Chapter 3 Steady-State Waves on Transmission Lines

We can still use the lossless transmission line equations.

$$\frac{\partial \mathcal{V}}{\partial z} = -L\frac{\partial \mathcal{I}}{\partial t}$$

$$\frac{\partial \mathcal{I}}{\partial z} = -C\frac{\partial \mathcal{V}}{\partial t}$$

use phasors for sinusoidal steady state

assume $\mathcal{V}(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\}$

$\mathcal{I}(z,t) = \mathrm{Re}\left\{I(z)e^{j\omega t}\right\}$

Substituting

$$\frac{dV(z)}{dz}e^{j\omega t} = -Lj\omega\, I(z)e^{j\omega t}$$

$$\frac{dI(z)}{dz}e^{j\omega t} = -Cj\omega\, V(z)e^{j\omega t}$$

Simplifying $\dfrac{dV(z)}{dz} = -j\omega L\, I(z)$

$$\frac{dI(z)}{dz} = -j\omega C\, V(z)$$

Combining

$$\frac{d^2V(z)}{dz^2} = -j\omega L\frac{dI(z)}{dz} = -j\omega L\left(-j\omega C\, V(z)\right)$$

$$\frac{d^2V(z)}{dz^2} = -\omega^2 LC\, V(z) = -\beta^2 V(z) \quad \text{where } \beta = \omega\sqrt{LC}$$

$$\frac{d^2V(z)}{dz^2} = -\beta^2 V(z)$$ complex wave equation

$\beta = \omega\sqrt{LC}$ phase constant

You also get

$$\frac{d^2I(z)}{dz^2} = -\beta^2 I(z)$$

The solutions are

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$I(z) = I^+ e^{-j\beta z} + I^- e^{+j\beta z} = \frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z}$$

$$\text{where } Z_0 = \frac{V^+}{I^+} = -\frac{V^-}{I^-} = \sqrt{\frac{L}{C}}$$

Equivalent time expressions are

$$\mathcal{V}(z,t) = \mathrm{Re}\left\{V(z)\, e^{j\omega t}\right\} = \mathrm{Re}\left\{V^+ e^{-j\beta z} e^{j\omega t} + V^- e^{+j\beta z} e^{j\omega t}\right\}$$

$$\mathcal{V}(z,t) = V^+ \cos(\omega t - \beta z) + V^- \cos(\omega t + \beta z)$$

You can also get

$$\mathcal{I}(z,t) = \frac{V^+}{Z_0} \cos(\omega t - \beta z) - \frac{V^-}{Z_0} \cos(\omega t + \beta z)$$

Consider an infinitely long transmission line

Then $\mathcal{V}(z,t) = V^+ \cos(\omega t - \beta z)$

$$\mathcal{I}(z,t) = \frac{V^+}{Z_0} \cos(\omega t - \beta z)$$

The wave propagates in the $+z$ direction at a velocity v_p

$$\omega t - \beta z = \text{constant}$$

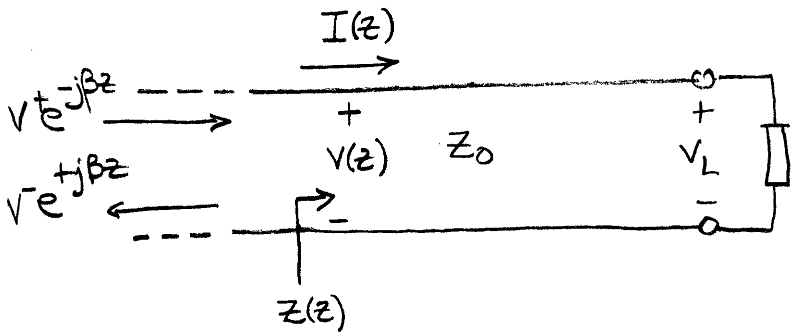
The is the velocity at which each point propagates.

$$\omega\, dt - \beta\, dz = 0 \quad \text{or} \quad \omega\, dt = \beta\, dz$$

$$v_p = \frac{dz}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}}$$

The phase velocity is usually <u>NOT</u> a function of the geometry of the transmission line, but is a function of the material.

3.2. Open & Short Circuited Lines



The total voltages and currents on the line are

$$V(z) = V^{+}e^{-j\beta z} + V^{-}e^{+j\beta z}$$

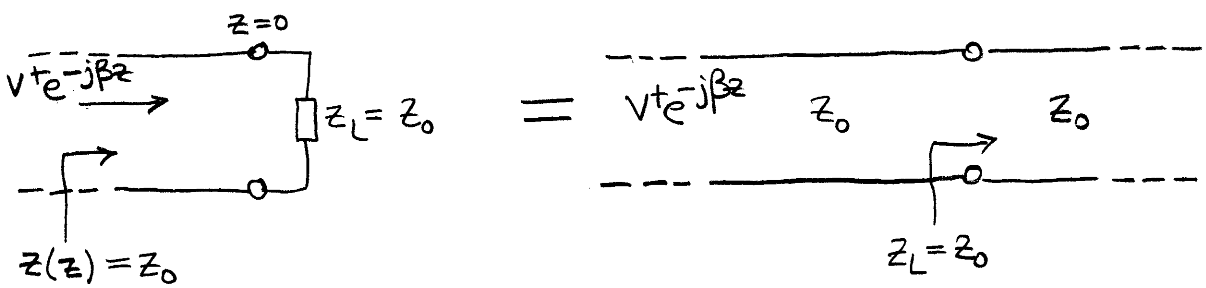
$$I(z) = \frac{V^{+}}{Z_0}e^{-j\beta z} - \frac{V^{-}}{Z_0}e^{+j\beta z}$$

We can write the line impedance anywhere z on the line looking towards the load

$$Z(z) \equiv \frac{V(z)}{I(z)} = Z_0 \frac{V^{+}e^{-j\beta z} + V^{-}e^{+j\beta z}}{V^{+}e^{-j\beta z} - V^{-}e^{+j\beta z}} = R(z) + jX(z)$$

Example 3-1 Matched Load

A transmission line is terminated with a load Z_0. Find the line impedance $Z(z)$ and the magnitude of the reflected wave V^{-}

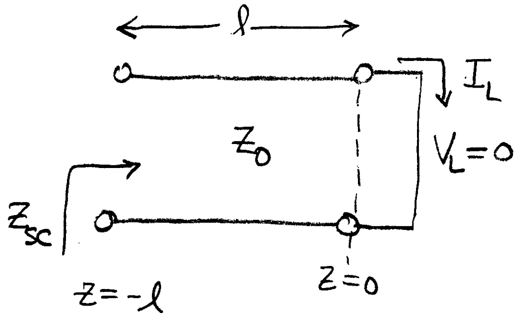


at $z=0$ the load boundary condition looks just like the input to an infinitely long transmission line of impedance Z_0

the impedance at any point $Z(z) = Z_0$

there is no reflected wave since the boundary condition at the load is satisified without it.

3.2.1 Short Circuited Line



at the load $V_L = \left[V^+ e^{-j\beta z} + V^- e^{+j\beta z} \right]_{z=0} = V^+ + V^- = 0 \Rightarrow V^- = -V^+$

at the load $I_L = \left[\frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z} \right]_{z=0} = \frac{1}{Z_0} V^+ - \frac{V^-}{Z_0} = \frac{2V^+}{Z_0}$

Everywhere else on the line

$$V(z) = V^+ \left(e^{-j\beta z} - e^{+j\beta z} \right) = -2V^+ j \sin(\beta z)$$

$$I(z) = \frac{V^+}{Z_0} \left(e^{-j\beta z} + e^{j\beta z} \right) = + \frac{2V^+}{Z_0} \cos(\beta z)$$

where we use the Euler identity $e^{j\theta} = \cos\theta + j\sin\theta$

The space-time function forms a standing wave

$$\mathcal{V}(z,t) = \mathrm{Re}\left\{ V(z) e^{j\omega t} \right\} = \mathrm{Re}\left\{ -2V^+ j \sin(\beta z)\, e^{j\omega t} \right\}$$

$$= \mathrm{Re}\left\{ +2V^+ e^{-j\frac{\pi}{2}} \sin\beta z\, e^{j\omega t} \right\}$$

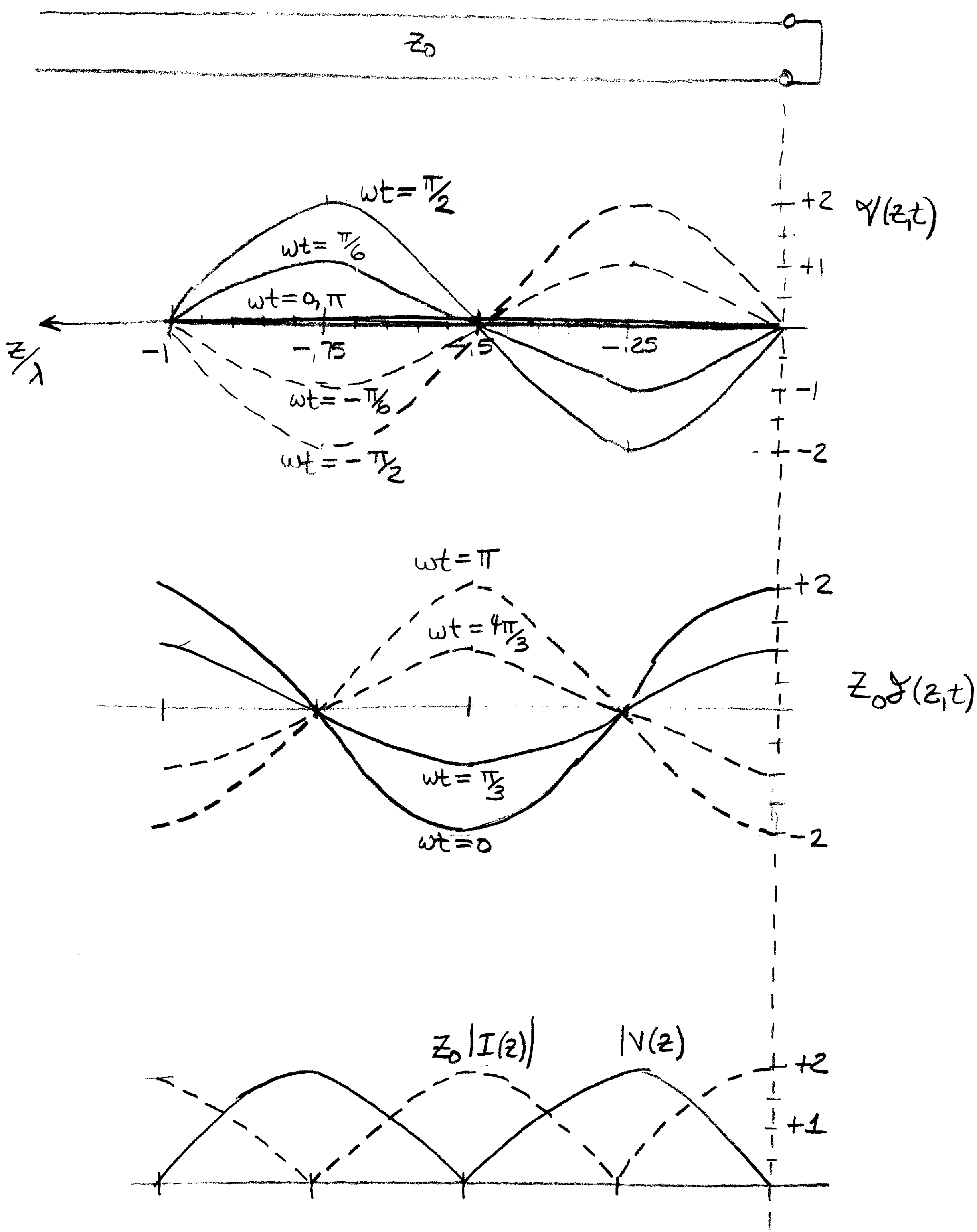
$$\mathcal{V}(z,t) = 2V^+ \sin\beta z \cos\left(\omega t - \frac{\pi}{2}\right)$$

If V^+ is complex, i.e. $V^+ = |V^+| e^{j\phi^+}$ this becomes

$$\mathcal{V}(z,t) = 2|V^+| \sin\beta z \cos\left(\omega t - \frac{\pi}{2} + \phi^+\right)$$

You can repeat this for the current to get

$$\mathcal{I}(z,t) = \frac{2|V^+|}{Z_0} \cos(\beta z) \cos(\omega t)$$

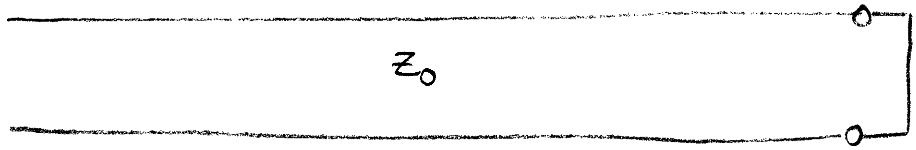


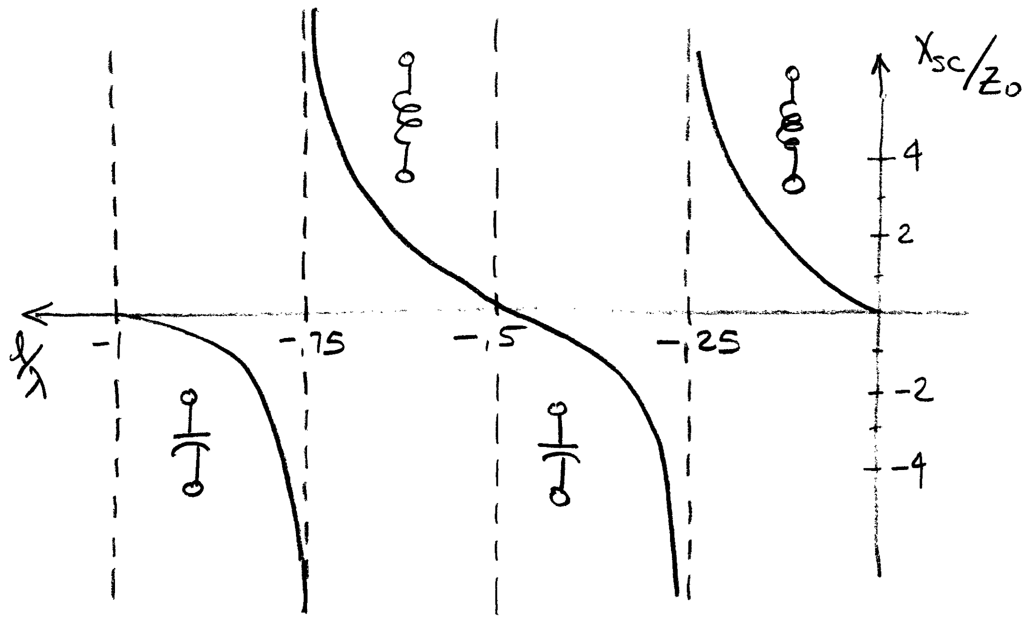
The line impedance is

$$Z(z) = \frac{V(z)}{I(z)} = Z_0 \frac{-2jV^+\sin(\beta z)}{2V^+\cos(\beta z)} = -jZ_0\tan(\beta z)$$

The input impedance where $z = -\ell$ is given by

$$Z_{sc} = jZ_0 \tan(\beta\ell) = jX_{sc}$$





Example 3-2 short circuit antenna twin-lead
$\ell = 10\text{cm}$, $Z_0 = 300\Omega$, shorted at one end
Find the input impedance at 300 MHz

$$\lambda = \frac{v_p}{f} = \frac{3\times10^8}{300\times10^6} = 1\text{m}.$$

$$\frac{\ell}{\lambda} = \frac{0.1\text{m}}{1\text{m}} = 0.1$$

$$Z_{sc} = jZ_0\tan(\beta\ell) = j(300)\tan\left(\frac{2\pi}{\lambda}\cdot 0.1\lambda\right) \cong j218\,\Omega$$

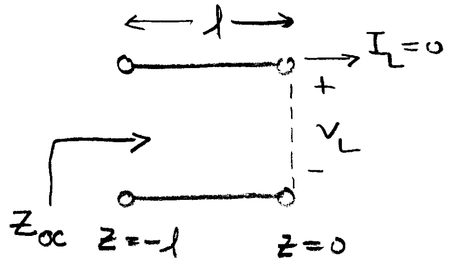
The equivalent lumped impedance is

$$X_{sc} = \omega L_{sc} = 218$$

$$L_{sc} = \frac{X_{sc}}{\omega} = \frac{218}{2\pi(300\times10^6)} = 0.116\,\mu\text{H}$$

3.2.2. Open circuited line

Analysis similar to short circuited line



at the load $I_L = I(z=0) = \left[\frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z} \right]_{z=0}$

$$I_L = \frac{V^+ - V^-}{Z_0} = 0 \quad \Rightarrow V^+ = V^-$$

The load voltage can be computed as

$$V_L = V(z=0) = \left[V^+ e^{-j\beta z} + V^- e^{+j\beta z} \right]_{z=0} = V^+ + V^- = 2V^+$$

We can use $V^+ = V^-$ to determine V and I anywhere along the line

$$V(z) = V^+ e^{-j\beta z} + V^+ e^{+j\beta z} = 2V^+ \cos(\beta z)$$

$$I(z) = \frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^+}{Z_0} e^{+j\beta z} = -2j \frac{V^+}{Z_0} \sin(\beta z) = +2e^{-j\frac{\pi}{2}} \frac{V^+}{Z_0} \sin(\beta z)$$

In the time domain

$$\mathcal{V}(z,t) = \mathrm{Re}\left\{ V(z) e^{j\omega t} \right\} = 2|V^+| \cos(\beta z) \cos(\omega t)$$

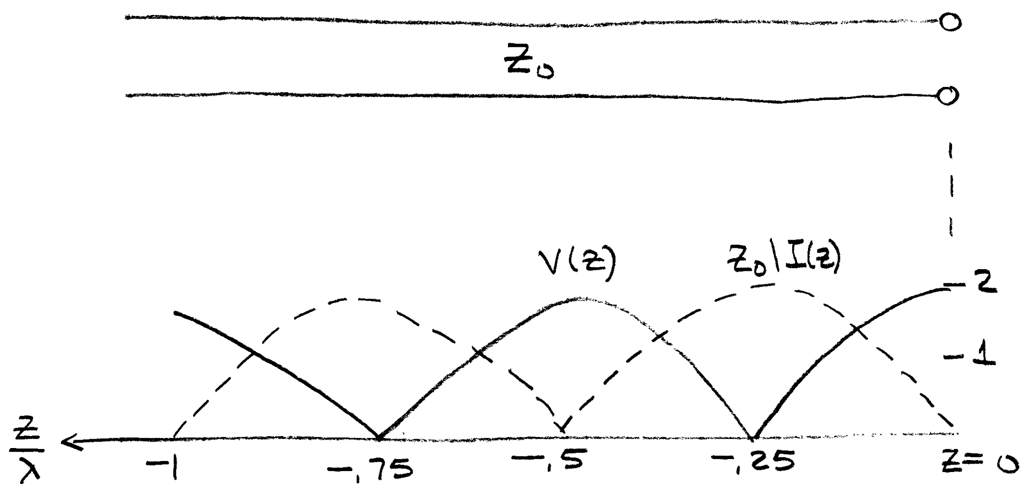
$$\mathcal{I}(z,t) = \mathrm{Re}\left\{ I(z) e^{j\omega t} \right\} = 2\frac{|V^+|}{Z_0} \sin(\beta z) \cos\left(\omega t - \frac{\pi}{2}\right)$$

where $V^+ = |V^+| e^{j\phi^+}$. Assumed $\phi^+ = 0$

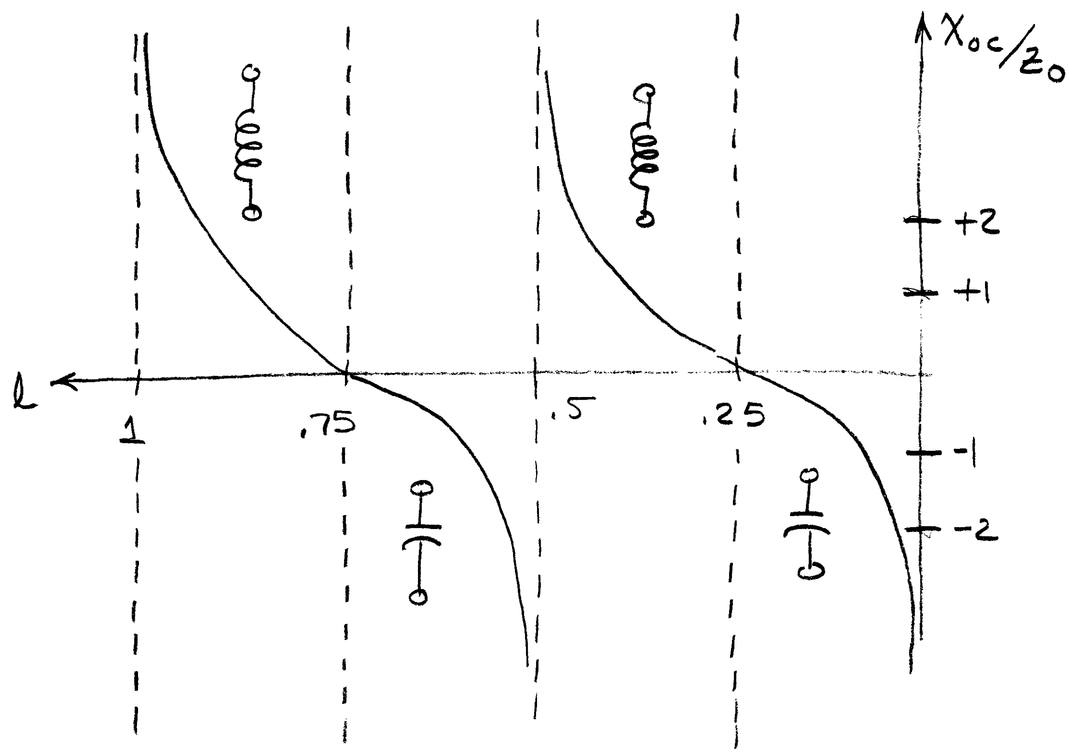
The line impedance looking towards the open is

$$Z(z) = \frac{V(z)}{I(z)} = +j Z_0 \cot(\beta z)$$

$$Z_{oc} = Z(z=-\ell) = -j Z_0 \cot(\beta \ell) = j X_{oc}$$



This is very similar to the results for the short circuited line, except they are 90° different. We have $I(z) = 0$ at $z = 0$



Just like for the short circuit, an open circuit can look capacitive or inductive depending upon the length ℓ of the line.

Example 3-3

Consider a $l = 20cm$ length of television lead-in ($Z_0 = 300\Omega$)
Assume it is connected to nothing (i.e., open circuit)
What is the input impedance of this line at 300 MHz?

For an open

$$Z_{oc} = -j Z_0 \cot(\beta l) = -j Z_0 \cot\left(\frac{2\pi}{\lambda} l\right)$$

The electrical length is calculated as

$$\lambda = \frac{v_p}{f} = \frac{3\times10^8}{300\times10^6} = 1m$$

$$\frac{l}{\lambda} = \frac{0.20\,m}{1m} = 0.2$$

$$\therefore Z_{oc} = -j\,300\cot(0.2\cdot 2\pi) = -j\,97.476\,\Omega$$

As the frequency increases we can start to use transmission lines to replace lumped inductors and capacitors
Also as f increases the size of the transmission lines decreases. Typically we can use transmission lines as circuit elements from 1 to 100 GHz.

Example 3-4

A short-circuited transmission line with $v_p = 2.07\times10^8$ m/s is to be designed to provide a 15nH inductance at 3GHz.
(a) Find the shortest possible length l if $Z_0 = 50\Omega$.

The impedance of a 15nH inductor at 3 GHz is

$$Z = j\omega L = j\,2\pi(3\times10^9)(15\times10^{-9}) = j\,282.75\,\Omega$$

The impedance at the input of a short circuited line is

$$Z_{sc} = j Z_0 \tan\left(\frac{2\pi}{\lambda}l\right)$$

The wavelength is $\lambda = \frac{v_p}{f} = \frac{2.07\times10^8}{3\times10^9} = 0.069\,m$

We can then calculate l as

$$j282.75 = j50\tan(91.061\,l)$$
$$\tan(91.061\,l) = 5.655$$
$$91.061\,l = 1.396$$
$$l = 1.53\,cm$$

(b) What is the equivalent lumped element at 4GHz of this line?
At 4GHz $\lambda = \frac{v_p}{f} = \frac{2.07\times10^8}{4\times10^9} = 0.052\,m$

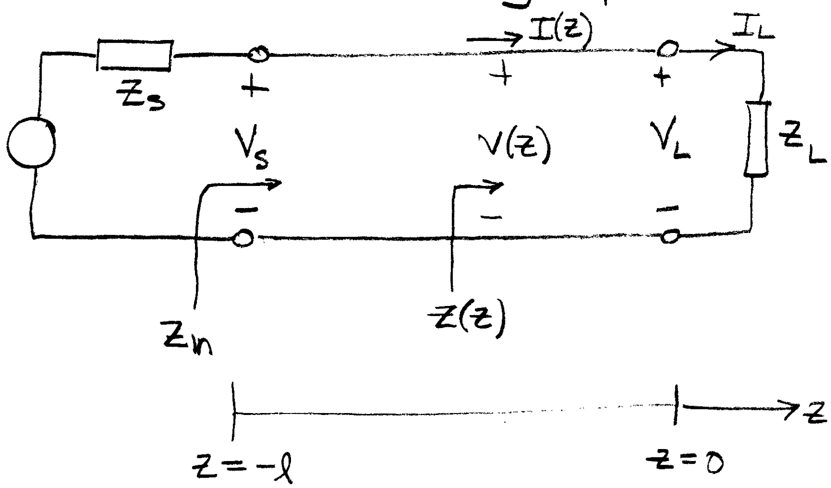
$$Z_{sc} = j50\tan\left(\frac{2\pi}{0.052}\cdot .0153\right) = j\,50\,(-3.51) = -j175.26\,\Omega$$

The equivalent capacitance is

$$\frac{1}{j\omega C} = -j\frac{1}{2\pi(4\times10^9)C} = -j175.26$$

$$\therefore C = \frac{1}{2\pi(4\times10^9)(175.26)} = \frac{1}{4.40\times10^9} = 0.227\,pf.$$

3.3 Lines terminated in an arbitrary impedance



The voltage and current anywhere on the line are

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$I(z) = \frac{1}{Z_0}\left(V^+ e^{-j\beta z} - V^- e^{+j\beta z}\right)$$

At the load

$$V_L = Z_L I_L$$

$$V(z)\Big|_{z=0} = Z_L\, I(z)\Big|_{z=0}$$

$$V^+ + V^- = Z_L \frac{1}{Z_0}(V^+ - V^-)$$

$$Z_L = Z_0 \frac{V^+ + V^-}{V^+ - V^-}$$

The voltage reflection coefficient at the load is defined as $\Gamma_L = \frac{V^-}{V^+}$

Solving the load relationship for $\frac{V^-}{V^+}$ gives

$$Z_L V^+ - Z_L V^- = Z_0 V^+ + Z_0 V^-$$

$$Z_L V^+ - Z_0 V^+ = Z_L V^- + Z_0 V^-$$

$$\frac{Z_L - Z_0}{Z_L + Z_0} = \frac{V^-}{V^+} = \Gamma_L$$

Since Z_L and Z_0 can be complex, Γ_L is in general complex

We can use Γ_L to re-write the expressions for the phasors on the line as

$$V(z) = V^+\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right) = V^+ e^{-j\beta z}\left(1 + \Gamma_L e^{j2\beta z}\right)$$

$$V(z) = V^+ e^{-j\beta z}\left(1 + \Gamma(z)\right)$$

where we define $\Gamma(z) = \dfrac{V^- e^{j\beta z}}{V^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)}$

The current can be written in the same way

$$I(z) = \frac{V^+}{Z_0}\left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right) = \frac{V^+}{Z_0} e^{-j\beta z}\left(1 - \Gamma_L e^{j2\beta z}\right)$$

$$I(z) = \frac{V^+}{Z_0} e^{-j\beta z}\left(1 - \Gamma(z)\right)$$

$$\Gamma(z) = \Gamma_L e^{j2\beta z}$$

is a generalized reflection coefficient which is valid at every point z on a transmission line.

A very useful relationship exists between the line impedance and the reflection coefficient at any position

$$Z(z) \equiv \frac{V(z)}{I(z)} = \frac{V^+ e^{-j\beta z}\left(1 + \Gamma(z)\right)}{\frac{V^+ e^{-j\beta z}}{Z_0}\left(1 - \Gamma(z)\right)} = Z_0 \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

This will be useful for understanding Smith Charts.

3.3.1 Voltage and current standing wave patterns

$$V(z) = V^+\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right)$$

Now let the load be resistive so that $\Gamma_L = \rho e^{j\psi} \to \rho$

$$V(z) = V^+\left(e^{-j\beta z} \pm \rho e^{j\beta z}\right)$$

↑

can be positive or negative even with R loads

$$V(z) = V^+\left(e^{-j\beta z} \underbrace{\pm \rho e^{-j\beta z} \mp \rho e^{-j\beta z}}_{\text{add + and − quantities}} \pm \rho e^{j\beta z}\right)$$

Reorganize

$$V(z) = V^+\left[e^{-j\beta z}(1 \pm \rho) \pm \rho\left(-e^{-j\beta z} + e^{j\beta z}\right)\right]$$

Multiply by $e^{j\omega t}$ and convert to time domain

$$v(z,t) = \underbrace{|V^+|(1 \pm \rho)\cos(\omega t - \beta z + \phi^+)}_{\text{This is a traveling (propagating) wave}} \pm \underbrace{|V^+|\, 2\rho \sin\beta z \cos\left(\omega t + \phi^+ + \frac{\pi}{2}\right)}_{\text{This is a standing wave. It is stationary in space.}}$$

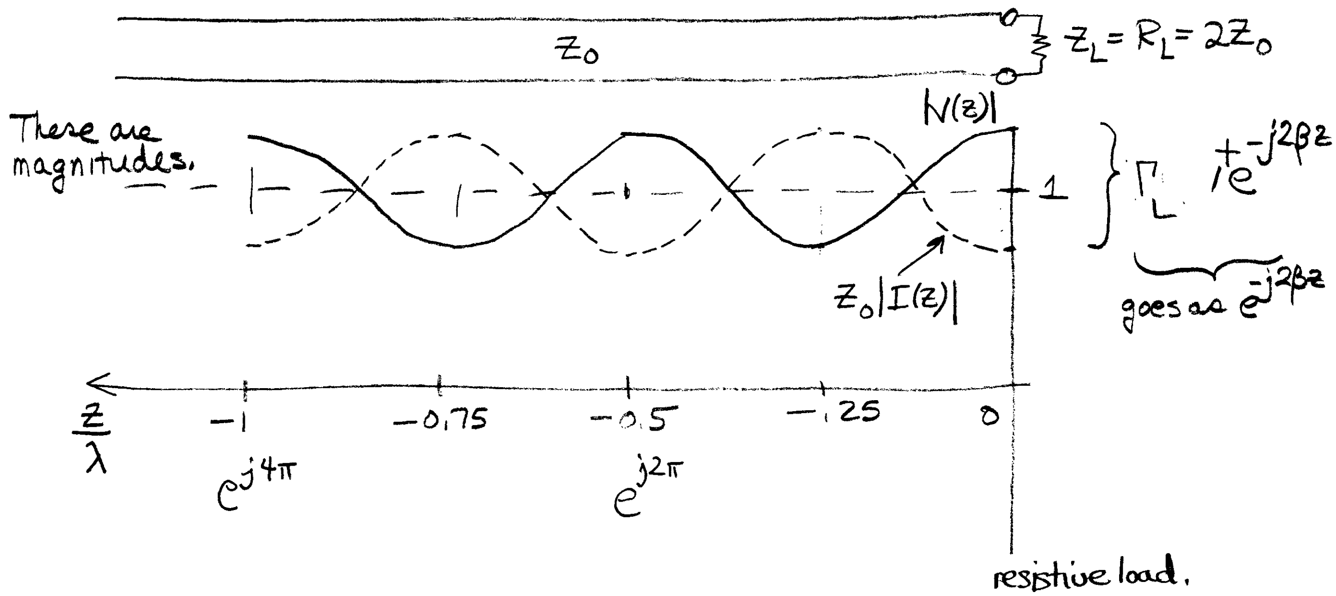
where we have assummed $V^+ = |V^+| e^{j\phi^+}$

It is important to consider the maximum and minimum values of $|V(z)|$

We can write $V(z) = V^+ e^{-j\beta z}\left(1 + \Gamma(z)\right)$ where $\Gamma(z) = \Gamma_L e^{j2\beta z} = \rho e^{j(\psi + 2\beta z)}$

$$\left.\begin{aligned} V_{max} &= |V(z)|_{max} = |V^+|\left(1 + |\Gamma_L|\right) = |V^+|(1+\rho) \\ V_{min} &= |V(z)|_{min} = |V^+|\left(1 - |\Gamma_L|\right) = |V^+|(1-\rho) \end{aligned}\right\} \text{ since } |\Gamma(z)| = |\Gamma_L| = \rho$$

The magnitude of V will vary sinusoidally from the load with a variation $\pm\rho$



For $Z_L = 2Z_0 = R_L$ we have

$$\Gamma_L = \frac{2Z_0 - Z_0}{2Z_0 + Z_0} = \frac{Z_0}{3Z_0} = \frac{1}{3} = \rho$$

$$V(z) = V^+ e^{-j\beta z}\left[1 + \Gamma(z)\right]$$

$$\Gamma(z) = \Gamma_L e^{j2\beta z} = \frac{1}{3} e^{j2\beta z}$$ where $\psi = 0$ since Z_L = real

$$\underbrace{e^{j2\frac{2\pi z}{\lambda}}} = e^{j\frac{4\pi}{\lambda}z}$$

You can also plot I

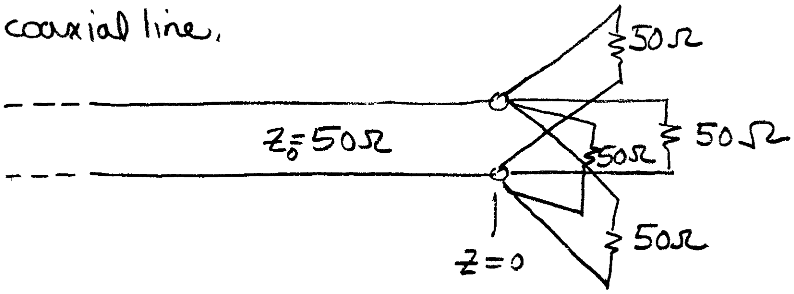
$$I(z) = \frac{V^+}{Z_0} e^{-j\beta z} \underbrace{\left[1 - \Gamma(z)\right]}$$

This has the same variation as $V(z)$ but the sign of Γ is reversed

Plot $I(z)Z_0$ since this is to the same scale as $V(z)$

Example 3-5

Four Yagi antennas with a feed point impedance of 50Ω are stacked in parallel on an antenna tower and connected to the transmitter by a 50Ω coaxial line.



(a) Calculate the load reflection coefficient Γ_L

put sign in exponent

$$Z_L = \frac{50}{4} = 12.5\Omega$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{12.5-50}{12.5+50} = \frac{-37.5}{62.5} = -0.6 = 0.6\,e^{j180^\circ}$$

(b) Calculate V_{max}, V_{min}, I_{max}, I_{min} assuming $V^+ = 1V$

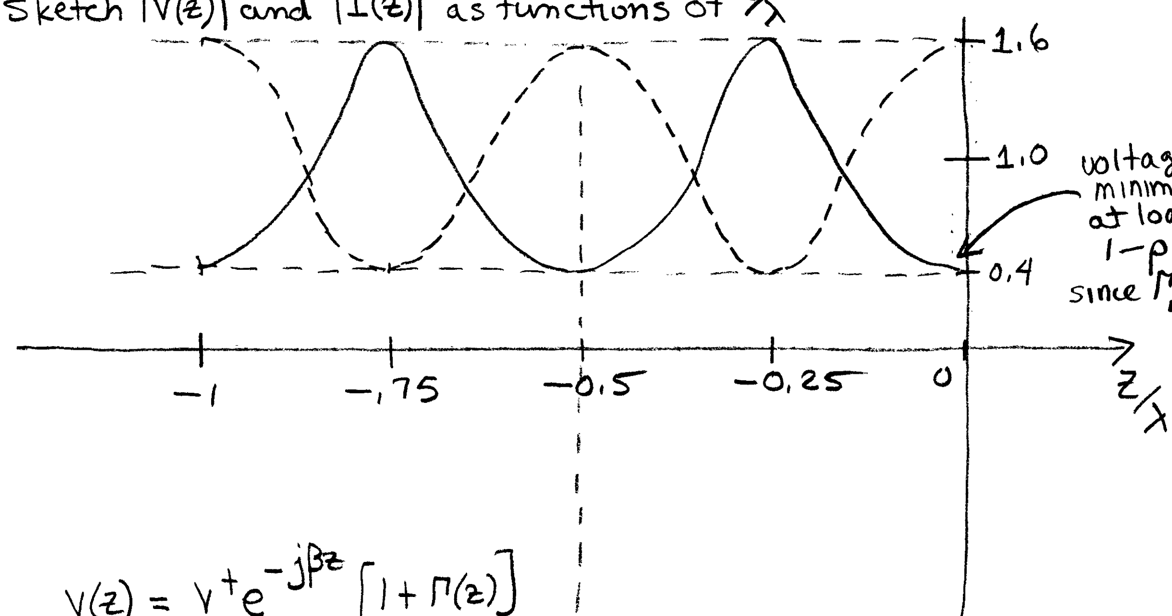
$$V_{max} = V^+(1+\rho) = 1(1+.6) = 1.6V$$

$$V_{min} = V^+(1-\rho) = 1(1-.6) = 0.4V$$

$$I_{max} = \frac{V^+}{Z_0}(1+\rho) = \frac{1(1+.6)}{50} = 32\,mA$$

$$I_{min} = \frac{V^+}{Z_0}(1-\rho) = \frac{1(1-.4)}{50} = 8\,mA$$

(c) Sketch $|V(z)|$ and $|I(z)|$ as functions of z/λ



voltage at minimum at load $1-\rho$ since $\Gamma_L < 0$

$$V(z) = V^+ e^{-j\beta z}\left[1 + \Gamma(z)\right]$$

$$V_L = V(z=0) = 1\left[1-0.6\right] = 0.4 \text{ volts.}$$

Current is out of phase so it starts at

$$I_L = I(z=0) = \frac{1}{50}\left[1+.6\right] = 32\,mA$$

$$Z_0 I_L = 1.6$$

Example 3-6 UHF Blade Antenna

The measured values of the feed point impedance are

Z_L	f
$22.5-j51$	225 MHz
$35-j16$	300
$45-j2.5$	400

The antenna is fed with a 50Ω coaxial line.

Calculate the load reflection coefficient and the standing wave ratio S on the line at

(a) 225 MHz

$$\Gamma_L = \frac{Z_L-Z_0}{Z_L+Z_0} = \frac{22.5-j51-50}{22.5-j51+50} = \frac{-27.5-j51}{72.5-j51} = +0.08-j0.65$$

$$\Gamma_L = 0.65e^{-j83.2^\circ}$$

The standing wave ratio is the ratio of V_{max} to V_{min} and is given by

$$S = \frac{1+\rho}{1-\rho} = \frac{1+0.65}{1-0.65} = 4.71$$

magnitude

(b) 300 MHz

$$\Gamma_L = \frac{Z_L-Z_0}{Z_L+Z_0} = \frac{35-j16-50}{35-j16+50} = \frac{-15-j16}{85-j16} = -0.14-j0.21$$

$$\Gamma_L = 0.25\,e^{-j122.5^\circ}$$

$$S = \frac{1+\rho}{1-\rho} = \frac{1+.25}{1-.25} = \frac{1.25}{.75} = 1.67$$

(c) $$\Gamma_L = \frac{Z_L-Z_0}{Z_L+Z_0} = \frac{45-j2.5-50}{45-j2.5+50} = \frac{-5-j2.5}{95-j2.5} = -.05-j0.03$$

$$\Gamma_L = 0.06e^{-j151.93}$$

$$S = \frac{1+\rho}{1-\rho} = \frac{1+.06}{1-.06} = 1.13$$

The standing wave ratio (SWR) or S is defined as $S = \dfrac{V_{max}}{V_{min}} = \dfrac{1+\rho}{1-\rho}$

inverting this

$$S(1-\rho) = 1+\rho$$
$$S - S\rho = 1+\rho$$
$$S-1 = S\rho + \rho = (S+1)\rho$$
$$\rho = \frac{S-1}{S+1}$$

The standing ratio S can be measured. If we can also measure the location of the first minimum we can estimate Z_L.

$$V(z) = V^+(e^{-j\beta z} + \Gamma_L e^{+j\beta z}) = V^+ e^{-j\beta z}(1 + \Gamma_L e^{j2\beta z})$$
$$V(z) = V^+ e^{-j\beta z}(1 + \rho e^{j\psi} e^{j2\beta z})$$

Now think about a minimum for $V(z)$. The minimum $|V(z)| = V_{min}$ will occur when

$$|V(z)| = |V^+||1-\rho|$$

or $$e^{j\psi} e^{j2\beta z} = e^{j(\psi + 2\beta z)} = -1$$

This occurs when the exponent is an odd multiple of π, i.e. $\cos\pi = -1$

$$\psi + 2\beta z_{min} = -(2m+1)\pi$$

The location of the first minimum is given by

$$\psi + 2\beta z_{min} = -\pi$$

We can write an expression for $|V(z)|$ as a function of ρ and ψ

$$V(z) = V^+ e^{-j\beta z}\left[1+\Gamma(z)\right]$$

$$V(z) = V^+ e^{-j\beta z}\left[1+\rho e^{j(\psi+2\beta z)}\right]$$

$$= V^+ e^{-j\beta z} + \rho V^+ e^{j(\psi+\beta z)}$$

$$V(z) = V^+\left[\cos\beta z + j\sin\beta z + \rho\cos(\psi+\beta z) + j\rho\sin(\psi+\beta z)\right]$$

$$|V(z)| = |V^+|\sqrt{\left[\cos(\beta z)+\rho\cos(\psi+\beta z)\right]^2 + \left[\sin(\beta z)+\rho\sin(\psi+\beta z)\right]^2}$$

This is readily plotted as a function of ρ and ψ

If $\psi = 0$ or π (i.e. purely resistive)

$$|V(z)| = |V^+|\sqrt{(1\pm\rho)^2\cos^2(\beta z) + \sin^2(\beta z)(1\mp\rho)^2}$$

lower sign is $\psi = \pi$

From the plots the load is at a voltage minimum or maximum when $Z_L = R_L + j0$

For the general case $Z_L = R_L + jX_L$ the minimum is displaced from $z = 0$.

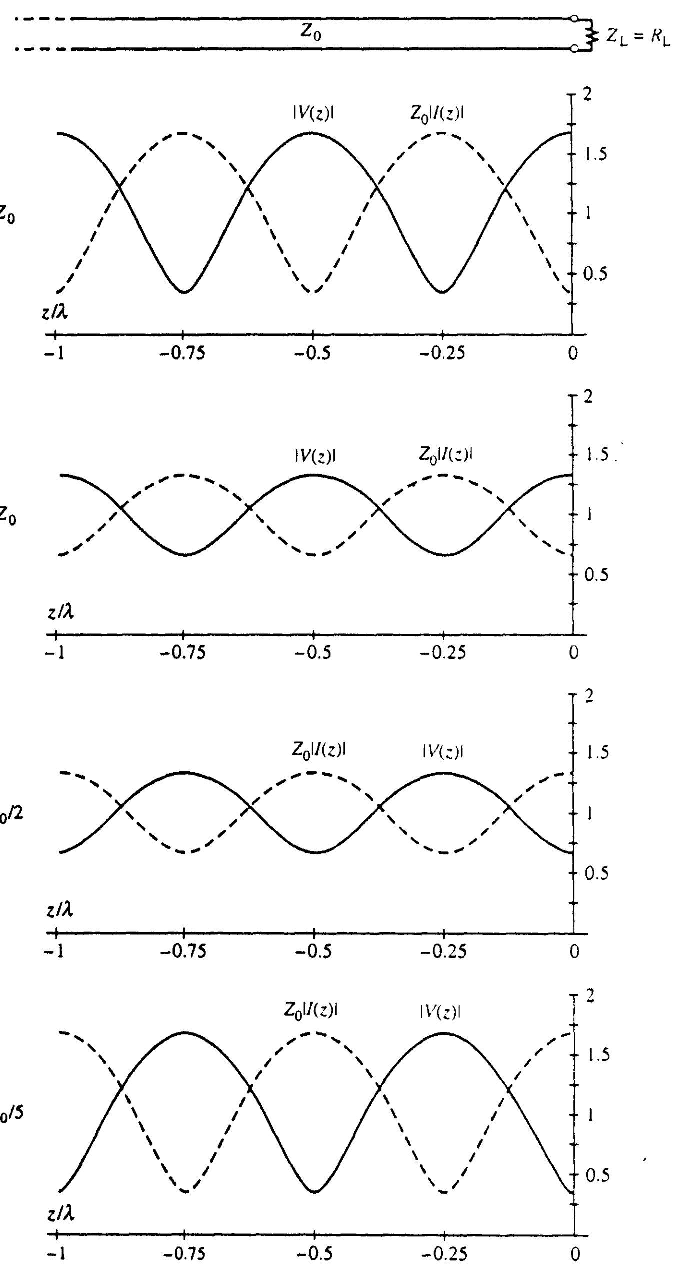


FIGURE 3.15. Voltage and current standing-wave patterns for different purely resistive loads. Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load position z/λ, for $V^+ = 1$ and for $R_L = 5Z_0, 2Z_0, Z_0/2$, and $Z_0/5$.

The key things to observe from these plots are that

(a) a maxima or a minima occurs at the load for a resistive termination

(b) that the S ratio decreases the closer you get to a matched load

(c) V has a maximum at the load if $R_L > Z_0$
a minimum at the load if $R_L < Z_0$

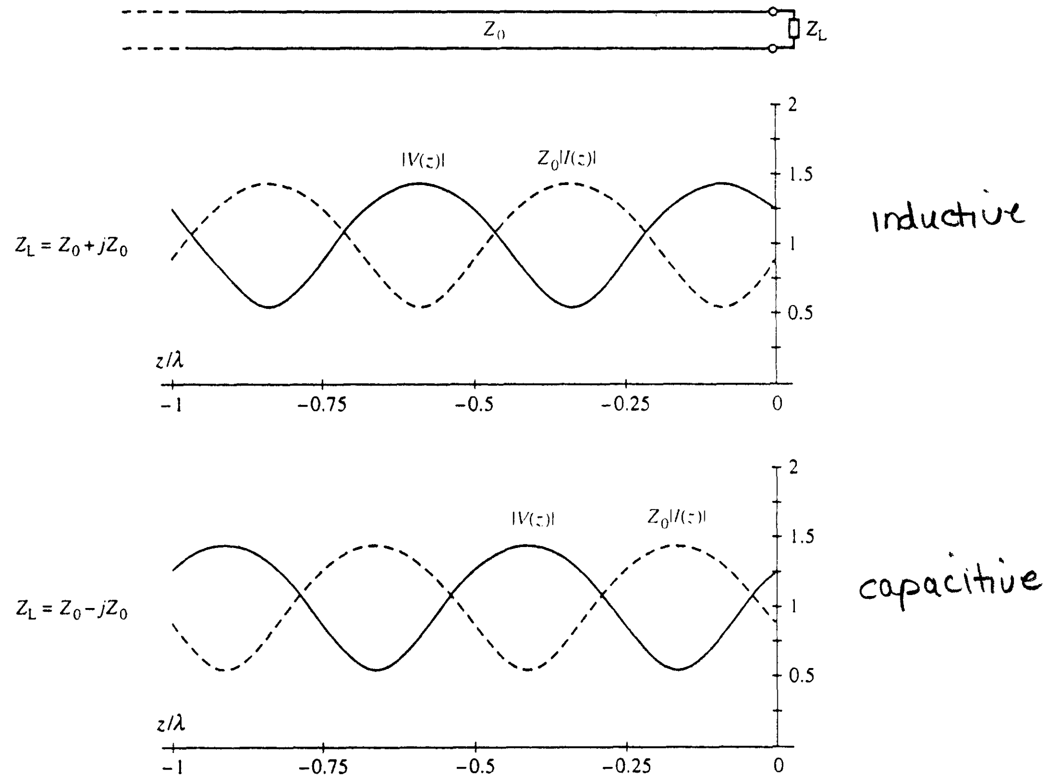


FIGURE 3.16. Voltage and current standing-wave patterns for complex loads. Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load z/λ for $V^+ = 1$ and for $Z_L = Z_0 + jZ_0$ (inductive load) and $Z_L = Z_0 - jZ_0$ (capacitive load).

Complex loads are harder to analyze. They also have standing wave patterns but you need to locate the first minimum to determine if the load is capacitive or inductive. If $-z_{min} < \frac{\lambda}{4}$ the load is capacitive

If $-z_{min} > \frac{\lambda}{4}$ then the load is inductive.

We can use $|1+\Gamma|$ to understand the behavior of $|V|$ near the load.

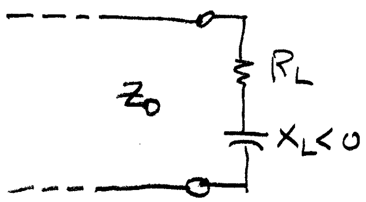
$$V(z) = V^+ e^{-j\beta z}\left[1+\Gamma(z)\right]$$

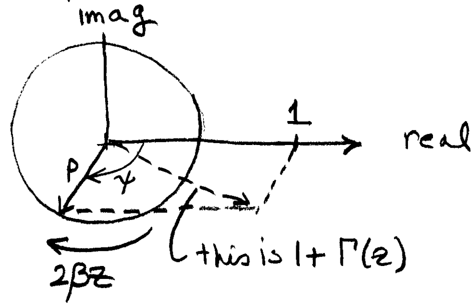
$$|V(z)| = \underbrace{|V^+|}_{\text{constant}}\ \underbrace{|1+\Gamma(z)|}_{\text{behavior of } |V| \text{ comes from this vector sum}} \qquad \text{since } \left|e^{-j\beta z}\right| = 1$$

$$|1+\Gamma(z)| = \left|1+\Gamma_L e^{j2\beta z}\right| = \underbrace{\left|1+\rho e^{j\psi} e^{j2\beta z}\right|}_{\text{look at magnitude of vector sum}}$$

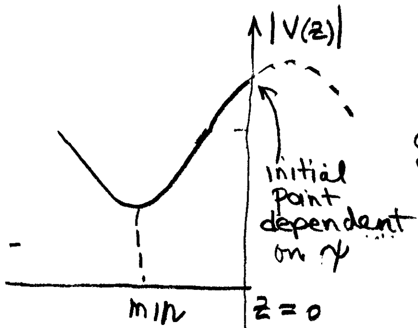
consists of "1" + vector of magnitude ρ and angle $\psi + 2\beta z$

We already know $-\pi < \psi < 0$ for capacitive loads



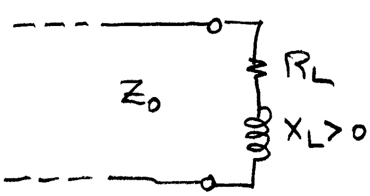


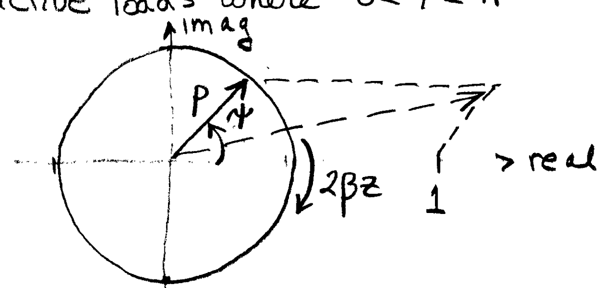
Remember
z is negative so $2\beta z$ goes clockwise

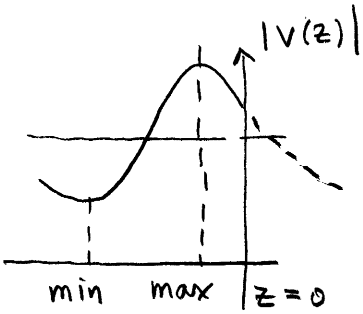


As z increases the angle $\psi + 2\beta z$ increases going clockwise. This causes $|1+\Gamma(z)|$ to decrease to a minimum at $\psi + 2\beta z = -\pi$

We can repeat this for inductive loads where $0 < \psi < \pi$







As z increases the angle actually decreases (going clockwise) until $|1+\Gamma|$ reaches a maximum when $\psi + 2\beta z = 0$

Note: inductive minima nearer load!

3.3.2. Transmission line impedance

$$Z(z) \equiv \frac{V(z)}{I(z)} = \frac{V^+ e^{-j\beta z}\left[1+\Gamma(z)\right]}{\frac{V^+}{Z_0} e^{-j\beta z}\left[1-\Gamma(z)\right]} = Z_0 \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

but $\Gamma(z) = \dfrac{V^- e^{+j\beta z}}{V^+ e^{-j\beta z}} = \Gamma_L e^{j2\beta z}$

$$\therefore\ Z(z) = Z_0 \frac{1+\Gamma_L e^{j2\beta z}}{1-\Gamma_L e^{j2\beta z}} = Z_0 \frac{1+\frac{Z_L-Z_0}{Z_L+Z_0} e^{j2\beta z}}{1-\frac{Z_L-Z_0}{Z_L+Z_0} e^{j2\beta z}}$$

but $\Gamma_L = \dfrac{Z_L-Z_0}{Z_L+Z_0}$

gives

$$Z(z) = Z_0 \frac{Z_L+Z_0+Z_L e^{j2\beta z} - Z_0 e^{j2\beta z}}{Z_L+Z_0-Z_L e^{j2\beta z}+Z_0 e^{j2\beta z}} \cdot \frac{e^{-j\beta z}}{e^{-j\beta z}}$$

$$= Z_0 \frac{Z_L e^{-j\beta z} + Z_0 e^{-j\beta z} + Z_L e^{j\beta z} - Z_0 e^{+j\beta z}}{Z_L e^{-j\beta z} + Z_0 e^{-j\beta z} - Z_L e^{+j\beta z} + Z_0 e^{j\beta z}}$$

$$= Z_0 \frac{Z_L\left(e^{-j\beta z}+e^{+j\beta z}\right) + Z_0\left(e^{-j\beta z} - e^{+j\beta z}\right)}{Z_L\left(e^{-j\beta z}-e^{+j\beta z}\right) + Z_0\left(e^{-j\beta z}+e^{j\beta z}\right)}$$

We know

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{j\theta}+e^{-j\theta} = 2\cos\theta$$
$$e^{j\theta}-e^{-j\theta} = 2j\sin\theta$$

Using these identities

$$Z(z) = Z_0 \frac{Z_L\, 2\cos\beta z + Z_0(-2j\sin\beta z)}{Z_L(-2j\sin\beta z) + Z_0\, 2\cos\beta z}$$

$$Z(z) = Z_0 \frac{Z_L \cos\beta z - j Z_0 \sin\beta z}{Z_0 \cos\beta z - j Z_L \sin\beta z} \cdot \frac{\frac{1}{\cos\beta z}}{\frac{1}{\cos\beta z}} = Z_0 \frac{Z_L - j Z_0 \tan\beta z}{Z_0 - j Z_L \tan\beta z}$$

It is common to normalize this expression to Z_o

$$\bar{Z}(z) = \frac{Z(z)}{Z_o}$$

$$\bar{Z}_L(z) = \frac{Z_L}{Z_o}$$

The normalized expression becomes

$$\bar{Z}(z) = \frac{\bar{Z}_L - j\tan\beta z}{1 - j\bar{Z}_L \tan\beta z}$$ Use with Smith Charts

We would like to be able to determine Z_L from measurements.

From previous page

$$Z(z) = Z_o \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

@ $z=0$

$$Z_L = Z(z=0) = Z_o \left.\frac{1+\Gamma_L e^{j2\beta z}}{1-\Gamma_L e^{j2\beta z}}\right|_{z=0} = Z_o \frac{1+\Gamma_L}{1-\Gamma_L} = Z_o \frac{1+\rho e^{j\psi}}{1-\rho e^{j\psi}}$$

Normalizing

$$\bar{Z}_L = \frac{Z_L}{Z_o} = \frac{1+\Gamma_L}{1-\Gamma_L} = \frac{1+\rho e^{j\psi}}{1-\rho e^{j\psi}}$$

but $\rho = \frac{S-1}{S+1}$ and $\psi = -\pi - 2\beta z_{min}$ are known and measurable

$$\bar{Z}_L = \frac{1+\frac{S-1}{S+1}e^{j\psi}}{1-\frac{S-1}{S+1}e^{j\psi}} = \frac{(S+1)+(S-1)e^{j\psi}}{(S+1)-(S-1)e^{j\psi}} = \frac{(S+1)+(S-1)\overbrace{e^{-j\pi}}^{-1}e^{-j2\beta z_{min}}}{(S+1)-(S-1)\underbrace{e^{-j\pi}}_{-1}e^{-j2\beta z_{min}}}$$

$$\bar{Z}_L = \frac{(S+1)-(S-1)e^{-j2\beta z_{min}}}{(S+1)+(S-1)e^{-j2\beta z_{min}}}$$

$$\bar{Z}_L = \frac{(S+1)e^{j\beta z_{min}} - (S-1)e^{-j\beta z_{min}}}{(S+1)e^{+j\beta z_{min}} + (S-1)e^{-j\beta z_{min}}}$$

$$= \frac{(e^{j\beta z_{min}} + e^{-j\beta z_{min}}) + S(e^{j\beta z_{min}} - e^{-j\beta z_{min}})}{(e^{j\beta z_{min}} - e^{-j\beta z_{min}}) + S(e^{j\beta z_{min}} + e^{-j\beta z_{min}})}$$

$$= \frac{2\cos\beta z_{min} + S\, 2j\sin\beta z_{min}}{2j\sin\beta z_{min} + S 2\cos\beta z_{min}} = \frac{1+jS\tan\beta z_{min}}{S+j\tan\beta z_{min}}$$

With Smith charts it is useful to use admittances

$$Y(z) = \frac{1}{Z(z)} = Y_0 \frac{1-\Gamma_L e^{j2\beta z}}{1+\Gamma_L e^{j2\beta z}} = Y_0 \frac{Y_L - jY_0 \tan\beta z}{Y_0 - jY_L \tan\beta z}$$

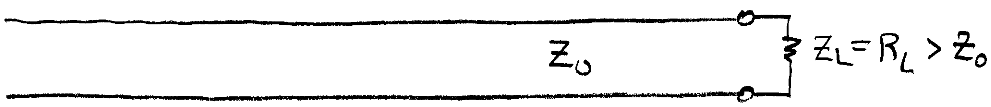
These are just inverses of previous expressions.

Using normalized line admittances.

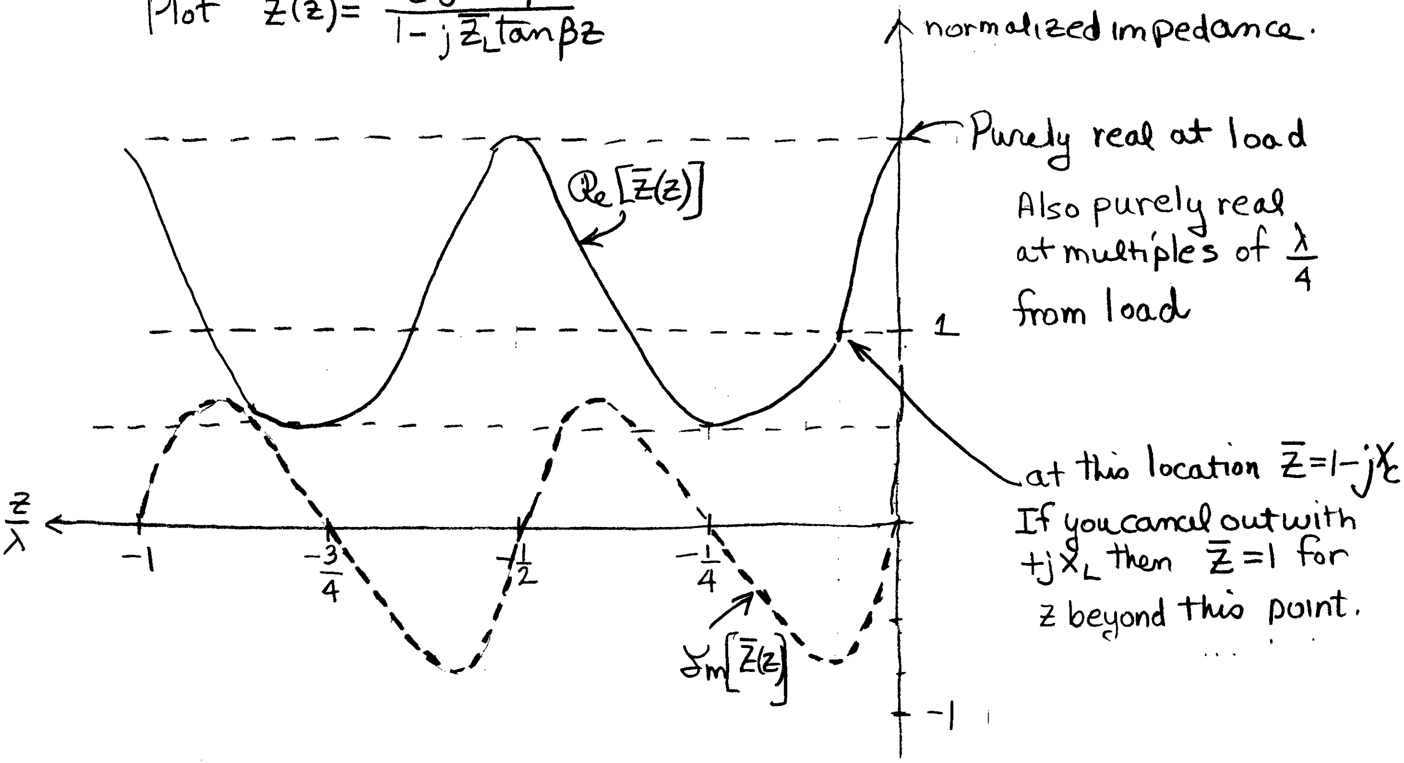
$$\overline{Y}(z) = \frac{\overline{Y}_L - j\tan\beta z}{1 - j\overline{Y}_L \tan\beta z}$$

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L}$$

Line impedance for resistive loads



Plot $\overline{Z}(z) = \dfrac{\overline{Z}_L - j\tan\beta z}{1 - j\overline{Z}_L \tan\beta z}$



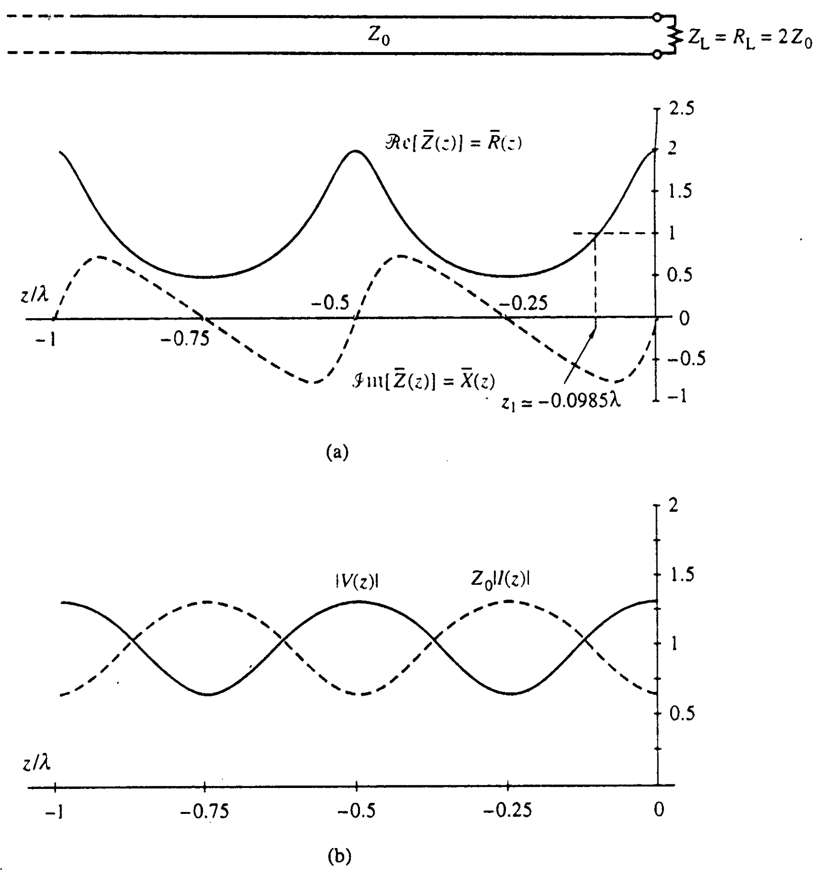


FIGURE 3.19. **Impedance along a line terminated with $Z_L = R_L = 2Z_0$.** (a) The real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ are shown as functions of z/λ. (b) Magnitudes of the voltage and current phasors (i.e., $|V(z)|$ and $Z_0|I(z)|$) are shown as functions of electrical distance from the load z/λ, for $V^+ = 1$.

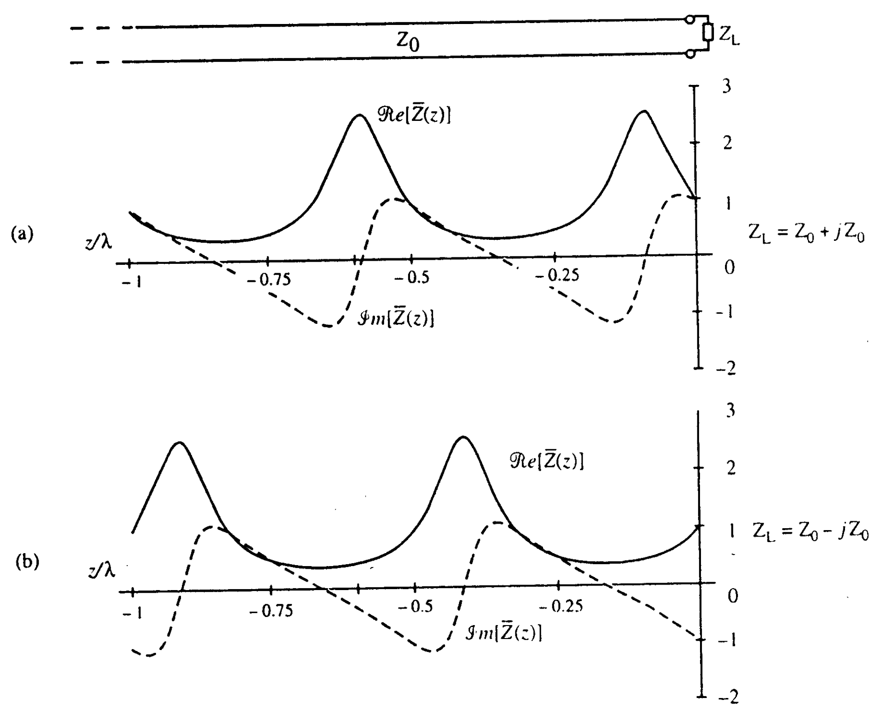


FIGURE 3.22. **Line impedance for two different complex load impedances.** The real and imaginary parts of the normalized line impedance $\bar{Z}(z)$ are shown as functions of electrical distance z/λ along the line for (a) $Z_L = Z_0 + jZ_0$ and (b) $Z_L = Z_0 - jZ_0$.

Some observations for a resistive load:

line impedance at a voltage maximum is real

$$V(z) = V^+ e^{-j\beta z} \Big[\underbrace{1 + \rho e^{j(\psi + 2\beta z)}}_{\text{this will be a maximum at}} \Big] \rightarrow |V^+|(1+\rho)$$

$\psi + 2\beta z = -m\,2\pi$

$m = 0, 1, 2, 3$

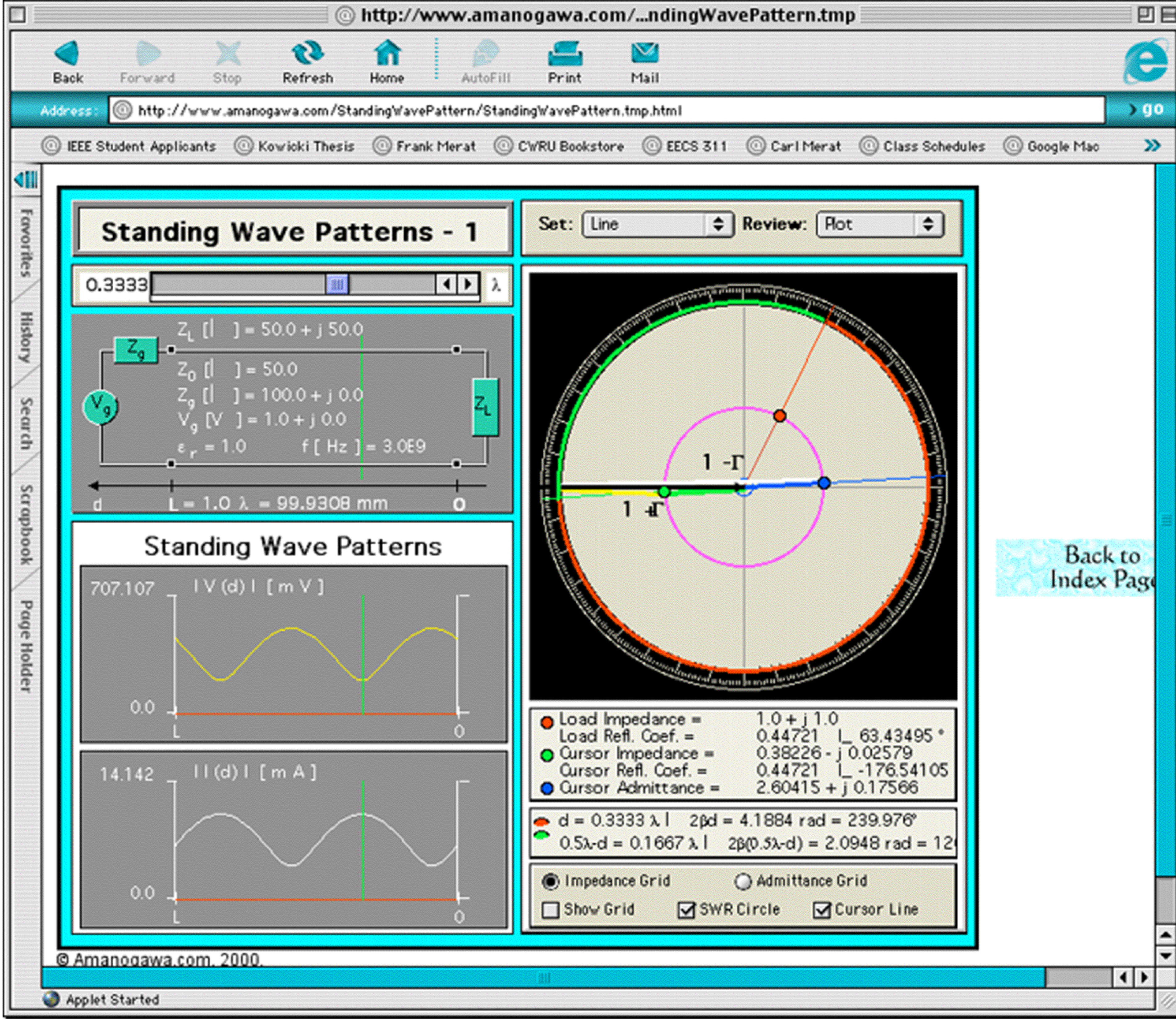
At the same position the current is

$$I(z) = \frac{V^+}{Z_0} e^{-j\beta z}\left[1 - \rho\, e^{j(\psi + 2\beta z)}\right] \rightarrow \frac{|V^+|}{Z_0}(1-\rho)$$

At this point the impedance is

$$\left| Z\,(z = \text{voltage maximum}) \right| = \frac{|V^+|(1+\rho)}{\frac{|V^+|}{Z_0}(1-\rho)} = Z_0 \frac{1+\rho}{1-\rho} = Z_0 S$$

For a complex load the maximina and minima of the voltage or the Line impedance do <u>NOT</u> occur at the load.



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Standing Wave Patterns - 1
Set: Line
Review: Plot
0.3333
λ
Z_L [Ω] = 50.0 + j 50.0
Z_0 [Ω] = 50.0
Z_g [Ω] = 100.0 + j 0.0
V_g [V] = 1.0 + j 0.0
ε_r = 1.0
f [Hz] = 3.0E9
Z_g
V_g
Z_L
d
L = 1.0 λ = 99.9308 mm
0
Standing Wave Patterns
707.107
\| V (d) \| [m V]
0.0
L
0
14.142
\| I (d) \| [m A]
0.0
L
0
1 -Γ
1 +Γ
Load Impedance = 1.0 + j 1.0
Load Refl. Coef. = 0.44721 ∠ 63.43495 °
Cursor Impedance = 0.38226 - j 0.02579
Cursor Refl. Coef. = 0.44721 ∠ -176.54105
Cursor Admittance = 2.60415 + j 0.17566
d = 0.3333 λ \| 2βd = 4.1884 rad = 239.976°
0.5λ-d = 0.1667 λ \| 2β(0.5λ-d) = 2.0948 rad = 12
Impedance Grid
Admittance Grid
Show Grid
SWR Circle
Cursor Line
Back to Index Page
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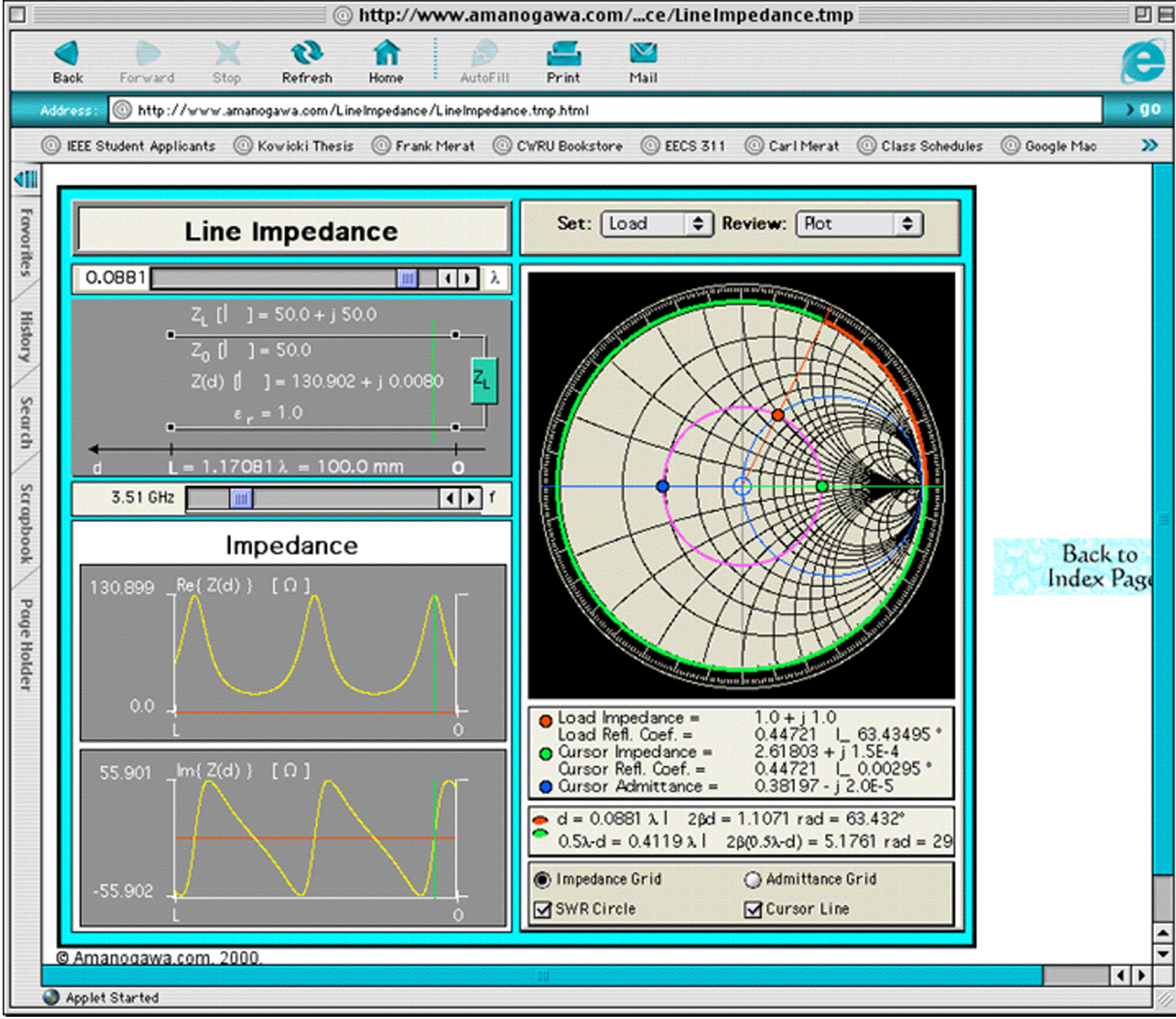
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Line Impedance
Set: Load
Review: Plot
0.0881
λ
Z_L [Ω] = 50.0 + j 50.0
Z_0 [Ω] = 50.0
Z(d) [Ω] = 130.902 + j 0.0080
ε_r = 1.0
Z_L
d
L = 1.17081 λ = 100.0 mm
0
3.51 GHz
f
Impedance
130.899
Re{ Z(d) } [Ω]
0.0
L
0
55.901
Im{ Z(d) } [Ω]
-55.902
L
0
Load Impedance = 1.0 + j 1.0
Load Refl. Coef. = 0.44721 ∠ 63.43495 °
Cursor Impedance = 2.61803 + j 1.5E-4
Cursor Refl. Coef. = 0.44721 ∠ 0.00295 °
Cursor Admittance = 0.38197 - j 2.0E-5
d = 0.0881 λ | 2βd = 1.1071 rad = 63.432°
0.5λ-d = 0.4119 λ | 2β(0.5λ-d) = 5.1761 rad = 29
Impedance Grid
Admittance Grid
SWR Circle
Cursor Line
Back to Index Page
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Applet Started

Example 3-7 Input impedance of a line

Find the input impedance of a 75-cm long transmission line where $Z_0 = 70\Omega$, terminated with a $Z_L = 140\Omega$ load at 50, 100, 150 and 200 MHz. Assume the phase velocity v_p to be equal to the speed of light in free space.

(a) $f = 50\text{MHz}$ so $\lambda = \dfrac{3\times10^8\text{ m/sec}}{50\times10^6/\text{sec}} = 6\text{m}.$

the electrical length of the line is $\ell/\lambda = \dfrac{0.75\text{m}}{6\text{m}} = 0.125$

$$Z_{in} = Z_0 \frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}$$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = 2\pi(0.125)$$

$$Z_{in} = 70\,\frac{140 + j70\tan(2\pi(\frac{1}{8}))}{70 + j140\tan(2\pi\frac{1}{8})} = 70\,\frac{140+j70}{70+j140} = 56 - j42\Omega$$

capacitive

(b) $f = 100\text{ mHz}$ $\lambda = \dfrac{3\times10^8\text{ m/s}}{100\times10^6/\text{s}} = 3\text{m}$

$$\frac{\ell}{\lambda} = \frac{0.75}{3} = \frac{1}{4} \qquad \tan\beta\ell = \tan\left(\frac{2\pi}{\lambda}\cdot\frac{1}{4}\lambda\right) = \tan\left(\frac{\pi}{2}\right) = \infty$$

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell} \rightarrow Z_0\frac{jZ_0}{jZ_L} = \frac{(Z_0)^2}{Z_L} = \frac{(70)^2}{140} = 35\Omega$$

resistive

(c) $f = 150\text{ mHz}$ $\lambda = \dfrac{3\times10^8\text{ m/s}}{150\times10^6/\text{s}} = 2\text{m}$

$$\frac{\ell}{\lambda} = \frac{0.75}{2} = \frac{3}{8} \qquad \tan\beta\ell = \tan\left(\frac{2\pi}{\lambda}\cdot\frac{3}{8}\lambda\right) = -1$$

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell} = 70\,\frac{140 - j70}{70 - j140} = 56 + j42\Omega$$

inductive

(d) $f = 200\text{ mHz}$ $\lambda = \dfrac{3\times10^8\text{ m/s}}{200\times10^6/\text{s}} = 1.5\text{m}$

$$\frac{\ell}{\lambda} = \frac{0.75\text{m}}{1.5\text{ m}} = \frac{1}{2} \qquad \tan\beta\ell = \tan\left(\frac{2\pi}{\lambda}\cdot\frac{1}{2}\lambda\right) = 0$$

$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan\beta\ell}{Z_0 + jZ_L\tan\beta\ell} = 70\,\frac{140 + j0}{70 + j0} = 140\Omega$$

resistive (equal to load impedance)

Example 3-8 The following measurements were carried out on a 100Ω transmission line terminated with an unknown load Z_L. The voltage standing wave ratio $S = 5$, the distance between successive voltage minima is 25cm, and the distance from Z_L to the first voltage minimum is 8cm.

(a) Determine the load reflection coefficient Γ_L.

From S we can determine the magnitude ρ of Γ_L

$$\rho = \frac{S-1}{S+1} = \frac{5-1}{5+1} = \frac{4}{6} = 0.667$$

Since the spacing between minima is $\frac{\lambda}{2}$ we identify $\frac{\lambda}{2} = 25\text{cm}$ or $\lambda = 50\text{cm}$.

Now, we can identify ψ, the angle of the load coefficient

Since $8\text{cm} < \frac{\lambda}{4} = 12.5\text{cm}$ this is a capacitive load

and $\psi = -\pi - 2\beta z$ (See pgs. 21-22 of these notes).

$$\psi = -\pi + 2\,\frac{2\pi}{\lambda}\cdot\frac{8}{50}\lambda = -1.13 \text{ radians} = -64.8^\circ$$

don't forget z is negative

$$\therefore\ \Gamma_L = \rho e^{j\psi} = 0.667\, e^{-j64.8^\circ}$$

(b) Determine the unknown load impedance.

$$\beta z_{min} = \frac{-2\pi}{\lambda}\,\frac{8}{50}\lambda = -1.0053 \text{ radians}$$

again note sign

$$\tan(\beta z_{min}) = \tan(-1.0053) \cong -1.575$$

$$Z_L = Z_0\,\frac{1 + jS\tan(\beta z_{min})}{S + j\tan(\beta z_{min})} = 100\,\frac{1 + j5(-1.575)}{5 + j(-1.575)}$$

$$= 100\,\frac{1 - j7.8787}{5 - j1.575} = 63.35 - j137.62\ \Omega$$

(c) Determine the location of the first voltage <u>maximum</u> with respect to the load.

The first voltage maximum is <u>always</u> $\frac{\lambda}{4}$ away from the voltage minimum.

$$z_{max} = z_{min} - \frac{\lambda}{4} = -8 - 12.5\text{cm} = -20.5\text{cm}.$$

note sign

Example 3-9 Inverted-V antenna

A 50Ω coaxial line filled with Teflon ($v_p = 21$ cm/nsec) is connected to an inverted-V antenna represented by Z_L as shown below. At f = 29.6 MHz the feed-point impedance of the antenna is approximately measured to be $Z_L \simeq 75\Omega + j25\Omega$. Find the two closest positions to the antenna along the line where the real part of the line impedance is equal to the characteristic impedance of the line (i.e., Z_0).

$$Z(z) = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}$$

Define $\eta = -\tan\beta l$ and substitute values (We assume $z = -l$)

$$Z(z) = 50\,\frac{(75+j25) - j50\zeta}{50 - j(75+j25)\zeta} = 50\,\frac{3 + j(1-2\zeta)}{(2+\zeta) - j3\zeta}$$

We want the real part of this. Using old fashion algebra

$$\mathrm{Re}\{Z(z)\} = \mathrm{Re}\left\{50\,\frac{3+j(1-2\zeta)}{(2+\zeta)-j3\zeta} \cdot \frac{(2+\zeta)+j3\zeta}{(2+\zeta)+j3\zeta}\right\}$$

$$= \mathrm{Re}\left\{50\,\frac{3(2+\zeta) - (1-2\zeta)(3\zeta) + j(1-2\zeta)(2+\zeta) + j9\zeta}{(2+\zeta)^2 + 9\zeta^2}\right\}$$

$$= 50\,\frac{3(2+\zeta) - 3\zeta(1-2\zeta)}{(2+\zeta)^2 + 9\zeta^2} = 50$$

$$3(2+\zeta) - 3\zeta(1-2\zeta) = (2+\zeta)^2 + 9\zeta^2$$

$$6 + 3\zeta - 3\zeta + 6\zeta^2 = 4 + 4\zeta + \zeta^2 + 9\zeta^2$$

$$4\zeta^2 + 4\zeta - 2 = 0$$

$$2\zeta^2 + 2\zeta - 1 = 0$$

$$\zeta = \frac{-2 \pm \sqrt{(2)^2 - 4(-1)(2)}}{4} = \frac{-2 \pm \sqrt{4+8}}{4} = \frac{-1 \pm \sqrt{3}}{2} =$$

$$\zeta = -1.367,\ 0.366$$

$\zeta = \tan(\beta z)$

$$\lambda = \frac{v_p}{f} = \frac{2.1\times10^{8}\ m/s}{29.6\times10^{6}/s} = 7.095\,m.$$

$$\therefore \tan(\beta z) = \tan\left(\frac{2\pi}{\lambda}\cdot\frac{z}{7.095}\lambda\right) = \tan\left(\frac{2\pi z}{7.095}\right)$$

Solving for z_1

$$\tan\left(\frac{2\pi z_1}{\lambda}\right) = -1.367$$

$$\frac{2\pi z_1}{\lambda} = -0.9392$$

$$z_1 = -0.149\lambda = -1.06\text{meters}$$

Solving for z_2

$$\tan\left(2\pi\left(\frac{z_2}{\lambda} + \frac{\frac{\lambda}{2}}{\lambda}\right)\right) = .35086$$

the second real point
must be $\frac{\lambda}{2}$ away from the first real point

$$\tan\left(2\pi\left(\frac{z_2}{\lambda} + \frac{1}{2}\right)\right) = .35086$$

$$2\pi\left(\frac{z_2}{\lambda} + \frac{1}{2}\right) = 0.33744$$

$$\frac{z_2}{\lambda} + \frac{1}{2} = 0.05371$$

$$z_2 = -0.4463\lambda = -3.167\text{meters}.$$

At $z = -1.06\,m$ and $-3.167\,m$ the real part of Z is equal to $Z_0 = 50\Omega$

Example 3-10

A radio transmitter is connected to an antenna having a feed-point impedance of $Z_L = 70 + j100\,\Omega$ with a $50\,\Omega$ coaxial line. Find

(a) the load reflection coefficient

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{70 + j100 - 50}{70 + j100 + 50} = \frac{20 + j100}{120 + j100} = .508 + j0.410$$

$$\Gamma_L = 0.65\, e^{j37.41^\circ}$$

(b) the standing wave ratio

$$S = \frac{1+\rho}{1-\rho} = \frac{1 + .653}{1 - .653} = 4.76$$

(c) the two positions closest to the load along the line where the line impedance is purely real, and their corresponding line impedance values.

This is an inductive load since $X_L > 0$

The maximum voltage occurs when the line impedance is purely real (See Fig 3.19) At this point the magnitude of $Z(z)$ is also a maximum.

This then reduces to finding the maximum voltage position

$$\text{i.e. } \psi + 2\beta z_{max} = 0$$

$$37.41^\circ = 0.6529 \text{ radians}$$

$$.6529 + 2\frac{2\pi}{\lambda} z_{max} = 0$$

$$z_{max} = \frac{-0.652}{4\pi}\lambda = -0.052\lambda$$

At this point $R(z)$ is a maximum and the impedance is $Z_0 S$

$$R_{max} = Z_0 S = (50)(4.76) = 238\,\Omega$$

The other point occurs at a voltage minimum.

$$\text{At a minimum} \quad \psi + 2\beta z_{min} = -\pi$$

$$2\frac{2\pi}{\lambda} z_{min} = -\pi - .6529 = -3.795$$

$$z_{min} = -0.302\lambda$$

$$\text{at this point } R_{min} = \frac{Z_0}{S} = \frac{50}{4.76} = 10.5\,\Omega.$$

3.3.3 Calculation of V^+

We need to include the source to get actual values for V^+.

The voltage at the input end of the transmission line is given by

$$V_s = V(z=-\ell) = V^+ e^{j\beta\ell}(1+\Gamma_L e^{-j2\beta\ell})$$

However this voltage can also be given by the input voltage divider.

$$V_s = \frac{Z_{in}}{Z_{in}+Z_s} V_0$$

Equate these two expressions for V_s and solve for V^+

$$V^+ e^{j\beta\ell}(1+\Gamma_L e^{-j2\beta\ell}) = \frac{Z_{in} V_0}{Z_{in}+Z_s}$$

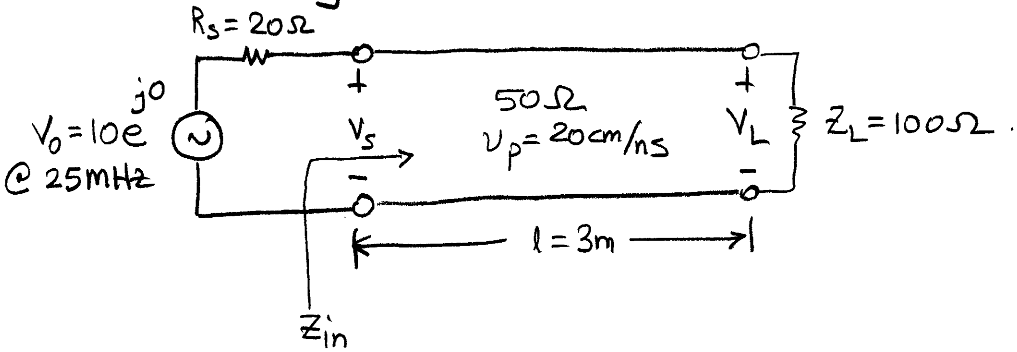
$$V^+ = \frac{Z_{in} V_0}{(Z_{in}+Z_s)\, e^{j\beta\ell}(1+\Gamma_L e^{-j2\beta\ell})}$$

Example 3-11

A sinusoidal voltage source $\mathcal{V}_0(t) = 10\cos(5\pi\times10^7 t)$ volts and $R_s = 20\Omega$ is connected to an antenna with feed point impedance $Z_L = 100\Omega$ through a 3-m long, lossless coaxial transmission line filled with polyethylene ($v_p = 20\,\text{cm/ns}$) and with a characteristic impedance $Z_0 = 50\Omega$ as shown below. Find

(a) the voltage and current phasors, $V(z)$ and $I(z)$, at any location on the line, and

(b) the corresponding instantaneous expressions $\mathcal{V}(z,t)$ and $\mathcal{I}(z,t)$



Solution:

$$f = \frac{\omega}{2\pi} = \frac{5\pi\times10^7}{2\pi} = 25\times10^6\ \text{Hz}.$$

$$\lambda = \frac{v_p}{f} = \frac{2\times10^8\,\text{m/s}}{25\times10^6} = 8\ \text{meters}.$$

The electrical length of the line is then $\frac{l}{\lambda} = \frac{3\text{m}}{8\text{m}} = 0.375$

$$\beta l = \frac{2\pi}{\lambda}\cdot\frac{3}{8}\lambda = \frac{3\pi}{4} \qquad \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{8} = \frac{\pi}{4}$$

$$\tan(\beta l) = -1$$

We then compute the input impedance.

$$Z_{in} = Z(z=-l) = Z_0\,\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l} = 50\,\frac{100 + j50(-1)}{50 + j(100)(-1)}$$

$$Z_{in} = 40 + j30\,\Omega$$

We compute the input voltage V_s from the input voltage divider

$$V_s = \frac{Z_{in}}{R_s + Z_{in}}V_0 = \frac{40 + j30}{60 + j30}(10) = 7.45e^{j10.3^\circ} = 7.45e^{j0.180}$$

But V_s can also be written as

$$V_s = V(z=-3m) = V^+ e^{j\beta l}\left(1+\Gamma_L e^{-j2\beta l}\right)$$

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100-50}{100+50} = \frac{1}{3}$$

so

$$V_s = V^+ e^{j\frac{3\pi}{4}}\left(1+\frac{1}{3}e^{-j\frac{3\pi}{2}}\right) = V^+ e^{j\frac{3\pi}{4}}\left(1+j\frac{1}{3}\right)$$

Equating the two expressions for V_s gives V^+

$$V^+ e^{j\frac{3\pi}{4}}\left(1+j\frac{1}{3}\right) = 7.45\, e^{j10.3^\circ}$$

$$V^+ = \frac{7.45}{1+j\,1/3}\, e^{j10.3^\circ} e^{-j\frac{3\pi}{4}} = 7.07 e^{-j0.32} e^{j0.18} e^{-j\frac{3\pi}{4}} = 7.07 e^{-j2.496}$$

$$\therefore V(z) = \underbrace{7.07\, e^{-j143^\circ}}_{V^+}\ \underbrace{e^{-j\frac{\pi z}{4}}}_{e^{j\beta z}}\ \underbrace{\left(1+\frac{1}{3}e^{j\cdot\frac{\pi z}{2}}\right)}_{1+\Gamma(z)}$$

remember this is $2\beta z$

$$I(z) = \frac{V(z)}{Z_0} = 0.141\, e^{-j143}\, e^{-j\frac{\pi z}{4}}\left(1-\frac{1}{3}e^{+j\frac{\pi z}{2}}\right)$$

(b) The corresponding time expressions are

$$\mathcal{V}(z,t) = \text{Re}\left\{V(z)e^{j\omega t}\right\} = \text{Re}\left\{\left(7.07e^{-j143^\circ}e^{-j\frac{\pi z}{4}} + 7.07e^{-j143^\circ}e^{-j\frac{\pi z}{4}}\frac{1}{3}e^{+j\frac{\pi z}{2}}\right)e^{j\omega t}\right\}$$

$$= \text{Re}\left\{\left(7.07e^{-j143^\circ}e^{-j\frac{\pi z}{4}} + 2.36e^{-j143^\circ}e^{+j\frac{\pi z}{4}}\right)e^{j\omega t}\right\}$$

$$= 7.07\cos\left(5\pi\times10^7 t - \frac{\pi z}{4} - 143^\circ\right) + 2.36\cos\left(5\pi\times10^7 t + \frac{\pi z}{4} - 143^\circ\right)$$

$$\mathcal{I}(z,t) = \text{Re}\left\{\left(0.141e^{-j143^\circ}e^{-j\frac{\pi z}{4}} - 0.141e^{-j143^\circ}e^{-j\frac{\pi}{4}z}\left(\frac{1}{3}\right)e^{-j\frac{\pi z}{2}}\right)e^{j\omega t}\right\}$$

$$= \text{Re}\left\{\left(0.141e^{-j143^\circ}e^{-j\frac{\pi z}{4}} - 0.047\,e^{-j143^\circ}e^{+j\frac{\pi z}{4}}\right)e^{j\omega t}\right\}$$

$$= 0.141\cos\left(5\pi\times10^7 t - \frac{\pi z}{4} - 143^\circ\right) - 0.047\cos\left(5\pi\times10^7 t + \frac{\pi z}{4} - 143^\circ\right)$$