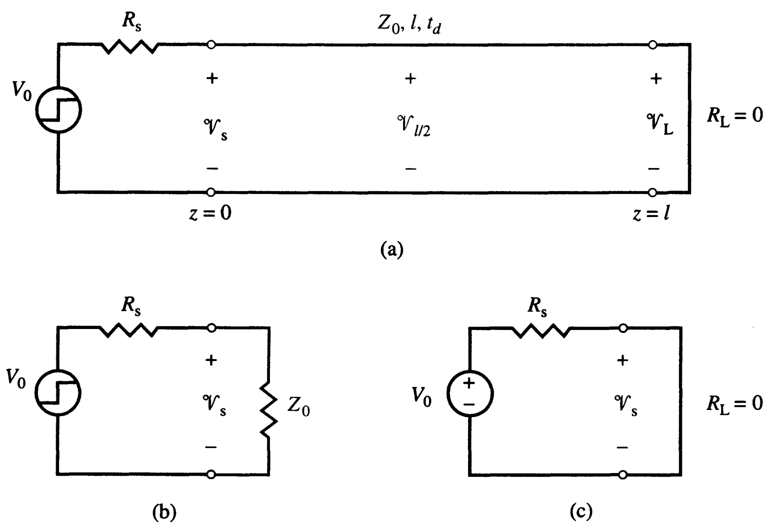


FIGURE 2.11. A short-circuited lossless line. (a) Step voltage applied to a short-circuited lossless line. (b) The initial equivalent circuit. (c) Steady-state (final) equivalent circuit.

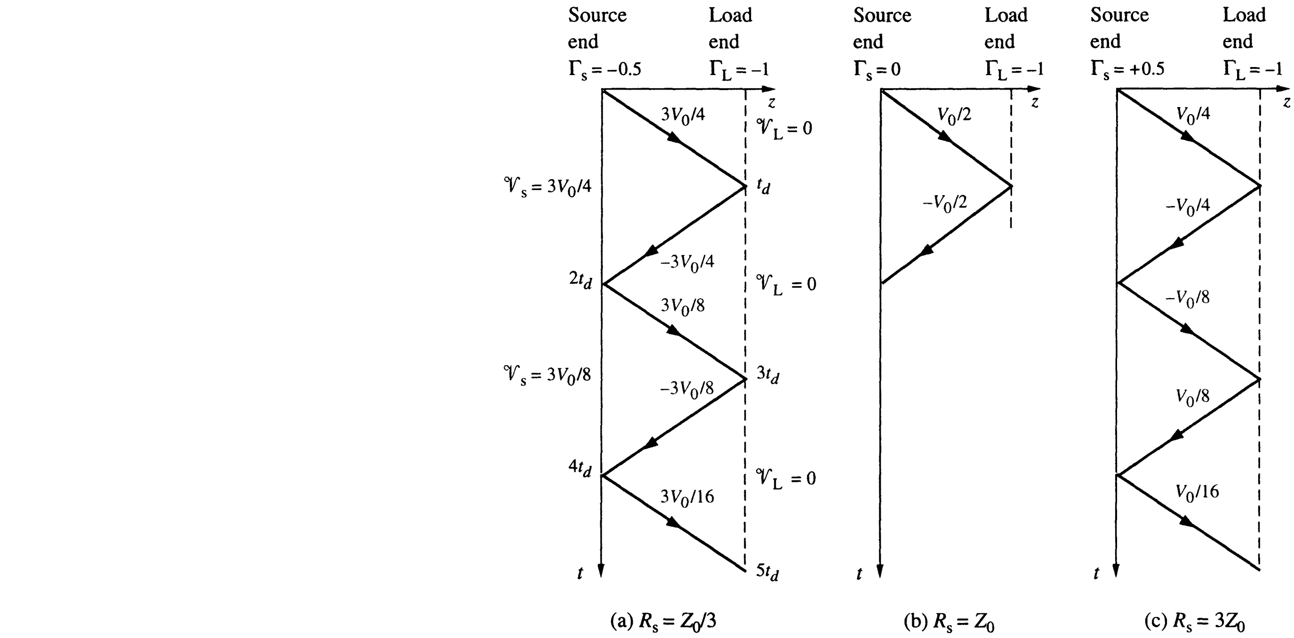


FIGURE 2.12. Bounce diagram for Example 2-2 (Figure 2.11). (a) $R_s = Z_0/3$, (b) $R_s = Z_0$, and (c) $R_s = 3Z_0$.

current is $\mathcal{I}_1^+(z,t) = \mathcal{V}_1^+(z,t)/Z_0$ and has an amplitude of $\mathcal{I}_1^+(0,0) = V_0/(Z_0 + R_s) = \mathcal{V}_1^+(0,0)/Z_0$. The equivalent circuit initially presented to the source by the line is thus simply a resistance of Z_0, as shown in Figure 2.11b. Eventually, when all transients die out, the equivalent circuit of the line is a short circuit (Figure 2.11c); thus, the voltage everywhere on the line (e.g., $\mathcal{V}_s$, $\mathcal{V}_{l/2}$, and $\mathcal{V}_L$) must eventually approach zero.

To analyze the behavior of the line voltage, we use a bounce diagram as shown in Figure 2.12. When the voltage disturbance $\mathcal{V}^+(z,t)$ reaches the end of the line (at $t = t_d = l/v_p$), a reflected voltage of $\mathcal{V}^-(z,t) = -\mathcal{V}^+(z,t)$ is generated,[22] since the total voltage at the short circuit ($R_L = 0$) has to be $\mathcal{V}_L(t) = \mathcal{V}^+(l,t) + \mathcal{V}^-(l,t) = 0$. In other words, the reflection coefficient at the load end is

$$\Gamma_L = \frac{\mathcal{V}_1^-(l,t)}{\mathcal{V}_1^+(l,t)} = \frac{0 - Z_0}{0 + Z_0} = -1$$

[22]The reflection process at the short circuit occurs rather differently than that from an open circuit as discussed in connection with Figure 2.10. As the next-to-the-last capacitance (C_y) is charged to V_0, L_z begins to carry current; however, C_z cannot take any charge, since it is short-circuited. Current flows freely from L_z through the short circuit and into the return conductor (just like the flow of water from the open end of a pipe), until the charge on C_y is exhausted. As a result, the current through L_z becomes twice as much as its normal value ($2I_0$), and the voltage across C_y drops to zero. In this way, "reflection" reduces the voltage from V_0 to zero and increases current from I_0 to $2I_0$.

However, the current $\mathcal{I}_1^-(z, t)$ associated with the voltage traveling in the $-z$ direction is $-\mathcal{V}_1^-(z, t)/Z_0$, resulting in a reflected current of $V_0/(Z_0 + R_s)$, which adds directly to the incident current $\mathcal{I}_1^+(z, t) = V_0/(Z_0 + R_s)$ traveling in the $+z$ direction, doubling the total current on the line.

As it travels toward the source during $t_d < t < 2t_d$, the reflected voltage makes the total voltage everywhere on the line zero and the total current on the line equal to $2V_0/(Z_0 + R_s)$. When this disturbance reaches the source end at $t = 2t_d$, the source presents an impedance of R_s, and the reflection coefficient at the source end (i.e., Γ_s) depends on the value of R_s relative to Z_0. For $R_s = Z_0$, we have $\Gamma_s = 0$, and no further reflections occur, so the voltage on the line remains zero; in other words, a steady state is reached. However, for $R_s \neq Z_0$, it takes further reflections to eventually reach steady state, as illustrated in Figure 2.13, where the time evolution of the voltages at the source end ($\mathcal{V}_s(t)$) and at the center ($\mathcal{V}_{l/2}(t)$) of the transmission line are shown. Note that the load voltage $\mathcal{V}_L(t)$ is identically zero at all times, as dictated by the short-circuit termination. Note also that the voltage everywhere on the line, including at the center and at the source end, eventually approaches zero; however, we see from Figure 2.13 that the particular way in which $\mathcal{V}_s(t)$ and $\mathcal{V}_{l/2}(t)$ approach their final value of zero depends critically on the ratio R_s/Z_0.

Example 2-3: Overshoot and ringing effects. The distributed nature of a high-speed digital logic board commonly leads to *ringing,* the fluctuations of the voltage and current about an asymptotic value. Ringing results from multiple reflections, especially when an unterminated (i.e., open-circuited)[23] transmission line is driven by a low-impedance buffer. Consider the circuit shown in Figure 2.14a, where a step voltage source of amplitude V_0 and a source resistance $R_s = Z_0/4$ drives a lossless transmission line of characteristic impedance Z_0 and a one-way propagation delay of t_d seconds. Sketch $\mathcal{V}_s$, $\mathcal{V}_L$, and $\mathcal{I}_s$ as a function of t.

Solution: At $t = 0$, the source voltage rises from 0 to V_0, and a voltage $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = Z_0V_0/(R_s + Z_0) = 0.8V_0$ is applied from the source end of the line. During $0 < t < t_d$, the line charges as the front of this voltage disturbance travels toward the load. At $t = t_d$, the disturbance front reaches the open end of the line, and a reflected voltage of amplitude $\mathcal{V}_1^-(l, t_d) = \Gamma_L\mathcal{V}_1^+(l, t_d) = 0.8V_0$ is launched back toward the source, since $\Gamma_L = 1$. (In other words, the total current at the open end of the line remains zero, so we have $\mathcal{I}_1^+(l, t) + \mathcal{I}_1^-(l, t) = 0$, and thus $\mathcal{V}_1^-(l, t) = \mathcal{V}_1^+(l, t) = 0.8V_0$.) Note that as long as the source voltage does not change, $\mathcal{V}_1^+(z, t)$ remains constant in time and also constant with z once it reaches the end of the line ($z = l$) at $t = t_d$. Once $\mathcal{V}_1^-(z, t)$ is launched (at $t = t_d^+$), it travels toward the source, reaching the source end at $t = 2t_d$, after which time it also remains constant and coexists along the

[23]Note that if the input impedance of a load device is very high compared to Z_0 (i.e., $R_L \gg Z_0$), R_L can be approximated as an open circuit.

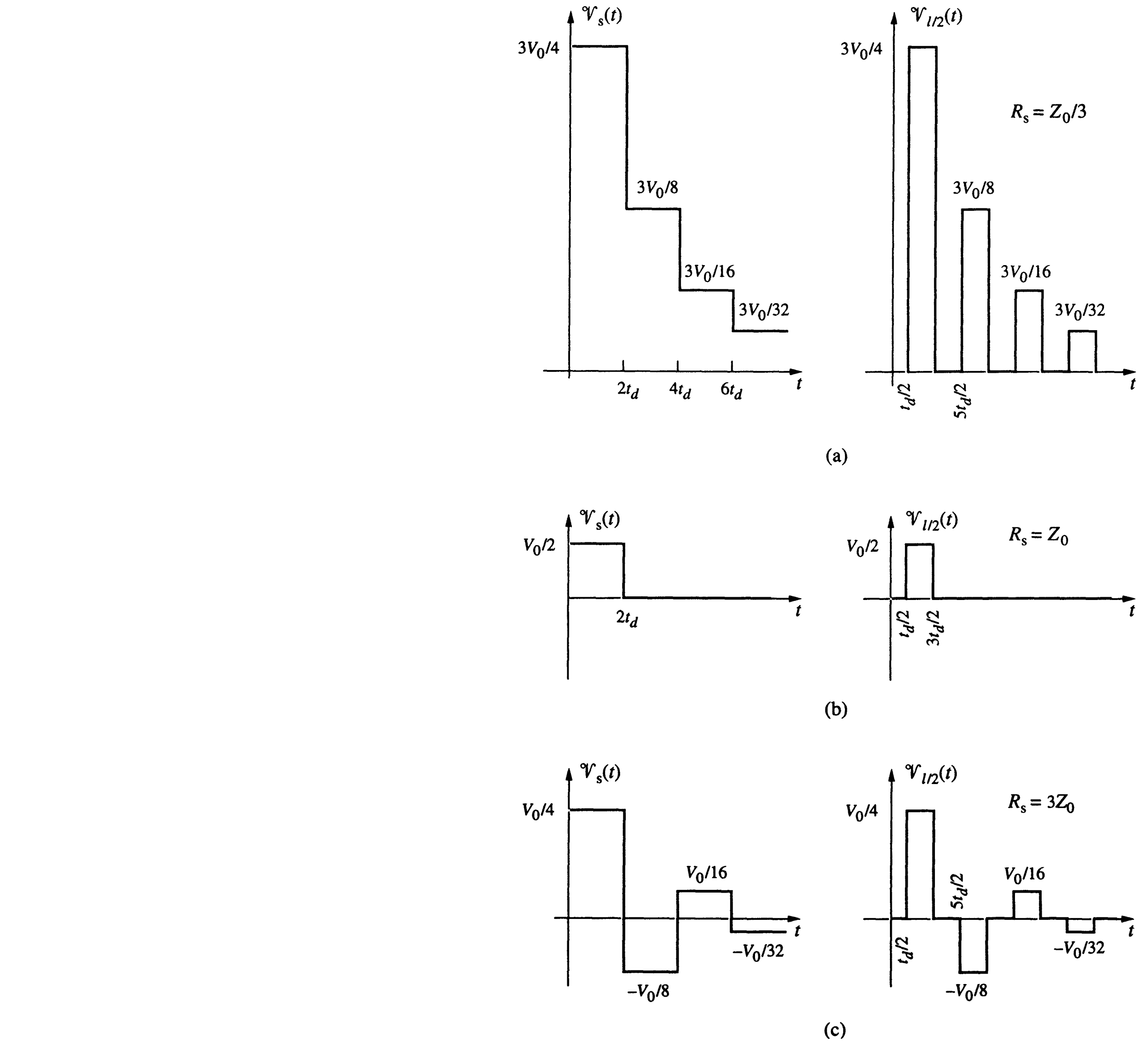


FIGURE 2.13. Step voltage applied to a short-circuited lossless line. Voltages at the source end ($\mathcal{V}_s$) and at the center ($\mathcal{V}_{l/2}$) of a short-circuited transmission line for source impedances of (a) $R_s = Z_0/3$, (b) $R_s = Z_0$, and (c) $R_s = 3Z_0$.

line with $\mathcal{V}_1^+(z, t)$. The arrival of the reflected voltage $\mathcal{V}_1^-(z, t)$ at the source end of the line at $t = 2t_d$ leads to the generation of a new reflected voltage disturbance of amplitude $\mathcal{V}_2^+(0, 2t_d) = \Gamma_s\mathcal{V}_1^-(0, 2t_d) = -0.48V_0$ [since $\Gamma_s = (R_s - Z_0)/(R_s + Z_0) = -0.6$] is launched toward the load. Note that $\mathcal{V}_2^+(z, t)$ is now superimposed on top of $\mathcal{V}_1^+(z, t)$ and $\mathcal{V}_1^-(z, t)$.

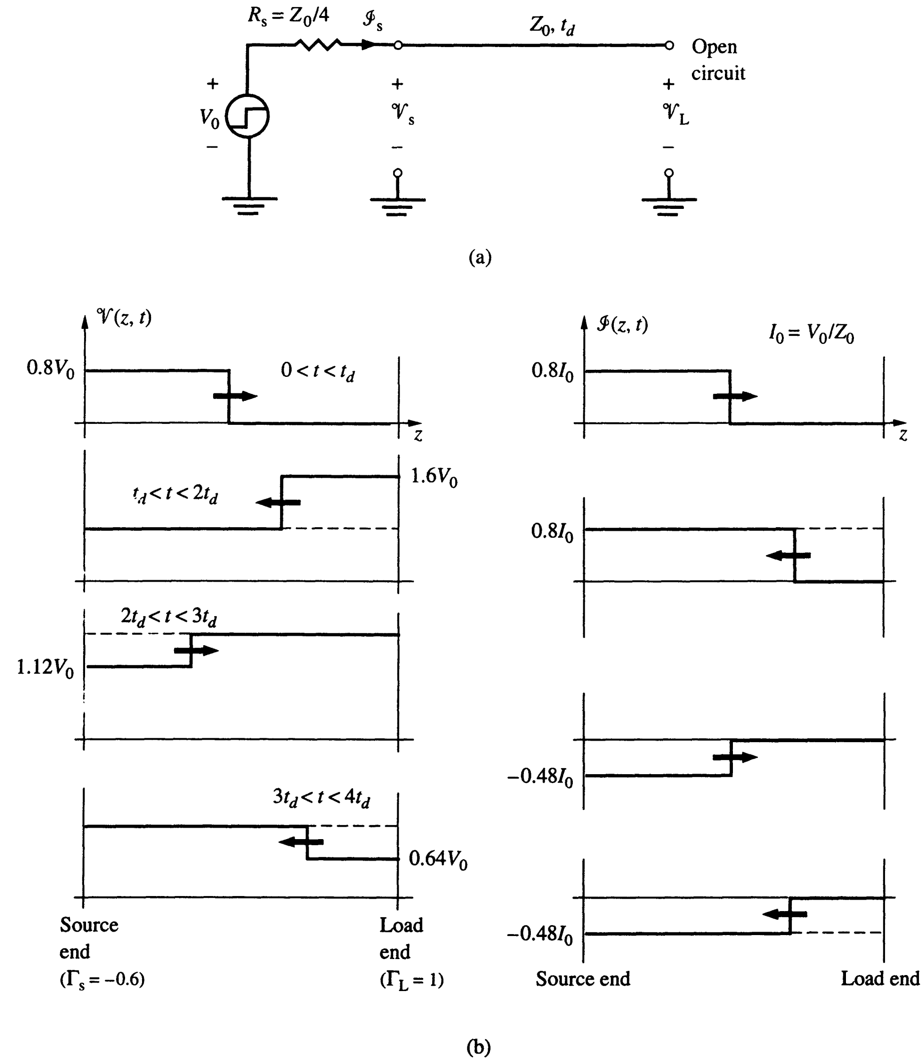
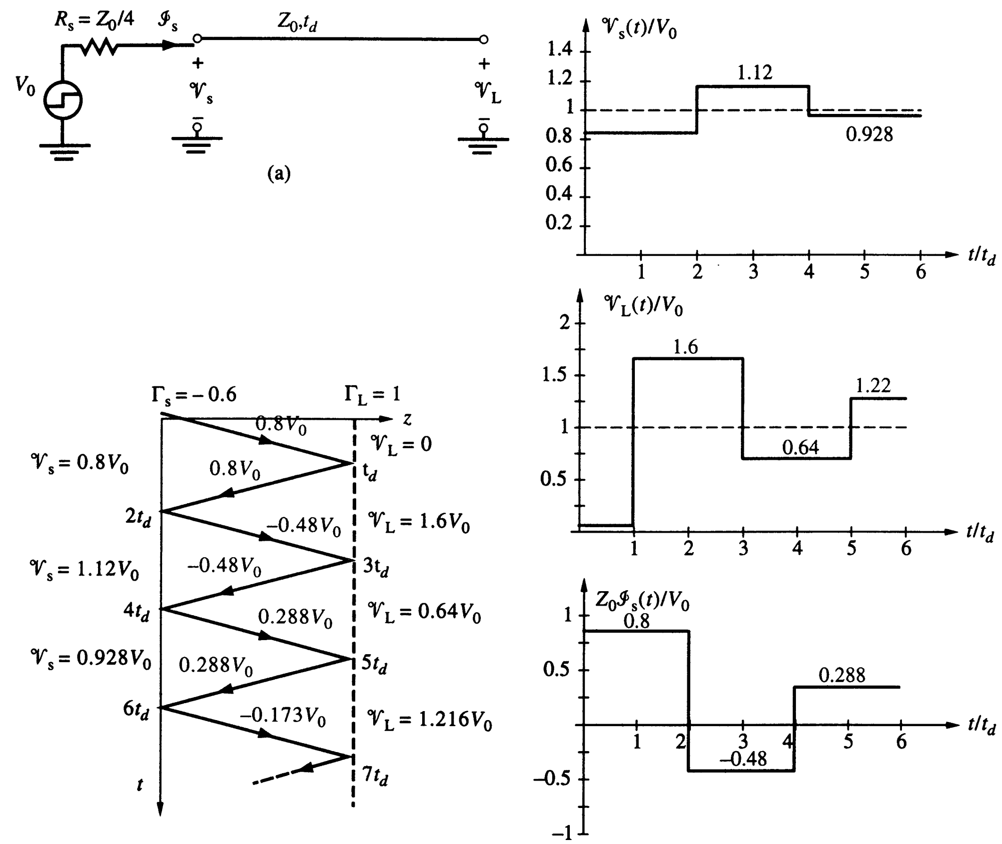


FIGURE 2.14. **Step response of an open-circuited line.** (a) Circuit for Example 2-3. (b) Voltage and current distributions along the line at different time intervals.

This process continues indefinitely, with the total voltage and current on the line gradually approaching their steady-state values. The voltage and current distribution along the line is shown in Figure 2.14b for various time intervals. The variation of the voltage (and thus the current) with time can be quantitatively determined by means of a bounce diagram, as shown in Figure 2.15b, which specifies the values of the source- and load-end voltages at any given time. The temporal variations of the source- and load-end voltages and the source-end current, as derived from the bounce diagram, are also shown in Figure 2.15c. Both voltage waveforms oscillate about and asymptotically approach their final



(b) (c)

FIGURE 2.15. **Step response of an open-circuited lossless line.** (a) Circuit for Example 2-3. (b) Bounce diagram. (c) Normalized source- and load-end voltages and source-end current as a function of t/t_d.

value V_0 (shown as dashed lines)—the process referred to earlier as ringing. The source-end current waveform eventually approaches zero, as expected for an open-circuited termination. For the case shown, the percentage maximum overshoot, defined as the percentage difference between the maximum value and the asymptotic value, for $\mathcal{V}_L$ is $[(1.6V_0 - V_0)/V_0] \times 100 = 60\%$.

2.4 TRANSIENT RESPONSE OF TRANSMISSION LINES WITH RESISTIVE TERMINATIONS

Our discussions in the preceding section served to introduce the concepts of reflection at discontinuities in the context of the relatively simple open- and short-circuited terminations. In this section, we study the response of transmission lines terminated with an arbitrary resistance R_L to excitations in the form of a step change in voltage (e.g., an applied voltage changing from 0 to V_0 at $t = 0$) or a short pulse of a given duration. Step excitation represents such cases as the output voltage of a driver gate changing from LOW to HIGH or HIGH to LOW state at a specific time, while pulse excitations are relevant to a broad class of computer communication problems. We consider two cases of resistive terminations: (i) single lossless transmission lines terminated in resistive loads and (ii) lossless transmission lines terminated in other lossless transmission lines. Resistively terminated lines and transmission lines terminated in other lines are encountered very often in practice. In digital communication applications, for example, logic gates are often connected via an interconnect to other gates with specific input resistances, and interconnects often drive combinations of other interconnects.

2.4.1 Single Transmission Lines with Resistive Terminations

We start with a general discussion of the step response of transmission lines with resistive terminations. Consider the circuit shown in Figure 2.16a where a step voltage source (0 to V_0 at $t = 0$) with a source resistance R_s drives a lossless transmission line of characteristic impedance Z_0 and one-way time delay t_d, terminated in a load resistance R_L. At $t = 0$, when the source voltage jumps to V_0, a voltage disturbance of amplitude $\mathcal{V}_1^+(0, 0) = Z_0V_0/(R_s + Z_0)$ is launched at the source end of the line; it travels down the line (during $0 < t < t_d$) and arrives at the load end at $t = t_d$, when a reflected voltage of amplitude $\mathcal{V}_1^-(l, t_d) = \Gamma_L\mathcal{V}_1^+(l, t_d)$, where $\Gamma_L = (R_L - Z_0)/(R_L + Z_0)$, is launched back toward the source. Note that $\mathcal{V}_1^+(z, t)$ remains constant in time (and also with z once it reaches $z = l$ at $t = t_d$), with its value given by $Z_0V_0/(R_s + Z_0)$. The reflected voltage travels along the line (during $t_d < t < 2t_d$) and reaches the source end at $t = 2t_d$, when a new voltage is reflected toward the load. The amplitude of the new reflected voltage is $\mathcal{V}_2^+(0, 2t_d) = \Gamma_s\mathcal{V}_1^-(0, 2t_d)$, where $\Gamma_s = (R_s - Z_0)/(R_s + Z_0)$. As this process continues with successive reflections at both ends, the total voltage at any time and at any particular position along the line

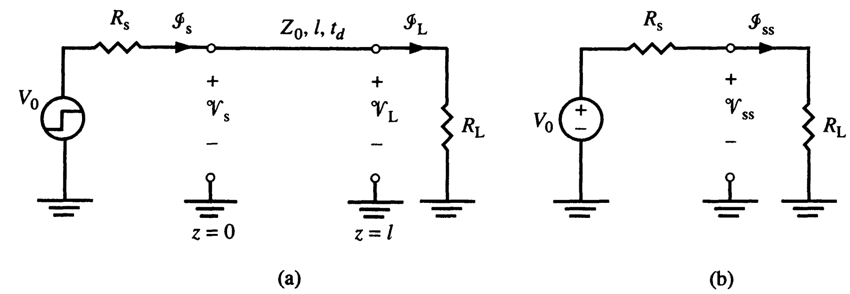


FIGURE 2.16. Resistively terminated line. (a) Step excitation of a lossless transmission line terminated with a resistive load. (b) Steady-state equivalent circuit seen by the source.

is given as an algebraic sum of all the voltages at that particular location and at that time. For example, at $t = 4t_d$, the total voltage at the center of the line is given by

$$\begin{aligned}\mathcal{V}_{l/2}(4t_d) &= \mathcal{V}_1^+(l/2, 4t_d) + \mathcal{V}_1^-(l/2, 4t_d) + \mathcal{V}_2^+(l/2, 4t_d) + \mathcal{V}_2^-(l/2, 4t_d) \\ &= \mathcal{V}_1^+(l/2, 4t_d)[1 + \Gamma_L + \Gamma_s\Gamma_L + \Gamma_s\Gamma_L^2] \\ &= \frac{Z_0 V_0}{R_s + Z_0}[1 + \Gamma_L + \Gamma_s\Gamma_L + \Gamma_s\Gamma_L^2]\end{aligned}$$

where we have used the fact that $\mathcal{V}_1^+(l/2, 4t_d) = \mathcal{V}_1^+(0, 0) = Z_0V_0/(R_s + Z_0)$; in other words, as long as the source voltage does not change again, $\mathcal{V}_1^+(z, t)$ remains constant in time and also is the same everywhere (i.e., at all z) once it reaches the end of the line $z = l$ at $t = t_d$. Similarly, the total current at the center of the line at $t = 4t_d$ is

$$\begin{aligned}\mathcal{I}_{l/2}(4t_d) &= \mathcal{I}_1^+(l/2, 4t_d) + \mathcal{I}_1^-(l/2, 4t_d) + \mathcal{I}_2^+(l/2, 4t_d) + \mathcal{I}_2^-(l/2, 4t_d) \\ &= \frac{V_0}{R_s + Z_0}[1 - \Gamma_L + \Gamma_s\Gamma_L - \Gamma_s\Gamma_L^2]\end{aligned}$$

In general, since $|\Gamma_L| \leq 1$ and $|\Gamma_s| \leq 1$, we have $|\mathcal{V}_{i+1}^{\pm}| \leq |\mathcal{V}_i^{\pm}|$, so the contribution of new individual reflected components to the total voltage or current at any position along the line diminishes as $t \to \infty$. The sum of the contributions of the voltage components traveling in both directions converges[24] to a finite steady-state value for the voltage at any position z, given as

$$\begin{aligned}\mathcal{V}(z, \infty) &= \mathcal{V}_1^+(z, \infty) + \mathcal{V}_1^-(z, \infty) + \mathcal{V}_2^+(z, \infty) + \mathcal{V}_2^-(z, \infty) + \mathcal{V}_3^+(z, \infty) + \mathcal{V}_3^-(z, \infty) + \cdots \\ &= \mathcal{V}_1^+(z, \infty)[1 + \Gamma_L + \Gamma_s\Gamma_L + \Gamma_s\Gamma_L^2 + \Gamma_s^2\Gamma_L^2 + \Gamma_s^2\Gamma_L^3 + \Gamma_s^3\Gamma_L^3 + \cdots] \\ &= \mathcal{V}_1^+(z, \infty)\{[1 + (\Gamma_s\Gamma_L) + (\Gamma_s^2\Gamma_L^2) + \cdots] + \Gamma_L[1 + (\Gamma_s\Gamma_L) + (\Gamma_s^2\Gamma_L^2) + \cdots]\} \\ &= \mathcal{V}_1^+(z, \infty)\left[\left(\frac{1}{1 - \Gamma_s\Gamma_L}\right) + \left(\frac{\Gamma_L}{1 - \Gamma_s\Gamma_L}\right)\right] = \mathcal{V}_1^+(z, \infty)\left(\frac{1 + \Gamma_L}{1 - \Gamma_s\Gamma_L}\right)\end{aligned}$$

This expression can be further simplified by substituting for $\mathcal{V}_1^+(z, \infty) = Z_0V_0/(R_s + Z_0)$, $\Gamma_L = (R_L - Z_0)/(R_L + Z_0)$, and $\Gamma_s = (R_s - Z_0)/(R_s + Z_0)$, yielding

$$\mathcal{V}(z, \infty) = V_{ss} = \left(\frac{R_L}{R_s + R_L}\right)V_0$$

a result that is expected, on the basis of the steady-state equivalent circuit shown in Figure 2.16b.

[24]Noting that $|\Gamma_s\Gamma_L| < 1$ and using the fact that for $|x| < 1$ we have

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x}$$

At steady state it appears from Figure 2.16b as if the transmission line is simply not there and that the source is directly connected to the load. While this is essentially true, the transmission line is of course still present and is in fact fully charged, with all of its distributed capacitors charged to a voltage V_{ss} and all of its distributed inductors carrying a current V_{ss}/R_L. If the source were to be suddenly disconnected, the energy stored on the line would eventually be discharged through the load resistance, but only after a sequence of voltages propagating back and forth, reflecting at both ends and becoming smaller in time (see Example 2-5).

We now consider three specific examples. Example 2-4 illustrates the step response of a resistively terminated line, whereas Example 2-5 illustrates the process of discharging of a charged transmission line. Example 2-6 illustrates the pulse response of a resistively terminated line.

Example 2-4: Step response of a resistively terminated lossless line. Consider the circuit shown in Figure 2.17a for the specific case of $R_s = 3Z_0$ and $R_L = 9Z_0$. Sketch $\mathcal{V}_s$, $\mathcal{V}_L$, $\mathcal{I}_s$, and $\mathcal{I}_L$ as a function of t.

Solution: Based on the above discussion, an incident voltage $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = Z_0V_0/(3Z_0 + Z_0) = V_0/4$ is launched on the line at $t = 0$. When this disturbance reaches the load at $t = t_d$, a reflected voltage $\mathcal{V}_1^-(z, t)$ of amplitude $\mathcal{V}_1^-(l, t_d) = \Gamma_L\mathcal{V}_1^+(l, t_d) = V_0/5$, where $\Gamma_L = (9Z_0 - Z_0)/(9Z_0 + Z_0) = 4/5$, is launched toward the source. The reflected voltage arrives the source end at $t = 2t_d$, and a new voltage $\mathcal{V}_2^+(z, t)$ of amplitude $\mathcal{V}_2^+(0, 2t_d) = \Gamma_s\mathcal{V}_1^-(0, 2t_d) = V_0/10$, where $\Gamma_s = (3Z_0 - Z_0)/(3Z_0 + Z_0) = 1/2$, is produced, traveling toward the load. At $t = 3t_d$, a voltage $\mathcal{V}_2^-(z, t)$ of amplitude $\mathcal{V}_2^-(l, 3t_d) = \Gamma_L\mathcal{V}_2^+(l, 3t_d) = 2V_0/25$ is launched from the load end, traveling toward the source, and so on. The bounce diagram is shown in Figure 2.17b. The source- and load-end voltages and the source- and load-end currents are shown in Figure 2.17d. The steady-state circuit seen by the source is also shown in Figure 2.17c.

Example 2-5: A charged line connected to a resistor. Consider a transmission line that is initially charged to a constant voltage $\mathcal{V}(z, t) = V_0$ (such as the steady-state condition of the circuit in Example 2-3), as shown in Figure 2.18a. At $t = 0$, the switch is moved from position 1 to position 2. Analyze and sketch the variation of the source- and load-end voltages $\mathcal{V}_s$ and $\mathcal{V}_L$ as a function of t for three different cases: (a) $R_{s2} = Z_0/3$, (b) $R_{s2} = Z_0$, and (c) $R_{s2} = 3Z_0$.

Solution: Before $t = 0$, the steady-state condition holds, and $\mathcal{V}_s(0^-) = \mathcal{V}_L(0^-) = V_{ss} = V_0$ and $\mathcal{I}_s(0^-) = \mathcal{I}_L(0^-) = I_{ss} = 0$. At $t = 0$, the switch moves to position 2, which causes both $\mathcal{V}_s$ and $\mathcal{I}_s$ to change immediately. The change in the source-end voltage $\mathcal{V}_s$ (and the source-end current $\mathcal{I}_s$) can be interpreted as a new voltage disturbance $\mathcal{V}_1^+(z, t)$ (and a new current disturbance

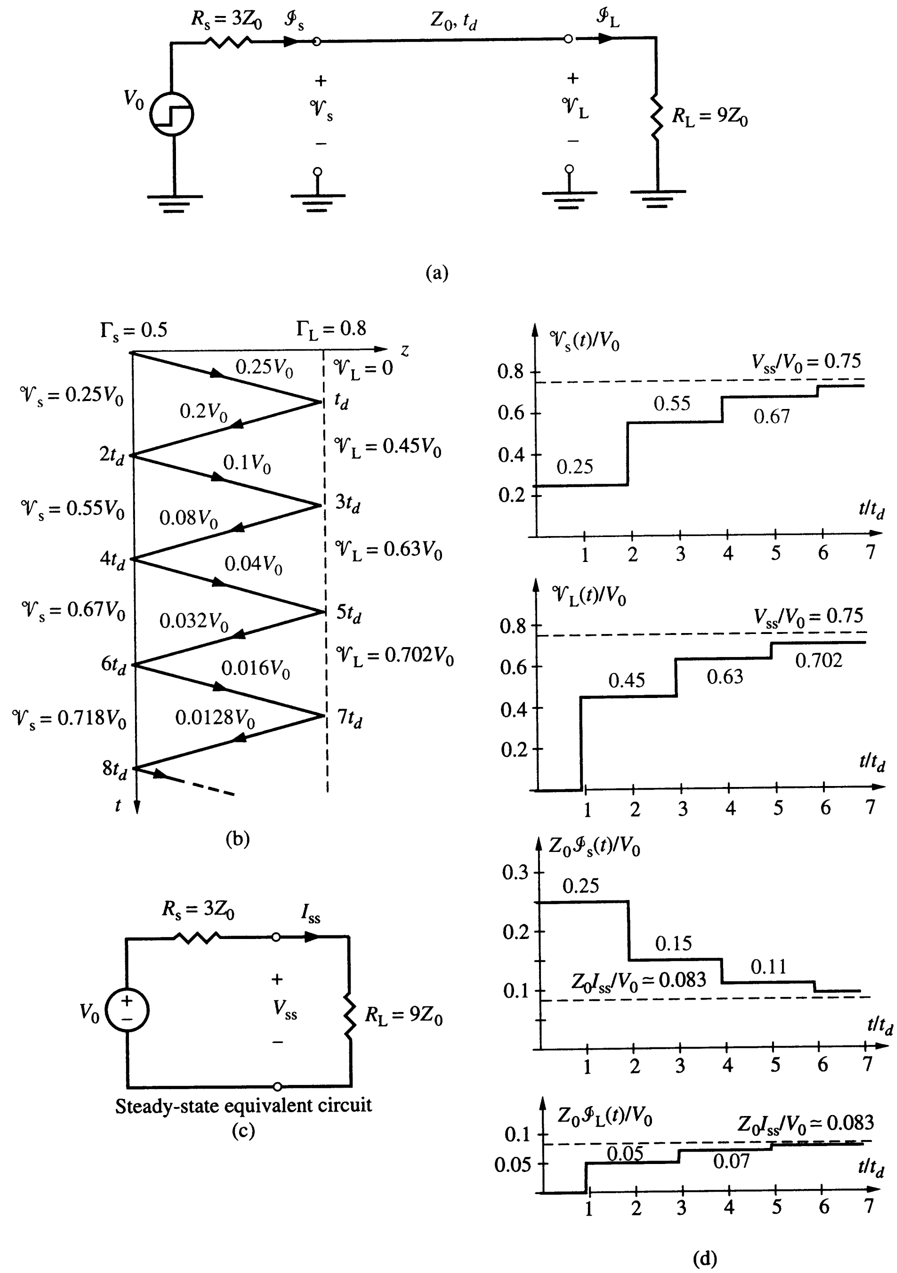


FIGURE 2.17. Step excitation of a resistively terminated lossless line. (a) The circuit configuration. (b) Bounce diagram. (c) Steady-state equivalent circuit seen by the source. (d) Normalized source- and load-end voltages and source- and load-end currents as a function of t/t_d.

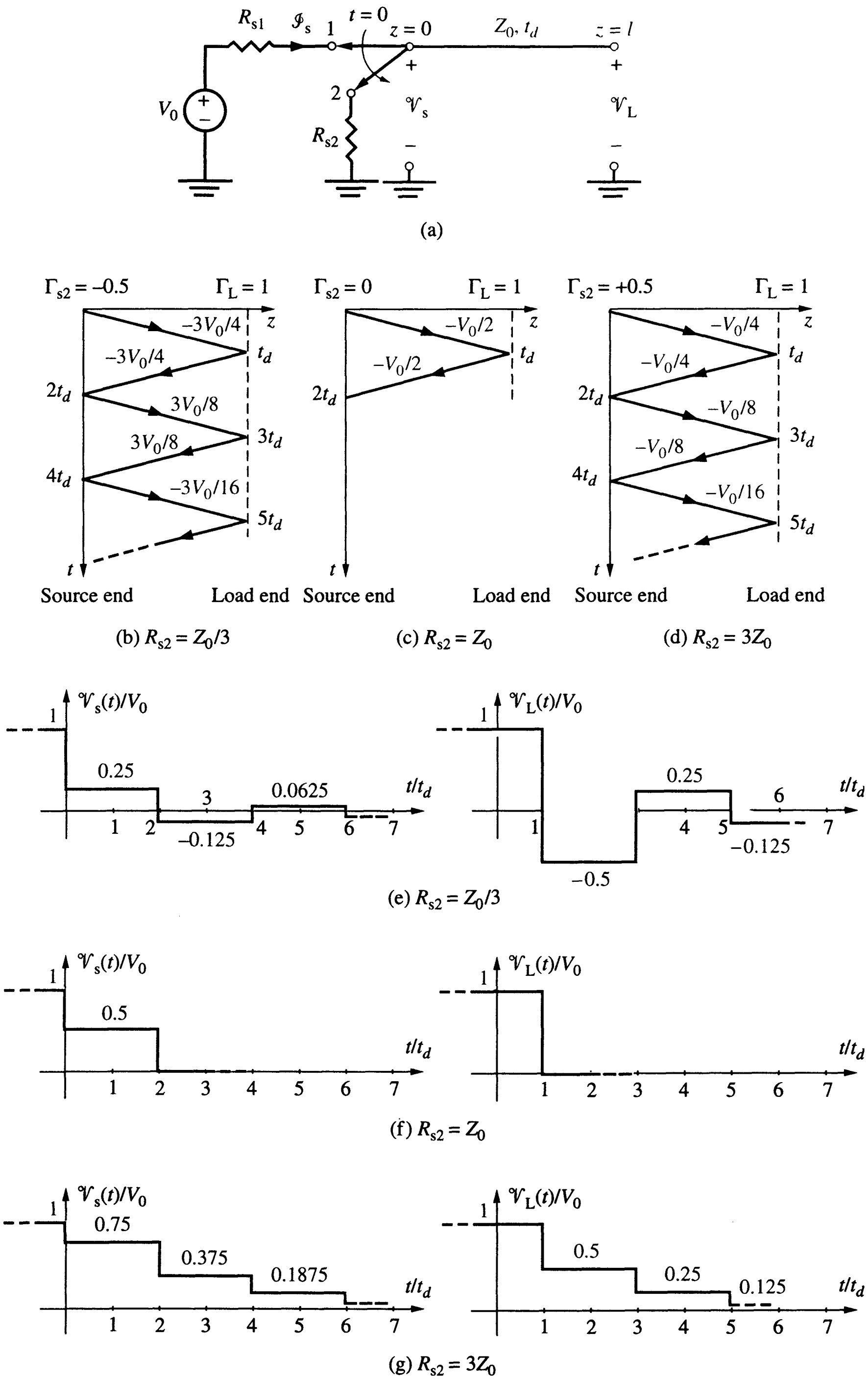


FIGURE 2.18. Discharging of a charged line. (a) A charged line connected to a resistor R_{s2} for Example 2-5. (b), (c), and (d) Bounce diagrams for $R_{s2} = Z_0/3$, $R_{s2} = Z_0$, and $R_{s2} = 3Z_0$. (e), (f), and (g) Normalized source- and load-end voltages as a function of t for the three different cases.

$\mathcal{I}_1^+(z, t)$) launched on the line from the source end. The amplitude of this new voltage $\mathcal{V}_1^+(z, t)$ (and the new current $\mathcal{I}_1^+(z, t)$) is determined by the change in $\mathcal{V}_s$ (or in $\mathcal{I}_s$) between $t = 0^-$ and $t = 0^+$, namely,

$$\mathcal{V}_1^+(0, 0) = \mathcal{V}_s(0^+) - \mathcal{V}_s(0^-) = \mathcal{V}_s(0^+) - V_0$$

and

$$\mathcal{I}_1^+(0, 0) = \mathcal{I}_s(0^+) - \mathcal{I}_s(0^-) = \mathcal{I}_s(0^+)$$

Using the new boundary condition at the source end imposed by R_{s2}, namely,

$$\mathcal{V}_s(0^+) = -R_{s2}\mathcal{I}_s(0^+) = -R_{s2}\mathcal{I}_1^+(0, 0) = -R_{s2}\mathcal{V}_1^+(0, 0)/Z_0$$

we can write

$$\mathcal{V}_1^+(0, 0) = -R_{s2}\frac{\mathcal{V}_1^+(0, 0)}{Z_0} - V_0 \rightarrow \mathcal{V}_1^+(0, 0) = -\frac{Z_0 V_0}{R_{s2} + Z_0}$$

Note that the negative sign in the source-end boundary condition $\mathcal{V}_s = -R_{s2}\mathcal{I}_s$ is due to the defined direction of $\mathcal{I}_s$, with positive current coming out of the terminal of positive voltage. At $t = t_d$, the new voltage disturbance reaches the open end of the line, where a reflected voltage $\mathcal{V}_1^-(z, t)$ with amplitude $\mathcal{V}_1^-(l, t_d) = \mathcal{V}_1^+(l, t_d)$ is produced, traveling toward the source end. At $t = 2t_d$, $\mathcal{V}_1^-(z, t)$ arrives at the source end, and a reflected voltage $\mathcal{V}_2^+(z, t)$ with amplitude $\mathcal{V}_2^+(0, 2t_d) = \Gamma_s\mathcal{V}_1^-(0, 2t_d) = (R_{s2} - Z_0)\mathcal{V}_1^-(0, 2t_d)/(R_{s2} + Z_0)$ is launched back toward the load. This process continues until a new steady-state condition is reached, when the line voltage eventually becomes zero. Figures 2.18b, c, and d show the bounce diagrams for three different values of R_{s2}, namely $Z_0/3$, Z_0, and $3Z_0$. Figures 2.18e, f, and g show the variation of the source- and load-end voltages for all three cases as a function of time t.

Example 2-6: Pulse excitation of a transmission line. A high-speed logic gate represented by a pulse voltage source of amplitude 1 V, pulse width 200 ps, and output impedance 900Ω drives a load of 25Ω through a 100Ω line, as shown in Figure 2.19. Assuming a lossless line with a one-way delay of $t_d = 400$ ps, sketch the voltage waveforms $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$ at the two ends of the line.

Solution: At $t = 0$, an initial voltage pulse $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = V_0Z_0/(R_s + Z_0) = 100$ mV is launched at the driver end of the line. The front of the 100 mV pulse reaches the load (at $z = l$) at $t = 400$ ps, and a reflected pulse of amplitude $\mathcal{V}_1^-(l, t_d) = \Gamma_L\mathcal{V}_1^+(l, t_d) = -60$ mV starts traveling back toward the driver. Both pulses $\mathcal{V}_1^+(z, t)$ and $\mathcal{V}_1^-(z, t)$ exist at the load end for a period of only 200 ps, adding up to a total voltage of 40 mV. The reflected

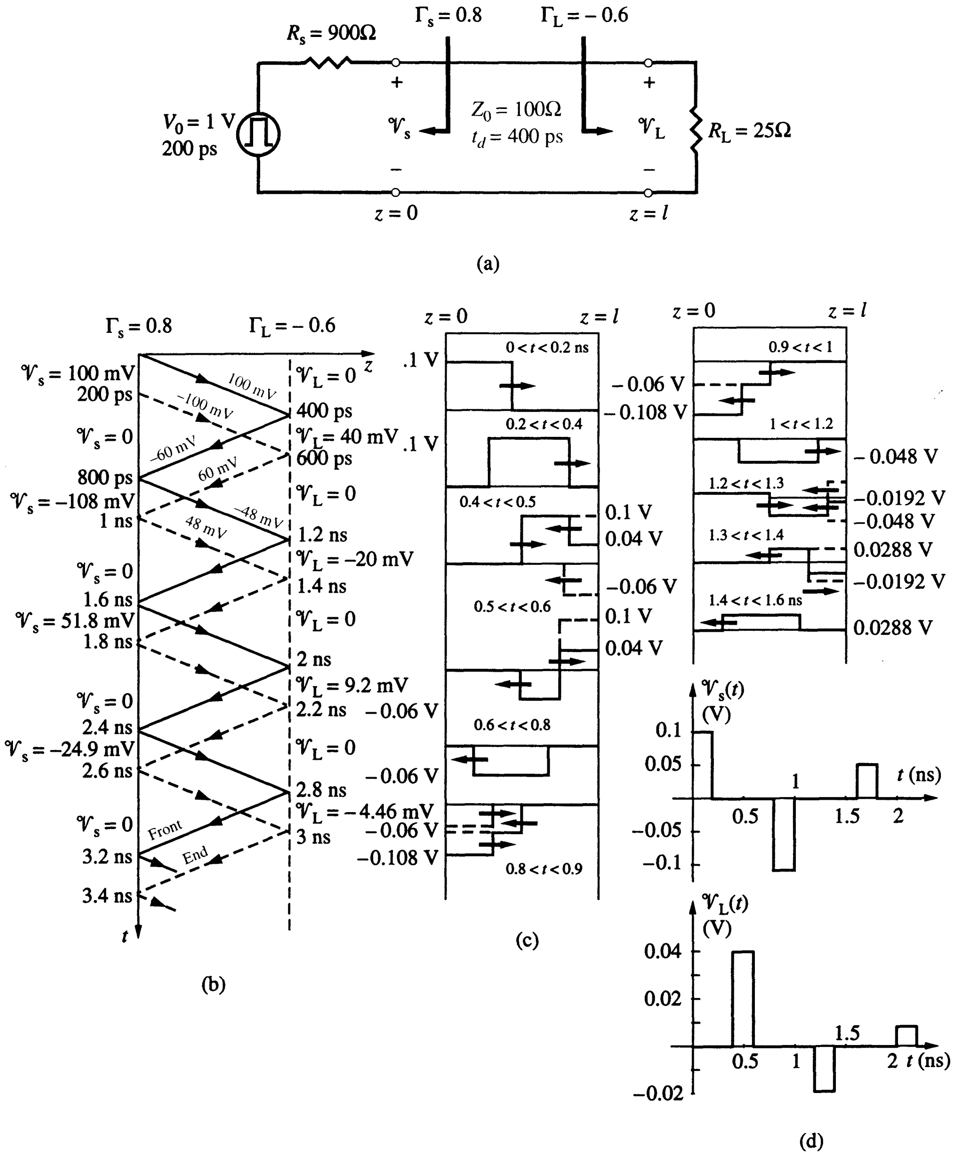


FIGURE 2.19. Pulse excitation of a transmission line (Example 2-6). (a) Circuit diagram. (b) Bounce diagram. (c) Distribution of voltage along the line at different times. (d) The source- and load-end voltages as a function of time.

pulse arrives at the driver end at $t = 800$ ps and launches a pulse of amplitude $\mathcal{V}_2^+(0, 2t_d) = \Gamma_s \mathcal{V}_1^-(0, 2t_d) = -48$ mV back toward the load end. The two pulses $\mathcal{V}_1^-(z, t)$ and $\mathcal{V}_2^+(z, t)$ exist at the driver end only for 200 ps, totaling to a voltage of -108 mV. This process continues on, with the pulse amplitude reduced by 40% and inverted at the load end and reduced by 20% at the driver end each time. Figure 2.19b shows the bounce diagram, and Figure 2.19c shows the snapshots of the voltages along the line at various time intervals. The variation with time of the driver- and load-end voltages are also shown in Figure 2.19d.

The Transmission Line as a Linear Time-Invariant System In some cases it is useful to think of the transmission line as a linear time-invariant system, with a defined input and output. For this purpose, the input $\mathcal{V}_{\text{in}}(t)$ can be defined as the voltage or current at the input of the line, while the output $\mathcal{V}_{\text{out}}(t)$ can be a voltage or current somewhere else on the line—for example, the load voltage $\mathcal{V}_{\text{L}}(t)$ as indicated in Figure 2.20.

Note that since the fundamental differential equations ([2.1] and [2.2]) that govern the transmission line voltage and current are linear, the relationship between $\mathcal{V}_{\text{in}}(t)$ and $\mathcal{V}_{\text{out}}(t)$ is linear. In other words, for two different input signals $\mathcal{V}_{\text{in}_1}$ and $\mathcal{V}_{\text{in}_2}$, which individually produce two different output signals $\mathcal{V}_{\text{out}_1}$ and $\mathcal{V}_{\text{out}_2}$, the response due to a linear superposition of the two inputs, $\mathcal{V}_{\text{in}_{1+2}} = a_1 \mathcal{V}_{\text{in}_1} + a_2 \mathcal{V}_{\text{in}_2}$, is

$$\mathcal{V}_{\text{out}_{1+2}} = a_1 \mathcal{V}_{\text{out}_1} + a_2 \mathcal{V}_{\text{out}_2}$$

Since the physical properties of the transmission line (L, C, t_d, Z_0) do not change with time, the relationship between $\mathcal{V}_{\text{in}}(t)$ and $\mathcal{V}_{\text{out}}(t)$ is also time-invariant. In

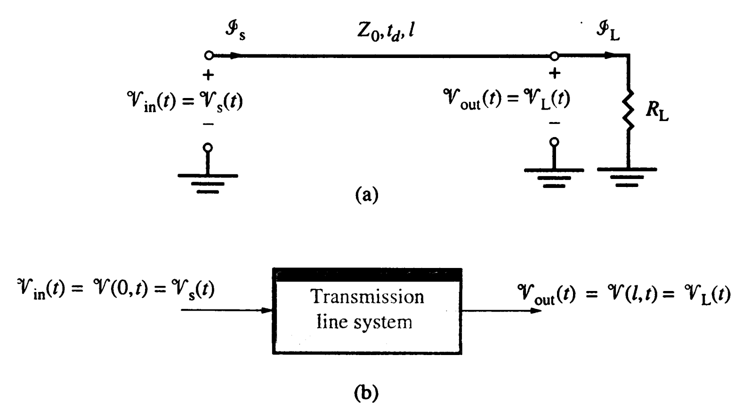


FIGURE 2.20. The transmission line as a linear time-invariant system. The input $\mathcal{V}_{\text{in}}(t)$ to the system can be defined as the line input voltage $\mathcal{V}_s(t)$, whereas the output $\mathcal{V}_{\text{out}}(t)$ could be any voltage or current of interest anywhere on the line, such as the load voltage $\mathcal{V}_{\text{L}}(t)$.

other words, if the output due to an input $\mathcal{V}_{\text{in}_1}(t)$ is $\mathcal{V}_{\text{out}_1}(t)$, then the output due to a time-shifted version of the input, namely $\mathcal{V}_{\text{in}_1}(t-\tau)$, is simply a similarly shifted version of the output, namely $\mathcal{V}_{\text{out}_1}(t-\tau)$.

As with any linear time-invariant system, the response of a transmission line to any arbitrary excitation signal can be determined from its response to an impulse excitation. In the transmission line context, an input pulse can be considered to be an impulse if its duration is much shorter than any other time constant in the system or the one-way travel time t_d in the case of lossless lines with resistive terminations. In most applications, however, it is necessary to determine the response of the line to step inputs, as were illustrated in Examples 2-3 through 2-5. For this purpose, it is certainly easier to determine the step response directly rather than to determine the pulse (or impulse) response first and then use it to determine the step response.

Treatment of a transmission line as a linear time-invariant system can sometimes be useful in determining its response to pulse inputs, as illustrated in Example 2-7.

Example 2-7: The transmission line as a linear time-invariant system. Consider the transmission line system of Figure 2.21a, the step response of which was determined in Example 2-4. Determine the load voltage $\mathcal{V}_{\text{L}}(t)$ for an input excitation in the form of a single pulse of amplitude V_0 and duration $0.5t_d$ (i.e., $\mathcal{V}_{\text{in}}(t) = V_0[u(t) - u(t - t_d/2)]$).

Solution: For the circuit of Figure 2.21a, it is convenient to define the input signal to be the excitation (source) voltage and the output as the load voltage, as indicated. As shown in Figure 2.21b, the input pulse of amplitude V_0 and duration $t_d/2$ can be viewed as a superposition of two different input signals: a step input starting at $t = 0$, namely $\mathcal{V}_{\text{in}_1}(t) = V_0 u(t)$, and a shifted negative step input, namely, $\mathcal{V}_{\text{in}_2}(t) = -V_0 u(t - t_d/2)$, where $u(\zeta)$ is the unit step function, $u(\zeta) = 1$ for $\zeta > 0$, and $u(\zeta) = 0$ for $\zeta < 0$. The output $\mathcal{V}_{\text{out}_1}(t)$ due to the input $\mathcal{V}_{\text{in}_1}(t)$ was determined in Example 2-4 and is plotted in the top panel of Figure 2.21c. Since the transmission line is a linear time-invariant system, the response $\mathcal{V}_{\text{out}_2}(t)$ due to the input $\mathcal{V}_{\text{in}_2}(t)$ is simply a flipped-over and shifted version of $\mathcal{V}_{\text{out}_1}(t)$, as shown in the middle panel of Figure 2.21c. The bottom panel of Figure 2.21c shows the superposition of the two responses, which is the desired pulse response.

Note that our solution of this problem using the linear time-invariant system treatment was simplified by the fact that the step response of the system was already in hand from Example 2-4. Note also that the pulse response in the case of Example 2-6 could also have been determined by a similar method. However, one then has to first determine the step response, and whether or not the system approach is easier in general depends on the particular problem.

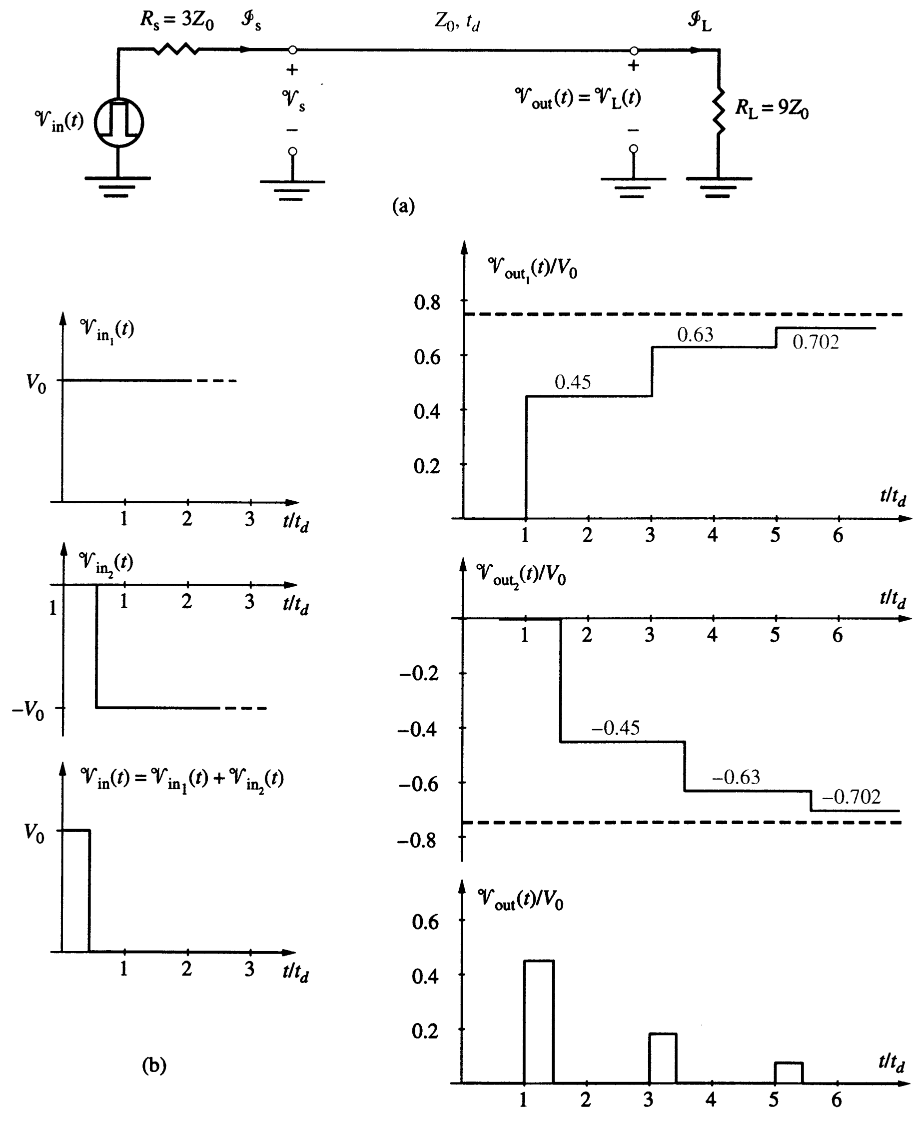


FIGURE 2.21. **Pulse response of a transmission line.** (a) The line configuration and source and load impedances. (b) The input pulse of amplitude V_0 and duration $t_d/2$ is represented as a superposition of a $\mathcal{V}_{in_1}$ and $\mathcal{V}_{in_2}$. (c) The response is computed as a superposition of the responses due to the individual step inputs. Note that $\mathcal{V}_{out_1}(t)$ was already computed in Example 2-4.

2.4.2 Junctions between Transmission Lines

We have seen that reflections from terminations at the source and load ends of transmission lines lead to ringing and other effects. Reflections also occur at discontinuities at the interfaces between transmission lines, connected either in cascade or in parallel, as shown, for example, in Figures 2.22a and 2.23a and as often encountered in practice. For example, consider the case of the two lossless transmission lines A and B (with characteristic impedances Z_{0A} and Z_{0B}) connected in tandem (i.e., in series) as shown in Figure 2.22a. Assume a voltage disturbance of amplitude $\mathcal{V}_{1A}^{+}(z, t)$ (with an associated current of $\mathcal{I}_{1A}^{+}(z, t) = \mathcal{V}_{1A}^{+}/Z_{0A}$) to arrive at the junction between lines A and B (located at $z = l_j$) from line A at $t = t_0$. A voltage $\mathcal{V}_{1A}^{-}(z, t)$ of amplitude $\mathcal{V}_{1A}^{-}(l_j, t_0) = \Gamma_{AB}\mathcal{V}_{1A}^{+}(l_j, t_0)$ reflects back to line A, where the reflection coefficient Γ_{AB} is given by $\Gamma_{AB} = (Z_{0B} - Z_{0A})/(Z_{0B} + Z_{0A})$, since line B presents a load impedance of Z_{0B} to line A. In addition, a voltage $\mathcal{V}_{1B}^{+}(z, t)$ of amplitude $\mathcal{V}_{1B}^{+}(l_j, t_0) = \mathcal{T}_{AB}\mathcal{V}_{1A}^{+}(l_j, t_0)$ is transmitted into line B, where $\mathcal{T}_{AB}$ is called the *transmission coefficient,* defined as the ratio of the transmitted voltage to the incident voltage, i.e., $\mathcal{T}_{AB} \equiv \mathcal{V}_{1B}^{+}(l_j, t_0)/\mathcal{V}_{1A}^{+}(l_j, t_0)$. To find $\mathcal{T}_{AB}$, we apply the boundary condition at the junction, which states that the total voltages on the left and right sides of the junction must be equal:

$$\mathcal{V}_{1A}^{+}(l_j, t_0) + \mathcal{V}_{1A}^{-}(l_j, t_0) = \mathcal{V}_{1B}^{+}(l_j, t_0)$$

yielding $\mathcal{T}_{AB} = 1 + \Gamma_{AB} = 2Z_{0B}/(Z_{0B} + Z_{0A})$. The transmission coefficient $\mathcal{T}_{AB}$ represents the fraction of the incident voltage that is transferred from line A to line B. Note that depending on the value of the reflection coefficient, the transmitted voltage can actually be larger in amplitude than the incident voltage, so that we may have $\mathcal{T}_{AB} > 1$ (in those cases when $\Gamma_{AB} > 0$).

Similarly, if a voltage disturbance $\mathcal{V}_{1B}^{-}(z, t)$ of amplitude $\mathcal{V}_{1B}^{-}(l_j, t_1)$ (produced by reflection when $\mathcal{V}_{1B}^{+}(z, t)$ reaches the end of line B) arrives at the same junction between A and B from line B at $t = t_1$, a voltage $\mathcal{V}_{2B}^{+}(z, t)$ of amplitude $\mathcal{V}_{2B}^{+}(l_j, t_1) = \Gamma_{BA}\mathcal{V}_{1B}^{-}(l_j, t_1)$ reflects back to line B and a voltage $\mathcal{V}_{2A}^{-}(l_j, t_1) = \mathcal{T}_{BA}\mathcal{V}_{1B}^{-}(l_j, t_1)$ is transmitted into line A, where Γ_{BA} and $\mathcal{T}_{BA}$ are given by

$$\Gamma_{BA} = \frac{Z_{0A} - Z_{0B}}{Z_{0A} + Z_{0B}} = -\Gamma_{AB}$$

$$\mathcal{T}_{BA} = 1 + \Gamma_{BA} = \frac{2Z_{0A}}{Z_{0A} + Z_{0B}} = \frac{Z_{0A}}{Z_{0B}}\mathcal{T}_{AB}$$

The following two examples are both associated with junctions between transmission lines.

Example 2-8: Cascaded transmission lines. Consider the transmission line system shown in Figure 2.22a, where a step voltage source of amplitude 1.5 V and source resistance 50Ω excites two cascaded lossless transmission lines (A and B) of characteristic impedances 50Ω and 25Ω and lengths 5 cm and 2 cm, respectively. The speed of

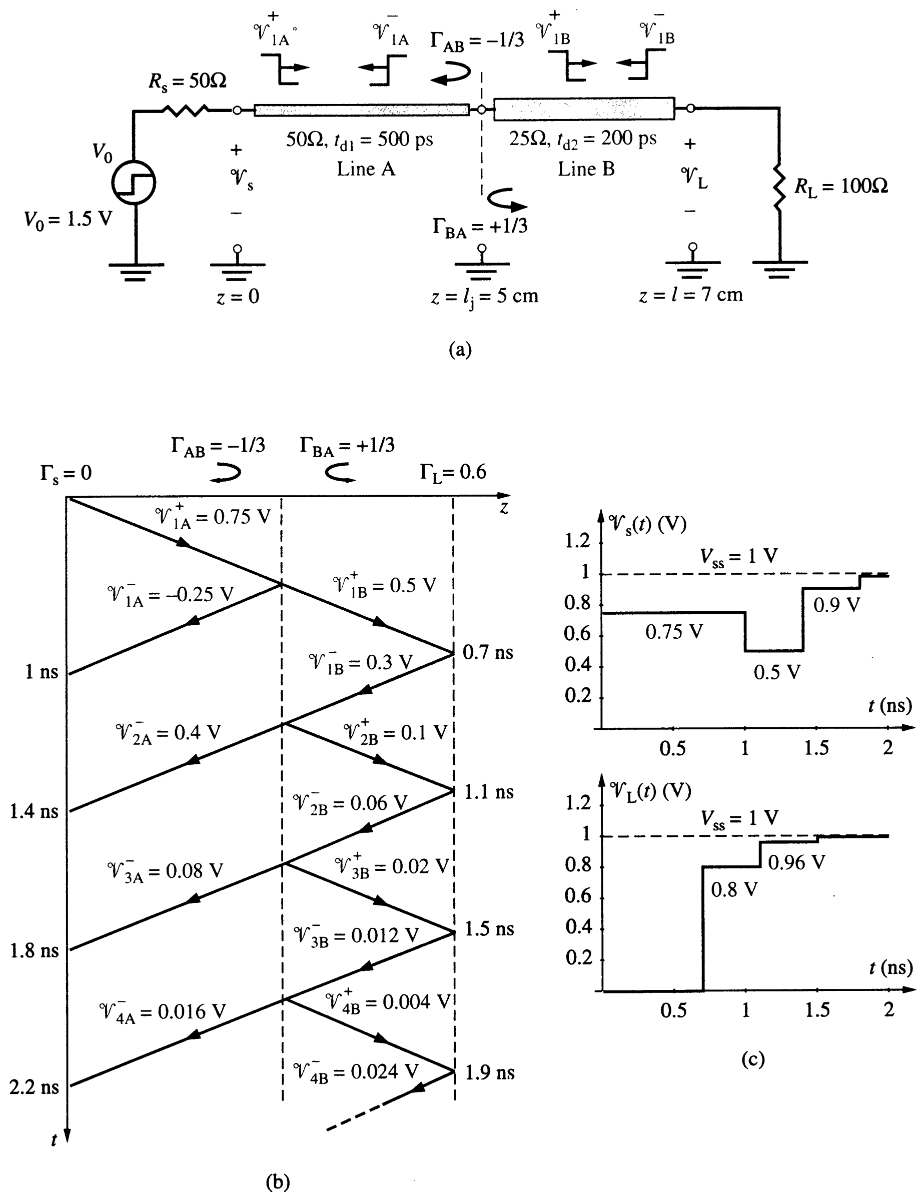


FIGURE 2.22. Cascaded transmission lines. (a) Circuit diagram for Example 2-8. (b) Bounce diagram. (c) Source-end voltage $\mathcal{V}_s$ and load-end voltage $\mathcal{V}_L$ as a function of t.

propagation in each line is 10 cm-ns^{-1}.[25] The second line (B) is terminated with a load impedance of 100Ω at the other end. Draw the bounce diagram and sketch the voltages $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$ as functions of time.

Solution: With respect to Figure 2.22a, we note that $l_j = 5$ cm and $l = 7$ cm. At $t = 0^+$, voltage $\mathcal{V}_{1A}^+(z, t)$ of amplitude $\mathcal{V}_{1A}^+(0, 0) = 0.75$ V is launched on line A. This voltage reaches the junction between the two lines at $t = t_{d1} = 5$ cm/(10 cm/ns) $= 500$ ps, when a reflected voltage $\mathcal{V}_{1A}^-(z, t)$ of amplitude $\mathcal{V}_{1A}^-(l_j, t_{d1}) = \Gamma_{AB}\mathcal{V}_{1A}^+(l_j, t_{d1}) = (-1/3)(0.75) = -0.25$ V, and a transmitted voltage $\mathcal{V}_{1B}^+(z, t)$ of amplitude $\mathcal{V}_{1B}^+(l_j, t_{d1}) = \mathcal{T}_{AB}\mathcal{V}_{1A}^+(l_j, t_{d1}) = (2/3)(0.75) = 0.5$ V are created. The reflected disturbance arrives at the source end at $t = 2t_{d1} = 1$ ns and is absorbed completely since $\Gamma_s = 0$. The transmitted wave reaches the load at $t = t_{d1} + t_{d2} = 700$ ps, and a reflected voltage $\mathcal{V}_{1B}^-(z, t)$ of amplitude $\mathcal{V}_{1B}^-(l, 700 \text{ ps}) = \Gamma_L\mathcal{V}_{1B}^+(l, 700 \text{ ps}) = (0.6)(0.5) \text{ V} = 0.3$ V is launched toward the source. This reflected disturbance arrives at the junction from line B at $t = t_{d1} + 2t_{d2} = 900$ ps, and reflected and transmitted voltages $\mathcal{V}_{2B}^+(z, t)$ and $\mathcal{V}_{2A}^-(z, t)$ of amplitudes respectively $\mathcal{V}_{2B}^+(l_j, 900 \text{ ps}) = \Gamma_{BA}\mathcal{V}_{1B}^-(l_j, 900 \text{ ps}) = (1/3)(0.3) = 0.1$V and $\mathcal{V}_{2A}^-(l_j, 900 \text{ ps}) = \mathcal{T}_{BA}\mathcal{V}_{1B}^-(l_j, 900 \text{ ps}) = (4/3)(0.3) = 0.4$ V are launched respectively toward the load and the source. The continuation of this process can be followed by means of a bounce diagram, as shown in Figure 2.22b. The source- and load-end voltages are plotted as a function of t in Figure 2.22c.

Example 2-9: Three parallel transmission lines. Consider three identical lossless transmission lines, each with characteristic impedance Z_0 and one-way time delay t_d, connected in parallel at a common junction as shown in Figure 2.23a. The main line is excited at $t = 0$ by a step voltage source of amplitude V_0 and a source resistance of $R_s = Z_0$. Find and sketch the voltages $\mathcal{V}_s(t)$, $\mathcal{V}_{L1}(t)$, and $\mathcal{V}_{L2}(t)$ for the following two cases: (a) $R_{L1} = R_{L2} = Z_0$ and (b) $R_{L1} = Z_0$ and $R_{L2} = \infty$.

Solution: For any voltage disturbance $\mathcal{V}_i(z, t)$ of amplitude V_i arriving at the junction from any one of the three lines, the parallel combination of the characteristic impedances of the other two lines acts as an equivalent load impedance at the junction. Once a voltage is incident at the junction, a voltage of amplitude $V_r = \Gamma_j V_i = (Z_0/2 - Z_0)V_i/(Z_0/2 + Z_0) = -V_i/3$ is reflected and a voltage of amplitude $V_t = \mathcal{T}_j V_i = (1 + \Gamma_j)V_i = 2V_i/3$ is transmitted to both of the other lines. Note that the reflection coefficient at the junction is denoted simply as Γ_j, because all of the transmission lines are identical and the reflection coefficient at the junction is thus the same regardless of which line the wave is incident from. For the circuit shown in Figure 2.23a, the bounce diagram and the sketches of voltages $\mathcal{V}_s$, $\mathcal{V}_{L1}$, and $\mathcal{V}_{L2}$ for both cases are shown in Figure 2.23b and c. When $R_{L1} = R_{L2} = Z_0$, only one of lines 1 and 2 is shown in the bounce diagram, since

[25]Note from Table 2.1 that 10 cm-ns^{-1} is approximately the speed of propagation in alumina (Al_2O_3), a ceramic commonly used for electronics packaging.

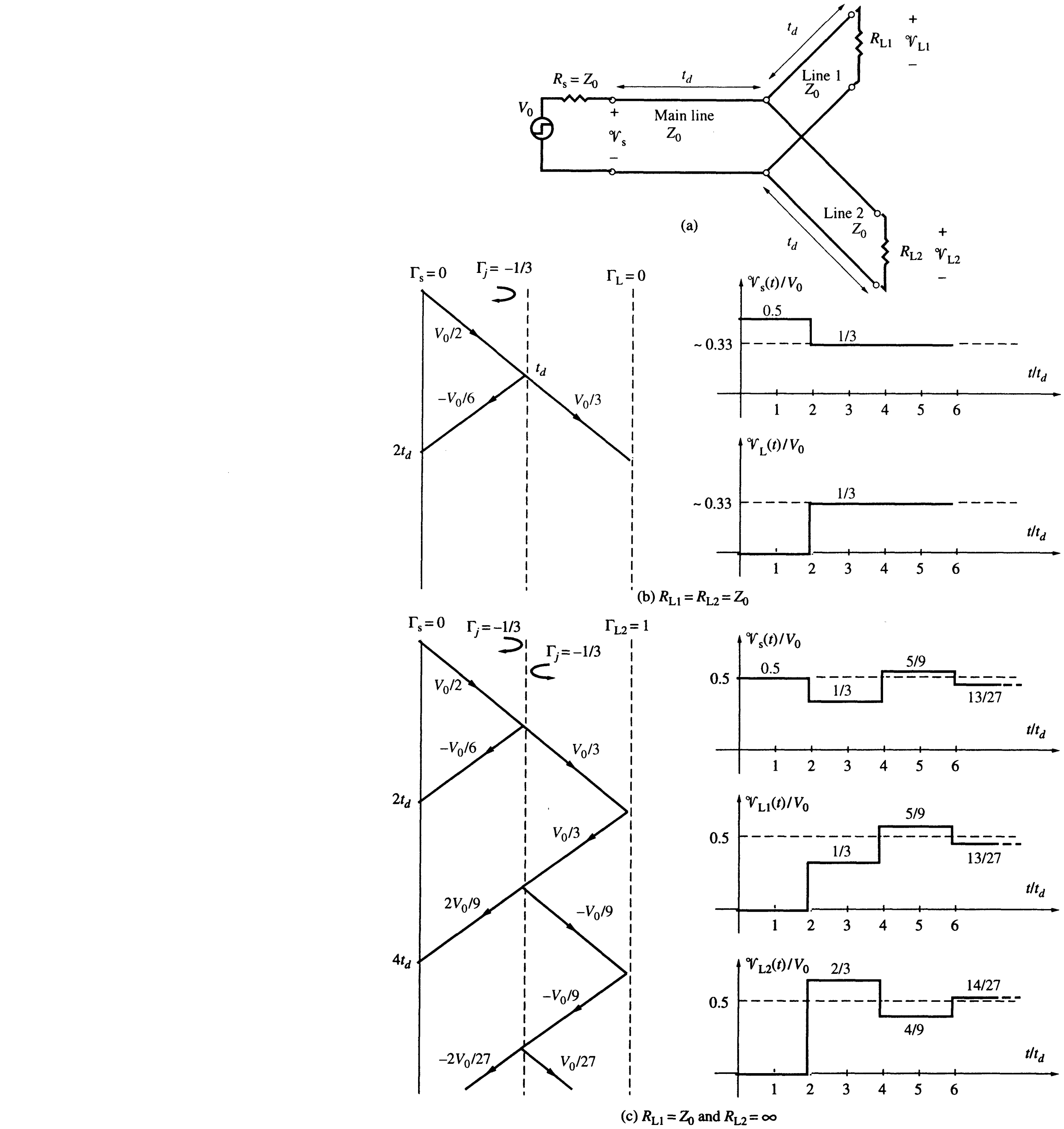


FIGURE 2.23. Three transmission lines connected in parallel at a common junction. (a) Circuit diagram for Example 2-9. (b) Bounce diagram and the variation of the $\mathcal{V}_s$ and $\mathcal{V}_{L1} = \mathcal{V}_{L2} = \mathcal{V}_L$ voltages as a function of t for $R_{L1} = R_{L2} = Z_0$. (c) Bounce diagram and the variation of the $\mathcal{V}_s$, $\mathcal{V}_{L1}$, and $\mathcal{V}_{L2}$ voltages as a function of t for $R_{L1} = Z_0$ and $R_{L2} = \infty$.

what happens on the other is identical, as seen in Figure 2.23b. When $R_{L1} = Z_0$ and $R_{L2} = \infty$, only line 2 is shown in the bounce diagram in Figure 2.23c, since no reflection occurs on line 1.

2.5 TRANSIENT RESPONSE OF TRANSMISSION LINES WITH REACTIVE OR NONLINEAR TERMINATIONS

Up to now, we have studied only transmission lines with resistive terminations. In this section, we consider reactive loads and loads with nonlinear current-voltage characteristics.

2.5.1 Reactive Terminations

Reactive loads are encountered quite often in practice; in high-speed bus designs, for example, capacitive loading by backplanes (consisting of plug-in cards having printed circuit board traces and connectors) often becomes the bottleneck when high-speed CPUs communicate with shared resources on the bus. Inductive loading due to bonding wire inductances is also important in many integrated-circuit packaging technologies. Packaging pins, vias between two wiring levels, and variations in line width can often be modeled as capacitive and inductive discontinuities. The capacitances and inductances of these various packaging components can range between 0.5 and 4 pF and between 0.1 and 35 nH, respectively.[26]

For transmission lines with resistive loads, the reflected and transmitted voltages and currents have the same temporal shape as the incident ones and do not change their shape as a function of time. For a step excitation, for example, the reflected voltage produced by a resistive termination remains constant in time, as discussed in preceding sections. However, in the case of capacitive or inductive terminations, the reflected and transmitted voltages and currents do not have the same temporal shape as the incident ones. The terminal boundary condition at the reactive termination must now be expressed as a differential equation whose general solution can be exceedingly complicated, whether the solutions are carried out in the time domain or by the use of Laplace transformation. We illustrate the basic principles by considering a line terminated at an inductance, as shown in Figure 2.24.

When the traveling disturbance $\mathcal{V}_1^+(z, t)$ (taken in Figure 2.24b, c as a constant voltage V_0) and its associated current $\mathcal{I}_1^+(z, t)$ (taken in Figure 2.24b, c as a constant current $I_0 = V_0/Z_0$) first reach the end of the line, the inductive load acts as an open circuit, since its current cannot change instantaneously. Thus the disturbance is initially reflected in the same manner as an open circuit, the terminal voltage jumping

[26]See Chapter 6 of H. B. Bakoglu, *Circuits, Interconnections, and Packaging for VLSI,* Addison Wesley, Reading, Massachusetts, 1990.

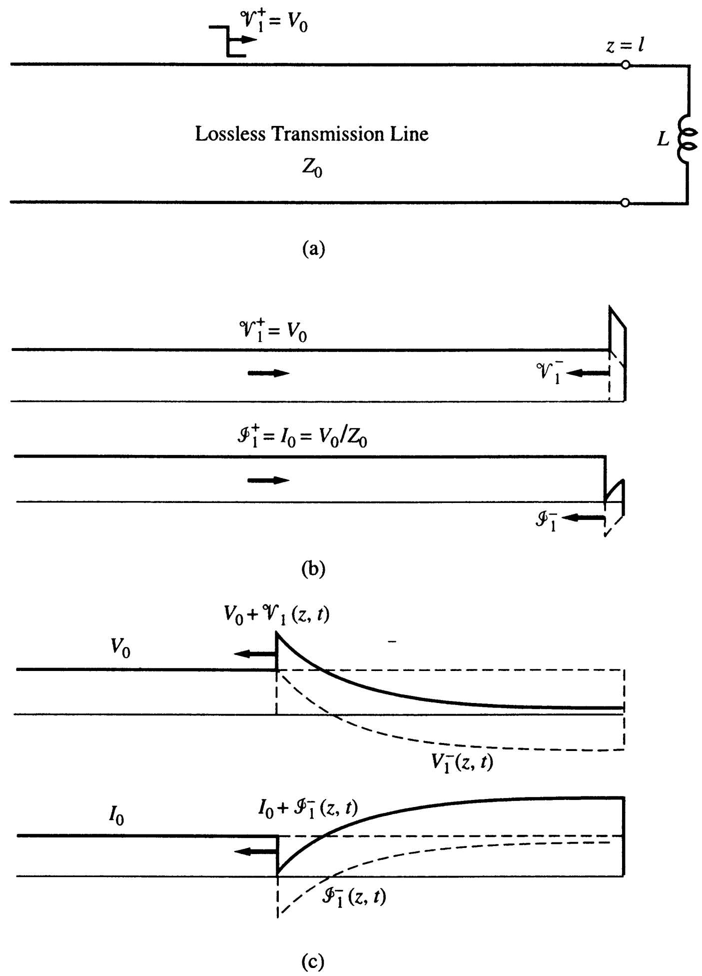


FIGURE 2.24. Reflection from a purely inductive load. The source is assumed to be matched (i.e., no reflections back from the source) and to supply a constant voltage V_0. The distributions of the voltage and current are shown at two different times: (b) immediately after reflection when the inductor behaves like an open circuit, and (c) later in time when the inductor behaves as a short circuit.

to $2V_0$ and the terminal current being zero, for that instant (Figure 2.24b). However, since a voltage of $2V_0$ now exits across the inductor, its current builds up, flowing more and more freely in time, until it is practically equivalent to a short circuit. At steady state, when the voltage across the inductance reduces to zero, the current through it becomes $2I_0$ (Figure 2.24c), similar to the case of a short-circuit termination (see Example 2-2).

Between an initial open circuit and an eventual short circuit, the voltage across the inductive load goes through all intermediate values, and the reflected voltage changes accordingly. To determine the analytical expression describing the variation of the voltage across the inductance, we need to simultaneously solve the transmission line equations (or the general solutions dictated by them, namely [2.8] and [2.9]) along with the differential equation describing the boundary condition imposed by the inductive load as

$$\mathcal{V}_L(t) = L\frac{d\mathcal{I}_L(t)}{dt}$$

For a general incident voltage $\mathcal{V}_1^+(z, t)$, the load voltage $\mathcal{V}_L(t)$ and current $\mathcal{I}_L(t)$ are given by

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(l, t) + \mathcal{V}_1^-(l, t)$$

$$\mathcal{I}_L(t) = \mathcal{I}_1^+(l, t) + \mathcal{I}_1^-(l, t) = \frac{\mathcal{V}_1^+(l, t)}{Z_0} - \frac{\mathcal{V}_1^-(l, t)}{Z_0}$$

where the location of the load (i.e., the inductive termination) is assumed to be at $z = l$. We thus have

$$\mathcal{V}_1^+(l, t) + \mathcal{V}_1^-(l, t) = L\frac{d}{dt}\left(\frac{\mathcal{V}_1^+(l, t)}{Z_0} - \frac{\mathcal{V}_1^-(l, t)}{Z_0}\right)$$

or

$$\frac{d\mathcal{V}_1^-(l, t)}{dt} + \frac{Z_0}{L}\mathcal{V}_1^-(l, t) = \frac{d\mathcal{V}_1^+(l, t)}{dt} - \frac{Z_0}{L}\mathcal{V}_1^+(l, t)$$

which is a differential equation with $\mathcal{V}_1^-(l, t)$ as the dependent variable, since $\mathcal{V}_1^+(l, t)$ is presumably known, it being the incident voltage disturbance arriving from the source end of the line. The right-hand side is therefore a known function of time, and the equation is simply a first-order differential equation with constant coefficients. Note that we assume the source to be either far enough away or matched, so that the voltage $\mathcal{V}_1^-(z, t)$ does not reach the source end and generate a reflected voltage $\mathcal{V}_2^+(z, t)$ before $\mathcal{V}_1^-(l, t)$ reaches its "steady-state" value (nearly zero in the case when $\mathcal{V}_1^+(z, t) = V_0$ as shown in Figure 2.24c).

To study the simplest case, let us consider an incident voltage with a constant amplitude V_0 (i.e., $\mathcal{V}_1^+(l, t) = V_0$) launched by a step source reaches the inductor at $t = 0$. The above differential equation then simplifies to

$$\frac{d\mathcal{V}_1^-(l, t)}{dt} + \frac{Z_0}{L}\mathcal{V}_1^-(l, t) = -\frac{Z_0 V_0}{L}$$

The solution of this first-order differential equation is[27]

$$\mathcal{V}_1^-(l, t) = -V_0 + Ke^{-(Z_0/L)t}$$

[27]The solution can be found via Laplace transformation or as a superposition of the homogeneous solution and the particular solution; the validity of the solution can be shown by simply substituting it into the differential equation.

where the coefficient K needs to be determined by the known initial conditions as they relate to $\mathcal{V}_1^-(l, t)$. At the instant of the arrival of the voltage disturbance ($t = 0$), the current through the inductance $\mathcal{I}_L(t = 0) = 0$, and we thus have the incident voltage fully reflected, or $\mathcal{V}_1^-(l, t = 0) = V_0$. Thus, we must have $K = 2V_0$. The solution for the reflected voltage is then

$$\mathcal{V}_1^-(l, t) = -V_0 + 2V_0 e^{-(Z_0/L)t}$$

which varies from its initial value of V_0 to an eventual value of $-V_0$, as shown in Figure 2.24c.

In Examples 2-10 and 2-11, we illustrate two specific cases of reactive loads, in which relatively simple time-domain solutions are possible and provide useful insight.

Example 2-10: Lossy capacitive load. Consider the transmission line system shown in Figure 2.25a where a step voltage source of amplitude V_0 and source resistance $R_s = Z_0$ excites a lossless transmission line of characteristic impedance Z_0 and one-way time delay t_d connected to a load consisting of a parallel combination of R_L and C_L. Find and sketch the source- and load-end voltages as a function of time.

Solution: At $t = 0$, an incident voltage $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = V_0/2$ is launched at the source end of the line. This disturbance reaches the capacitive load at $t = t_d$, when a voltage $\mathcal{V}_1^-(z, t)$ of initial amplitude $\mathcal{V}_1^-(l, t_d)$ reflects toward the source. For $t \geq t_d$, the total load voltage $\mathcal{V}_L(t)$ and current $\mathcal{I}_L(t)$ are given by

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(l, t) + \mathcal{V}_1^-(l, t)$$

$$\mathcal{I}_L(t) = \mathcal{I}_1^+(l, t) + \mathcal{I}_1^-(l, t) = \frac{\mathcal{V}_1^+(l, t)}{Z_0} - \frac{\mathcal{V}_1^-(l, t)}{Z_0}$$

where $\mathcal{V}_1^+(l, t) = \mathcal{V}_1^+(0, 0) = V_0/2$. These two equations are related by the boundary condition imposed by the load; that is,

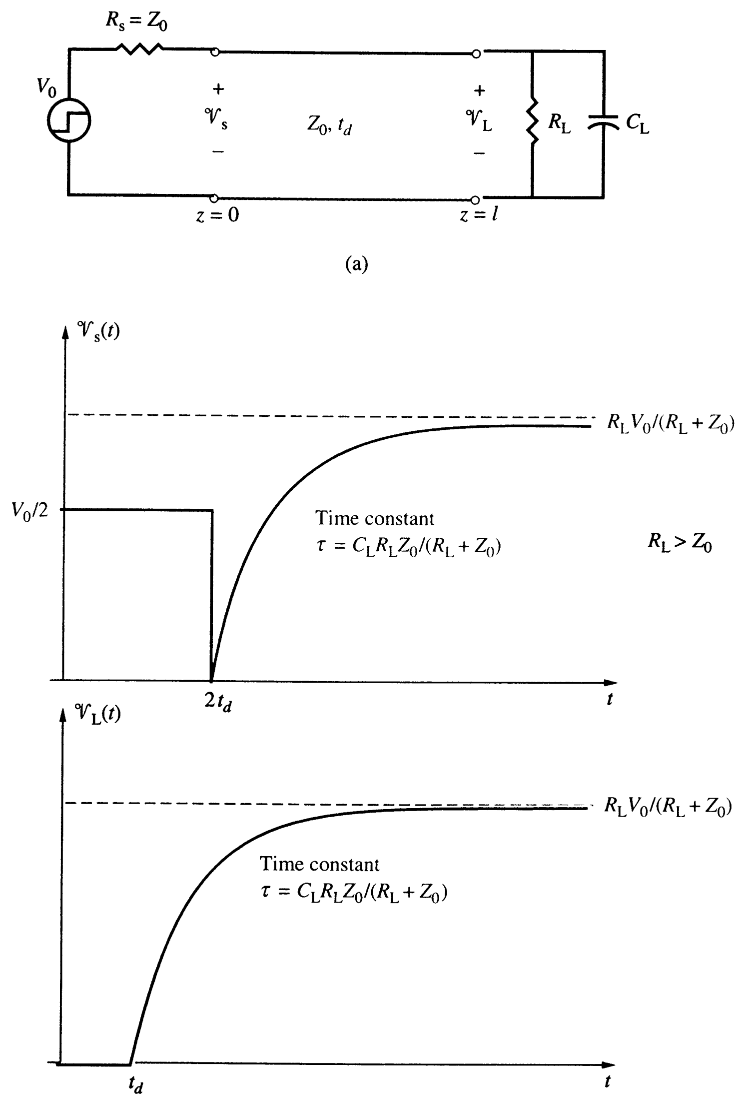
$$\mathcal{I}_L(t) = \frac{\mathcal{V}_L(t)}{R_L} + C_L \frac{d\mathcal{V}_L(t)}{dt}$$

Substituting the first two equations into the third equation yields

$$\frac{d\mathcal{V}_1^-(l, t)}{dt} + \left[\frac{R_L + Z_0}{R_L Z_0 C_L}\right]\mathcal{V}_1^-(l, t) = \left[\frac{R_L - Z_0}{R_L Z_0 C_L}\right]\mathcal{V}_1^+(l, t)$$

which is a first-order differential equation for $\mathcal{V}_1^-(t)$. Note that in deriving it, we have used the fact that the incident voltage is constant in time, so that $d\mathcal{V}_1^+(l, t)/dt = 0$. The solution of this first-order differential equation can be found by noting that $\mathcal{V}_1^+(l, t) = V_0/2$ and by writing the general solution as

$$\mathcal{V}_1^-(l, t) = K_1 + K_2 e^{-[(R_L + Z_0)/(R_L Z_0 C_L)](t - t_d)}$$



(b)

FIGURE 2.25. Step response of a capacitively loaded line. (a) Circuit diagram. (b) Time variation of the source- and load-end voltages.

in which case the constants K_1 and K_2 can be determined by using the initial and final conditions. Note that the reflected voltage must vary exponentially from $-\mathcal{V}_1^+(l, t)$ at $t = t_d$ to $(R_L - Z_0)\mathcal{V}_1^+(l, t)/(R_L + Z_0)$ for $t \to \infty$. We thus have

$$\mathcal{V}_1^-(l, t) = \mathcal{V}_1^+(l, t)\left[\frac{R_L - Z_0}{R_L + Z_0} - \frac{2R_L}{R_L + Z_0}e^{-[(R_L+Z_0)/(R_L Z_0 C_L)](t-t_d)}\right]$$

which is valid for $t \geq t_d$. This behavior can be understood as follows: When the incident voltage reaches the capacitive load at $t = t_d$, the capacitor C_L is initially uncharged and acts like a short circuit, resulting in $\mathcal{V}_1^-(l, t_d) = -\mathcal{V}_1^+(l, t_d) = -V_0/2$. However, at steady state the capacitor is fully charged and acts like an open circuit, resulting in $\mathcal{V}_1^-(l, \infty) = (R_L - Z_0)\mathcal{V}_1^+(l, \infty)/(R_L + Z_0) = (R_L - Z_0)V_0/[2(R_L + Z_0)]$, as expected. Note also that the time constant of the exponential variation is $\tau = R_L Z_0 C_L/(R_L + Z_0) = R_{Th} C_L$, where R_{Th} is the Thévenin equivalent resistance,[28] as seen from the terminals of the capacitor. Substituting $\mathcal{V}_1^-(l, t)$ into $\mathcal{V}_L(t)$ yields

$$\mathcal{V}_L(t) = \frac{R_L V_0}{R_L + Z_0}[1 - e^{-[(R_L+Z_0)/(R_L Z_0 C_L)](t-t_d)}]$$

valid for $t \geq t_d$. When the front of the reflected voltage reaches the source end at $t = 2t_d$, it is completely absorbed, since the source end of the line is matched (i.e., $R_s = Z_0$). The voltage at the source end is given by $\mathcal{V}_s(t) = \mathcal{V}_1^+(0, t) = V_0/2$ for $t < 2t_d$, and

$$\mathcal{V}_s(t) = \frac{R_L V_0}{R_L + Z_0}[1 - e^{-[(R_L+Z_0)/(R_L Z_0 C_L)](t-2t_d)}]$$

is valid for $t \geq 2t_d$. Sketches of $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$ are shown in Figure 2.25b.

Example 2-11: A lumped series inductor between two transmission lines. Microstrip transmission lines on printed circuit boards are often connected together with bonding wires, which are inherently inductive. In this example, we consider a typical model for such a connection, namely a lumped series inductor between two different transmission lines. The measurement of the bonding-wire inductance is considered later in Example 2-16. Two microstrip transmission lines having equal line parameters of $L = 4$ nH-(cm)$^{-1}$ and $C = 1.6$ pF-(cm)$^{-1}$ and lengths 15 cm and 10 cm are connected by a wire represented by a series lumped inductance of $L_w = 5$ nH, shown in Figure 2.26a. The end of the shorter line is matched with a 50Ω load, and the circuit is excited at $t = 0$ by a unit step voltage source of $R_s = 50\Omega$. Find and sketch the variations with time of the source and load end voltages. Assume lossless lines.

Solution: Using the given line parameters L and C, the characteristic impedance and the phase velocity of the microstrip lines can be calculated as

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{4 \times 10^{-9}}{1.6 \times 10^{-12}}} = 50\Omega$$

[28] In this case being simply equal to the parallel combination of the load resistance R_L and the characteristic impedance Z_0 of the line.

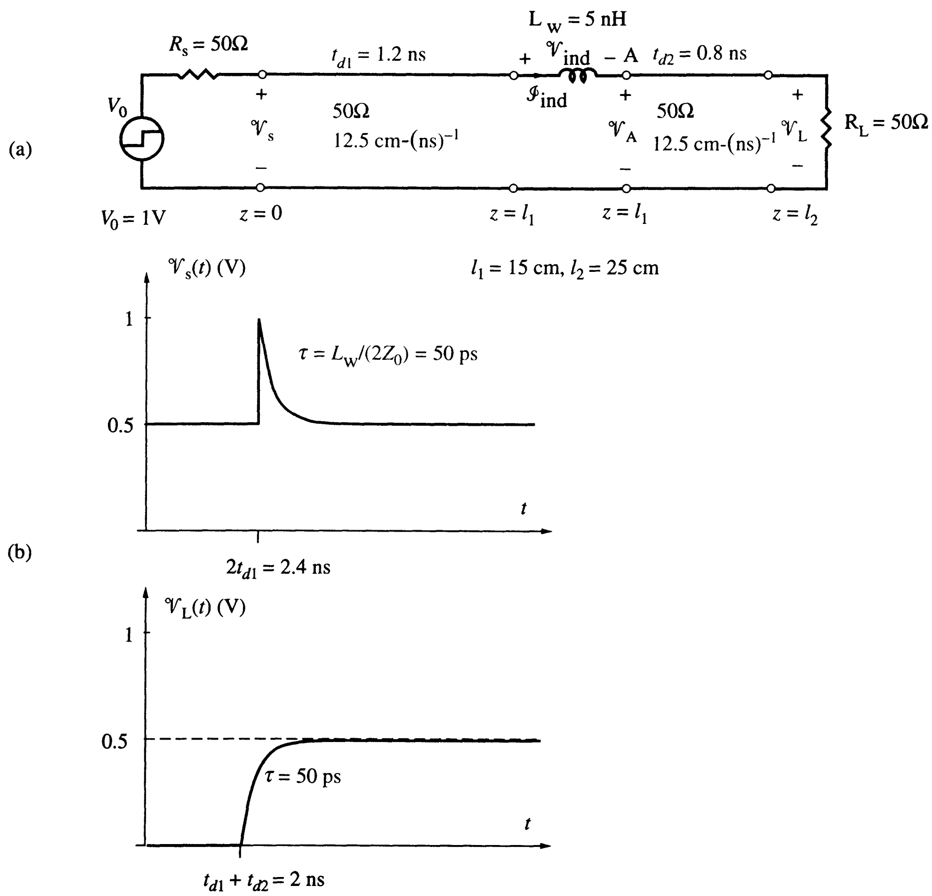


FIGURE 2.26. A lumped series inductor between two different transmission lines. (a) Circuit diagram. (b) Time variation of source- and load-end voltages for Example 2-11.

and

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times 10^{-9} \times 1.6 \times 10^{-12}}} = 12.5 \times 10^9 \text{ cm-s}^{-1}$$

Therefore, the one-way delay times of the two lines are respectively $t_{d1} = 15 \text{ cm}/(12.5 \text{ cm-(ns)}^{-1}) = 1.2$ ns and $t_{d2} = 10 \text{ cm}/(12.5 \text{ cm-ns}^{-1}) = 0.8$ ns, as indicated in Figure 2.26a.

As in the previous example, at $t = 0$, an incident voltage $\mathcal{V}_1^+(z, t)$ of amplitude $\mathcal{V}_1^+(0, 0) = 0.5$ V is launched from the source end of the line. Note that the amplitude of this incident voltage remains constant in time, as long as it is supplied by the source. When this disturbance arrives at the junction at $t = t_{d1} = 1.2$ ns, the uncharged inductor initially acts like an open circuit (i.e., it resists the flow of current), producing a reflected voltage $\mathcal{V}_1^-(z, t)$ of initial amplitude $\mathcal{V}_1^-(l_1, t_{d1}) = \mathcal{V}_1^+(l_1, t_{d1}) = 0.5$ V, the same as one would have if the two lines were not connected. Current flows into the second line through the inductor as the inductor charges exponentially and eventually behaves like a short circuit at steady state. The following equations apply for $t \geq t_{d1}$:

$$\mathcal{V}_1^+(l_1, t) + \mathcal{V}_1^-(l_1, t) = \mathcal{V}_{\text{ind}}(t) + \mathcal{V}_{\text{A}}(t) = L_{\text{w}}\frac{d\mathcal{I}_{\text{ind}}(t)}{dt} + Z_0\mathcal{I}_{\text{ind}}(t)$$

where $\mathcal{V}_1^+(l_1, t) = 0.5$ V and $\mathcal{V}_{\text{A}}(t)$ is the voltage at position A, at which the impedance seen looking toward the load is $Z_0 = 50\Omega$. We also have

$$\mathcal{I}_{\text{ind}}(t) = \mathcal{I}_1^+(l_1, t) + \mathcal{I}_1^-(l_1, t) = \frac{\mathcal{V}_1^+(l_1, t)}{Z_0} - \frac{\mathcal{V}_1^-(l_1, t)}{Z_0}$$

where $\mathcal{I}_{\text{ind}}(t)$ and $\mathcal{V}_{\text{ind}}(t)$ are, respectively, the current through and the voltage across the inductor, as defined in Figure 2.26a. Substituting the second equation into the first yields

$$\frac{d\mathcal{V}_1^-(l_1, t)}{dt} + \frac{2Z_0}{L_{\text{w}}}\mathcal{V}_1^-(l_1, t) = 0$$

which is a first-order differential equation for $\mathcal{V}_1^-(l_1, t)$. The solution of this equation is

$$\mathcal{V}_1^-(l_1, t) = \mathcal{V}_1^+(l_1, t_{d1})e^{-(2Z_0/L_{\text{w}})(t-t_{d1})} = 0.5e^{-2\times10^{10}(t-1.2\times10^{-9})} \text{ V}$$

which is valid for $t \geq t_{d1} = 1.2$ ns. Note that we have used the fact that $\mathcal{V}_1^-(l_1, t_{d1}) = 0.5$ and that the reflected voltage varies exponentially from $\mathcal{V}_1^-(l_1, t_{d1}) = \mathcal{V}_1^+(l_1, t_{d1})$ at $t = t_{d1}$ (when the inductor initially behaves like an open circuit) to zero at $t \to \infty$ (when the fully energized inductor eventually behaves like a short circuit). The time constant of the exponential variation is $\tau = L_{\text{w}}/R_{\text{Th}} = 50$ ps, where $R_{\text{Th}} = 2Z_0 = 100\Omega$ is the Thévenin equivalent resistance as seen from the terminals of the inductor.

To satisfy the boundary condition at the junction, and noting that the inductor L_{w} is a lumped element, the current on both sides of the inductor must be the same. Thus, the inductor current $\mathcal{I}_{\text{ind}}(t)$ can be written as

$$\mathcal{I}_{\text{ind}}(t) = \mathcal{I}_1^+(l_1, t_{d1}) + \mathcal{I}_1^-(l_1, t) = \frac{\mathcal{V}_1^+(l_1, t_{d1})}{Z_0}[1 - e^{-(2Z_0/L_{\text{w}})(t-t_{d1})}]$$

which is valid for $t \geq t_{d1}$. At $t = t_{d1}$, a transmitted voltage $\mathcal{V}_{\text{A}}^+(z, t) = Z_0\mathcal{I}_{\text{A}}^+(z, t)$ is launched at position A on the second line, where $\mathcal{I}_{\text{A}}^+(z, t) = \mathcal{I}_{\text{ind}}(t)$. Therefore, the transmitted voltage $\mathcal{V}_{\text{A}}^+(z, t)$ can be written as

$$\mathcal{V}_{\text{A}}^+(z, t) = Z_0\mathcal{I}_{\text{ind}}(t) = \mathcal{V}_1^+(l_1, t_{d1})[1 - e^{-(2Z_0/L_{\text{w}})(t-t_{d1})}]$$

valid for $t \geq t_{d1}$.

The voltage $\mathcal{V}_{\text{A}}^+(z, t)$ arrives at the load end at $t = t_{d1} + t_{d2} = 2$ ns, where it is completely absorbed, since $R_{\text{L}} = Z_0 = 50\Omega$. The load voltage is given by

$$\mathcal{V}_L(t) = \mathcal{V}_A^+(l_2, t) = \mathcal{V}_1^+(l_1, t_{d1})[1 - e^{-(2Z_0/L_w)(t-(t_{d1}+t_{d2}))}]$$
$$= 0.5[1 - e^{-2\times 10^{10}(t-2\times 10^{-9})}] \text{ V}$$

valid for $t \geq (t_{d1} + t_{d2}) = 2$ ns. Note that $\mathcal{V}_1^+(l_1, t_{d1}) = 0.5$ V for $t \geq t_{d1}$, since the incident voltage remains constant unless the source changes. Similarly, the source-end voltage is given by $\mathcal{V}_s(t) = \mathcal{V}_1^+(0, t) = 0.5$ V for $t < 2t_{d1} = 2.4$ ns and

$$\mathcal{V}_s(t) = \mathcal{V}_1^+(0, t)[1 + e^{-(2Z_0/L_w)(t-2t_{d1})}] = 0.5[1 + e^{-2\times 10^{10}(t-2.4\times 10^{-9})}] \text{ V}$$

for $t \geq 2t_{d1} = 2.4$ ns. Sketches of the time variations of source- and load-end voltages are shown in Figure 2.26b.

In the preceding two examples we used time-domain methods to determine the step response of transmission lines with terminations or discontinuities involving reactive elements. The basis for our analyses was the simultaneous solution of [2.10], describing the transmission line voltage and current, together with the differential equations that describe the terminal voltage-current relationship of the reactive load. These time-domain solutions were tractable partly because the excitation voltage was a simple step function and also because the reactive discontinuities that we analyzed involved only one energy storage element (i.e., a single capacitor or inductor). When the input voltage function is more complicated, or when the reactive discontinuity involves more than one energy storage element, it is often easier to use Laplace transform methods to determine the response of the line. We demonstrate the use of the Laplace transform method in Example 2-12.

Example 2-12: Reflections due to inductance of resistor leads. Consider the transmission line system shown in Figure 2.27, where the source voltage amplitude increases linearly from zero to V_0 over a time[29] of t_r. The output

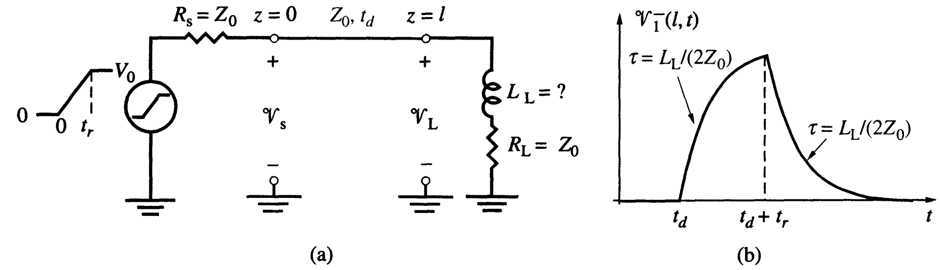


FIGURE 2.27. Reflections due to inductance of resistor leads. (a) Circuit diagram. (b) Time variation of the reflected voltage $\mathcal{V}_1^-(l, t)$ for the case $R_L = Z_0$.

[29]Note that t_r is not exactly the rise time discussed in Section 1.1, which was defined as the time required for the signal to change from 10% to 90% of its final value.

resistance of the source is $R_s = Z_0$, while the transmission line having a characteristic impedance Z_0 and a one-way time delay t_d is terminated in a reactive load consisting of a series combination of R_L and L_L. Find an expression for the reflected voltage at the load [i.e., $\mathcal{V}_1^-(l, t)$], and determine its maximum value for $R_L = Z_0$.

Solution: From Figure 2.27 we note that, starting at $t = 0$, an incident voltage is launched at the source end of the line, given by

$$\mathcal{V}_1^+(0, t) = \frac{V_0}{2t_r}[tu(t) - (t - t_r)u(t - t_r)]$$

where $u(\cdot)$ is the unit step function. The Laplace transform of this voltage waveform is

$$\tilde{\mathcal{V}}_1^+(s) = \frac{V_0}{2t_r}\frac{1 - e^{-t_r s}}{s^2}$$

This incident voltage propagates to the load end, and no reflected voltage exists until it arrives there at $t = t_d$. For $t \geq t_d$, the total load voltage and current $\mathcal{V}_L(t)$ and $\mathcal{I}_L(t)$ are given by

$$\mathcal{V}_L(t) = \mathcal{V}_1^+(l, t) + \mathcal{V}_1^-(l, t)$$

$$\mathcal{I}_L(t) = \mathcal{I}_1^+(l, t) + \mathcal{I}_1^-(l, t) = \frac{\mathcal{V}_1^+(l, t)}{Z_0} - \frac{\mathcal{V}_1^-(l, t)}{Z_0}$$

and are related by the boundary condition imposed by the load,

$$\mathcal{V}_L(t) = \mathcal{I}_L(t)R_L + L_L\frac{d\mathcal{I}_L(t)}{dt}$$

Substituting the first two equations into the third equation yields

$$L_L\frac{d\mathcal{V}_1^-(l, t)}{dt} + (R_L + Z_0)\mathcal{V}_1^-(l, t) = L_L\frac{d\mathcal{V}_1^+(l, t)}{dt} + (R_L - Z_0)\mathcal{V}_1^+(l, t)$$

Note that unlike the case of Example 2-10, the derivative of the incident voltage is not zero, since $\mathcal{V}_1^+(l, t)$ is not constant in time during the time interval $t_d \leq t \leq (t_d + t_r)$. To solve this differential equation for the reflected voltage $\mathcal{V}_1^-(l, t)$ using the given functional form of $\mathcal{V}_1^+(l, t)$, we can take its Laplace transform,

$$(sL_L + R_L + Z_0)\tilde{\mathcal{V}}_1^-(s) = (sL_L + R_L - Z_0)\tilde{\mathcal{V}}_1^+(s)$$

where $\tilde{\mathcal{V}}_1^\pm(s)$ is the Laplace transform of $\mathcal{V}_1^\pm(l, t)$. Using the previously noted form of $\tilde{\mathcal{V}}_1^+(s)$, we then have

$$\tilde{\mathcal{V}}_1^-(s) = \left[\frac{sL_L + R_L - Z_0}{sL_L + R_L + Z_0}\right]\left[\frac{V_0}{2t_r}\frac{1 - e^{-t_r s}}{s^2}\right] = \left[\frac{s + (R_L - Z_0)/L_L}{s + (R_L + Z_0)/L_L}\right]\left[\frac{V_0}{2t_r}\frac{1 - e^{-t_r s}}{s^2}\right]$$

which can be expanded into its partial fractions as

$$\tilde{\mathcal{V}}_1^-(s) = \frac{K_1}{s + (R_L + Z_0)/L_L} + \frac{K_2}{s} + \frac{K_3}{s^2}$$

where the coefficients are

$$K_2 = -K_1 = \frac{V_0 Z_0 L_L}{t_r (R_L + Z_0)^2} \quad \text{and} \quad K_3 = \frac{V_0}{2t_r}\frac{R_L - Z_0}{R_L + Z_0}$$

Taking the inverse Laplace transform[30] yields

$$\mathcal{V}_1^-(l,t) = \frac{V_0}{t_r}\frac{Z_0 L_L}{(R_L + Z_0)^2}\{[1 - e^{-(R_L+Z_0)t'/L_L}]u(t') + [1 - e^{-(R_L+Z_0)(t'-t_r)/L_L}]u(t' - t_r)\}$$
$$+ \frac{V_0}{2t_r}\frac{R_L - Z_0}{R_L + Z_0}[t'u(t') - (t' - t_r)u(t' - t_r)]$$

where $t' = t - t_d$. Note that the solution for $\mathcal{V}_1^-(l,t)$ is valid only for $t \geq t_d$, or $t' \geq 0$.

A practical case of interest is that in which $R_L = Z_0$. When a microstrip is terminated at a matched load resistance to avoid reflections, the nonzero inductance of the resistor leads may nevertheless produce reflections. To determine the maximum reflection voltage due to the inductance of the resistor leads, we substitute $R_L = Z_0$ in the solution for $\mathcal{V}_1^-(l,t)$ to find

$$\mathcal{V}_1^-(l,t) = \frac{V_0}{t_r}\frac{L_L}{4Z_0}\{[1 - e^{-(2Z_0/L_L)(t-t_d)}]u(t - t_d) - [1 - e^{-(2Z_0/L_L)(t-t_d-t_r)}]u(t - t_d - t_r)\}$$

The time variation of $\mathcal{V}_1^-(l,t)$ is plotted in Figure 2.27b, showing that the reflected voltage rises and falls exponentially with a time constant of $L/(2Z_0)$. The maximum reflected voltage occurs at $t = t_d + t_r$ and is given by

$$[\mathcal{V}_1^-(l,t)]_{\max} = \frac{V_0}{t_r}\frac{L_L}{4Z_0}[1 - e^{-(2Z_0/L_L)t_r}]$$

Note that in practice the maximum reflected voltage can easily be measured—for example, by using a time-domain reflectometer (see Section 2.6.1), from which the value of the inductance L_L of the resistor leads can be calculated, since Z_0 and t_r are known in most cases.

---

[30]We use the following Laplace transform pairs:

$$e^{-a(t-b)}u(t-b) \Longleftrightarrow \frac{e^{-bs}}{s+a}$$

$$(t-b)u(t-b) \Longleftrightarrow \frac{e^{-bs}}{s^2}$$

where a and b are constants.

2.5.2 Nonlinear Terminations

Up to now, we have considered responses of transmission lines with linear sources and linear loads, and we have analyzed reflections using the simultaneous analytical solution of the transmission line voltage and current expressions together with the load characteristics. In some high-speed digital circuits, both the sources (driver gates) and loads (receiving gates) can have nonlinear current-voltage characteristics. This is particularly the case for transistor-transistor logic (TTL) and complementary metal oxide semiconductor (CMOS) logic gates. In cases where transmission lines are terminated in or driven by nonlinear loads, a graphical technique known as the Bergeron method[31] is quite useful.

The Bergeron graphical technique can be applied to transmission line circuits that involve linear or nonlinear devices. It provides the same basic information as the bounce diagram (voltage and current versus time) but with fewer calculations. It relies on a graphical means of describing the reflections on the transmission line. The graphs involved in using this approach can become quite complex, especially if reactive elements are present. Also, the graphical technique requires accurate knowledge of the current-voltage characteristics of the nonlinear devices.

To illustrate the graphical methodology before we apply it to a nonlinear load, we consider in Example 2-13 the step response of a simple transmission line terminated in a linear resistive load.

Example 2-13: Graphical solution of the step response of a resistively terminated lossless line. Consider a transmission line with a characteristic impedance of 100Ω, driven by a 1 V step source with source resistance of 25Ω, and terminated in a 300Ω load, as shown in Figure 2.28a. Use the graphical Bergeron method to analyze the effects of reflections on the source- and load-end voltages and currents.

Solution: We start by plotting the current-voltage characteristics of the source and load ends (i.e., $\mathcal{I}_s$ versus $\mathcal{V}_s$ and $\mathcal{I}_L$ versus $\mathcal{V}_L$), using the same set of axes. At the source end we have

$$\mathcal{V}_s = V_0 - R_s\mathcal{I}_s = 1 - 25\mathcal{I}_s$$

which is a straight line (referred to as the source-end line) with slope $-1/R_s = -(1/25)$ S, as shown in Figure 2.28b. At the load end we have

$$\mathcal{V}_L = R_L\mathcal{I}_L = 300\mathcal{I}_L$$

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[31] L. Bergeron was a French hydraulic engineer and developed this graphical method in 1949 to study water hammer waves in pipes. For an English translation of Bergeron's original work see L. B. J. Bergeron, *Water Hammer in Hydraulics and Wave Surges in Electricity,* John Wiley & Sons, Inc., New York, 1961. The use of this method in high-speed digital switching circuits was initially suggested in 1968. See R. S. Singleton, No need to juggle equations to find reflection—just draw three lines, *Electronics,* pp. 93–99, October 28, 1968.

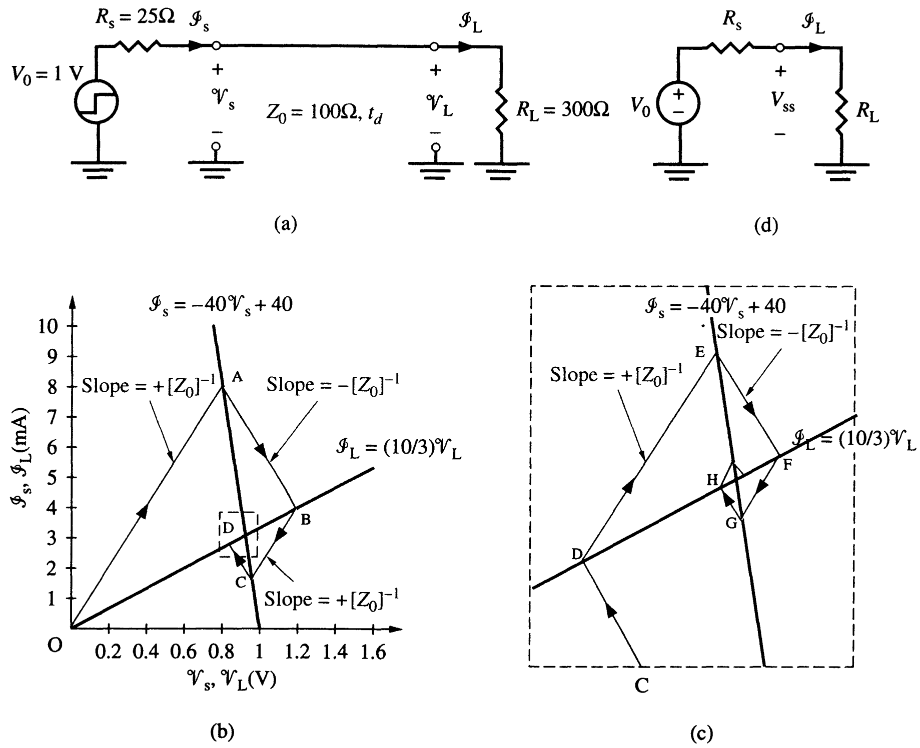

FIGURE 2.28. Bergeron method applied to a line terminated with a linear resistor. (a) Circuit diagram. (b) Current versus voltage characteristics at the source and load ends are shown as heavy lines. The lighter lines are lines with slopes of $+[Z_0]^{-1}$ and $-[Z_0]^{-1}$ in units of mA-(V)$^{-1}$, or mS. The intersection points of the lighter lines with the source- and load-end current-voltage characteristics represent respectively the values of the source- or load-end voltages at the corresponding times. (c) Enlarged view of the rectangular dashed-line region in (b). (d) Steady-state equivalent circuit.

which is also a straight line (the load-end line) with slope $1/R_L = (1/300)$ S, as shown in Figure 2.28b. These two straight lines, sometimes also referred to simply as the "load lines," define the relation between current and voltage at the source and load ends. At $t = 0$, a voltage $\mathcal{V}_1^+$ is launched from the source end. This voltage disturbance is accompanied by its associated current given by $\mathcal{I}_1^+ = [Z_0]^{-1}\mathcal{V}_1^+ = [100]^{-1}\mathcal{V}_1^+$. The relationship between $\mathcal{I}_1^+$ and $\mathcal{V}_1^+$ is another straight line with slope $[100]^{-1}$ S, shown as the line segment OA in Figure 2.28b. Noting that at $t = 0$, $\mathcal{V}_s = \mathcal{V}_1^+$ and $\mathcal{I}_s = \mathcal{I}_1^+$, the intersection of the line OA with the source-end line at point A determines the amplitude of the initial voltage and current launched at the source end of the transmission line. This intersection point essentially is the solution of the two linear equations describing the source-end voltage and current, with the resultant voltage $V_A = \mathcal{V}_s(t = 0) = \mathcal{V}_1^+$ being what we would have if we simply divided the applied voltage V_0 between the source resistance and the characteristic impedance of the line. The resultant voltage and current values can easily be read from point A in Figure 2.28b as $\mathcal{V}_1^+ = V_A = 0.8$ V and $\mathcal{I}_1^+ = I_A = 8$ mA, respectively.

At $t = t_d$, when the voltage disturbance reaches the load end of the line, a reflected voltage $\mathcal{V}_1^-$ is generated, which is accompanied by its associated current $\mathcal{I}_1^-$. Namely, we have

$$\mathcal{I}_1^- = -[Z_0]^{-1}\mathcal{V}_1^- \quad \rightarrow \quad \mathcal{I}_L - \mathcal{I}_1^+ = -[Z_0]^{-1}(\mathcal{V}_L - \mathcal{V}_1^+)$$

which represents a straight line passing through point A and having a slope $-[Z_0]^{-1} = -[100]^{-1}$, if we now view the axes in Figure 2.28b as $\mathcal{I}_L$ versus $\mathcal{V}_L$. The intersection of this straight line (shown in Figure 2.28b) with the load-end line (i.e., $\mathcal{V}_L = 300\mathcal{I}_L$) at point B determines the values of $\mathcal{V}_L$ and $\mathcal{I}_L$ at $t = t_d$, which can easily be read from the graph as $\mathcal{V}_L(t = t_d) = \mathcal{V}_1^+ + \mathcal{V}_1^- = V_B = 1.2$ V and $\mathcal{I}_L(t = t_d) = \mathcal{I}_1^+ + \mathcal{I}_1^- = I_B = 4$ mA, respectively. The value of $\mathcal{V}_L$ at $t = t_d$, for example, can in turn be used to determine the amplitude of the reflected voltage as $\mathcal{V}_1^- = (\mathcal{V}_L - \mathcal{V}_1^+) = (1.2 - 0.8) = 0.4$ V.

The reflected disturbance $\mathcal{V}_1^-(z, t)$ reaches the source end at $t = 2t_d$, at which time a new voltage $\mathcal{V}_2^+$ is created, accompanied by its corresponding current $\mathcal{I}_2^+ = [Z_0]^{-1}\mathcal{V}_2^+$. Noting that at $t = 2t_d$, we have

$$\mathcal{V}_s = \mathcal{V}_1^+ + \mathcal{V}_1^- + \mathcal{V}_2^+ = V_B + \mathcal{V}_2^+$$
$$\mathcal{I}_s = \mathcal{I}_1^+ + \mathcal{I}_1^- + \mathcal{I}_2^+ = I_B + \mathcal{I}_2^+$$

the relationship between $\mathcal{I}_2^+$ and $\mathcal{V}_2^+$ can be rewritten as $[Z_0]^{-1}(\mathcal{V}_s - V_B) = (\mathcal{I}_s - I_B)$, representing a straight line passing through the point B and having a slope $[Z_0]^{-1} = [100]^{-1}$ S, as shown in Figure 2.28b. The intersection of this straight line with the source-end line ($\mathcal{V}_s = 1 - 25\mathcal{I}_s$) at point C determines the values of $\mathcal{I}_s$ and $\mathcal{V}_s$ at $t = 2t_d$, which can be read from Figure 2.28b as $\mathcal{I}_s = \mathcal{I}_1^+ + \mathcal{I}_1^- + \mathcal{I}_2^+ = I_C = 1.6$ mA and $\mathcal{V}_s = \mathcal{V}_1^+ + \mathcal{V}_1^- + \mathcal{V}_2^+ = V_C = 0.96$ V, respectively. From the value of $\mathcal{V}_s$ at $t = 2t_d$, one can determine the amplitude of the reflected voltage $\mathcal{V}_2^+ = \mathcal{V}_s(t = 2t_d) - V_B = 0.96 - 1.2 = -0.24$ V.

We can continue this procedure by drawing a straight line with slope $-[Z_0]^{-1}$ passing through the point C and find its intersection with the load-end line at point D and from there read $\mathcal{I}_L$ and $\mathcal{V}_L$ values at $t = 3t_d$ to be 2.8 mA and 0.84 V, respectively. A straight line with slope $[Z_0]^{-1}$ passing through the point D can then be drawn to intersect the source-end line at point E, representing $\mathcal{I}_s$ and $\mathcal{V}_s$ values at $t = 4t_d$ of 3.52 mA and ~0.912 V, respectively. The process continues in this manner until both the source- and the load-end voltages and currents converge to the steady-state values as determined by the intersection of the source- and load-end lines. At that point, we have $\mathcal{V}_s = \mathcal{V}_L$ and $\mathcal{I}_s = \mathcal{I}_L$, the transmission line is charged to its final voltage $V_{ss} \simeq 0.923$ V, and the source and load ends are essentially connected together, as shown in the steady-state equivalent circuit shown in Figure 2.28d.

The graphical technique illustrated in Example 2-13 is not particularly needed when dealing with transmission lines with linear terminations. However, it becomes very useful for cases in which the source or load terminations are nonlinear. In Example 2-14, we consider a transmission line terminated by a diode, which in general has a nonlinear current-voltage characteristic.

Example 2-14: A nonlinear termination. Consider a transmission line with characteristic impedance of 50Ω driven by a 0.7 V step voltage source with an internal impedance of 25Ω and terminated by a diode, as shown in Figure 2.29a. The diode has a current-voltage characteristic given by

$$\mathcal{I}_L = I_0(e^{\mathcal{V}_L/V_T} - 1)$$

where I_0 is called the saturation current, given as $I_0 = 10^{-15}$ A, and V_T is the thermal voltage, having the value $V_T \simeq 26$ mV at room temperature. Use the graphical

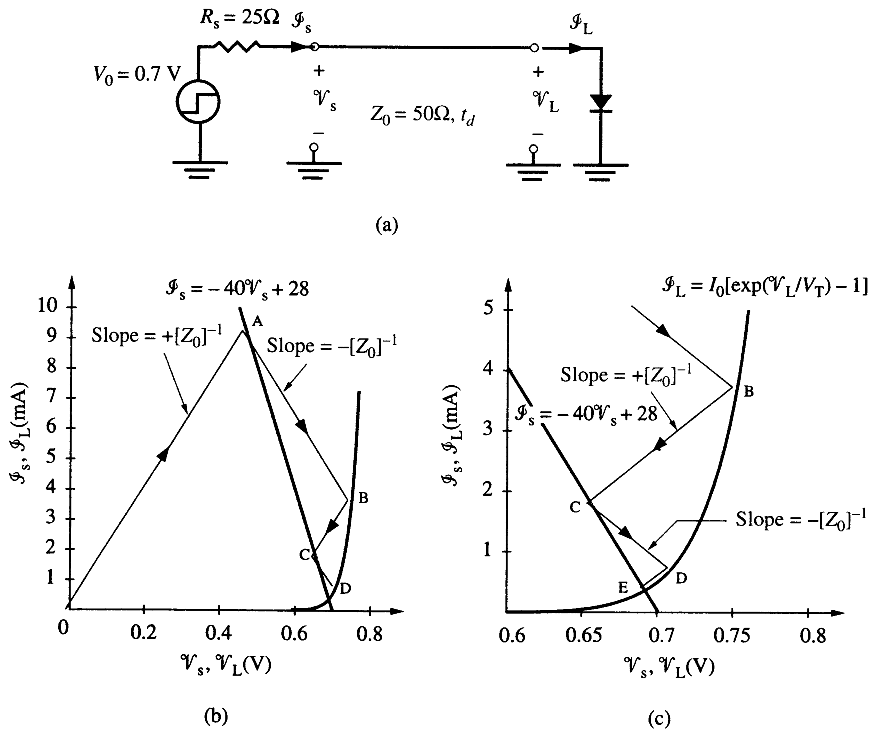

FIGURE 2.29. Step response of a line terminated in a nonlinear load. (a) Circuit diagram. (b) Current-voltage characteristics on the source- and load-ends are shown as heavy colored lines. The lighter lines are lines with slopes of $+[Z_0]^{-1}$ and $-[Z_0]^{-1}$. The intersection points of the lighter lines with the source- and load-end current-voltage characteristics represent the values of the source- and load-end voltages, respectively, at the corresponding times.
(c) Enlarged view of the region around the steady-state point (i.e., the intersection of the source- and load-end current-voltage characteristics).

Bergeron technique to determine how long it will take for the circuit to reach steady state. Neglect the charging effects of the diffusion capacitance of the diode.

Solution: The solution of this problem can proceed in a manner quite similar to Example 2-13, except that the current-voltage characteristic of the load is nonlinear instead of a straight line. Following a procedure similar to that in Example 2-13, we first plot the source-end current-voltage characteristic, which is a straight line given by $\mathcal{V}_s = 0.7 - 25\mathcal{I}_s$, and then the load-end characteristic (i.e., the nonlinear diode equation) on the same graph, as shown in Figure 2.29b. At $t = 0$, the voltage $\mathcal{V}_1^+$ and its associated current $\mathcal{I}_1^+ = [Z_0]^{-1}\mathcal{V}_1^+$ are launched from the source end of the line. Since at $t = 0$ we have $\mathcal{V}_s = \mathcal{V}_1^+$ and $\mathcal{I}_s = \mathcal{I}_1^+$, the relationship between $\mathcal{I}_1^+$ and $\mathcal{V}_1^+$ is a straight line given as $\mathcal{V}_s = 50\mathcal{I}_s$ with a slope $[Z_0]^{-1} = [50]^{-1}$ S, which is plotted on the same graph. The intersection of this line with the source-end line is denoted by A, as shown in Figure 2.29b. The coordinates of point A can be read from Figure 2.29b as $I_A \simeq 9.33$ mA and $V_A \simeq 0.467$ V, which are also the values of the incident current and associated voltage disturbances $\mathcal{I}_1^+$ and $\mathcal{V}_1^+$, respectively. Next, we draw a straight line with slope $-[Z_0]^{-1} = -[50]^{-1}$ passing through point A and find its intersection with the diode characteristic, which is denoted as point B, as shown. The coordinates of point B are approximately given by $V_B = \mathcal{V}_L(t = t_d) \simeq 0.7519$ V and $I_B = \mathcal{I}_L(t = t_d) \simeq 3.63$ mA, respectively. Using the value of $\mathcal{V}_L(t_d)$, the amplitude of the reflected voltage can be determined as $\mathcal{V}_1^- = \mathcal{V}_L(t_d) - \mathcal{V}_1^+ \simeq 0.285$ V. We then draw a straight line with slope $+[Z_0]^{-1}$ and passing through point B and find its intersection with the source-end line at point C, where we have $V_C \simeq 0.657$ V and $I_C \simeq 1.73$ mA. This process continues until we reach the intersection point of the source-end line with the diode characteristic approximately at time $t = 4t_d$ and at point E, which is indistinguishably close to the intersection point of the source- and load-end current-voltage characteristics. Thus, in this particular case, it takes approximately $4t_d$ units of time for the circuit to reach steady state. At steady state (i.e., $t \to \infty$), we have $\mathcal{V}_s = \mathcal{V}_L \simeq 0.6912$ V and $\mathcal{I}_s = \mathcal{I}_L \simeq 0.3514$ mA.

The accuracy of the Bergeron plot technique depends heavily on the accuracy of the current-voltage characteristics of the nonlinear devices, which are usually provided in terms of their typical values by the device manufacturers. Nevertheless, in spite of potential inaccuracies in device characteristics, the graphical Bergeron plot method is a powerful technique for gaining insight into the response of a line terminated by a nonlinear device, for example to estimate the approximate duration of ringing effects. Finally, it should be noted that the graphical method is entirely suitable to circumstances where both the load- and the source-end current-voltage characteristics might be nonlinear. Such cases may arise in high-speed digital applications, since both the driver gates and the receiving gates are inherently nonlinear transistor devices, especially when used in on/off modes.

2.6 SELECTED PRACTICAL TOPICS

In this section, we discuss two selected practical topics, namely (a) time-domain reflectometers, and (b) the effects of source rise time. We also briefly comment on transients on lossy transmission lines. A brief discussion on time-domain reflectometry and two associated examples serve to introduce this simplest and most direct method of measuring characteristic impedance of a line, the nature of its termination, and the presence of discontinuities on the transmission line. A discussion of source rise time effects is necessary because up to now we have primarily (except in Example 2-12) considered responses to ideal step function excitations. In most practical applications, the sources used and the outputs of driver gates have finite rise and fall times.

2.6.1 Time-Domain Reflectometry

In practice, it is often necessary to make a number of measurements on a given transmission line system to characterize its transient response. The quantities that need to be measured include the nature (capacitive, inductive, or resistive) of the load termination, the characteristic impedance of the line, the maximum voltage level at which the line can be used, and others of a more specialized character. A time-domain reflectometer[32] (commonly abbreviated as TDR) is an instrument which is used to test, characterize, and model a system involving transmission lines and their accessories. In general, it consists of a very-fast-rise-time (typically less than 50 ps) step pulse source and a display oscilloscope in a system that operates like a closed-loop radar, as shown in Figure 2.30. The source produces an incident step voltage, which travels down the transmission line under investigation, and the incident and the reflected voltages at a particular point (typically the source end) on the line are monitored by the display oscilloscope using a high-impedance probe. The output impedance of the step source is typically well matched to the nominal characteristic impedance of the line to eliminate reflections from the source end.

The most common use of time-domain reflectometry involves the measurement of the characteristics of an unknown load termination or a discontinuity on the line. The former application is illustrated in Example 2-15 for resistive loads. A discontinuity on a transmission line could, for example, be a point of breakage on a buried coaxial line, an unwanted parasitic capacitance on an interconnect, or the inductance of a bonding wire between two interconnects. The latter case is illustrated in Example 2-16.

[32]B. M. Oliver, Time domain reflectometry, *Hewlett-Packard Journal,* 15 (6), February 1964.

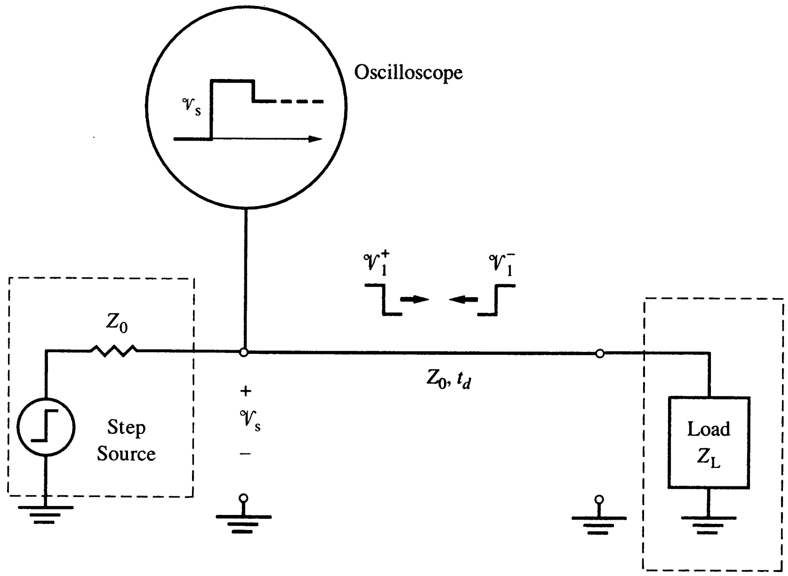

FIGURE 2.30. **Time-domain reflectometry.** Essential components of a typical TDR system.

Example 2-15: TDR displays for resistive loads. A TDR system (represented by a step pulse source of amplitude V_0 and output impedance $R_s = Z_0$) is connected to a transmission line of characteristic impedance Z_0 terminated with a resistive load R_L, as shown in Figure 2.31. Three TDR waveforms monitored at the source end are shown for three different values of R_L. Find the load resistance R_L for each case.

Solution: The initial value (immediately after the application of the step input) of the source-end voltage $\mathcal{V}_s(0)$ is equal to

$$\mathcal{V}_s(0) = \mathcal{V}_1^+(z = 0, 0) = \frac{V_0}{2} = 0.2 \text{ V}$$

from which the amplitude of the step voltage is found to be $V_0 = 0.4$ V. At $t = 1$ ns, the reflected voltage arrives at the source end and is completely absorbed. So

$$\mathcal{V}_s(1\text{ns}) = \mathcal{V}_1^+(0, 1 \text{ ns}) + \mathcal{V}_1^-(0, 1 \text{ ns}) = \mathcal{V}_1^+(0, 1 \text{ ns})(1 + \Gamma_L)$$

where $\Gamma_L = (R_L - Z_0)/(R_L + Z_0)$. For $R_L = R_{L1}$, we have

$$0.2(1 + \Gamma_{L1}) = 0.1$$

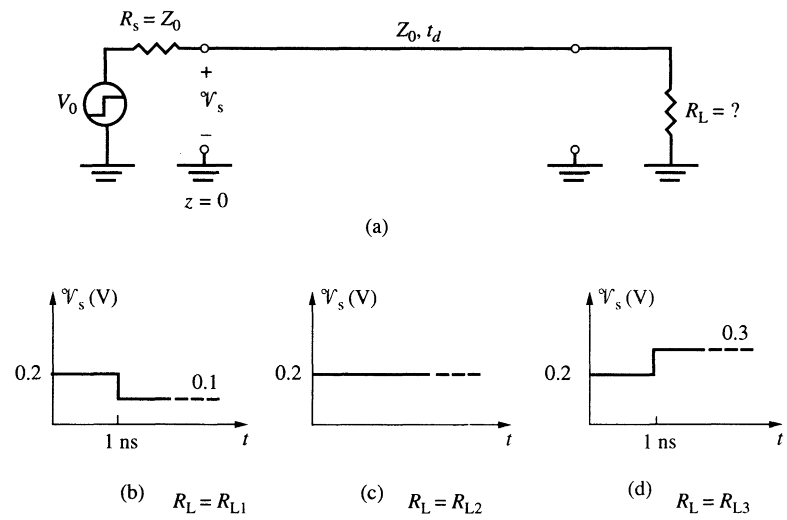

FIGURE 2.31. TDR displays for resistive loads. (a) A TDR system connected to a transmission line terminated with an unknown load resistor R_L. (b) $R_L = R_{L1}$. (c) $R_L = R_{L2}$. (d) $R_L = R_{L3}$.

from which $\Gamma_{L1} = -0.5$, yielding $R_{L1} = Z_0/3$. Similarly, we find $R_{L2} = Z_0$ and $R_{L3} = 3Z_0$.

A simple summary of the TDR waveforms observed at the source end for purely resistive, capacitive, and inductive terminations is provided in Figure 2.32. Note that the case of a resistive termination was discussed in the preceding example, while a simple inductive termination was discussed in Section 2.5.1 and in connection with Figure 2.24. The result for the capacitive termination case corresponds to that of Example 2-10 for $R_L = \infty$.

We now illustrate (Example 2-16) the use of the TDR technique for the measurement of the value of a reactive element connected between two transmission lines.

Example 2-16: TDR measurement of the inductance of a bonding wire connecting two transmission lines. Consider a bonding wire between two microstrip interconnects (each with characteristic impedance Z_0) on an integrated circuit board, as shown in Figure 2.33a. To measure the value of the bonding-wire inductance L_w, the circuit is terminated with a matched load (Z_0) on one side and is excited by a matched TDR system (i.e., $R_s = Z_0$) at the input side, as shown in Figure 2.33b. The TDR waveform $\mathcal{V}_s(t)$ measured is shown in Figure 2.33c, which is similar to Figure 2.26b (Example 2-11). Determine the value of the bonding-wire inductance in terms of the area under the "glitch" seen in the TDR waveform.

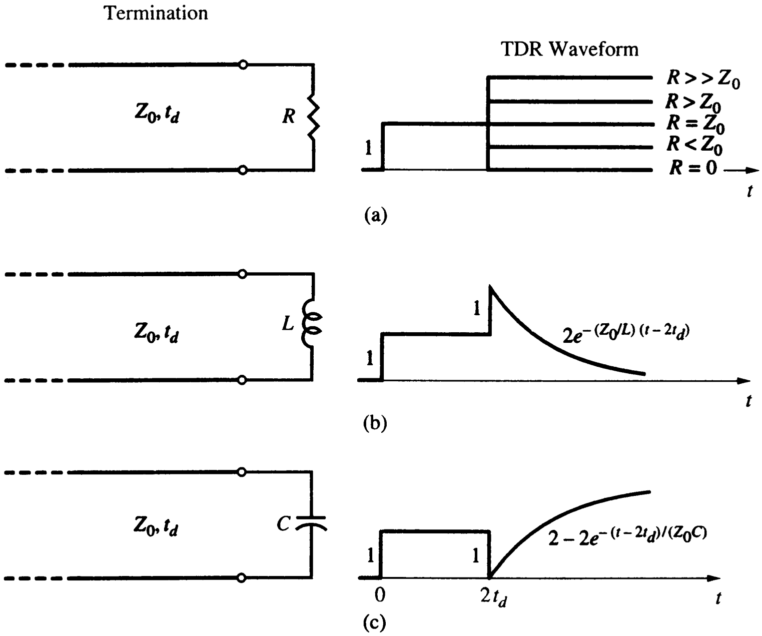

FIGURE 2.32. TDR signatures produced by simple terminations. Source-end TDR voltage signatures for purely (a) resistive, (b) inductive, and (c) capacitive terminations. In terms of excitation by a step voltage source of amplitude V_0 and source resistance R_s, the TDR traces shown are drawn for $V_0 = 2$ V and $R_s = Z_0$. (This figure was adapted from Figure 5 of B. M. Oliver, Time domain reflectometry, *Hewlett-Packard Journal,* 15 (6), pp. 14–9 to 14–16, February 1964. ©Hewlett-Packard Company 1964. Reproduced with permission.)

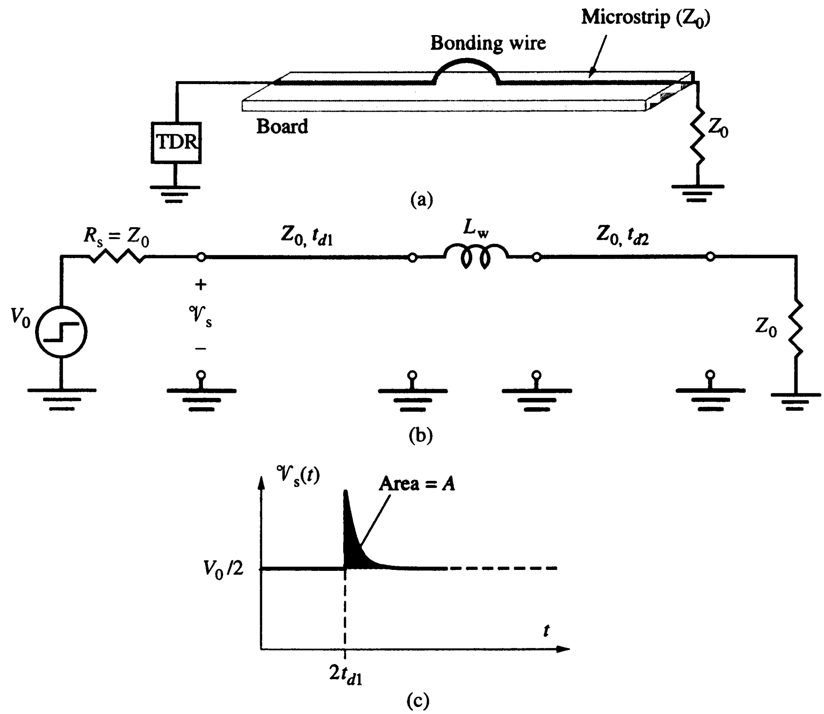

FIGURE 2.33. Measurement of bonding-wire inductance using a TDR system. (a) Actual circuit configuration. (b) Equivalent transmission line circuit model. (c) Measured TDR waveform.

Solution: In principle, the bonding-wire inductance L_w can be determined from the curvature of the glitch by accurately fitting an exponential function. However, a more accurate method is to determine L_w from the area under the curve, which can be measured more accurately. Following an approach similar to that used in Example 2-11, it can be shown that the source-end voltage is

$$\mathcal{V}_s(t) = \begin{cases} \dfrac{V_0}{2} & 0 < t < 2t_{d1} \\ \dfrac{V_0}{2}[1 + e^{-(2Z_0/L_w)(t-2t_{d1})}] & t \geq 2t_{d1} \end{cases}$$

To find the area A under the glitch, we integrate $(\mathcal{V}_s(t) - V_0/2)$ from $t = 2t_{d1}$ to $t = \infty$:

$$A = \int_{2t_{d1}}^{\infty} \frac{V_0}{2} e^{-(2Z_0/L_w)(t-2t_{d1})}\, dt = \frac{V_0}{2}\int_0^{\infty} e^{-(2Z_0/L_w)t'}\, dt' = -\frac{L_w V_0}{4Z_0} e^{-(2Z_0/L_w)t'}\Big|_0^{\infty} = \frac{L_w V_0}{4Z_0}$$

where $t' = t - 2t_{d1}$. Therefore, the bonding-wire inductance L_w is given in terms of the area A as

$$L_w = \frac{4Z_0 A}{V_0}$$

A simple summary of TDR signatures of purely resistive or purely reactive discontinuities on a transmission line is provided in Figure 2.34. The signatures of discontinuities involving combinations of reactive and resistive elements are dealt with in several problems at the end of this chapter.

2.6.2 Effects of Source Rise Time

Up to now, as we considered the consequences of propagation time delays that result from transmission line effects, we assumed that the excitations are ideal step sources that rise and fall instantaneously, with rise times and fall times (t_r, t_f) being identically zero. In practice, however, driver devices possess finite rise and fall times, which can be comparable to delays due to propagation effects. As discussed briefly in Chapter 1, the ratio of the source rise time and the one-way time delay (along a transmission line) can often be a useful determinant of whether or not lumped analyses are applicable. For example, a trace of length l on a printed circuit board behaves mostly in a lumped fashion as long as t_r and $t_f > 6t_d$, where $t_d = l/v_p$ is the one-way propagation delay of the signal on the trace. However, for high-speed drivers (small t_r and t_f) or longer trace lengths l, the trace behaves like a transmission line, or a distributed circuit. Example 2-17 illustrates the relationship between signal rise or fall time and the one-way time delay along a printed circuit board trace.

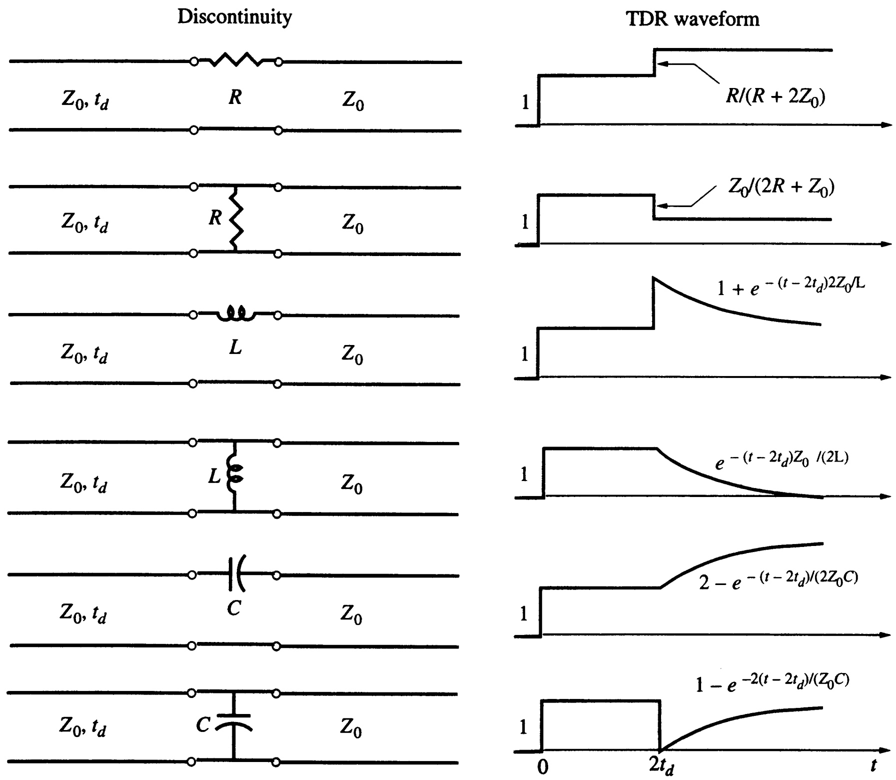

FIGURE 2.34. TDR signatures produced by simple discontinuities. Source-end TDR voltage signatures for shunt or series purely resistive, inductive, and capacitive discontinuities. In terms of excitation by a step voltage source of amplitude V_0 and source resistance R_s, the TDR voltage waveforms shown are drawn for $V_0 = 2$ V and $R_s = Z_0$, and one-way travel time t_d from the source to the disconuity. (This figure was adapted from Figure 6 of B. M. Oliver, Time domain reflectometry, *Hewlett-Packard Journal,* 15(6), pp. 14–9 to 14–16, February, 1964. ©Hewlett-Packard Company 1964. Reprinted with permission.)

Example 2-17: Rise time versus one-way time delay. A digital integrated circuit chip with 1 ns rise and fall times drives another chip with a very large input impedance (100kΩ) through a microstrip trace with characteristic impedance 60Ω, a phase velocity of 20 cm/ns, and length 6 cm on a printed circuit board. Does the line need to be terminated (at a matched impedance) to reduce transmission line effects (such as ringing)?

Solution: Comparing the rise and fall times of the driver with the one-way propagation delay along the trace (i.e., $t_d = 6/20 = 0.3$ ns), we find $t_r = t_f \simeq$

$3.33t_d$, indicating that transmission line effects may not be neglected.[33] Accordingly, it might be useful in this case to terminate the line with a 60Ω resistive load to eliminate reflections and possible ringing.

We can also examine the interplay between t_r and t_d from a more quantitative perspective. When the length of a transmission line is short enough so that t_r/t_d is large, the shape of the output waveform (i.e., the voltage at the end of the line) strongly depends on the finite rise time of the source signal. To see this, consider that it is possible for a new component voltage to arrive at the load or source positions before the previous voltage rises to its final value; for example, $\mathcal{V}_1^-(z, t)$ could arrive at $z = 0$ before the input voltage has risen to its full value. In such cases, the temporal variation of the total voltage or current at any position along the line can be found by summing all of the component voltages $\mathcal{V}_i^-(z, t)$ and $\mathcal{V}_i^+(z, t)$, $i = 1, 2, 3, \ldots$, each of which, as mentioned before, exists for all time after its generation. As an example, consider the circuit of Example 2-3, where the source was considered to be an ideal step voltage source with a rise time $t_r = 0$. We now assume that the same circuit, shown again in Figure 2.35a, is driven by a source having a finite rise time[34] $t_r \neq 0$, such that the source voltage changes linearly from 0 to V_0 in t_r seconds. Note that we have chosen $R_s = Z_0/4$. At $t = 0$, a voltage of $\mathcal{V}_1^+(z, t)$ given by

$$\mathcal{V}_1^+(0, t) = \begin{cases} \left[\dfrac{Z_0 V_0}{(R_s + Z_0)t_r}\right]t = \left(\dfrac{0.8V_0}{t_r}\right)t & t \leq t_r \\ \dfrac{Z_0 V_0}{(R_s + Z_0)} = 0.8V_0 & t \geq t_r \end{cases}$$

is applied from the source side of the line. This voltage reaches the open-circuited end of the line ($z = l$) at $t = t_d$, and a voltage $\mathcal{V}_1^-(z, t)$ of amplitude given by

$$\mathcal{V}_1^-(l, t) = \begin{cases} \Gamma_L \mathcal{V}_1^+(l, t) = \left(\dfrac{0.8V_0}{t_r}\right)(t - t_d) & t_d \leq t \leq t_d + t_r \\ 0.8V_0 & t \geq t_d + t_r \end{cases}$$

reflects toward the source, since $\Gamma_L = 1$. This reflected voltage arrives at the source end of the line at $t = 2t_d$, and a new voltage $\mathcal{V}_2^+(z, t)$ of amplitude given by

$$\mathcal{V}_2^+(0, t) = \begin{cases} \Gamma_s \mathcal{V}_1^-(0, t) = -\left(\dfrac{0.48V_0}{t_r}\right)(t - 2t_d) & 2t_d \leq t \leq 2t_d + t_r \\ -0.48V_0 & t \geq 2t_d + t_r \end{cases}$$

---

[33] Note that, as mentioned in Chapter 1, a rule-of-thumb criterion for interconnects between integrated circuit chips is that lumped analysis is appropriate only if $t_r/t_d > 6$, and inappropriate for $t_r/t_d < 2.5$, with the applicability and the required accuracy being the determining factors in the intervening range ($2.5 < t_r/t_d < 6$) depending on the particular application in hand.

[34] Note that t_r is not exactly the rise time discussed in Section 1.1, which was defined as the time required for the signal to change from 10% to 90% of its final value.

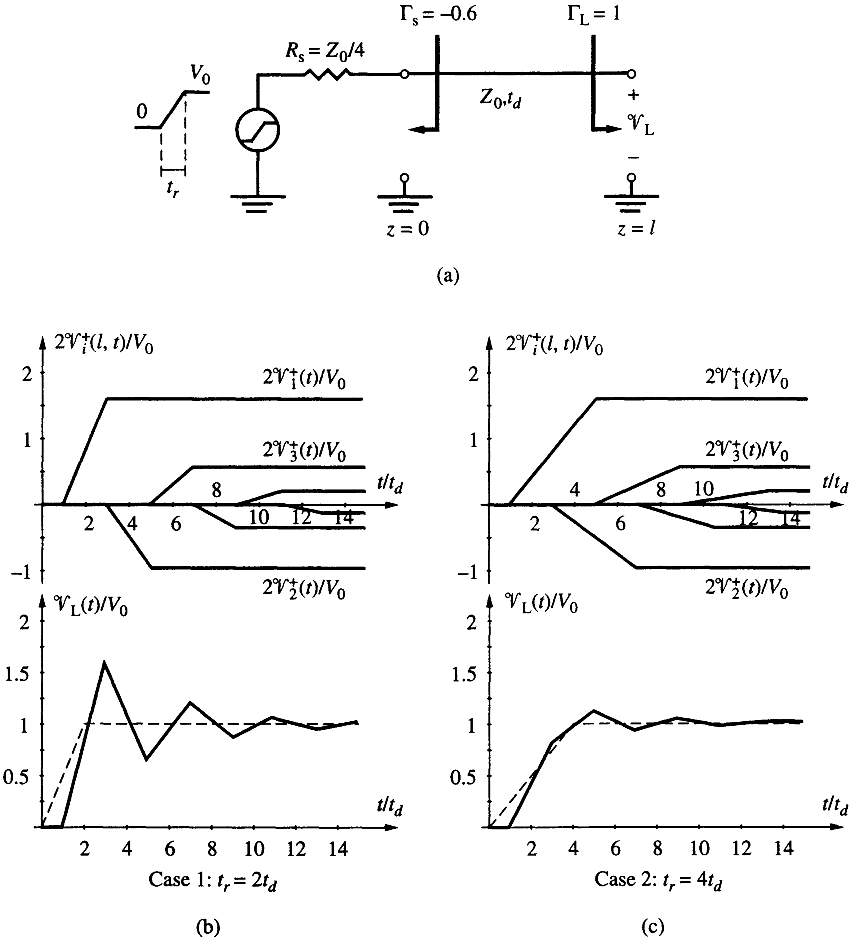

FIGURE 2.35. Effect of source rise time. (a) Circuit diagram for a transmission line excited with a step source of amplitude V_0 and rise time t_r and terminated with an open circuit at the load end. (b) The individual component voltages at the load end (top panel) and the load voltage $\mathcal{V}_L(t)$ (bottom panel) versus time for $t_r = 2t_d$. (c) The individual component voltages at the load end (top panel) and the load voltage $\mathcal{V}_L(t)$ (bottom panel) versus time for $t_r = 4t_d$.

reflects toward the load, since $\Gamma_s = -0.6$. At $t = 3t_d$, a new reflected voltage $\mathcal{V}_2^-(z,t)$ of amplitude given by

$$\mathcal{V}_2^-(l,t) = \begin{cases} \Gamma_L \mathcal{V}_2^+(l,t) = -\left(\dfrac{0.48V_0}{t_r}\right)(t - 3t_d) & 3t_d \le t \le 3t_d + t_r \\ -0.48V_0 & t \ge 3t_d + t_r \end{cases}$$

is launched toward the source. This process continues indefinitely, with the total voltages and currents gradually approaching their steady-state values. The top panels of Figure 2.35b and 2.35c show the individual contributions of the various component voltages $\mathcal{V}_i^+(z,t)$ and $\mathcal{V}_i^-(z,t)$, where $i = 1, 2, 3, \ldots$, that are generated as a result of reflections at both ends of the line, for two different rise times, $t_r = 2t_d$ and $t_r = 4t_d$. Note that since $\Gamma_{\text{L}} = 1$ and thus $\mathcal{V}_i^-(l,t) = \mathcal{V}_i^+(l,t)$, the quantity actually plotted in the top panels of Figure 2.35b and 2.35c is $2\mathcal{V}_i^+(l,t)$. The bottom panels show the time variation of the total load voltage, as determined by the summation of the component voltages. Shown in dashed lines in the lower panels are the load voltage waveforms predicted by a simple lumped analysis (i.e., neglecting transmission line effects). Note that $\mathcal{V}_{\text{L}}(t)$ is simply the superposition of all the component voltages at $z = l$, namely $\mathcal{V}_i^{\pm}(l,t)$, for $i = 1, 2, 3, \ldots$.

It is clear from Figure 2.35b that for $t_r = 2t_d$, the output voltage waveform is substantially different from that expected based on a lumped treatment (i.e., by neglecting transmission line effects or in effect assuming that $l = 0$ or $t_d = 0$), shown for comparison as a dashed line in the lower panel of Figure 2.35b. For the case of $t_r = 4t_d$, however, the load voltage variation deviates only slightly from the lumped case, as shown in Figure 2.35c. For larger values of t_r/t_d, $\mathcal{V}_{\text{L}}(t)$ are even more similar to that predicted based on a lumped assumption, and transmission line effects can be neglected for these applications.

2.6.3 Transients on Lossy Transmission Lines

At a qualitative level, losses on a transmission line lead to *distortion* of the information being transmitted. Distortion is defined as the change in the shape of the signal, as a function of distance, as it travels down the line. For example, a signal that is in the form of a rectangular pulse at the beginning of the line does not retain its rectangular shape as it propagates further; a steplike change in the input voltage is rounded off when observed at points further down the line. The distortion is a result of the fact that the general solutions of the transmission line equations for the lossy case ($R, G \neq 0$) are no longer in the form $f(t - z/v_p)$. In Chapter 3, we shall see that the phase velocity v_p for sinusoidal signals, which is independent of frequency on a lossless line, becomes a function of frequency for lossy lines. If we imagine a transient signal (e.g., a pulse) at some point $z = z_1$ to consist of a superposition of its Fourier components, each of these components travels to a new point $z = z_2$ at a different speed. In addition, each frequency component is in general attenuated by a different amount. Even if there were no reflected pulses (i.e., an infinitely long or a matched line), the differently attenuated and time-shifted sinusoidal components of the signal at $z = z_2$ do not add up to reconstruct the original shape of the signal at $z = z_1$.

The treatment of the propagation of transient signals on lossy lines is a difficult problem, generally requiring extensive analyses using Laplace transformation methods or numerical time domain solutions.[35] The special case of RC lines (i.e., lines with $L = 0$ and $G = 0$), which represents most on-chip interconnect structures and some thin-film package wires that exhibit small inductance but significant resistance, can be treated analytically.[36]

2.7 TRANSMISSION LINE PARAMETERS

We have seen in previous sections that the response of a lossless transmission line to a given excitation depends on its characteristic impedance Z_0 and the propagation speed v_p (or the one-way travel time $t_d = l/v_p$), which in turn depends on the line inductance L and capacitance C per unit line length. The response of lossy lines is additionally influenced by the values per unit line length of the series resistance R and shunt conductance G. In general, the values of these transmission line parameters depend on (i) the geometric shapes, physical dimensions, and proximity of the two conductors that form the line; (ii) the electromagnetic properties of the material surrounding the conductors; and (iii) the electrical conductivity of the conductors and the frequency of operation. In later chapters, after we have introduced the governing electromagnetic equations, we will discuss methods by which the line capacitance, inductance, resistance, and conductance per unit length can be defined and determined from basic principles. In the case of the common transmission lines shown in Figure 2.36, we will be able to find convenient analytical expressions for the line parameters. For other, more complicated, structures R, L, C, and G can be either

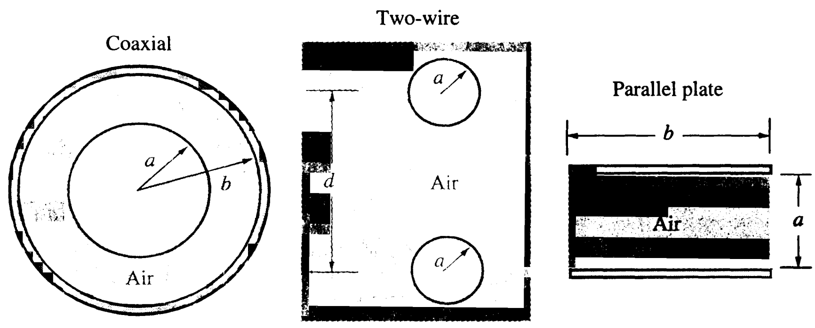

FIGURE 2.36. Cross-sectional view of three common uniform transmission lines. Expressions for the circuit parameters L, R, C, and G for these coaxial, two-wire, and parallel-plate lines are provided in Table 2.2.

[35] As an example, see F. Chang, Transient analysis of lossy transmission lines with arbitrary initial potential and current distributions, *IEEE Trans. Circuits Syst.—I: Fundamental Theory and Applications,* 39(3), pp.180–198, March 1992.

[36] See H. B. Bakoglu, *Circuits, Interconnections, and Packaging for VLSI,* Addison-Wesley, 1990; and A. Wilnai, Open-ended RC line model predicts MOSFET IC response, *EDN,* pp. 53–54, December 1971.

TABLE 2.2. Transmission line parameters for some uniform two-conductor transmission lines surrounded by air

	Coaxial	Two-wire	Parallel-plate*
L (μH-m^{-1})	$0.2\ln(b/a)$	$0.4\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]$	$\frac{1.26a}{b}$
C (pF-m^{-1})	$\frac{55.6}{\ln(b/a)}$	$\frac{27.8}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$	$\frac{8.85b}{a}$
R (Ω-m^{-1})	$\frac{4.15\times10^{-8}(a+b)\sqrt{f}}{ab}$	$\frac{8.3\times10^{-8}\sqrt{f}}{a}$	$\frac{5.22\times10^{-7}\sqrt{f}}{b}$
G** (S-m^{-1})	$\frac{7.35\times10^{-4}}{\ln(b/a)}$	$\frac{3.67\times10^{-4}}{\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]}$	$\frac{1.17\times10^{-4}b}{a}$
Z_0 (Ω)	$60\ln(b/a)$	$120\ln\left[\frac{d}{2a}+\sqrt{\left(\frac{d}{2a}\right)^2-1}\right]$	$\frac{377a}{b}$

*Valid for $b \gg a$.
**For polyethylene at 3 GHz.

evaluated using numerical techniques or measured. Parameters for many different transmission lines are also extensively available in handbooks.[37]

Expressions for L, R, C parameters and for Z_0 for the common uniform transmission lines shown in Figure 2.36 are given in Table 2.2. The characteristic impedances (Z_0) provided are for lossless lines (i.e., $Z_0 = \sqrt{L/C}$). In Table 2.2, we have assumed the transmission line conductors to be made of copper and the surrounding medium to be air. Note that the parameters depend on the geometric shapes and the physical dimensions of the lines (d, a, and b). The line capacitance C and characteristic impedance Z_0 for the case when the surrounding medium is a nonmagnetic[38] material other than air can be derived from those given in Table 2.2 by using the propagation speed v_p for these media as given in Table 2.1. Specifically we have

$$[C]_{\text{material}} = \frac{c^2}{v_p^2}[C]_{\text{air}} \quad \text{and} \quad [Z_0]_{\text{material}} = \frac{v_p}{c}[Z_0]_{\text{air}}$$

where c is the speed of light in free space, or $c \simeq 3\times10^8$ m-s^{-1}. The line inductance L remains the same, since it is governed by the magnetic properties of the surrounding material.

[37] *Reference Data for Engineers,* 8th ed., Sams Prentice Hall Computer Publishing, Carmel, Indiana, 1993.

[38] Magnetic properties of materials are discussed in Section 6.8. In the transmission line context, all materials can be considered nonmagnetic except for iron, nickel, cobalt, a few of their alloys, and some special compounds involving mixtures of magnetic materials with barium titanate.

TABLE 2.3. Relative conductivities of metals versus copper

Material	Relative conductivity	Material	Relative conductivity
Aluminum	0.658	Silver	1.06
Brass	0.442	Sodium	0.375
Copper	1.00	Stainless steel	0.0192
Gold	0.707	Tin	0.151
Lead	0.0787	Titanium	0.0361
Magnesium	0.387	Tungsten	0.315
Nickel	0.250	Zinc	0.287

The series resistance (R) is inversely proportional to the electrical *conductivity* of the particular metal that the conductors are made of, with the values given in Table 2.2 being relative to that of copper. The physical underpinnings of electrical conductivity are discussed in Chapter 5. For now, it suffices to know that it is a quantitative measure of the ability of a material to conduct electrical current and that the values of conductivity for different materials are tabulated extensively in various handbooks (see Table 5.1). A brief list of conductivities of some common metals relative to that of copper is provided in Table 2.3. The series resistance R is proportional to the square root of the frequency because of the so-called *skin effect,* which results from the nonuniform distribution of electrical current in a metal at higher frequencies, and which is discussed in Chapter 8.

With air as the surrounding medium, the shunt conductance $G = 0$, since air is an excellent insulator and leakage losses through it are generally negligible. In the case of other surrounding media for which leakage losses may not be negligible, the value of G depends on the geometrical layout of the conductors (as do the values of C, L, and R) but is more strongly determined by the loss properties of the insulating medium surrounding the conductors and is, in general, a rather complicated function of the frequency of operation. Table 2.2 provides a representative expression for G for polyethylene as the surrounding medium at an operating frequency of 3 GHz. High-frequency losses in insulating materials are discussed in Sections 7.4 and 8.1.

Examples 2-18, 2-19, and 2-20 illustrate the use of the formulas given in Table 2.2 for selected transmission lines.

Example 2-18: Television antenna lead-in wire. A student measures the dimensions of a television antenna lead-in wire made of two copper wires. The diameter of the wires is found to be $\sim$1 mm each, while the spacing between the centers of the conductors is $\sim$0.7 cm. Assume the conductors to be surrounded by air, although they might in fact be held together by some plastic material. Determine the values of the line parameters and the characteristic impedance at an operating frequency of 200 MHz.

Solution: We can directly use the formulas given in Table 2.2 for the two-wire line. Noting that $d = 0.7$ cm and $2a = 1$ mm, we have

$$L = 0.4\ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] \simeq 1.05\ \mu\text{H-m}^{-1}$$

$$C = \frac{27.8}{\ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right]} \simeq 10.6\ \text{pF-m}^{-1}$$

$$R = \frac{8.3 \times 10^{-8}\sqrt{200 \times 10^6}}{a} \simeq 2.35\Omega\text{-m}^{-1}$$

$$Z_0 = 120\ln\left[\frac{d}{2a} + \sqrt{\left(\frac{d}{2a}\right)^2 - 1}\right] \simeq 316\Omega$$

Note that the characteristic impedance of this line is quite close to 300Ω, within the tolerances of the measurement. Indeed, the household television lead-in line is usually referred to as a 300-ohm line. Note also that $G \simeq 0$ for this wire, since the leakage losses are negligible.

Example 2-19: Coaxial line. A coaxial line consists of inner and outer conductors made of copper and having radii of $a = 0.65$ mm and $b = 2.75$ mm, the space between the conductors being filled with air. The line is to be used at 1 GHz. Find the values of the distributed parameters and the characteristic impedance of this line.

Solution: We can directly use the formulas given in Table 2.2 for the coaxial line:

$$L = 0.2\ln\left(\frac{b}{a}\right) \simeq 0.289\ \mu\text{H-m}^{-1}$$

$$C = \frac{55.6}{\ln\left(\frac{b}{a}\right)} \simeq 38.5\ \text{pF-m}^{-1}$$

$$R = \frac{4.15 \times 10^{-8}(a + b)}{ab}\sqrt{10^9} \simeq 2.5\Omega\text{-m}^{-1}$$

$$Z_0 = 60\ln\left(\frac{b}{a}\right) \simeq 86.5\Omega$$

Note once again that $G = 0$ for this air-filled coaxial line.

Example 2-20: RG58/U coaxial line. RG58/U is a commonly used coaxial line with an inner conductor of diameter 0.45 mm and an outer conductor of inside diameter 1.47 mm constructed using copper conductors and filled with polyethylene as its insulator. The line is to be used at 3 GHz. Find the line parameters (i.e., R, L, C, G, and Z_0). Note from Table 2.1 that the propagation speed for polyethylene at 3 GHz is $v_p = 20$ cm-(ns)$^{-1} = 2 \times 10^8$ m-s^{-1}.

Solution: We use the expressions provided in Table 2.2, except for the multipliers needed for C and Z_0 to correct for the fact that the filling is polyethylene rather than air. Note that we can also use the expression from Table 2.2 for G, since it was also given for polyethylene and for 3 GHz. We have

$$R = \frac{4.15 \times 10^{-8}(0.45 + 1.47) \times 10^{-3}\sqrt{3 \times 10^9}}{(0.45 \times 10^{-3})(1.47 \times 10^{-3})} \simeq 6.6\Omega/\text{m}$$

$$L = 0.2 \ln\left(\frac{1.47}{0.45}\right) \simeq 0.237\ \mu\text{H/m}$$

$$C = \left(\frac{c^2}{v_p^2}\right)\frac{55.6}{\ln\left(\frac{b}{a}\right)} = \left(\frac{3}{2}\right)^2 \frac{55.6}{\ln\left(\frac{1.47}{0.45}\right)} \simeq 106\ \text{pF/m}$$

$$G = \frac{7.35 \times 10^{-4}}{\ln\left(\frac{1.47}{0.45}\right)} \simeq 6.21 \times 10^{-4}\ \text{S/m}$$

$$Z_0 = \left(\frac{v_p}{c}\right) 60 \ln\left(\frac{b}{a}\right) \simeq 47.4\Omega$$

Note that the value of Z_0 is close to the nominal 50-ohm impedance of this coaxial line. Note also that the difference between 47.4Ω and 50Ω is within the range of accuracy (i.e., two digits after the decimal point) by which the physical quantities were specified (e.g., the radii of conductors, the value of v_p, etc.).

2.8 SUMMARY

This chapter discussed the following topics:

- **Transmission line parameters.** A transmission line is commonly characterized by its distributed parameters R (in Ω/m), L (in H/m), G (in S/m), and C (in F/m), whose values are determined by the line geometry, the conductivity of the metallic conductors, the electrical and magnetic properties of the surrounding insulating material, and the frequency of excitation. Formulas for calculating R, L, and C are provided in Table 2.2 for coaxial, two-wire, and parallel-plate transmission lines surrounded by air. Formulas for G are also provided in Table 2.2 for the same lines surrounded by polyethylene.

- **Transmission line equations.** The distributed parameters of the line are used in an equivalent lumped-circuit model to represent a differentially short segment of the line. Using this model as a basis, the transmission line equations are derived from Kirchhoff's laws in the limit where the length of the line segment approaches zero. The differential equations governing the behavior of voltage and current on a lossless ($R = 0$ and $G = 0$) transmission line, and the wave equation for voltage derived from them, are

$$\left.\begin{aligned}\frac{\partial \mathcal{V}}{\partial z} &= -L\frac{\partial \mathcal{I}}{\partial t}\\ \frac{\partial \mathcal{I}}{\partial z} &= -C\frac{\partial \mathcal{V}}{\partial t}\end{aligned}\right\} \longrightarrow \quad \frac{\partial^2 \mathcal{V}}{\partial z^2} = LC\frac{\partial^2 \mathcal{V}}{\partial t^2}$$

- **Propagating-wave solutions, characteristic impedance, and phase velocity.** The general solution of the transmission line equations leads to mathematical expressions for voltage and current along the line that are wave equations in nature, depending on both distance and time. These are

$$\mathcal{V}(z,t) = f^+\left(t - \frac{z}{v_p}\right) + f^-\left(t + \frac{z}{v_p}\right)$$

$$\mathcal{I}(z,t) = \frac{1}{Z_0}\left[f^+\left(t - \frac{z}{v_p}\right) - f^-\left(t + \frac{z}{v_p}\right)\right]$$

where Z_0 is the characteristic impedance of a lossless line, which is defined as the voltage-to-current ratio of a single disturbance propagating in the $+z$ direction and is given by $Z_0 = \sqrt{L/C}$. The characteristic impedance is one of the most important quantities that determine the response of a transmission line, and its value is tabulated for most practical transmission lines. Formulas for calculating Z_0 for three different types of lines (coaxial, two-wire, and parallel-plate) are provided in Table 2.2 for lossless transmission lines surrounded by air. The velocity with which waves on a transmission line propagate is called the phase velocity and is given by $v_p = (LC)^{-1/2}$. For a lossless transmission line, the phase velocity is determined by the properties of the material surrounding the transmission line conductors and is equal to the speed of light in free space ($c = 3 \times 10^8$ m/s) if the conductors constituting the line are surrounded by free space or air. For uniform transmission lines, v_p is a constant, regardless of the shape of the voltage (or current) signal traveling down the line. Values of v_p for selected materials are tabulated in Table 2.1.

- **Transmission lines terminated in resistive loads, reflection coefficient.** Transient response of a lossless transmission line to step or pulse excitation involves reflections from discontinuities along the line or from loads at its termination. Reflection effects are described in terms of the reflection coefficient Γ, defined as the ratio of the reflected to the incident voltage at a given point. The reflection coefficients at the load and source ends of a transmission line are given by

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \qquad \text{Load end}$$

$$\Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} \qquad \text{Source end}$$

where R_L is the resistance terminating the load end of the line, the line's characteristic impedance is Z_0, and R_s is the source resistance. In the special case of a matched termination, we have $R_L = Z_0$, so $\Gamma_L = 0$, and thus no reflection occurs from the termination. Similarly, when $R_s = Z_0, \Gamma_s = 0$, and no reflection occurs from the source end. In general, when a voltage disturbance is launched from the source end of a transmission line (e.g., due to a step change in input voltage), a sequence of reflections from both the load and source ends of the line occurs. The process of multiple reflections from the load and source ends of a transmission line can be described using a bounce diagram.

- **Transmission lines terminated in reactive or nonlinear loads.** To determine the transient behavior of lossless lines terminated in reactive or nonlinear elements, it is necessary to solve the differential equations that determine the voltage-current relationships of the terminations subject to the appropriate initial conditions. For reactive loads, the reflected voltage due to a step excitation is no longer a simple step function but, in general, varies continuously at a fixed position with respect to time depending on the nature of the reactive termination. For nonlinear terminations, a graphical approach known as the Bergeron plot can be used to find the voltage and current following each reflection, using the known current-voltage characteristics of the nonlinear device.

2.9 PROBLEMS

2-1. Open-circuited line. Consider the circuit shown in Figure 2.37, with an ideal unit step source connected to a lossless line of characteristic impedance $Z_0 = 50\Omega$ having an open-circuited termination at the other end. Assuming a one-way propagation delay

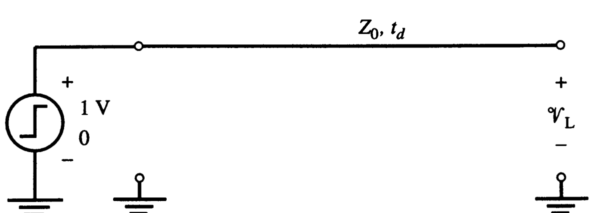

FIGURE 2.37. Open-circuited line. Problem 2-1.

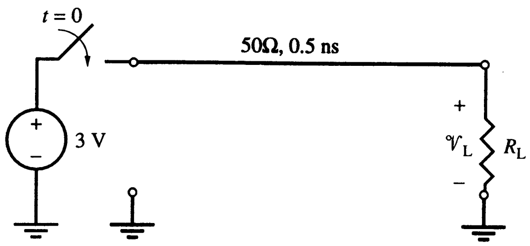

FIGURE 2.38. **Resistive loads.** Problems 2-2 and 2-4.

across the line of t_d, use a bounce diagram to sketch the voltage $\mathcal{V}_L(t)$ versus time for $0 \le t \le 10t_d$.

2-2. **Resistive loads.** The circuit shown in Figure 2.38 consists of an uncharged transmission line connected to a load resistance R_L. Assuming that the switch closes at $t = 0$, sketch the load voltage $\mathcal{V}_L(t)$ over the time interval $0 \le t \le 3$ ns for the following load resistances: (a) $R_L = 25\Omega$, (b) $R_L = 50\Omega$, (c) $R_L = 100\Omega$.

2-3. **Ringing.** The transmission line system shown in Figure 2.39 is excited by a step-voltage source of amplitude 3.6 V and source impedance 15Ω at one end, and is terminated with an open circuit at the other end. The line is characterized by the line parameters $L = 4.5$ nH-(cm)$^{-1}$, $C = 0.8$ pF-(cm)$^{-1}$, $R = 0$, $G = 0$, and has a length of $l = 30$ cm. Sketch the load voltage $\mathcal{V}_L(t)$ over $0 \le t \le 10$ ns with the steady-state value indicated.

2-4. **Discharging of a charged line.** For the circuit of Problem 2-2, assume that the switch has been closed for a long time before it opens at $t = 0$. Sketch the load voltage $\mathcal{V}_L(t)$ over $0 \le t \le 3$ ns for the same three cases.

2-5. **Pulse excitation.** The circuit shown in Figure 2.40 is excited by an ideal voltage pulse of 1 V amplitude starting at $t = 0$. Given the length of the line to be $l = 10$ cm and the propagation speed to be 20 cm-(ns)$^{-1}$, (a) sketch the voltage at the source end of

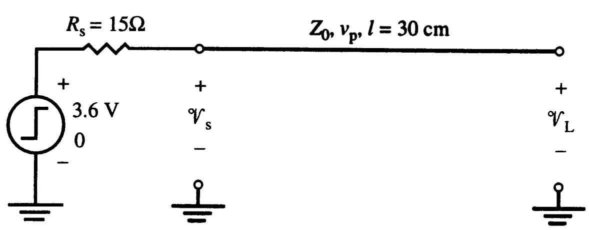

FIGURE 2.39. **Ringing.** Problem 2-3.

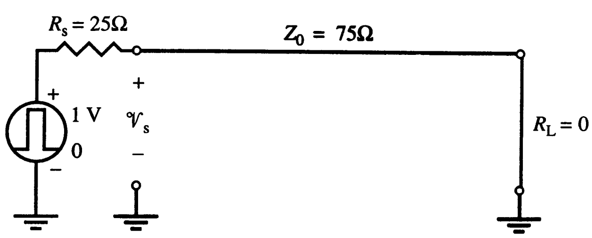

FIGURE 2.40. **Pulse excitation.** Problem 2-5.

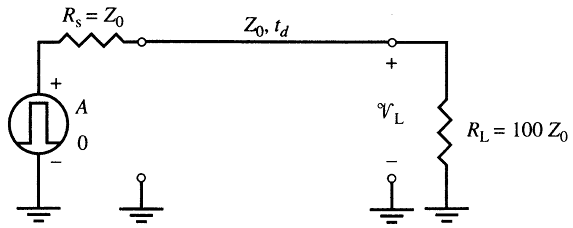

FIGURE 2.41. Pulse excitation. Problem 2-6.

the line, $\mathcal{V}_s(t)$, for an input pulse duration of 10 ns; (b) repeat part (a) for an input pulse duration of 1 ns.

2-6. Pulse excitation. The circuit shown in Figure 2.41 is excited with a voltage pulse of amplitude A and pulse width t_w. Assuming the propagation delay of the line to be t_d, sketch the load voltage $\mathcal{V}_L(t)$ versus t for $0 \leq t \leq 10t_d$ for (a) $t_w = 2t_d$, (b) $t_w = t_d$, and (c) $t_w = t_d/2$.

2-7. Observer on the line. A transmission line with an unknown characteristic impedance Z_0 terminated in an unknown load resistance R_L, as shown in Figure 2.42, is excited by a pulse source of amplitude 1 V and duration $t_w = 3t_d/4$, where t_d is the one-way flight time of the transmission line. An observer at the center of the line observes the voltage variation shown. (a) Determine Z_0 and R_L. (b) Using the values found in (a), sketch the voltage variation (up to $t = 4t_d$) that would be observed by the same observer if the pulse duration were $t_w = 1.5t_d$.

2-8. Cascaded transmission lines. For the transmission line circuit shown in Figure 2.43, sketch $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$ over $0 \leq t \leq 5$ ns.

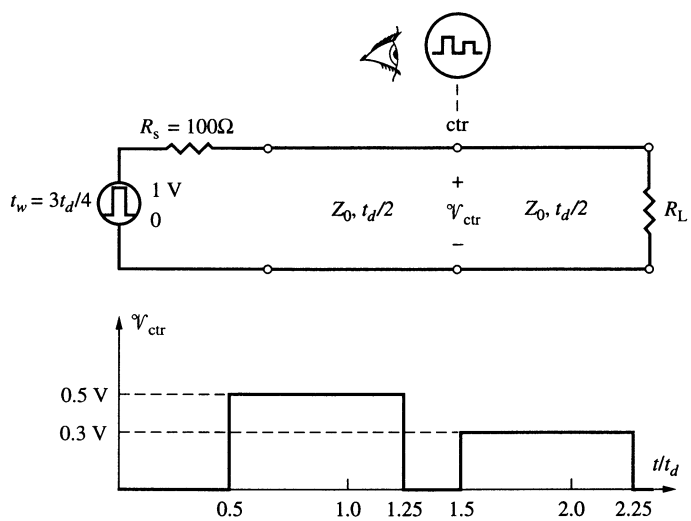

FIGURE 2.42. Observer on the line. Problem 2-7.

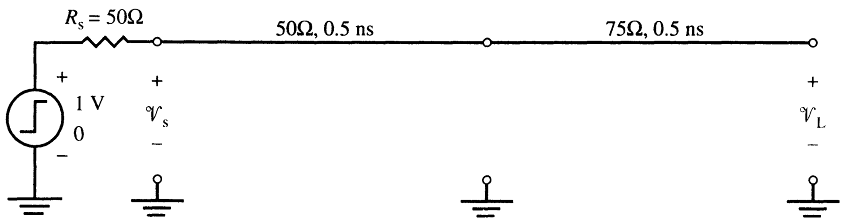

FIGURE 2.43. Cascaded lines. Problem 2-8.

2-9. Time-domain reflectometry (TDR). A TDR is used to test the transmission line system shown in Figure 2.44. Using the sketch of $\mathcal{V}_s(t)$ observed on the TDR scope as shown, determine the values of Z_{01}, l_1, and R_1. Assume the phase velocity of the waves to be 20 cm-(ns)$^{-1}$ on each line. Plot $\mathcal{V}_{R1}(t)$ versus t for $0 \leq t \leq 4$ ns.

2-10. Time-domain reflectometry (TDR). TDR measurements can also be used in cases with more than one discontinuity. Two transmission lines of different characteristic impedances and time delays terminated by a resistive load are being tested by a TDR, as shown in Figure 2.45. (a) Given the TDR display of the source-end voltage due to a 3-V, 100Ω step excitation starting at $t = 0$, find the characteristic impedances (Z_{01} and Z_{02}) and the time delays (t_{d1} and t_{d2}) of both lines, and the unknown load R_L. (b) Using the values found in part (a), find the time and magnitude of the next change in the source-end voltage $\mathcal{V}_s(t)$, and sketch it on the display.

2-11. Time-domain reflectometry (TDR). Consider the circuit shown in Figure 2.46. The two line segments are of equal length l. Assuming the propagation speeds on the two lines are equal to 15 cm-(ns)$^{-1}$ each, find Z_{01}, Z_{02}, R_L and l using the TDR display of the source voltage $\mathcal{V}_s(t)$, as shown.

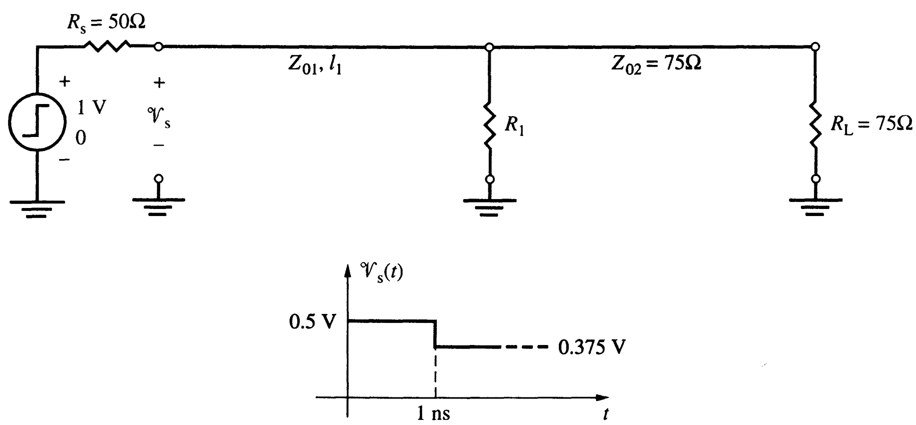

FIGURE 2.44. Time-domain reflectometry. Problem 2-9.

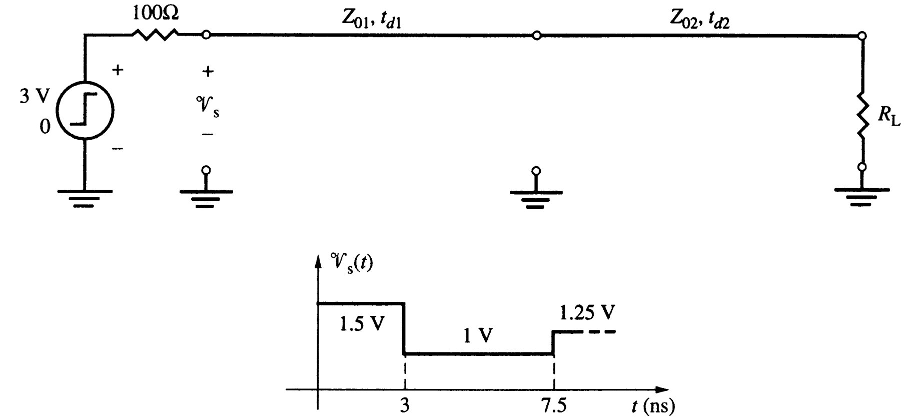

FIGURE 2.45. **Time-domain reflectometry.** Problem 2-10.

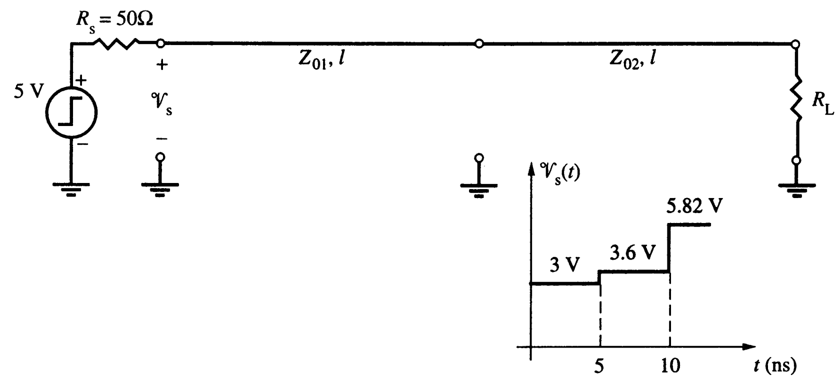

FIGURE 2.46. **Time-domain reflectometry.** Problem 2-11.

2-12. Multiple lines. For the distributed transmission line system shown in Figure 2.47, sketch the voltages $\mathcal{V}_1(t)$, $\mathcal{V}_2(t)$, and $\mathcal{V}_3(t)$ versus t for $0 \le t \le 7t_d$. Assume the one-way propagation delay to be t_d on each transmission line.

2-13. Digital IC chips. Two impedance-matched, in-package-terminated Integrated Circuit (IC) chips are driven from an impedance-matched IC chip, as shown in Figure 2.48. Assuming the lengths of the interconnects to be 15 cm each and the propagation velocity on each to be 10 cm-(ns)$^{-1}$, do the following: (a) Sketch the voltages $\mathcal{V}_{L1}$ and $\mathcal{V}_{L2}$ for a time interval of 10 ns. Indicate the steady-state values on your sketch. (b) Repeat part (a) if one of the load chips is removed from the end of the interconnect connected to it (i.e., the lead point A is left open-circuited).

2-14. Multiple lines. For the distributed interconnect system shown in Figure 2.49, and for $Z_{01} = Z_{02} = 50\Omega$, find and sketch the three load voltages $\mathcal{V}_1(t)$, $\mathcal{V}_2(t)$, and $\mathcal{V}_3(t)$ for a time interval of 5 ns. Assume each interconnect to have a one-way time delay of 1 ns. (b) Repeat part (a) for $Z_{01} = Z_{02} = 25\Omega$.

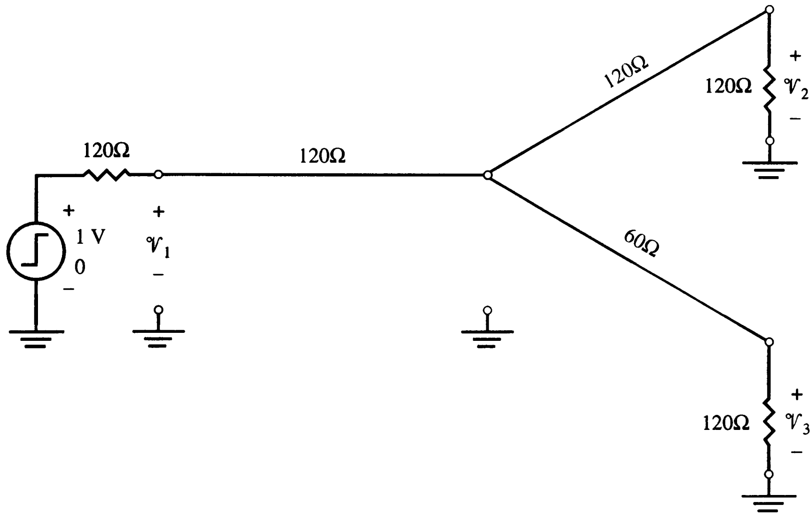

FIGURE 2.47. **Multiple lines.** Problem 2-12.

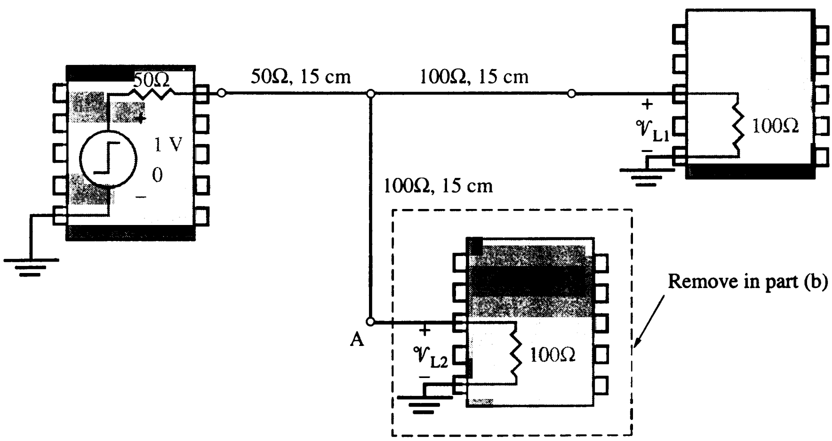

FIGURE 2.48. **Digital IC chips.** Problem 2-13.

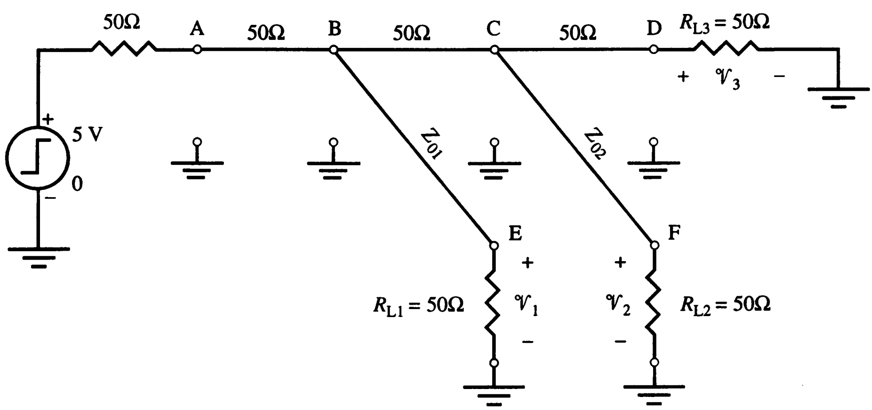

FIGURE 2.49. **Multiple lines.** Problem 2-14.

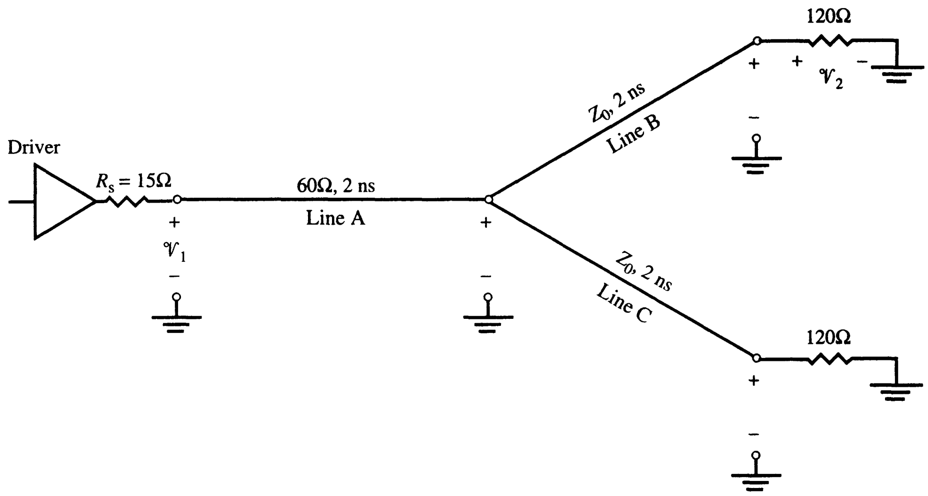

FIGURE 2.50. **Reflections due to mismatches.** Problem 2-15.

2-15. Reflections due to parasitic effects. The circuit shown in Figure 2.50 consists of a low-impedance driver driving a distributed interconnect system that was intended to be impedance-matched, with $Z_0 = 120\Omega$. An engineer performs some tests and measurements and observes reflections due to parasitic effects associated with the two interconnects terminated at the 120Ω loads. Assuming that the effective characteristic impedances of these interconnects (i.e., taking parasitic effects into account) is such that we have $Z_0 = 80\Omega$ instead of 120Ω, find and sketch the voltages $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ for $0 \leq t \leq 12$ ns, assuming the one-way time delay on each interconnect to be 2 ns. Comment on the effects of the mismatch caused by parasitic effects. Assume the initial incident wave launched at the driver end of the 60Ω line to be $\mathcal{V}_1^+ = 4$ V.

2-16. Parallel multiple lines. The transmission line system shown in Figure 2.51 consists of three lines, each having $Z_0 = 50\Omega$ and a one-way propagation delay of 1 ns. (a) Find and sketch the voltages $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$ versus t for $0 \leq t \leq 10$ ns.

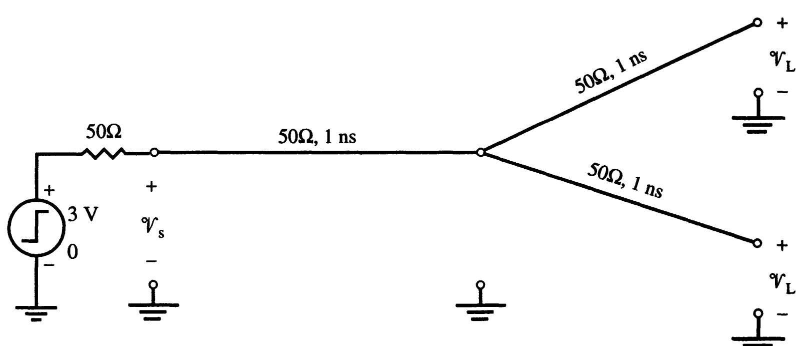

FIGURE 2.51. **Parallel multiple lines.** Problem 2-16.

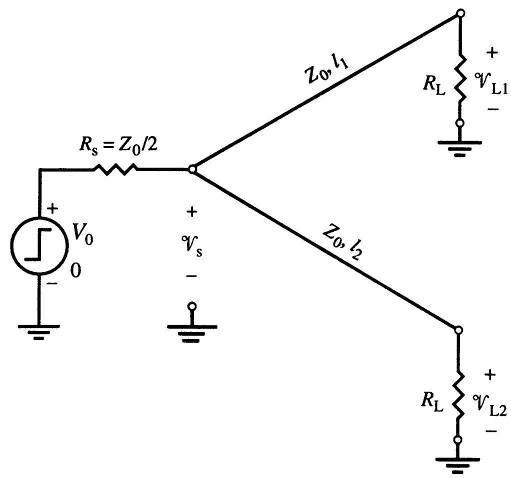

FIGURE 2.52. Optimized multiple lines. Problem 2-17.

(b) Repeat part (a) when the open-circuited ends are terminated with a load resistance of 50Ω each.

2-17. Optimized multiple lines. The transmission line system shown in Figure 2.52 consists of an ideal step source of amplitude V_0 and output impedance R_s connected to two identical parallel lossless transmission lines terminated by equal resistive loads R_L. The parameters of the lines are $L = 2.5$ nH/cm and $C = 1$ pF/cm. (a) Calculate Z_0 and v_p for the lines. (b) For $R_s = Z_0/2$, $R_L \gg Z_0$, and line lengths of $l_2 = 1.5l_1 = 30$ cm, sketch the voltages $\mathcal{V}_s$, $\mathcal{V}_{L1}$, and $\mathcal{V}_{L2}$ versus t for the time interval $0 \leq t \leq 7.5$ ns. (c) Repeat part (b) if the line lengths are optimized to be $l_1 = l_2 = 25$ cm each to minimize ringing effects, and compare with the results of part (b).

2-18. Optimized multiple lines. A multisection transmission line consists of three lossless transmission lines used to connect an ideal step source of 5-V amplitude and 6Ω output impedance to two separate load resistances of 66Ω each, as shown in Figure 2.53.

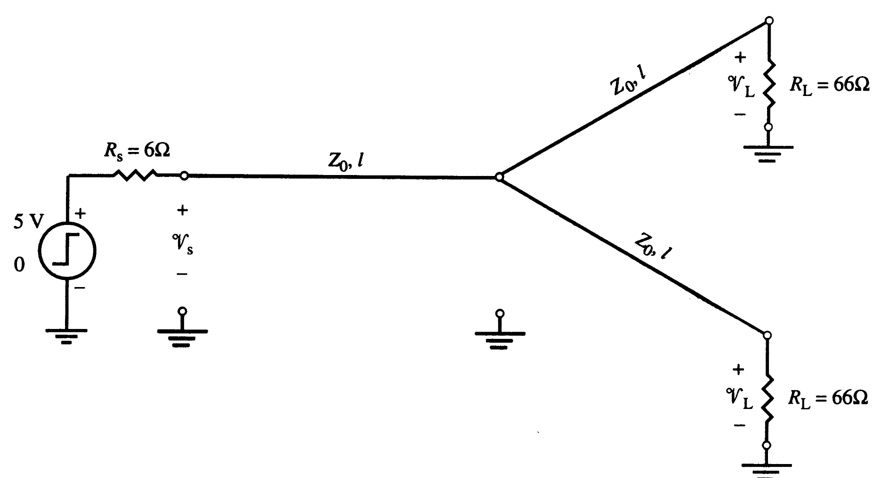

FIGURE 2.53. Optimized multiple lines. Problem 2-18.

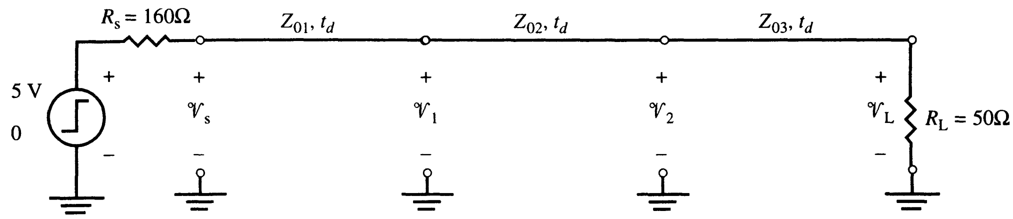

FIGURE 2.54. Multiple lines. Problem 2-19.

All three lines are characterized by line parameters $L = 364.5$ nH-m^{-1}, and $C =$ 125 pF-m^{-1}. To minimize ringing effects, the line lengths are optimized to be of equal length. If each line length is $l = 40$ cm, sketch the voltages $\mathcal{V}_s$ and $\mathcal{V}_L$ versus t for $0 \leq t \leq 20$ ns, and comment on the performance of the circuit in minimizing ringing.

2-19. Minimizing ringing on multiple lines. The three-section equal-delay lossless transmission line system shown in Figure 2.54 is to be designed to minimize ringing effects at the load due to impedance discontinuities along the transmission path. For a step source of 5-V amplitude and 160Ω source impedance and a load impedance of 50Ω, one design consists of three lines with $Z_{01} = 148\Omega$, $Z_{02} = 200\Omega$, and $Z_{03} = 69\Omega$. Assuming a one-way time delay of $t_d = 250$ ps for each of the lines, sketch the voltages $\mathcal{V}_s$, $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{V}_L$ versus t for $0 \leq t \leq 1.5$ ns, and comment on the performance of the design.

2-20. Charging and discharging of a line. For the transmission line system shown in Figure 2.55, the switch S_2 is closed at $t = 2t_d$ (where t_d is the propagation delay of each line) after the switch S_1 is closed at $t = 0$. Find and sketch the voltage $\mathcal{V}_L$ versus t for $0 \leq t \leq 6t_d$.

2-21. Digital IC interconnect. The circuit shown in Figure 2.56 consists of a logic gate driving another logic gate via a 50Ω interconnect. At $t = 0$, the driver output voltage switches from LOW to HIGH state. The Thévenin equivalent of the driver gate at its output at LOW and HIGH states can be approximated respectively as a -1.67 V or a -0.85 V voltage source, each in series with a 7Ω resistor. The input impedance of the load gate can be approximated by a very large impedance (say, 50 kΩ). Assume the signal delay and the length of the interconnect to be 200 ps-(in.)$^{-1}$ and 6 in., respectively. (a) Sketch the voltage at the load end and comment on the performance of the circuit. (b) Connect an additional terminating network at the load end as shown. Select $R_T = 50\Omega$ and $V_T = -2$ V and repeat part (a).

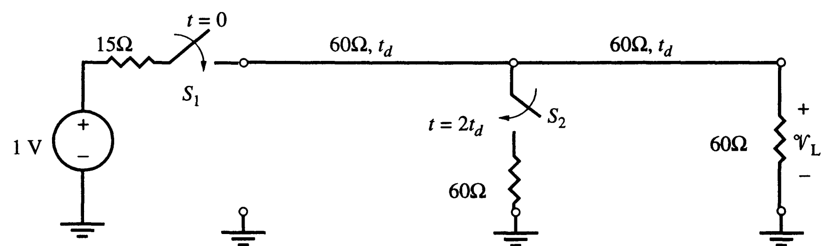

FIGURE 2.55. Charging and discharging of a line. Problem 2-20.

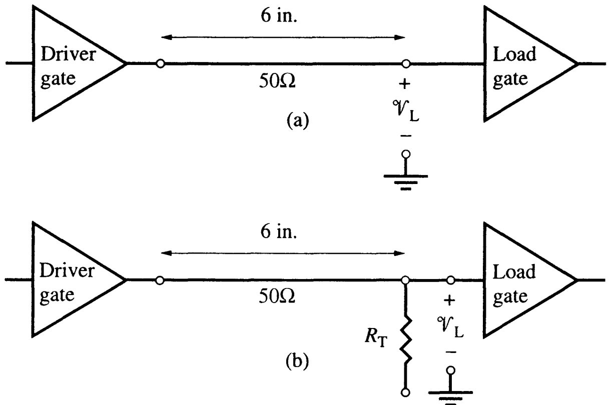

FIGURE 2.56. Digital IC interconnect. Problem 2-21.

2-22. Digital IC interconnect. A circuit consists of one logic gate driving an identical gate via a 1-ft-long, 50Ω interconnect. Before $t = 0$, the output of the driver gate is at LOW voltage state and can simply be approximated as a 14Ω resistor. At $t = 0$, the output of the driver gate goes from LOW to HIGH state and can be approximated with a 5-V voltage source in series with a 14Ω resistor. The input of the load gate can be approximated to be an open circuit (i.e., $R_L = \infty$). Assuming that a minimum load voltage of 3.75 V is required to turn and keep the load gate on, (a) find the time at which the load gate will turn on for the first time. (b) Find the time at which the load gate will turn on permanently. (Assume a signal time delay of 1.5 ns-(ft)$^{-1}$ along the interconnect for both parts.)

2-23. Terminated IC interconnects. The logic circuit of Problem 2-22 needs to be modified to eliminate ringing. Two possible solutions are to terminate the line in its characteristic impedance at either the source (series termination) or receiver (parallel termination) end. Both of these circuits are shown in Figure 2.57. (a) Select the value of the termi-

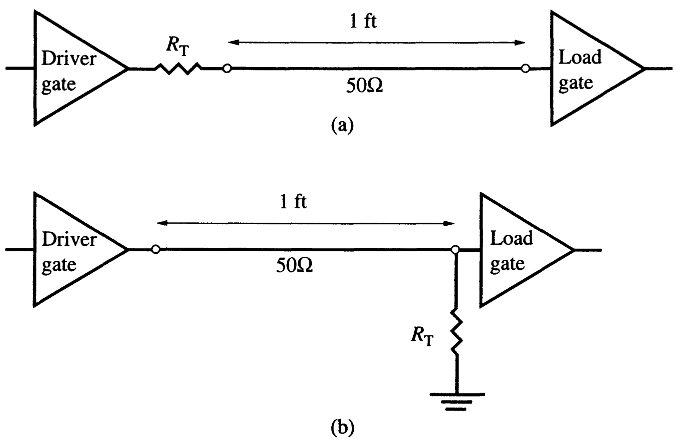

FIGURE 2.57. Terminated IC interconnects. (a) Series termination. (b) Parallel termination. Problem 2-23.

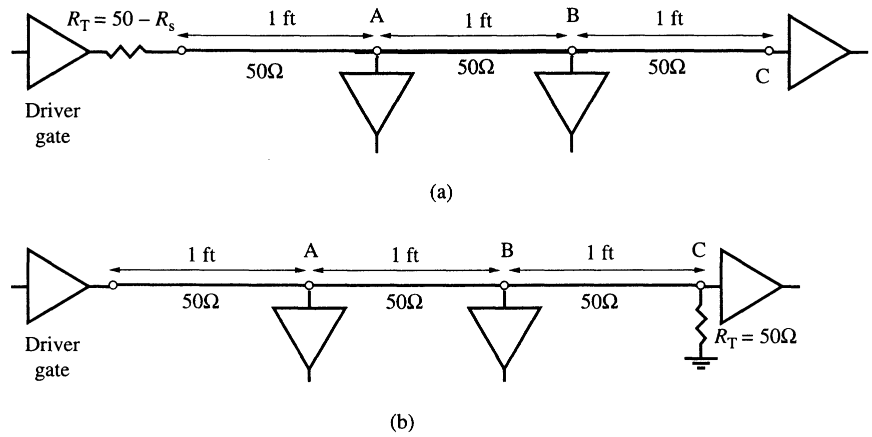

FIGURE 2.58. **IC gate interconnects.** (a) Series termination. (b) Parallel termination. Problem 2-24.

nation resistance R_T in both circuits to eliminate ringing. (b) Compare the performance of these two circuits in terms of their speed and dc power dissipation. Which technique is the natural choice for a design to achieve low-power dissipation at steady state?

2-24. Digital IC gate interconnects. A disadvantage of the series termination scheme in Problem 2-23 is that the receiver gate or gates must be near the end of the line to avoid receiving a two-step signal. This scheme is not recommended for terminating distributed loads. The two circuits shown in Figure 2.58 have three distributed loads equally positioned along a 3-ft-long 50Ω interconnect on a pc board constructed of FR4 material (take $v_p \simeq 14.3$ cm-(ns)$^{-1}$). Each circuit uses a different termination scheme. Assuming the driver and all the loads to be the same gates as in Problem 2-22, find the times at which each load gate changes its logic state after the output voltage of the driver gate switches to HIGH state at $t = 0$. Comment on the performance of both circuits and indicate which termination scheme provides faster speed. (Use some of the data provided in Problem 2-22.)

2-25. Open-ended stub. An electrical engineer is assigned the task of designing the circuit in Problem 2-24 that has the parallel termination scheme. After the design is complete, she performs some tests and measurements on the circuit. Noticing some peculiar effects in the test results, she decides to check the design. She realizes that she used a 4-ft-long interconnect, where the extra foot extends beyond the farthest element away from the driver and is not terminated at the end (i.e., an open-circuited stub). (See Figure 2.59.) Does this open stub affect the overall performance of the circuit? Explain.

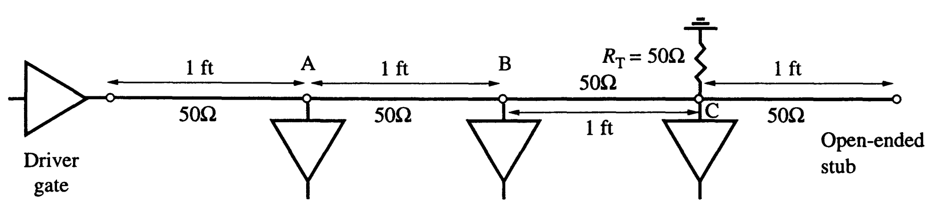

FIGURE 2.59. **Open-ended stub.** Problem 2-25.

2-26. Digital IC circuit. For the digital IC circuit shown in Figure 2.60, the driver gate goes from LOW to HIGH state at $t = 0$, and its Thévenin equivalent circuit (including the series termination resistance) can be approximated as a 5-V voltage source in series with a 50Ω resistor. If the time delay for all interconnects is given to be 2 ns-(ft)$^{-1}$, find the time(s) at which each receiver gate changes its state permanently. Assume each load gate to change state when its input voltage exceeds 4 V. Also assume each load gate to appear as an open circuit at its input. Support your solution with sketches of the two load voltages $\mathcal{V}_1$ and $\mathcal{V}_2$ as functions of time for a reasonable time interval.

2-27. Two driver gates. Two identical logic gates drive a third identical logic gate (load gate), as shown in Figure 2.61. All interconnects have the same one-way time delay t_d and characteristic impedance Z_0. When any one of these driver gates is at HIGH state, its Thévenin equivalent as seen from its output terminals consists of a voltage source with voltage V_0 in series with a resistance of value $R_s = Z_0$. At LOW state, its Thévenin equivalent is just a resistance of value $R_s = Z_0$. The input impedance of the load gate is very high compared to the characteristic impedance of the line (i.e., $Z_{in} \gg Z_0$). (a) Assuming steady-state conditions before both driver gates change to HIGH state at $t = 0$, sketch the load voltage $\mathcal{V}_L$ as a function of time for $0 \leq t \leq 7t_d$.

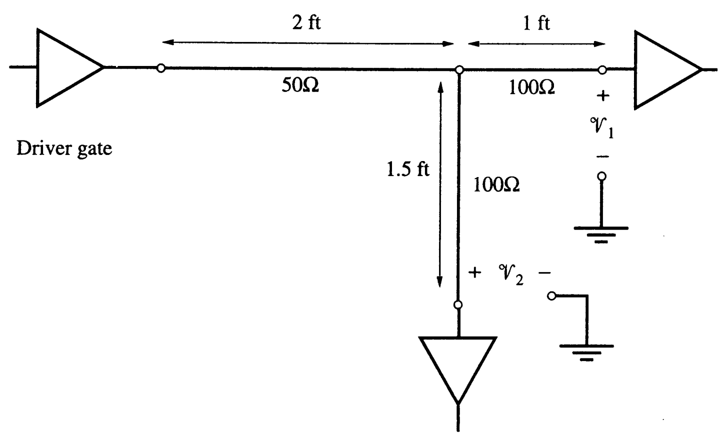

FIGURE 2.60. **Digital IC circuit.** Problem 2-26.

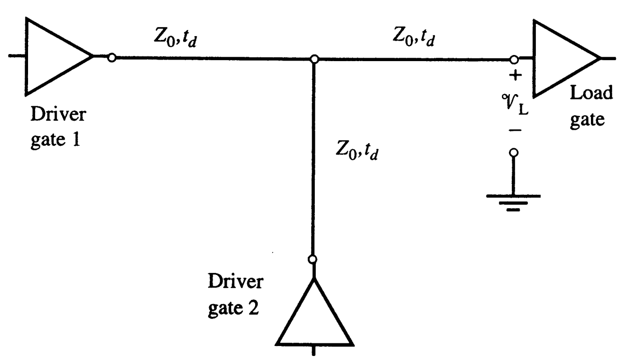

FIGURE 2.61. **Two driver gates.** Problem 2-27.

What is the eventual steady-state value of the load voltage? (b) Assume steady-state conditions before $t = 0$ to be such that driver gate 1 is at HIGH state and gate 2 is at LOW state. At $t = 0$, gate 1 and gate 2 switch states. Repeat part (a).

2-28. Capacitive load. For the transmission line system shown in Figure 2.62, the switch is closed at $t = 0$. Each of the two transmission lines has a one-way time delay of 2 ns. Assuming both transmission lines and the 5 pF capacitor to be initially uncharged, find and sketch the voltage $\mathcal{V}_1(t)$ across the resistor R_1.

2-29. Inductive load. The circuit shown in Figure 2.63 consists of a voltage source of amplitude 5 V and a source resistance of 50Ω driving a lossless 50Ω transmission line having a one-way time delay of 3 ns terminated with an ideal inductor of 25 nH. The circuit has been in steady state for a long time with the switch at position A. At $t = 0$, the switch is moved to position B. (a) Find the mathematical expressions and sketch the voltages at the source and load ends of the line. (b) Repeat part (a) for the case of the same line terminated with a 25 nH inductor in series with a 50Ω resistor.

2-30. Unknown lumped element. The transmission line circuit has an unknown lumped element, as shown in Figure 2.64. With the source-end voltage due to step excitation measured to be as plotted, determine the type of the unknown element, and find its value in terms of the shaded area A.

2-31. Unknown lumped element. The following circuit consists of two transmission lines of characteristic impedances Z_{01} and Z_{02} connected with an unknown lumped series element, as shown in Figure 2.65. The circuit is excited by a step source of amplitude V_0 and a source resistance $R_s = 50\Omega$, starting at $t = 0$. The source-end voltage is observed as a function of time, as shown. (a) Assuming the second line to be terminated

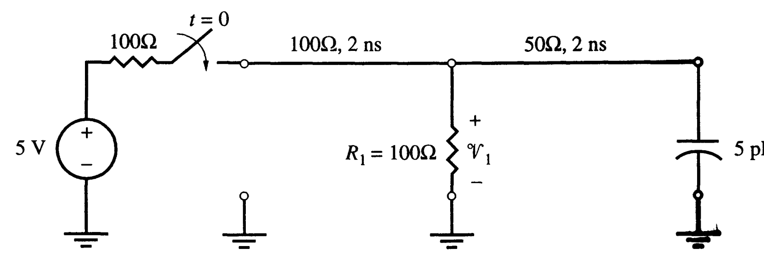

FIGURE 2.62. Capacitive load. Problem 2-28.

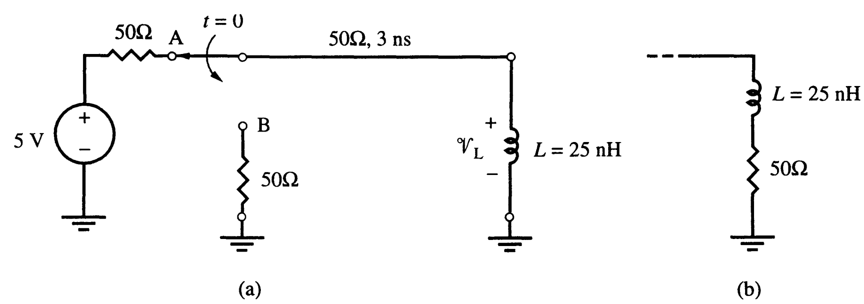

FIGURE 2.63. Inductive load. Problem 2-29.

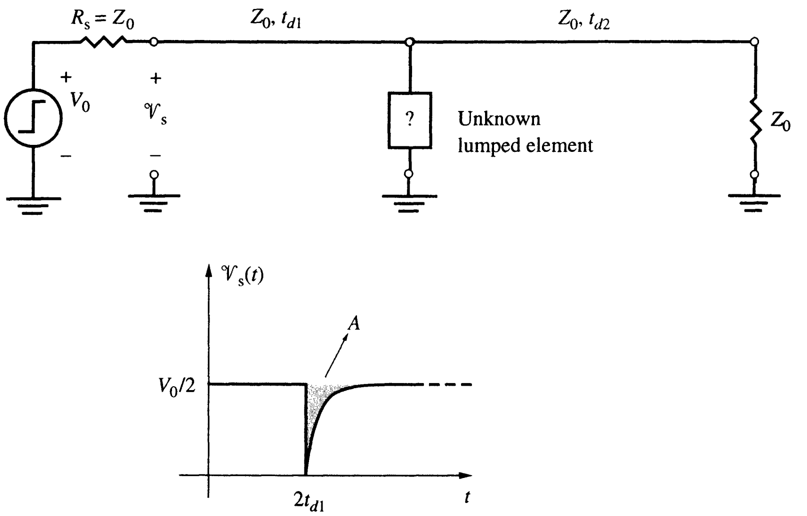

FIGURE 2.64. Unknown lumped element. Problem 2-30.

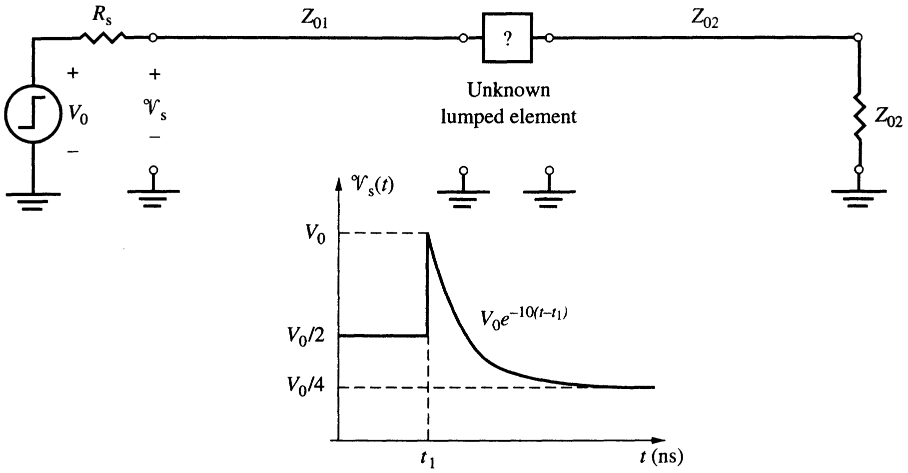

FIGURE 2.65. Unknown lumped element. Problem 2-31. Both t and t_1 are in ns.

with $R_L = Z_{02}$ at the far end, determine Z_{01}, Z_{02}, the type (e.g., inductance, capacitance, resistance) of the unknown circuit element, and its value (i.e., nH, pF, or Ω). (b) Find and sketch the voltage across this element.

2-32. Capacitive load. Two transmission lines of characteristic impedances 75Ω and 50Ω are joined by a connector that introduces a shunt resistance of 150Ω between the lines, as shown in Figure 2.66. The load end of the 50Ω line is terminated with a capacitive load with a 30 pF capacitor initially uncharged. The source end of the 75Ω line is excited by a step function of amplitude 3.6 V and a series resistance of 75Ω, starting

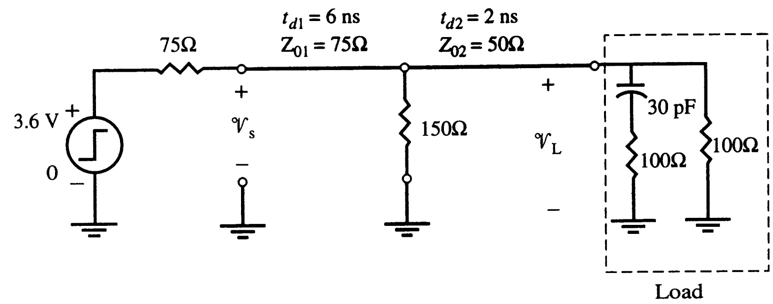

FIGURE 2.66. **Capacitive load.** Problem 2-32.

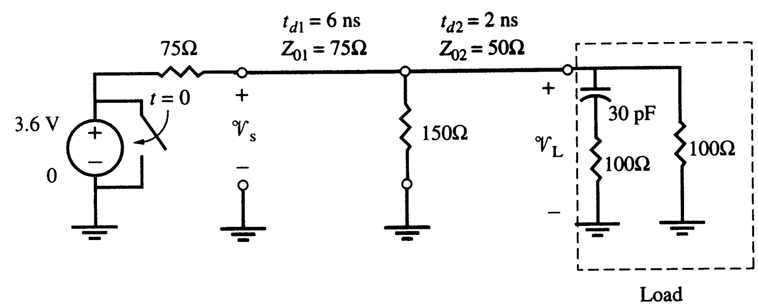

FIGURE 2.67. **Discharging with a capacitive load.** Problem 2-33.

at $t = 0$. Assuming the total time delay of each line to be $t_{d1} = 6$ ns and $t_{d2} = 2$ ns, respectively, find and sketch (a) the voltage $\mathcal{V}_L(t)$ at the load end of the 50Ω line and (b) the voltage $\mathcal{V}_s(t)$ at the source end of the 75Ω line.

2-33. Discharging with a capacitive load. In the circuit shown in Figure 2.67, the 3.6 V source voltage is shorted by a switch at $t = 0$, after being connected to the circuit for a long time. Find and sketch the source- and the load-end voltages $\mathcal{V}_s(t)$ and $\mathcal{V}_L(t)$.

2-34. Inductive load. Two transmission lines of characteristic impedances 50Ω and 75Ω are joined by a connector that introduces a series resistance of 25Ω between the lines, as shown in Figure 2.68. The load end of the 75Ω line is terminated with an inductive load. The inductor is initially uncharged. Find and sketch the voltage $\mathcal{V}_L$. First determine the initial and final values and accurately mark all points of your sketch.

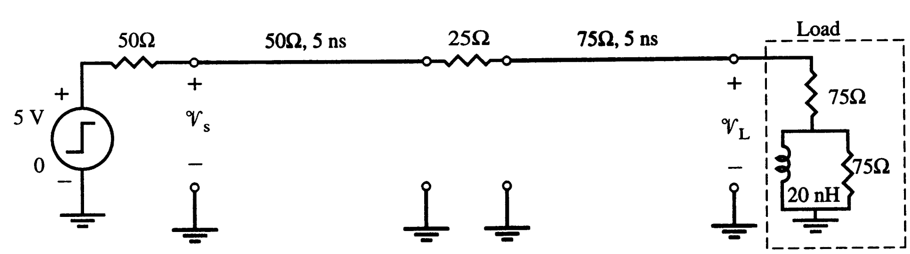

FIGURE 2.68. **Inductive load.** Problem 2-34.

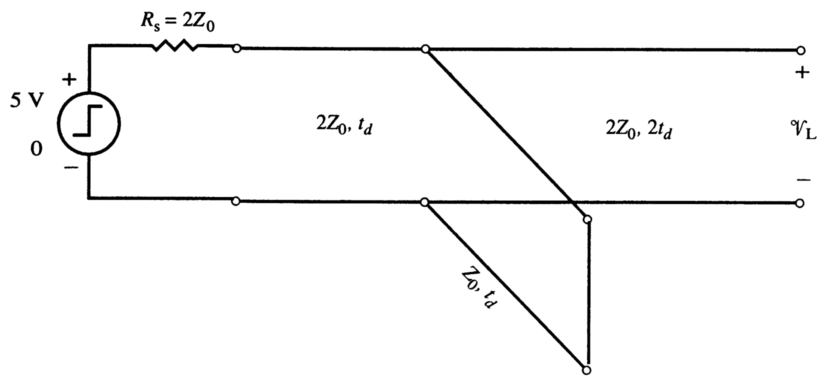

FIGURE 2.69. Step excitation. Problem 2-35.

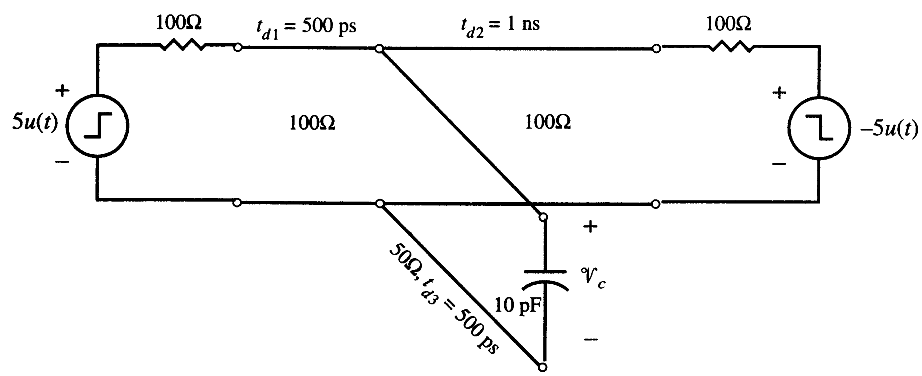

FIGURE 2.70. Capacitive load excited by two sources. Problem 2-36.

2-35. Step excitation. The circuit shown in Figure 2.69 is excited by a step-voltage source of amplitude 5 V and source resistance $R_s = 2Z_0$, starting at $t = 0$. Note that the characteristic impedance of the shorted stub is half that of the main line and that the second segment of the main line is twice as long, so its one-way time delay is $2t_d$. (a) Assuming the load to be an open circuit (i.e., a very large resistance), sketch the load voltage $\mathcal{V}_L(t)$ versus t for $0 \leq t \leq 11t_d$. (b) Repeat part (a) for the case when the input is a pulse of duration $t_w = 4t_d$.

2-36. Capacitive load excited by two sources. For the transmission line system shown in Figure 2.70, find the mathematical expression for the capacitor voltage $\mathcal{V}_c(t)$ and sketch it for $t > 0$. Assume the capacitor to be initially uncharged.

2-37. Two sources. For the circuit shown in Figure 2.71, sketch the voltages $\mathcal{V}_{s1}(t)$ and $\mathcal{V}_{s2}(t)$ for $0 \leq t \leq 7t_d$.

2-38. Nonlinear termination. Consider a 50Ω, 2-ns transmission line used to connect a driver logic gate to a load gate. At $t = 0$, the driver gate switches from LOW to HIGH state and can be modeled at HIGH state with an output voltage of 5 V in series with an output impedance of 10Ω. The load gate has a nonlinear voltage-current characteristic represented by

$$\mathcal{I}_L = 0.35(1 - e^{-\mathcal{V}_L/2})$$

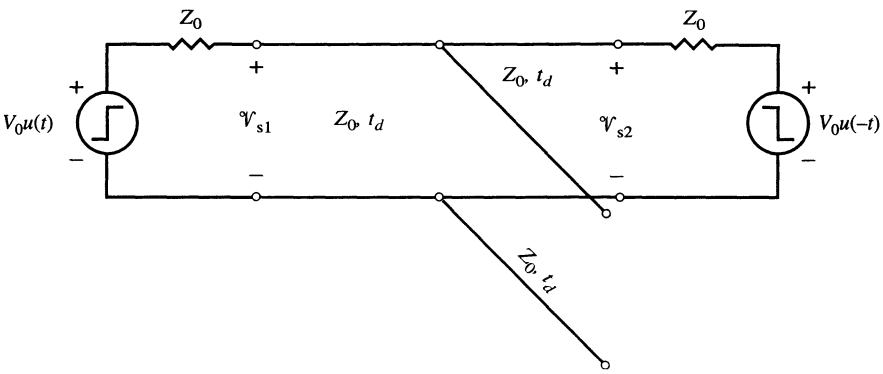

FIGURE 2.71. **Two sources.** Problem 2-37.

where $\mathcal{I}_L$ is in A and $\mathcal{V}_L$ is in V. Use the graphical Bergeron technique to determine approximately the time it would take for the circuit to reach a steady state.

2-39. Nonlinear source. A circuit consists of a driver gate connected to a load gate via a 50Ω, 2-ns transmission line. At $t = 0$, the driver gate switches from LOW state to HIGH state. The output of the driver at HIGH state has a nonlinear voltage-current characteristic represented by

$$\mathcal{I}_s = 8(5 - \mathcal{V}_s) - (5 - \mathcal{V}_s)^2$$

where $\mathcal{I}_s$ is in mA and $\mathcal{V}_s$ is in V. The load gate has a very large input impedance (~10 MΩ). Use the graphical Bergeron technique to (a) sketch the load voltage $\mathcal{V}_L$ versus t and (b) determine the approximate time it takes for the circuit to reach steady state. Assume the line to be uncharged at $t = 0$.

2-40. Effects of source rise time. Consider a lossless transmission line trace excited by a voltage source with output impedance 12.5Ω at one end and terminated by a short circuit at the other end. The characteristic impedance, the propagation delay, and the length of the trace are equal to 50Ω, 80 ps-(cm)$^{-1}$, and 25 cm, respectively. The source voltage increases linearly from zero at $t = 0$ to an amplitude of 5 V at $t = t_r$. (a) Find and sketch the source end voltage of the trace if the source rise time $t_r = 1$ ns. (b) Repeat part (a) for $t_r = 250$ ps.

2-41. Effects of source rise time. Consider a step voltage source of 3 V amplitude, 1 ns rise time, and 25Ω output impedance connected to a transmission line with $Z_0 = 50\Omega$ characteristic impedance and $C = 1$ pF-(cm)$^{-1}$ line capacitance terminated with a load resistance $R_L \gg Z_0$. Find and sketch the voltages at the two ends of the transmission line for a line length of (a) $l = 5$ cm, and (b) $l = 50$ cm. Compare the results and comment on the difference.

2-42. RG 8 coaxial line. A student buys an RG 8 low-loss coaxial cable from Radio Shack for a VHF antenna project. He looks up the specifications of the RG 8 coax in the Radio Shack product catalog and finds out that its characteristic impedance is $Z_0 = 50\ \Omega$, its velocity factor is $v_p/c = 0.66$, and its line capacitance is $C = 26.4$ pF-(ft)$^{-1}$. He then cuts a portion of this coax and measures the diameter of the inner conductor and the outer diameter of the dielectric to be approximately 2 mm and 7.5 mm, respectively. Using these values, find or verify the values of the unit length line parameters L, C, R, and G and the characteristic impedance Z_0 of this cable at 100 MHz. Note that the dielectric inside RG 8 coax is polyethylene and that the leakage conductance per unit length of a polyethylene-filled coaxial line at 100 MHz is approximately given by $G \simeq 1.58 \times 10^{-5}/\ln(b/a)$ S-m^{-1}.

2-43. Two-wire line. Calculate the per-unit-length line parameters L, C, R, and G and the characteristic impedance Z_0 of an air-insulated two-wire line made of copper wires with wire separation of 2.1 cm and wire diameter of 1.2 mm at a frequency of 200 MHz.

2-44. Distributed capacitive load. A transmission line system consists of a driver gate, a transmission line trace, and a load gate. The transmission line trace is $l = 25$ cm long and is characterized by the trace parameters $L = 2.46$ nH-(cm)$^{-1}$ and $C = 0.984$ pF-(cm)$^{-1}$. The driver output resistance is 20Ω for driving a HIGH-to-LOW signal and 25Ω for a LOW-to-HIGH one, and its driver output voltage is between 3.5 V (at HIGH state) and 0 V (at LOW state). The load gate has a very large input resistance of ~50 MΩ. Consider the case when the driver changes from HIGH to LOW state at $t = 0$. (a) Sketch the voltage at each end of the trace by using a bounce diagram. Neglect the fall time of the output voltage of the driver gate. (b) Repeat part (a) for the case in which the same circuit has an additional total load capacitance of $C_L = 15$ pF that is uniformly distributed along the length of the trace, and comment on the difference. (Hint: Combine the load capacitance C_L with the line capacitance C as if the combination is the new "effective" line capacitance of the trace.)