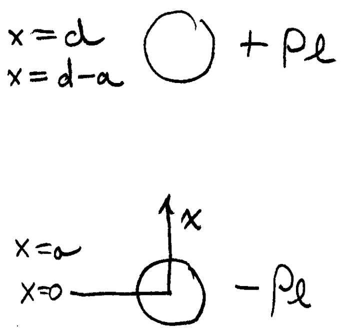
Example 4-29. Capacitance of a Two-Wire Line



Very useful problem for telephony, radio transmission lines, and power transmission.

General solution is complicated because "proximity effect" causes charge densities to be larger on facing sides

We will solve for $d \gg a$ to avoid this effect

This is a very difficult problem to solve for the potential directly.

See Inan & Inan, Engineering Electromagnetics, Example 4-11

The potential from a finite length of charge $-l$ to $+l$ is given by

$$\Phi(\underline{P}) = -\frac{\rho_l}{4\pi\epsilon_0} \ln\left[\frac{z-a+\left[r^2+(z-a)^2\right]^{\frac{1}{2}}}{z+a+\left[r^2+(z+a)^2\right]^{\frac{1}{2}}}\right]$$

You could compute $\underline{E} = -\underline{\nabla}\Phi$ from this function but it is VERY complex as shown in this example.

Furthermore, we are interested in infinite length lines where $l \to \infty$

See Paris & Hurd, Basic Electromagnetic Theory, Example 3-3

As $l \to \infty$ the arguement of the $\ln[\ ]$ function increases without limit and Φ is undefined. The problem is that our previous expressions for Φ, i.e. $\Phi = \frac{1}{4\pi\epsilon}\int_V \frac{\rho}{r}\,dv$, work only for bounded charge distributions.

The best method to find the potential associated with an infinite line charge is the integral form of Gauss' Law.

If we use this expression for the E field between the two conductors we can write an expression for the E field between the two conductors as

$$E_x(x,0,0) = \frac{-\rho_\ell}{2\pi\epsilon_0 x} \overset{\text{direction}}{-} \frac{\rho_\ell}{2\pi\epsilon_0 (d-x)}$$

ignores a

We will need to use the most general definition of capacitance

$$C \triangleq \frac{Q}{\Phi_{12}} = \frac{\oint \underline{D}\cdot d\underline{s}}{-\int_L \underline{E}\cdot d\underline{l}}$$

Gauss Law

defn of potential

Since these are infinite lines Q is readily given as ρ_ℓ, the charge per unit length.

Φ_{12} can be integrated as follows (remember we can use <u>any path</u>)

$$\Phi_{12} = -\frac{1}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{-\rho_\ell}{x} - \frac{\rho_\ell}{d-x}\right]dx$$

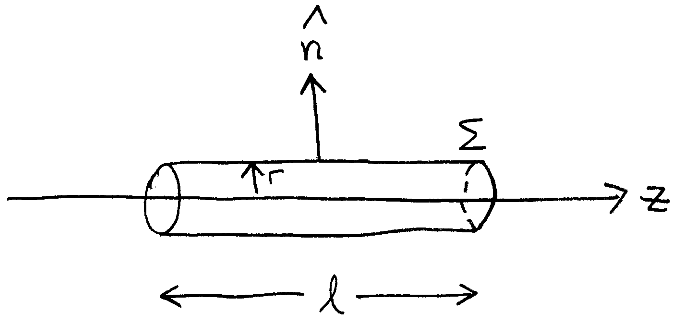
$$= \frac{\rho_\ell}{2\pi\epsilon_0}\int_a^{d-a}\left[\frac{1}{x} - \frac{1}{d-x}\right]dx = \frac{\rho_\ell}{2\pi\epsilon_0}\Big[\ln x - \ln(d-x)\Big]_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left(\frac{x}{d-x}\right)\Bigg|_{x=a}^{x=d-a}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\left[\ln\left(\frac{d-a}{+a}\right) - \ln\left(\frac{a}{d-a}\right)\right] = \frac{\rho_\ell}{2\pi\epsilon_0}\, 2\ln\left(\frac{d-a}{a}\right)$$

and, for $d \gg a$,

$$\Phi_{12} \cong \frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)$$



$$\oint_S \underline{D} \cdot \hat{n}\, ds = \int \rho\, d\tau = Q_{enclosed}$$

By symmetry $\quad D_r \cdot 2\pi r \cdot l = \rho_l\, l \quad$ where ρ_l is the line charge density

$$D_r = \frac{\rho_l}{2\pi r}$$

$$\underline{D} = \frac{\rho_l}{2\pi r}\hat{r}$$

$$\underline{E} = \frac{\rho_l}{2\pi \epsilon_0 r}\hat{r}$$

Sine $\underline{E} = -\underline{\nabla}\Phi$ and the field is circularly symmetric, i.e.

$\frac{\partial \Phi}{\partial \phi} \to 0$ and $\frac{\partial \Phi}{\partial z} = 0$ since it does not matter where

we put our origin.

$$\frac{\rho_l}{2\pi\epsilon_0 r}\hat{r} = \underline{E} = -\underline{\nabla}\Phi = -\frac{\partial \Phi}{\partial r}\hat{r}$$

or, $\quad \Phi(r) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r + c$

If we pick a reference potential $\Phi(r=r_0)=0$

$$0 = \Phi(r=r_0) = -\frac{\rho_l}{2\pi\epsilon_0}\ln r_0 + c$$

and $\Phi(r) = \frac{\rho_l}{2\pi\epsilon_0}\ln\left(\frac{r_0}{r}\right)$

We can compute the capacitance per unit length as

$$C \cong \frac{\rho_\ell}{\frac{\rho_\ell}{\pi\epsilon_0}\ln\left(\frac{d}{a}\right)} = \frac{\pi\epsilon_0}{\ln\left(\frac{d}{a}\right)}$$

For example, a 115kV transmission line uses two 1.407cm aluminum conductors separated by 3 meters

$$\text{or } C \cong \frac{\pi\left(8.854\times 10^{-12}\ F/m\right)}{\ln\left(\frac{3}{.01407}\right)} = 5.19\ nF/km.$$

Hint for future problems

You can compute C for conductor configurations for which you have derived $\underline{E}$ or Φ. For example, a single cylindrical conductor above a ground plane is that of this twin line with a infinitely large conducting sheet between the two conductors. You can also use the "method of images".

Vector potential

Others have proven that Maxwell's Equations are satisified if

$$\underline{\nabla} \cdot \underline{A} = -\mu\epsilon \frac{\partial \Phi}{\partial t}$$

For static (time-independent) fields $\frac{\partial}{\partial t} \to 0$ and $\underline{\nabla} \cdot \underline{A} = 0$

Then, the vector potential is defined by

$$\underline{\nabla} \times \underline{B} = -\nabla^2 \underline{A} = \mu \underline{J}$$

We get three component equations

$$\nabla^2 A_x = -\mu J_x$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

which is similar to Poisson's Equation.

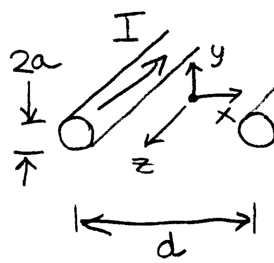
The general solution is $A_x = \int \frac{\mu J_x \, dv}{4\pi r}$

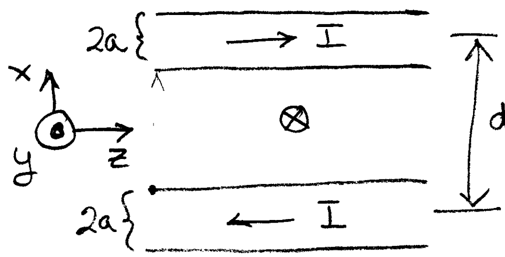
etc.

In general $\underline{A} = \int \frac{\mu \underline{J} \, dv}{4\pi r}$

This is known as the Green's function solution for $\underline{A}$

6-40 Determine the inductance per unit length of a two-wire transmission line in air as shown below, designed for an amateur radio transmitter, with conductor radius $a = 1mm$ and spacing $d = 6cm$.





Superimpose the field from each conductor modeling the fields as those from two infinitely long parallel lines

$$B_y \cong -\frac{\mu_0 I}{2\pi \underbrace{(x+\frac{d}{2})}_{\text{positive}}} + \frac{\mu_0 I}{2\pi \underbrace{(x-\frac{d}{2})}_{\text{negative}}}$$

The flux linked by this circuit of two conductors over a length ℓ is approximated by

$$\Phi = \int_S \underline{B} \cdot d\underline{s} = \frac{\mu_0 I \ell}{2\pi} \int_{-\frac{d}{2}+a}^{+\frac{d}{2}-a} \left[\frac{1}{x+\frac{d}{2}} - \frac{1}{x-\frac{d}{2}} \right] dx$$

$$= \frac{\mu_0 I \ell}{2\pi} \left[\ln(x+\frac{d}{2}) - \ln(x-\frac{d}{2}) \right]_{a-\frac{d}{2}}^{-a+\frac{d}{2}}$$

$$= \frac{\mu_0 I \ell}{2\pi} \left[\ln(-a+d) - \ln(a) - \ln(-a) + \ln(a-d) \right]$$

$$= \frac{\mu_0 I \ell}{2\pi} \left[\ln(d-a) + \ln(a-d) - \ln(a) - \ln(-a) \right]$$

$$= \frac{\mu_0 I \ell}{2\pi} \, 2\left[\ln(d-a) - \ln(a) \right] = \frac{\mu_0 I \ell}{\pi} \ln\left(\frac{d-a}{a}\right)$$

Since $d \gg a$ $\quad \Psi = \frac{\mu_0 I l}{\pi} \ln\left(\frac{d}{a}\right)$

$$\therefore L = \frac{\Psi}{I} = \frac{\mu_0 l}{\pi} \ln\left(\frac{d}{a}\right)$$

(a) $\frac{L}{l} = \frac{\mu_0}{\pi} \ln\left(\frac{d}{a}\right) = \frac{4\pi \times 10^{-7}}{\pi} \ln\left(\frac{60\,mm}{1\,mm}\right) = 1.64 \times 10^{-6} \frac{H}{m}.$

(b) Repeat part (a) if the conductor spacing is doubled (i.e.) $d = 12cm$.

$$\frac{L}{l} = \frac{\mu_0}{\pi} \ln\left(\frac{d}{a}\right) = \frac{4\pi \times 10^{-7}}{\pi} \ln\left(\frac{120\,mm}{1\,mm}\right) = 1.915 \times 10^{-6} \frac{H}{m}.$$