Mathematical Appendix

1. Delta Function

For our purposes we will define the Dirac delta function by the operational properties

$$\delta(x-x') = 0 \quad , \quad x \neq x' \tag{1a}$$

$$\int_{x_1}^{x_2} \delta(x-x')f(x')dx' = \begin{cases} f(x) \ , & x_1 < x < x_2 \\ 0 \ , & x \text{ not in } x_1 - x_2 \end{cases} \tag{1b}$$

for any function $f(x')$ which is continious at $x'= x$. If $f(x') = 1$ we obtain $\int_{x_1}^{x_2} \delta(x-x')dx' = 1$ if $x_1 < x < x_2$ and zero otherwise.

If we take a Fourier transform of $\delta(x-x')$ we obtain

$$F\delta(x-x') = \int_{-\infty}^{\infty} e^{-jwx}\delta(x-x')dx = e^{-jwx'} \tag{2}$$

The inverse transform relation is

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jwx}\ e^{-jwx'}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jw(x-x')}dw \tag{3}$$

which is a useful operational expression for $\delta(x-x')$. An heuristic proof of (3) is based on the following $\lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} e^{jw(x-x')}dw = \lim_{T \to \infty} \frac{T}{\pi} \frac{\sin T(x-x')}{T(x-x')}$ which can be shown to have the properties expressed by (1b) .

A unit source localized in space-time can be represented by

$$\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-t') = \delta(\vec{r}-\vec{r}\,')\delta(t-t') \tag{4}$$

where $\delta(\vec{r}-\vec{r}\,')$ is an abbreviation for the product of the three one dimensional spatial delta functions. In an orthogonal curvilinear coordinate system u_1, u_2, u_3

with metric coefficients h_1, h_2, h_3 we have

$$\delta(\vec{r}-\vec{r}\,') \quad \frac{\delta(u_1 - u_1')}{h_1}\,\frac{\delta(u_2 - u_2')}{h_2}\,\frac{\delta(u_3 - u_3')}{h_3} \tag{5}$$

Division by the h_i is required since the volume element $dV = h_1 h_2 h_3 du_1 du_2 du_3$ and we require

$$\int_V \delta(\vec{r}-\vec{r}\,')dV' = 1 = \int_V \prod_{i=1}^{3} \frac{\delta(u-u_i)}{h_i} h_i du_i \tag{6}$$

Often we have to evaluate integrals of the form $\int_{x_1}^{x_2} \delta[g(x)-p(x')]dx'$. Let $u' = p(x')$ and then $dp/dx' = du'/dx'$ so we obtain $\int_{u'(x_1)}^{u'(x_2)} \delta[g(x)-u'] \frac{du'}{dp/dx'} = \left.\frac{1}{dp/dx'}\right|_{u'=g(x)=p(x')}$.

<u>Example</u>

Let $p(x') = x'^2$, $g(x) = ax^2 + b$. Then $g(x) = p(x')$ can be solved for x' to give $x' = \sqrt{ax^2 + b}$ and $\int_{u'(x_1)}^{u'(x_2)} \delta[g-p]dx' = \frac{1}{2\sqrt{ax^2 + b}}$ provided $x' = \sqrt{ax^2 + b}$ is within the interval x_1 to x_2 . Otherwise the value of the integral is zero.

<u>Reference</u>

The above heuristic use of the delta function can be made rigorous by means of distribution theory. An excellent reference is A. H. Zemanian "Distribution Theory and Transform Analysis", McGraw-Hill Book Co., 1965.

Fourier Transform in the Complex Domain

Contour Integration

Consider a function $f(x, y) = f(Z)$ of the complex variable $Z = x + jy$. f is analytic at all points at which it has a unique derivative. This means that $\lim_{\Delta Z \to 0} \frac{f(Z+\Delta Z) - f(Z)}{\Delta Z}$ must be independent of the direction of ΔZ in the complex plane (see Fig. 1). If we choose $\Delta Z = \Delta x$ we get $\frac{df}{dZ} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$ where we have written $f = u(x, y) + jv(x, y)$. If we choose $\Delta Z = j\Delta y$ we get

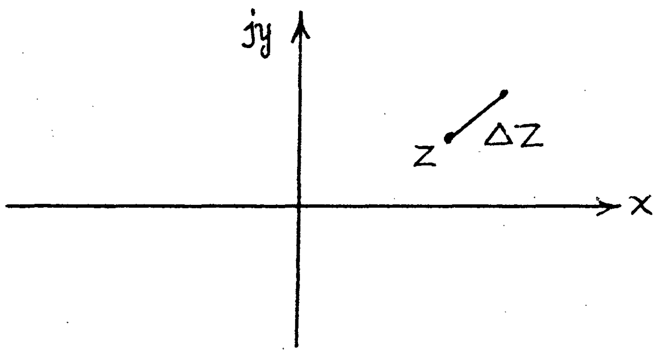


Fig. 1

$\frac{df}{dZ} = \frac{1}{j}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$. For df/dZ to have a unique value we thus see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{7}$$

which are called the Cauchy-Riemann equations. It may be shown that the Cauchy-Riemann equations are also sufficient to guarantee that $f(Z)$ be analytic.

Cauchy Integral Formula

Let $f(Z) = u + jv$ be analytic within and on a closed contour. Then we can show that $\oint_C f dZ = 0$ where C is arbitrary (see Fig. 2). To show the validity of this result consider a vector $\vec{A} = \vec{a}_x A_x(x, y) + \vec{a}_y A_y(x, y)$ and use Stokes' law to get $\oint_C \vec{A}\cdot d\vec{\ell} = \int_S \nabla \times \vec{A}\cdot d\vec{S}$ or $\oint_C (A_x d_x + A_y d_y) = \int_S (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) dx\, dy$. We have

$\oint_C f\, dZ = \oint_C (u+jv)(dx+jdy) = \oint_C (udx-vdy) + j\oint_C (vdx + udy)$. In the first integral

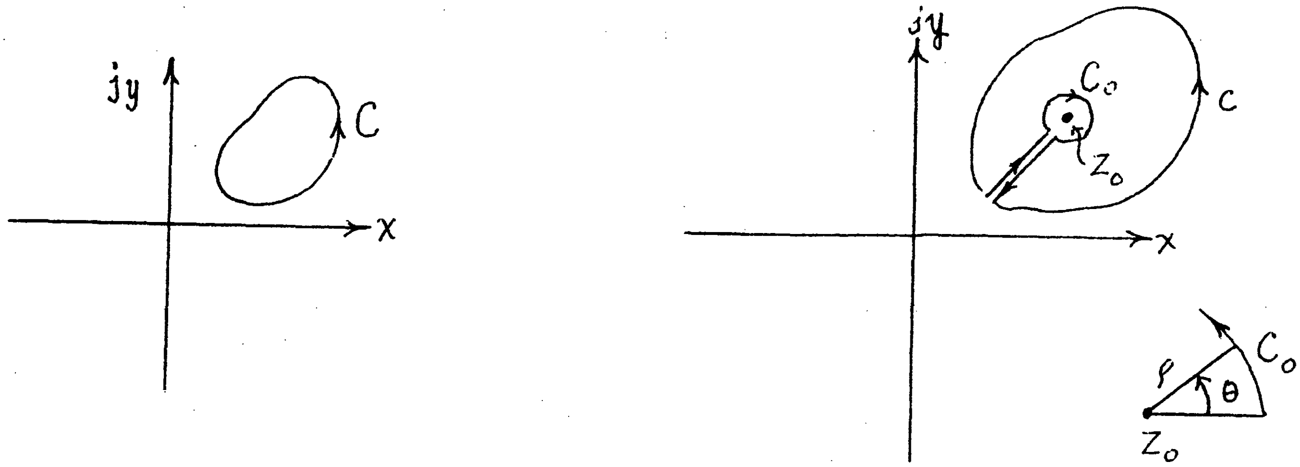


Fig. 2

let $u = A_x$, $v = -A_y$ and apply Stokes' law to obtain $\oint_C (udx-vdy) = \int_S (\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y})dxdy =$ 0 because $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ from the Cauchy-Riemann equations. To show that the second integral vanishes let $v = A_x$ and $u = A_y$ and apply Stokes' law again.

Consider now $f(Z) = g(Z)/(Z-Z_o)^n$ where g is analytic inside and on C and Z_o is inside C. Consider a modified contour that includes a circle C_o around the singular point Z_o as in Fig. 2 . Then within and on $C + C_o$ f is analytic and hence

$$\oint_{C+C_o} fdZ = 0 = \oint_C fdZ + \oint_{C_o} fdZ$$

since the integral along the cut is traversed twice in opposite directions and vanishes. Thus we have

$$\oint_C fdZ = \oint_C \frac{g}{(Z-Z_o)^n} dZ = -\oint_{C_o} \frac{g}{(Z-Z_o)^n} dZ$$

On C_o, $Z-Z_o = \rho e^{j\theta}$, $dZ =-j\rho e^{j\theta} d\theta$, hence

$$\oint_C fdZ = \int_0^{2\pi} \frac{g(Z) j\rho e^{j\theta} d\theta}{(\rho e^{j\theta})^n}$$

But g is analytic at Z_o and we can choose ρ so small that $g(Z) \approx g(Z_o)$ everywhere on C_o. Thus

$$\oint_C f dZ = g(Z_o) j\rho^{-n+1} \int_0^{2\pi} e^{-j\theta(n-1)} d\theta = \begin{cases} 0, & n \neq 1 \\ 2\pi j g(Z_o), & n=1 \end{cases} \tag{8}$$

Residue Theory

Consider a function f that has a pole of order n at Z_o. This means that near Z_o, f behaves as constant$/(Z-Z_o)^n$. The function $g = (Z-Z_o)^n f(Z)$ will be analytic at Z_o. A Taylor expansion about Z_o gives $g(Z) = g(Z_o) + g'(Z_o)(Z-Z_o) + \frac{1}{2!} g''(Z_o)(Z-Z_o)^2 + \ldots$. Hence we can write

$$f(Z) = \frac{g(Z_o)}{(Z-Z_o)^n} + \ldots \frac{g^{n-1}(Z_o)}{(n-1)!(Z-Z_o)} + \frac{g^n(Z_o)}{n!} + \frac{g^{n+1}(Z_o)(Z-Z_o)}{(n+1)!} + \ldots\ldots$$

The coefficient of $(Z-Z_o)^{-1}$ is $g^{n-1}(Z_o)/(n-1)!$ and is called the residue in the series expansion of $f(Z)$, the latter being called a Laurent series. We now see that if we form the contour integral of $f(Z)$ about any contour C enclosing Z_o we get

$$\oint_C f(Z) dZ = 2\pi j \text{ (residue at } Z_o)$$

$$= 2\pi j \frac{1}{(n-1)!} \frac{d^{n-1}}{dZ^{n-1}} (Z-Z_o)^n f(Z) \Big|_{Z_o} \tag{9}$$

since only the term involving $(Z-Z_o)^{-1}$ gives a non-zero result. In general, if f has many pole singularities within C we get

$$\oint_C f dz = 2\pi j \, \Sigma \text{ (residues of } f \text{ within } C) \tag{10}$$

The other type of singularity commonly encountered is the branch point singularity which arises in connection with multi-valued functions. For example $f = \sqrt{Z}$ has a branch point at $Z = 0$ since if we evaluate f on a small circular contour C enclosing $Z = 0$ we find that after going through an angle 2π that f becomes $-f$, i.e. $f(\rho e^{2j\pi}) = -f(\rho)$. The contour integration theory given above must be modified when f has branch points. However, we will not discuss these modifications here.

Let $f = g(Z)/h(Z)$ where g is analytic at Z_o. If h has a zero of order one at Z_o then f has a pole of order one. The residue at Z_o is given by: residue $= \lim_{Z \to Z_o} \frac{g(Z)}{h(Z)} (Z-Z_o)$. By L'Hospitals' rule the limit is given by $\{d[g(Z-Z_o)]/dZ\ /(dh/dZ)\}$ at Z_o or $[(Z-Z_o)g' + g]/h' = g(Z_o)/h'(Z_o)$ which is a convenient formula for the residue in this special case.

Fourier Transforms in the Complex Plane

Let $f(t)$ be of exponential order at $t = \pm\infty$,

i.e. $f(t) < e^{\alpha_+ t}$ as $t \to +\infty$

$f(t) < e^{-\alpha_- t}$ as $t \to -\infty$

Define $f_+(t) \equiv f(t)\ ,\ t \geq 0$ $\qquad$ $f_-(t) \equiv f(t)\ ,\ t \leq 0$

$\equiv 0\ ,\ t < 0$ $\qquad$ $\equiv 0\ ,\ t > 0$

Let $F_+(w) = \mathcal{F}f_+(t) = \int_{-\infty}^{\infty} e^{-jwt} f_+(t)dt = \int_{0}^{\infty} e^{-jwt} f_+(t)dt$, $w = \omega + j\sigma$.

SINCE $f_+(t)$ IS DEFINED ONLY FOR $t \geq 0$

The integral is uniformly convergent for all w in the lower half plane $\sigma < -\alpha_+$ and defines an analytic function in the lower half plane, i.e. F_+ is analytic for $\sigma < -\alpha_+$. To recover f_+ we use the inversion formula

$$f_+(t) = \frac{1}{2\pi} \int_{C_+} F_+(w) e^{jwt}\, dw$$

The contour C_+ must be chosen to be parallel with the w axis in the lower half plane $\sigma < -\alpha_+$ for the following reasons (see Fig. 3):

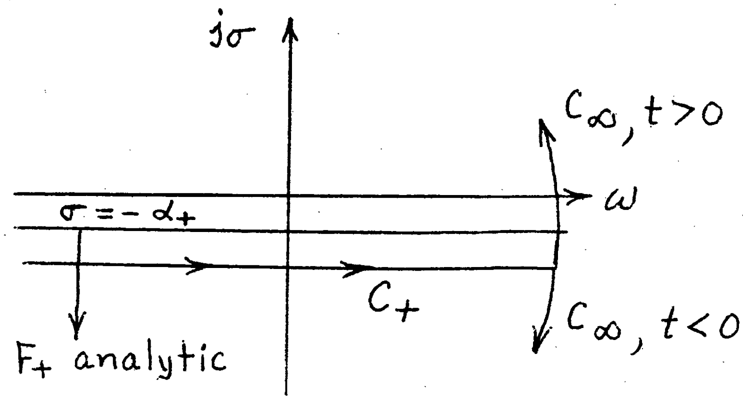


Fig. 3

For $t < 0$, e^{jwt} becomes exponentially small in the lower half plane. Thus $\int_{C_\infty} F_+ e^{jwt} dw = 0$. Hence $\int_{C_+} F_+ e^{jwt} dw$ can be replaced by a contour integral $\oint_{C_+ + C_\infty} F_+ e^{jwt} dw$. But to obtain $f_+(t) \equiv 0$ for $t < 0$ we see that the above integral must vanish and hence C_+ must be chosen so that F_+ is analytic within $C_+ + C_\infty$. For $t > 0$ we can close the contour in the upper half plane to obtain $f_+ = \frac{1}{2\pi} \int_{C_+} F_+ e^{jwt} dw = \frac{1}{2\pi} \oint F_+ e^{jwt} dw = j\, \Sigma$ (residues of F_+ in upper half plane). The contour C_+ can be distorted in any arbitrary manner as long as it is not moved across a singularity of F_+.

The transform of f_- is handled in a similar way. Thus (see Fig. 4)

$$F_-(w) = \int_{-\infty}^{0} f_-(t) e^{-jwt} dt \quad , \quad f_- = \frac{1}{2\pi} \int_{C_-} F_-(w) e^{jwt} dw$$

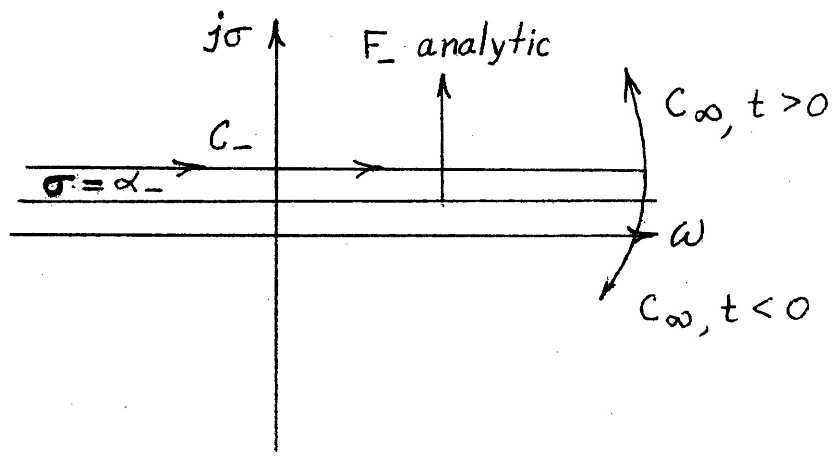


Fig. 4

If α_+ and α_- are such that both F_+ and F_- have a common strip in which both are analytic then we can choose $C_+ = C_- = C$ which is a common inversion contour as in Fig. 5. Then $f = f_+ + f_- = \frac{1}{2\pi} \int_{C_+} F_+ e^{jwt} dw + \frac{1}{2\pi} \int_{C_-} F_- e^{jwt} dw = \frac{1}{2\pi} \int_C F e^{jwt} dw$ where $F = F_+ + F_-$.

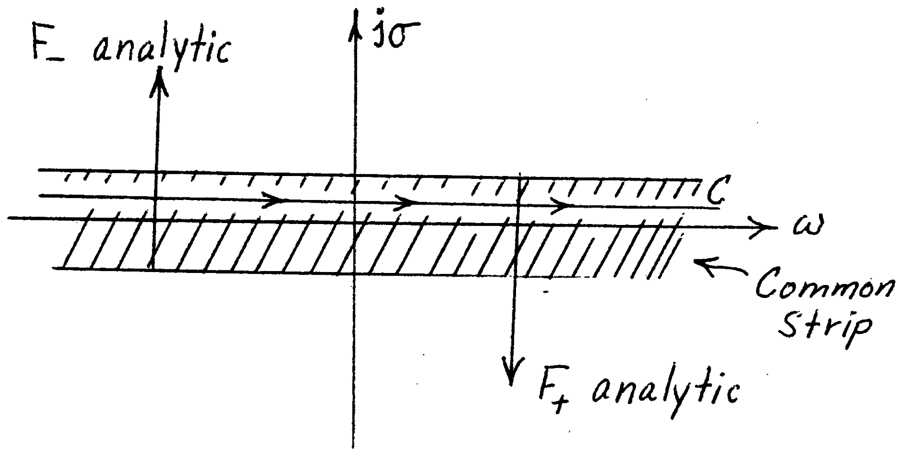


Fig. 5

<u>Eample 1</u>

Let $f(t) = e^{-jk|t|}$, then

$$F_+ = \int_0^{\infty} e^{-jkt-jwt}dt = \left.\frac{e^{-jkt-jwt}}{-j(k+w)}\right|_0^{\infty} = \frac{1}{j(k+w)} \quad , \quad \sigma < 0$$

$$F_- = \int_{-\infty}^{0} e^{jkt-jwt}dt = \left.\frac{e^{jkt-jwt}}{-j(w-k)}\right|_{-\infty}^{0} = \frac{1}{-j(w-k)} \quad , \quad \sigma > 0$$

A common inversion contour C can be chosen as in Fig. 6 by deforming C_+ and C_- without crossing the poles at $w = -k$ for F_+ and $w = k$ for F_-. Then we can write $f = \frac{1}{2\pi}\int_C e^{jwt} F(w)dw$ where $F = F_+ + F_- = \frac{1}{j}(\frac{1}{w+k} - \frac{1}{w-k}) = \frac{-2k}{j(w^2 - k^2)}$.

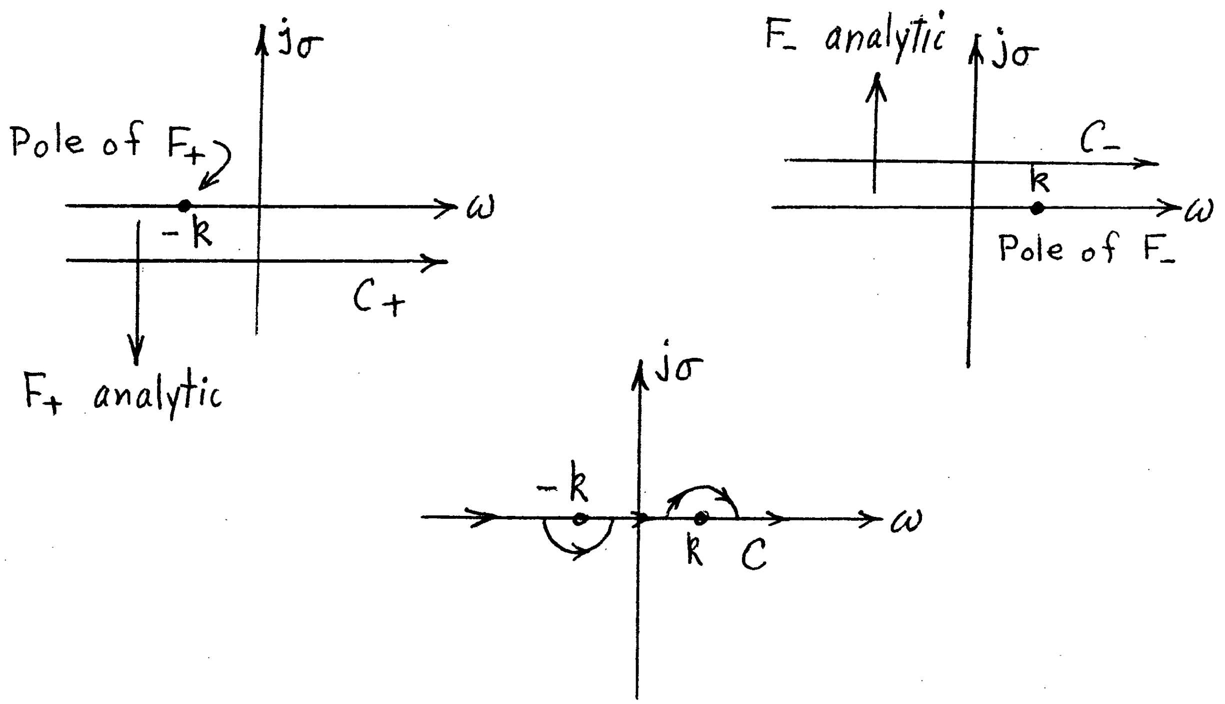


Fig. 6

Example 2

Let $f(t) = e^{-jkt}$, $F_+ = \frac{1}{j(w+k)}$ as before, but $F_- = \int_{-\infty}^{0} e^{-jkt} e^{-jwt} dt = \frac{1}{-j(w+k)}$, $\sigma > 0$.

The inversion contours are illustrated below in Fig. 7 .

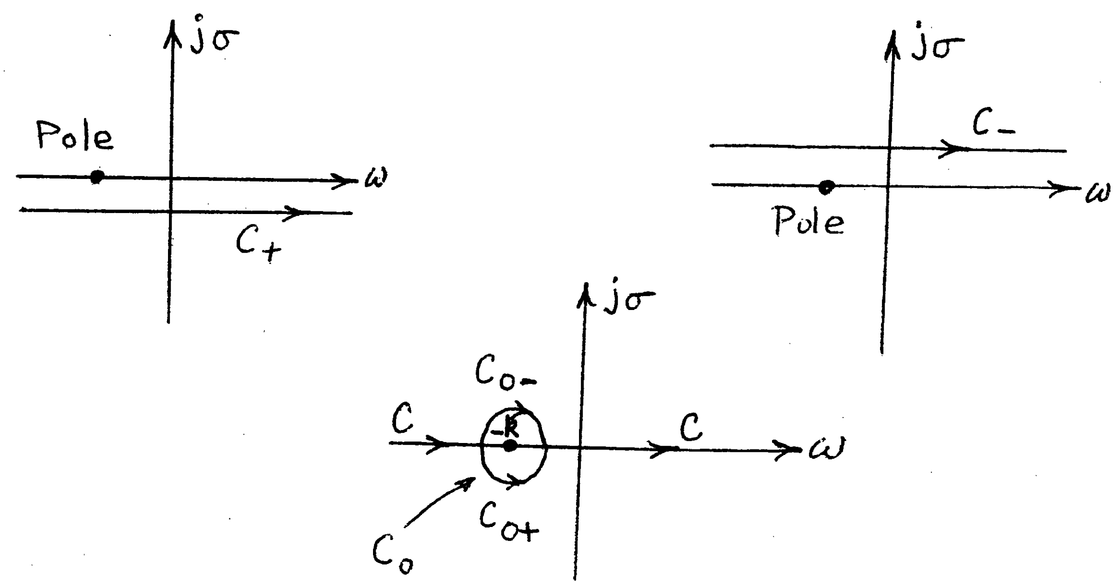


Fig. 7

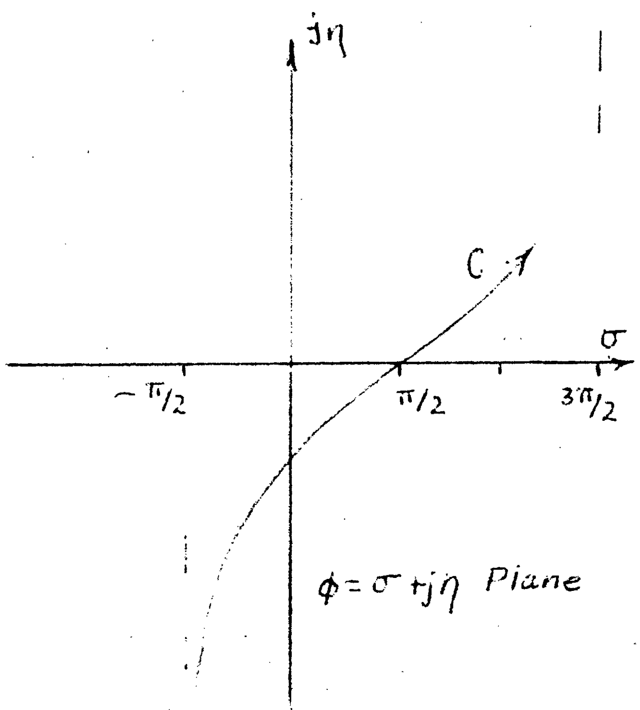
We can deform C_+ and C_- into the common contour C as shown plus the contour C_o around $w = -k$. Thus $f = \frac{1}{2\pi}\int_C (F_+ + F_-)e^{jwt}dw + \frac{1}{2\pi}\int_{C_{0-}} F_- e^{jwt}dw + \frac{1}{2\pi}\int_{C_{0+}} F_+ e^{jwt}dw$. The integrals along C cancel and those along C_{0+} and C_{0-} combine to give $\frac{1}{2\pi}\oint_{C_o} \frac{1}{j(w+k)} e^{jwt}dw$ = residue of $e^{jwt}/(w+k)$ at $-k$ which equals e^{-jkt}. Thus in the complex Fourier transform theory a simple pole shows up in the transform of e^{-jkt} instead of the delta function $2\pi\delta(\omega+k) = \int_{-\infty}^{\infty} e^{-j\omega t-jkt}dt$ which occurs if ω is treated as a real variable.

Asymptotic Evaluation of Integrals by the Saddle Point Method

The saddle point method or method of steepest descents is a generalization of Rayleigh's method of stationary phase for the asymptotic evaluation of certain types of integrals that commonly occur in diffraction and radiation problems. A typical integral is

$$I = \int_C F(\phi) e^{-jk_o r \cos(\phi-\theta)} d\phi$$

where C is the contour illustrated below. The method consists of deforming the contour C into a steepest descent contour along which $e^{-jk_o r \cos(\phi-\theta)}$ will be exponentially decreasing at a maximum rate. The major contribution to I will then come from a short portion of the transformed contour along which F can be represented by a simple series expansion and such that the integral can be evaluated term by term. The first term turns out to be the dominant one.



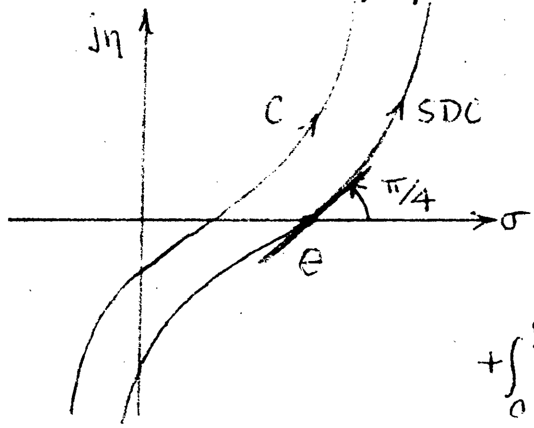
We will consider two important cases, (1) $F(\phi)$ has no singularity in the near vicinity of the saddle point, (2) $F(\phi)$ has a simple pole near the saddle point.

Case I, $F(\phi)$ analytic near saddle point

Let $f(\phi) = -jk_o r \cos(\phi-\theta)$. The saddle points are the values of ϕ for which $df/d\phi = 0$. We have $\frac{df}{d\phi} = jk_o r \sin(\phi-\theta)$ so $\phi = \theta$ is a saddle point. A complex function such as f can have no maxima or minima and hence the stationary points are saddle points. Near the saddle point a Taylor series expansion gives $f = f_1 + jf_2 = -jk_o r + \frac{jk_o r}{2}(\phi-\theta)^2 + \cdots$ since the first derivative vanishes. Let $\phi - \theta = \rho e^{j\psi}$.

Then $f \approx -jk_o r + j\frac{k_o r}{2}\rho^2 \cos 2\psi - \frac{k_o r}{2}\rho^2 \sin 2\psi$.

If we deform C into the steepest descent contour SDC passing through the saddle point θ in such a direction that $\cos 2\psi = 0$, $\sin 2\psi = 1$ the exponential term $e^{f(\phi)}$ becomes $e^{-jk_o r}\, e^{-k_o r\rho^2/2}$. For $k_o r$ very large the exponential term decreases very fast and for $\rho > \rho_1$ becomes negligible. For $k_o r$ large enough ρ_1 is so small that $F(\phi)$ is well approximated by $F(\theta)$ for $\rho \le \rho_1$. Thus $I \sim F(\theta) e^{-jk_o r}\, 2\int_0^{\rho_1} e^{-k_o r\rho^2/2} e^{j\pi/4} d\rho$



since $d\phi = e^{j\pi/4} d\rho$ in quadrant 1 and $-e^{j\pi/4} d\rho$ in quadrant 3, and where we have equated $\int_{\rho_1}^{0} -e^{j\pi/4} e^{-k_o r\rho^2/2} d\rho + \int_0^{\rho_1} e^{j\pi/4} e^{-k_o r\rho^2/2} d\rho$ to $2\int_0^{\rho_1} e^{j\pi/4} e^{-k_o r\rho^2/2} d\rho$

Because of the rapid decay of the exponential we can approximate

$$\int_0^{\rho_1} e^{-k_o r\rho^2/2} d\rho \quad \text{by} \quad \int_0^{\infty} e^{-k_o r\rho^2/2} d\rho = (\pi/2k_o r)^{1/2}$$

with the error $-\int_{\rho_1}^{\infty} e^{-k_o r\rho^2/2} d\rho$ vanishing as $r \to \infty$. Hence

$$I \sim 2F(\theta) e^{-jk_o r + j\pi/4} \left(\frac{\pi}{2k_o r}\right)^{1/2}$$

When $F(\phi) = \frac{1}{\pi}$ we obtain the asymptotic value of the Hankel function $H_o^2(k_o r)$, which is

$$H_o^2(k_o r) \sim \frac{e^{-jk_o r + j\pi/4}}{\sqrt{\pi k_o r/2}}$$

To obtain additional terms in the asymptotic evaluation of I a more careful analysis is required. Very near the saddle point the SDC passes through the saddle point at an angle $\pi/4$ relative to the σ axis. In general the SDC contour is specified by the condition Imag. $[f(\phi) - f(\theta)] = 0$. Thus Imag. $[-jk_o r \cos(\sigma-\theta+j\eta) + jk_o r] = jk_o r\,[1 - \cos(\sigma-\theta) \cosh\eta] = 0$. Along the SDC $e^{f(\phi)} = e^{-jk_o r \cos(\phi-\theta)} = e^{-jk_o r} e^{-k_o r \sin(\sigma-\theta)\sinh\eta}$

At this point it is convenient to change variables according to

$$w^2 = 2j[\cos(\phi-\theta)-1] \quad \text{or} \quad w = 2e^{-j\pi/4} \sin\frac{\phi-\theta}{2}, \text{ i.e.}$$

$$w^2 = -\frac{2}{k_o r}\left[f(\phi) - f(\theta)\right]$$

We then have

$$dw = e^{-j\pi/4} \cos\frac{\phi-\theta}{2}\, d\phi \quad \text{or} \quad d\phi = \frac{e^{j\pi/4}\, dw}{\sqrt{1-jw^2/4}}$$

and our integral becomes

$$I = \int_C F(w)\, e^{-jk_o r + j\pi/4}\, e^{-\frac{k_o r}{2} w^2} \left(1 - jw^2/4\right)^{-1/2} dw$$

where C is the SDC consisting of the real u axis in the w plane ($w = u + jv$). We now expand $F(w)\,(1-jw^2/4)^{-1/2}$ in a Taylor series about the saddle point $w = 0$. Thus let

$$F(w)(1-jw^2/4)^{-1/2} = \sum_{n=0}^{\infty} a_n w^n$$

where

$$a_n = \frac{1}{n!} \frac{d^n}{dw^n}\left[F(w)(1-jw^2/4)^{-1/2}\right]\Bigg|_{w=0}$$

Using the result

$$\int_{-\infty}^{\infty} e^{-k_o r w^2/2}\, w^{2n}\, dw = \left(\frac{2}{k_o r}\right)^{n+1/2} \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{2\pi}\ 1\cdot 3\cdot 5\cdot 7 \cdots (2n-1)}{(k_o r)^{n+1/2}}$$

and noting that terms involving odd powers of w integrate to zero we get

$$I = e^{-jk_0 r + j\pi/4} \sum_{n=0}^{\infty} a_{2n} \left(\frac{2}{k_0 r}\right)^{n+1/2} \Gamma(n+1/2)$$

where Γ is the gamma function.

In the above evaluation of I it was assumed that the Taylor series expansion could be used for all w, i.e. that $F(w)(1-jw^2/4)^{-1/2} = \sum_{n=0}^{\infty} a_n w^n$ converges for all w. Actually the function has branch points at $w = \pm 2\sqrt{-j}$ and F may also have singularities. However, for $k_o r$ sufficiently large the error made in u ing this Taylor series expansion for all w can be shown to be exponentially small. (see B. L. an der Waerden, "On the Method of Saddle Points", Appl. Sci. Research, vol. 2B, pp. 33-43, 1950).

As an example we have

$$H_0^2(x) = \frac{1}{\pi} e^{-jx + j\pi/4} \int_{-\infty}^{\infty} e^{-xw^2/2} (1 - jw^2/4)^{-1/2} dw.$$

But $(1-jw^2/4)^{-1/2} = 1 + j\frac{w^2}{8} - \frac{3}{8}\frac{w^4}{16} - j\frac{15w^6}{48\cdot 64} + \cdots$ and

hence we obtain

$$H_0^2(x) \sim \sqrt{\frac{2}{\pi x}}\, e^{-jx + j\pi/4} \left[1 + \frac{j}{8x} - \frac{9}{128x^2} - j\frac{15^2}{48\cdot 64 x^3} + \cdots\right]$$

which is the standard asymptotic expansion of the Hankel function. Note that this form could not be obtained from the approximate method presented first since in that case we had

$$H_0^2(x) = \frac{1}{\pi} e^{-jx + j\pi/4}\, 2\int_0^{\infty} e^{-x\rho^2/2} d\rho$$

and only the first term is obtained since $F(\phi)$ is a constant.

Case II, F(ϕ) has a pole near saddle point

In some problems F(ϕ) may have a pole very close to the saddle point, in addition the parameters involved in the problem may be such that the pole can be made to coincide with the saddle point. In the w plane it is therefore of interest to consider the case when F(w) has a pole at w_o and w_o lies very close to the saddle point at w = 0. If we write

$$\frac{F(w)}{\sqrt{1-jw^2/4}} \text{ as } \frac{F(w)}{\sqrt{1-jw^2/4}} - \frac{R(w_o)}{\sqrt{1-jw_o^2/4}}\frac{1}{w-w_o} + \frac{R(w_o)}{\sqrt{1-jw_o^2/4}}\frac{1}{w-w_o} \text{ where}$$

$$R(w_o) = \lim_{w \to w_o} (w-w_o)F(w)$$

is the residue of F(w) at the pole w_o then

$$F_1(w) = \frac{F(w)}{\sqrt{1-jw^2/4}} - \frac{R(w_o)}{(w-w_o)\sqrt{1-jw_o^2/4}}$$

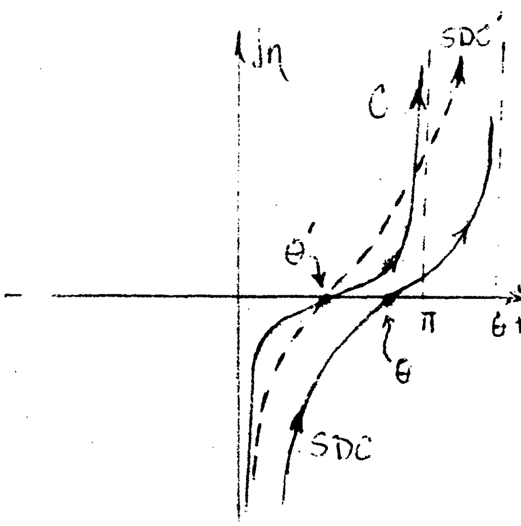
is analytic at w_o. The asymptotic evaluation of

$$I_1 = e^{-jk_or+j\pi/4}\int_{-\infty}^{\infty} F_1(w)\, e^{-k_o r w^2/2}\, dw$$

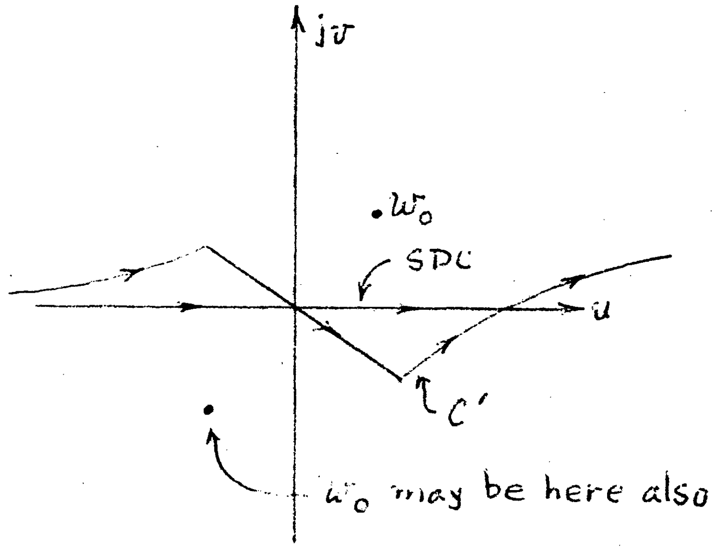
may be carried out according to the method outlined under Case I. The original integral

$$I = I_1 + I_2 \quad \text{where} \quad I_2 = \frac{R(w_o)}{\sqrt{1-jw_o^2/4}}\, e^{-jk_or+j\pi/4}\int_{-\infty}^{\infty}\frac{e^{-k_o r w^2/2}}{w-w_o}\, dw$$

is thus split into two parts. The integral $I_p = \int_{-\infty}^{\infty}\frac{e^{-k_o r w^2/2}}{w-w_o}\, dw$ involved in I_2 can be evaluated exactly. The original contour C and the SDC in the ϕ plane and w plane are illustrated below.



φ Plane



w Plane

The contour C may or may not cross the SDC contour depending on the value of θ . In the φ plane we show a SDC labelled SDC' corresponding to the saddle point θ'. The contour C' in the w plane is then the contour C since the mapping $w = 2e^{-j\pi/4}\sin\frac{\phi-\theta}{2}$ depends on θ.

Initially we will assume that the point w_o is not crossed in deforming C into the SDC. Then if Imag. $w_o > 0$ we have

$$I_p = j\pi e^{-k_0 r w_0^2/2}\left[1 + \operatorname{erf}\left(j\sqrt{\frac{k_0 r}{2}}\, w_0\right)\right], \quad Imag.\ w_0 > 0$$

where the error function erf x is given by

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt. \quad \text{For Imag. } w_0 < 0$$

we have

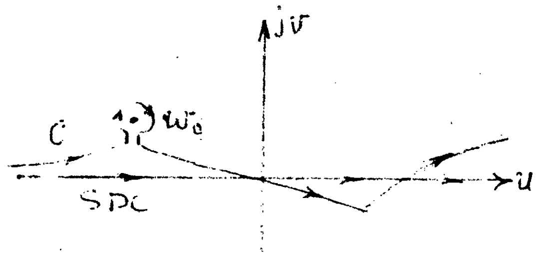
$$I_p = j\pi e^{-k_0 r w_0^2/2}\left[-1 + \operatorname{erf}\left(j\sqrt{\frac{k_0 r}{2}}\, w_0\right)\right]$$

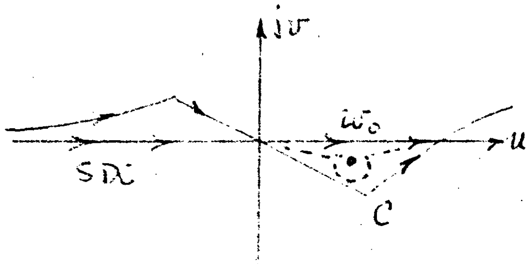
$$= -j\pi e^{-k_0 r w_0^2/2}\operatorname{erfc}\left(j\sqrt{\frac{k_0 r}{2}}\, w_0\right), \quad Imag.\ w_0 < 0$$

where $\operatorname{erfc} x = 1 - \operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ is the complement of the error function. When Im. w = 0 the Cauchy principal value is taken to give

$$I_p = j\pi e^{-k_o r w_o^2/2} \operatorname{erf}\left(j\sqrt{\tfrac{k_o r}{2}}\, w_o\right), \quad \text{Imag. } w_o = 0$$

Consider now what happens if w_o is located such that C crosses w_o when it is deformed into the SDC contour. If Im. $w_o > 0$ initially then when C crosses w_o we must add a term $-2\pi j\, e^{-k_o r w_o^2/2}$ to I_p (see Figure and note that the integral around w_o is in a clockwise direction.) If Im. $w_o < 0$ then we must add a term $2\pi j\, e^{-k_o r\, w_o^2/2}$ to I_p since the integration around w_o will now be in the positive sense. For the original integral along C we now have the following results.





(1) $$\int_C = I_1 + j\pi e^{-\frac{k_o r w_o^2}{2}} A\left[1 + \operatorname{erfc}\left(j\sqrt{\tfrac{k_o r}{2}}\, w_o\right)\right], \quad \text{Imag. } w_o > 0$$

and pole not crossed in deforming C or when Im $w_o < 0$ and the pole <u>is</u> crossed in deforming C, where A is the constant $\dfrac{R(w_o)}{\sqrt{1-j w_o^2/4}}\, e^{-jk_o r + j\pi/4}$

(2) $$\int_C = I_1 - j\pi A e^{-k_o r w_o^2/2} \operatorname{erfc}\left(j\sqrt{\tfrac{k_o r}{2}}\, w_o\right), \quad \text{Imag. } w_o < 0$$

for C not crossing the pole or Im. $w_o > 0$ and C crossing the pole when it is deformed into the SDC.

$$(3) \int_C = I_1 + j\pi A e^{-k_o r w_o^2/2} \operatorname{erf}\left(j\sqrt{\frac{k_o r}{2}} w_o\right), \quad \text{Imag. } w_o = 0$$

Note that we may write I_p in the form

$$I_p = \int_{-\infty}^{\infty} \frac{e^{-k_o r w^2/2} - e^{-k_o r w_o^2/2}}{w - w_o} dw + e^{-k_o r w_o^2/2} \int_{-\infty}^{\infty} \frac{dw}{w - w_o}$$

The first integral is analytic at w_o and equal to

$j\pi e^{-k_o r w_o^2/2} \operatorname{erf}\left(j\sqrt{\frac{k_o r}{2}} w_o\right)$ while the second integral has the value

$$\int_{-\infty}^{\infty} \frac{dw}{w - w_o} = \begin{cases} j\pi, & \text{Imag. } w_o > 0 \\ 0, & \text{Imag. } w_o = 0 \\ -j\pi, & \text{Imag. } w_o < 0 \end{cases}$$

and accounts for the discontinuous behavior of I_p as a function of w_o.

Method of Stationary Phase

The method of stationary phase for finding asymptotic values of integrals is closely related to the method of steepest descents. To explore this relationship we will examine the behavior of $f(\phi)$ in the vicinity of a saddle point more closely. The integral of concern is of the form $I = \int_C F(\phi) e^{f(\phi)} d\phi$ where C is a given contour in the complex $\phi = \sigma + j\eta$ plane.

Saddle points occur when $\partial f/\partial\phi = 0$. Let θ be a particular saddle point and consider $g(\phi) = g_1 + jg_2 = f(\phi) - f(\theta)$. Near the saddle point g has the form

$$g(\phi) = g(\theta) + g'(\theta)(\phi-\theta) + \frac{1}{2} g''(\theta)(\phi-\theta)^2 + \cdots = \frac{1}{2} g''(\theta)(\phi-\theta)^2$$

since $g(\theta)$ and $g'(\theta)$ are zero. Let $\rho e^{j\omega}$ be the polar form for the variable $\phi-\theta$. Then $g(\phi) = \frac{1}{2} g''(\theta) \rho^2 e^{2j\omega}$

so we can write

$$g_1 = \frac{1}{2} g'' \rho^2 \cos 2\omega , \qquad g_2 = \frac{1}{2} g'' \rho^2 \sin 2\omega$$

where g" has been assumed to be real. There is no loss in generality in assuming g" real since if it is complex we have $g'' = |g''| e^{j\gamma}$ and γ can be absorbed in the angle ω by choosing a new origin (i.e. x axis orientation) for ω.

The contours g_1 = const. are orthogonal to the contours g_2 = const. because of the Cauchy-Riemann equations

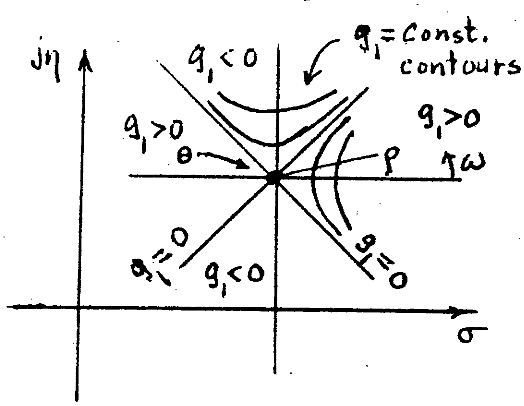
$$\frac{\partial g_1}{\partial \sigma} = \frac{\partial g_2}{\partial \eta} , \quad \frac{\partial g_1}{\partial \eta} = - \frac{\partial g_2}{\partial \sigma}$$

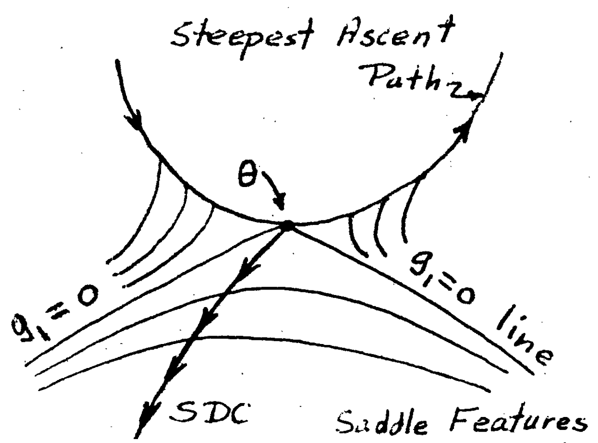
If we assume g" to be real and positive then g_1 is positive in quadrants

$$-\frac{\pi}{4} < \omega < \frac{\pi}{2} , \quad \frac{3\pi}{4} < \omega < \frac{5\pi}{4}$$

and negative in the remaining quadrants. Also g_1 increases in magnitude away from the saddle point and hence rises in the quadrants

$-\frac{\pi}{4}$ to $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ to $\frac{5\pi}{4}$ and falls in algebraic value in the other two quadrants. Thus if g_1 is viewed as the elevation of land around the saddle point it has the topological features associated with a saddle as shown in the Figure.





Saddle Features of g_1 around θ

Note that there is a steepest descent path through the saddle point along which g_1 increases negatively as rapidly as possible. Along this path the contours g_1 = cost. are cut at right angles so the path has the direction of

$$\nabla g_1 = \frac{\partial g_1}{\partial \sigma} + j \frac{\partial g_1}{\partial \eta}$$

and coincides with a g_2 = const. contour ($g_2 = 0$ in this example). If the contour C can be deformed into a steepest descent contour it is clear that $e^{f(\phi)}$ decreases rapidly along this path because it equals $e^{-\frac{1}{2} g'' p^2}$ along this contour.

Exactly the same discussion can be applied to the function g_2. Thus the paths of steepest ascent and descent for g_2 occur along the lines

$\nabla g_2 = \frac{\partial g_2}{\partial \sigma} + j \frac{\partial g_2}{\partial \eta} = g_1 =$ const. contour. The latter paths are located at 45° with respect to the corresponding steepest ascent or descent paths for g_1. Along a steepest descent path for g_2 our integral would become

$$I = \int_{\substack{SDC \\ \text{for } g_2}} F(\phi) e^{-j\frac{1}{2} g'' \rho^2} d\rho$$

The argument can now be made that because of the rapid variation in phase of the integrand the major contribution to the integral comes from a small region near the saddle point. However, this argument can be replaced by a different one, namely showing that the contour for the above integral can be deformed into a steepest descent contour for g_1 instead of g_2 and value of the integral justified on the basis that it agrees with what is obtained by integrating along the SDC for g_1.

Thus the method of stationary phase can be justified in terms of the more intuitively clear arguments used in the standard method of steepest descents by shifting the contour of integration. In deforming these contours any contributions from residues of poles swept across must be properly added on. Also the paths at infinity connecting the deformed contours must be considered. Usually these do not give a contribution. The contour yielding the maximum rate of change of phase is given exactly by

$$Re\left[f(\phi) - f(\epsilon) \right] = 0$$

for all values of ϕ.

Infinite Products

Some waveguide problems require the construction of infinite products. It is thus necessary to have an understanding of some of the properties of infinite products, conditions required for convergence, and the asymptotic behavior of infinite products.

A.5. Infinite Products and the Gamma Function

Let $f(z)$ be an integral function, i.e., a function with no singularities in the finite complex z plane. The logarithmic derivative of this function is then a meromorphic function, i.e., a function whose only singularities are poles. Let $f'(z)/f(z)$ have simple poles at $z = z_n$ with unit residues. The partial-fraction expansion is

$$\frac{f'(z)}{f(z)} = \frac{f'(z)}{f(z)}\bigg|_{z=0} + \sum_n \left(\frac{1}{z - z_n} + \frac{1}{z_n}\right)$$

Integrating with respect to z from 0 to z gives

$$\ln f(z)\Big|_0^z = \frac{f'(0)}{f(0)} z + \sum_n \left(\ln \frac{z - z_n}{-z_n} + \frac{z}{z_n}\right)$$

or

$$f(z) = f(0)e^{[f'(0)/f(0)]z} \prod_n \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

which is the infinite-product expansion of the integral function $f(z)$. When $f(z)$ is an even function $f_e(z)$ of z, $f_e'(0)$ equals zero, and we get

$$f_e(z) = f_e(0) \prod_n \left(1 - \frac{z^2}{z_n^2}\right)$$

Consider the function $\cos z$, which has zeros at $z = n\pi + \pi/2 = (n + \frac{1}{2})\pi$. Using the general formula gives the following infinite-product expansion:

$$\cos z = \prod_{n=0}^{\infty} \left[1 - \frac{z^2}{(n + \frac{1}{2})^2\pi^2}\right]$$

Similarly the infinite-product expansion of $(\sin z)/z$ is found to be

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right)$$

Gamma Function

The gamma function is defined by the following integral:

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1}\, du$$

The integral defines an analytic function of z for all z for which the real part is positive. An equivalent definition of the Γ function is the following infinite-product representation:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \left[\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}\right]^{-1}$$

where $\gamma = 0.57722$ and is Euler's constant. The infinite-product representation defines $\Gamma(z)$ as an analytic function throughout the complex z plane, except in the vicinity of the negative-real axis, where $\Gamma(z)$ has simple poles. The following useful properties of the gamma function have been established:

1. $\Gamma(z+1) = z\Gamma(z)$.
2. $\Gamma(n+1) = n!$, n an integer.
3. $\Gamma(\frac{1}{2}) = \pi^{1/2}$.
4. $2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \pi^{1/2}\Gamma(2z)$.
5. $\Gamma(z)\Gamma(1-z) = \left[z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\right]^{-1} = \frac{\pi}{\sin \pi z}$.
6. As $z \to \infty$, the asymptotic value of $\Gamma(z)$ is $(2\pi)^{1/2}e^{z \ln z}z^{-1/2}e^{-z+1}$, valid for all z except in the region of the negative-real axis, where $\Gamma(z)$ has poles at $z = 0, -1, -2, \ldots$, etc. From (5) an asymptotic expansion valid for z on the negative-real axis may be found.
7. For x finite and $|y|$ very large, $|\Gamma(x+jy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\pi|y|/2}e^{1-x}$.

Example 1

Construct a function with simple poles at $w = \Gamma_n = [(n\pi/a)^2 - k_0^2]^{1/2}$ and find its asymptotic behavior for large w. n takes on the values $1, 2, 3, \cdots \infty$

Solution

The series $\sum_{n=1}^{\infty}\left(\frac{1}{w-\Gamma_n} + \frac{1}{n\pi/a}\right)$ is convergent since Γ_n approaches $n\pi/a$ for n large. Hence $e^{wa/n\pi}$ are suitable convergence factors to use in an infinite product. Let

$$f(w) = \prod_{n=1}^{\infty}\left(1 - \frac{w}{\Gamma_n}\right)e^{wa/n\pi}$$

Then $1/f(w)$ has the required simple poles. The asymptotic behavior can be established by comparison with the gamma function. Consider

$$\frac{\prod_{n=1}^{\infty}\left(1 - \frac{wa}{n\pi}\right)e^{wa/n\pi}}{\prod_{n=1}^{\infty}\left(1 - w/\Gamma_n\right)e^{wa/n\pi}} \times \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{wa}{n\pi}\right)e^{wa/n\pi}} =$$

$$\prod_{n=1}^{\infty} \frac{(1 - wa/n\pi)}{(1 - w/\Gamma_n)} \quad \frac{\prod_{n=1}^{\infty}\left(1 + \frac{wa}{n\pi}\right) e^{-wa/n\pi}}{\prod_{n=1}^{\infty}\left(1 - \frac{w^2 a^2}{n^2\pi^2}\right)}$$

$$= \prod_{n=1}^{\infty} \frac{(1 - wa/n\pi)}{(1 - w/\Gamma_n)} \quad \frac{wa}{\sin wa} \quad \frac{e^{-\gamma wa}}{wa\,\Gamma(wa)} .$$

Since $\Gamma_n \to n\pi/a$ the first product approaches the constant $\prod_{n=1}^{\infty} \frac{\Gamma_n a}{n\pi}$ as $w \to \infty$. Thus we find that

$$\frac{1}{f(w)} \sim K \frac{e^{-\gamma wa}}{\sin wa} \quad \frac{\sqrt{wa}}{\sqrt{2\pi}} \quad \frac{e^{-wa \ln wa}}{e^{-wa+1}}$$

$$= \frac{K}{\sin wa} \frac{\sqrt{wa}}{e\sqrt{2\pi}} \, e^{-wa[\gamma - 1 + \ln wa]} \qquad \text{as } w \to \infty$$

where K is a suitable constant.

Example 2

Find the infinite product expansion of $J_0(w)$ and the asymptotic behavior of the part that has zeroes in the left half of the complex w plane.

Solution Let $\pm w_n$ be the zeroes of $J_0(w) = 0$. Since $J_0(w) = J_0(-w)$, $J_0'(0) = 0$, thus

$$J_0(w) = \prod_{n=1}^{\infty}\left(1 - w^2/w_n^2\right) = \prod_{n=1}^{\infty}\left(1 - \frac{w}{w_n}\right) e^{w/w_n}$$

$$\times \prod_{n=1}^{\infty}\left(1 + \frac{w}{w_n}\right) e^{-w/w_n} .$$ Other convergence

factors can also be used. For large w we know that $J_0(w) \sim \sqrt{\frac{2}{\pi w}} \cos(w - \pi/4)$. Hence w_n is asymptotic to $(2n-1)\pi/2 + \pi/4$, $n = 1, 2, 3, ----$ i.e. $w_n \sim (4n-1)\pi/4$. Hence we can also write

$$J_0(w) = \prod_{n=1}^{\infty}\left(1 - \frac{w}{w_n}\right)e^{\frac{4w}{4n\pi}} \prod_{n=1}^{\infty}\left(1 + \frac{w}{w_n}\right)e^{-\frac{4w}{4n\pi}}$$

The part whose asymptotic behavior we want is

$$f(w) = \prod_{n=1}^{\infty}\left(1 + \frac{w}{w_n}\right)e^{-\frac{4w}{4n\pi}}$$

$$= \prod_{n=1}^{\infty} \frac{(1 + w/w_n)}{\left(1 + \frac{4w}{(4n-1)\pi}\right)} \prod_{n=1}^{\infty}\left(1 + \frac{4w}{(4n-1)\pi}\right)e^{-\frac{w}{n\pi}}$$

The first product approaches a constant as $w \to \infty$. Hence we need to find the asymptotic behavior of

$$g(w) = \prod_{n=1}^{\infty}\left(1 + \frac{4w}{(4n-1)\pi}\right)e^{-\frac{w}{n\pi}}$$

which can not be directly compared with the gamma function. We note that $1 + 4w/(4n-1)\pi = \left(n - \frac{1}{4} + \frac{w}{\pi}\right)/(n - 1/4)$ $= \left(1 + \frac{\hat{w} - 1/4}{n}\right)/(1 - 1/4n)$ and also that $4w/4n\pi$ $= \frac{\hat{w} - 1/4}{n} + \frac{1}{4n}$ where $\hat{w} = w/\pi$. Hence

$$g(w) = \prod_{n=1}^{\infty} \frac{1 + \frac{\hat{w} - 1/4}{n}}{1 - \frac{1}{4n}} e^{-\frac{\hat{w} - 1/4}{n}} e^{-\frac{1}{4n}}$$

$$= \frac{\prod_{n=1}^{\infty}\left(1+\frac{\hat{w}-1/4}{n}\right)e^{-\frac{\hat{w}-1/4}{n}}}{\prod_{n=1}^{\infty}\left(1-\frac{1}{4n}\right)e^{1/4n}} = \frac{e^{-\gamma(\hat{w}-1/4)}}{(\hat{w}-\frac{1}{4})\Gamma(\hat{w}-\frac{1}{4})\prod_{n=1}^{\infty}(1-\frac{1}{4n})e^{1/4n}}$$

$$\sim K_1\,\hat{w}^{-1/2}e^{-\hat{w}[\gamma-1+\ln\hat{w}]+\frac{1}{4}[\gamma-1+\ln\hat{w}]}$$

$$= K_2\,\hat{w}^{-1/4}e^{-\hat{w}(\gamma-1+\ln\hat{w})}$$

since $e^{\frac{1}{4}\ln\hat{w}} = \hat{w}^{1/4}$. K_1 and K_2 are constants. If we had simply used $w_n \sim 4n\pi/4 = n\pi$ we would have gotten a $\hat{w}^{-1/2}$ instead of $\hat{w}^{-1/4}$ factor which would not have been correct. A shift in the zeroes of a function away from the integers n does effect the asymptotic behavior.

There is considerable latitude in the choice of exponential convergence factors that can be used. The requirement is that the series

$$S = \sum_{n=1}^{\infty}\left(\frac{1}{w-w_n}+\frac{1}{s_n}\right)$$

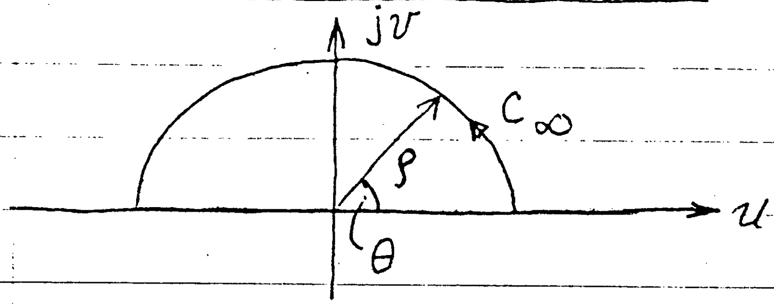
converges. If we choose $s_n = \frac{1}{n\pi}$ then

$$S \sim \sum_{n=1}^{\infty}\left(\frac{1}{w-\frac{4n-1}{4}\pi}+\frac{1}{n\pi}\right)$$

$$= \sum_{n=1}^{\infty}\frac{w+\pi/4}{n\pi\left(w-\frac{4n-1}{4}\pi\right)}$$

which is a convergent series.

JORDAN'S LEMMA



$$\lim_{\rho\to\infty} \int_{C_\infty} e^{j w t} F(w)\, dw = 0 \quad \text{provided} \lim_{|w|\to\infty} |F(w)| = K|w|^{\epsilon}$$

with $\epsilon < 0$, $t > 0$.

On semicircle $w = \rho\cos\theta + j\rho\sin\theta$ and

$$\left|\lim_{\rho\to\infty} \int_{C_\infty} e^{j w t} F\, dw\right| \leq \lim_{\rho\to\infty} \int_{C_\infty} |e^{j w t}||F|\, \rho\, d\theta$$

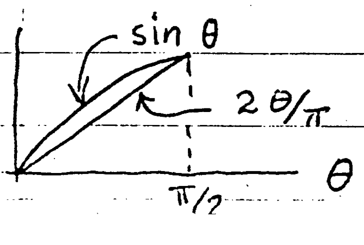
$$= \lim_{\rho\to\infty} \int_0^{\pi} e^{-\rho t \sin\theta} K \rho^{1+\epsilon}\, d\theta$$

$$= \lim_{\rho\to\infty} K\rho^{1+\epsilon}\, 2\int_0^{\pi/2} e^{-\rho t \sin\theta}\, d\theta$$

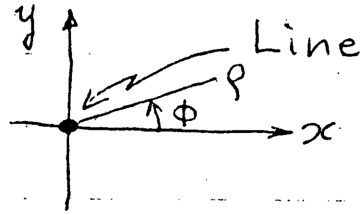
$$\leq \lim_{\rho\to\infty} K\rho^{1+\epsilon}\, 2\int_0^{\pi/2} e^{-\rho t \frac{2\theta}{\pi}}\, d\theta \quad \text{since } \sin\theta > \frac{2\theta}{\pi}$$

for $0 \leq \theta \leq \pi/2$. We thus obtain

$$\lim_{\rho\to\infty} K\rho^{1+\epsilon}\, 2\, \frac{e^{-\rho t} - 1}{-2\rho t/\pi} = 0 \quad \text{for } \epsilon < 0.$$



Fourier Transform With Branch Points



$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + k_0^2\psi = -\delta(x)\,\delta(y)$$

In cylindrical coord's we know that $\psi(\rho,\phi) = -\frac{j}{4}H_0^2(k_0\rho)$

We will solve problem in rectangular coord's.

Let $\hat{\psi}(y,w) = \int_{-\infty}^{\infty} \psi(y,x)\, e^{jwx}\, dx$

Then $\frac{\partial^2\hat{\psi}}{\partial y^2} + (k_0^2 - w^2)\hat{\psi} = -\delta(y)$

Assume $\hat{\psi} = C\, e^{-j\sqrt{k_0^2-w^2}\,|y|}$ where the branch used is $\text{Imag}\sqrt{k_0^2-w^2} < 0$ to ensure bounded solution as $|y| \to \infty$.

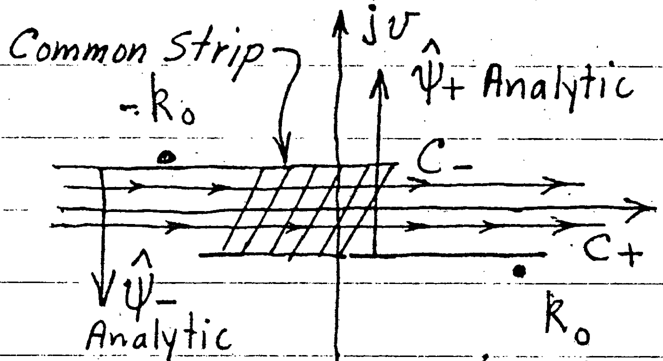
We need $\left.\frac{\partial\hat{\psi}}{\partial y}\right|_{0-}^{0+} = -1 = -2jC\sqrt{k_0^2-w^2}$

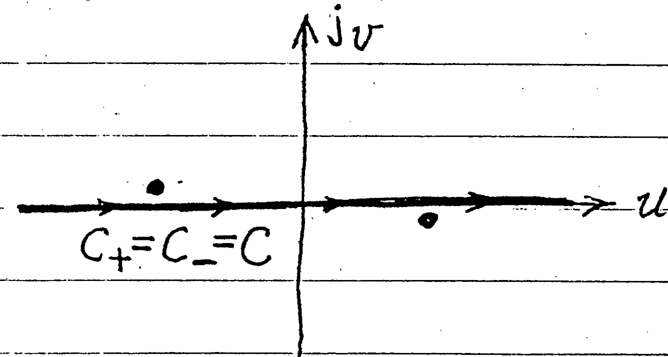
Hence $\hat{\psi} = -(j/2\sqrt{k_0^2-w^2})\, e^{-j\sqrt{k_0^2-w^2}\,|y|}$ and

$$\psi(y,x) = \frac{-j}{4\pi}\int_C \frac{e^{-j\sqrt{k_0^2-w^2}\,|y| - jwx}}{\sqrt{k_0^2-w^2}}\, dw$$

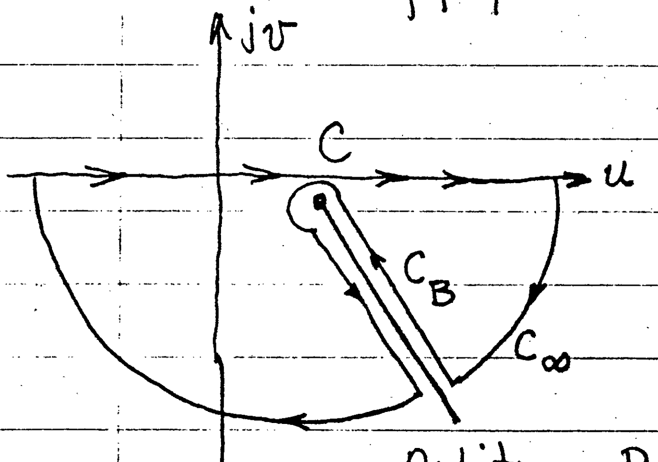
Now $\sqrt{k_0^2-w^2} = \sqrt{w-k_0}\,\sqrt{w+k_0}\,\sqrt{-1}$ so there are branch points at $w = \pm k_0$. We know that for $y=0$, $\psi \sim \frac{e^{-jk_0|x|}}{\sqrt{|x|}}$ as $|x| \to \infty$. ∴ if k_0 is real $\hat{\psi}_+(0,w) = \int_0^{\infty} \psi(0,x)\, e^{jwx}\, dx$ is analytic in UHP $\text{Imag}\; w > 0$. Similarly $\hat{\psi}_-(0,w)$ is analytic in LHP $\text{Imag}\; w < 0$.

The technique used to find the common inversion contour is to let $k_0 = k_0' - jk_0''$ which moves the branch points off the real axis. Now $\psi_+ \sim e^{-k_0'' x}$ as $x \to \infty$ so $\hat{\psi}_+$ is analytic for $\text{Imag}\, w > -k_0''$. Thus C_+ is chosen as shown in Figure below. Since $\hat{\psi}_-$ is analytic for $\text{Imag}\, w < k_0''$ C_- runs below the branch point at $-k_0$





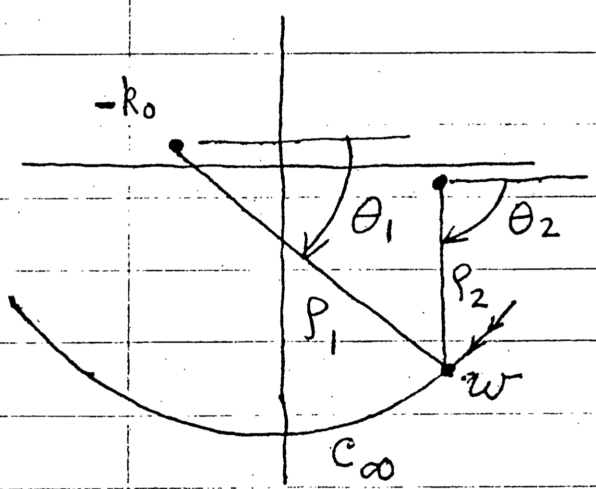
When $x > 0$, $e^{-jwx} = e^{-jux + vx}$ becomes small in lower half plane. We will close contour in lower half plane but <u>must</u> <u>not</u> <u>encircle</u> <u>a</u> <u>branch</u> <u>point</u> if we wish to apply residue theory. No poles are enclosed so $\int_C + \int_{C_B} + \int_{C_\infty} = 0$



Arbitrary Branch Line Running to ∞

Hence
$$\psi = -\left[\int_{C_B} + \int_{C_\infty} \frac{-j}{4\pi} e^{-jwx} \frac{e^{-j\sqrt{k_0^2 - w^2}\,|y|}}{\sqrt{k_0^2 - w^2}} dw\right]$$

It is desirable to choose the branch line such that $\int_{C_\infty} = 0$ if possible. This will be the case if $\text{Imag}\sqrt{k_0^2 - w^2} \leq 0$ everywhere on C_∞. Let us write $\sqrt{k_0^2 - w^2} = -j\sqrt{w^2 - k_0^2}$. Then we require $\text{Re}\sqrt{w^2 - k_0^2} \geq 0$ on C_∞.



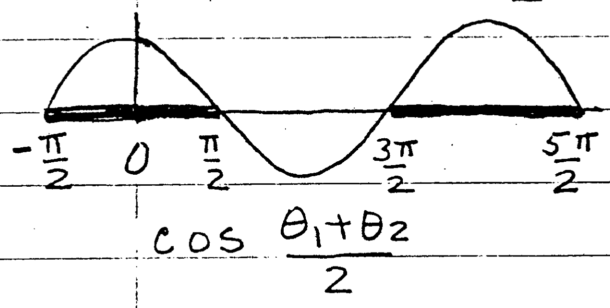
$$\sqrt{w + k_0} = \sqrt{\rho_1}\, e^{j\theta_1/2}$$

$$\sqrt{w - k_0} = \sqrt{\rho_2}\, e^{j\theta_2/2}$$

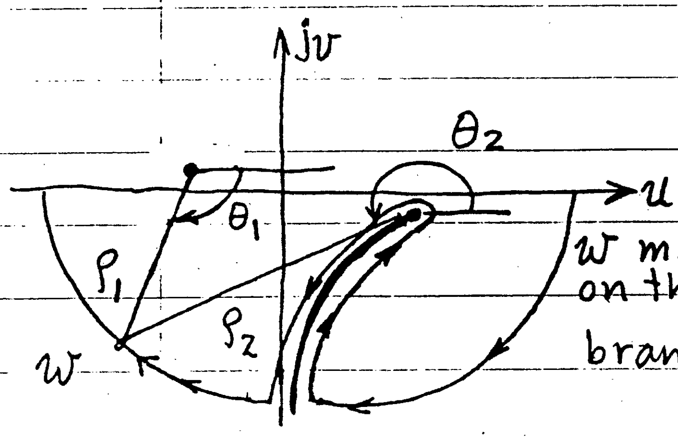
$$\sqrt{w^2 - k_0^2} = \sqrt{\rho_1 \rho_2}\, e^{j\frac{\theta_1 + \theta_2}{2}}$$

In order to have a positive real part $\cos\frac{\theta_1+\theta_2}{2} \geq 0$

Thus $-\frac{\pi}{2} \leq \frac{\theta_1+\theta_2}{2} \leq \frac{\pi}{2}$, $\frac{3\pi}{2} \leq \frac{\theta_1+\theta_2}{2} \leq \frac{5\pi}{2}$, etc.



Condition is violated if w moves on C_∞ past imaginary axis. Thus we must run branch line along imaginary axis at infinity.



w moves on this path and $\cos\frac{\theta_1+\theta_2}{2} \geq 0$ for branch cut chosen as in Figure.

We now have for $x > 0$

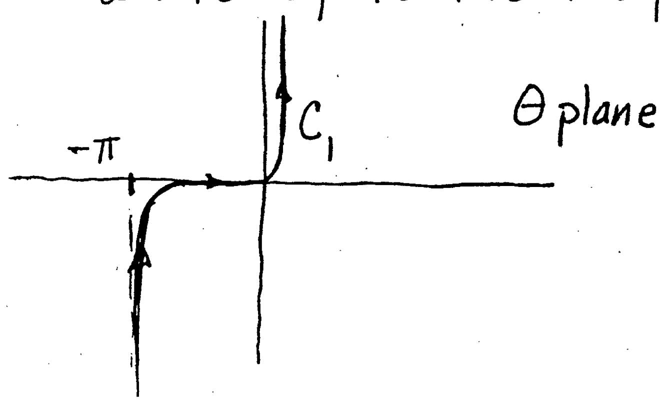
$$\psi(y, x) = -\frac{1}{4\pi} \int_{C_B} \frac{e^{-jwx - \sqrt{w^2 - k_0^2}\,|y|}}{\sqrt{w^2 - k_0^2}}\, dw$$

where C_B is the contour running around the branch cut shown in the Figure on Page 29. For $x < 0$ we can change to a branch cut integral in the upper half plane. If we then change w to $-w$ in that integral we find that this is equivalent to using the integral above with x replaced by $|x|$ and is valid for all x. The change from the contour C to C_B is useful only if the latter integral is easier to evaluate or if dominant residue waves are picked up so that the remaining branch cut integral is a small correction. For this problem no advantage is gained.

If we change to cylindrical coordinates $x = \rho \cos\phi$, $y = \rho \sin\phi$, $w = k_0 \cos\theta$, we obtain

$$\psi(\rho, \phi) = -\frac{j}{4\pi} \int_{C_1} e^{-jk_0\rho\cos(\theta - \phi)}\, d\theta$$

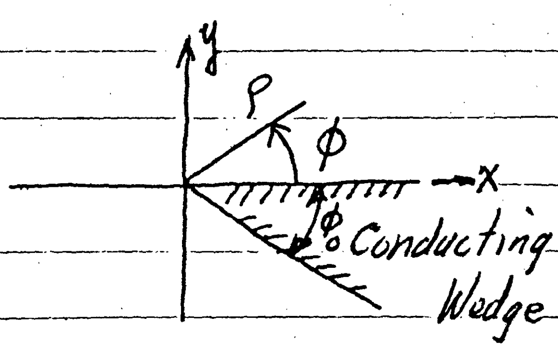
where C_1 is the mapping of C into the complex θ plane as shown.



The integral is a standard one for the Hankel function.

EDGE CONDITIONS

At a sharp edge or corner of a perfect conductor some of the field components may become infinite but the order of the singularity allowed is such that no net power is radiated from edge (no source at the edge is assumed). An equivalent statement is that the energy in the field in the small volume around the edge must be finite.



For a 2 dimensional wedge the field can be split into TE and TM fields relative to z axis.

TM Waves

Let $E_z = \psi_e(\rho,\phi)e^{-j\beta z}$. Solutions for E_z are of the form

$$\psi_e = \begin{Bmatrix} \sin \nu\phi \\ \cos \nu\phi \end{Bmatrix} \begin{Bmatrix} J_\nu(k\rho) \\ Y_\nu(k\rho) \end{Bmatrix}, \quad k = \sqrt{k_0^2 - \beta^2}$$

The Y_ν functions are too singular near $\rho = 0$ so they can not be used. In order to satisfy the boundary conditions at $\phi = 0, \phi_0$, $\sin \nu\phi = \sin \frac{n\pi\phi}{2\pi - \phi_0}$ so $\nu_n = \frac{n\pi}{2\pi - \phi_0}$

Since J_ν behaves like ρ^ν for small ρ we find that $E_z \sim \rho^\nu$. The smallest ν_n is $\nu_1 = \pi/(2\pi - \phi_0)$

For $\phi_0 = 0$ (half plane) $E_z \sim \rho^{1/2}$

For $\phi_0 = \frac{\pi}{2}$ (90° corner) $E_z \sim \rho^{2/3}$

From Maxwell's Eq's it is readily found that near $\rho=0$
E_ρ and $H_\phi \sim \rho^{\nu-1}$, E_ϕ and $H_\rho \sim \rho^{\nu}$
Thus for $\phi_0 = 0$ (half plane) $E_\rho \sim \rho^{-1/2}$, and $J_z \sim \rho^{-1/2}$
and for $\phi_0 = \pi/2$, $E_\rho \sim \rho^{-1/3}$, and $J_z \sim \rho^{-1/3}$.
The normal field components and parallel (to the edge) currents can be singular.

TE Waves

Let $H_z = \psi_h(\rho,\phi)\, e^{-j\beta z}$. Solutions for ψ_h are of the form
$$\psi_h = \cos\frac{n\pi\phi}{2\pi-\phi_0}\, J_{\nu_n}(k\rho)$$

Now $n=0$ is allowed so there is one solution with $H_z = J_0(k\rho)\, e^{-j\beta z}$, and a H_ρ and E_ϕ field also. For this solution H_z and $J_\rho \sim$ constant as $\rho \to 0$ and H_ρ and $E_\phi \sim 0$ as $\rho \to 0$. For $n \neq 0$ the fields and currents have the same singular behavior as for the TM waves.

Final Value Theorem

This theorem gives the asymptotic behavior of the Fourier transform as $|\omega| \to \infty$ in terms of the behavior of $f(x)$ as $x \to 0$. Consider a function $f(x) \equiv 0$ for $x<0$, $f(x) \neq 0$ for $x>0$, and let $f(x) \sim Kx^{\alpha}$ as $x \to 0$. Also let this be the dominant singularity of $f(x)$ if $\alpha < 0$. We have
$$F(\omega) = \int_0^{\infty} e^{j\omega x} f(x)\, dx$$
Let $\omega x = \lambda$, $dx = d\lambda/\omega$, then

$$F(\omega) = \int_0^\infty e^{j\lambda} f\left(\frac{\lambda}{\omega}\right) \frac{d\lambda}{\omega}$$

Now let $\omega \to \infty$ and use $f(\lambda/\omega) \sim K(\lambda/\omega)^\alpha$ to obtain

$$\lim_{\omega\to\infty} F(\omega) = \int_0^\infty e^{j\lambda} K \frac{\lambda^\alpha}{\omega^{\alpha+1}} d\lambda = \omega^{-\alpha-1}\left(\int_0^\infty K\lambda^\alpha e^{j\lambda} d\lambda\right)$$

Hence the asymptotic behavior of $F(\omega)$ is determined by that of $f(x)$ at $x=0$. If $f(x) \sim x^{1/2}$ then $F(\omega) \sim \omega^{-3/2}$ while if $f(x) \sim x^{-1/2}$ then $F(\omega) \sim \omega^{-1/2}$.

A more complex situation occurs when $f(x)$ is non-zero over only a finite closed interval $-\ell \le x \le \ell$. The asymptotic behavior of $F(\omega)$ is then governed by the most singular behavior (smallest α) of $f(x)$ on the given interval. Initially assume that $f(x)$ is even and that $\lim_{x\to\ell} f(x) = K(\ell - x)^\alpha$. We have

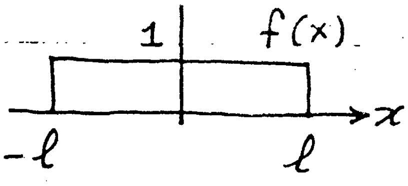
$$F(\omega) = \int_{-\ell}^{\ell} f(x) e^{j\omega x} dx = \int_{-\ell}^{\ell} f(x) e^{j\omega(x-\ell)} e^{j\omega\ell} dx$$

Now let $x - \ell = -x'$, then $F(\omega) = e^{j\omega\ell}\int_0^{2\ell} f(\ell - x') e^{-j\omega x'} dx'$

Let $\omega x' = \lambda$, $dx' = \frac{d\lambda}{\omega}$, and use $\lim_{\omega\to\infty} f\left(\ell - \frac{\lambda}{\omega}\right) = K\left(\frac{\lambda}{\omega}\right)^\alpha$

to obtain $F(\omega) \sim K e^{j\omega\ell} \omega^{-\alpha-1} \int_0^\infty \lambda^\alpha e^{-j\lambda} d\lambda$ since the limits on λ are 0 to $2\omega\ell$ which become 0 to ∞ as $\omega \to \infty$. The behavior of $f(x)$ at $x = \pm\ell$ determines the behavior of $F(\omega)$ at infinity. The following examples illustrate and extend the theory.

Example 1



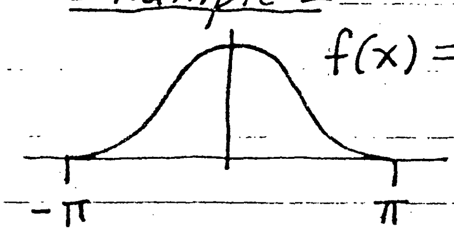
At $x=\pm l$, $f(x)\to 1$ so $\alpha=0$

By direct evaluation

$$F(\omega) = 2l\,\frac{\sin \omega l}{\omega l}$$

so F has the $\omega^{-\alpha-1} = \omega^{-1}$ behavior as expected.

Example 2



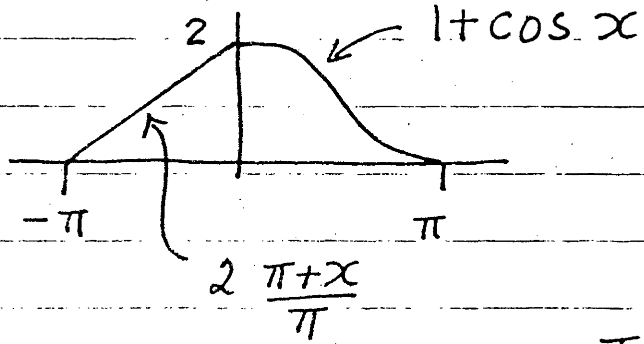
$f(x) = 1+\cos x$, At $x=\pm\pi$, $f(x)\to \frac{(\pi-|x|)^2}{2}$

so $\alpha = 2$. By direct evaluation

$$F(\omega) = \frac{-2\sin\omega\pi}{\omega(\omega^2-1)}$$

which has the $\omega^{-\alpha-1} = \omega^{-2-1} = \omega^{-3}$ behavior.

Example 3



$f(x)$ is continuous on $-\pi<x<\pi$ and $f(x)$ has $\alpha=1$ at $x\to-\pi$ and $\alpha=2$ at $x\to\pi$

Split $f(x)$ into $f_1 = 2\,\frac{\pi+x}{\pi}$, $-\pi$ to 0 and $f_2 = 1+\cos x$, 0 to π.

Then f_1 has $\alpha=1$ as $x\to-\pi$ and $\alpha=0$ as $x\to 0$ while f_2 has $\alpha=0$ as $x\to 0$ and $\alpha=2$ as $x\to\pi$.

By direct computation $F_1(\omega) = \frac{2}{j\omega} + \frac{2}{\pi\omega^2}(1-e^{-j\omega\pi})$

and has terms with ω^{0-1} and $\omega^{-1-1} = \omega^{-2}$. For $F_2(\omega)$ we get $F_2(\omega) = \frac{1-2\omega^2}{j(\omega^2-1)\omega} + \frac{je^{j\omega\pi}}{\omega(\omega^2-1)}$ which has the expected ω^{-1} and ω^{-3} behavior as $\omega\to 0$. For F_1+F_2 we get, for large ω, $\frac{j}{\omega^3}(e^{j\omega\pi}-1) + \frac{2}{\pi\omega^2}(1-e^{-j\omega\pi})$. This has the expected ω^{-2} and ω^{-3} behavior ($\alpha=1$, $\alpha=2$, at $x=-\pi,\pi$).

Let $f(x)$ be continuous on the interval $-\ell<x<\ell$ and let $f(x) \sim K_1(\ell+x)^{\alpha_1}$ as $x \to -\ell$ and $f(x) \sim K_2(\ell-x)^{\alpha_2}$ as $x \to \ell$. f can be written as the sum of an even function $f_e(x)$ and an odd function $f_o(x)$, i.e.

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{f_e(x)} + \underbrace{\frac{f(x)-f(-x)}{2}}_{f_o(x)}$$

We then have $f_e(x) \sim \frac{K_2}{2}(\ell-x)^{\alpha_2} + \frac{K_1}{2}(\ell-x)^{\alpha_1}, \; x \to \ell$;

$$F_e(w) = \int_{-\ell}^{\ell} e^{jwx} f_e(x)\,dx = \int_{-\ell}^{\ell} \cos wx \, f_e(x)\,dx$$

$$= \mathrm{Re} \int_{-\ell}^{\ell} e^{jwx} f_e(x)\,dx.$$ If we apply the procedure used on Pg. 33 we obtain

$$F_e(w) \sim \mathrm{Re}\left\{ \frac{K_1}{2} e^{jw\ell} w^{-\alpha_1-1} \int_0^\infty \lambda^{\alpha_1} e^{-j\lambda}\,d\lambda \right.$$

$$\left. + \frac{K_2}{2} e^{jw\ell} w^{-\alpha_2-1} \int_0^\infty \lambda^{\alpha_2} e^{-j\lambda}\,d\lambda \right\}$$

The edge behavior at both $x = \pm \ell$ determine the behavior of F_e as $w \to \infty$. The behavior of $F_o(w) = \mathcal{J} f_o(x)$ may be established in a similar way and will also have $w^{-\alpha_1-1}$ and $w^{-\alpha_2-1}$ terms.

Wiener-Hopf Integral Equation

$$\int_{-\infty}^{0} G(z-z')\, J(z')\, dz' = E(z), \quad -\infty \leq z \leq 0 \qquad (1)$$

where G and E are known and J is to be found. If the integration extended from $-\infty \leq z' \leq \infty$ and the equation held for all z then it could be solved using the convolution theorem and Fourier transforms to obtain $\hat{G}(w)\hat{J}(w) = \hat{E}(w)$ where the $(\wedge)$ denotes the Fourier transform. In order to put the equation into the required form define

$$J_-(z') \equiv \begin{cases} J(z'), & -\infty \leq z' \leq 0 \\ 0, & z' > 0 \end{cases}, \quad E_-(z) \equiv \begin{cases} E(z), & -\infty \leq z < 0 \\ 0, & z > 0 \end{cases}$$

and let $E_+(z)$ be given by (1) when $z > 0$, where $E_+(z) \equiv 0$ for $z < 0$. Then (1) becomes

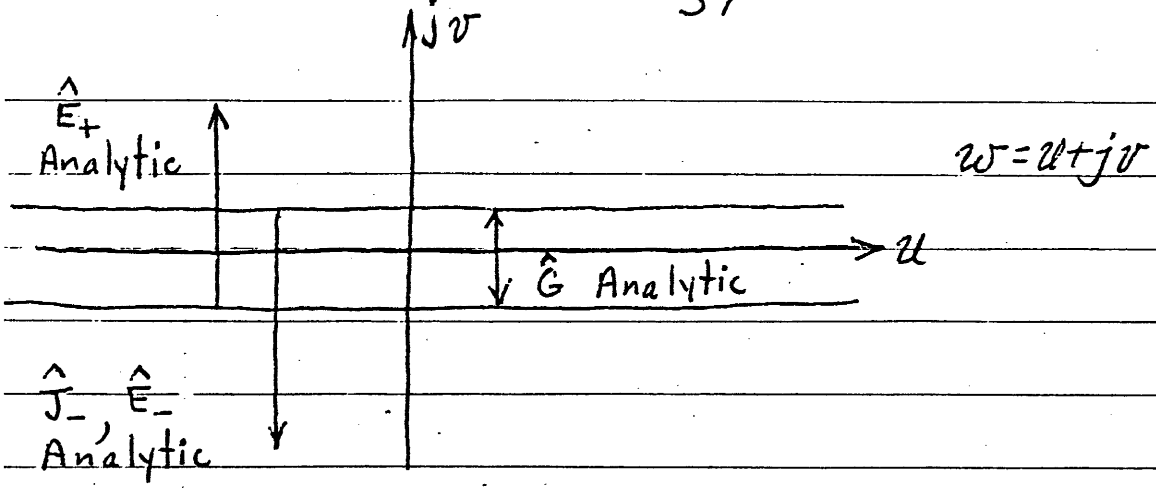
$$\int_{-\infty}^{\infty} G(z-z')\, J_-(z')\, dz' = E_+(z) + E_-(z), \quad \text{all } z \qquad (2)$$

Let

$$\hat{G}(w) = \int_{-\infty}^{\infty} G(z) e^{jwz} dz, \quad \hat{J}_-(w) = \int_{-\infty}^{0} J_-(z) e^{jwz} dz,$$

$$\hat{E}_+(w) = \int_{0}^{\infty} E_+(z) e^{jwz} dz, \quad \hat{E}_-(w) = \int_{-\infty}^{0} E_-(z) e^{jwz} dz$$

Assume that we can establish that all Fourier transforms exist and have a common strip parallel to Re w axis in which all are analytic. Then for w in this strip the Fourier transform of (2) may be taken to give



$$\hat{G}(w)\,\hat{J}_-(w) = \hat{E}_+(w) + \hat{E}_-(w) \qquad (3)$$

The objective now is to factor and rearrange (3) so that we obtain a relation of the form

$$f_+(w) = f_-(w) \quad \text{in common strip}$$

Then f_+ is the analytic continuation of f_- into the upper half plane. Together f_+ and f_- must equal a function $h(w)$ that is analytic everywhere in the complex plane. In practise we usually find that $h(w)$ is a simple low order polynomial such as $C_0 + C_1 w + C_2 w^2$. $h(w)$ is established from the edge conditions on E_-, E_+ and J_- at $z=0$ and the constants are determined by known incident fields. Very often only one C_i is non-zero.

Assume that we can factor $\hat{G}$ into $\hat{G} = \hat{G}_- / \hat{G}_+$. Then from (3)

$$\hat{G}_-\hat{J}_- = \hat{G}_+\hat{E}_+ + \hat{G}_+\hat{E}_-$$

Since $\hat{G}_+\hat{E}_-$ are known we now factor this into the form $\hat{G}_+\hat{E}_- = \hat{S}_+ + \hat{S}_-$. Then we obtain

$$\hat{G}_- \hat{J}_- - \hat{S}_- = \hat{G}_+ \hat{E}_+ + \hat{S}_+ = h(w)$$

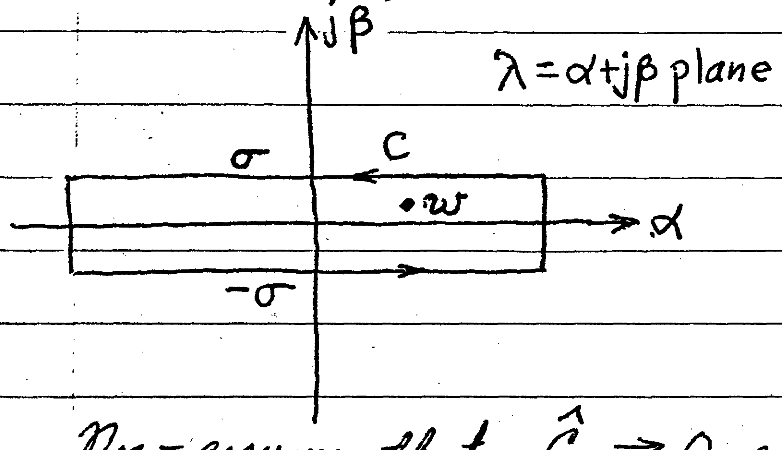
Then we get

$$\hat{J}_- = \frac{h(w) + \hat{S}_-}{\hat{G}_-} \qquad (4)$$

from which $J_-(z)$ can be found by means of the inverse Fourier transform.

In some problems the factorization can be done by inspection. When this is not the case a formal procedure to do the factorization, as described below, can be used.

Sum Factorization



Let $\hat{G}(\lambda)$ be analytic within and on C. Then

$$\hat{G}(w) = \frac{1}{2\pi j} \oint_C \frac{\hat{G}(\lambda)}{\lambda - w} d\lambda$$

Now assume that $\hat{G} \to 0$ as $\lambda \to \pm\infty$ for $-\sigma \le \beta \le \sigma$ then as we extend C to $\pm\infty$ the contributions to the integral from the end pieces of the contour vanish. Hence we get

$$\hat{G}(w) = \frac{1}{2\pi j} \int_{-\infty - j\sigma}^{\infty - j\sigma} \frac{\hat{G}(\lambda)}{\lambda - w} d\lambda - \frac{1}{2\pi j} \int_{-\infty + j\sigma}^{\infty + j\sigma} \frac{\hat{G}(\lambda)}{\lambda - w} d\lambda$$

where in the second integral we reversed the direction of integration. The first integral has its contour below the pole at $\lambda = \omega$ and is therefore analytic for all ω with $\text{Imag}\,\omega > -\sigma$ i.e. in the upper half plane. It represents $\hat{G}_+(\omega)$. Similary the second integral represents $\hat{G}_-(\omega)$.

$$\hat{G}_+(\omega) = \frac{1}{2\pi j} \int_{-\infty - j\sigma}^{\infty - j\sigma} \frac{\hat{G}(\lambda)}{\lambda - \omega} d\lambda \qquad (5a)$$

$$\hat{G}_-(\omega) = -\frac{1}{2\pi j} \int_{-\infty + j\sigma}^{\infty + j\sigma} \frac{\hat{G}(\lambda)}{\lambda - \omega} d\lambda \qquad (5b)$$

Example 1

$$\hat{G} = \frac{1}{\omega^2 + a^2} = \frac{1}{(\omega + ja)(\omega - ja)} = \frac{1}{2ja(\omega - ja)}$$

$$- \frac{1}{2ja(\omega + ja)}$$ so by inspection $\hat{G}_+ = \frac{-1}{2ja(\omega + ja)}$

and $\hat{G}_- = \frac{1}{2ja(\omega - ja)}$. Note that $\hat{G}_+$ has a pole at $\omega = -ja$ in lower half plane and hence is analytic in upper half plane. $\hat{G}_-$ has a pole in upper half plane. If we substitute $\hat{G}$ into (5) the integrand is proportional to λ^{-3}. Hence we can close the contour by a semi-circle and evaluate the integrals by residue theory. The contour can be closed in either half plane and the residue evaluation will give the same expressions for $\hat{G}_+$ and $\hat{G}_-$ as given above.

Example 2

$\hat{G}(w) = \frac{\sin wb}{(w^2+a^2)}$; We need to associate the term e^{jwb} with $\hat{G}_+$ since this decays in UHP while e^{-jwb} will grow in UHP. We also need to include the pole at $w=-ja$ with $\hat{G}_+$ and that at $w=ja$ with $\hat{G}_-$. By using (5a) we get

$$\hat{G}_+(w) = \frac{1}{2\pi j}\int_{-\infty-j\sigma}^{\infty-j\sigma} \frac{e^{j\lambda b}-e^{-j\lambda b}}{2j(\lambda^2+a^2)(\lambda-w)}\,d\lambda$$

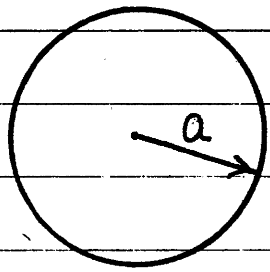
The integral with the factor $e^{j\lambda b}$ can be closed in the UHP while that with the $e^{-j\lambda b}$ can be closed in the LHP. The residue evaluation gives

$$\hat{G}_+ = \frac{e^{-ab}}{2j(2ja)(ja-w)} + \frac{e^{jwb}}{2j(w^2+a^2)} + \frac{e^{-ab}}{2j(2ja)(ja+w)}$$

The second term approaches $e^{-ab}/2j(2ja)(w-ja)$ as $w \to ja$ and cancels the first term at the point ja. Hence $w=ja$ is not a pole. The only pole is that at $w=-ja$ and e^{jwb} decays in the UHP. Hence $\hat{G}_+$ is analytic in UHP. For $\hat{G}_-$ we have $\hat{G}_- = \hat{G} - \hat{G}_+$ or by using (5b) we find

$$\hat{G}_- = -\frac{e^{-jwb}}{2j(w^2+a^2)} - \frac{e^{-ab}}{2j(2ja)(w+ja)} - \frac{e^{-ab}}{2j(2ja)(ja-w)}$$

Dyadic Green's Function For Spherical Cavity



$\vec{n}\times\vec{E}=0$ $\qquad$ $\nabla\times\nabla\times\bar{G}_e - k_0^2\bar{G}_e = \bar{I}\,\delta(\vec{r}-\vec{r}\,')$

Scalar Radial Green's Function

$$\frac{d}{dr}r^2\frac{dg_n}{dr} - [n(n+1)-k_0^2r^2]g_n = -\delta(r-r') \qquad (1)$$

Case 1 $\quad j_n(k_{ni}a)=0$, Required boundary condition for the $\vec{M}$ and $\vec{L}$ functions.

The two basic solutions are:

$$g_n = -\sum_{i=1}^{\infty}\frac{2j_n(k_{ni}r_<)\,j_n(k_{ni}r_>)}{a^3[j_n'(k_{ni}a)]^2(k_0^2-k_{ni}^2)} \qquad (2a)$$

$$g_n = -k_0\frac{j_n(k_0r_<)[j_n(k_0a)y_n(k_0r_>)-y_n(k_0a)j_n(k_0r_>)]}{j_n(k_0a)} \qquad (2b)$$

When $k_0\to 0$, (2b) gives

$$g_n \sim \frac{r_<^n}{(2n+1)a^n}\left[\frac{a^n}{r_>^{n+1}}-\frac{r_>^n}{a^{n+1}}\right] \qquad (3)$$

which is the sum in (2a) when $k_0=0$.

Case 2 $\quad \frac{d}{dr}[r j_n(l_{ni}r)]=0$ at $r=a$. Required boundary condition for the $\vec{N}$ functions.

$$g_n = -\sum_{i=1}^{\infty}\frac{2j_n(l_{ni}r_<)\,j_n(l_{ni}r_>)}{a^3 j_n^2(l_{ni}a)\left[1-\frac{n(n+1)}{l_{ni}^2a^2}\right](k_0^2-l_{ni}^2)} \qquad (4a)$$

$$g_n = \frac{-k_o j_n(k_o r_<)\left[y_n(k_o r_>)(k_o a\, j_n(k_o a))' - j_n(k_o r_>)(k_o a\, y_n(k_o a))'\right]}{[k_o a\, j_n(k_o a)]'} \quad (4b)$$

The prime means a derivative with respect to $k_o a$.
When $k_o \to 0$ (4b) gives

$$g_n \sim \frac{r_<^n}{a^n}\left[\frac{(n+1)a^n/r_>^{n+1} + n\, r_>^n/a^{n+1}}{(n+1)(2n+1)}\right] \quad (5)$$

This equals the sum in (4a) when $k_o = 0$. Note that (3) and (5) can be obtained by solving (1) directly with $k_o = 0$.

$\vec{L}_j$ Function Contribution to $\bar{G}_e$

$$-\sum_{n,m,e,o,i} \frac{\pi\, \vec{L}_j(\vec{r})\, \vec{L}_j(\vec{r}\,')}{k_o^2 Q_{nm}\, a^3 k_{ni}^2 [j_n'(k_{ni}a)]^2} \quad (6)$$

where $\vec{L}_j = \nabla P_n^m(\cos\theta) \left\{\begin{matrix}\cos\\ \sin\end{matrix} m\phi\right\} j_n(k_o r)$. The subscript j stands for a particular combination of n, m, e, o, i. The sum over i in (6) is the same as in (2a) with $k_o = 0$. Hence another form for (6) is

$$-\sum_{n,m,e,o} \frac{\pi}{2k_o^2 Q_{nm}} \nabla\nabla' P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, P_n^m(\cos\theta') \begin{matrix}\cos\\ \sin\end{matrix} m\phi'$$

$$\cdots \frac{r_<^n}{(2n+1)a^n}\left[\frac{a^n}{r_>^{n+1}} - \frac{r_>^n}{a^{n+1}}\right] \quad (7)$$

The $\vec{L}_j$ contribution to $\bar{G}_e$ is a "static like" field.

$\vec{M}_j$ Function Contribution to $\bar{G}_e$

$$-\sum_{n,m,e,o,i} \frac{\pi \; \vec{M}_j(\vec{r}) \; \vec{M}_j(\vec{r}\,')}{n(n+1) Q_{nm} \, a^3 k_{ni}^2 \left[j_n'(k_{ni}a)\right]^2 (k_0^2 - k_{ni}^2)} \tag{8}$$

Where $\vec{M}_j(\vec{r}) = \nabla \times \vec{a}_r \, k_{ni} r \, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi \; j_n(k_{ni} r)$.

The sum over i in (8) is the same as in (2a).
Hence another form is

$$-\sum_{n,m,e,o} \frac{\pi}{n(n+1) Q_{nm}} \left[\left(\nabla \times \vec{a}_r \, r P_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi\right)\left(\nabla' \times \vec{a}_r \, r' P_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi'\right)\right]$$

$$\cdots \left[\frac{k_0 j_n(k_0 r_<)\left[j_n(k_0 a) y_n(k_0 r_>) - y_n(k_0 a) j_n(k_0 r_>)\right]}{j_n(k_0 a)}\right] \tag{9}$$

$\vec{N}_j$ Function Contribution to $\bar{G}_e$

$$-\sum_{n,m,e,o,i} \frac{\pi \; \vec{N}_j(\vec{r}) \; \vec{N}_j(\vec{r}\,')}{n(n+1) Q_{nm} a^3 \left[1 - \frac{n(n+1)}{l_{ni}^2 a^2}\right] l_{ni}^2 \, j_n^2(l_{ni} a) \, (k_0^2 - l_{ni})^2} \tag{10}$$

Where $[l_{ni} a \, j_n(l_{ni} a)]' = 0$ is the equation for finding the l_{ni}. In (10)

$$\vec{N}_j(\vec{r}) = \nabla \times \nabla \times \vec{a}_r \, r \, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi \; j_n(l_{ni} r)$$

In (10) we note that

$$\frac{1}{l_{ni}^2 (k_0^2 - l_{ni}^2)} = \frac{1}{k_0^2 l_{ni}^2} + \frac{1}{k_0^2 (k_0^2 - l_{ni}^2)}$$

Thus the sum over i in (10) can be obtained from (4b) and (5) since one sum is the same as in (4a) and the other is the same as in (4a) with $k_0 = 0$.

Hence another form for (10) is

$$-\sum_{n,m,e,o} \frac{\pi}{2n(n+1)Q_{nm} k_0^2}\Big[(\nabla\times\nabla\times\vec{a}_r)(\nabla'\times\nabla'\times\vec{a}_r)\, r r' P_n^m(\cos\theta)\,{}^{\cos}_{\sin} m\phi$$

$$P_n^m(\cos\theta')\,{}^{\cos}_{\sin} m\phi'\Big]\Bigg[\frac{k_0 j_n(k_0 r_<)\big[y_n(k_0 r_>)(k_0 a j_n(k_0 a))' - j_n(k_0 r_>)(k_0 a y_n(k_0 a))'}{[k_0 a\, j_n(k_0 a)]'}$$

$$+ \frac{r_<^n}{a^n}\,\frac{(n+1)a^n/r_>^{n+1} + n r_>^n/a^{n+1}}{(n+1)(2n+1)}\Bigg] \qquad (11)$$

It is seen that part of the $\vec{N}_j$ contribution is a "static like" field.

The vector operator in (10) and (11) can be expanded into the form

$$\left[\frac{n(n+1)}{r^2}\vec{a}_r + \left(\vec{a}_\theta\frac{\partial}{\partial\theta} + \frac{\vec{a}_\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\frac{1}{r}\frac{\partial}{\partial r}\right]\left[\frac{n(n+1)}{r'^2}\vec{a}_r + \left(\vec{a}_\theta\frac{\partial}{\partial\theta'} + \frac{\vec{a}_\phi}{\sin\theta'}\frac{\partial}{\partial\phi'}\right)\frac{1}{r'}\frac{\partial}{\partial r'}\right]$$

operating on each term in (11).

Delta Function Terms

In the $\vec{L}_j$ contribution given by (7) we have an operator $\partial^2/\partial r\partial r'$ in the $\vec{a}_r\vec{a}_r$ term. This will produce a delta function. We note that

$$\frac{\partial}{\partial r} r^2 \frac{\partial g_n}{\partial r} = 2r\frac{\partial g_n}{\partial r} + r^2\frac{\partial^2 g_n}{\partial r^2} = 2r\frac{\partial g_n}{\partial r} - r^2\frac{\partial^2 g_n}{\partial r\partial r'}$$

and from (1) we have

$$\frac{\partial^2 g_n}{\partial r\partial r'} = \frac{\delta(r-r')}{r^2}$$

Hence the delta function term in (7) is

$$-\sum_{n,m,e,o} \frac{\pi}{2k_0^2 Q_{nm}} \vec{a}_r \vec{a}_r \; P_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi \; P_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi' \; \frac{\delta(r-r')}{r^2}$$

$$= - \frac{\delta(\vec{r}-\vec{r}')}{k_0^2} \vec{a}_r \vec{a}_r \qquad (12)$$

which is the expected result.

The $\vec{M}_j$ contribution to $\bar{G}_e$ does not contain a delta function. From the $\vec{N}_j$ contribution delta functions are produced by the $\frac{\partial}{\partial r}\frac{\partial}{\partial r'}$ operator but since (11) contains $-g_n$ from (4b) and g_n from (5) these cancel. Hence the only delta function term is that given by (12), which is the same as in the free space Green's dyadic.

Cancelling Terms

We expect the remaining contribution from the $\vec{L}_j$ functions to be cancelled by the "static like" part of the $\vec{N}_j$ contribution. It is relatively easy to show that this is indeed the case. For example the $\vec{a}_r \vec{a}_r$ term from the $\vec{L}_j$ contribution has the factor

$$\frac{1}{2n+1} \frac{\partial}{\partial r_<} \frac{\partial}{\partial r_>} \frac{r_<^n}{a^n}\left[\frac{a^n}{r_>^{n+1}} - \frac{r_>^n}{a^{n+1}}\right] = -\frac{n}{2n+1} \frac{r_<^{n-1}}{a^n}\left[\frac{(n+1)a^n}{r_>^{n+2}} + \frac{n r_>^{n-1}}{a^{n+1}}\right]$$

The corresponding term contributed from the "static like" $\vec{a}_r \vec{a}_r$ term from the $\vec{N}_j$ contribution is the negative of this and thus cancels this term. Likewise the other $\vec{L}_j$ terms are cancelled.

Green's Dyadic $\bar{G}_e$ for a Spherical Cavity

$\bar{G}_e$ is the sum of the three contributions (6), (8), and (10). The irrotational part is given by (6). When the summation over i is carried out $\bar{G}_e$ is given by the sum of (7), (9) and (11). When the delta function term is extracted and the cancelling terms are taken into account the following expression is obtained:

$$\bar{G}_e(\vec{r},\vec{r}\,') = -\sum_{n,m,e,o} \frac{\pi}{2n(n+1)Q_{nm}k_0}\left[\frac{\vec{M}_{j\sigma}(\vec{r})\,\vec{M}_{j\sigma}(\vec{r}\,')}{j_n(k_0 a)} + \frac{\vec{N}_{j\sigma}(\vec{r})\,\vec{N}_{j\sigma}(\vec{r}\,')}{[k_0 a\, j_n(k_0 a)]'}\right] - \frac{\vec{a}_r\vec{a}_r}{k_0^2}\,\delta(\vec{r}-\vec{r}\,') \qquad (13)$$

Where

$$\vec{M}_{j\sigma}(\vec{r}) = \nabla\times\vec{a}_r\, k_0 r\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi \begin{cases} j_n(k_0 r), & r<r' \\ z_n(k_0 r), & r>r' \end{cases}$$

$$\vec{M}_{j\sigma}(\vec{r}\,') = \nabla'\times\vec{a}_r\, k_0 r'\, P_n^m(\cos\theta') \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi' \begin{cases} j_n(k_0 r'), & r'<r \\ z_n(k_0 r'), & r'>r \end{cases}$$

$$k_0\vec{N}_{j\sigma}(\vec{r}) = \nabla\times\nabla\times\vec{a}_r\, k_0 r\, P_n^m \begin{matrix}\cos\\ \sin\end{matrix}\, m\phi \begin{cases} j_n(k_0 r), & r<r' \\ w_n(k_0 r), & r>r' \end{cases}$$

$$\vec{N}_{j\sigma}(\vec{r}\,') = \nabla' \times \nabla' \times \vec{a}_r k_0 r' P_n^m \begin{matrix}\cos\\ \sin\end{matrix} m\phi' \begin{cases} j_n(k_0 r'), & r' < r \\ w_n(k_0 r'), & r' > r \end{cases}$$

where

$$z_n(k_0 r) = j_n(k_0 a) y_n(k_0 r) - y_n(k_0 a) j_n(k_0 r)$$

$$w_n(k_0 r) = [k_0 a j_n(k_0 a)]' y_n(k_0 r) - [k_0 a y_n(k_0 a)]' j_n(k_0 r)$$

and σ is a label to indicate that the radial functions have a different form for $r < r'$ and $r > r'$. The normalization constant Q_{nm} is given by

$$Q_{nm} = \frac{\epsilon_{om}\,(n+m)!\,\pi^2}{(2n+1)(n-m)!}$$

It is seen that $\bar{G}_e$ can be expressed in terms of the $\vec{M}$ and $\vec{N}$ functions plus a reduced delta function term in complete analogy with the free space Green's dyadic. In (13) the resonant frequencies are determined by the vanishing of the Wronskian determinants.

R.E. Collin, July 1981

EIGENFUNCTION EXPANSIONS AND THE GREEN'S DYADIC

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EIGENFUNCTION EXPANSIONS AND DYADIC GREEN'S FUNCTIONS

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Introduction

In this article the main features of the eigenfunction expansion of the electric field is reviewed along with its relationship to the electric field Green's dyadic. Various representations for the Green's dyadic are presented and their inter-relationships pointed out. For many problems the eigenfunction expansion method provides the most useful solution for the field and gives immediately considerable insight into the basic physical properties of the field. For waveguides and cavities the eigenfunction expansion method is by far the preferable way to solve most problems involving current sources and wave scattering.

We also point out in this review the great amount of unity that exists among the various results on Green's dyadics as presented by different authors. A brief historical review of past difficulties and misconceptions that prevailed are also discussed along with comments on probable reasons why some of these misunderstandings were perpetuated and finally resolved.

Some basic but important mathematical relations for Green's functions are summarized in Appendix I and provide a basis for the resolution of some of the apparent but non-real differences in various solutions for Green's functions presented in the literature.

Direct Vector Potential Method

Consider a system of currents $\vec{J}(\vec{r})$ contained within a finite volume V_1 as shown in Figure 1. The electric field is a solution of

$$\nabla \times \nabla \times \vec{E} - k_o^2 \vec{E} = -j\omega\mu_o \vec{J} \tag{1}$$

and must satisfy appropriate boundary conditions. For currents radiating into free space this is the radiation condition at infinity.

A well known standard method of finding $\vec{E}$ is through the use of the vector potential. The relevant equations are:

$$\vec{E} = -j\omega \left(\vec{A} + \frac{\nabla\nabla\cdot\vec{A}}{k_o^2}\right) \tag{2a}$$

$$(\nabla^2 + k_o^2)\vec{A} = -\mu_o \vec{J} \tag{2b}$$

$$\vec{A}(\vec{r}) = \int_{V_1} \frac{e^{-jk_oR}}{4\pi R} \mu_o \vec{J}(\vec{r}') \, d\vec{r}' \tag{2c}$$

where $R = |\vec{r}-\vec{r}'|$. More often than not the integral (2c) cannot be evaluated analytically and so must be evaluated numerically. The integrand is a function of both $\vec{r}$ and $\vec{r}'$. Consequently for every different observation point $\vec{r}$ the evaluation of the integral must be done. In order to obtain a reasonably complete map of the electric field in space a great deal of numerical computation must be carried out. This large collection of numerical data, along with field amplitude plots which may also be computed, is often hard to interpret for its physical content as regards the basic properties of the electric field produced by the given system of currents. The eigenfunction method to be discussed

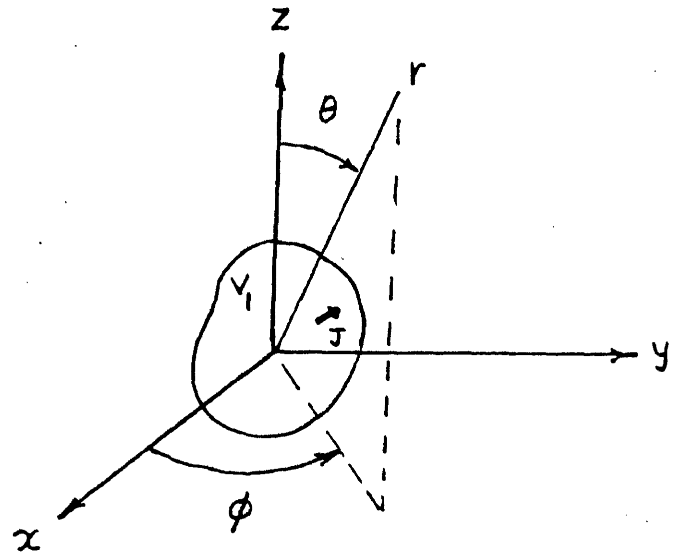

Fig. 1 The system of currents in V_1

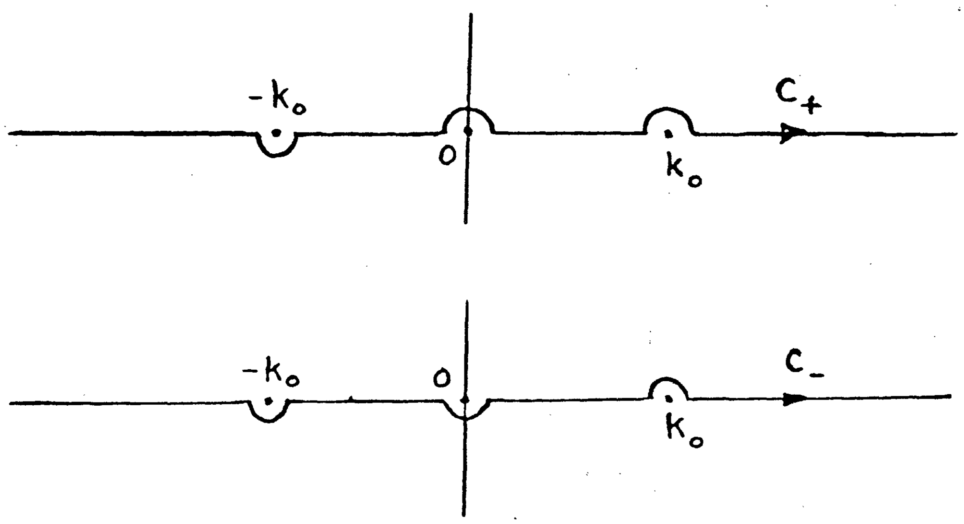

Fig. 2 The contours $C_{\pm}$ in the k plane

next overcomes many of the short comings of this classical solution based on the use of the vector potential and the set of Equations (2) given above.

Eigenfunction Expansion Method

In the eigenfunction expansion method the electric field is described in terms of a sum of propagating vector eigenmodes or waves. These discrete vector modes can be obtained from the solutions $\Psi_{nm_o^e}$ of the scalar Helmholtz equation[1],

$$(\nabla^2 + k_o^2)\ \Psi_{nm_o^e} = 0 \tag{3}$$

In spherical coordinates the appropriate solutions to (3) are

$$\Psi_{nm_o^e} = \Psi_j = P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}\ m\phi \quad z_n\ (k_o r) \tag{4}$$

where P_n^m is an associated Legendre polynomial, the even (odd) functions (e,o) involve cos $m\phi$ (sin $m\phi$), $z_n(k_o r)$ is a spherical Bessel function, and j is a shorthand summation index standing for a particular combination of n,m,e,o . For outward propagating spherical waves $z_n(k_o r)$ is the spherical Hankel function of the second kind $h_n^2(k_o r)$. For inward propagating waves z_n is the spherical Hankel function $h_n^1(k_o r)$ of the first kind while for standing waves z_n is the spherical Bessel function $j_n(k_o r)$. We will use superscripts ($\pm$) to designate the use of h_n^2 and h_n^1 respectively and no superscript to mean that j_n is used.

The Hansen vector wave functions are obtained from the Ψ_j as follows:

$$\vec{M}_j = \nabla \times \vec{a}_r\, k_o r\, \Psi_j \tag{5a}$$

$$k_o \vec{N}_j = \nabla \times \vec{M}_j \tag{5b}$$

$$\vec{L}_j = \nabla \Psi_j \tag{5c}$$

The $\vec{M}_j$ and $\vec{N}_j$ functions have zero divergence and are called solenoidal or transverse while $\vec{L}_j$ has zero curl and is called irrotational or longitudinal.[†] These vector wave functions satisfy the homogeneous equation obtained from (1) by setting $\vec{J} = 0$.

In the region $r > b$ where b is the radius of the smallest sphere that will enclose the volume V_1 the electric field is solenoidal and can be represented as a sum of outward propagating solenoidal modes in the form

$$\vec{E}(\vec{r}) = \sum_j (c_j \vec{M}_j^+ + d_j \vec{N}_j^+) \tag{6}$$

where the c_j and d_j are suitable amplitude coefficients. This representation, once the amplitude coefficients have been found, immediately provides a great deal of physical insight into the nature of the field. The $\vec{M}_j^+$ functions have only θ and ϕ components and are thus TE waves. The $\vec{N}_j^+$ represent TM waves since they have a radial component (note that $\nabla \times \vec{N}_j^+ = k_o \vec{M}_j^+$ so the magnetic field will be transverse. The various $\vec{M}_J^+$, $\vec{N}_j^+$ waves represent multipole fields, for example $\vec{N}_{11e}^+$, $\vec{N}_{11o}^+$, $\vec{N}_{10}^+$ are fields radiated from x, y, and z directed electric dipoles, $\vec{M}_{11e}^+$, $\vec{M}_{11o}^+$, $\vec{M}_{10}^+$ are fields radiated by corresponding magnetic dipoles. Each mode radiates a fixed amount of power independent of the presence of the

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[†]The origin of the terms transverse and longitudinal is discussed in Appendix II.

other modes. The total radiated power is the sum of that contributed by each mode. It is these properties as well as others that enables one to obtain considerable information about the field using the eigenfunction expansion (6), information that is much more difficult to deduce from the direct solution based on the vector potential. We will also see later on that the amplitude coefficients c_j and d_j do not depend on the observation point $\vec{r}$ so only one computation is needed for each coefficient. When the coefficients have been determined an analytical expression for the field everywhere outside the source region is obtained.

Inside the source region the field is not solenoidal so the $\vec{L}_j$ functions must also be included. Furthermore on any spherical surface $r < b$ the currents in the region 0 to r produce outward propagating waves at r while the currents in the region r to b produce inward propagating waves that are reflected at the origin to produce standing waves. Thus the expansion within the source region is more complex than that given by (6). The appropriate expansion will be given later.

The determination of the amplitude coefficients by means of a Fourier series type of analysis can only be carried out in a simple way provided the eigenfunctions are mutually orthogonal. The required set of mutually orthogonal vector eigenfunctions are generated from the scalar functions[2],

$$\Psi_j(k,\vec{r}) = P_n^m \begin{matrix} \cos \\ \sin \end{matrix} m\phi \quad j_n(kr) \tag{7}$$

In (7) k replaces the k_o in (4) and is treated as a continuous eigenvalue parameter. From these scalar functions corresponding vector eigenfunctions are obtained using the relations shown in (5). We then can show that

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \vec{L}_j \cdot \vec{M}_k \sin\theta \, d\theta \, d\phi \, r^2 dr = 0$$

or in general

$$\int_V \vec{L}_j \cdot \vec{M}_i \, d\vec{r} = \int_V \vec{L}_j \cdot \vec{N}_i \, d\vec{r} = \int_V \vec{N}_j \cdot \vec{M}_i \, d\vec{r} = 0 \qquad (8)$$

The normalization integrals are

$$\int_V \vec{L}_j \cdot \vec{L}_j \, d\vec{r} = Q_{nm}\delta(k-k') = \frac{\varepsilon_{om}\pi^2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta(k-k') \qquad (9a)$$

$$\int_V \vec{M}_j \cdot \vec{M}_j \, d\vec{r} = \int_V \vec{N}_j \cdot \vec{N}_j \, d\vec{r} = n(n+1)Q_{nm} \delta(k-k') \qquad (9b)$$

In (8) j may be equal to i and k may be the same in both functions. In (9) one function has k and the other has k' as the eigenvalue parameter. Also in (9) $\varepsilon_{om} = 2$ for $m = 0$ and equals 1 for $m > 0$ and $\delta(k-k')$ is the delta function.

The solution for $\vec{E}$ described by (1) is obtained by letting

$$\vec{E} = \int_0^\infty \sum_j (a_j \vec{L}_j + c_j \vec{M}_j + d_j \vec{N}_j) dk \qquad (10)$$

The sum over the continuous spectrum of k values is an integral. When this expansion is substituted into (1) we obtain

$$\int_0^\infty dk \sum_j [-a_j k_o^2 \vec{L}_j + (k^2-k_o^2)(c_j\vec{M}_j + d_j\vec{N}_j)] = -j\omega\mu_o \vec{J} \qquad (11)$$

since $\nabla \times \vec{L}_j = 0$, $\nabla \times \nabla \times \vec{M}_j = k^2\vec{M}_j$ and similarly for $\vec{N}_j$. We can now scalar multiply by $\vec{L}_j(k',\vec{r})$, $\vec{M}_j(k',\vec{r})$ and $\vec{N}_j(k',\vec{r})$ in turn and use (8) and (9) to obtain the amplitude coefficients a_j, c_j and d_j. This is a straightforward procedure and yields for the $\vec{E}$ field the solution

$$\vec{E}(\vec{r}) = \int_0^\infty dk \sum_j \left[\frac{-j\omega\mu_o}{n(n+1)Q_{nm}(k^2-k_o^2)} \right] [\vec{M}_j(k,\vec{r}) \int_V \vec{M}_j(k,\vec{r}\,') \cdot \vec{J}(\vec{r}\,')d\vec{r}\,'$$

$$+ \vec{N}_j(k,\vec{r}) \int_V \vec{N}_j(k,\vec{r}\,') \cdot \vec{J}(\vec{r}\,')d\vec{r}\,' - \frac{n(n+1)}{k_o^2} \vec{L}_j(k,\vec{r}) \int_V \vec{L}_j(k,\vec{r}\,') \cdot \vec{J}(\vec{r}\,')d\vec{r}\,'] \quad (12)$$

At this point the solution does not look like that given earlier by (6). A considerable amount of manipulation of (12) is required to reduce it to the simple form given earlier. The steps to be followed are outlined below but many of the details are not included.

We should also comment on the validity of taking the curl curl operation term by term and interchanging the order of summation and doing the volume integral on a term by term basis to obtain the a_j, c_j, and d_j from (11). In general sufficient conditions are known that will allow these interchanges. But sufficient conditions are not necessary conditions so there may be circumstances where known sufficient conditions are violated yet the interchange in the order of summation and differentiation or integration may still be allowed. When the current satisfies a Hölder condition, i.e., positive constants α and β exist such that[3],

$$|\vec{J}(\vec{r}) - \vec{J}(\vec{r}\,')| \leq \alpha|\vec{r}-\vec{r}\,'|^\beta \quad (13)$$

then the interchange is allowed. However, not all current systems that we might wish to consider satisfy a Hölder condition, the violation usually occuring at the boundary of V_1 where $\vec{J}$ might be assumed to be finite and non-zero. In such cases it is necessary to check that the electric field given by (12) satisfies Maxwell's equations, in particular that the normal component of $\epsilon_o\vec{E}$ at the boundary of V_1 has a discontinuity in accordance with Gauss' law and with the surface charge density ρ_s given by $j\omega\rho_s = \vec{n} \cdot \vec{J}$ on the boundary of V_1. It should also be kept in mind that like any Fourier series, an eigenfunction expansion converges in the mean square sense and at points of discontinuity converges to the average value. These problems do not arise for physical current systems but do arise from our mathematical modeling of physical current systems when we artificially postulate currents that have non-zero normal components on the boundary of V_1. For any reasonable system of currents the series solution given by (12) is convergent and well behaved and various simplifications may be carried out on a term by term basis. This is because the coefficients a_j, c_j, and d_j decrease rapidly in value with increasing j so that the field is well approximated by a finite number of terms.

Helmholtz Theorem and Simplication of Solution for $\vec{E}$

Physical insight into the possibility of simplifying the solution (12) for $\vec{E}$ may be obtained from a consideration of Helmholtz's theorem. Helmholtz's theorem states that a vector field such as $\vec{J}$ may be split into its solenoidal and irrotational parts by means of the following formulas[4],

$$\vec{J}(\vec{r}) = -\nabla \left[\int_{V_1} \frac{\nabla'\cdot\vec{J}(\vec{r}')}{4\pi R}\, d\vec{r}' - \oint_{S_1} \frac{\vec{n}\cdot\vec{J}}{4\pi R}\, ds' \right]$$

$$+ \nabla \times \left[\int_{V_1} \frac{\nabla'\times\vec{J}}{4\pi R}\, d\vec{r}' - \oint_{S_1} \frac{\vec{n}\times\vec{J}}{4\pi R}\, ds' \right], \quad \vec{r} \text{ in } V_1, \qquad (14a)$$

$$\vec{J}(\vec{r}) = 0, \quad \vec{r} \text{ not in } V_1, \qquad (14b)$$

On an individual basis the solenoidal part $\vec{J}_s$ obtained from the curl operation and the irrotational part $\vec{J}_I$ obtained from the gradient operation are, in general, non-zero outside V_1. Yet outside V_1 we have $\vec{J}_s = -\vec{J}_I$ so that these two distinct types of vector fields may cancel identically in certain regions of space. A similar cancellation may be expected in the solution for $\vec{E}$, that is, outside the source region the $\vec{L}_j$ terms might be cancelled.

If the $\vec{N}_j$ and $\vec{L}_j$ functions are examined it will be found that they have the same θ and ϕ dependence and are not orthogonal with respect to integration over θ and ϕ only. Indeed it is this property that enables the part of the series solution involving the $\vec{N}_j$ functions to cancel the contribution from the $\vec{L}_j$ functions outside the source region and also to cancel most of the $\vec{L}_j$ contribution within the source region.

The first step in the simplification of (12) is to perform the integral (sum) over k. For the $\vec{M}_j$ functions this involves an integral of the form

$$\int_0^\infty \frac{k^2 j_n(kr)\, j_n(kr')}{k^2 - k_o^2}\, dk \tag{15a}$$

while for the $\vec{N}_j$ and $\vec{L}_j$ functions an integral of the form

$$\int_0^\infty \frac{j_n(kr)\, j_n(kr')}{k^2 - k_o^2}\, dk \tag{15b}$$

occurs. Actually in some terms derivatives with respect to r and r' occur but these may be brought outside the integral. The integral is evaluated by converting it to a contour integral. We first replace $j_n(kr')$ by $\frac{1}{2}\,[h_n^1(kr') + h_n^2(kr')]$. When k approaches zero $j_n(kr)$ behaves like $(2kr)^n n!/(2n+1)!$ while $h_n^{1,2}(kr')$ behaves like $\mp j\, 2(2n)!/n!(2kr')^{n+1}$. Thus the product $j_n h_n^{1,2}$ has a first order pole at $k = 0$ with a residue $\mp j\, r^n/(2n+1)(r')^{n+1}$. But in the sum $h_n^1 + h_n^2$

this pole term is cancelled. If we make use of the circuit relations $h_n^1(-kr') = e^{-jn\pi} h_n^2(kr')$ and $j_n(-kr) = e^{jn\pi} j_n(kr)$ we can write

$$\int_0^{\infty} j_n(kr)\, h_n^1(kr')dk = -\int_0^{-\infty} j_n(-kr)h_n^1(-kr')\,dk = \int_{-\infty}^{0} j_n(kr)\, h_n^2(kr')dk$$

so in place of (15b) we have

$$\lim_{\eta\to 0} \int_{\eta}^{\infty} \frac{j_n(kr)\, j_n(kr')}{k^2-k_o^2}\,dk = \lim_{\eta\to o} \left[\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty} \frac{j_n(kr)\, h_n^2(kr')}{2(k^2-k_o^2)}\,dk \right] =$$

$$P\int_{-\infty}^{\infty} \frac{j_n(kr)\, h_n^2(kr')}{2(k^2-k_o^2)}\,dk \qquad (16)$$

Note that in this conversion it is the Cauchy principal value, denoted by the symbol P, that must be taken in order to get the original pole cancellation that had occurred from the sum $h_n^1 + h_n^2$. In the conversion of the integral in (15a) the Cauchy principal value is not required since the integrand has a k^2 factor in the numerator which cancels the pole at $k = 0$.

The principal value integral in (16) may be written as a contour integral plus or minus one half of the pole contribution, thus

$$P\int_{-\infty}^{\infty} \frac{j_n(kr)\, h_n^2(kr')}{k^2-k_o^2}\,dk = \int_{C\pm} (\quad)dk \pm \pi j \text{ (Residue at } k=0) \qquad (17)$$

where $C_{\pm}$ are the contours shown in Figure 2. Either contour may be used. The integrand in (17) is well behaved at infinity so the contour may be closed by a semicircle in the lower half complex k plane for $r < r'$. When $r > r'$ the role of r and r' in (17) is interchanged and the contour is closed again in the lower half plane. The integral is evaluated using residue theory. By this means we find that (15a) gives

$$\int_0^{\infty} \frac{j_n j_n k^2}{k^2 - k_o^2} dk = \frac{-j\pi k_o}{2} j_n(k_o r_<) h_n^2 (k_o r_>) \tag{16a}$$

while (15b) gives

$$\int_0^{\infty} \frac{j_n j_n}{k^2 - k_o^2} dk = \frac{-j\pi}{2k_o} j_n(k_o r_<) h_n^2 (k_o r_>) - \frac{\pi}{2(2n+1)k_o^2} \frac{r_<^n}{r_>^{n+1}} \tag{16b}$$

The k = 0 pole contribution is the zero frequency or static radial eigenfunction. The notation $r_<$ and $r_>$ means the following:

$r_<$, smaller of r,r'

$r_>$, greater of r,r'

The functions described by (16) are continuous at r=r' with discontinuous first derivatives. A second derivative with respect to r or r' results in a delta function term. For example

$$\frac{d}{dr_<} \frac{d}{dr_>} \frac{r_<^n}{r_>^{n+1}} = -n(n+1) \frac{r_<^{n-1}}{r_>^{n+2}} + (2n+1) \frac{\delta(r-r')}{r^2} \tag{17}$$

The relations given by (16) may be used to evaluate the integrals over k in the solution (12) for $\vec{E}$. In the contribution from the $\vec{N}_j\vec{N}_j$ functions a delta function $\delta(r-r')$ occurs in the $\vec{a}_\theta\vec{a}_\theta$ and $\vec{a}_\phi\vec{a}_\phi$ terms from the residue contributions at $k = \pm k_o$ and also from the residue term at k = 0 since these terms involve

$$\frac{d\, rj_n(kr)}{dr} \quad \frac{d\, r'j_n(kr')}{dr'} .$$

However, these delta function terms cancel. The $\vec{a}_r\vec{a}_r$ contribution **from the $\vec{N}_j\vec{N}_j$ functions do not produce a delta function but the** $\vec{a}_r\vec{a}_r$ **contribution from the $\vec{L}_j\vec{L}_j$ functions is found to contribute a delta function term which is**

$$-j\omega\mu_o \int_{V_1} d\vec{r}' \sum_j \frac{P_n^m(\cos\ )\ P_n^m(\cos\ ')}{-k_o^2\ Q_{nm}} \begin{matrix}\cos\\ \sin\end{matrix} m\phi \begin{matrix}\cos\\ \sin\end{matrix} m\phi'$$

$$\frac{\pi}{2}\ \frac{\delta(r-r')}{r^2}\ \vec{a}_r\vec{a}_r \cdot \vec{J}(\vec{r}')$$

$$= \frac{j\omega\mu_o}{k_o^2} \int_{V_1} \vec{a}_r\vec{a}_r \cdot \vec{J}(\vec{r}')\ \ \delta(\vec{r}-\vec{r}')d\vec{r}' = \frac{j\omega\mu_o}{k_o^2}\ \vec{a}_r\vec{a}_r \cdot \vec{J}(\vec{r}) \tag{18}$$

since

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{e,o} \frac{\pi}{2} \frac{P_n^m(\cos\theta)\ P_n^m(\cos\theta')}{Q_{nm}} \begin{matrix}\cos\\ \sin\end{matrix} m\phi \begin{matrix}\cos\\ \sin\end{matrix} m\phi' = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta} .$$

All of the remaining residue contributions from the $\vec{L}_j\vec{L}_j$ terms at the $k = 0$ pole are cancelled by the residue contributions from the $\vec{N}_j\vec{N}_j$ terms at the $k = 0$ pole. The simplified expression for the electric field becomes

$$\vec{E}(\vec{r}) = -j\omega\mu_o \sum_j \frac{-j\pi}{2k_o\ n(n+1)Q_{nm}} [\vec{M}_j^{+}(k_o,\vec{r}) \int_{V_1} \vec{M}_j(r_o,\vec{r}') \cdot \vec{J}(\vec{r}')d\vec{r}'$$

$$+ \vec{N}_j^{+}(k_o,\vec{r}) \int_{V_1} \vec{N}_j(k_o,\vec{r}') \cdot \vec{J}(\vec{r}')d\vec{r}'\ ]\ ,\ \vec{r} \text{ not in } V_1\ , \tag{19}$$

Inside the source region the integral over V_1 is split into an integral from 0 to r plus an integral from r to b. In the first integral the functions $\vec{M}_j^+(k_o, \vec{r})\ \vec{M}_j(k_o, \vec{r}\,')$ and $\vec{N}_j^+(k_o, \vec{r})\ \vec{N}_j(k_o, \vec{r}\,')$ are used while in the second integral the functions $\vec{M}_j(k_o, \vec{r})\ \vec{M}_j^+(k_o, \vec{r}\,')$ and $\vec{N}_j(k_o, \vec{r})\ \vec{N}_j^+(k_o, \vec{r}\,')$ are used. In addition the contribution

$\frac{j\omega\mu_o}{k_o^2}\ \vec{a}_r\vec{a}_r \cdot \vec{J}(\vec{r})$ must be added.

Inside the source region the eigenfunction expansion of the electric field involves both outward propagating spherical waves as well as spherical standing waves. The amplitudes of these waves are dependent on the observation coordinate r and hence must be computed for each value of r of interest. Of course, the additional term $\frac{j\omega\mu_o}{k_o^2}\ J_r(\vec{r})\vec{a}_r$ must be included. Outside of the source region the amplitude coefficients are not dependent on r so only one computation is needed. It is also seen that explicit dependence of the field on the $\vec{L}_j$ functions can be eliminated by cancellation coming from the $\vec{N}_j$ functions. The only remaining contribution from the $\vec{L}_j$ functions is the simple $\frac{j\omega\mu_o}{k_o^2}\ J_r(\vec{r})\vec{a}_r$ term.

The whole procedure for finding the eigenfunction expansion of the electric field can be organized in a systematic way by introducing the Green's dyadic which is discussed next.

Electric Field Green's Dyadic

The eigenfunction expansion of the electric field can be expressed in the form

$$\vec{E}(\vec{r}) = -j\omega\mu_o \int_{V_1} \bar{G}_e(\vec{r},\vec{r}\,') \cdot \vec{J}(\vec{r}\,')d\vec{r}\,' \qquad (20)$$

where $\bar{G}_e$ is called the electric field Green's dyadic. From (12) we see that one representation for $\bar{G}_e$ is

$$\bar{G}_e = \int_0^{\infty} dk \sum_j \left[\frac{\vec{M}_j(k,\vec{r})\vec{M}_j(k,\vec{r}\,') + \vec{N}_j(k,\vec{r})\vec{N}_j(k,\vec{r}\,')}{n(n+1)\; Q_{nm}\; (k^2-k_o^2)} - \frac{\vec{L}_j(k,\vec{r})\vec{L}_j(k,\vec{r}\,')}{k_o^2\; Q_{nm}} \right] \quad (21)$$

Another representation for $\bar{G}_e$ is obtained from the simplified solution for $\vec{E}$, which is equivalent to evaluating the integral over k in (21) on a term by term basis, and is

$$\bar{G}_e = \frac{-j\pi}{2k_o} \sum_j \frac{\vec{M}_{j\sigma}(k_o,\vec{r})\vec{M}_{j\sigma}(k_o,\vec{r}\,') + \vec{N}_{j\sigma}(k_o,\vec{r})\vec{N}_{j\sigma}(k_o,\vec{r}\,')}{n(n+1)\; Q_{nm}} - \frac{\vec{a}_r\vec{a}_r}{k_o^2}\,\delta(\vec{r}-\vec{r}\,') \quad (22)$$

where the notation

$$\begin{aligned} \vec{M}_{j\sigma}(k_o,\vec{r}) &= \vec{M}_j(k_o,\vec{r}) , & r < r' \\ &= \vec{M}_j^{+}(k_o,\vec{r}) , & r > r' \\ \vec{M}_{j\sigma}(k_o,\vec{r}\,') &= \vec{M}_j^{+}(k_o,\vec{r}\,'), & r' > r \\ &= \vec{M}_j(k_o,\vec{r}\,') , & r' < r \end{aligned}$$

and similarly for $\vec{N}_{j\sigma}$, has been introduced.

The use of (21) and (22) in (20) to find $\vec{E}$ will be defined to mean term by term integration over $\vec{r}\,'$ so as to yield the same results as given by (12) or (19). The connection between the two representations (21) and (22) is closely related to two fundamental methods for solving a Green's function problem and is discussed in Appendix I.

A number of interesting properties of the Green's dyadic as given by (21) can be deduced. The Green's dyadic is the solution of the equation

$$(\nabla \times \nabla \times - k_o^2)\ \bar{G}_e = \bar{I}\ \delta(\vec{r}-\vec{r}') \tag{23}$$

where $\bar{I}$ is the unit dyadic. The scalar Green's function $g_o = \frac{1}{4\pi R}$ which is a solution of Poisson's equation

$$\nabla^2 g_o = -\delta(\vec{r}-\vec{r}') \tag{24}$$

can be expanded in an eigenfunction series and it is then found that

$$\int_0^\infty dk\ \frac{\vec{L}_j\ \vec{L}_j}{k_o^2\ Q_{nm}} = \frac{1}{k_o^2} \nabla\ \nabla'\ g_o \tag{25}$$

The unit source $\bar{I}\ \delta(\vec{r}-\vec{r}')$ can be expanded in the form

$$\bar{I}\ \delta(\vec{r}-\vec{r}') = \int_0^\infty dk \sum_j \left[\frac{\vec{M}_j\vec{M}_j + \vec{N}_j\vec{N}_j}{n(n+1)Q_{nm}} + \frac{\vec{L}_j\vec{L}_j}{Q_{nm}} \right] \tag{26}$$

Since

$$\int_{V_1} \bar{I}\ \delta(\vec{r}-\vec{r}') \cdot \vec{J}(\vec{r}')d\vec{r}' = \vec{J}(\vec{r}) \tag{27}$$

the use of (26) is another way of splitting a vector field into its solenoidal and irrotational parts. We may use (26) to eliminate the $\vec{L}_j\vec{L}_j$ terms in (21). Thus another representation for $\bar{G}_e$ is

$$\bar{G}_e = \int_0^\infty dk \sum_j \frac{k^2\ (\vec{M}_j\vec{M}_j + \vec{N}_j\vec{N}_j)}{k_o^2(k^2-k_o^2)\ n(n+1)\ Q_{nm}} - \frac{1}{k_o^2}\ \bar{I}\ \delta(\vec{r}-\vec{r}') \tag{28}$$

In this form the expression for $\bar{G}_e$ represents

$$\bar{G}_e = \frac{1}{k_o^2} [\nabla \times \bar{G}_m - \bar{I}\ \delta(\vec{r}-\vec{r}')\ ] \tag{29}$$

Professor C. T. Tai used this form and carried out the integration over k to obtain the representation given by (22) above.[5] In this evaluation there is no pole at $k=0$ because of the k^2 factor in the numerator.

One final representation for $\bar{G}_e$ will be given but there are many other possible forms obtained by carrying out partial summations of the series. We may regroup the terms in (21) into the following arrangement:

$$\bar{G}_e = \int_0^\infty dk \sum_j \left[\frac{\vec{M}_j\vec{M}_j + \vec{N}_j\vec{N}_j + n(n+1)\vec{L}_j\vec{L}_j}{n(n+1)\ Q_{nm}\ (k^2-k_o^2)} - \frac{k^2}{k^2-k_o^2}\ \vec{L}_j\vec{L}_j \right] \qquad (30)$$

In this form the last group of $\vec{L}_j\vec{L}_j$ terms represents $-\nabla\nabla'\, e^{-jk_oR}/(4\pi k_o^2 R)$ while the first group of terms is the eigenfunction expansion of $\bar{I}\, e^{-jk_oR}/4\pi R$. Consequently (30) is the eigenfunction expansion of the usual free space Green's dyadic

$$\bar{G}_e = (\bar{I} - \frac{1}{k_o^2}\ \nabla\nabla')\ \frac{e^{-jk_oR}}{4\pi R} \qquad (31)$$

For the type of problem being discussed here the representation (22), first derived by Professor Tai, appears to be the most compact and useful form. It yields directly the spherical wave expansion formulas given by Wood for use in parabolic reflector analysis.[6]

At points $\vec{r} \neq \vec{r}'$ the series expansion for $\bar{G}_e$ converges to a finite value given in closed form by (31). At the singular point $\vec{r} = \vec{r}'$ all of the series diverge. A partial summation, such as integration over k, allows one to represent part of the Green's dyadic as an explicit delta function but the remaining series are still divergent at the singular point. Some of these series diverge more rapidly than the series representation for the delta function so the delta function singularity is not the dominant one. The divergent character of the series expansion of $\bar{G}_e$ is of little consequence in practice since $\bar{G}_e$ is used in a term by term integration over $\vec{r}'$ involving the scalar product $\bar{G}_e \cdot \vec{J}(\vec{r}')$ and the resultant series is a convergent one.

Waveguides and Cavities

For waveguide problems the natural way to represent the electric field is in terms of propagating and evanescent waves. For cavities an expansion of the field in terms of the resonant modes of the cavity is the preferred representation. Any other method is both more cumbersome and lacking in physical insight.

The most convenient representation for $\bar{G}_e$ in a waveguide is a form analogous to (22) for free space. The series part is a discontinuous residue series made up from the TE and TM normal waveguide modes. The remaining part is the residual contribution from the $\vec{L}_j$ functions in the source region and has the form $-\vec{a}_z\vec{a}_z\ \delta(\vec{r}-\vec{r}')/k_o^2$ where z is along the axis of the waveguide.[7,8,9] Many other representations are possible with various delta function terms exhibited in explicit form but the above is often the most useful one in practice (there are exceptions, such as occurs for a bifurcation problem where an alternative form is more useful.)

The electric field in a cavity and also the electric field Green's dyadic can be expanded in terms of the resonant eigenmodes of the cavity. For a simply connected cavity with a single surface the solenoidal and irrotational modes are described by the following equations.[10]

$$(\nabla^2 + k_n^2)\, \vec{E}_n = 0 \tag{32a}$$

$$\nabla \cdot \vec{E}_n = 0 \tag{32b}$$

$$\vec{n} \times \vec{E}_n = 0 \quad \text{on } s \tag{32c}$$

$$\ell_n \vec{F}_n = \nabla \psi_n \tag{33a}$$

$$(\nabla^2 + \ell_n^2)\, \psi_n = 0 \tag{33b}$$

$$\nabla \times \vec{F}_n = 0 \tag{33c}$$

$$\psi_n = 0 \quad \text{on } s \tag{33d}$$

The solenoidal modes $\vec{E}_n$ and irrotational modes $\vec{F}_n$ are mutually orthogonal. If we assume that the modes are normalized then we readily find that

$$\bar{G}_e = \sum_n \left[\frac{\vec{E}_n(\vec{r})\vec{E}_n(\vec{r}\,')}{k_n^2 - k_o^2} - \frac{\vec{F}_n(\vec{r})\vec{F}_n(\vec{r}\,')}{k_o^2} \right] \tag{34}$$

$$\bar{I}\, \delta(\vec{r}-\vec{r}\,') = \sum_n (\vec{E}_n\vec{E}_n + \vec{F}_n\vec{F}_n) \tag{35}$$

These expressions are analogous to (21) and (26) for the free space problem. For a cavity the most important feature of (34) is that it shows that whenever k_o is close to one of the resonant wave numbers k_n that mode is excited with a very large amplitude relative to that of the other modes. In this circumstance the field in the cavity can be approximated by this one single mode alone. This provides a great simplification. In practice the k_n, which are obtained for the loss free cavity, must be replaced by[10]

$$\frac{k_n}{[1 + Q_n^{-1} - j\, Q_n^{-1}]^{1/2}} \approx \left(1 - \frac{1}{2Q_n} + \frac{j}{2Q_n}\right) k_n$$

where Q_n is the Q of the n'th mode. For a microwave cavity Q_n is typically greater than 1000 so the amplitude of the n'th mode when $k_o = k_n$ may be 1000 times or more than that of any other excited mode.

$\bar{G}_e$ for a cavity may be represented in many other forms as well. If we use (35) to eliminate the $\vec{F}_n$ functions we find that

$$\bar{G}_e = \sum_n \frac{k_n^2 \vec{E}_n \vec{E}_n}{k_o^2 (k_n^2 - k_o^2)} - \frac{\bar{I}\,\delta(\vec{r}-\vec{r}\,')}{k_o^2} \tag{36}$$

which is analogous to (28) for the free space problem. This particular representation was discussed by Rahmat-Samii.[11] The sum in (34) and (36) is a triple sum. If the series is summed over one set of indices the result is a discontinuous double series plus a reduced delta function term. For example, in a rectangular cavity if the modal series representing the z dependence is summed the delta function term left over is $-\vec{a}_z \vec{a}_z \quad \delta(r-r')/k_o^2$. This is the form presented by Tai and Rozenfeld[12] and is identical to that given by Rahmat-Samii even though the form is different.† Tai and Rozenfeld also give the solution in the form (34). A technique for carrying out the modal sum is described in Appendix I. The solution presented by Tai and Rozenfeld is analogous to that given in (22) for the free space problem. The solution presented by Rahmat-Samii has the useful feature that it shows explicitly the resonance factor $k_n^2 - k_o^2$ which can be readily modified to account for the finite Q of the mode. In the solution given by Tai and Rozenfeld the resonance manifests itself by the vanishing of a Wronskian determinant and hence some additional algebra is required to extract the solution at a resonance.

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†Johnson T. H. Wang has shown that the two solutions are identical for a rectangular cavity. The author is grateful to Dr. Wang for sending a preprint of his paper in which this equivalence is demonstrated.

In the limit as k_o approaches k_n the n'th term in Tai and Rozenfeld's solution becomes continuous and passes over to the n'th eigenfunction or cavity mode of Rahmat-Samii's solution. This feature is also discussed in Appendix I.

For a cavity the Green's dyadic can be expressed in forms analogous to (21) or (30) and the $\vec{F}_n\vec{F}_n$ contributions can be identified with the expansion of $\nabla\nabla' g_o$ or $\nabla\nabla' g$ where g_o and g are scalar Green's functions and satisfy

$$\nabla^2 g_o = -\delta(\vec{r}-\vec{r}')$$

$$g_o = 0 \text{ on s}$$

$$(\nabla^2 + k_o^2)g = -\delta(\vec{r}-\vec{r}')$$

$$g = 0 \text{ on S}$$

in the cavity. Thus there is a great similarity as regards the basic composition of the free space and cavity Green's dyadic. A similar correspondence holds for the waveguide problems also.

Principal Volume Integration Method

If the solution for $\vec{E}$ as given by (20) is interpreted to mean the limit of an improper integral, i.e.

$$\lim_{V_o \to 0} -j\omega\mu_o \int_{V-V_o} \bar{G}_e \cdot \vec{J}(\vec{r}')d\vec{r}' \qquad (37)$$

where V_o is a small volume excluding the singular point at $\vec{r}' = \vec{r}$, it does not give the correct electric field inside the source region. The reason is that $\bar{G}_e$ contains a point singularity at $\vec{r}' = \vec{r}$ which is excluded. Van Bladel showed that if V_o was a spherical volume then $\vec{E}(\vec{r})$ is given by (37) plus an additional term $j\omega\mu_o \vec{J}(\vec{r})/3k_o^2$ which is non-zero only within the source region.[13] Yaghjian generalized this result to allow V_o to be an arbitrary shaped volume and obtained a correction term of the form $j\omega\mu_o \bar{L} \cdot \vec{J}/k_o^2$ where $\bar{L}$ depends on the shape of V_o, thus[14]

$$\vec{E}(\vec{r}) = -j\omega\mu_o \lim_{V_o \to 0} \int_{V-V_o} \bar{G}_e \cdot \vec{J} d\vec{r}' + j\omega\mu_o \frac{\bar{L} \cdot \vec{J}(\vec{r})}{k_o^2} \quad (38)$$

The volume integral converges non-uniformly and this is why $\bar{L}$ depends on the manner in which the limit is taken.† Yaghjian also showed that $\bar{L}$ is the classical depolarization factor associated with a volume V_o.

The motivation for Yaghjian's analysis was to cast (20) into a form that could be evaluated on a computer by keeping V_o small but finite. This computational procedure is useful for free space problems using the explicit closed form solution for $\bar{G}_e$ given by (31). However, when the eigenfunction expansion of $\bar{G}_e$ is used one does not want to use (38) since it implies that the eigenfunction series should be summed first. But this is contrary to the purpose of using the eigenfunction expansion in the first place which was to develop the solution for $\vec{E}(\vec{r})$ as a series of modes. In this latter case we can integrate each term in the expansion of $\bar{G}_e$ over the whole volume which is a much simpler procedure and has no computational difficulties associated with it.

If the $\nabla'\nabla' e^{-jk_o R}/4\pi R$ is evaluated explicitly in the vicinity of the point $\vec{r} = \vec{r}'$ and this expression is used in a volume integral that

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†Any of the eigenfunction expansions for $\bar{G}_e$ given earlier may be used in (38) to find $\vec{E}(\vec{r})$. However, the limit must be handled with care because of the non-uniform convergence.

includes the singular point different results are obtained depending on how the integration is done. This procedure however, is not a correct one since it does not include the contribution from the $\nabla'\nabla'$ term at the singular point itself.

The above feature along with the fact that $\vec{E}(\vec{r})$ is not given by the limit of the improper integral in (37) has led some author's to conclude that the integral

$$\int_V \bar{G}_e \cdot \vec{J}(\vec{r}\,')d\vec{r}\,' \tag{39}$$

has no meaning. The desire that $\vec{E}(\vec{r})$ should be given by the limit of an improper integral comes from classical potential theory where such limits for the scalar and vector potentials are appropriate. However, since $\bar{G}_e$ is the solution to a differential equation for a source $\bar{I}\ \ \delta(\vec{r}-\vec{r}\,')$ which has no place in a classical theory there is no a priori reason to expect or even require that $\vec{E}(\vec{r})$ be given by the limit of an improper integral, which of course it is not. This then leaves the interpretation of the meaning of (39) open. One approach is to define (39) in an operational sense. For example, when $\bar{G}_e$ is given as an eigenfunction expansion we can define (39) to mean term by term integration. This is a valid operational definition since it gives the correct expansion for $\vec{E}(\vec{r})$. We note that $\vec{E}(\vec{r})$ as given by (38) is also an operational method since it carries the rule that $\bar{L}$ is changed in accordance with the shape of V_o, and such a rule does not belong to the classical definition of the limit of the improper integral.

Another definition for (39) is that the integral of the $[\nabla'\nabla' e^{-jk_o R}/4\pi R] \cdot \vec{J}(\vec{r}')$ term is to be done by parts, which avoids the need to know the value of the derivatives at the singular point itself. This definition is a natural one when $\bar{G}_e$ is viewed as a distribution.† Distribution theory provides a mathematical framework within which entities such as delta functions and singular quantities such as $\bar{G}_e$ can be handled in accordance with specified rules in a non-ambiguous way.[15] Even though distribution theory provides a meaning for the integral in (39) the actual numerical evaluation still requires some regularization procedure if integration by parts is not done. A suitable regularization method has been given by Lee, Boersma, Law, and Deschamps and generalizes the formula (38) obtained by Yaghjian in that it allows V_o to be of any size without any error being incurred.[16] Although (38) and its more general version as given by Lee et al are interesting and useful in their own right many author's still prefer to integrate the $\nabla'\nabla'$ term by parts to obtain an expression that can then be evaluated numerically on a computer.

A well known example that illustrates some of the above comments is the function $\nabla^2 \frac{1}{4\pi R}$. Since the solution to Poisson's equation

$$\nabla^2 \Phi = -\delta(\vec{r}-\vec{r}')$$

is the function $1/4\pi R$ we see that the differential equation requires that $\nabla^2 \frac{1}{4\pi R}$ equals $-\delta(\vec{r}-\vec{r}')$. In the vicinity of the point $\vec{r} = \vec{r}'$

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†An excellent but brief summary of distribution theory may be found in Appendix 6 of Reference 19.

it is easily verified that $\nabla^2 \frac{1}{4\pi R} = 0$. If this value is used in a volume integral over a region including the singular point the result is zero which is not in accord with the differential equation. For this case we must interpret $\nabla^2 \frac{1}{4\pi R}$ as having the value $-\delta(\vec{r}-\vec{r}\,')$ at the singular point. We also note that

$$\lim_{V_o \to 0} \int_{V-V_o} \nabla^2 \frac{1}{4\pi R}\, d\vec{r} = 0$$

where V_o excludes the singular point. The interpretation of $\nabla^2 \frac{1}{4\pi R}$ as being equal to $-\delta(\vec{r}-\vec{r}\,')$ is outside the domain of classical analysis. The operational equivalence is established by integration by parts, e.g.

$$\lim_{V_o \to 0} \int_{V_o} \rho(\vec{r})\, \nabla^2 \frac{1}{4\pi R}\, d\vec{r} = \lim_{V_o \to 0} \int_{V_o} [\nabla \cdot (\rho\, \nabla \frac{1}{4\pi R}) - \nabla\rho \cdot \nabla \frac{1}{4\pi R}]\, d\vec{r}$$

$$= \lim_{V_o \to 0} \oint_S \rho\, \nabla \frac{1}{4\pi R} \cdot d\vec{s} - \lim_{V_o \to 0} \int_{V_o} \nabla\rho \cdot \nabla \frac{1}{4\pi R}\, d\vec{r}$$

The surface integral gives $-\rho(\vec{r}\,')$ and the remaining volume integral vanishes with V_o as long as $\nabla\rho$ is bounded at $\vec{r}\,'$. In view of this operational equivalence it seems reasonable to assume that a function such as $\nabla\nabla \frac{1}{4\pi R}$ should be interpreted in a similar way through an integration by parts, thus avoiding the need to specify the value of the derivatives at the singular point itself.

A Brief Historical Review

In the early history of waveguides it was established that the normal modes were E and H modes. For many years it was generally thought that these comprised a complete set for the expansion of an arbitrary field in a waveguide. There are many examples of author s discussing the expansion of arbitrary fields in waveguides in terms of E and H modes only.[17,18,19,20] In these, as well as in other treatments, the problems of interest centered around scattering by metallic obstacles or the fields radiated by arbitrary sources outside the source region. The question of the field within the source region was not examined and it is only within the source region that the E and H mode expansion fails to be complete. Morse and Feshbach discuss a Green's dyadic for the vector Helmholtz equation in a rectangular waveguide and give one expansion that includes a contribution from longitudinal modes.[21] They do not treat the vector wave equation (23) correctly (Ref. 21, p. 1783, p. 1876) or discuss when the longitudinal modes might be needed. These same author's (Ref. 21, p. 1781) give expansions for the transverse and longitudinal parts of the dyadic source function $\bar{I}\delta(\vec{r}-\vec{r}')$ and imply that each part is zero whenever $\vec{r}\neq\vec{r}'$. This interpretation is incorrect and erroneously led to the conclusion that the longitudinal modes only contribute within the source region. The conclusion is not in agreement with Helmholtz's theorem and resulted in the incorrect treatment of the vector wave equation noted above. Johnson, Howard, and Dudley gave a discussion of this point in their paper on the Green's dyadic singularity.[22]

Stratton in his classic book indicates that the $\vec{L}$ and $\vec{N}$ functions are not orthogonal.[23] However, he does not integrate over r which, as Tai has shown, is necessary to achieve orthogonality. Morse and Feshbach do not discuss the orthogonality of the $\vec{L}$, $\vec{M}$, and $\vec{N}$ functions in spherical coordinates for the free space problem. They do, however, give an expansion for the Green's dyadic for the vector potential and state that for the electromagnetic case the longitudinal part will usually disappear (Ref. 21, p 1875). The lack of orthogonality between the $\vec{L}$ and $\vec{N}$ functions with respect to θ and ϕ integrations suggests that perhaps the former are not needed. We now know that the $\vec{N}$ functions cancel much, but not all, of the $\vec{L}$ function contribution to the field.

In view of the existing ambiguity as regards the need for the longitudinal modes it is not surprising that Tai developed expansions for the Green's dyadics in his book which included only the $\vec{M}$ and $\vec{N}$ functions.[2] It is interesting to note that in the evaluation of the k integral for the contribution from the $\vec{N}$ functions, Tai missed the pole contribution from $k = 0$ (Ref. 2, p 173). If this pole contribution had been identified Professor Tai would have been alerted to the need for the $\vec{L}$ functions.

After the publication of his book Professor Tai discovered the need for the $\vec{L}$ functions although he approached the problem from the relation $k_o^2 \bar{G}_e = \nabla \times \bar{G}_m - \bar{I}\,\delta(\vec{r}-\vec{r}\,')$ and thus avoided having to deal with the $\vec{L}$ functions explicitly.[5,7] The author also discovered independently the lack of incompleteness for the E and H modes in a waveguide at about the same time.[9] However, a much earlier (but unknown to the above authors) contribution to the field expansion in waveguides which was complete was given by H. J. Butterweck [24] (the author is indebted to Georg Karawas for bringing this reference to his attention).

The complete expansion of the field in a cavity underwent a similar period of controversy during its development. The results of various investigations over a period of years finally culminated in a definitive paper on the subject by Kurokawa[25] (this paper contains references to many of the earlier contributions). In retrospect it is surprising that the results of the investigations on the eigenfunction expansions for cavities did not influence the work on expansion of fields in waveguides and the general theory of Green's dyadics until much later.

APPENDIX I

Some Mathematical Properties of Green's Functions

The basic equation encountered in the solution of boundary value problems is the Sturm-Liouville equation

$$\frac{d}{dx} p(x) \frac{d\, g(x)}{dx} + [q(x) + \lambda\, \sigma(x)]g(x) = - \delta(x-x') \tag{1}$$

Let the interval of interest be $0 \leq x \leq a$. For simplicity we will assume that $g(0) = g(a) = 0$. There are two fundamental ways of solving (1). The normalized eigenfunctions ψ_n associated with (1) satisfy

$$\frac{d}{dx}\, p\, \frac{d\psi_n}{dx} + (q + \sigma\, \lambda_n)\psi_n = 0 \tag{2a}$$

$$\int_0^a \psi_n\, \psi_m\, \sigma\, dx = \delta_{nm} = \begin{cases} 1, n=m \\ 0, n\neq m \end{cases} \tag{2b}$$

$$\psi_n(0) = \psi_n(a) = 0 \tag{2c}$$

In terms of these

$$g(x,x') = - \sum_n \frac{\psi_n(x)\, \psi_n(x')}{\lambda-\lambda_n} \tag{3}$$

The second way to solve for g is to let

$$g = C_1\, \Phi_1\, (x) , \qquad x \leq x'$$

$$g = C_2\, \Phi_2\, (x) , \qquad x \geq x'$$

where Φ_1 satisfies the boundary conditions at x = 0, Φ_2 satisfies the boundary conditions at x = a. The constants C_1 and C_2 are found such that g is continuous at x = x' and

$$p(x) \left. \frac{dg}{dx} \right|_{x'_-}^{x'_+} = -1$$

as obtained by integrating (1) over a vanishingly small interval centered on x'. It is found that

$$g = - \frac{\Phi_1(x_<)\, \Phi_2(x_>)}{p(x')W(x')} \tag{4}$$

where W is the Wronskian determinant

$$W = \Phi_1(x') \frac{d\Phi_2(x')}{dx'} - \Phi_2(x') \frac{d\Phi_1(x')}{dx'} \tag{5}$$

It is easy to show that p W equals a constant. The two solutions Φ_1 and Φ_2 are linearly independent whenever $W \neq 0$.

The character of the Green's function is determined by its spectrum. The spectrum is the set of discrete poles $\lambda = \lambda_n$ exhibited in (3). The solution given by (4) has the same spectrum since W has simple zeroes whenever $\lambda = \lambda_n$. The sum of the series in (3) is given by (4). As λ approaches λ_n, Φ_1 and Φ_2 become linearly dependent and in the limit as λ approaches λ_n the function g given by (4) becomes $-\psi_n(x)\,\psi_n(x')/(\lambda-\lambda_n)$ i.e., both solutions have the same residue at the pole $\lambda = \lambda_n$.

The above properties may be used to develop various representations of multi-dimensional Green's functions, to perform the sum over a set of eigenfunctions, and to demonstrate the equivalence between various representations of Green's functions.

It is known that multi-dimensional Green's functions can be synthesized from products of related one dimensional Green's functions for each coordinate variable by means of contour integration over the spectra.[26,27,28,29] This synthesis procedure leads automatically to various alternative representations.

As a concrete example consider the radial Green's function problem for a spherical coordinate system, namely:

$$\frac{d}{dr}\ r^2\ \frac{dg_n}{dr}\ - n(n+1)g_n + k_o^2 r^2 g_n\ = -\delta(r-r') \tag{6}$$

By using Hankel transforms the eigenfunction expansion method gives

$$g_n = \frac{2}{\pi}\int_0^{\infty} \frac{j_n(kr)\ j_n(kr')}{k^2-k_o^2}\, k^2 dk\ =\ \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{j_n(kr)\ j_n(kr')}{2k_o(k-k_o)}\, k^2 dk \tag{7}$$

In (7) k corresponds to λ_n and k_o corresponds to λ so this equation is analoguous to (3). The closed form solution analogous to (4) is

$$g_n\ =\ -jk_o j_n(k_o r_<)\ h_n^2(k_o r_>) \tag{8}$$

In (8) the Wronskian determinant was

$$j_n(k_o r')\ \frac{dh_n^2(k_o r')}{dr'}\ -\ h_n^2(k_o r')\ \frac{dj_n(k_o r')}{dr'}\ =\ \frac{1}{jk_o r'^2} \tag{9}$$

The solution (8) may be found from (7) by contour integration. When $k_o = 0$ we get the zero frequency Green's function given by

$$g_n\ =\ \frac{2}{\pi}\int_0^{\infty} j_n(kr) j_n(kr') dk \tag{10a}$$

and

$$g_n = \frac{r_<^n}{(2n+1) r_>^{n+1}} \tag{10b}$$

Many of the tedious manipulations used to construct different representations of Green's functions can often be greatly simplified by using the above results obtained from the general theory. In particular we note that the differential equation (6) can be written as

$$\frac{d}{dr} r^2 \frac{dg_n}{dr} = [n(n+1) - k_o^2 r^2] g_n - \delta(r-r') \tag{11}$$

so clearly a second derivative of g_n generates a delta function.

The usefulness of the above relations in summing series will be illustrated in connection with the spherical cavity problem. For the $\vec{M}$ functions the radial functions are $j_n(kr)$ and the boundary conditions at r = a are $j_n(ka) = 0$. This determines a set of eigenvalues k_{ni}, i = 1,2,3,..... . The normalized eigenfunctions for (6) for this problem are

$$\psi_n = \frac{j_n(k_{ni} r)}{\left[\frac{a^3}{2}\right]^{1/2} j_n'(k_{ni} a)} \tag{12}$$

One solution for g_n is thus

$$g_n = - \sum_{i=1}^{\infty} \frac{2 j_n(k_{ni} r_<)\, j_n(k_{ni} r_>)}{a^3 [j_n'(k_{ni} a)]^2 \, (k_o^2 - k_{ni}^2)} \tag{13}$$

The closed form solution is

$$g_n = - k_o \frac{j_n(k_o r_<) [j_n(k_o a) y_n(k_o r_>) - y_n(k_o a) j_n(k_o r_>)]}{j_n(k_o a)} \tag{14}$$

Note that the denominator equals zero when $k_o = k_{ni}$. Equation (14) is the sum of the series in (13). We can demonstrate the equivalence by finding the residue of (14) at $k_o = k_{ni}$. As k_o approaches k_{ni} the denominator becomes (Taylor series expansion with respect to k_o^2 about k_{ni}^2)

$$j_n(\sqrt{k_o^2}\,a) = j_n(k_{ni}a) + \frac{a}{2k_{ni}} j_n'(k_{ni}a)\,(k_o^2 - k_{ni}^2) + ---$$

Hence (14) approaches the value

$$g_n \rightarrow \frac{j_n(k_{ni}r_<)\,j_n(k_{ni}r_>)\;2k_{ni}^2\;Y_n(k_{ni}a)}{a\,j_n'(k_{ni}a)\,(k_o^2 - k_{ni}^2)}$$

By using the Wronskian relationship

$$j_nY_n' - Y_nj_n' = \frac{1}{k_o^2r^2}$$

which for $k_o = k_{ni}$, $r = a$, gives $Y_n = -(k_{ni}^2a^2j_n')^{-1}$ we see that

$$g_n \rightarrow -\frac{2j_n(k_{ni}r_<)\,j_n(k_{ni}r_>)}{a^3(j_n')^2(k_o^2 - k_{ni}^2)} \tag{15}$$

which is the same as the ith term in (13).

The same procedure may be applied in connection with the $\vec{N}$ and $\vec{L}$ functions in a spherical cavity. For the $\vec{N}$ functions the boundary conditions are $d\,rj_n(kr)/dr = 0$ at $r = a$ while for the $\vec{L}$ functions $j_n(ka) = 0$.

In the paper by Tai and Rozenfeld[12] it was indicated that the sum over the index i could not be done so as to express $\bar{G}_e$ for a spherical cavity in terms of a discontinuous series plus a delta function term. As shown above the general theory of Green's functions does give a method for summing the series to obtain a closed form solution involving a function with a discontinuous derivative at $r = r'$.

APPENDIX II

A three dimensional Fourier transform of the equation (23) for $\bar{G}_e$ gives

$$(-\vec{k} \times \vec{k} \times -k_o^2 \bar{I}) \cdot \hat{\bar{G}}_e = [(k^2-k_o^2) \bar{I} - \vec{k}\,\vec{k}] \cdot \hat{\bar{G}}_e = \bar{I}\, e^{j\vec{k}\cdot\vec{r}\,'} \qquad (1)$$

where $\hat{\bar{G}}_e$ is the Fourier transform of $\bar{G}_e$.

We now assume that the inverse operator is

$$A\,\vec{k}\,\vec{k} + B\,\bar{I} \text{ and use } (A\,\vec{k}\,\vec{k} + B\,\bar{I}) \cdot [(k^2-k_o^2)\bar{I} - \vec{k}\,\vec{k}] = \bar{I}$$

to find A and B. We then obtain

$$\hat{\bar{G}}_e = \left[\frac{\bar{I}}{k^2-k_o^2} - \frac{\vec{k}\,\vec{k}}{k_o^2(k^2-k_o^2)} \right] e^{j\vec{k}\cdot\vec{r}\,'} \qquad (2)$$

We now group the terms into transverse and longitudinal components relative to $\vec{k}$, thus

$$\hat{\bar{G}}_e = \left[\frac{k^2\,\bar{I}-\vec{k}\,\vec{k}}{(k^2-k_o^2)k^2} - \frac{\vec{k}\,\vec{k}}{k^2 k_o^2} \right] e^{j\vec{k}\cdot\vec{r}\,'} \qquad (3)$$

The first part is transverse and has zero divergence while the second part is longitudinal and has zero curl. This is the origin of the terminology transverse and longitudinal used to describe solenoidal and irrotational vector fields. It is not difficult to show that the inverse of (3) gives the free space solution (31).

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